Numerical Optimization 2024 - Homework 7

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Problem 1.

Verify that neither the LICQ nor the MFCQ holds for the constraint set defined by

$$c_1(x) = 1 - x_1^2 - (x_2 - 1)^2,$$

$$c_2(x) = -x_2 \geq 0 \text{ at the point } x^* = (0, 0)^T$$

Solution:

Gradients of the constraints:

$$\nabla c_1(x) = \begin{bmatrix} -2x_1 \\ -2(x_2 - 1) \end{bmatrix},$$

$$\nabla c_2(x) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

At point $x^* = (0, 0)^T$:

$$\nabla c_1(x^*) = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\nabla c_2(x^*) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Checking if LICQ is satisfied:

The 2 vectors are not linearly independent (multiply with 0.5 and get the other vector), therefore LICQ is not satisfied.

Checking if MFCQ is satisfied:

For direction $w = (1, 1)$ and $\nabla c_i(x^*)^T w > 0$:

$$\nabla c_1(x^*)^T w = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 2 > 0,$$

$$\nabla c_2(x^*)^T w = \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -1 < 0$$

therefore MFCQ is not satisfied.

Problem 2.

Show that for the feasible region defined by

$$(x_1 - 1)^2 + (x_2 - 1)^2 \leq 2,$$
$$(x_1 - 1)^2 + (x_2 + 1)^2 \leq 2,$$
$$x_1 \geq 0$$

the MFCQ is satisfied at $x^* = (0,0)^T$ but the LICQ is not satisfied.

Solution:
Rearranging the inequalities, we get:

$$(x_1 - 1)^2 + (x_2 - 1)^2 - 2 \leq 0,$$
$$(x_1 - 1)^2 + (x_2 + 1)^2 - 2 \leq 0,$$
$$x_1 \geq 0$$

Multiplying both sides with -1 in the first two inequalities, we get:

$$-(x_1 - 1)^2 - (x_2 - 1)^2 + 2 \geq 0,$$
$$-(x_1 - 1)^2 - (x_2 + 1)^2 + 2 \geq 0$$

The constraints after rearranging:

$$
\begin{aligned}
c_1(x) &= -(x_1-1)^2-(x_2-1)^2+2 \geq 0, \\
c_2(x) &= -(x_1-1)^2-(x_2+1)^2+2 \geq 0, \\
c_3(x) &= x_1 \geq 0
\end{aligned}
$$

Finding the gradients of the constraints:

$$
\begin{aligned}
\nabla c_1(x) &= \begin{bmatrix} -2(x_1-1) \\ -2(x_2-1) \end{bmatrix}, \\
\nabla c_2(x) &= \begin{bmatrix} -2(x_1-1) \\ -2(x_2+1) \end{bmatrix}, \\
\nabla c_3(x) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

The gradients at $x^* = (0,0)^T$:

$$
\begin{aligned}
\nabla c_1(x^*) &= \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \\
\nabla c_2(x^*) &= \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \\
\nabla c_3(x^*) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
$$

We see that we have 3 active constraints.

Checking if LICQ is satisfied:

Since the dimension of the problem is 2, which is less than the number of active constraints, the set of equality constraint gradients is not linearly independent, LICQ is not satisfied.

Checking if MFCQ is satisfied:

For direction $w = (2,1)$ and $\nabla c_i(x^*)^T w > 0$:

$$
\begin{aligned}
\nabla c_1(x^*)^T w &= \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 6 > 0, \\
\nabla c_2(x^*)^T w &= \begin{bmatrix} 2 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 > 0, \\
\nabla c_3(x^*)^T w &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2 > 0
\end{aligned}
$$

therefore MFCQ is satisfied.

Problem 3.

Prove that when the KKT conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0,$$
$$c_i(x^*) = 0, \text{ for all i} \in \mathcal{E},$$
$$c_i(x^*) \geq 0, \text{ for all i} \in \mathcal{I},$$
$$\lambda_i^* \geq 0, \text{ for all i} \in \mathcal{I},$$
$$\lambda_i^* c_i(x^*) = 0, \text{ for all i} \in \mathcal{E} \cup \mathcal{I}$$

and the LICQ are satisfied at x^*, then the Lagrange multiplier λ^* in the KKT conditions is unique.

Solution:
Assume that there are two sets of Lagrange multipliers λ_1^* and λ_2^* that satisfy the KKT conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda_1^*) = 0 \text{ and } \nabla_x \mathcal{L}(x^*, \lambda_2^*) = 0,$$
$$\nabla_x \mathcal{L}(x^*, \lambda_2^*) - \nabla_x \mathcal{L}(x^*, \lambda_1^*) = 0 \text{ using } d \text{ can be rewritten as:}$$
$$\nabla f(x^*) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} (\lambda_2^* - \lambda_1^*) \nabla c_i(x^*) = 0,$$
$$\nabla f(x^*) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} d \nabla c_i(x^*) = 0$$

which implies that d is orthogonal to the gradients of the equality constraints as $d = \lambda_2^* - \lambda_1^*$.

For the inequality constraints, we have:

$$(\lambda_{2,i}^* - \lambda_{1,i}^*) c_i(x^*) \geq 0 \text{ which holds for } d_i c_i(x^*) = 0$$

Direction d cannot be orthogonal to all gradients and be a linear combination at the same time, which means that there can be only one vector that satisfies the KKT conditions.