2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 3

MAT281E Linear Algebra and Applications HW 3

Instructions: Scan or take photos of clearly handwritten solutions, combine them in one doc, docx or pdf file and submit by 16:00 December 14, 2020. Late homeworks are not accepted. 4-5 problems will be checked in detail which will contribute 80% to the final mark. The rest will be checked for completeness which will contribute 20% to the final mark.

Q1)

let uxv be equal to $\vec{w}$. $\vec{w}$ is orthagonal to both $\vec{u}$ and $\vec{v}$

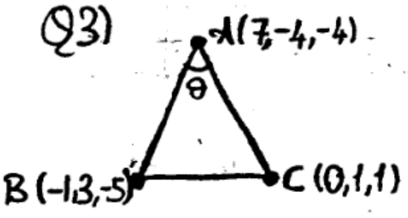
$$\vec{u} . \vec{w} = |u| . |w| . \underbrace{\cos 90}_{0} = 0$$

Q2)

$u = \langle 1,4,1 \rangle \quad V = \langle 0,-2,3 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 4 & 1 \\ 0 & -2 & 3 \end{vmatrix} = \underbrace{\langle 14, -3, -2 \rangle}_{\text{This vector is orthagonal to both } \vec{u} \text{ and } \vec{v}}$$

Q3)



$\vec{AB} = \langle -8, 7, -1 \rangle$

$\vec{AC} = \langle -7, 5, 5 \rangle$

$Area = |\vec{AB}| . |\vec{AC}| . \sin\theta . \frac{1}{2}$

$Area = |\vec{AB} \times \vec{AC}| . \frac{1}{2}$

$$Area = \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ -8 & 7 & -1 \\ -7 & 5 & 5 \end{vmatrix} \right\| = |\langle 40, 47, 9 \rangle| . \frac{1}{2} = 31,2$$

Q4) $\vec{AB} = \langle -8, 7, -1 \rangle$
$\vec{AC} = \langle -7, 5, 5 \rangle$
$\vec{AD} = \langle -7, 4, 4 \rangle$

$$\left.\begin{aligned} \vec{AB} &= \langle -8,7,-1\rangle \\ \vec{AC} &= \langle -7,5,5\rangle \\ \vec{AD} &= \langle -7,4,4\rangle \end{aligned}\right\} V = |\vec{AD}\cdot(\vec{AB}\times\vec{AC})| = |\langle -7,4,4\rangle \cdot \langle 40,47,9\rangle| = 56$$

Since $V \neq 0$, these vectors are NOT coplanar

5. Find all unit vectors in the plane determined by vectors **u** = (0 1 1) and **v** = (2 -1 3) that are perpendicular to the vector **w** = (5 7 -4)

Q5)

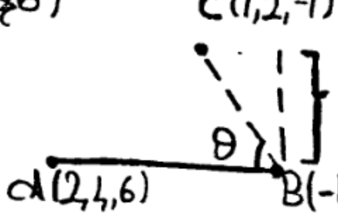
$$\vec{N} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = \langle 4, 2, -2 \rangle \Big\} \text{Normal vector of the determined plane}$$

let $\vec{p} = \langle a, b, c \rangle$, $\vec{p} \perp \vec{N} \Rightarrow \vec{p}\cdot\vec{N} = 0$, $\vec{p} \perp \vec{w} \Rightarrow \vec{p}\cdot\vec{w} = 0$

$$\left.\begin{aligned} 4a + 2b - 2c &= 0 \\ 5a + 7b - 4c &= 0 \end{aligned}\right\} \begin{aligned} a &= b \\ c &= 3a \end{aligned} \Rightarrow \vec{p_1} = \langle a, a, 3a \rangle, \ \vec{p_2} = \langle -a, -a, -3a \rangle$$

convert $\vec{p}$ into unit vector $\Rightarrow \vec{p_1}' = \left\langle \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle$, $\vec{p_2}' = \left\langle \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-3}{\sqrt{11}} \right\rangle$

Q6)



Distance $= |\vec{BC}| \cdot \sin\theta$

$\vec{AB} = \langle -3, -1, -2 \rangle \Rightarrow |\vec{AB}| = \sqrt{14}$
$\vec{BC} = \langle 2, -1, -5 \rangle \Rightarrow |\vec{BC}| = \sqrt{30}$

$|\vec{AB} \times \vec{BC}| = |\vec{AB}| \cdot |\vec{BC}| \cdot \sin\theta$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ -3 & -1 & -2 \\ 2 & -1 & -5 \end{vmatrix} = \langle 3, -19, 5 \rangle, \quad |\vec{AB} \times \vec{BC}| = \sqrt{395}$$

$$\sqrt{395} = \sqrt{14} \cdot \sqrt{30} \cdot \sin\theta \Rightarrow \sin\theta = \sqrt{0{,}94}$$

$$\text{Distance} = |\vec{BC}| \cdot \sin\theta = \sqrt{30} \cdot \sqrt{0{,}94} = 5{,}31$$

a) $N_1 = \langle 1,2,-1 \rangle$ and $N_2 = \langle 2,-1,1 \rangle$

$$\vec{N_1} \cdot \vec{N_2} = |N_1| . |N_2| . \cos\theta = \sqrt{6} . \sqrt{6} . \cos\theta = -1 \quad \Big\} \cos\theta = \frac{-1}{6}$$

$$\vec{N_1} \times \vec{N_2} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \langle 1,-3,-5 \rangle , \; |N_1 \times N_2| = \sqrt{35} = \cancel{N_1} N_2 . \sin\theta$$

$$\sqrt{35} = \sqrt{6} . \sqrt{6} . \sin\theta \quad \Big\} \sin\theta = \frac{\sqrt{35}}{6}$$

∴ The planes are neither parallel nor perpendicular

b) $N_1 = \langle 3,1,-1 \rangle$ and $N_2 = \langle 1,2,5 \rangle$

$$\vec{N_1} \cdot \vec{N_2} = |N_1| . |N_2| . \cos\theta = \sqrt{11} . \sqrt{30} . \cos\theta = 0 \quad \Big\} \begin{array}{l} \cos\theta = 0 \\ \theta = 90^\circ \end{array}$$

∴ The planes are perpendicular

c) $N_1 = \langle 1,1,-3 \rangle$ and $N_2 = \langle 2,2,-6 \rangle$

$$\vec{N_1} \times \vec{N_2} = \begin{vmatrix} i & j & k \\ 1 & 1 & -3 \\ 2 & 2 & -6 \end{vmatrix} = \langle 0,0,0 \rangle , \; |\vec{N_1} \times \vec{N_2}| = 0$$

$$|\vec{N_1} \times \vec{N_2}| = |N_1| . |N_2| . \sin\theta = \sqrt{11} . \sqrt{44} . \sin\theta = 0 \quad \Big\} \begin{array}{l} \sin\theta = 0 \\ \theta = 0^\circ \end{array}$$

∴ The planes are parallel.

Q8)

$P = (1,0,-1)$ and $M = (x,y,z)$

$\vec{PM} = \langle x-1, y, z+1 \rangle$ and $\vec{n} \perp \vec{PM}$

$\vec{PM} \cdot \vec{n} = |PM| \cdot |n| \cdot \cos 90 = 0$

$5(x-1) + 1(y) - 2(z+1) = 0$ } point normal form of the plane

$5x - 5 + y - 2z - 2 = 0 \Rightarrow 5x + y - 2z - 7 = 0$ } General eq. of the plane

Q9) Find the position vector by calculating $\vec{QP}$

$\vec{QP} = \langle 0,2,3 \rangle \rightarrow$ Position vector

The point $P = (0,2,1)$ is on the plane

} Find the parametric equation

$$\left.\begin{aligned} x &= 0 + 0.t \\ y &= 2 + 2.t \\ z &= 1 + 3.t \end{aligned}\right\}$$

line eq. : $(0, 2+2t, 1+3t)$

Q10)

let the plane equation be : $ax+by+cz-d=0$

$\underbrace{ax+by+cz-d=0}_{\vec{N_1}=\langle a,b,c\rangle} \;//\; \underbrace{x+2y+5z-3=0}_{\vec{N_2}=\langle 1,2,5\rangle}$

Since these lines are parallel,

$\frac{a}{1}=\frac{b}{2}=\frac{c}{5}=k \Rightarrow$ plane: $kx+2ky+5kz-d=0$

P is on the plane $\Rightarrow 5k-4k+5k-d = 6k-d=0$

! With every $k \in R$ ve get a different plane

For $k=1$, $d=6$. and plane: $x+2y+5z-6=0$

Q11) Let the plane equation be : $ax+by+cz+d=0$

Normal of the plane is $\vec{N}=\langle a,b,c\rangle$

line equation $\Rightarrow (1+3t,-1-2t,3-t)=P_0+\vec{V}.t$ $\left\{ \begin{array}{l} P_0=(1,-1,3) \\ \vec{V}=\langle 3,-2,-1\rangle \end{array} \right.$

① $\vec{N}\perp\vec{V} \Rightarrow \vec{N}.\vec{V}=3a-2b-c=0$

② P_0 is on the plane $\Rightarrow a-b+3c+d=0$

The plane is orthagonal to the plane given by $x+y-z=0$ those normal vector is $\vec{N_1}=\langle 1,1,-1\rangle$

③ $\vec{N}\perp\vec{N_1}=\vec{N}.\vec{N_1}=a+b-c=0$

Using the equations ①, ② and ③:

$2a=3b$, $c=a+b$ $a-b+3(a+b)+d=0$ $4a+2b+d=0$	$8b+d=0$ * Since we do not have enough data, there is an infinite amount of planes that can be written	for $b=2$; The plane equation is: $3x+2y+5z-16=0$

Q12)

let the equation of the plane be $ax+by+cz+d=0$

Normal vector of the plane is $\vec{N}=\langle a,b,c\rangle$

$\overrightarrow{P_2P_1}=\langle 1,0,4\rangle$, $\overrightarrow{P_2P_1}\perp\vec{N}\Rightarrow\overrightarrow{P_2P_1}\cdot\vec{N}$

(1) $a+4c=0$

The plane is perpendicular to the plane $2x+2y+3z=1$ those normal vector is $\vec{N_1}=\langle 2,2,3\rangle$

$\vec{N}\perp\vec{N_1}\Rightarrow\vec{N}\cdot\vec{N_1}=0$

(2) $2a+2b+3c=0$

The points P_1 and P_2 are on the plane

(3) $a+2b+3c+d=0$

(4) $2b-c+d=0$

Using (1) and (2) $\Rightarrow 2b=5c$ ... (5)

Using (5) and (4) $\Rightarrow 4c+d=0$

Since we do not have enough data, there is an infinite amount of planes that can be written

For $c=-2\Rightarrow$ The plane eq: $8x-5y-2z+8=0$

Q13)

The line described $\Rightarrow (1-2t, 1+t, 5-2t) = P_0 + \vec{V}.t \left\{ \begin{array}{l} P_0 = (1,1,5) \\ \vec{V} = \langle -2,1,-2 \rangle \end{array} \right.$

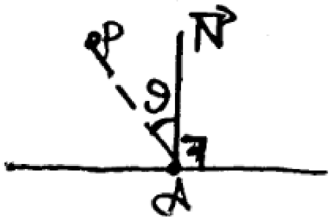
The plane given has the normal vector of $\vec{N} = \langle 1,2,0 \rangle$

$\vec{V} \cdot \vec{N} = |\vec{V}| . |\vec{N}| . \cos\theta = 3 . \sqrt{3} . \cos\theta = 0 \Rightarrow \cos\theta = 0 \Big\} \theta = 90°$

The line and the normal are perpendicular $\Rightarrow$ The line and the plane are parallel

Q14) let $A = (0,0,-3)$ be a point on the plane $\Rightarrow \vec{AP} = \langle 1,3,3 \rangle$

The normal vector of the plane is $\vec{N} = \langle 5,4,-1 \rangle$

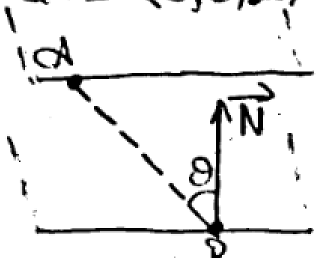


$\vec{AP} . \vec{N} = |\vec{AP}| . |\vec{N}| . \cos\theta = 14$

$\cos\theta = \frac{14}{\sqrt{19} . \sqrt{42}}$, distance is $|\vec{AP}| . \cos\theta = 2,16$

Q15) let $A = (0,0,-3)$ and $B = (0,0,-5)$ be points on the planes respectively.

$\vec{BA} = \langle 0,0,2 \rangle$ and the normal vectors for the planes $\vec{N} = \langle 5,4,-1 \rangle$



$\vec{BA} . \vec{N} = |\vec{BA}| . |\vec{N}| . \cos\theta = -2 \Big\} \cos\theta = \frac{-2}{2 . \sqrt{42}}$

distance $= |\vec{AB}| . \cos\theta$

$= \left| \cancel{2} . \frac{-2}{\cancel{2} . \sqrt{42}} \right|$

$= 0,31$

Q16) let $P = (a, b, c)$

$$\left.\begin{array}{l} ① \ a-4b-c=-3 \\ ② \ 2a-3b+2c=0 \end{array}\right\} \quad 1) \text{ for } c=-1 \quad P_1=(4,2,-1) \qquad 2) \text{ for } c=-6 \quad P_2=(15,6,-6)$$

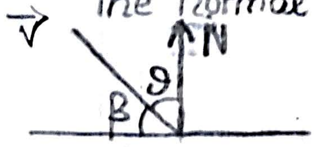
$$\overrightarrow{P_1P_2} = \langle 11, 4, -5 \rangle \} \text{ Location vector}$$

The equations for the line are:

$$x = 4 + 11t$$
$$y = 2 + 4t$$
$$z = -1 - 5t$$

Q17) The location vector of the given line is $\vec{V} = \langle -2, 1, -2 \rangle$

The normal vector of the given plane is $\vec{N} = \langle 1, 1, -1 \rangle$



$$\theta + \beta = 90^\circ \qquad \vec{V}\cdot\vec{N} = \underbrace{|\vec{V}|}_{3}.\underbrace{|\vec{N}|}_{\sqrt{3}}.\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{3\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right) \approx 79^\circ \Rightarrow \beta \approx 90 - 79 = 11^\circ$$

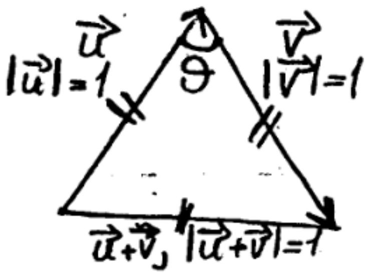
18) If the lines intersect, they must have a common point

$$\begin{aligned} 1-2t &= 3s \\ 1+2t &= 1-s \\ 5-2t &= 1+2s \end{aligned} \Rightarrow \left.\begin{aligned} 3s+2t &= 1 \\ s+2t &= 0 \\ 2s+2t &= 4 \end{aligned}\right\} \text{Solve for } s,t \begin{bmatrix} 3 & 2 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \end{array}\right] \begin{array}{l} E_{2,1}(-3) \\ E_{2,3}(-2) \end{array} \longrightarrow \left[\begin{array}{cc|c} 0 & -4 & 1 \\ 1 & 2 & 0 \\ 0 & -2 & 4 \end{array}\right] E_3(-1/2) \longrightarrow \left[\begin{array}{cc|c} 0 & -4 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{array}\right]$$

$$E_{3,1}(4) \longrightarrow \left[\begin{array}{cc|c} 0 & 0 & -7 \\ 1 & 2 & 0 \\ 0 & 1 & -2 \end{array}\right] \Big\} \text{We got an inconsistent System} \Rightarrow \text{The lines don't intersect}$$

Q19)



When we add the two vectors, we get a triangle those sides are equal.

Therefore all of the interior angles of the triangle is θ, and since $3\theta = 180$, θ is 60

$$|\vec{v} \times \vec{u}| = |\vec{v}| . |\vec{u}| . \sin\theta = 1.1.\sin 60 = \frac{\sqrt{3}}{2}$$

Q20) Gauss-Schwarz Inequality : $|\vec{u}.\vec{v}| \leq \|\vec{u}\|.\|\vec{v}\|$

Verify the inequality using $\vec{u} = \langle 2,0,-1,4 \rangle$ and $\vec{v} = \langle 0,2,1,1 \rangle$

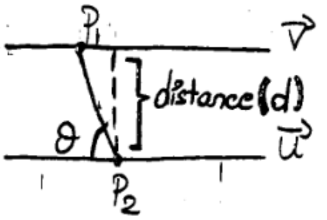
$$|\vec{u}.\vec{v}| = 2.0 + 0.2 + (-1).(1) + 4.1 = 3 = \sqrt{9}$$

$$|\vec{u}| = \sqrt{21} \, , \, |\vec{v}| = \sqrt{6} \Rightarrow |\vec{u}|.|\vec{v}| = \sqrt{126}$$

As shown above, $|\vec{u}.\vec{v}| = \sqrt{9} < \sqrt{126} = |\vec{u}|.|\vec{v}|$

Q21) location vector for line 1 = $\vec{v} = \langle -2,3,-2 \rangle$ and $P_1 = (1,1,5)$

location vector for line 2 = $\vec{u} = \langle 3,-1,2 \rangle$ and $P_2 = (0,1,1)$



$\overrightarrow{P_2P_1} = \langle 1,0,4 \rangle$, $\vec{u} \cdot \overrightarrow{P_2P_1} = |\vec{u}| \cdot |\overrightarrow{P_2P_1}| \cdot \cos\theta$

$\sqrt{14} \cdot \sqrt{17} \cdot \cos\theta = 11 \Rightarrow \cos\theta = \frac{11}{\sqrt{14} \cdot \sqrt{17}} \Rightarrow \sin\theta = 0,7$

$d = |\overrightarrow{P_2P_1}| \cdot \sin\theta = 2,89$

Q22) * I have assumed you meant to write:

$\underline{\underline{c_1}}(1,0,2,0) + \underline{\underline{c_2}}(1,-2,-2,0) + \underline{\underline{c_3}}(0,1,1,-2) = (2,-1,3,1)$

$$\left.\begin{array}{l} c_1 + c_2 = 2 \\ -2c_2 + c_3 = -1 \\ 2c_1 - 2c_2 + c_3 = 3 \\ -2c_3 = 1 \end{array}\right\} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 2 & -2 & 1 & 3 \\ 0 & 0 & -2 & 1 \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_4(-\frac{1}{2})} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

$$\xrightarrow[E_{4,2}(-1)]{E_3(\frac{1}{2})} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -\frac{1}{2} \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow[E_2(-\frac{1}{2})]{E_{3,1}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array}\right]$$

The system is inconsistent. therefore there are no c_1, c_2, c_3 that that meet the requirements.

Q23)

The x variables differ between $x_1 \dots x_3$ } Domain is R^3
The w results differ betwen $w_1 \dots w_3$ } Codomain is R^3 } $f: R^3 \to R^3$

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & -1 \\ 5 & 3 & -3 \end{bmatrix}}_{\text{Standard Matrix}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

∇ To find the range, find the constraint on w_i when they are in reduced echolon form.

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & w_1 \\ 1 & -3 & -1 & w_2 \\ 5 & 3 & -3 & w_3 \end{array}\right] \xrightarrow[E_{2,3}(-5)]{E_{2,1}(-2)} \left[\begin{array}{ccc|c} 0 & 9 & 1 & w_1-2w_2 \\ 1 & -3 & -1 & w_2 \\ 0 & 18 & 2 & w_3-5w_2 \end{array}\right] \xrightarrow{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 0 & 9 & 1 & w_1-2w_2 \\ 1 & -3 & -1 & w_2 \\ 0 & 0 & 0 & w_3-w_2-2w_1 \end{array}\right]$$

$$E(1/9) \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1/9 & w_1-2w_2/9 \\ 1 & -3 & -1 & w_2 \\ 0 & 0 & 0 & w_3-w_2-2w_1 \end{array}\right] \xrightarrow{E_{1,2}(3)} \left[\begin{array}{ccc|c} 0 & 1 & 1/9 & (w_1-2w_2)/9 \\ 1 & 0 & -2/3 & (w_1+w_2)/3 \\ 0 & 0 & 0 & w_3-w_2-2w_1 \end{array}\right]$$

$$E_{1,2} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2/3 & (w_1+w_2)/9 \\ 0 & 1 & 1/9 & (w_1-2w_2)/3 \\ 0 & 0 & 0 & w_3-w_2-2w_1 \end{array}\right]$$

} $w_3 - w_2 - 2w_1 = 0$
∇ Constraint

Q24)

a) Standart matrix for the equation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

} Result is $(2,-1,3)$

b) Standard matrix for the equation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right\}$$ Result is $(1,0,5)$

c) Standard matrix for the operation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$ Result is $(1,0,0)$

d) The standard matrix for the operation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix}$

$$\left.\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3}-3/2 \\ -1+3\sqrt{3}/2 \end{bmatrix} \right\}$$ Result: $\left(1, -\sqrt{3}-\frac{3}{2}, -1+\frac{3\sqrt{3}}{2}\right)$

e)
$$\underbrace{\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\substack{\text{Reflection ar.} \\ x=z}} \cdot \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\substack{\text{Dilation by} \\ 2}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix}}_{\substack{\text{Rotation by } 30^\circ \\ \text{around } x \text{ axis}}}}_{} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Standard matrix for the operation $= \begin{bmatrix} 0 & 1 & \sqrt{3} \\ 0 & -\sqrt{3} & 1 \\ 2 & 0 & 0 \end{bmatrix}$

The result : $\begin{bmatrix} 0 & 1 & \sqrt{3} \\ 0 & -\sqrt{3} & 1 \\ 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\sqrt{3}-2 \\ 3+2\sqrt{3} \\ 2 \end{bmatrix}$

1

f)

$$= \underbrace{\begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix}}_{\text{Rotation around y axis by } 90^\circ} \cdot \underbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{\text{Contr. with factor 2}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Orthogonal pr. onto x-axis}} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -1/2 \end{bmatrix}$$

, Standard matrix for the operation is $\Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/2 & 0 & 0 \end{bmatrix}$

Q25)

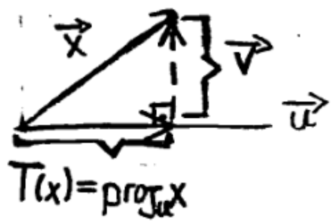
$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $T_2 \cdot T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

! $T_1 \cdot T_2$ is firstly projects to x-z plane, then projects to x-y plane
$T_2 \cdot T_1$ is firstly projects to x-y plane, then projects to x-z plane

Since those two operations give the same results, $T_1 \cdot T_2 = T_2 \cdot T_1$.

Q26)



$\vec{v} \perp \vec{u}$ and $T(x) \parallel \vec{u} \Rightarrow \vec{v} \perp T(x)$ ①

$T(x) + \vec{v} = x$, $\text{proj}_u x + \vec{v} = x$

$v = x - \text{proj}_u x$, and according to ①.

$(x - \text{proj}_u x) \perp \text{proj}_u x$

$a_1)$ $\underbrace{\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ for the $T(x): R^3 \to R^3$ to be one-to-one, A must be invertible. Therefore $|A| \neq 0$

$$|A| = \begin{vmatrix} 1 & 3 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{vmatrix} \xrightarrow[E_{1,3}(-5)]{E_{1,2}(-2)} \begin{vmatrix} 1 & 3 & -1 \\ 0 & -9 & 1 \\ 0 & -9 & 1 \end{vmatrix} \xrightarrow{E_{2,3}(-1)} \begin{vmatrix} 1 & 3 & -1 \\ 0 & -9 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

A is <u>not</u> invertible, Therefore, the transformation is <u>not</u> one-to-one

$a_2)$ Find the range

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 2 & -3 & -1 & w_2 \\ 5 & 6 & -4 & w_3 \end{array}\right] \xrightarrow[E_{1,3}(-5)]{E_{1,2}(-2)} \left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 0 & -9 & 1 & w_2-2w_1 \\ 0 & -9 & 1 & w_3-5w_1 \end{array}\right] \xrightarrow{E_{2,3}(-1)} \left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 0 & -9 & 1 & w_2-2w_1 \\ 0 & 0 & 0 & w_3-3w_1-w_2 \end{array}\right]$$

For the system to be consistent, $w_3 - w_2 - 3w_1$ must be 0

The constraint for the range is $w_3 - w_2 - 3w_1 = 0$

There is not an inverse transformation for this transformation

b) $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & -3 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ $\Bigg\}$ For the $T(x) : R^3 \to R^3$ to be one-to-one A must be invertible. Therefore $|A| \neq 0$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 0 & 3 \end{vmatrix} \xrightarrow[E_{1,3}(-2)]{E_{1,2}(-1)} \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} -2 & -2 \\ -2 & 1 \end{vmatrix} = -6 \neq 0$$

A is invertible, therefore the transformation is one-to-one

Range of the transformation is R^3

b2) To find the inverse transformation, first find A^{-1}

$$A^{-1} = \frac{adj(A)}{|A|} = -\frac{1}{6} \cdot \begin{bmatrix} -3 & -3 & 0 \\ -5 & 1 & 2 \\ 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 5/6 & -1/6 & -1/3 \\ -1/3 & -1/3 & 1/3 \end{bmatrix}$$

Therefore the inverse transformation $T_{A^{-1}} \; R^3 \to R^3$ is :

$$w_1 = \frac{x_1}{2} + \frac{x_2}{2} \qquad w_2 = \frac{5x_1}{6} - \frac{x_2}{6} - \frac{x_3}{3} \qquad w_3 = \frac{-x_1}{3} - \frac{x_2}{3} + \frac{x_3}{3}$$

Q28) We can notate the given transformation as:

$w_1 = 2x_1 + 3x_2$
$\underbrace{\forall \; w_2 = 1 \; \forall}_{}$ $\Bigg\}$ We can not define a matrix A where $A \cdot x = w$, so the transformation is not linear

w_2 doesn't get effected as x_1, x_2 changes

Q29) let the base vectors be: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

① Rotation around z axis by 45°

$$e_1' = \begin{bmatrix} \cos 45 \\ \sin 45 \\ 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \quad e_2' = \begin{bmatrix} -\sin 45 \\ \cos 45 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} \quad e_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

② Orthogonal projection onto the x axis

$$e_1'' = \begin{bmatrix} \sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix} \quad e_2'' = \begin{bmatrix} -\sqrt{2}/2 \\ 0 \\ 0 \end{bmatrix} \quad e_3'' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To find the standard matrix, write the e_i'' together

$$\text{St. Matrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q30) Let the standard matrices for the transformations be A, B respectively

$T_1 \rightarrow$ One-to-one $\Rightarrow |A| \neq 0$
$T_2 \rightarrow$ Not one-to-one $\Rightarrow |B| = 0$

Let the st. matrix for $T_1.T_2$ be C
Let the st. matrix for $T_2.T_1$ be D

$|C| = |A.B| = |A|.\underbrace{|B|}_{0} = 0 \Rightarrow T_1.T_2$ is not one-to-one

$|D| = |B.A| = \underbrace{|B|}_{0}.|A| = 0 \Rightarrow T_2.T_1$ is not one-to-one

Q31)

a) $V: R^2 \to (2x, x)$ } Test the axioms one by one

✓ 1- let $u=(2a,a)$ and $v=(2b,b) \Rightarrow u+v = (2(a+b),(a+b)) \in V$

✓ 2- let $u=(2a,a)$, $v=(2b,b)$ and $w=(2c,c)$

$$\left.\begin{array}{l} u+(v+w) = (2a,a)+(2(b+c),(b+c)) = (2(a+b+c),(a+b+c)) \\ (u+v)+w = (2(a+b),(a+b))+(2c,c) = (2(a+b+c),(a+b+c)) \end{array}\right\} \text{equal}$$

✓ 3- $u+v = (2(a+b), a+b)$, $v+u = (2(b+a), b+a)$ } equal

✓ 4- $u=(2a,a)$ and $0=(0,0) \in V \Rightarrow u+0 = 0+u = (2a,a) = u$

✓ 5- $u=(2a,a)$ and $-u=(-2a,-a) \in V \Rightarrow u+(-u) = (0,0) = 0$

✓ 6- $u=(2a,a)$ and $ku=(2ak,ak) \in V$

$$\left.\begin{array}{l} ✓\ 7-\ u+v=(2(a+b), a+b),\ k(u+v) = (2k(a+b), k(a+b)) \\ k.u=(2ka,ka),\ kv=(2kb,kb),\ ku+kv = (2k(a+b), k(a+b)) \end{array}\right\} \text{equal}$$

✓ 8- $(k+m)u = (2(k+m)a, (k+m)a)$, $ku=(2ka,ka)$, $mu=(2ma,ma)$

$ku+mu = (2(k+m)a,(k+m)a) = (k+m)u$

$$\left.\begin{array}{l} ✓\ 9-\ k(mu) = k.(2ma,ma) = (2kma, kma) \\ m(ku) = m.(2ka,ka) = (2mka, mka) \end{array}\right\} \text{equal}$$

✓ 10- $1.u = (1.2a, 1.a) = (2a,a) = u$

$\therefore$ axioms are true for every $u \in V \Rightarrow V$ is a vector space

b) $V: R^2 \to (2x+1, x)$ } Test axioms one by one

X 1- $u=(2a+1,a)$, $v=(2b+1,b) \Rightarrow u+v = (2(a+b+1),(a+b)) \notin V$

$\therefore 2(a+b+1) = 2a+2b+2 \neq 2(a+b)+1$. Since the axiom is not true for V, it is <u>not</u> a vector space

c) $V: R^2 \to (x,y),\ x \geq 0$ } Test axioms one by one

✓1- $u=(a,b)$, $v=(d,e)$ and $a \geq 0, d \geq 0$

$u+v=(a+d,\ b+e) \in V$

✓2- $u+v=(a+d,\ b+e)$ and $v+u=(d+a,\ e+b)$ } equal

✓3- $u=(a,b)$, $v=(d,e)$, $w=(f,g)$ and $a \geq 0,\ d \geq 0,\ f \geq 0$

$$\left.\begin{aligned} u+(v+w) &= (a,b)+(d+f,\ e+g) = (a+d+f,\ b+e+g) \\ (u+v)+w &= (a+d,\ b+e)+(f,g) = (a+d+f,\ b+e+g) \end{aligned}\right\} \text{equal}$$

✓4- $u=(a,b)$ and $0=(0,0) \in V$

$u+0=0+u=(a,b)=u$

X5- $u=(a,b)$, $-u=(-a,-b) \notin V$ (if $a \neq 0$)

$\therefore$ for any $a>0$ the axiom is not met.

V is <u>not</u> a vector space

d) $V: R^2 \to (1,y)$ } Test axioms one by one

X1- $u=(1,b)$ $v=(1,e)$

$u+v=(2,\ b+e) \notin V$

$\therefore$ Since $2 \neq 1$ the axiom is not met

V is <u>not</u> a vector space

Q32) $V: M_{2x2} \rightarrow \begin{bmatrix} x & 2x \\ 3x & 4x \end{bmatrix}, x \in R$ } Test the axioms

✓1- $u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}, v = \begin{bmatrix} b & 2b \\ 3b & 4b \end{bmatrix}$ and $u+v = \begin{bmatrix} (a+b) & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} \in V$

✓2- $u+v = \begin{bmatrix} (a+b) & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix}, v+u = \begin{bmatrix} (b+a) & 2(b+a) \\ 3(b+a) & 4(b+a) \end{bmatrix}$ } equal

✓3- $u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}, v = \begin{bmatrix} b & 2b \\ 3b & 4b \end{bmatrix}, w = \begin{bmatrix} c & 2c \\ 3c & 4c \end{bmatrix}$

$$\left.\begin{aligned} u+(v+w) &= \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} b+c & 2(b+c) \\ 3(b+c) & 4(b+c) \end{bmatrix} = \begin{bmatrix} a+b+c & 2(a+b+c) \\ 3(a+b+c) & 4(a+b+c) \end{bmatrix} \\ (u+v)+w &= \begin{bmatrix} a+b & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} + \begin{bmatrix} c & 2c \\ 3c & 4c \end{bmatrix} = \begin{bmatrix} a+b+c & 2(a+b+c) \\ 3(a+b+c) & 4(a+b+c) \end{bmatrix} \end{aligned}\right\} \text{equal}$$

✓4- $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in$ and $u+0 = 0+u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = u$

✓5- $u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}$ and $-u = \begin{bmatrix} -a & -2a \\ -3a & -4a \end{bmatrix} \in V$ $u+(-u) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

✓6- $k.u = \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} \in V$

✓7-
$$\left.\begin{aligned} k.(u+v) &= k. \begin{bmatrix} a+b & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} = \begin{bmatrix} k(a+b) & 2k(a+b) \\ 3k(a+b) & 4k(a+b) \end{bmatrix} \\ ku+kv &= \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} + \begin{bmatrix} kb & 2kb \\ 3kb & 4kb \end{bmatrix} = \begin{bmatrix} k(a+b) & 2k(a+b) \\ 3k(a+b) & 4k(a+b) \end{bmatrix} \end{aligned}\right\} \text{equal}$$

✓8-
$$\left.\begin{aligned} (k+m).u &= (k+m). \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = \begin{bmatrix} (k+m)a & 2(k+m)a \\ 3(k+m)a & 4(k+m)a \end{bmatrix} \\ ku+mu &= \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} + \begin{bmatrix} ma & 2ma \\ 3ma & 4ma \end{bmatrix} = \begin{bmatrix} (k+m)a & 2(k+m)a \\ 3(k+m)a & 4(k+m)a \end{bmatrix} \end{aligned}\right\} \text{equal}$$

✓9-
$$\left.\begin{aligned} k(mu) &= k. \begin{bmatrix} ma & 2ma \\ 3ma & 4ma \end{bmatrix} = \begin{bmatrix} kma & 2kma \\ 3kma & 4kma \end{bmatrix} \\ (km).(u) &= \begin{bmatrix} kma & 2kma \\ 3kma & 4kma \end{bmatrix} \end{aligned}\right\} \text{equal}$$

✓10- $1.u = 1. \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = u$

∇ All axioms are met for every $u \in V$

V is a vector space

Q33) Test the axioms

✓ 1- $u = a$, $v = b$ and $a, b \in R \Rightarrow u+v = \sqrt{u^2+v^2} \in V$

✓ 2- $u+v = \sqrt{a^2+b^2}$, $v+u = \sqrt{b^2+a^2} \Rightarrow u+v = v+u$

✓ 3- $u = a$, $v = b$, $w = c \Rightarrow u+(v+w) = \sqrt{a^2+(\sqrt{b^2+c^2})^2} = \sqrt{a^2+b^2+c^2}$ $\Big\}$ equal

$(u+v)+w = \sqrt{(\sqrt{a^2+b^2})^2+c^2} = \sqrt{a^2+b^2+c^2}$.

X 4- $\vec{0} = 0 \in V$ and $u+0 = \sqrt{a^2+0} = \sqrt{a^2} = |a| \stackrel{?}{=} a$

The axiom is not met when $a < 0$

Ex: $u = -1 \Rightarrow u+0 = \sqrt{(-1)^2+0} = \sqrt{1} = 1 \neq -1$

$\therefore$ Since all axioms are not met the set is not a vector space

Q34) a) $W = M_{n \times n}$, $|M| = 0$

let $u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}_{n \times n}$ and $v = \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}_{n \times n}$

$|u| = 1 . 0 . \ldots . 0 = 0$ and $|v| = 0 . 1 . \ldots . 1 = 0$

$u+v = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}_{n \times n}$ and $|u+v| = 1 \ldots 1 = 1 \neq 0$.

Since $u+v \notin W$, W is not a subspace

b) $W = M_{nn},\ |M| \neq 0$

let $u = \begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}$ and $|u| \neq 0$

let $k = 0$

$k.u = 0.\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} = \vec{0}$

$|k.u| = 0.\ 0 \ldots 0 . 0 = 0$

Since $|k.u| \notin W$, W is not a subspace

Q35)

a) $W = M_{n \times m},\ a_{11} = a_{12} = \ldots = a_{1m} = 0$

let $u = \begin{bmatrix} 0 & \cdots & 0 \\ u_{21} & \cdots & u_{2n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nm} \end{bmatrix}$ and $v = \begin{bmatrix} 0 & \cdots & 0 \\ v_{21} & \cdots & v_{2n} \\ \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{nm} \end{bmatrix}$

Since W is closed under addition and mult, W is a subspace

$u+v = \begin{bmatrix} 0 & \cdots & 0 \\ u_{21}+v_{21} & \cdots & u_{2n}+v_{2n} \\ \vdots & \ddots & \vdots \\ u_{n1}+v_{n1} & \cdots & u_{nm}+v_{nm} \end{bmatrix} \in W$

$k.u = k.\begin{bmatrix} 0 & \cdots & 0 \\ u_{21} & \cdots & u_{2n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nm} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ k.u_{21} & \cdots & k.u_{2n} \\ \vdots & \ddots & \vdots \\ k.u_{n1} & \cdots & k.u_{nm} \end{bmatrix} \in W$

b) $W = M_{n\times m} : a_{11} \dots a_{1m} \neq 0$

let $\vec{u} = \begin{bmatrix} 1 & \dots & 1 \\ u_{21} & \dots & u_{2m} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nm} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 & \dots & -1 \\ v_{21} & \dots & v_{2m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \dots & v_{nm} \end{bmatrix}$

Test the axioms :

$u + v = \begin{bmatrix} 0 & \dots & 0 \\ u_{21}+v_{21} & \dots & u_{2m}+v_{2m} \\ \vdots & \ddots & \vdots \\ u_{n1}+v_{n1} & \dots & u_{nm}+v_{nm} \end{bmatrix} \notin W$ } Since W is not closed under addition, it is not a subspace

Q36)

$2\times3 \quad 3\times1$

a) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow[E_{1,3}(-1)]{E_{1,2}(-2)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & -1 & 5 & 0 \end{array}\right] \xrightarrow{E_{2,3}(1)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 13 & 0 \end{array}\right]$

$E_3(\frac{1}{13}) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \xrightarrow[E_{3,1}(2)]{E_{3,2}(-8)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Big\} \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array}$

Solution space for the given part is $W_{(x_1,x_2,x_3)} : x_1 = x_2 = x_3 = 0$

$W = \langle (0,0,0) \rangle$

b) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-2)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \end{array}\right]$

$\left.\begin{array}{l} x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \\ x_2 + 8x_3 = 0 \Rightarrow x_2 = -8x_3 \end{array}\right\}$ let $x_3 = t \Rightarrow \begin{array}{l} x_1 = 2t \\ x_2 = -8t \\ x_3 = t \end{array}$ Infinite solutions for t

Solution space for the given part is $W(x_1, x_2, x_3) : x_1 = 2t, x_2 = -8t, x_3 = t, t \in \mathbb{R}$

Ex: for $t=1$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 1 & 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 1 & 8 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-2)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 1 & 8 & 0 \end{array}\right] \xrightarrow{E_{2,3}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\left.\begin{array}{l} x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \\ x_2 + 8x_3 = 0 \Rightarrow x_2 = -8x_3 \end{array}\right\}$ let $x_3 = t \Rightarrow \begin{array}{l} x_1 = 2t \\ x_2 = -8t \\ x_3 = t \end{array}$ Infinite solution for t

Solution space for the given part is $W(x_1, x_2, x_3) : x_1 = 2t, x_2 = -8t, x_3 = t, t \in \mathbb{R}$

Ex: for $t=2$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -16 \\ 2 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 0 & -4 \\ -1 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 0 & -4 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right] \xrightarrow[E_{1,3}(1)]{E_{1,2}(-2)} \left.\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\right\} \text{Infinite solutions}$$

$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$.

The solution space for the given part is $W_{(x_1,x_2,x_3)} : x_1 = 2x_3$

Ex: for $x_3 = 1$ and $x_3 = 2$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$