Numerical Optimization 2024 - Homework 8

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Problem 1.

Problem 2.

Consider the following modification of (12.36), where t is a parameter to be fixed prior to solving the problem:

$$\min_x \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 \quad \text{s.t.} \quad \begin{bmatrix} 1 - x_1 - x_2 \\ 1 - x_1 + x_2 \\ 1 + x_1 - x_2 \\ 1 + x_1 + x_2 \end{bmatrix} \geq 0$$

Lagrangian of the problem is:

$$L(x, \lambda) = \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 - \Big(\lambda_1(1 - x_1 - x_2) + \lambda_2(1 - x_1 + x_2) + \lambda_3(1 + x_1 - x_2) + \lambda_4(1 + x_1 + x_2)\Big)$$

$$= \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 - \lambda_1(1 - x_1 - x_2) - \lambda_2(1 - x_1 + x_2) - \lambda_3(1 + x_1 - x_2) - \lambda_4(1 + x_1 + x_2)$$

$$= \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 - \lambda_1 + \lambda_1 x_1 + \lambda_1 x_2 - \lambda_2 + \lambda_2 x_1 - \lambda_2 x_2 - \lambda_3 - \lambda_3 x_1 + \lambda_3 x_2 - \lambda_4 - \lambda_4 x_1 - \lambda_4 x_2$$

$$= \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - t)^4 - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 + x_1(\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4) + x_2(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4)$$

Derivative of the Lagrangian w.r.t x_1:

$$\frac{\partial L(x, \lambda)}{\partial x_1} = 2(x_1 - 3/2) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

w.r.t x_2:

$$\frac{\partial L(x, \lambda)}{\partial x_2} = 4(x_2 - t)^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

KKT conditions are:

$$2(x_1 - 3/2) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

$$4(x_2 - t)^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

a) Plugging in $x^* = (1, 0)^T$ we get:

$$2(1 - 3/2) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4$$

$$-1 + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

$$4(0 - t)^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4$$

$$-4t^3 + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

$$\lambda_1(1 - 1 - 0) = 0,$$

$$\lambda_2(1 - 1 + 0) = 0,$$

$$\lambda_3(1 + 1 - 0) = 0,$$

$$\lambda_4(1 + 1 + 0) = 0$$

We see that λ_1 and λ_2 are non zero and λ_3 and λ_4 are zero.
Substituting $\lambda_3 = \lambda_4 = 0$ into the equations:

$$\lambda_1 + \lambda_2 = 1$$

$$\lambda_1 - \lambda_2 = 4t^3$$

Solving the system of equations we get:

$$\lambda_1 = \frac{1 + 4t^3}{2}, \lambda_2 = \frac{1 - 4t^3}{2}$$

Since λ_1 and λ_2 are non-negative, t^3 must be in the range $[-1/4, 1/4]$, which implies:

$$t \in [-1/4^{1/3}, 1/4^{1/3}]$$

b) For $t = 1$, we get:

$$\min_x \left(x_1 - \frac{3}{2}\right)^2 + (x_2 - 1)^4$$

Since only the first constraint is active, $\lambda_1 > 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = 0$:

$$2(x_1 - 3/2) + \lambda_1 = 0,$$

$$4(x_2 - 1)^3 + \lambda_1 = 0,$$

$$2x_1 - 3 + 4(x_2 - 1)^3 = 0$$

$$x_1 = 3/2 - 2(x_2 - 1)^3$$

? I don't understand how to proceed after this point.

Problem 3.

Consider the problem of finding the point on the parabola $y = \frac{1}{5}(x-1)^2$ that is closest to $(x, y) = (1, 2)$ in the Euclidean norm sense. We can formulate this problem as:

$$\min f(x, y) = (x-1)^2 + (y-2)^2 \text{ subject to } (x-1)^2 = 5y$$

Lagrangian of the problem is:

$$L(x, y, \lambda) = (x-1)^2 + (y-2)^2 - \lambda((x-1)^2 - 5y)$$

$$= (x-1)^2 + (y-2)^2 - \lambda(x-1)^2 + 5\lambda y$$

$$= (1-\lambda)(x-1)^2 + (y-2)^2 + 5\lambda y$$

Derivative of the Lagrangian w.r.t x:

$$\frac{\partial L(x, y, \lambda)}{\partial x} = 2(x-1)(1-\lambda)$$

w.r.t y:

$$\frac{\partial L(x, y, \lambda)}{\partial y} = 2(y-2) + 5\lambda$$

KKT conditions:

$$2(x^* - 1)(1 - \lambda^*) = 0$$

$$2(y^* - 2) + 5\lambda^* = 0$$

$$(x^* - 1)^2 - 5y^* = 0$$

Solving the equations we get:

$$x^* = 1, y^* = 0, \lambda^* = 4/5 \text{ since } (x^* - 1)^2 = -5/2 \text{ can not be a solution}$$

a) (All the KKT as found above), Checking if LICQ holds:

$$\nabla c(x^*, y^*) = \begin{bmatrix} 2(x^* - 1) \\ -5 \end{bmatrix} \text{ which is non-zero, therefore LICQ holds.}$$

Finding the w values for $\nabla c(x^*, y^*)^T w = 0$ at the point $(1, 0)$:

$$\begin{bmatrix} 2(x^* - 1) \\ -5 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 2(1 - 1) \\ -5 \end{bmatrix}^T \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & -5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$$

$$-5w_2 = 0, w_2 = 0$$

For w_1 and $w^T \nabla^2 L(x^*, y^*, \lambda^*) w > 0$:

$$w^T \nabla^2 L(x^*, y^*, \lambda^*) w = \begin{bmatrix} w_1 & 0 \end{bmatrix} \begin{bmatrix} 2(1 - \lambda^*) & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} w_1 \\ 0 \end{bmatrix} = 2(1 - \lambda^*) w_1^2 > 0$$

$$= 2(1 - 4/5) w_1^2 = 2(1/5) w_1^2 > 0$$

b) Checking the second order condition (the result above), we see that the point $(1, 0)$ is a solution.

c) Substituting $(x - 1)^2 = 5y$ into the objective function we get:

$$\min f(x, y) = (x - 1)^2 + (y - 2)^2 = 5y + (y - 2)^2 = 5y + y^2 - 4y + 4 = y^2 + y + 4$$

Solving the new objective function, min $f(y) = y^2 + y + 4$:

Taking the derivative of the objective function with respect to y:

$$f'(y) = 2y + 1, y = -1/2$$

At $y = -1/2$, $f(y) = 1/4 - 1/2 + 4 = 15/4$ which suggests that solutions of this problem cannot be solutions of the original problem.