Numerical Optimization 2024 - Homework 9

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Problem 1.

Consider the problem:

$$\min_{x \in \mathbb{R}^2} f(x) = -2x_1 + x_2 \quad \text{subject to} \begin{cases} (1-x_1)^3 - x_2 \geq 0 \\ x_2 + 0.25x_1^2 - 1 \geq 0 \end{cases}$$

The optimal solution is $x^* = (0,1)^T$, where both constrains are active.

Solution:

(a)

$$\nabla g_1(x) = \begin{pmatrix} -3(1-x_1)^2 \\ -1 \end{pmatrix} \quad \text{and} \quad \nabla g_2(x) = \begin{pmatrix} 0.5x_1 \\ 1 \end{pmatrix}$$

At the optimal solution $x^* = (0,1)^T$, we get:

$$\nabla g_1(x^*) = \begin{pmatrix} -3 \\ -1 \end{pmatrix} \quad \text{and} \quad \nabla g_2(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

We see that the gradients are not linearly independent, LICQ is not satisfied.

(b)

The Lagrangian is:

$$\mathcal{L}(x,\lambda) = -2x_1 + x_2 - \lambda_1((1-x_1)^3 - x_2) - \lambda_2(x_2 + 0.25x_1^2 - 1)$$

The KKT conditions are:

$$
\begin{aligned}
-2 - 3\lambda_1(1-x_1)^2 - 0.5\lambda_2 x_1 &= 0 \\
1 + \lambda_1 - \lambda_2 &= 0 \\
\lambda_1((1-x_1)^3 - x_2) &= 0 \\
\lambda_2(x_2 + 0.25x_1^2 - 1) &= 0 \\
\lambda_1 &\geq 0 \\
\lambda_2 &\geq 0
\end{aligned}
$$

For the optimal solution $x^* = (0,1)^T$, all KKT conditions are not satisfied:

$$-2 - 3\lambda_1(1-0)^2 - 0.5\lambda_2(0) \neq 0$$

(c)

(d)

I do not understand how to solve **c** and **d**, I looked at the examples in the book multiple times, I still do not understand how to solve them.

Problem 2.

$$\min_{x\in\mathbb{R}^2} f(x) = x_1x_2 \text{ subject to } x_1^2 + x_2^2 = 1$$

Solution:

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda) = x_1x_2 - \lambda(1 - x_1^2 - x_2^2)$$

The KKT conditions:

$$
\begin{aligned}
x_2 + 2\lambda x_1 &= 0 \\
x_1 + 2\lambda x_2 &= 0 \\
1 - x_1^2 - x_2^2 &= 0 \\
\lambda &\geq 0
\end{aligned}
$$

Solving the equations, we get:

if $\lambda = 0$, then $x_1 = x_2 = 0$,

if $\lambda = 1/2$, then $x_1 = x_2 = \pm 1/\sqrt{2}$.

The Lagrangian Hessian is:

$$\nabla^2_{xx}\mathcal{L}(x_1, x_2, \lambda) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}$$

For the point $(0,0)$ the matrix is not positive definite, therefore, function minimum is found at $x_1, x_2 \in \left\{(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})\right\}$. The plot of the problem is shown in Figure 1.

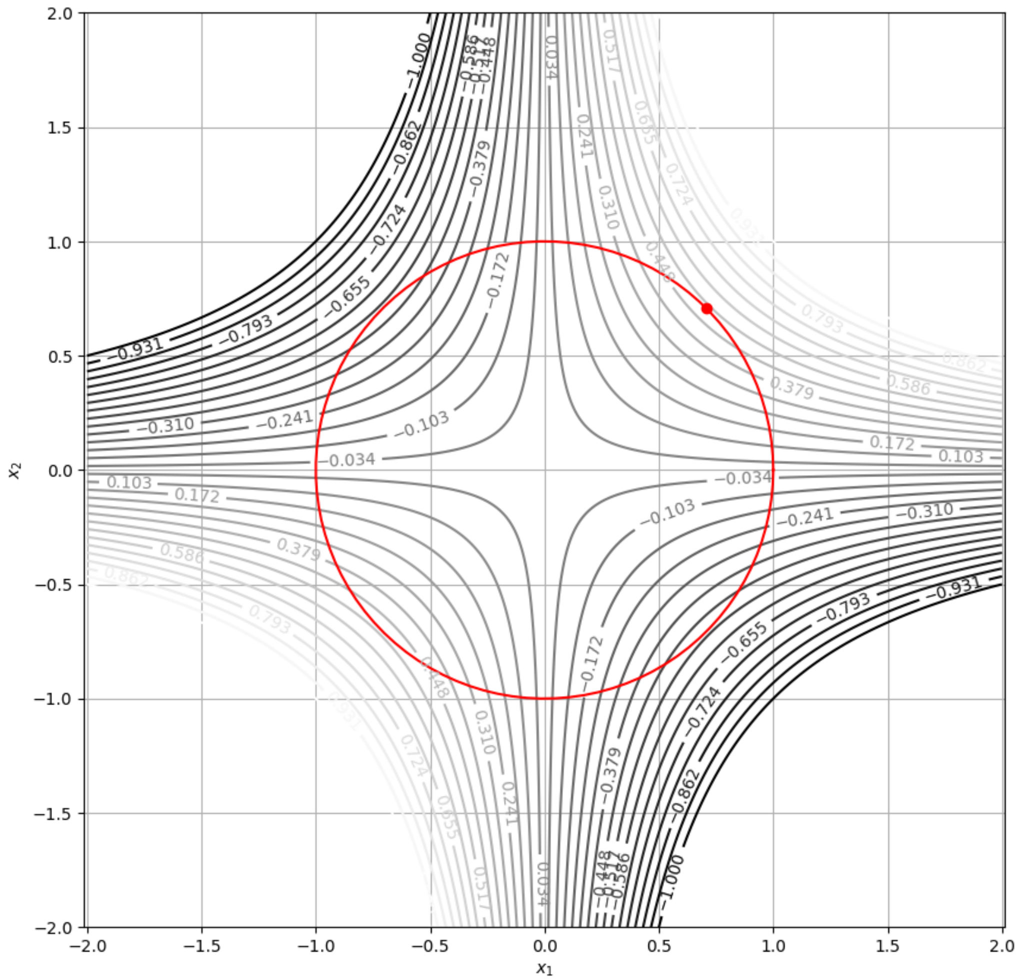

Figure 1: Plot of $f(x) = x_1 x_2$ over the unit circle.

Problem 3.

Find the *maxima* of $f(x) = x_1 x_2$ over the unit disk defined by the inequality constraint $1 - x_1^2 - x_2^2 \geq 0$.

Solution:

The Lagrangian is:

$$\mathcal{L}(x_1, x_2, \lambda) = -x_1 x_2 - \lambda(1 - x_1^2 - x_2^2)$$

The KKT conditions are:

$$
\begin{aligned}
-x_2 - \lambda(-2x_1) &= 0 \\
-x_1 - \lambda(-2x_2) &= 0 \\
\lambda(1 - x_1^2 - x_2^2) &= 0 \\
\lambda &\geq 0
\end{aligned}
$$

Solving the equations, we get:

if $\lambda = 0$, then $x_1 = x_2 = 0$,
if $\lambda = 1/2$, then $x_1 = x_2 = \pm 1/\sqrt{2}$.

The points are:

$$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}), (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{1}{2}), (0, 0, 0)$$

The Lagrangian Hessian is:

$$\nabla^2_{xx}\mathcal{L}(x_1, x_2, \lambda) = \begin{pmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}$$

Plugging in the points, we get:

$$\nabla^2_{xx}\mathcal{L}(\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, \frac{1}{2}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which are positive definite matrices.

For the point $(0, 0, 0)$, we get:

$$\nabla^2_{xx}\mathcal{L}(0, 0, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is not positive definite, therefore, the optimal points are $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$ and $(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.