2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = v_1 . c_1 + v_2 c_2 + v_3 . c_3$

$$\left.\begin{matrix} 4c_1 - 2c_2 = a \\ 3c_1 + 3c_3 = b \\ 2c_1 - c_2 = c \\ c_1 + c_3 = d \end{matrix}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 3 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow{\substack{E_{1,4}(-4) \\ E_{2,4}(-3) \\ E_{3,4}(-2)}} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

For the system to be consistent,
$a - 2c = 0 \Rightarrow a = 2c$
$b - 3d = 0 \qquad b = 3d$

Given vectors span a subspace consisting of vector K, that

$$K = \begin{bmatrix} 2c \\ 3d \\ c \\ d \end{bmatrix} = -c . \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{V_2} + d . \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{V_3}$$

b) The vectors are linearly independent if and only if the eq. $v_1 . c_1 + v_2 . c_2 + v_3 . c_3 = 0$ only has the trivial solution

$$\left.\begin{matrix} 4c_1 - 2c_2 = 0 \\ 3c_1 + 3c_3 = 0 \\ 2c_1 - c_2 = 0 \\ c_1 + c_3 = 0 \end{matrix}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Same operations used in the section a}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

since there are an infinite amount of solutions, the vectors are linearly dependent

c) Basis of the space spanned by these vectors is : $\left\{ \begin{bmatrix}-2\\0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\3\\0\\1\end{bmatrix} \right\}$
R^4's basis has 4 vectors, therefore, we need to find 2 additional vectors that are not in our initial span

let $v_4 = (0,0,0,1)$ and $v_5 = (1,0,0,0)$ $\exists$ v_4 and v_5 are lin. ind. and not in the span

The basis we get as a result is $= \left\{ \underbrace{\begin{bmatrix}-2\\0\\-1\\0\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}0\\3\\0\\1\end{bmatrix}}_{v_3}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$

Q2) ∇ $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow f_1 = \cos 2x$ and $f_2 = \cos 2x$

Applying Wronskian :

$$\begin{vmatrix} \cos 2x & \cos 2x \\ (\cos 2x)' & (\cos 2x)' \end{vmatrix} = \begin{vmatrix} \cos 2x & \cos 2x \\ -2\sin 2x & -2\sin 2x \end{vmatrix} = -2\sin 2x \cos 2x + 2\sin 2x \cos 2x = 0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of R^3 is 3, therefore basis for R^3 consist of 3 vectors

Choose $u_1, u_3, u_4 \Rightarrow$ Test for linear independence :

$u_1.c_1 + u_3.c_3 + u_4.c_4 = 0$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 0 \\ -2c_1 &= 0 \\ -c_3 + 7c_4 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_{2,1}(2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_2(1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right]$$

$$\xrightarrow[E_{3,2}(1)]{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \left.\begin{aligned} c_1 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}\right\} \Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1, u_3, u_4 form a basis without testing for their span.

Base $= \{(1,-2,0), (1,0,-1), (0,0,7)\}$

∇ Express the remaining vector, u_2, by the base vectors.

$u_1.c_1 + u_3.c_3 + u_4.c_4 = u_2$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 1 \\ -2c_1 &= 2 \\ -c_3 + 7c_4 &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{aligned} c_1 &= -1 \\ c_3 &= 2 \\ c_4 &= 6/7 \end{aligned}\right\} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

The dimension of the subspace spanned by these vectors is 4

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^4 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

~~Q6) The vectors u and v, are the basis for the $u \times v$ plane.~~

~~The vector resulting from $proj_{u \times v} w$ lies in the $u \times v$ plane, therefore it can be expressed as a linear combination of the basis vectors u and v.~~

~~Therefore u, v and $proj_{u \times v} w$ are not linearly independent~~

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.

Q8)

a) The dimension of R^6 is 6, so the basis for R^6 has 6 vectors. Therefore any amount of vectors that is less than 6, cannot form a basis

b) let $v=(-x,y)$ and $m=(x,-y)$

Since $v=-1\cdot m$, these vectors are linearly dependent and therefore, cannot form a basis

c) We're given 3 vectors which is the same amount of vectors in the basis of R^3, so we only need to show whether these vectors are linearly independent or not.

$v_1=(1,2,3)$ $v_2=(1,2,0)$ $v_3=(-1,2,6)$

$c_1v_1+c_2v_2+c_3v_3=0$

$$\left.\begin{aligned} c_1+c_2-c_3&=0\\ 2c_1+2c_2+2c_3&=0\\ 3c_1+\qquad 6c_3&=0\end{aligned}\right\}\begin{bmatrix}1&1&-1\\2&2&2\\3&0&6\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\2&2&2&0\\3&0&6&0\end{array}\right]\xrightarrow[E_{3,1}(-3)]{E_{2,1}(-2)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&6&0\\0&-3&9&0\end{array}\right]\xrightarrow[E_3(-1/3)]{E_2(1/6)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&1&0\\0&1&-1&0\end{array}\right]$$

$$\xrightarrow[E_{1,2}(1)]{E_{3,2}(1)}\left[\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\xrightarrow{E_{1,3}(-1)}\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\Big\}\ c_1=c_2=c_3=0$$

$\therefore$ The vectors are linearly independent, therefore they are a basis for R^3

d) Any set of vectors that include the zero vector is linearly dependent Therefore they can't form a basis.

Q9)

$v_1\cdot c_1+v_2\cdot c_2+v_3\cdot c_3=w$

$2c_1\quad +c_3=1$ / $-2c_1+c_2+5c_3=1$ / $-c_2+4c_3=1$

$$\begin{bmatrix}2&0&1\\-2&1&5\\0&-1&4\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}2&0&1&1\\-2&1&5&1\\0&-1&4&1\end{array}\right]\xrightarrow{E_{2,1}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&-1&4&1\end{array}\right]$$

$$\xrightarrow{E_{3,2}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&10&3\end{array}\right]\xrightarrow{E_3(1/10)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&1&3/10\end{array}\right]\xrightarrow[E_{1,3}(-1)]{E_{2,3}(-6)}\left[\begin{array}{ccc|c}2&0&0&0{,}7\\0&1&0&0{,}2\\0&0&1&0{,}3\end{array}\right]$$

$$-E_1(\tfrac{1}{2})\rightarrow\left[\begin{array}{ccc|c}1&0&0&0{,}35\\0&1&0&0{,}20\\0&0&1&0{,}30\end{array}\right]\Bigg\}\begin{array}{l}c_1=0{,}35\\c_2=0{,}20\\c_3=0{,}30\end{array}$$

$w = 0{,}35\cdot v_1 + 0{,}20\cdot v_2 + 0{,}30\cdot v_3 \Rightarrow$ Coordinates of w relative to the basis is $(0{,}35, 0{,}20, 0{,}30)$

Q10)

a) R^3 has a dimension of 3, because of that, the maximum amount of linearly independent vectors we can express is 3. Therefore, the given vectors can't be linearly independent

b)

① Find the space spanned by v_1, v_2, v_3, v_4

let $u=(a,b,c)$

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = u$

$$\left.\begin{array}{l}2c_1 + c_3 + c_4 = a\\-c_1 + c_2 - c_3 = b\\-c_1 + 3c_2 - 2c_3 + c_4 = c\end{array}\right\}\begin{bmatrix}2&0&1&1\\-1&1&-1&0\\-1&3&-2&1\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix}=\begin{bmatrix}a\\b\\c\end{bmatrix}$$

$$\left[\begin{array}{cccc|c}2&0&1&1&a\\-1&1&-1&0&b\\-1&3&-2&1&c\end{array}\right]\xrightarrow[E_{1,3}(2)]{E_{2,3}(-1)}\left[\begin{array}{cccc|c}0&6&-3&3&a+2c\\0&-2&1&-1&b-c\\-1&3&-2&1&c\end{array}\right]$$

$$\begin{array}{l}E_1(1/6)\\E_3(-1)\end{array}\rightarrow\left[\begin{array}{cccc|c}0&1&-1/2&1/2&(a+2c)/6\\0&-2&1&-1&b-c\\1&-3&2&-1&-c\end{array}\right]\xrightarrow[E_{3,1}(3)]{E_{2,1}(2)}\left[\begin{array}{cccc|c}0&1&-1/2&1/2&(a+2c)/6\\0&0&0&0&(a+3b-c)/3\\1&0&1/2&-1/2&a/2\end{array}\right]$$

$\frac{a+3b-c}{3}=0 \Rightarrow a+3b-c=0 \Big\}$ let $a=t$, $b=k$

The given vectors span a subspace of R^3 in which $v_i=\begin{bmatrix}t\\k\\t+3k\end{bmatrix}$

$v_i = t\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}+k\cdot\begin{bmatrix}0\\1\\3\end{bmatrix} \Rightarrow$ Basis for the subspace is $= (\underbrace{\langle 1,0,1\rangle}_{v_4}, \underbrace{\langle 0,1,3\rangle}_{v_2})$

c) The dimension of the space spanned by those vectors is 2, since it has 2 basis vectors.

d) * I already calculated the span in section "b"

e) Test if we can express w as a linear combination of the basis vectors

$c_1\cdot v_4 + c_2\cdot v_2 = w$

$$\left.\begin{array}{l}c_1 + 0\cdot c_2 = 2\\0\cdot c_1 + c_2 = 2\\c_1 + 3c_2 = 0\end{array}\right\}\left[\begin{array}{cc|c}1&0&2\\0&1&2\\1&3&0\end{array}\right]\xrightarrow[E_{3,2}(-3)]{E_{3,1}(-1)}\left[\begin{array}{cc|c}1&0&2\\0&1&2\\0&0&-8\end{array}\right]$$

System is inconsistent, w is not in the span

Q11) let $x=t$ and $y=k$ then we can say that our plane consists of Vectors with the rule of:

$$V=\begin{bmatrix} t \\ k \\ 2+k-t \end{bmatrix} = t.\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

The basis for this plane and ONLY for this plane is:

$\{(1,0,-1),(0,1,1),(0,0,2)\}$

Q12) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] - E_{2,1}(-2) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & -3 & 0 \end{array}\right] \rightarrow E_2(-1/3) - \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$

$E_{1,2}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$ $\quad w+\frac{y}{3}=0 \quad x-\frac{4y}{3}+z=0$ let $x=k$, $y=3t$

$w=-t,\ x=k,\ y=3t,\ z=4t-k$

Vectors in this subspace can be expressed as

$$\begin{bmatrix} -t \\ k \\ 3t \\ 4t-k \end{bmatrix} = t.\begin{bmatrix} -1 \\ 0 \\ 3 \\ 4 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Basis for the Subspace

Q13) $\frac{x-1}{3}=t \quad y+\frac{1}{3}=t \quad \frac{1-z}{3}=t, \quad \frac{2x-2}{3}=2t$

$\frac{2x-2}{3}-y-\frac{1}{3}+\frac{z}{3}-\frac{1}{3}=2t-t-t=0$

$\frac{2x}{3}-y+z=\frac{4}{3} \Rightarrow 2x-3y+3z=4$, let $x=3t$ and $y=2k \Rightarrow z=2k-2t+\frac{4}{3}$

The subspace spanned by this equation consist of vectors with the rule of:

$$V=\begin{bmatrix} 3t \\ 2k \\ 2k-2t+4/3 \end{bmatrix} = t.\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4/3 \end{bmatrix}$$

The basis for this line and only this line is:

$\{(3,0,-2),(0,2,2),(0,0,4/3)\}$

Q14) let $p(x) = ax^2 + bx + d$

$$c_1 \cdot p_1(x) + c_2 \cdot p_2(x) + c_3 p_3(x) + c_4 p_4 = p(x)$$

$$c_1 \cdot x^2 + c_1 + c_2 \cdot x^2 - 2c_2 x + 3c_2 + c_3 x - c_3 + 3c_4 x^2 + 2c_4 x + c_4 = ax^2 + bx + d$$

$$x^2(c_1 + c_2 + 3c_4) + x(-2c_2 + c_3 + 2c_4) + (c_1 + 3c_2 - c_3) = ax^2 + bx + d$$

$$\left.\begin{aligned} c_1 + c_2 \quad\quad + 3c_4 &= a \\ -2c_2 + c_3 + 2c_4 &= b \\ c_1 + 3c_2 - c_3 + c_4 &= d \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 1 & 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 1 & 3 & -1 & 1 & d \end{array}\right] \xrightarrow{E_{3,1}(-1)} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 2 & -1 & -2 & d-a \end{array}\right]$$

$$\xrightarrow{E_{2,3}(1)} \left.\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & d-a+b \end{array}\right]\right\} d - a + b = 0 \quad \text{let } a = k,\ b = t \Rightarrow d = k - t$$

These vectors span a subspace of P^2 with the rule of:

$$P(x) = \begin{bmatrix} k \\ t \\ k-t \end{bmatrix} = k \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \text{Basis is} : \langle (1,0,1), (0,1,-1) \rangle$$

∇ Dimension of this subspace is 2 (2 vectors in basis)

∇ Since we have $2 < 3$ vectors in our basis, it does not span the space of second order polynomials.

Let $P_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, P_0 is not in the span of our vectors.

By the add-minus theorem, if we add this vector to our basis, the new basis will span all P^2 (3 basis vectors)

Q15) Columnvectors of A are $= \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{V_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}}_{V_3} \right\}$, let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$c_1 \cdot V_1 + c_2 \cdot V_2 + c_3 V_3 = V$$

$$\left.\begin{aligned} 2c_1 - c_2 \quad\quad &= a \\ c_1 + 2c_2 + 5c_3 &= b \\ c_1 + c_2 + 3c_3 &= c \end{aligned}\right\} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & a \\ 1 & 2 & 5 & b \\ 1 & 1 & 3 & c \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 0 & -3 & -6 & a-2c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right] \xrightarrow{E_{1,2}(2)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & a+2b-4c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right]$$

$$a + 2b - 4c = 0, \text{ let } a = 4t,\ b = 2k \Rightarrow c = t + k$$

These vectors span a subspace of $\mathbb{R}^3$ with the rule:

$$V = \begin{bmatrix} 4t \\ 2k \\ t+k \end{bmatrix} = t \cdot \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix} + k \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \Bigg\} \text{Basis is } \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

V The dimension of the columnspace is 2

V For $Ax = b$ to be consistent for every b, A must be invertible

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{vmatrix} \xrightarrow[E_{1,3}(-2)]{E_{2,3}(-1)} \begin{vmatrix} 0 & -3 & -6 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} -3 & -6 \\ 1 & 2 \end{vmatrix} = 0$$

Since $|A| = 0$, A is not invertible. Therefore, $Ax = b$ is not consistent for every b

Q16)

a) columnvectors of A are : $\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{V_1}, \underbrace{\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{V_3} \right\}$, $b = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$c_1 V_1 + c_2 V_2 + c_3 V_3 = b$

$$\left.\begin{aligned} c_1 - 2c_2 + c_3 &= -1 \\ c_1 + 2c_2 + 2c_3 &= 1 \\ 2c_1 - 2c_2 + 2c_3 &= 1 \end{aligned}\right] \begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow[E_{3,1}(-2)]{E_{2,1}(-1)} \left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 0 & 4 & 1 & 2 \\ 0 & 2 & 0 & 3 \end{array}\right] \xrightarrow[E_{2,3}(-2)]{E_{1,3}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -4 \\ 0 & 2 & 0 & 3 \end{array}\right]$$

$$\xrightarrow[E_{1,2}(-1)]{E_3(1/2)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3/2 \end{array}\right] \Bigg\} \begin{aligned} c_1 &= 6 \\ c_2 &= 3/2 \\ c_3 &= -4 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}_{b} = 6 \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}}_{V_1} + \frac{3}{2} \underbrace{\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}}_{V_2} - 4 \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{V_3}$$

b) $$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & -1 & 2 \end{bmatrix} \xrightarrow{\text{Same operations as in section "a"}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow E_{2,3} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank = no. of pivots in row echelon form = 3

c) Nullity = col. no − rank = 3 − 3 = 0

d)

→ $Ax=0$ } Find the homogeneous solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\text{Same operations as in section "a"}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Bigg\} X=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ $Ax=b$ } Find the general solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow{\text{Same operations as in Section "a"}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3/2 \end{array}\right] \Bigg\} X=\begin{bmatrix} 6 \\ 3/2 \\ -4 \end{bmatrix}$$

Q17)

a) $Ax=0$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 4 & -4 & -4 & 0 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{E_1(1/2)} \left[\begin{array}{cccc|c} 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} x_2 - x_3 - x_4 = 0 \\ x_1 + x_3 + 3x_4 = 0 \end{array} \Rightarrow \begin{array}{l} \text{let } x_3 = t \text{ and } x_4 = k \\ \star\ x_2 = t+k,\ x_1 = -t-3k \end{array}$$

The space spanned by these vectors is the nullspace

$$NS(A) = \begin{bmatrix} -t-3k \\ t+k \\ t \\ k \end{bmatrix} = t.\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k.\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b) Nullity = Dimension of the nullspace = 2

c) Rank = No. of non-zero rows in reduced echelon form $= 2$,

d) $Ax = b$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 4 & -4 & -4 & -2 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} 2x_2 - 2x_3 - 2x_4 = -1 \\ x_1 + x_3 + 3x_4 = 3 \end{array}\right\} \begin{array}{l} \text{Let } x_3 = t \text{ and } x_4 = k \\ x_1 = 3 - t - 3k \text{ and } x_2 = -1/2 + t + k \end{array}$$

$$\text{Solution vector } x = \begin{bmatrix} 3 - t - 3k \\ -1/2 + t + k \\ t \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Homogeneous solution}}}_{\text{General Solution}}$$

e) For $Ax = b$ to be consistent for every b, A must be invertible. Since A is not a square matrix, it is not invertible.

Q18)

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix} \xrightarrow[E_{3,1}(-2)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{bmatrix} \xrightarrow{E_{3,2}(-2)} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to v_1 \\ \to v_2 \\ \end{array} \\ \begin{array}{cccc} \downarrow & & \downarrow & \downarrow \\ v_3 & & v_4 & v_5 \end{array} \end{array}$$

Basis for row space $= \{(1,0,1,-4), (0,0,1,7)\}$

Basis for col. space $= \{(1,0,0), (1,1,0), (-4,7,0)\}$

Q19)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow[E_{3,1}(-1)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{E_{2,3}(1)} \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array} \end{array}$$

Basis for col. space $= \{(1,0,0,0), (2,0,-2,0), (3,0,-5,0)\}$

§20)

$A=\begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, $E_{2,1}(-1)$ can be expressed as $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=A'$

↳ Elementary matrix

A' can be expressed as $E.A$

① Find the col. space of A

$c_1.\begin{bmatrix}1\\ -1\end{bmatrix}+c_2.\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix}$ $\left.\begin{matrix}c_1+c_2=a\\ -c_1-c_2=b\end{matrix}\right\}\Rightarrow$ let $c_1=t$, $c_2=k$ $a=t+k$ / $b=-t-k$

Col. space of A consists from vectors with the rule: $\begin{bmatrix}t+k\\ -t-k\end{bmatrix}=t.\begin{bmatrix}1\\ -1\end{bmatrix}+k.\begin{bmatrix}1\\ -1\end{bmatrix}$

② Find the col. space of $A'=E.A$

$c_1.\begin{bmatrix}1\\ 0\end{bmatrix}+c_2.\begin{bmatrix}-1\\ 1\end{bmatrix}=\begin{bmatrix}a\\ b\end{bmatrix}$ $\left.\begin{matrix}c_1-c_2=a\\ c_2=b\end{matrix}\right\}\Rightarrow$ let $c_1=t$, $c_2=k$ $a=t-k$ / $b=k$

Col. space of A consists from vectors with the rule : $\begin{bmatrix}t-k\\ k\end{bmatrix}=t.\begin{bmatrix}1\\ 0\end{bmatrix}+k.\begin{bmatrix}-1\\ 1\end{bmatrix}$

∇ The rules are different, so A and EA don't have the same columnspace.

§21)

$A^T=\begin{bmatrix}1 & 2 & 1 & 0\\ 2 & -4 & 0 & 1\\ 4 & 0 & -2 & 3\end{bmatrix}$

$\begin{bmatrix}1 & 2 & 1 & 0\\ 2 & -4 & 0 & 1\\ 4 & 0 & -2 & 3\end{bmatrix} \xrightarrow[E_{3,1}(-4)]{E_{2,1}(-2)} \begin{bmatrix}1 & 2 & 1 & 0\\ 0 & -8 & -2 & 1\\ 0 & -8 & -6 & 3\end{bmatrix} \xrightarrow{E_{3,2}(-1)} \begin{bmatrix}1 & 2 & 1 & 0\\ 0 & -8 & -2 & 1\\ 0 & 0 & -4 & 2\end{bmatrix}$

$\xrightarrow[E_3(-1/4)]{E_2(-1/8)} \begin{bmatrix}1 & 2 & 1 & 0\\ 0 & 1 & 1/4 & -1/8\\ 0 & 0 & 1 & -1/2\end{bmatrix} \xrightarrow{E_{1,2}(-2)} \begin{bmatrix}1 & 0 & 1/2 & 1/4\\ 0 & 1 & 1/4 & -1/8\\ 0 & 0 & 1 & -1/2\end{bmatrix} \xrightarrow[E_{1,3}(-1/2)]{E_{2,3}(-1/4)} \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$

Lin. Ind. Col.

A basis for rowspace A is : $\{(1,2,4),(2,-4,0)\}$, Q22) $x=2t,\ y=t,\ z=t/2$

The basis for the solution space $AX=0$ is : $t.\begin{bmatrix}2\\1\\1/2\end{bmatrix}$

The nullity is $=1$. Rank = row n. − nullity $= 3-1=2$

$x=2t,\ y=t,\ z=t/2 \Rightarrow x-2y=0,\ y-2z=0,\ x-4z=0$

* Write an augmented matrix for $A.X=0$

$$A=\left[\begin{array}{ccc|c}1&-2&0&0\\0&1&-2&0\\1&0&-4&0\end{array}\right]$$

Q23)

$$\left[\begin{array}{ccc|c}1&1&-1&b_1\\2&-1&3&b_2\\-1&3&1&b_3\\0&2&-1&b_4\end{array}\right]\xrightarrow[E_{3,1}(1)]{E_{2,1}(-2)}\left[\begin{array}{ccc|c}1&1&-1&b_1\\0&-3&5&b_2-2b_1\\0&4&0&b_1+b_3\\0&2&-1&b_4\end{array}\right]\xrightarrow{E_3(1/4)}\left[\begin{array}{ccc|c}1&1&-1&b_1\\0&-3&5&b_2-2b_1\\0&1&0&(b_1+b_3)/4\\0&2&-1&b_4\end{array}\right]$$

$$\xrightarrow[E_{4,3}(-2)]{E_{2,3}(3)}\left[\begin{array}{ccc|c}1&1&-1&b_1\\0&0&5&(-5b_1+4b_2+3b_3)/4\\0&1&0&(b_1+b_3)/4\\0&0&-1&(-b_1-b_3+2b_4)/2\end{array}\right]\xrightarrow{E_{2,4}(5)}\left[\begin{array}{ccc|c}1&1&-1&b_1\\0&0&0&(-15b_1+4b_2-7b_3+20b_4)/4\\0&1&0&(b_1+b_3)/4\\0&0&-1&(-b_1-b_3+2b_4)/2\end{array}\right]*$$

$$\xrightarrow[E_{1,4}(-1)]{E_{1,3}(-1)}\left[\begin{array}{ccc|c}1&0&0&(5b_1+b_3-4b_3)/4\\0&0&0&(-15b_1+4b_2-7b_3+20b_4)/4\\0&1&0&(b_1+b_3)/4\\0&0&-1&(-b_1-b_3+2b_4)/2\end{array}\right]$$

* → For the system to be consistent,
$-15b_1+4b_2-7b_3+20b_4=0$

* In the case the system is consistent, there is a unique solution which:

$x_1=\dfrac{5b_1+b_3-b_4}{4}\qquad x_2=\dfrac{b_1+b_3}{4}\qquad x_3=\dfrac{b_1+b_3-2b_4}{2}$

Q24) We have 6 variables but only 3 equations. Therefore, there is $6-3=3$ free variables for the system $Ax=0$

Nullity = row number − rank $= 3-3=0$

Since there is not a row of zeroes in the reduced echolon form.

Q25) Assume we have a matrix shown as:

$$A_{4\times3} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A \cdot X = 0 \Rightarrow X = \begin{bmatrix} -ax_3 \\ -bx_3 \\ x_3 \end{bmatrix}$$

① Nullspace of A, defines a line which $x = -a \cdot t$; $y = -b \cdot t$, $z = t$
And for $t=0$, we can see the line passes through origin.

② The columnspace of A is the subspace of R^4 spanned by the pivots.

$$\text{Col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$ } This subspace defines a plane (Not a line)

! Therefore, the nullspace, column space and the row space cannot all be a line through origin

Assume we have a matrix shown as:

$$A_{2\times4} = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} \quad A \cdot X = 0 \Rightarrow X = \begin{bmatrix} (-x_4 - x_3)/a \\ (-x_4 - x_3)/b \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -1/a \\ -1/b \\ 1 \\ 0 \end{bmatrix} + x_4 \cdot \begin{bmatrix} -1/a \\ -1/b \\ 0 \\ 1 \end{bmatrix}$$

① Nullspace of A defines a plane. And for $x_3 = x_4 = 0$, we can see it passes through origin

② The columnspace for A is the subspace of R^2 spanned by the pivots

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$ } This subspace spans a plane passing through origin

③ The row space of A is the subspace of R^4 spanned by the non-zero rows

$$\text{row}(A) = \text{span}\{(1,0,a,c), (0,1,b,d)\}$$ } This subspace spans a plane passing through origin

! Therefore, the nullspace, column space and the row space can all be a plane through origin

★ For the nullspace to define a line, there must be exactly one free variable. And it can only be achieved by a matrix which can be reduced to:

★ For the nullspace to define a plane there must be two free variables And it can only be achieved by a matrix which can be reduced to:

Q26)

① $\begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & 1/2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow[E_{3,1}(-3)]{E_{2,1}(-t)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1-t/2 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow{E_{2,3}(1/2)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/4 \\ 0 & t-3/2 \end{bmatrix}$

$\xrightarrow{E_2(4)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow[E_{3,2}(3/2-t)]{E_{1,2}(-1/2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ } Since the remaining matrix does not have an element with t, the column space is the same for every t

② $A^T = \begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \Rightarrow$ Find the col. space for A^T (row sp. for A)

$\begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_{2,1}(-2)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 0 & 1-t/2 & t-3/2 \end{bmatrix}$ } Can't be reduced further

∇ Since the remaining matrix still has elements with t, the row space for A is not the same for every t.

(Note: We didn't use $E(1/t)$ because since $t \in \mathbb{R}$, it can be 0)

③ $Ax = b \Rightarrow \begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 4 & 2 & b_1 \\ t & 1 & b_2 \\ 3 & t & b_3 \end{array}\right] \xrightarrow[\text{in step ①}]{\text{Same op.}} \left[\begin{array}{cc|c} 1 & 0 & b_1' \\ 0 & 1 & b_2' \\ 0 & 0 & b_3' \end{array}\right]$ } ∇ Assume the condition $b_3'=0$ is met for the system to have a solution.

$x = \begin{bmatrix} b_1' \\ b_2' \end{bmatrix}$ } Our solution space is in $\mathbb{R}^2$ but our column space is in $\mathbb{R}^3$. Therefore the solution space is not in the span of our column vectors.

Q27)

b) $A.x = \lambda.x \Rightarrow (A-\lambda I).x = 0 \Rightarrow |A-\lambda I| = 0$

$= \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} \xrightarrow{E_{3,2}(1)} \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix} \xrightarrow{E_{1,2}(-\lambda)} \begin{vmatrix} 0 & -\lambda+\lambda^2 & 1+\lambda \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix}$

$= \cancel{-1.(-1)^3} \cdot \begin{vmatrix} -\lambda+\lambda^2 & 1+\lambda \\ -\lambda & -1-\lambda \end{vmatrix}$

$= \lambda(\lambda-1).-1.(\lambda+1)+\lambda^2+\lambda = -\lambda^3+\lambda^2+2\lambda = \lambda(-\lambda^2+\lambda-2) = 0$ } Characteristic equation

Solutions for the char. eq. $\Rightarrow \lambda = 0, \lambda = 2, \lambda = -1$

$\lambda=0$ / $(A-\lambda I).x = A.x = 0$ } Find the eigenvector "$\vec{x}$"

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$ } $x_3 = 0$, $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$, Let $x_1 = x_2 = t$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Basis for the eigenspace $= \{(1,1,0)\}$

$$\lambda I - A = -A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{E_{3,2}(1)} \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = 2$ // $A - \lambda I = A - 2I = 0$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(1)]{E_{1,3}(2)} \left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right]$$

$$E_2(-1/2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & 0 & -1/2 & 0 \end{array}\right] \left\} \begin{array}{l} x_1 - \frac{1}{2}x_3 = 0 \\ x_2 + \frac{3}{2}x_3 = 0 \\ \text{Let } x_3 = 2t \end{array} \right.$$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Basis for the eigenspace $= \{(1, -3, 2)\}$

$$\lambda I - A = 2I - A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -3/2 \\ -1 & 0 & 1/2 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = -1$ // $A - \lambda I = A + I = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[E_{3,1}(-1)]{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(2)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \left\} \begin{array}{l} x_1 + x_3 = 0 \\ x_2 = 0 \\ \text{let } x_1 = t \end{array} \right.$$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Basis for the eigenspace $= \{(1, 0, -1)\}$

$$\lambda I - A = -I - A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{as above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

a) $A \cdot x = x \cdot \lambda \Rightarrow (A-\lambda I)x = 0 \Rightarrow |A-\lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 1-2\lambda+\lambda^2-9 = \boxed{\lambda^2-2\lambda-8=0} \Bigg\} \boxed{\begin{matrix}\lambda_1=4 \\ \lambda_2=-2\end{matrix}}$$

Characteristic eq. *Eigenvalues*

$\lambda=4$ $(A-4I) \cdot x = 0$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{matrix} -3x_1+3x_2=0 \\ \text{let } x_1=t \end{matrix}$$

Eigenvectors $= X = \begin{bmatrix} t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(1,1)\}$

$$\lambda I - A = 4I - A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \xrightarrow{E_{2,1}(+1)} \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

$\lambda=-2$ $(A+2I) \cdot x = 0$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-1)} \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} 3x_1+3x_2=0 \\ \text{let } x_1=t \end{matrix}$$

Eigenvectors $= X = \begin{bmatrix} t \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda^3 - 3\lambda + 2 - 1$

$-1 + (2 - 2\lambda - \lambda + \lambda^2)$

Basis for the eigenspace $= \{(1,-1)\}$

$$\lambda I - A = -2I - A = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

c) $A \cdot x = x \cdot \lambda \Rightarrow (A-\lambda I) \cdot \lambda = 0 \Rightarrow |A - \lambda I| = 0$

$$= \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ 1-2\lambda & -\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(-2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$\xrightarrow{E_{1,2}(1-\lambda)} \begin{vmatrix} 0 & -1+(2-\lambda)(1-\lambda) & 3-5\lambda \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} = -1(-1)^3 \cdot \begin{vmatrix} \lambda^2-3\lambda+1 & 3-5\lambda \\ 1 & 3-\lambda \end{vmatrix}$$

$= -\lambda^3+6\lambda^2-5\lambda=0$ } Characteristic equation

$-\lambda(\lambda^2-6\lambda+5)=0 \Rightarrow \lambda_1=0,\ \lambda_2=5,\ \lambda_3=1$ } Eigenvalues

$\lambda=0$ / $(A-\lambda I)X = A.X = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 2 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{1,2}(+1)]{E_{3,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\left.\begin{array}{l} x_1+x_3=0 \\ x_2+3x_3=0 \end{array}\right\}$ let $x_3=t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ -3t \\ t \end{bmatrix} = t.\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1,-3,1)\}$

$\lambda I - A = -A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & -5 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda I - A$ has the rank of "2"

$\lambda=5$ / $(A-\lambda I)X = (A-5I).X=0$

$$\left[\begin{array}{ccc|c} -4 & -1 & -2 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-4)} \left[\begin{array}{ccc|c} 0 & 11 & -22 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(3)]{E_{1,3}(-11)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$\left.\begin{array}{l} -x_1-x_3=0 \\ x_2-2x_3=0 \end{array}\right\}$ let $x_3=t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t.\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1,2,1)\}$

$\lambda I - A = 5I - A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\lambda I - A$ has the rank of "2"

$\lambda=1$ / $(A-\lambda I)X = (A-I).X=0$

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ -1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \quad \left.\begin{array}{l} -x_1+3x_3=0 \\ x_2+2x_3=0 \end{array}\right\} \text{Let } x_3=t$$

Eigenvectors $= X = \begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} = t.\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(3,-2,1)\}$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{as above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & -1 & -2 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

Q28) $A^5 . x = A.(A.(A.(A.(\underbrace{A.x}_{x.\lambda})))) = ... = \lambda^5 . x$

Eigenvalues for A^5 are: $\lambda_1' = 0^5 = 0$, $\lambda_2' = 2^5 = 32$, $\lambda_3' = (-1)^5 = -1$

Eigenvectors for A^5 are the same:

★ For $\lambda_1' = 0$: $X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ Basis $= \{(1,1,0)\}$, Dimension: 1

★ For $\lambda_2' = 32$: $X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t.\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \Rightarrow$ Basis $= \{(1,-3,2)\}$, Dimension: 1

★ For $\lambda_3' = -1$: $X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t.\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow$ Basis $= \{(1,0,-1)\}$, Dimension: 1

3 eigenspaces

Q29)

a) Since we have a 2^{nd} degree polynomial, the dimension of the matrix is 2×2 (it has to be a square matrix for it to have a ch. pol.)

b) The characteristic polynomial is equal to $\det(\lambda I - A)$ for $\lambda = 0$;

$\det(-A) = (-1)^2 . \det(A) = \cancel{0^2} + \cancel{6.0} + 9 = 9$

c) $A^{-1}/A.x = A^{-1}/\lambda.x$

$I.x = A^{-1}.x.\lambda$

$I.\frac{1}{\lambda}.x = A^{-1}.x$

For the characteristic polynomial, $\lambda' = \frac{1}{\lambda}$

$\frac{1}{\lambda^2} + \frac{6}{\lambda} + 9 = p'(\lambda)$

d) The characteristic polynomial is calculated by $\det(\lambda I - A) = 0$. If we take A^T instead of A, the ch. pol. won't change for the following reasons:

1) With $\lambda I - A^T$, we substract λ from values in the main diagonal of A^T. And by definition, taking the transpoze of a matrix doesn't change it's main diagonal.

2) $\det(A) = \det(A^T)$

Q30) Find the eigenvalues of A: $\Rightarrow |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-1 & 1 & 4 \\ 0 & \lambda-2 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\cdot\begin{vmatrix} \lambda-2 & -4 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\cdot(\lambda-1)\cdot(\lambda-2)$$

$$\lambda_{1,2} = 1 \quad , \lambda_3 = 2$$

∇ Since we only have two unique eigenvalues, we can't find three linearly independent eigenvectors. Therefore, A is not diagonalizable

Find the eigenvectors:

$\lambda=1$/

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} x_2 + 4x_3 = 0 \\ \text{Let } x_1 = t,\ x_2 = 4k \end{array}$$

$$\text{Eigenvectors} = \begin{bmatrix} t \\ 4k \\ -k \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k\cdot\begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix}$$

★ For $t=t$, $k=0$, an eigenvector is $= \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \Big\}$ Rank $=1$

$\lambda=2$/

$$\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[E_{1,3}(-4)]{E_{2,3}(4)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Bigg\} \begin{array}{l} x_1 + x_2 = 0 \\ x_3 = 0 \\ \text{Let } x_1 = t \end{array}$$

$$\text{Eigenvectors} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

★ We can't write an eigenvector whose rank is 1

Try to express A with rank=1 eigenvectors

$$c\cdot\begin{bmatrix} t_1 & t_2 & t_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} ; \quad t_1, t_2, t_3 \neq 0 \ (\text{rank} = 1)$$

∇ It is obvious that this system is inconsistent, therefore we can't write A as a linear combination of rank 1 matrices formed by it's eigenvectors.