Numerical Optimization 2024 - Homework 1

Deadline: Wednesday, March 20, 15:30.

Let $f(x) = x^4 - 10x^2 + 9$ and consider the minimization problem

$$\min_{x \in \mathbb{R}} f(x).$$

In this homework, you will explore some issues related to finding minimizers of f numerically.

Problem 1. Finding minimizers analytically.

(a) Find all 4 solutions of $f(x) = 0$ (you can e.g. try some whole numbers or you can use a substitution $y = x^2$).

(b) Draw a sketch of the graph of f. How many stationary point do you expect f to have?

(c) Compute the derivative f' and find the stationary points (points x such that $f'(x) = 0$).

(d) Compute the second derivative f'' to check which stationary point are local minimizers ($f''(x) > 0$) and which are local maximizers ($f''(x) < 0$).

Problem 2. Iterative approach to finding the stationary points - 1. trial.
Let $x_{k+1} = x_k - f'(x_k)$.

(a) Choose 5 starting points x_0 and perform 5 iterations. Write down numbers $x_0, x_1, \ldots, x_5$.

(b) Draw a sketch of the graph of function $h(x) = x - f'(x)$ and for one of your starting points x_0, draw also several first iterations $x_0, x_1, x_2, x_3, \ldots$. Does it help to start very close to a stationary point?

(c) Draw a sketch of the graph of function $h(x)$ again and draw also the graph of the minus identity, i.e., the graph of function $x \to -x$ (the line $y = -x$). These two graphs should meet at three points $(0,0)$, $(-q,q)$ and $(q,-q)$ for some number $q > 0$. Add also a square with corners at points $(-q,-q)$, $(-q,q)$, $(q,-q)$, and (q,q) to the drawing.

(d) What happens if you start the iteration with $x_0 = -q$ or $x_0 = q$? If an iterate x_k is outside of the interval $[-q,q]$, what happens with the next iterate x_{k+1}? What is the sign of x_{k+1} and what is its size compared to x_k? You can answer by just trying several x_k or you can explain it geometrically.

We will now try to remedy the situation by considering the iterates $x_{k+1} = x_k - \gamma f'(x_k)$ for some suitably chosen number $\gamma > 0$.

Problem 3. Computations and setting of γ:

(a) Compute the value q from the previous problem by solving the equation $-x = h(x) := x - f'(x)$.

(b) Using derivatives of $h(x) = x - f'(x)$ find its stationary points and then also the maximal (and minimal value) of function h over the interval $[-q, q]$ and compare it with q. Does the graph of $h(x)$ over the interval $[-q, q]$ stays within the square from the previous problem or does it escape? What does it mean for the iteration process?

(c) Solve $\min_{x \in \mathbb{R}} f''$ using the third and the fourth derivative of f. Denote the minimal value s.

(d) Solve $\max_{x \in [-3,3]} f''$. Note that the feasible set is only the interval. Where will the maximal value (let us denote it t) be attained? Write down number $\mu = \max\{|s|, |t|\}$ and set $\gamma = 1/\mu$.

Iterative approach to finding the stationary points - 2. trial. Let $x_{k+1} = x_k - \gamma f'(x_k)$:

(e) Try the same 5 starting points x_0 and perform 10 iterations. Write down numbers $x_0, x_1, \ldots, x_{10}$.

(f) Draw a sketch of the graph of function $x \to x - (1/\mu) f'(x)$ and for one of your starting points x_0, draw also several first iterations $x_0, x_1, x_2, x_3, \ldots$.

(g) In case none of your starting points generated a sequence converging to a stationary point of f, look at the sketch of f and try to start very close to a stationary point. Does it help?