Numerical Optimization 2024 - Homework 2

Deadline: Wednesday, April 10, 15:30.

In this homework, you will examine various rates of convergence (linear, superlinear, quadratic, cubic), see pages 619-620 of the Nocedal-Wright book or the lecture notes. Note that, in order to find the rate of convergence of a sequence x_k, one needs to first find the limit x^* of the sequence. Then, one looks into what happens with the fractions $l_k := \|x_{k+1} - x^*\| / \|x_k - x^*\|$ to see if the sequence converges linearly (if $l_k \leq r$ for large enough k and some number $r < 1$). If $l_k \to 1$, such convergence is called sublinear (worse/slower than linear) and if $l_k \to 0$, we get the superlinear (better/faster than linear) convergence. In the latter case, one looks further into "quadratic" fractions $q_k := \|x_{k+1} - x^*\| / \|x_k - x^*\|^2$ to check if the convergence is even quadratic (if $q_k \leq M$ for large enough k and some number M). If even $q_k \to 0$, one considers "cubic" fractions $c_k := \|x_{k+1} - x^*\| / \|x_k - x^*\|^3$ (the convergence is cubic if $c_k \leq M$ for large enough k and some number M). Hence, cubic convergence is a special case of quadratic convergence, which is a special case of superlinear convergence, which is a special case of linear convergence.

Problem 1. Determine the rate of convergence (linear, superlinear, quadratic, cubic) for the following sequences:

(a) $x_k = 1/k$;

(b) $x_k = 1/k!$;

(c) $x_k = a^k$ for some $a \in (0,1)$;

(d) $x_k = 1 + (0.5)^{2^k}$.

In the following two problems, you can determine the rate convergence either "theoretically" by computing the corresponding limits of sequences, or, alternatively, you can do it "numerically" by iterating, i.e., producing the actual sequences and the numbers l_k, q_k, and c_k, and observing where these quantities seem to converge.

Problem 2. In Homework 1, Problem 3, we showed that the sequence given by $x_{k+1} = x_k - \gamma(4x_k^3 - 20x_k)$ for $\gamma = 1/88$ converges to $x^* = \sqrt{5}$ if we start e.g. with $x_0 = 1$. Determine the rate of this convergence:

- theoretically: denote $\tilde{h}(x) = x - (x^3 - 5x)/22$ and compute the limit of $|\tilde{h}(x) - x^*| / |x - x^*|$ as $x \to x^* = \sqrt{5}$.
  Hint: use the substitution $z = x - x^*$ and/or note that $\tilde{h}(x) = x - x(x - x^*)(x + x^*)/22$.

- numerically: try to produce 100 iterates and write the value l_k (or q_k and c_k if needed). Be careful about the rounding error - use at least 10 digits of numbers or even more if needed.

Problem 3. In the first tutorial, we showed that the sequence given by $x_{k+1} = x_k + \gamma \cos(x_k)$ for any $\gamma > 0$ either does not converge or converges to $x^* = \pi/2$ if we start e.g. with $x_0 = 1$. Determine the rate of this convergence depending on γ:

- theoretically: compute the limit of $|x + \gamma \cos(x) - x^*|/|x - x^*|$ as $x \to x^* = \pi/2$. For the value $\hat{\gamma}$ that gives superlinear convergence (the above limit is zero), compute also the limit of $|x + \hat{\gamma}\cos(x) - x^*|/|x - x^*|^j$ for $j = 2, 3$ as $x \to x^* = \pi/2$.
  Hint: use the substitution $z = x - x^*$ and/or note that $\cos(x + x^*) = -\sin(x)$ and/or use the Taylor expansion of function sin.

- numerically: do the same as in Problem 2 for the values $\gamma = 1/2$, $\gamma = 3$, $\gamma = 2$ and $\gamma = 1$: try to produce 100 iterates and write the value l_k (or q_k and c_k if needed). Be careful about the rounding error - use at least 10 digits of numbers or even more if needed. If the convergence seems to be very fast, produce less iterates (maybe less than 10) and be particularly careful about the rounding error. Btw, you may use $\pi = 3.14159265358979323846264338327950288419716939937 51$.