Numerical Optimization 2024 - Homework 2

Aral Cimcim, k11720457

Artificial Intelligence, JKU Linz

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Problem 1.

(a)

$$x_k = 1 \,/\, k$$

$$x_{k+1} = 1 \,/\, (k+1)$$

$$x^* = 0$$

$$l_k = \frac{\parallel x_{k+1} - x^* \parallel}{\parallel x_k - x^* \parallel}$$

$$l_k = \frac{\parallel 1 \,/\, (k+1) - 0 \parallel}{\parallel 1 \,/\, k - 0 \parallel}$$

$$l_k = 1 - \frac{1}{k+1}$$

$$\lim_{k \to \infty} l_k = 1, \text{ sublinear}$$

(b)

$$x_k = 1 \,/\, k!$$

$$\lim_{k \to \infty} x_k = 0$$

$$l_k = \frac{\parallel 1 \,/\, (k+1)! - 0 \parallel}{\parallel 1 \,/\, k! - 0 \parallel}$$

$$l_k = \frac{k!}{(k+1)!}$$

$$l_k = \frac{1}{k+1}$$

$$\lim_{k\to\infty} l_k = 0 \quad \text{superlinear}$$

$$\text{Check for quadratic fractions:} \quad \frac{1}{(k+1)^2}$$

$$\lim_{k\to\infty} \frac{1}{(k+1)^2} = 0$$

(c)

$$x_k = a^k \text{ for some } a \in (0,1)$$

$$x^* = 0$$

$$l_k = \frac{\| a^{k+1} - 0 \|}{\| a^k - 0 \|} = \frac{a^{k+1}}{a^k} = a, \text{ converges linearly}$$

(d)

$$x_k = 1 + (0.5)^{2^k}$$

$$x^* = 1$$

$$l_k = \frac{\| 1 + (0.5)^{2^{k+1}} - 1 \|}{\| 1 + (0.5)^{2^k} - 1 \|} = \frac{(0.5)^{2^{k+1}}}{(0.5)^{2^k}} = (0.5)^{2^k}, \text{ superlinear}$$

$$\text{Check for quadratic fractions:} \quad \frac{\| 1 + (0.5)^{2^{k+1}} - 1 \|}{\| 1 + (0.5)^{2^k} - 1 \|^2} = \frac{(0.5)^{2^{k+1}}}{((0.5)^{2^k})^2} = 1, \text{ quadratic}$$

Problem 2.

$$x_{k+1} = x_k - \gamma(4x_k^3 - 20x_k) \text{ for } \gamma = 1/88 \text{ converges to } x^* = \sqrt{5}$$

Convergence of x_{k+1} is shown in Figure: 1, the l_k values for 100 iterations are in Figure: 2 and the corresponding plot is in Figure: 3. The values indicate linear convergence approaching to the constant 0.55 after 10 iterations.

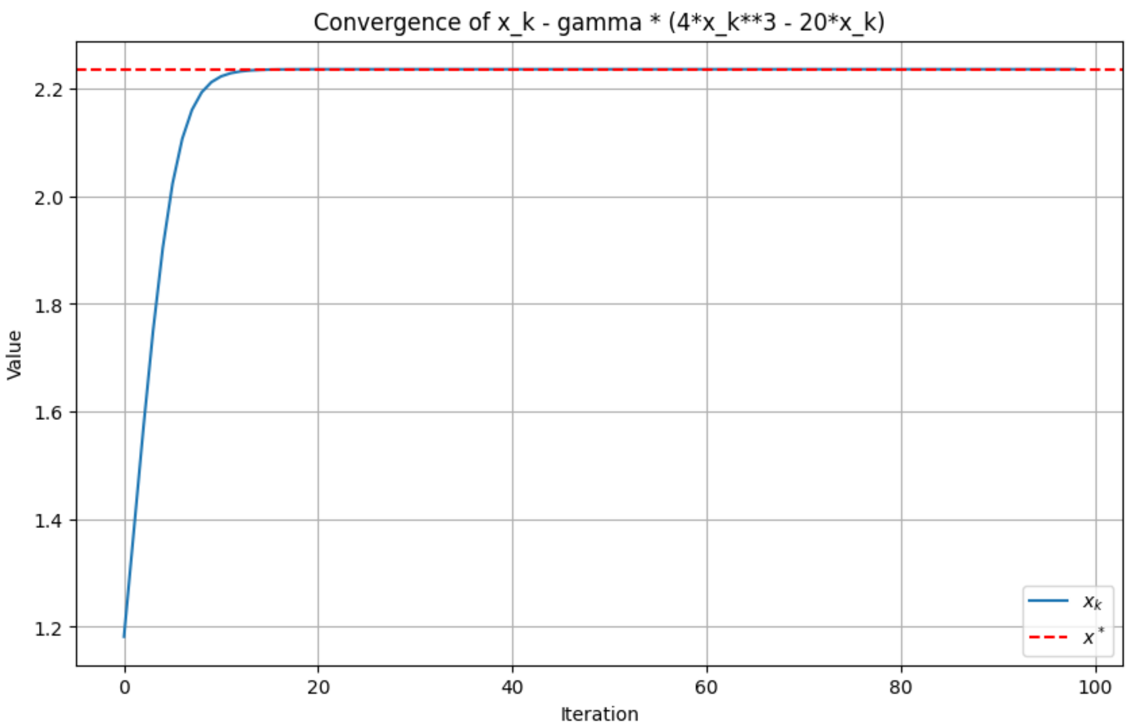

Figure 1: Convergence of $x_k - \gamma(4x_k^3 - 20x_k)$

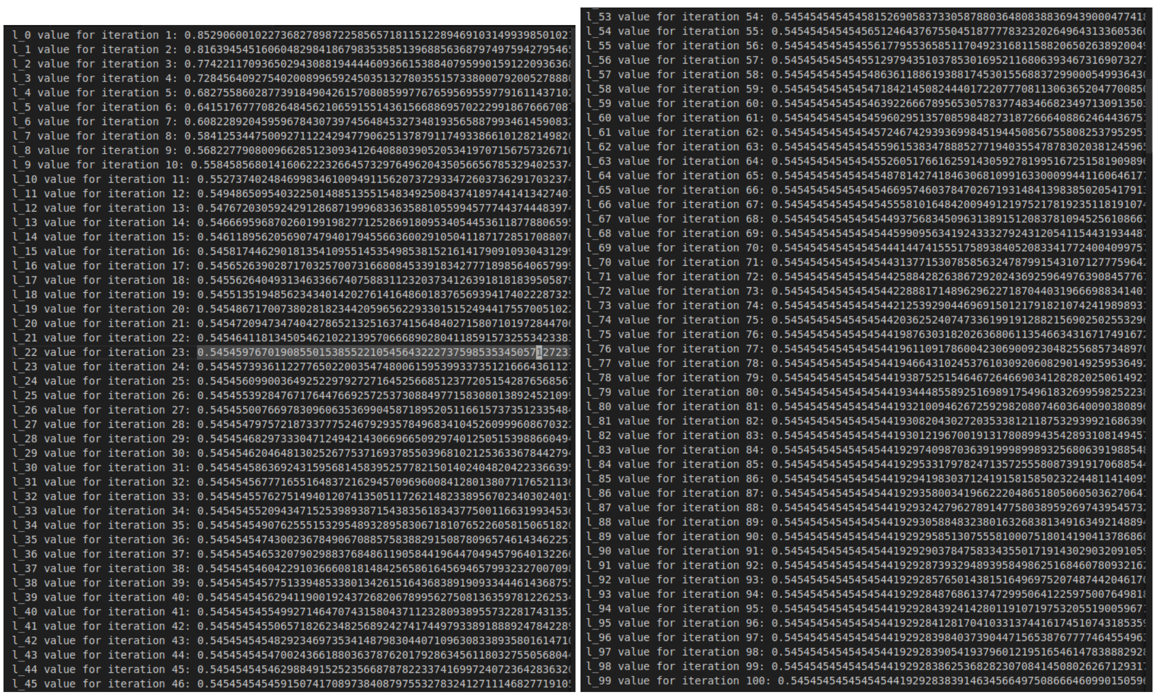

Figure 2: 100 iterations for l_k

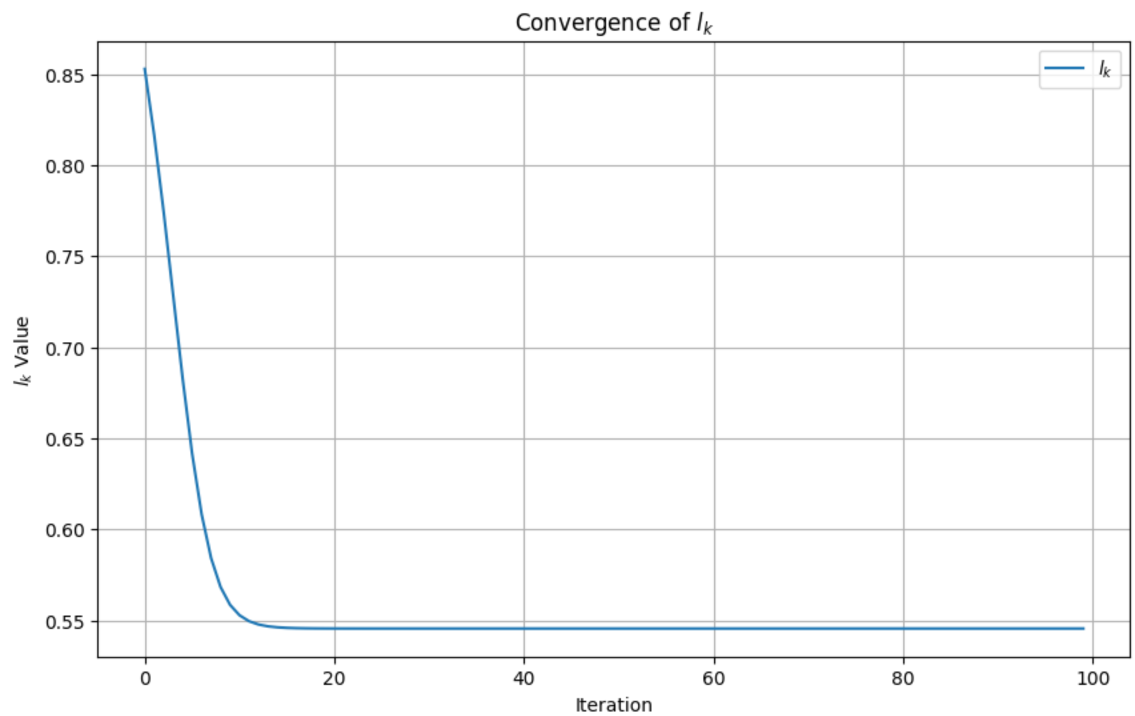

Figure 3: Convergence of l_k

Problem 3.

$$x_{k+1} = x_k + \gamma \cos(x_k) \text{ for any } \gamma > 0 \text{ either converges to } x^* = \pi/2 \text{ or does not converge}$$

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Figure: 4 shows 100 iterations for $\gamma = 0.5$

I plotted the graph (Figure: 5) for the first 10 iterations where it is easier to see that l_k linearly converges to 0.5.

Figure: 6 shows 100 iterations for $\gamma = 3$ and Figure: 7 shows the convergence of l_k for $\gamma = 3$ reaching to 1.

Figure: 8 shows 100 iterations for $\gamma = 2$ and Figure: 9 shows the convergence of l_k for $\gamma = 2$ reaching to 1 slower than $\gamma = 3$.

Figure: 10 shows 100 iterations for $\gamma = 1$ and Figure: 11 shows the convergence of l_k for $\gamma = 1$ reaching to 1 faster than $\gamma = 3$ and $\gamma = 2$.

For gamma values 3, 2 and 1 l_k converges sublinearly.

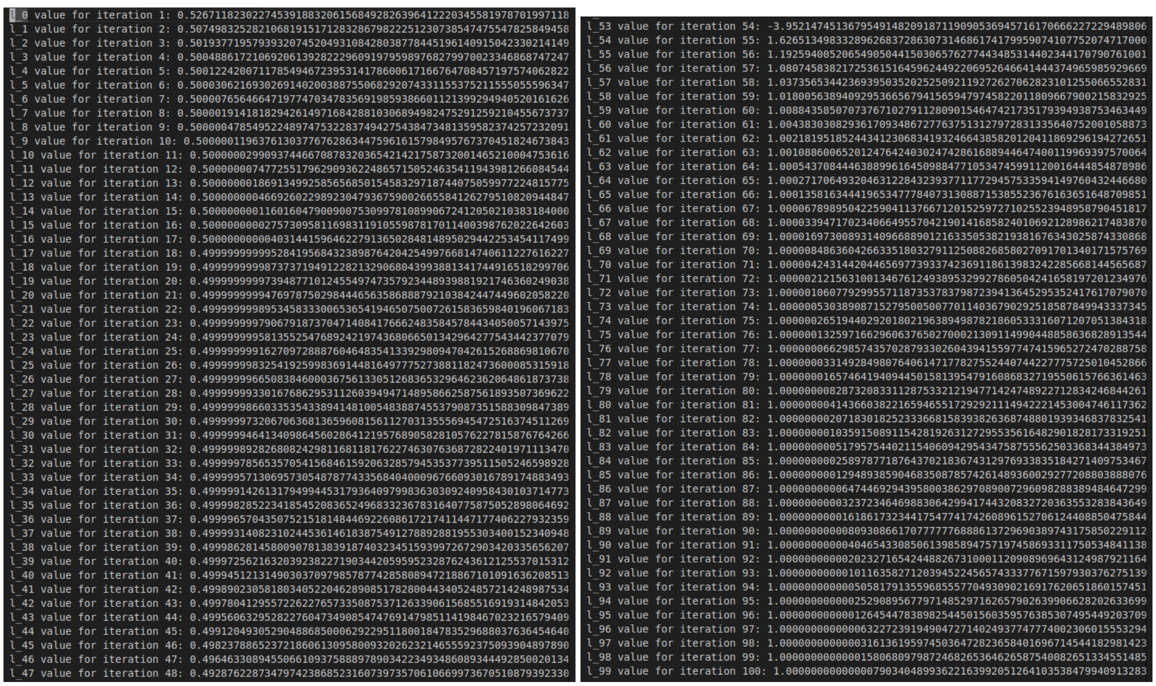

Figure 4: 100 iterations for $\gamma = 1/2$

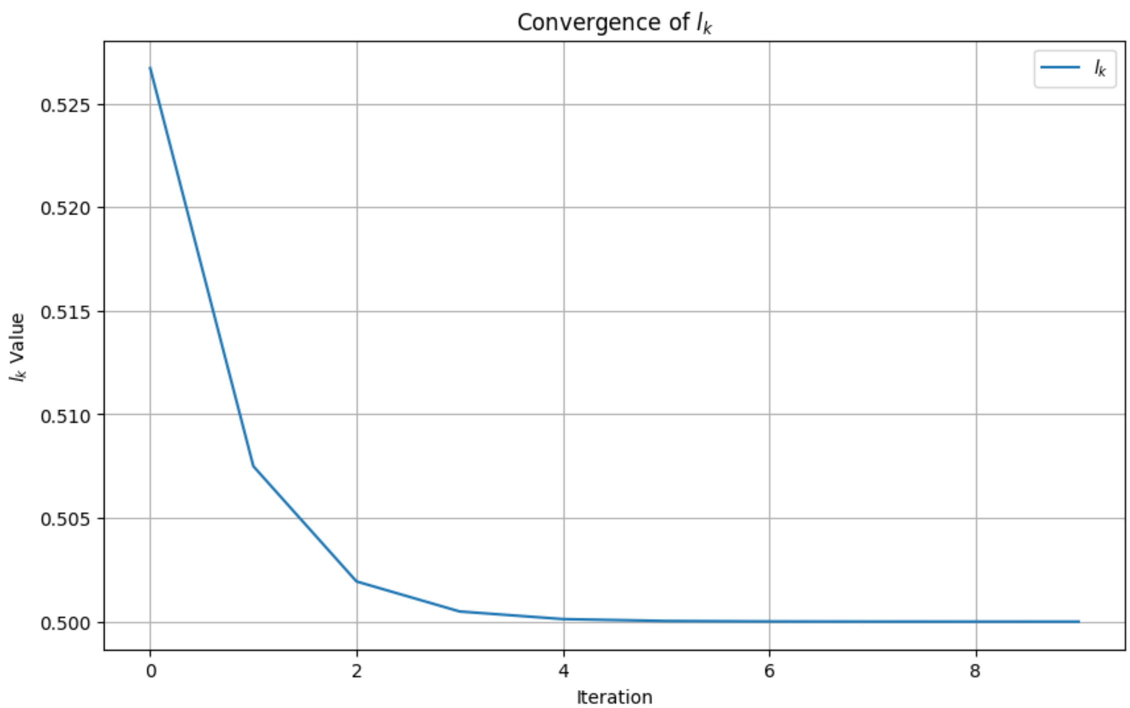

Figure 5: Convergence of l_k for $\gamma = 1/2$

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l_0 value for iteration 1: 1.83972906186352764870076305890430416215266779265081277
l_1 value for iteration 2: 1.47824825689754777733362371464282097043430782892241012
l_2 value for iteration 3: 0.93225634079314193675157072351152870360518686169982419
l_3 value for iteration 4: 1.05719718441762887723978606902781659463894667941154026
l_4 value for iteration 5: 0.95922757730905848420594552057340333809530725642963403
l_5 value for iteration 6: 1.03332757718724061627772096884648179046771858951262488
l_6 value for iteration 7: 0.97536154921727013755964873927987012780728378166270558
l_7 value for iteration 8: 1.01973323783935996988439907109012476284161623659070278
l_8 value for iteration 9: 0.98511985500732167927770269308225732530405988182342193
l_9 value for iteration 10: 1.0117671578720953920417748417484137679673092147346273
l_10 value for iteration 11: 0.9910259560354549582910187610944811468021403922782 95
l_11 value for iteration 12: 1.0070412302698460628249139269376084239386611515212 25
l_12 value for iteration 13: 0.9945945454300478685459966698533628322198820366147 73
l_13 value for iteration 14: 1.0042209710328263557981904947234738820374047618422 93
l_14 value for iteration 15: 0.9967469445658959230587078514753687404853724086670 84
l_15 value for iteration 16: 1.0025328432567427379150868455540752707536512664742 55
l_16 value for iteration 17: 0.9980434239505129732595134425379582432768593995328 40
l_17 value for iteration 18: 1.0015207208237216144473005344457912644058992385028 84
l_18 value for iteration 19: 0.9988236377179912936241108623885094345909168504613 39
l_19 value for iteration 20: 1.0009133408415412737020909378762504165522718929159 57
l_20 value for iteration 21: 0.9992928920123902592839466307949678861046684996927 65
l_21 value for iteration 22: 1.0005486556614743528096312083675385508262135399865 37
l_22 value for iteration 23: 0.9995750191221359695705423185328128226673321687978 35
l_23 value for iteration 24: 1.0003296221753789110980790964585591259854708851541 64
l_24 value for iteration 25: 0.9997446028879783698978331780063626589366801204735 55
l_25 value for iteration 26: 1.0001980443803104205288286870980071527158594660346 14
l_26 value for iteration 27: 0.9998465241077713127325676164049202102974793304508 47
l_27 value for iteration 28: 1.0001189943367920463271565559027584512623585534364 61
l_28 value for iteration 29: 0.9999077745388968143854230904477017254990590198553 59
l_29 value for iteration 30: 1.0000714991118377768927152264916620891840866203587 35
l_30 value for iteration 31: 0.9999445816790804293344539701436751746612711072104 94
l_31 value for iteration 32: 1.0000429616893225781143394748717780285475367929867 86
l_32 value for iteration 33: 0.9999666994817000583406221269391341177774734472899 61
l_33 value for iteration 34: 1.0000258146275963221888320660780096832221648545658 98
l_34 value for iteration 35: 0.9999799900650333540298490689942050460526577737995 16
l_35 value for iteration 36: 1.0000155114596267349941103822678080759066967619077 35
l_36 value for iteration 37: 0.9999879762873031656089902884789868964066165905781 23
l_37 value for iteration 38: 1.0000093205350487499640061690198241406897424779857 49
l_38 value for iteration 39: 0.9999927751233110749851114463050008694165370042639 77
l_39 value for iteration 40: 1.0000056005392699075928616832408030978866630118678 05
l_40 value for iteration 41: 0.9999956586815072635760916897129664743523701682158 75
l_41 value for iteration 42: 1.0000033652655969590112975345205003621908166222662 90
l_42 value for iteration 43: 0.9999973913700698811562861533449647243611187562346 82
l_43 value for iteration 44: 1.0000020221303280458301076405489782411044496552962 05
l_44 value for iteration 45: 0.9999984325161805341427032143014638102661491693768 02
l_45 value for iteration 46: 1.0000012150639027065552839809148952793065134073725 85
l_46 value for iteration 47: 0.9999990581244543237620226111826521788800408455061 23
l_47 value for iteration 48: 1.0000007301115229833733114531865805059322815106762 49
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l_53 value for iteration 54: 1.0000001584016674592980188064211306530266415679512 42
l_54 value for iteration 55: 0.9999998772123244696801598488818987141040880082770 15
l_55 value for iteration 56: 1.0000000951809364922362365613723922163003339695550 33
l_56 value for iteration 57: 0.9999999262189145989554822586637044052690990153856 51
l_57 value for iteration 58: 1.0000000571926482254200365692759401707896554387647 18
l_58 value for iteration 59: 0.9999999556661655501035054018209645338642247485155 56
l_59 value for iteration 60: 1.0000000343661154060287199393554879579027188851521 96
l_60 value for iteration 61: 0.9999999733605320711266103379226641153186526626685 82
l_61 value for iteration 62: 1.0000000206500298950429307547071383507206352115025 03
l_62 value for iteration 63: 0.9999999839927843845315770520072294277016217701437 10
l_63 value for iteration 64: 1.0000000124082611891276951296900079808499496149201 61
l_64 value for iteration 65: 0.9999999903815289682073937119402907386195798813570 75
l_65 value for iteration 66: 1.0000000074559187875486856441510633624856845964706 70
l_66 value for iteration 67: 0.9999999942204199142666959700151075394501632412453 62
l_67 value for iteration 68: 1.0000000044801381951830758452187343528662289360457 99
l_68 value for iteration 69: 0.9999999965271459785884969788589124102611886778005 52
l_69 value for iteration 70: 1.0000000026920408793288175710244200035147193087554 50
l_70 value for iteration 71: 0.9999999979132195088209020862188242164745917865899 07
l_71 value for iteration 72: 1.0000000016176027894224838854477814302748572763005 85
l_72 value for iteration 73: 0.9999999987460881613878284380482811399083199367196 08
l_73 value for iteration 74: 1.0000000009719907190835543783392209202177646902343 03
l_74 value for iteration 75: 0.9999999992465451550653305109767143086657535048246 35
l_75 value for iteration 76: 1.0000000005840531082494352993118850900899382356761 39
l_76 value for iteration 77: 0.9999999995472614824793576304875329955219352810590 25
l_77 value for iteration 78: 1.0000000003509478166020314270954058152026932473505 59
l_78 value for iteration 79: 0.9999999997279569482108451792345633452233550790864 59
l_79 value for iteration 80: 1.0000000002108787037167202607233774900588219434260 84
l_80 value for iteration 81: 0.9999999998365338639103357625599112080113578669491 99
l_81 value for iteration 82: 1.0000000001267134913010268858472895981998922060011 56
l_82 value for iteration 83: 0.9999999999017759361120011591613800456974539193652 26
l_83 value for iteration 84: 1.0000000000761400077146298940994065832264239049082 20
l_84 value for iteration 85: 0.9999999999409788178435194804520000898333844312773 45
l_85 value for iteration 86: 1.0000000000457512381890810031500392689913449552910 73
l_86 value for iteration 87: 0.9999999999645351808489069670252402591732256119677 30
l_87 value for iteration 88: 1.0000000000274911293768757749752766166343256424593 31
l_88 value for iteration 89: 0.9999999999786898096292898604648446126796945389274 72
l_89 value for iteration 90: 1.0000000000165189321130731778568930133213352888728 70
l_90 value for iteration 91: 0.9999999999871950918692647621058516153838421503577 38
l_91 value for iteration 92: 1.0000000000099259203169765149182842981807741218430 75
l_92 value for iteration 93: 0.9999999999923057752438359145845551132584625508799 84
l_93 value for iteration 94: 1.0000000000059642886128961532187859544179559622121 56
l_94 value for iteration 95: 0.9999999999953767003252222636192612690992572707150 09
l_95 value for iteration 96: 1.0000000000035838096687939538578535708499487122250 31
l_96 value for iteration 97: 0.9999999999997221968396883111504588733254986167216 598
l_97 value for iteration 98: 1.0000000000021534192573156617720051861072795337573 03
l_98 value for iteration 99: 0.9999999999983307594504078146036520577853329973111 17
l_99 value for iteration 100: 1.000000000001293921325472752975163031521411141306 96
```

Figure 6: 100 iterations for $\gamma = 3$

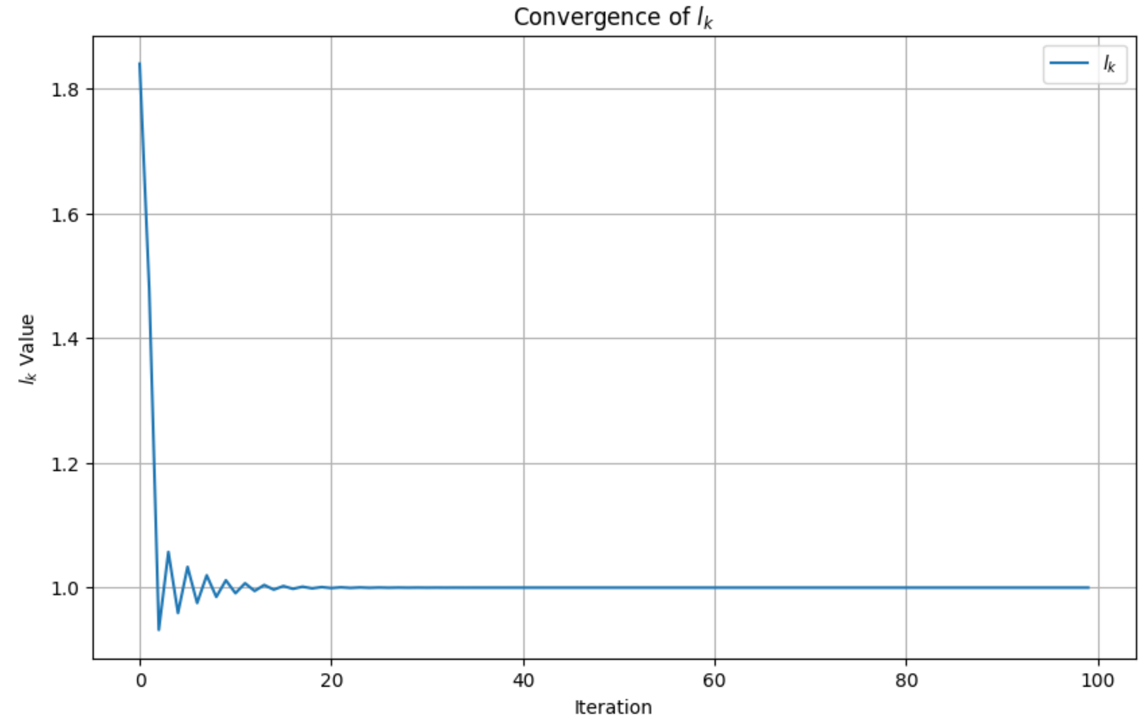


Figure 7: Convergence of l_k for $\gamma = 3$

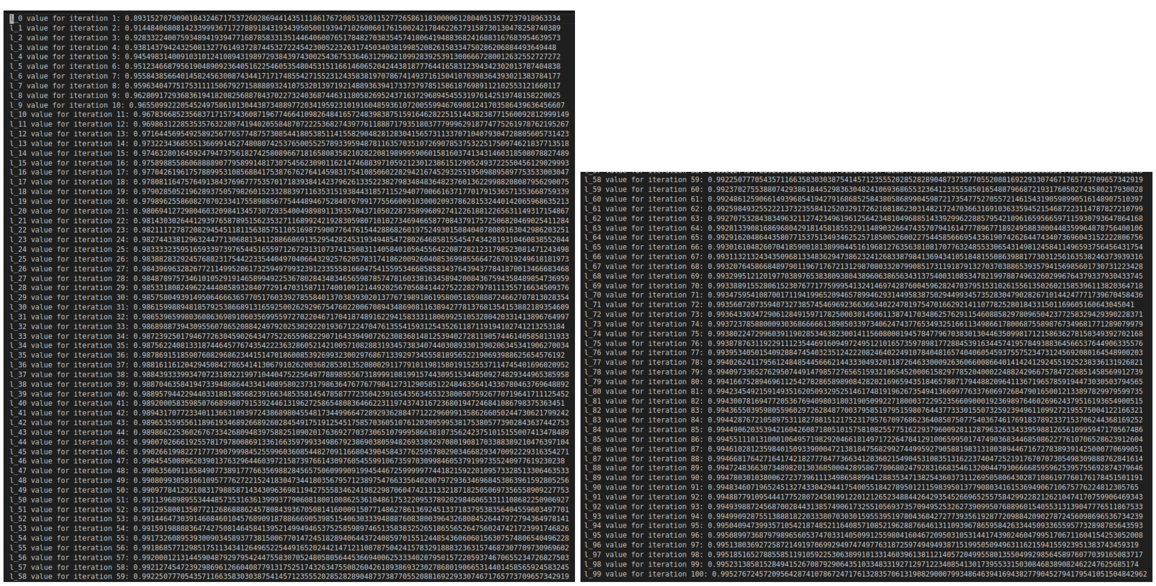


Figure 8: 100 iterations for $\gamma = 2$

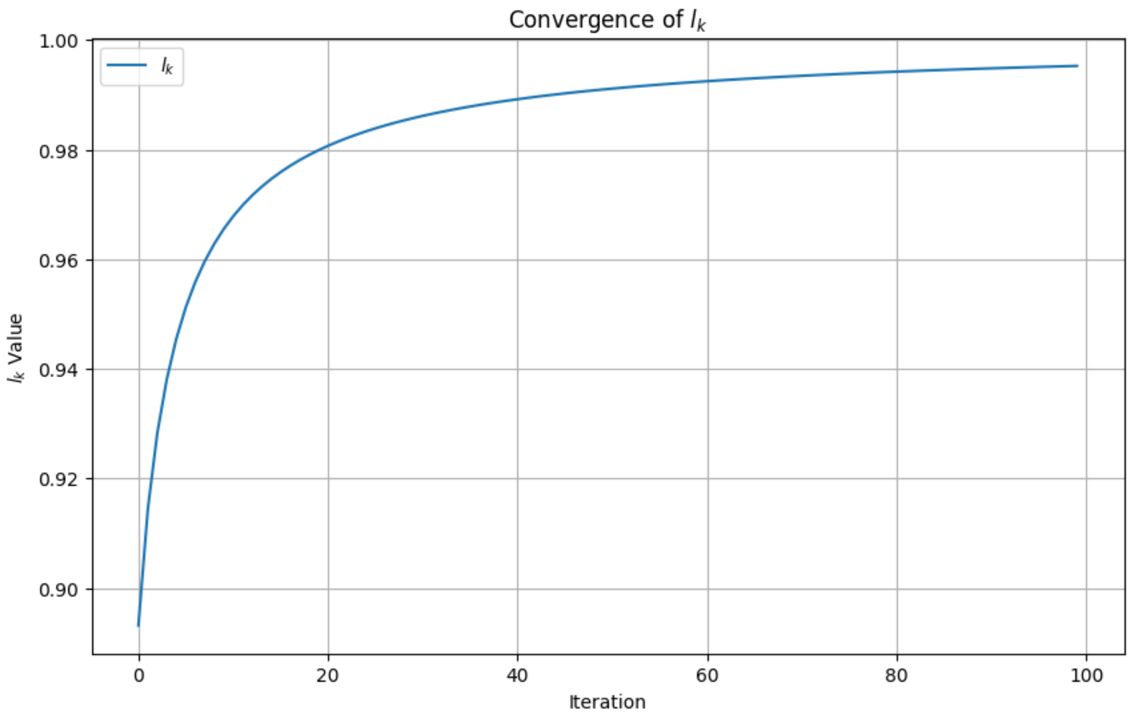
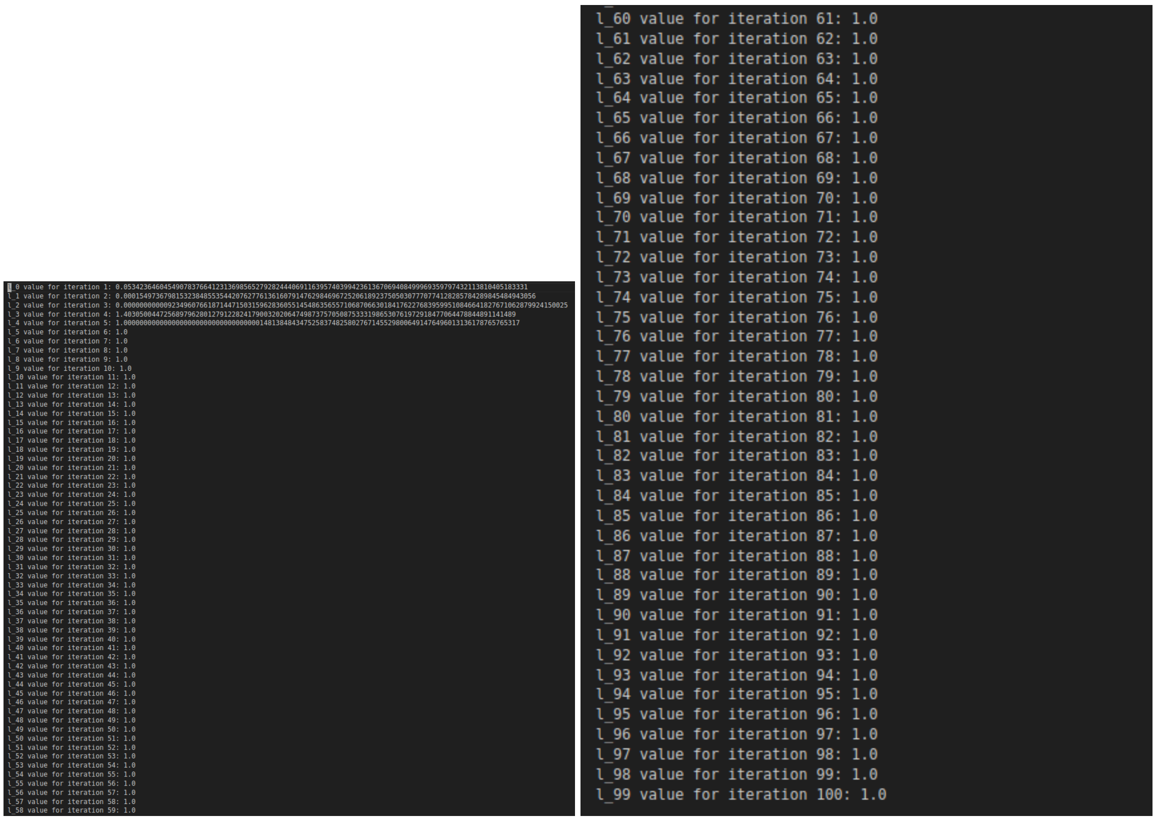


Figure 9: Convergence of l_k for $\gamma = 2$



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l_0 value for iteration 1: 0.053423646045490783766412313698565279282444069116395740399423613670694084999693597974321138104051 83331
l_1 value for iteration 2: 0.000154973679815323848553544207627761361607914762984696725206189237505030777077412828578428984548494 3056
l_2 value for iteration 3: 0.00000000000923496076618714471503159628360551454863565571068706630184176227683959951084664182767106287 9924150025
l_3 value for iteration 4: 1.403050044725689796280127912282417900320206474987375705087533319865307619729184770644788448911 41489
l_4 value for iteration 5: 1.000000000000000000000000000000001481384843475258374825802767145529800649147649601313617876576 5317
l_5 value for iteration 6: 1.0
l_6 value for iteration 7: 1.0
l_7 value for iteration 8: 1.0
l_8 value for iteration 9: 1.0
l_9 value for iteration 10: 1.0
l_10 value for iteration 11: 1.0
l_11 value for iteration 12: 1.0
l_12 value for iteration 13: 1.0
l_13 value for iteration 14: 1.0
l_14 value for iteration 15: 1.0
l_15 value for iteration 16: 1.0
l_16 value for iteration 17: 1.0
l_17 value for iteration 18: 1.0
l_18 value for iteration 19: 1.0
l_19 value for iteration 20: 1.0
l_20 value for iteration 21: 1.0
l_21 value for iteration 22: 1.0
l_22 value for iteration 23: 1.0
l_23 value for iteration 24: 1.0
l_24 value for iteration 25: 1.0
l_25 value for iteration 26: 1.0
l_26 value for iteration 27: 1.0
l_27 value for iteration 28: 1.0
l_28 value for iteration 29: 1.0
l_29 value for iteration 30: 1.0
l_30 value for iteration 31: 1.0
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l_51 value for iteration 52: 1.0
l_52 value for iteration 53: 1.0
l_53 value for iteration 54: 1.0
l_54 value for iteration 55: 1.0
l_55 value for iteration 56: 1.0
l_56 value for iteration 57: 1.0
l_57 value for iteration 58: 1.0
l_58 value for iteration 59: 1.0
```

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l_60 value for iteration 61: 1.0
l_61 value for iteration 62: 1.0
l_62 value for iteration 63: 1.0
l_63 value for iteration 64: 1.0
l_64 value for iteration 65: 1.0
l_65 value for iteration 66: 1.0
l_66 value for iteration 67: 1.0
l_67 value for iteration 68: 1.0
l_68 value for iteration 69: 1.0
l_69 value for iteration 70: 1.0
l_70 value for iteration 71: 1.0
l_71 value for iteration 72: 1.0
l_72 value for iteration 73: 1.0
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l_84 value for iteration 85: 1.0
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l_93 value for iteration 94: 1.0
l_94 value for iteration 95: 1.0
l_95 value for iteration 96: 1.0
l_96 value for iteration 97: 1.0
l_97 value for iteration 98: 1.0
l_98 value for iteration 99: 1.0
l_99 value for iteration 100: 1.0
```

Figure 10: 100 iterations for $\gamma = 1$

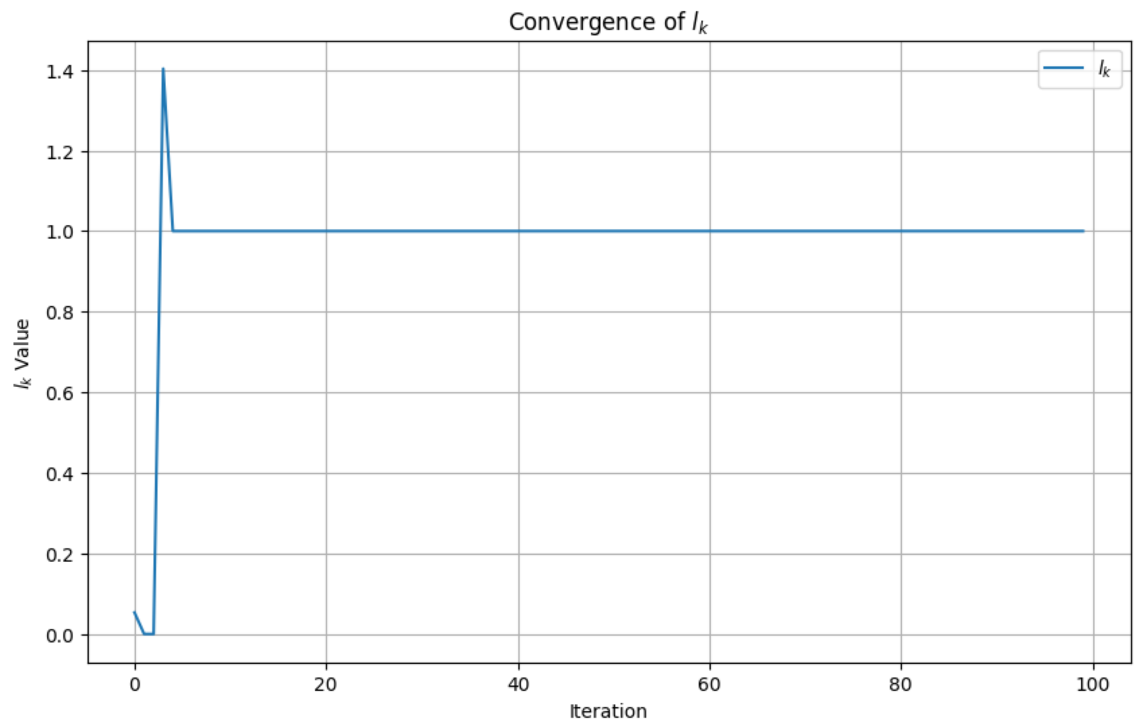

Figure 11: Convergence of l_k for $\gamma = 1$