2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = V_1 \cdot c_1 + V_2 c_2 + V_3 \cdot c_3$

$$\left.\begin{array}{llll} 4c_1 & -2c_2 & & = a \\ 3c_1 & & +c_3 & = b \\ 2c_1 & -c_2 & & = c \\ c_1 & & +c_3 & = d \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 1 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow[E_{3,4}(-2)]{E_{1,4}(-4) \; E_{2,4}(-3)} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & -2 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$\xrightarrow{E_{3,1}(-2)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & -2 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow[E_{3,2}(-1)]{E_{4,2}(\frac{1}{2})} \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & -2 & b-3d \\ 0 & -1 & 0 & c-b+3d \\ 1 & 0 & 0 & b/2-d/2 \end{array}\right]$$

For the system to be consistent, $a-2c$ must be 0

Given vectors span a subspace consisting of vectors K, that

$$K = \begin{bmatrix} 2c \\ b \\ c \\ d \end{bmatrix} \; ; \; b, c, d \in \mathbb{R}$$

b) The vectors are linearly independent if and only if the eq. $V_1 \cdot c_1 + V_2 \cdot c_2 + V_3 \cdot c_3 = 0$ only has the trivial solution

$$\left.\begin{array}{llll} 4c_1 & -2c_2 & & = 0 \\ 3c_1 & & +c_3 & = 0 \\ 2c_1 & -c_2 & & = 0 \\ c_1 & & +c_3 & = 0 \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Same operations used in the section a}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

Since $c_1 = c_2 = c_3 = 0$, the vectors are linearly independent

c) The dimension for R^4 is 4, therefore any linearly ind. 4 vectors form a basis.
We already have 3 vectors which are linearly independent, but they don't span R^4, if we can find a fourth vector "v" which is not in the span, we can write the basis for R^4 (add-minus theorem)

We found that subspace spanned of the given vectors consists of vectors such:

$$K=\begin{bmatrix}2c\\b\\c\\d\end{bmatrix};\ b,c,d\in\mathbb{R} \text{ therefore } v=\begin{bmatrix}1\\2\\3\\4\end{bmatrix} \text{ is not in the span}$$

A basis for R is $=\{(4,3,2,1),(-2,0,-1,0),(0,3,0,1),(1,2,3,4)\}$

Q2) ∇ $\cos^2x-\sin^2x=\cos 2x \Rightarrow f_1=\cos 2x$ and $f_2=\cos 2x$

Applying Wronskian:

$$\begin{vmatrix}\cos 2x & \cos 2x\\ (\cos 2x)' & (\cos 2x)'\end{vmatrix}=\begin{vmatrix}\cos 2x & \cos 2x\\ -2\sin 2x & -2\sin 2x\end{vmatrix}=-2\sin 2x\cos 2x+2\sin 2x\cos 2x=0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of R^3 is 3, therefore basis for R^3 consists of 3 vectors

Choose u_1,u_3,u_4 $\Rightarrow$ Test for linear independence:

$u_1.c_1+u_3.c_3+u_4.c_4=0$, find c_1,c_3,c_4

$$\left.\begin{matrix}c_1+c_3=0\\ -2c_1\qquad=0\\ -c_3+7c_4=0\end{matrix}\right\}\begin{bmatrix}1&1&0\\-2&0&0\\0&-1&7\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&0\\-2&0&0&0\\0&-1&7&0\end{array}\right]-E_{2,1}(2)\rightarrow\left[\begin{array}{ccc|c}1&1&0&0\\0&2&0&0\\0&-1&7&0\end{array}\right]-E_2(1/2)\rightarrow\left[\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&-1&7&0\end{array}\right]$$

$$\begin{matrix}E_{1,2}(-1)\\E_{3,2}(1)\end{matrix}\rightarrow\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&7&0\end{array}\right]\left.\right\}\begin{matrix}c_1=0\\c_3=0\\c_4=0\end{matrix}\Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1,u_3,u_4 form a basis without testing for their span.

Base $=\{(1,-2,0),(1,0,-1),(0,0,7)\}$

∇ Express the remaining vector, u_2, by the base vectors

$u_1.c_1+u_3.c_3+u_4.c_4=u_2$, find c_1,c_3,c_4

$$\left.\begin{array}{l} c_1 + c_3 = 1 \\ -2c_1 = 2 \\ -c_3 + 7c_4 = 4 \end{array}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{array}{l} c_1 = -1 \\ c_3 = 2 \\ c_4 = 6/7 \end{array}\right\} \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

The dimension of the subspace spanned by these vectors is 4

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^4 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

Q6) The vectors u and v, are the basis for the $u \times v$ plane.

The vector resulting from $\mathrm{proj}_{u \times v} w$ lies in the $u \times v$ plane, therefore it can be expressed as a linear combination of the basis vectors u and v.

Therefore u, v and $\mathrm{proj}_{u \times v} w$ are not linearly independent

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.