Numerical Optimization 2024 - Homework 1

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Let $f(x) = x^4 - 10x^2 + 9$ for the minimization problem $\min_{x \in \mathbb{R}} f(x)$.

Problem 1.

(a)

$$y = x^2$$

$$y^2 - 10y + 9 = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{10 \pm \sqrt{100 - 36}}{2}$$

$$y = \frac{10 \pm \sqrt{64}}{2}$$

$$y = \frac{10 \pm 8}{2}$$

$$y = 9, x = \{3, -3\}$$

$$y = 1, x = \{1, -1\}$$

(b) and (c)

First derivative of $f(x)$:

$$f'(x) = 4x^3 - 20x = 0$$

$$4x^3 = 20x$$

$$x^2 = 5$$

$$x = \pm\sqrt{5}$$

$$x = \{0, \sqrt{5}, -\sqrt{5}\}$$

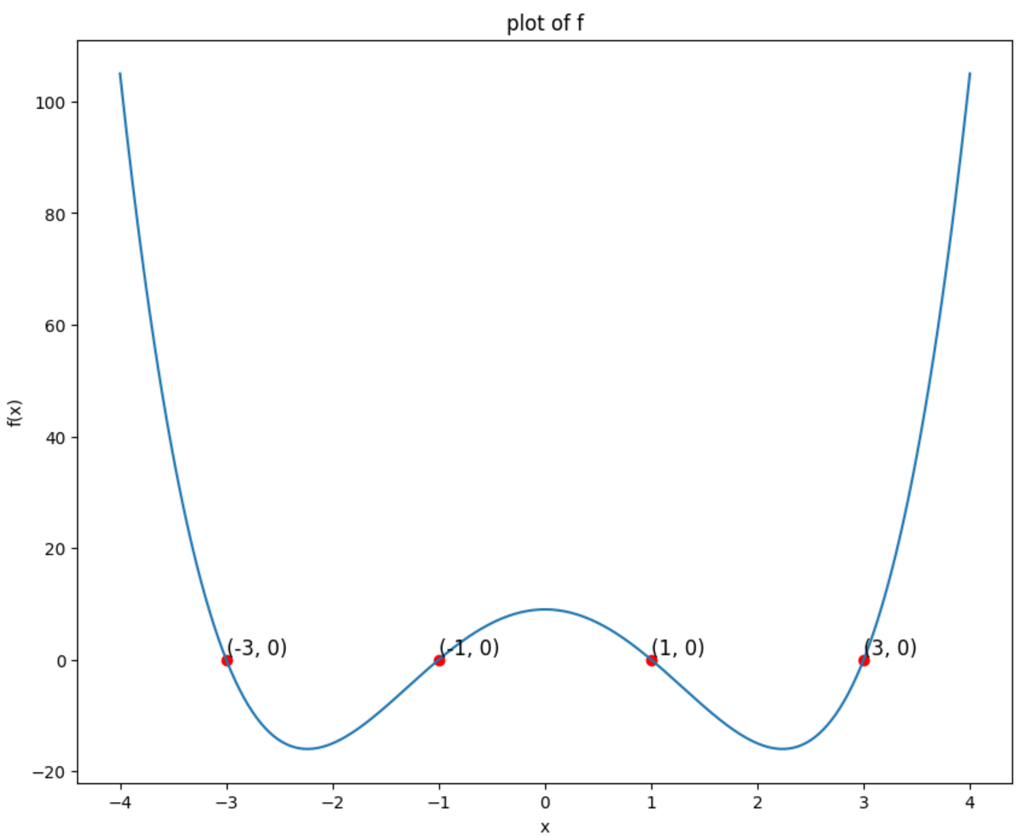

Figure 1: Graph of $f(x)$ and stationary points where $f'(x) = 0$

I expect 3 stationary points for local min and max.

(d)

Second derivative of $f(x)$:

$$f''(x) = 12x^2 - 20$$
$$f''(0) = -20$$
$$f''(\sqrt{5}) = 40$$
$$f''(-\sqrt{5}) = 40$$

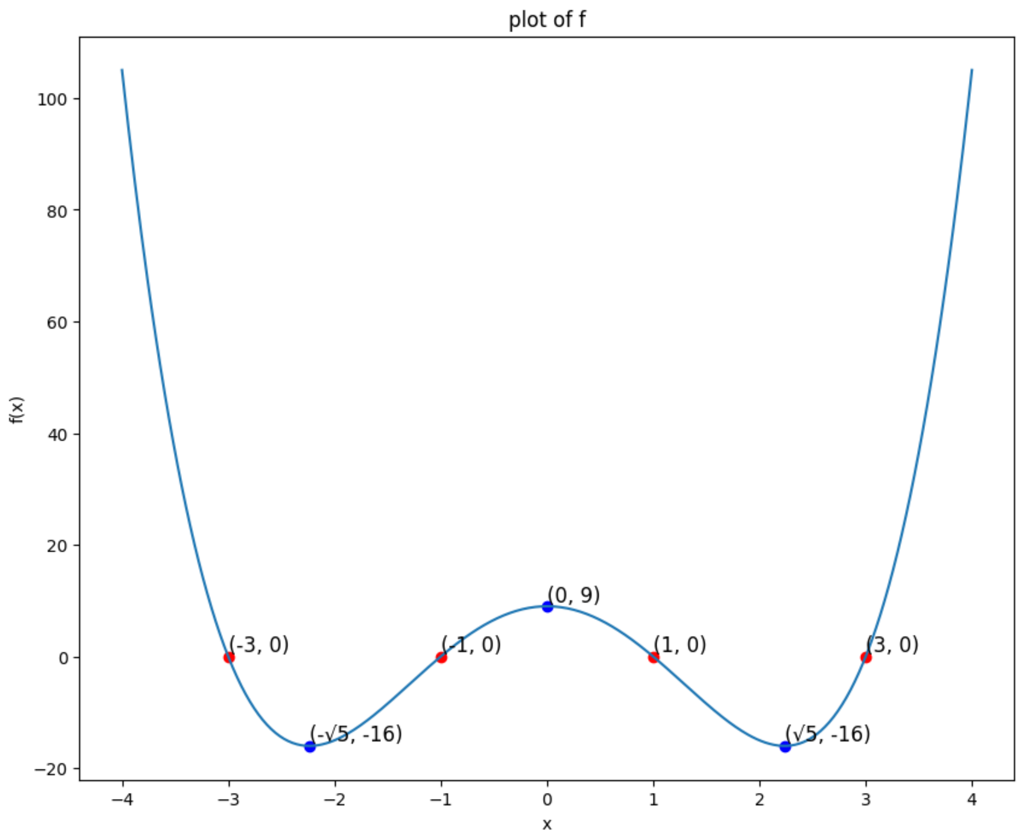

Figure 2: Graph of $f(x)$ with local minimizers and local maximizers where $f''(x) = 0$

Problem 2.

Let $x_{k+1} = x_k - f'(x_k)$

(a)

5 starting points: $\{-0.2, -0.001, 0, 0.0001, 0.02\}$

(b) and (c)

$$x_0 = -0.2$$
$$x_1 = -4.168$$
$$x_2 = 202.101718528$$
$$x_3 = -33015219.222220108$$
$$x_4 = 1.4394697653346336e + 23$$
$$x_5 = -1.1930746922429726e + 70$$

$$x_0 = -0.001$$
$$x_1 = -0.020999996$$
$$x_2 = -0.44096287202116796$$
$$x_3 = -8.917242469387194$$
$$x_4 = 2649.0349925834794$$
$$x_5 = -74357152805.09055$$

$$x_0 = 0$$
$$x_1 = 0$$
$$x_2 = 0$$
$$x_3 = 0$$
$$x_4 = 0$$
$$x_5 = 0$$

$$x_0 = 0.0001$$
$$x_1 = 0.002099999996$$
$$x_2 = 0.04409996287200021$$
$$x_3 = 0.9257561566944865$$
$$x_4 = 16.26729660331447$$
$$x_5 = -16877.31624241661$$

$$x_0 = 0.02$$

$$x_1 = 0.41996800000000006$$

$$x_2 = 8.523043732439174$$

$$x_3 = -2297.549209977365$$

$$x_4 = 48512541317.80819$$

$$x_5 = -4.566905953253297e + 32$$

No, it does not help. (graphs for (b) and (c) are Figure 3 and Figure 4)

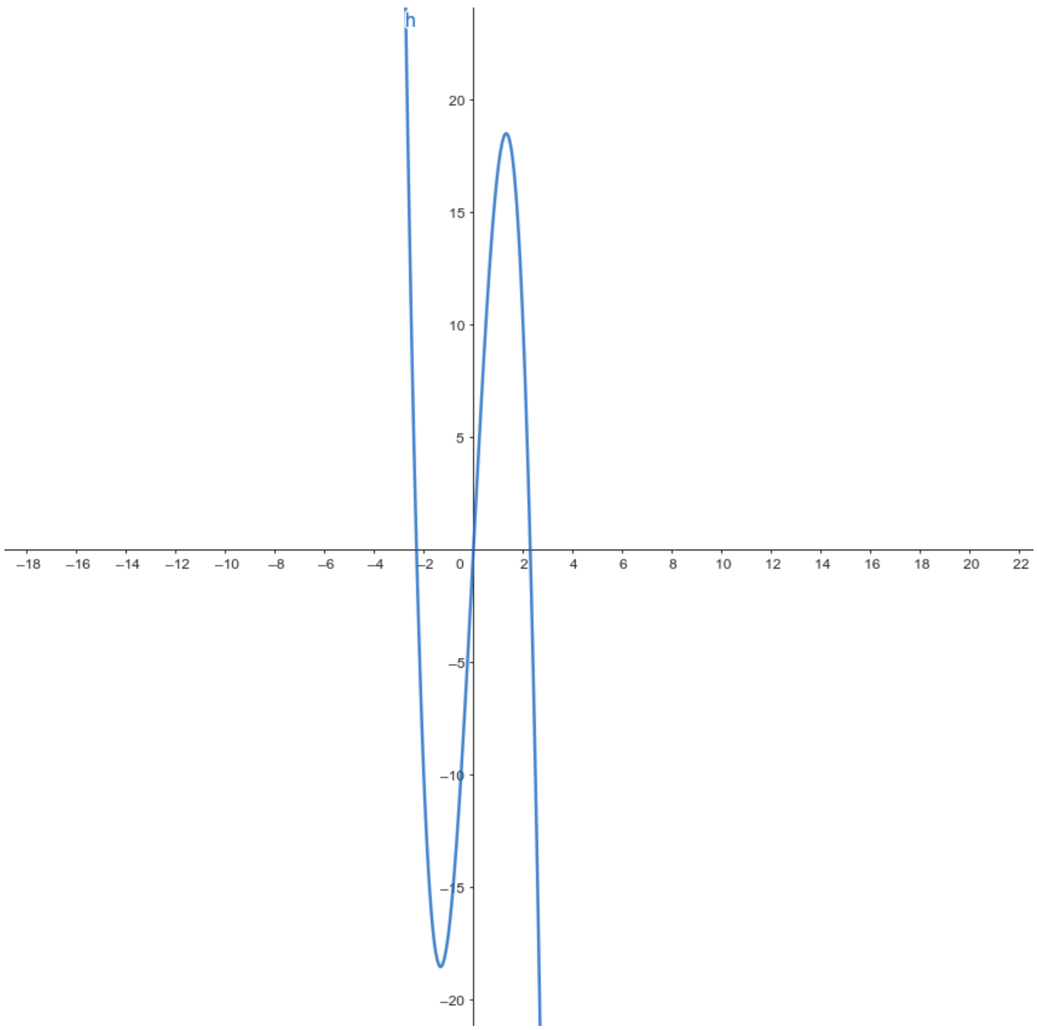

Figure 3: Graph of $h(x)$

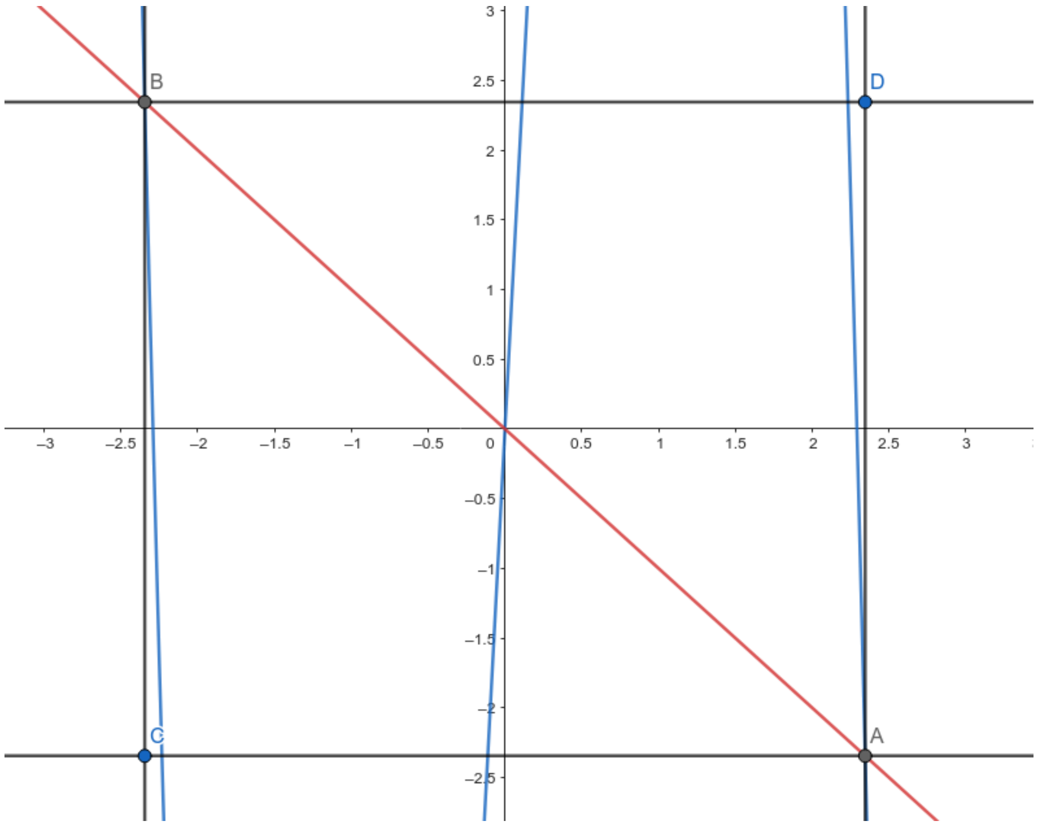

Figure 4: Graph of $h(x)$ and line $y = -x$

(d)

The 3 intersecting points are $A = \{2.3452, -2.3452\}, B = \{-2.3452, 2.3452\}, \{0, 0\}$

Starting at $-q$:

$$x_0 : -2.3452$$

$$x_1 : 2.344853285632011$$

$$x_2 : -2.3292546757024457$$

$$x_3 : 1.6344598476569043$$

$$x_4 : 16.8580869619733$$

$$x_5 : -18809.922745112937$$

Starting at q:

$$x_0 : 2.3452$$

$$x_1 : -2.344853285632011$$
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In all cases, the values diverge either to positive or to negative infinity. For the value outside the range, the sign changes to negative and the size decreases, the rate of divergence seems to be exponential.

Problem 3.

Compute q by solving the equation $-x = h(x) := x - f'(x)$

(a)

$$h(x) = x - f'(x)$$
$$-x = x - f'(x)$$
$$2x - f'(x) = 0$$
$$f'(x) = 4x^3 - 20x$$
$$2x = 4x^3 - 20x$$
$$22x = 4x^3$$
$$q = \sqrt{\frac{11}{2}}$$

(b)

$$h(x) = x - 4x^3 + 20x = -4x^3 + 21x$$

The stationary points for $h'(x) = -12x^2 + 21$ are $x = \pm\frac{\sqrt{7}}{2}$
For the interval $[-\sqrt{\frac{11}{2}}, \sqrt{\frac{11}{2}}]$ max and min values for $h(\frac{\sqrt{7}}{2})$ and $h(-\frac{\sqrt{7}}{2})$ are min: $-7\sqrt{7}$ and max: $7\sqrt{7}$.

It stays outside the square, it means that the iterations are not reaching to a specific finite number.

(c)

$$f''(x) = 12x^2 - 20, for\ x = 0, f''(0) = -20$$
$$f'''(x) = 24x, for\ x = 0, f'''(0) = 0$$
$$f''''(x) = 24, for\ x = 0, f''''(0) = 24\ where\ x = 0, f''(0) = -20 = s$$

(d)

$$f''(-3) = 108 - 20 = 88$$
$$f''(3) = 108 - 20 = 88$$
$$\text{t is attained at } x = \pm 3$$
$$\mu = \max\{20, 88\} = 88, \gamma = \frac{1}{88}$$

Let $x_{k+1} = x_k - \gamma f'(x_k)$

(e)

$$x_0 : -0.2$$
$$x_1 : -0.2450909090909091$$
$$x_2 : -0.3001241838147668$$
$$x_3 : -0.3671054281585156$$
$$x_4 : -0.4482896856818773$$
$$x_5 : -0.5460787084261626$$
$$x_6 : -0.6627856085743156$$
$$x_7 : -0.8001845369349272$$
$$x_8 : -0.9587558229360171$$
$$x_9 : -1.1365957574442003$$
$$x_{10} : -1.3281715598815835$$

$$x_0 : -0.001$$
$$x_1 : -0.001227272681818182$$
$$x_2 : -0.001506198207298862$$
$$x_3 : -0.0018485158263662842$$
$$x_4 : -0.0022686327725223274$$
$$x_5 : -0.0027842305991883035$$
$$x_6 : -0.00341700929976798$$
$$x_7 : -0.004193600508949864$$
$$x_8 : -0.005146688181445074$$
$$x_9 : -0.006316383844158441$$

$x_{10} : -0.007751914172244357$

$x_0 : 0$

$x_1 : 0.0$

$x_2 : 0.0$

$x_3 : 0.0$

$x_4 : 0.0$

$x_5 : 0.0$

$x_6 : 0.0$

$x_7 : 0.0$

$x_8 : 0.0$

$x_9 : 0.0$

$x_{10} : 0.0$

$x_0 : 0.0001$

$x_1 : 0.00012272727726818182$

$x_2 : 0.00015061983457093523$

$x_3 : 0.00018485161499991995$

$x_4 : 0.00022686334539461075$

$x_5 : 0.00027842319608993273$

$x_6 : 0.00034170119422022565$

$x_7 : 0.00041936055472950736$

$x_8 : 0.0005146697683612081$

$x_9 : 0.0006316401640647424$

$x_{10} : 0.0007751947353519551$

$x_0 : 0.02$

$x_1 : 0.024545090909090912$

$x_2 : 0.030122848503339496$

$x_3 : 0.03696770802450434$

$x_4 : 0.04536716346222939$

$x_5 : 0.05567363816879307$

$x_6 : 0.06831889395955722$

$$x_7 : 0.08383142092878017$$

$$x_8 : 0.10285723738191233$$

$$x_9 : 0.12618441907830039$$

$$x_{10} : 0.15477137012454759$$

(f)

The graphs for (f) are Figure 5 and Figure 6.

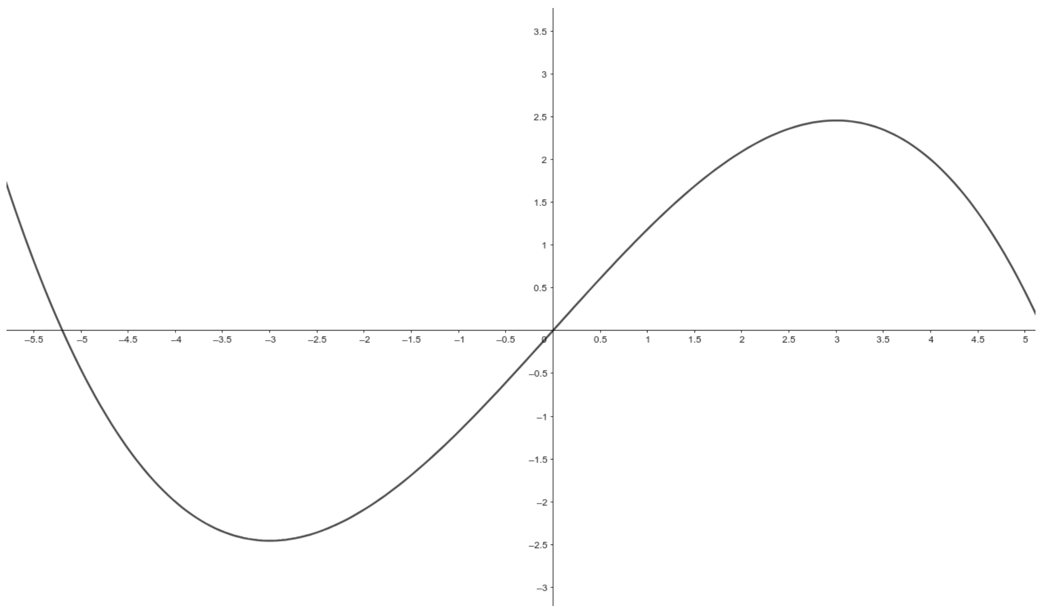

Figure 5: Graph of $x \to x - \frac{1}{\mu} f'(x)$

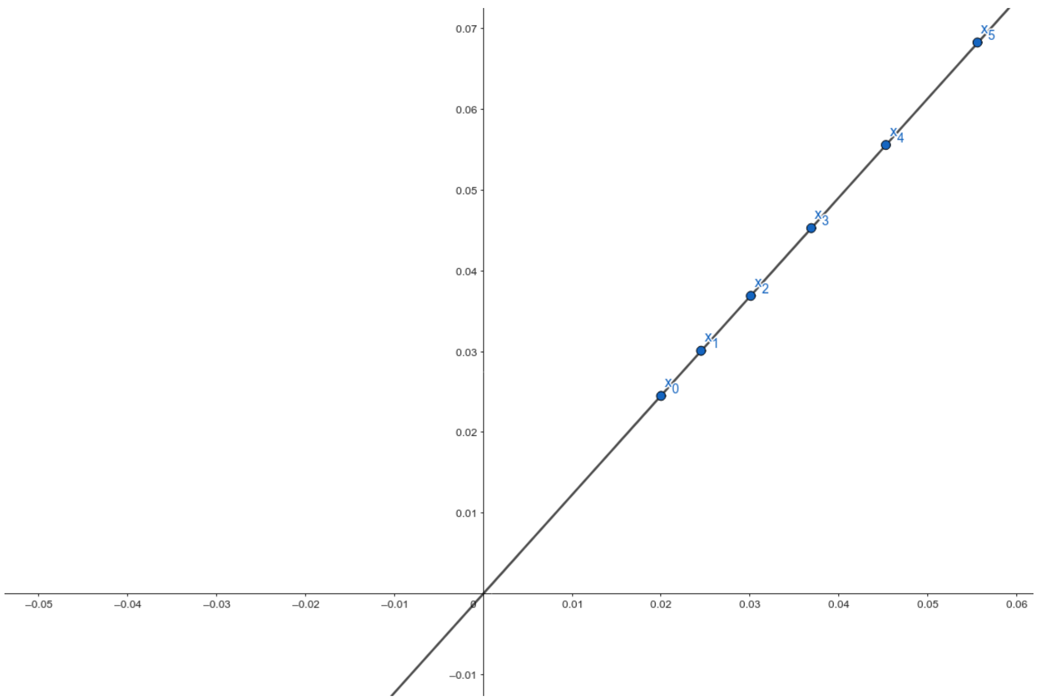

Figure 6: Multiple iterations for $x \to x - \frac{1}{\mu} f'(x)$