Partitioning Undirected Graphs

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Problem Definition

Partition a graph into 2 equal sized subgraphs such that the number of edges cut is minimized. Edges can have weights.

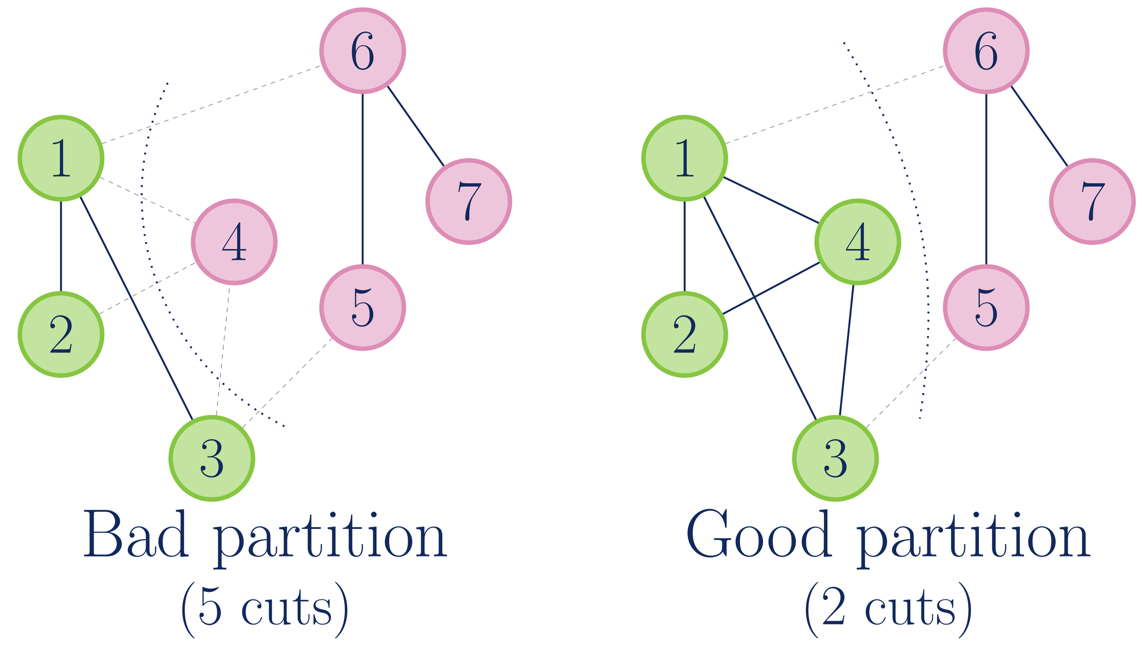
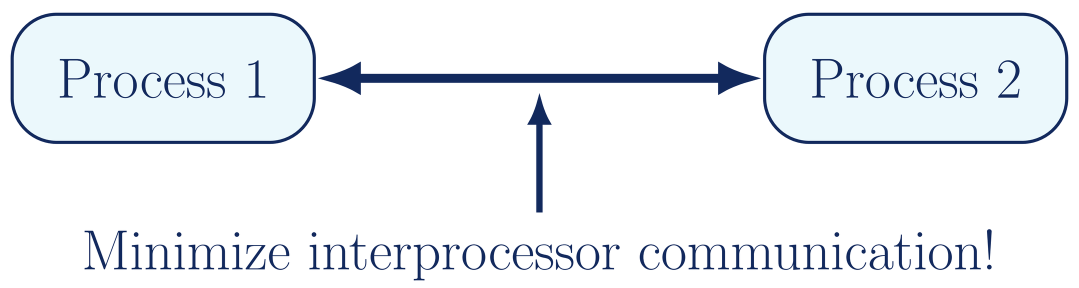


Figure 1: Different partitions of a graph

Motivation

- It is usually better to divide the data into partitions that have *equal load* so that algorithms on data can be operated in a parallel fashion by multiple processors.
- Parallelism brings the problem of interprocessor communication, which is time consuming.
- It is crucial to divide the data in a way that *the communication between processors is minimized*.
- This can be modeled as a **graph partitioning problem**.



Methodology

Integer Programming (IP)

- IP is an optimization method with integer-valued decision variables.
- Usually addresses problems involving discrete choices.

How to Solve IP Models?

1 Branch and Bound

- Tree-based recursive solution algorithm.

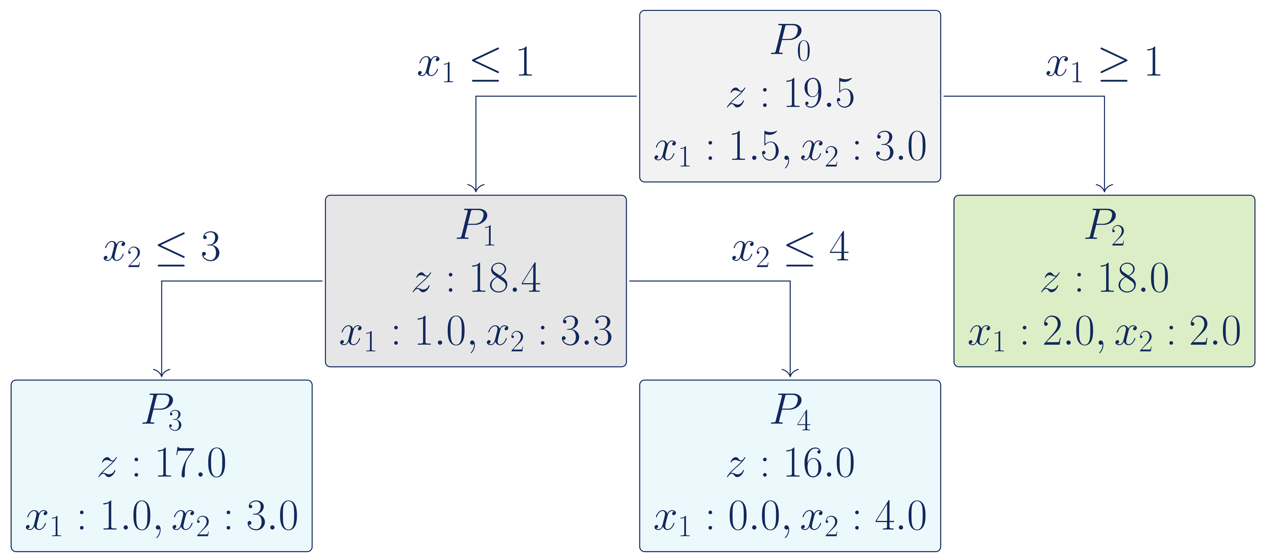


Figure 2: An IP problem solved with branch and bound.

2 Cutting Plane

- Iteratively refines the feasible region.

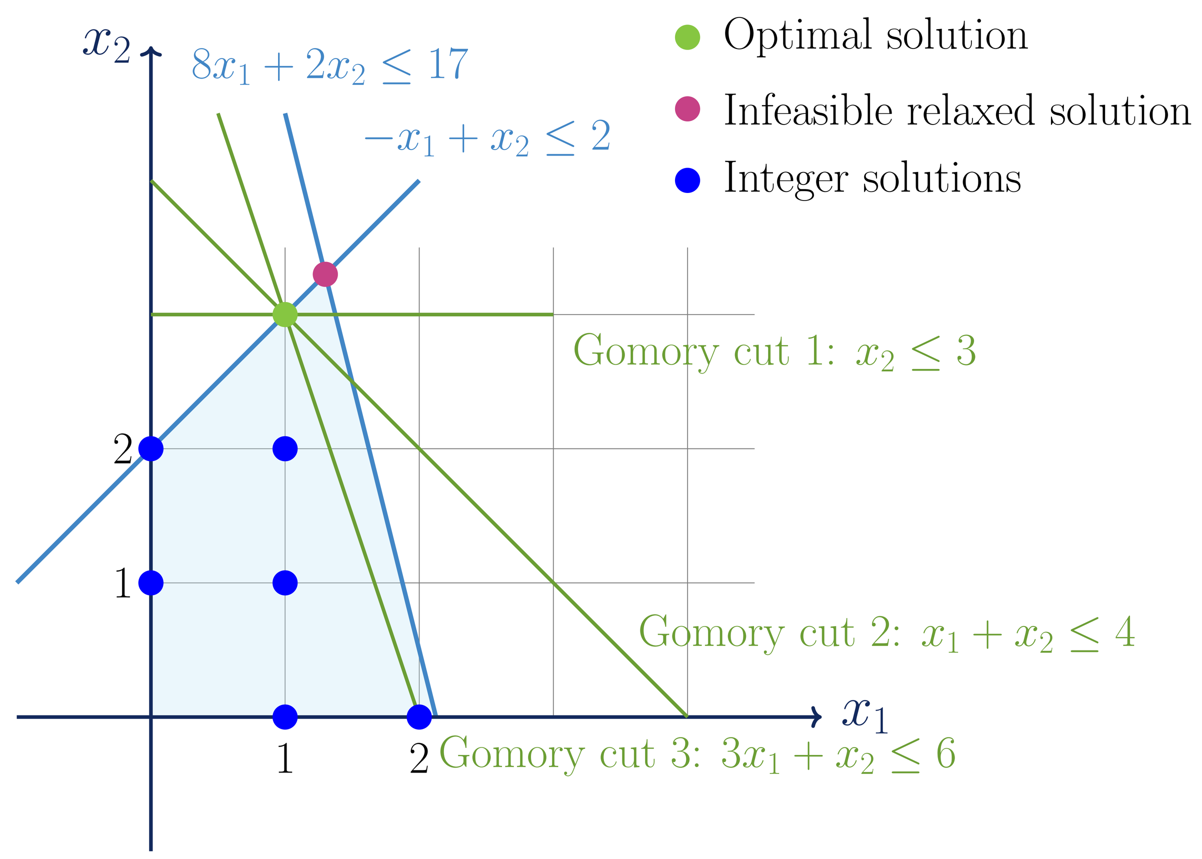


Figure 3: An IP problem solved with cutting plane.

3 Gurobi

- Parallel optimization solver
- Uses both branch & bound and cutting plane.
- Efficient for linear, integer, quadratic programming.
- Gurobi is used in the software [1].

Integer Programming Model

1 Sets

- $V = \{1, \ldots, n\}$: the set of vertices.

2 Parameters

- w_{ij} = weight of edge i-j, $\forall i, j \in V, \quad i < j$

3 Decision Variables

- $x_{ij} = \begin{cases} 1, & \text{edge from } i \text{ to } j \text{ is “cut”} \\ 0, & \text{otherwise} \end{cases}, \quad \begin{matrix} \forall i, j \in V \\ i < j \\ w_{ij} \neq 0 \end{matrix}$
- $p_i = \begin{cases} 1, & \text{vertex } i \text{ is in partition \#1} \\ 0, & \text{vertex } i \text{ is in partition \#0} \end{cases}, \quad \forall i \in V$

4 Objective Function

$$\min Z = \sum_{i=1}^{n-1} \sum_{j \in V | i < j, w_{ij} \neq 0} w_{ij} \times x_{ij}$$

5 Constraints

1. Adjacent nodes are in the same partition
   - $p_i - p_j \leq x_{ij} \quad \forall i, j \in V, \quad i < j \quad w_{ij} \neq 0$
   - $p_i - p_j \geq -x_{ij} \quad \forall i, j \in V, \quad i < j \quad w_{ij} \neq 0$
2. Equal partition sizes
   - $\sum_{i=1}^{n} p_i = \left\lfloor \frac{n}{2} \right\rfloor$
3. Binary variable constraints
   - $x_{ij} = \{0, 1\} \quad \forall i, j \in V, \quad i < j \quad w_{ij} \neq 0$
   - $p_i = \{0, 1\} \quad \forall i \in V$

Testing & Results

The software [1] is tested on various graph sizes/densities:

Key Findings

- The software can partition graphs of any density when the graph's size is less than 50.
- When the graph's size is greater than 50, complexity of the problem increases very quickly and the software can only partition sparse graphs.
- On average, weightless version of the problem takes slightly more time to solve than the weighted version.

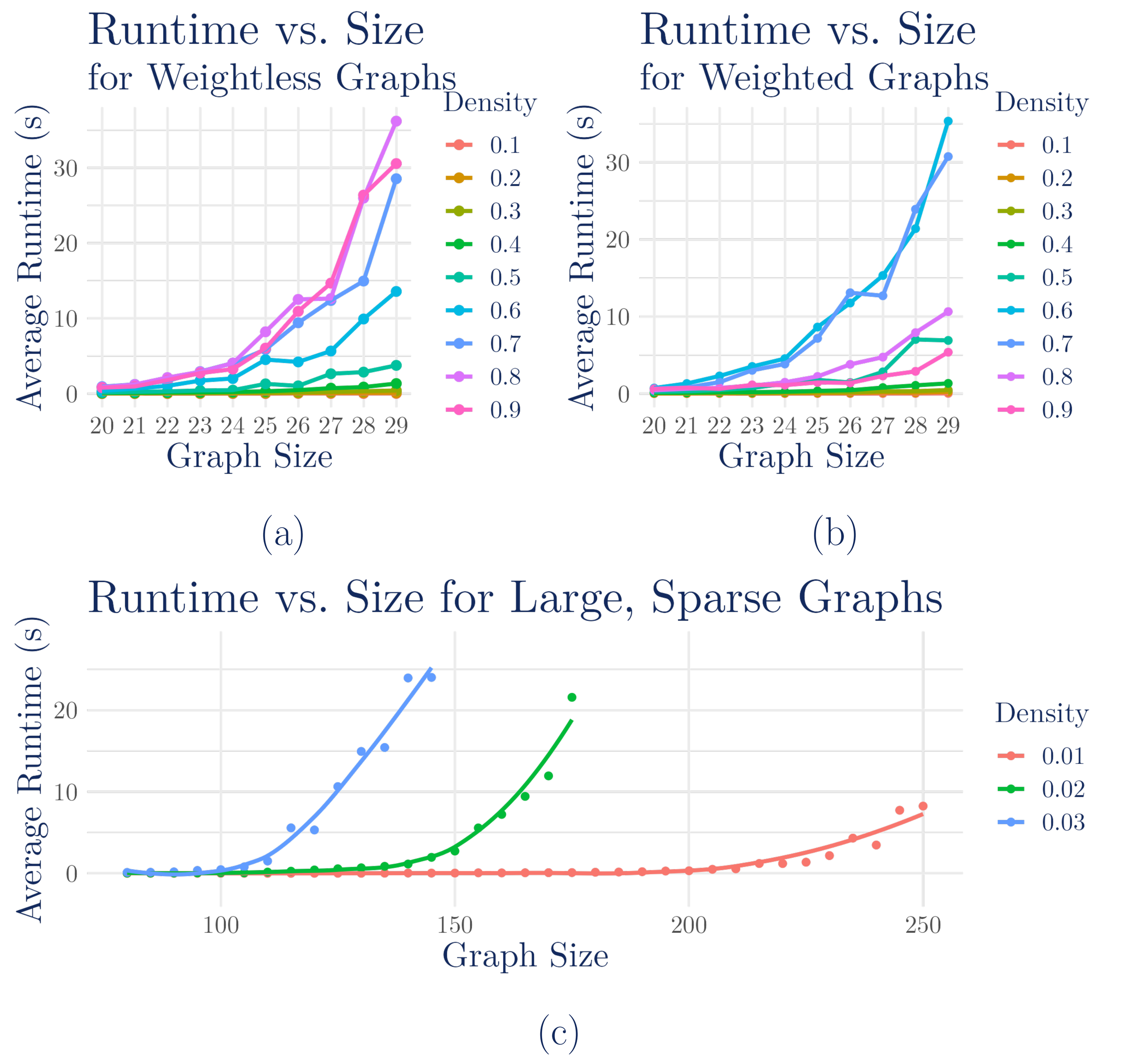
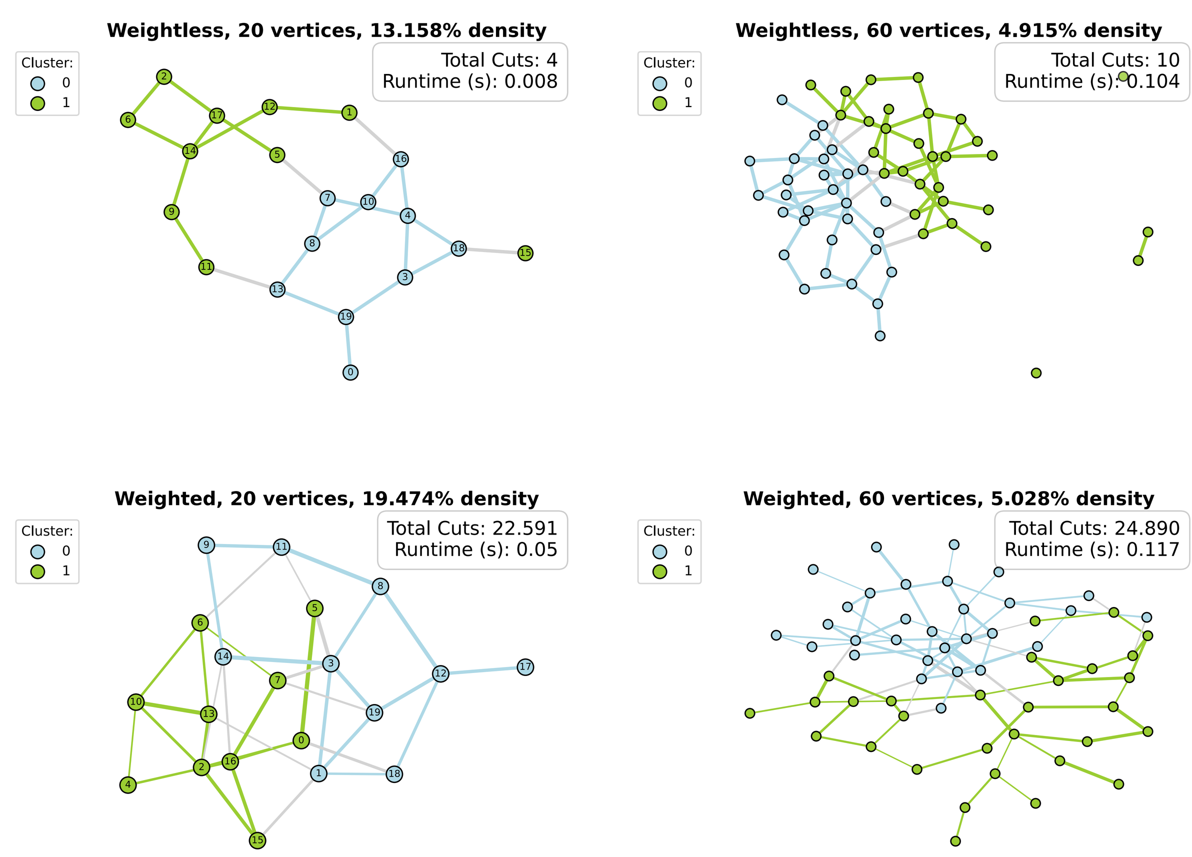


Figure 4: Plots of the runtime of the algorithm

Showcase



References

[1] Aral Dörtoğul. *Graph Partitioner*. URL: https://github.com/araldortogul/CMPE492_Partitioning_Undirected_Graphs.