

2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 2

1.

$$\det(A) = \cancel{a_{14} \cdot M_{14}} + \cancel{a_{24} \cdot M_{24}} + \cancel{a_{34} \cdot (-M_{34})} + \cancel{a_{44} \cdot M_{44}}$$

$\downarrow 0$ $\downarrow 8$ $\downarrow 0$ $\downarrow -1$

$$M_{24} = \begin{vmatrix} 1 & -2 & 4 \\ 1 & 0 & 6 \\ 2 & 3 & 0 \end{vmatrix} = 2 \cdot \begin{vmatrix} 0 & 6 \\ 3 & 0 \end{vmatrix} - 1 \cdot \begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} + 2 \cdot \begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}$$
$$= -48$$

$$M_{44} = \begin{vmatrix} 2 & -2 & 4 \\ 4 & 1 & 0 \\ 1 & 0 & 6 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} - 4 \cdot \begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix} + 1 \cdot \begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix}$$
$$= 56$$

$$\begin{aligned}\det(A) &= 8 \cdot (-48) + (-1) \cdot 56 \\ &= -440,\end{aligned}$$

* To calculate the determinant, the last column is used because it had two zeroes which made the calculations easier.

2.

Q2)

$$\left[\begin{array}{cccc} 2 & -2 & 1 & 0 \\ 4 & 1 & 0 & 8 \\ 4 & 0 & 6 & 0 \\ 0 & 3 & 0 & -1 \end{array} \right] \rightarrow \left[\begin{array}{cccc} 10 & 0 & 1 & 16 \\ 4 & 1 & 0 & 8 \\ -10 & 8 & 0 & -25 \end{array} \right] \left. \begin{array}{l} E_{2,1}(2) \\ E_{2,4}(-3) \end{array} \right\}$$

$$\det(A) = \cancel{\alpha_{12} \cdot (-M_{12})} + \alpha_{22} \cdot M_{22} + \cancel{\alpha_{32} \cdot (-M_{32})} + \cancel{\alpha_{42} \cdot M_{42}}$$

$$M_{22} = \begin{vmatrix} 10 & 1 & 16 \\ 4 & 0 & 0 \\ -10 & 8 & -25 \end{vmatrix} = 10 \cdot \begin{vmatrix} 6 & 0 \\ 0 & -25 \end{vmatrix} - 1 \cdot \begin{vmatrix} 4 & 16 \\ 0 & -25 \end{vmatrix} - 10 \cdot \begin{vmatrix} 4 & 16 \\ 6 & 0 \end{vmatrix} \\ = -440$$

$$\det(A) = 1 \cdot (-440) = -440,$$

3.

i)

Q3)

$$\text{i) } \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -6 \\ 8 & 1 & 6 \\ 4 & 7 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 2 & 6 \\ -6 & 6 & 6 \end{bmatrix}$$

$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & -1 \\ -6 & 2 \end{vmatrix} = -2 \quad C_{21} = 8 \quad C_{31} = 4$

$C_{12} = -2 \quad C_{22} = 2 \quad C_{32} = 1$

$C_{13} = -6 \quad C_{23} = 6 \quad C_{33} = 6$

$\det(A) = \begin{vmatrix} 1 & -2 & 0 \\ 0 & -6 & 2 \end{vmatrix} = 1 \cdot (-2) + (-1) \cdot (-8) = 6$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \begin{bmatrix} -1/3 & 4/3 & 2/3 \\ -1/3 & 1/3 & 1/6 \\ -7 & 7 & 7 \end{bmatrix}$$

ii)

ii)

$$A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -6 & 2 \\ 0 & -6 & 1 \end{bmatrix} \xrightarrow[E_{2 \leftrightarrow 3}]{} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & -6 & 2 \end{bmatrix} \xrightarrow[E_{2+3}]{} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 8 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}, \quad \det(A) = 0 \quad (\text{row of zeroes})$$

* The inverse of the matrix A doesn't exist since $\det(A)=0$
 Which means such inverse can't be calculated by method
 of adjoints.

4.

Q4)

$$\begin{bmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 1 & -1 & -5 \\ 1 & 4 & 1 & 1 & -1 \\ 2 & -4 & 4 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} \text{ and } x_3 = \frac{\det(A_3)}{\det(A)}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 1 & -1 & -5 \\ 1 & 4 & 1 & 1 & -1 \\ 2 & -4 & 4 & 4 & 0 \end{vmatrix} \stackrel{E_{3,4}(-2)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & -1 & 0 & 4 & -12 \end{vmatrix} \stackrel{E_{3,3}(-3)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & 2 & 2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & -1 & 0 & 4 & -12 \end{vmatrix} \stackrel{E_{3,3}(-1)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & 1 & 0 & 4 & -12 \end{vmatrix} \stackrel{E_{2,5}(-2)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & 1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & 1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot (-1) \cdot \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 1 & 4 & 2 & 1 & -7 \\ 0 & 1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot (-1) \cdot 1 = -1$$

$$\stackrel{E_{4,2}(-2)}{\rightarrow} -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 17 & -55 \\ 1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 0 & 2 & 17 & -55 \\ 2 & 17 & -55 & 1 \\ 1 & 0 & 4 & -12 \end{vmatrix}$$

$$= (2 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -55 \end{vmatrix} + 2 \cdot \begin{vmatrix} 6 & -12 \\ -9 & 1 \end{vmatrix}) \cdot -1$$

$$= 456$$

$$\det(A_1) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 1 & -1 & -5 \\ 1 & 4 & 1 & 1 & -1 \\ 2 & -4 & 4 & 4 & 0 \end{vmatrix} \stackrel{E_{2,4}(-2)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & 0 & -1 & -17 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix} \stackrel{E_{2,3}(-2)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 0 & 2 & 0 & -1 & -17 \\ 0 & -2 & 0 & 1 & -13 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix} \stackrel{E_{3,4}(-2)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 0 & 2 & 0 & -1 & -17 \\ 0 & -2 & 0 & 1 & -13 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix}$$

$$\stackrel{E_{3,4}(1)}{\rightarrow} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 0 & 2 & 0 & -1 & -17 \\ 0 & -2 & 0 & 1 & -13 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 0 & 2 & 0 & -1 & -17 \\ 0 & -2 & 0 & 1 & -13 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot (-1) \cdot \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 0 & 2 & 0 & -1 & -17 \\ 0 & -2 & 0 & 1 & -13 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -4 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot (-1) \cdot 1 = -1$$

$$= (8 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -13 \end{vmatrix} + 16 \cdot \begin{vmatrix} 6 & -12 \\ -9 & -17 \end{vmatrix}) \cdot -1$$

$$= 2352$$

$$\det(A_3) = \begin{vmatrix} 2 & 0 & 2 & 6 & 0 \\ 1 & 0 & 1 & 0 & 6 \\ 3 & -2 & 2 & -1 & -5 \\ 1 & 4 & 2 & 1 & 1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 3 & -2 & 2 & -1 & -5 \\ 2 & -1 & 0 & 4 & 0 \\ 1 & 4 & 2 & 1 & 1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{\substack{E_{1,2}(-1) \\ E_{1,3}(-3)}} \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 \\ 2 & -1 & 0 & 4 & 0 \\ 1 & 4 & 2 & 1 & 1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{\substack{E_{2,3}(1-3) \\ E_{2,4}(1-0)}} \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & -2 & 1 & 10 & -5 \\ 1 & 4 & 2 & 1 & 1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{E_{2,5}(-2)} \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & -2 & 1 & 10 & -5 \\ 0 & 4 & 1 & 12 & -1 \\ 0 & -1 & 2 & -2 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -3 & 6 \\ -2 & -1 & -10 & -5 \\ -4 & -2 & -2 & 0 \end{vmatrix} \xrightarrow{E_{1,3}(4)} \begin{vmatrix} 0 & 0 & -3 & 6 \\ 0 & 3 & -6 & -5 \\ -7 & -2 & -2 & 0 \end{vmatrix} \xrightarrow{E_{3,2}(-2)} \begin{vmatrix} 0 & 0 & -3 & 6 \\ 0 & 3 & -6 & -5 \\ 1 & -2 & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 0 & -3 & 6 \\ 3 & -6 & -5 \\ -7 & -10 & 1 \end{vmatrix} = 2 \cdot (-3) \begin{vmatrix} -3 & 6 \\ -10 & -1 \end{vmatrix} - 7 \cdot \begin{vmatrix} -3 & 6 \\ -6 & -5 \end{vmatrix}$$

$$= -1092$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{2352}{456} \approx 5.16 \text{ and } x_3 = \frac{|A_3|}{|A|} = \frac{-1092}{456} \approx -2.39$$

5.

Q5)

* For the given matrix to be non-invertible, its determinant must be equal to zero.

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ x & 1 & -2 \end{vmatrix} = E_{2,3}(-x) \rightarrow \begin{vmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ 0 & 1-x^2 & -2-5x \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 1 \\ 1-x^2 & -2-5x \end{vmatrix} = 1-x^2$$

$$1-x^2 = 0 \Rightarrow x = -1 \text{ or } x = 1$$

6.

Q6)

$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$, Since A is an upper-triangular matrix, determinant equals to the diagonal product

$$\star \det(A) = 2 \cdot 1 \cdot 6 \cdot (-1) = -12 \star$$

$\text{adj}(A) = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}^T$ Since A is an upper-triangular matrix, cofactors which are placed above the diagonal are all equal to zero

$$C_{11} = \begin{vmatrix} 1 & 2 & 8 \\ 0 & 6 & 5 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

$$C_{21} = 12 \quad C_{22} = -12 \quad C_{31} = 8 \quad C_{32} = 4 \quad C_{33} = -2$$

$$C_{41} = -56 \quad C_{42} = -76 \quad C_{43} = -10 \quad C_{44} = 12$$

$$\text{adj}(A) = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 12 & -12 & 0 & 0 \\ 8 & 4 & -2 & 0 \\ -56 & -76 & -10 & 12 \end{bmatrix}^T = \begin{bmatrix} -6 & 12 & 8 & -56 \\ 0 & 8 & -2 & -76 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \begin{bmatrix} \frac{1}{2} & -1 & -\frac{2}{3} & \frac{14}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{19}{3} \\ 0 & 0 & \frac{1}{6} & \frac{5}{6} \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

7.

Q7)

$$= \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 2 & 12 & -1 \end{vmatrix} = E_{1,3}(-1) \rightarrow \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix}$$

$$= E_{2,1}(2) \\ = E_{2,1}(2) \rightarrow 1 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -8 & 9 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & -8 & 9 \end{vmatrix}$$

$$E_{3,2}(-1) \\ = E_{3,1}(-6) \rightarrow -10 \cdot \begin{vmatrix} 1 & 0 & 0 & \frac{28}{5} \\ 0 & 1 & 0 & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & -\frac{69}{5} \end{vmatrix} = 138 \cdot \begin{vmatrix} 1 & 0 & 0 & \frac{28}{5} \\ 0 & 1 & 0 & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 1 \end{vmatrix} = 138$$

8.

Q8)

$$= \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 0 & 8 & 12 \end{vmatrix} E_{1,3}(-1) \rightarrow = \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & 3 & 5 \\ 0 & 6 & 4 & 12 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 5 \\ 0 & 2 & 3 & 5 \\ 0 & 6 & 4 & 12 \end{vmatrix}$$

$$E_{2,1}(2) \\ E_{2,3}(-2) \rightarrow = 2 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -10 & 6 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & -10 & 6 \end{vmatrix}$$

$$E_{3,2}(-1) \\ E_{3,1}(-6) \rightarrow = -10 \cdot \begin{vmatrix} 1 & 0 & 0 & \frac{28}{5} \\ 0 & 1 & 0 & \frac{8}{5} \\ 0 & 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & 0 \end{vmatrix} = -10 \cdot 0 = 0$$

9.

Q9)

$$A^T = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & -5 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & -5 & 1 \end{vmatrix} = E_{3,1}(-1) \begin{vmatrix} 0 & -9 & 2 \\ 1 & 2 & -3 \\ 1 & -5 & 1 \end{vmatrix} = -23,$$

$$\det((A^T)^6) = \det(\underbrace{A^T \cdot A^T \cdot A^T \cdot A^T \cdot A^T \cdot A^T}_{\text{Square matrices of the same size}}) = \det A^T \cdot \det A^T \dots \det A^T = (\det A^T)^6$$

Square matrices of the
same size

$$\det((A^T)^6) = (\det(A^T))^6 = (-23)^6 = 118035889,$$

10.

Q10)

$$\det(A) = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 3 & 0 & 8 & 12 \end{vmatrix} = E_{1,2}(1) \rightarrow \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 3 & 0 & 8 & 12 \end{vmatrix}$$

$$E_{2,3}(-1) \rightarrow = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 8 & 12 \end{vmatrix} = 0$$

Since $\det(A) = 0$, we can surely say that A^{-1} doesn't exist.

11.

Q11)

$$\begin{aligned}
 \det(B^{-2}CA^{-3}) &= \det(B^{-2}) \cdot \det(C) \cdot \det(A^{-3}) \\
 &= (3)^{-2} \cdot (2)^{-3} \\
 &= \frac{1}{9} \cdot \frac{1}{8} \\
 &= \frac{1}{72}
 \end{aligned}$$

12.

Q12)

let $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$, $w = \langle w_1, w_2, w_3 \rangle$

a) $u \cdot (v \times w)$

$$\begin{aligned}
 &= u \cdot \left(\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right) \\
 &= u \cdot [i(v_2w_3 - v_3w_2) - j(v_1w_3 - v_3w_1) + k(v_1w_2 - v_2w_1)] \\
 &= \underline{u_1 \cdot v_2 \cdot w_3} - \underline{u_1 \cdot v_3 \cdot w_2} - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1
 \end{aligned}$$

b) $v \cdot (u \times w)$

$$\begin{aligned}
 &= v \cdot \left(\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right) \\
 &= v \cdot [i(u_2w_3 - u_3w_2) - j(u_1w_3 - u_3w_1) + k(u_1w_2 - u_2w_1)] \\
 &= \underline{v_1 \cdot u_2 \cdot w_3} - \underline{v_1 \cdot u_3 \cdot w_2} - \underline{v_2 \cdot u_1 \cdot w_3} + \underline{v_2 \cdot u_3 \cdot w_1} + \underline{v_3 \cdot u_1 \cdot w_2} - \underline{v_3 \cdot u_2 \cdot w_1} \\
 &= -(u_1v_2w_3 - u_1v_3w_2 - u_2v_1w_3 + u_2v_3w_1 + u_3v_1w_2 - u_3v_2w_1)
 \end{aligned}$$

c) $w \cdot (u \times v)$

$$= w \cdot \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right)$$

$$= w \cdot [\hat{i}(u_2v_3 - u_3v_2) - \hat{j}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1)]$$

$$= u_1u_2v_3 - u_1u_3v_2 - u_2u_1v_3 + u_2u_3v_1 + u_3u_1v_2 - u_3u_2v_1$$

$$= -(u_1v_2w_3 - u_1u_3w_2 - u_2u_1w_3 + u_2v_3w_1 + u_3v_1w_3 - u_3v_2w_1)$$

* a, b, c have the same magnitude

13.

Q13)

$$\begin{vmatrix} a+b & e+f \\ c-d & g+h \end{vmatrix} = \boxed{ag + ah + bg + bh - ac - ad - cf - fh}$$

$$\begin{vmatrix} a & c \\ e & g \end{vmatrix} = a.g - c.e \quad \begin{vmatrix} b & -d \\ e & g \end{vmatrix} = b.g + d.e$$

$$\begin{vmatrix} a & f \\ -g & h \end{vmatrix} = a.h + c.f \quad \begin{vmatrix} b & -d \\ -g & h \end{vmatrix} = b.h - d.f$$

$$* \begin{vmatrix} a+b & e+f \\ c-d & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & -d \\ e & g \end{vmatrix} + \begin{vmatrix} a & f \\ -g & h \end{vmatrix} + \begin{vmatrix} b & -d \\ -g & h \end{vmatrix}$$

14.

Q14)

$$\begin{vmatrix} k+1 & -4 \\ \frac{k}{2} & k-1 \end{vmatrix} = (k+1)(k-1) - 8$$

$$= k^2 - 1 - 8 = k^2 - 9$$

For the matrix to be non-invertible, $k^2 - 9$ must be 0

$$k^2 - 9 = 0 \Rightarrow k^2 = 9 \Rightarrow k = 3 \quad k = -3$$

15.

Q15)

let $P_1 = (x_1, y_1, z_1)$ and $\vec{OP}_1 = \langle x_1, y_1, z_1 \rangle$

let $P_2 = (x_2, y_2, z_2)$ and $\vec{OP}_2 = \langle x_2, y_2, z_2 \rangle$

$\vec{O} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right)$ and $\vec{OP} = \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right\rangle$

Find the volume of the parallelepiped by $\vec{OP}_1 \cdot (\vec{OP}_2 \times \vec{OP})$

$$V = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} & \frac{z_1+z_2}{2} \end{vmatrix} = x_1 \cdot \left(\frac{y_2 z_1 + y_2 z_2 - z_2 y_1 - z_2 y_2}{2} \right) - x_2 \cdot \left(\frac{y_1 z_1 + y_1 z_2 - z_2 y_1 - z_2 y_2}{2} \right) + \frac{x_1+x_2}{2} \cdot (y_1 z_2 - z_1 y_2) \quad \left. \begin{array}{l} \text{Since } V=0, \\ \text{the vectors} \\ \text{are coplanar} \end{array} \right\} = 0$$

16.

$$x = (4, 0, -2) \quad y = (2, 3, -1) \quad z = (0, 0, 1) \quad \text{plane: } ax+by+cz+d=0$$

$$4a-2c+d=0 \quad 2a+3b-c+d=0 \quad c+d=0 \quad \left. \begin{array}{l} \text{Solve the} \\ \text{equations} \end{array} \right\}$$

$$\left[\begin{array}{ccc|cc} 4 & 0 & -2 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{E_{2,1}(-2)} \left[\begin{array}{ccc|cc} 0 & -6 & 9 & -1 & 0 \\ 0 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

$$E_2\left(\frac{1}{2}\right) \rightarrow \left[\begin{array}{ccc|cc} 0 & -6 & 0 & -1 & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \xrightarrow{E_1\left(-\frac{1}{6}\right)} \left[\begin{array}{ccc|cc} 0 & 1 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right]$$

$$E_{3,2}\left(\frac{1}{2}\right) \rightarrow \left[\begin{array}{ccc|cc} 0 & 1 & 0 & \frac{1}{6} & 0 \\ 0 & \frac{3}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array} \right] \quad \left. \begin{array}{l} b+\frac{d}{6}=0 \\ a+\frac{3}{2}b+d=0 \end{array} \right\} \quad \left. \begin{array}{l} c+d=0 \\ * \text{let } d=12 \\ * \end{array} \right. \\ \downarrow \\ d=12, c=-12, b=-2, a=-9$$

$$\text{plane: } -9x - 2y - 12z + 12 = 0$$

17.

Q(7) let $A = \langle 0, 1 \rangle$ and $B = \langle 2, 0 \rangle$

$$A+B = \langle 2, 1 \rangle$$

$$|A|=1, |B|=2 \text{ and } |A+B|=\sqrt{5}$$

$$|A+B| < |A| + |B|$$

let $C = \langle 0, 0 \rangle$ and $D = \langle 0, 0 \rangle$

$$C+D = \langle 0, 0 \rangle$$

$$|C|=0, |D|=0 \text{ and } |C+D|=0$$

$$|C+D| = |C| + |D|$$

18.

Q(8)

let $A = \langle 0, 2 \rangle$ and $B = \langle 0, 1 \rangle$

$$A-B = \langle 0, 1 \rangle$$

$$|A-B| = |A| - |B|$$

let $C = \langle 0, 3 \rangle$ and $D = \langle 3, 0 \rangle$

$$C-D = \langle -3, 3 \rangle$$

$$|C-D| \geq |C| - |D|$$

19.

$A = (0, 1, 0)$, $B = (0, 0, -1)$, $C = (2, -1, 1)$ are on the plane
 $\vec{AB} = \langle 0, -1, -1 \rangle$ $\vec{AC} = \langle 2, -2, 1 \rangle$

i)

$$\text{i) } \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \underbrace{\langle -3, -2, 2 \rangle}_{\vec{N} = \text{normal vector}}$$

$$\left. \begin{array}{l} u_a = k \cdot \vec{N}, \vec{N} \cdot (u - u_a) = 0 \\ k = \frac{\vec{u} \cdot \vec{N}}{|\vec{N}|^2} = -\frac{7}{17} \end{array} \right\} \quad \left. \begin{array}{l} u_a = \left\langle \frac{21}{17}, \frac{14}{17}, \frac{14}{17} \right\rangle \\ u - u_a = \left\langle \frac{-4}{17}, \frac{20}{17}, \frac{16}{17} \right\rangle \end{array} \right\}$$

ii)

$$\text{i) } \vec{AB} \times \vec{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \underbrace{\langle -3, -2, 2 \rangle}_{\vec{N}}$$

$$\left. \begin{array}{l} u_a = k \cdot \vec{N}, (u - u_a) \cdot \vec{N} = 0 \\ u = k \cdot \vec{N} + (u - u_a) \\ \vec{u} \cdot \vec{N} = k |\vec{N}|^2 + 0 \\ k = \frac{\vec{u} \cdot \vec{N}}{|\vec{N}|^2} = -\frac{7}{17} \end{array} \right\} \quad \left. \begin{array}{l} u_a = \left\langle \frac{21}{17}, \frac{14}{17}, \frac{14}{17} \right\rangle \\ \text{Projection onto plane normal vector} \end{array} \right\}$$

20.

let $P = (x, y, z)$ and $\vec{P_0 P} = \langle x, y-1, z-2 \rangle$

$\vec{v} \cdot \vec{P_0 P}$ should be equal to 0 since they are
+ orthogonal

$$x - (y-1) + (2z-4) = 0$$

$$x - y + 2z = 3 \Rightarrow \text{This is the equation of the plane whose normal vector is } \vec{v}$$

21.

i)

Q21)

$$\text{i) } \vec{d} = \langle x, y, z \rangle$$

$$\vec{d} \cdot \langle 1, 0, -1 \rangle = x - z = 0 \text{ (orthogonality)}$$

$$x = z = 0 \Rightarrow x = z$$

$$\vec{d} = \langle x, y, x \rangle \text{ and } |\vec{d}| = \sqrt{x^2 + y^2}$$

For all values of x, y ; the vector $\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \rangle$ meets the requirements.

ii)

ii) The product of two vectors is orthogonal to both of them

$$\langle 1, 0, -1 \rangle \times \langle 1, 1, -1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1, 0, 1 \rangle$$

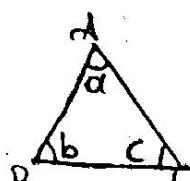
Since $\langle 1, 0, 1 \rangle$ is orthogonal, $-1 \cdot \langle 1, 0, 1 \rangle = \langle -1, 0, -1 \rangle$ is also orthogonal.

Convert the vectors into unit norm;

$$\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle \text{ and } \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$$

22.

Q22)



$$\vec{AB} \times \vec{AC} = |\vec{AB}| \cdot |\vec{AC}| \cdot \sin(\alpha) \Rightarrow \frac{|\vec{AB} \times \vec{AC}|}{|\vec{AB}| \cdot |\vec{AC}|} = \sin \alpha$$

$$\vec{CA} \times \vec{CB} = |\vec{CA}| \cdot |\vec{CB}| \cdot \sin(c) \Rightarrow \frac{|\vec{CA} \times \vec{CB}|}{|\vec{CA}| \cdot |\vec{CB}|} = \sin c$$

$$\vec{BA} \times \vec{BC} = |\vec{BA}| \cdot |\vec{BC}| \cdot \sin(b) \Rightarrow \frac{|\vec{BA} \times \vec{BC}|}{|\vec{BA}| \cdot |\vec{BC}|} = \sin b$$

Since the value of $\sin \theta$ gets higher as θ goes, and since $\theta < 180^\circ$ in a triangle, we can use the calculations you've to compare the sin values of the angles. The angle with the largest sine value is the largest angle of the triangle.

23.

Q23)

$$\hat{u} \times \hat{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & -2 \end{vmatrix} = \langle -4, 0, -4 \rangle \quad \left. \begin{array}{l} \text{Orthogonal to both} \\ u \text{ and } v \end{array} \right\}$$

If $\langle -4, 0, -4 \rangle$ is orthogonal to the vectors above,
we can say that $-\langle -4, 0, -4 \rangle = \langle 4, 0, 4 \rangle$ is also orthogonal.

• Convert $\langle -4, 0, -4 \rangle$ into unit norm

$$|\langle -4, 0, -4 \rangle| = 4\sqrt{2}$$

$$\text{Result} = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \right\rangle$$

• Convert $\langle 4, 0, 4 \rangle$ into unit norm

$$|\langle 4, 0, 4 \rangle| = 4\sqrt{2}$$

$$\text{Result} = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

24.

Q24)

let $a = (1, 0)$ be a point on the line $x+2y=1$

The distance between the point a and the line

$x+2y=3$ gives us also the distance between those lines

$$D = \frac{|1 \cdot 1 + 2 \cdot 0 - 3|}{\sqrt{1^2 + 2^2}} = \frac{2}{\sqrt{5}}$$

25.

$$\text{Q25) } A = (0, 1, -4)$$

let $B = (x_0, y_0, z_0)$ be a point on the plane

The normal vector of the plane $\vec{N} = \langle 1, 0, 1 \rangle$

$$\vec{AB} = \langle -x_0, 1-y_0, -5-z_0 \rangle, \vec{AB} \parallel \vec{N}$$

\vec{AB} 's projection over \vec{N} is equal to \vec{AB} because $\vec{AB} \parallel \vec{N}$

$$\begin{array}{|c|c|c|} \hline \vec{AB}_\alpha = k \cdot \vec{N} & k = \frac{-1}{\sqrt{1+0+1}} & \vec{AB}_\alpha = \left\langle -\frac{3}{2}, 0, \frac{3}{2} \right\rangle \\ \hline k = \frac{\vec{AB} \cdot \vec{N}}{|\vec{N}|^2} & k = -\frac{3}{2} & |\vec{AB}_\alpha| = \frac{3}{\sqrt{2}} \\ \hline k = \frac{-x_0 - 5 - z_0}{2} & \vec{AB}_\alpha = k \cdot \vec{N} & \\ \hline \end{array}$$

26.

$$\text{Q26) let } u = \langle u_1, u_2, \dots, u_n \rangle \text{ and } v = \langle v_1, v_2, \dots, v_n \rangle$$

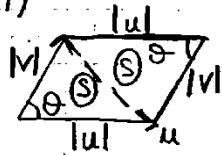
$$\begin{aligned} |u \cdot v| &= |u| \cdot |v| \cdot \cos 90^\circ & |u \times v| &= |u| \cdot |v| \cdot \sin 90^\circ \\ &= 0 & &= |u| \cdot |v| \end{aligned}$$

$$\begin{aligned} |u + v| &= |(u_1 + v_1), (u_2 + v_2), \dots, (u_n + v_n)| = \sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2 + \dots + (u_n + v_n)^2} \\ &= \sqrt{\underbrace{|u|^2 + |v|^2 + \dots + |u|^2}_{|u|^2} + \underbrace{|v|^2 + |v|^2 + \dots + |v|^2}_{|v|^2} + \underbrace{2(u_1 v_1 + u_2 v_2 + \dots + u_n v_n)}_{2 \cdot u \cdot v}} \\ &= \sqrt{|u|^2 + |v|^2} \end{aligned}$$

$$\begin{aligned} |u - v| &= |(u_1 - v_1), (u_2 - v_2), \dots, (u_n - v_n)| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2} \\ &= \sqrt{\underbrace{|u|^2 + |v|^2 + \dots + |u|^2}_{|u|^2} + \underbrace{|v|^2 + |v|^2 + \dots + |v|^2}_{|v|^2} - \underbrace{2(u_1 v_1 + u_2 v_2 + \dots + u_n v_n)}_{-2 \cdot u \cdot v}} \\ &= \sqrt{|u|^2 + |v|^2} \end{aligned}$$

27.

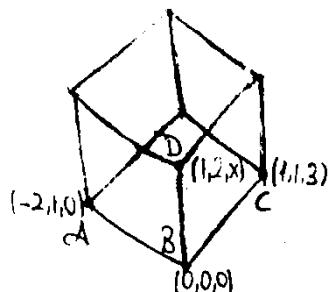
27)



$$S = l|u| \cdot l|v| \cdot \sin \theta \cdot \frac{1}{2}$$

$$\text{Area} = 2S = l|u| \cdot l|v| \cdot \sin \theta = |u \times v|$$

28.



$$\begin{aligned} \vec{BA} &= \langle -2, 1, 0 \rangle \\ \vec{BC} &= \langle 1, 1, 3 \rangle \\ \vec{BD} &= \langle -1, 2, x \rangle \end{aligned} \quad \left. \begin{array}{l} \vec{V} = \vec{BD} \cdot (\vec{BA} \times \vec{BC}) \\ V = \begin{vmatrix} 1 & 2 & x \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 3 - 2(6) + x(-3) \\ = 15 - 3x \end{array} \right\}$$

$$V=0 = 15 - 3x \Rightarrow x = 5,$$

29.

Q29)

If A has more than one solutions, $|A|$ must be equal to 0, because if it wasn't we would only have one unique solution.

Because of that reason, there is NOT a solution for b in $Ax=b$.

30.

If A is a square matrix which has more than one solutions for $AX = b$ just like in the question 29, then it is NOT possible for $\det(A)$ to be 9 because $\det(A)$ must be 0 for the equation above to be satisfied.

But if A is just a regular square matrix, and if $\det(A)$ is 9, then:

$$A^{-1}/AX = A^{-1}/b, A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{\text{adj}(A)}{9}$$

$$X = A^{-1} \cdot b \Rightarrow X = b \cdot \frac{\text{adj}(A)}{9}$$

In this case, the equation has a solution for every b

31.

Q31)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 2 & 5 & -5 & b_3 \end{array} \right] \xrightarrow{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_3 - 2b_1 \end{array} \right]$$

$$E_{2,3}(-1) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & b_1 - 2b_2 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 - 2b_1 - b_2 \end{array} \right] \quad \left. \begin{array}{l} \text{For the system to be consistent,} \\ \text{the last row should be zero} \\ b_3 - 2b_1 - b_2 = 0 \end{array} \right\}$$