2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = V_1 . c_1 + V_2 . c_2 + V_3 . c_3$

$$\left.\begin{array}{l} 4c_1 - 2c_2 = a \\ 3c_1 + 3c_3 = b \\ 2c_1 - c_2 = c \\ c_1 + c_3 = d \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 3 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow{\begin{array}{c} E_{1,4}(-4) \\ E_{2,4}(-3) \\ E_{3,4}(-2) \end{array}} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

For the system to be consistent,
$a - 2c = 0 \Rightarrow a = 2c$
$b - 3d = 0 \quad\;\; b = 3d$

Given vectors span a subspace consisting of vector K, that

$$K = \begin{bmatrix} 2c \\ 3d \\ c \\ d \end{bmatrix} = -c . \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{V_2} + d . \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{V_3}$$

b) The vectors are linearly independent if and only if the eq. $V_1 . c_1 + V_2 . c_2 + V_3 . c_3 = 0$ only has the trivial solution

$$\left.\begin{array}{l} 4c_1 - 2c_2 = 0 \\ 3c_1 + 3c_3 = 0 \\ 2c_1 - c_2 = 0 \\ c_1 + c_3 = 0 \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Same operations used in the section a}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

Since there are an infinite amount of solutions, the vectors are linearly dependent

c) Basis of the space spanned by these vectors is : $\left\{ \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\mathbb{R}^4$'s basis has 4 vectors, therefore, we need to find 2 additional vectors that are not in our initial span

let $V_4 = (0,0,0,1)$ and $V_5 = (1,0,0,0)$ $\exists$ V_4 and V_5 are lin. ind. and not in the span

The basis we get as a result is $= \left\{ \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{V_3}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right.$

Q2) ∇ $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow f_1 = \cos 2x$ and $f_2 = \cos 2x$

Applying Wronskian :

$$\begin{vmatrix} \cos 2x & \cos 2x \\ (\cos 2x)' & (\cos 2x)' \end{vmatrix} = \begin{vmatrix} \cos 2x & \cos 2x \\ -2\sin 2x & -2\sin 2x \end{vmatrix} = -2\sin 2x \cos 2x + 2\sin 2x \cos 2x = 0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of $\mathbb{R}^3$ is 3, therefore basis for $\mathbb{R}^3$ consists of 3 vectors

Choose u_1, u_3, u_4 $\Rightarrow$ Test for linear independence :

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = 0$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 0 \\ -2c_1 &= 0 \\ -c_3 + 7c_4 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_{2,1}(2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_2(1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right]$$

$$\xrightarrow[E_{3,2}(1)]{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \left.\begin{aligned} c_1 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}\right\} \Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1, u_3, u_4 form a basis without testing for their span.

Base $= \{(1,-2,0), (1,0,-1), (0,0,7)\}$

∇ Express the remaining vector, u_2, by the base vectors.

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = u_2$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 1 \\ -2c_1 &= 2 \\ -c_3 + 7c_4 &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{aligned} c_1 &= -1 \\ c_3 &= 2 \\ c_4 &= 6/7 \end{aligned}\right\} - \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^5 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

Q6) Let's assume that u, v and w are linearly independent vectors in R^n where $n>3$

In that case, the space R^n has $n>3$ basis vectors and therefore the maximum amount of linearly ind. vectors we can write is also $n>3$

Since the product of $u \times v$ is orthogonal to both u and v, it cannot be expressed as a linear combination of u, v. So; u, v and $(u \times v)$ are linearly independent.

For this newly formed vector, there are 2 possibilities:

1) It can be parallel to w, since w is also linearly ind. from u, v

2) It can also be orthogonal to w, since for our vector space R^n, $n>3$ linearly ind. vectors can be written.

For 1, $\text{proj}_{u \times v} w$ is equal to w, since they are parallel. In this case we can say u, v and $\text{proj}_{u \times v} w = w$ are linearly independent.

However, for 2 $\text{proj}_{u \times v}$ is $\vec{0}$, since they are orthogonal. In this case u, v and $\text{proj}_{u \times v} = \vec{0}$ are linearly dependent, because a set of vectors which contains the 0 vector is always linearly dependent.

As a conclusion, no, we can't necessarily say that u, v and $\text{proj}_{u \times v} w$ are linearly independent.

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.

Q8)

a) The dimension of R^6 is 6, so the basis for R^6 has 6 vectors. Therefore any amount of vectors that is less than 6, cannot form a basis

b) let $v=(-x,y)$ and $m=(x,-y)$

Since $v=-1.m$, these vectors are linearly dependent and therefore, cannot form a basis

c) We're given 3 vectors which is the same amount of vectors in the basis of R^3, so we only need to show whether these vectors are linearly independent or not.

$v_1=(1,2,3)$ $v_2=(1,2,0)$ $v_3=(-1,2,6)$

$c_1v_1+c_2v_2+c_3v_3=0$

$$\left.\begin{aligned} c_1+c_2-c_3&=0\\ 2c_1+2c_2+2c_3&=0\\ 3c_1+6c_3&=0\end{aligned}\right\}\begin{bmatrix}1&1&-1\\2&2&2\\3&0&6\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\2&2&2&0\\3&0&6&0\end{array}\right]\xrightarrow[E_{3,1}(-3)]{E_{2,1}(-2)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&6&0\\0&-3&9&0\end{array}\right]\xrightarrow[E_3(-1/3)]{E_2(1/6)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&1&0\\0&1&-3&0\end{array}\right]$$

$$\xrightarrow[E_{1,2}(1)]{E_{3,2}(3)}\left[\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\xrightarrow{E_{1,3}(-1)}\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\Big\}\ c_1=c_2=c_3=0$$

! The vectors are linearly independent, therefore they are a basis for R^3

d) Any set of vectors that include the zero vector is linearly dependent Therefore they can't form a basis.

Q9)

$V_1.c_1+V_2.c_2+V_3.c_3=w$

$2c_1+c_3=1$ / $-2c_1+c_2+5c_3=1$ / $-c_2+4c_3=1$

$$\begin{bmatrix}2&0&1\\-2&1&5\\0&-1&4\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}2&0&1&1\\-2&1&5&1\\0&-1&4&1\end{array}\right]\xrightarrow{E_{2,1}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&-1&4&1\end{array}\right]$$

$$\xrightarrow{E_{3,2}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&10&3\end{array}\right]\xrightarrow{E_3(1/10)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&1&3/10\end{array}\right]\xrightarrow[E_{1,3}(-1)]{E_{2,3}(-6)}\left[\begin{array}{ccc|c}2&0&0&0,7\\0&1&0&0,2\\0&0&1&0,3\end{array}\right]$$

$$-E_1\left(\tfrac{1}{2}\right) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0{,}35 \\ 0 & 1 & 0 & 0{,}20 \\ 0 & 0 & 1 & 0{,}30 \end{array}\right] \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \begin{array}{l} c_1 = 0{,}35 \\ c_2 = 0{,}20 \\ c_3 = 0{,}30 \end{array}$$

$w = 0{,}35 \cdot v_1 + 0{,}20 \cdot v_2 + 0{,}30 \cdot v_3 \Rightarrow$ Coordinates of w relative to the basis is

$$(0{,}35,\ 0{,}20,\ 0{,}30)$$

Q10)

a) $\mathbb{R}^3$ has a dimension of '3', because of that, the maximum amount of linearly independent vectors we can express is 3. Therefore, the given vectors can't be linearly independent

b)

1) Find the space spanned by v_1, v_2, v_3, v_4

let $u = (a, b, c)$

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = u$

$$\left.\begin{array}{rrrrl} 2c_1 & & +c_3 & +c_4 & = a \\ -c_1 & +c_2 & -c_3 & & = b \\ -c_1 & +3c_2 & -2c_3 & +c_4 & = c \end{array}\right\} \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & a \\ -1 & 1 & -1 & 0 & b \\ -1 & 3 & -2 & 1 & c \end{array}\right] \xrightarrow[E_{1,3}(2)]{E_{2,3}(-1)} \left[\begin{array}{cccc|c} 0 & 6 & -3 & 3 & a+2c \\ 0 & -2 & 1 & -1 & b-c \\ -1 & 3 & -2 & 1 & c \end{array}\right]$$

$$\begin{array}{l} E_1(1/6) \rightarrow \\ E_3(-1) \end{array} \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & -2 & 1 & -1 & b-c \\ 1 & -3 & 2 & -1 & -c \end{array}\right] \xrightarrow[E_{3,1}(3)]{E_{2,1}(2)} \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & 0 & 0 & 0 & (a+3b-c)/3 \\ 1 & 0 & 1/2 & -1/2 & a/2 \end{array}\right] !$$

$$\left.\frac{a+3b-c}{3} = 0 \Rightarrow a+3b-c=0 \right\} \text{let } a = t,\ b = k$$

The given vectors span a subspace of $\mathbb{R}^3$ in which $v_i = \begin{bmatrix} t \\ k \\ t+3k \end{bmatrix}$

$$v_i = t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \text{Basis for the subspace is} = (\underbrace{\langle 1,0,1 \rangle}_{v_1}, \underbrace{\langle 0,1,3 \rangle}_{v_2})$$

c) The dimension of the space spanned by those vectors is 2, since it has 2 basis vectors.

d) * I already calculated the span in section "b"

e) Test if we can express w as a linear combination of the basis vectors

$$c_1 \cdot v_1 + c_2 \cdot v_2 = w$$

$$\left.\begin{array}{rl} c_1 + 0 \cdot c_2 & = 2 \\ 0 \cdot c_1 + c_2 & = 2 \\ c_1 + 3c_2 & = 0 \end{array}\right\} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-3)]{E_{3,1}(-1)} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{array}\right]$$

System is inconsistent, w is not in the span

Q11) $x-y+z=2 \Rightarrow x=(2+y)-z$, let $2+y=t$ and $z=k$

$$\text{Vectors} = \begin{bmatrix} t-k \\ t-2 \\ k \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k\cdot\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

Since our last vector $[0,-2,0]$ is a constant, we cannot express vectors on the plane using a scalar multiple of it other than "1". Therefore, we cannot form a basis.

Q12) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] - E_{2,1}(-2) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & -3 & 0 \end{array}\right] \rightarrow E_2(-1/3) - \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$

$E_{1,2}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$ $\quad w+\frac{y}{3}=0 \quad x-\frac{4y}{3}+z=0$ let $x=k$, $y=3t$

$$w=-t,\ x=k,\ y=3t,\ z=4t-k$$

Vectors in this subspace can be expressed as

$$\begin{bmatrix} -t \\ k \\ 3t \\ 4t-k \end{bmatrix} = t\cdot\underbrace{\begin{bmatrix} -1 \\ 0 \\ 3 \\ 4 \end{bmatrix} + k\cdot\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\text{Basis for the Subspace}}$$

Dimension of this subspace is 2

Q13) $\frac{x}{3}+\frac{1}{3}=t \quad y+\frac{1}{3}=t \quad \frac{1}{3}-\frac{z}{3}=t$

① $\frac{x}{3}=y$ ② $y=-\frac{z}{3}$ ③ $\frac{x}{\cancel{3}}=-\frac{z}{\cancel{3}}$ $\Big\}$ Let $z=3k$, $x=-3k$, $y=-k$

Our vectors have the form of : $\begin{bmatrix} -3k \\ -k \\ 3k \end{bmatrix} = k\cdot\begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$

Basis for the line is $= \{(-3,-1,3)\}$

Q14) let $p(x) = ax^2 + bx + d$

$c_1 \cdot p_1(x) + c_2 \cdot p_2(x) + c_3 p_3(x) + c_4 p_4 = p(x)$

$c_1 \cdot x^2 + c_1 + c_2 \cdot x^2 - 2c_2 x + 3c_2 + c_3 x - c_3 + 3c_4 x^2 + 2c_4 x + c_4 = ax^2 + bx + d$

$x^2(c_1 + c_2 + 3c_4) + x(-2c_2 + c_3 + 2c_4) + (c_1 + 3c_2 - c_3) = ax^2 + bx + d$

$$\left.\begin{aligned} c_1 + c_2 \quad\quad +3c_4 &= a \\ -2c_2 + c_3 + 2c_4 &= b \\ c_1 + 3c_2 - c_3 + c_4 &= d \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 1 & 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 1 & 3 & -1 & 1 & d \end{array}\right] - E_{3,1}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 2 & -1 & -2 & d-a \end{array}\right]$$

$$E_{2,3}(1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & d-a+b \end{array}\right] \Big\} \; d-a+b=0 \quad \text{let } a=k,\ b=t \Rightarrow d=k-t$$

These vectors span a subspace of P^2 with the rule of:

$$P(x) = \begin{bmatrix} k \\ t \\ k-t \end{bmatrix} = k \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \text{Basis is}: \langle (1,0,1), (0,1,-1) \rangle$$

- ∇ Dimension of this subspace is 2 (2 vectors in basis)
- ∇ Since we have $2 < 3$ vectors in our basis, it does not span the space of second order polynomials.

Let $P_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, P_0 is not in the span of our vectors.

By the add-minus theorem, if we add this vector to our basis, the new basis will span all P^2 (3 basis vectors)

Q15) Columnvectors of A are $= \Big\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{V_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}}_{V_3} \Big\}$, let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$c_1 \cdot V_1 + c_2 \cdot V_2 + c_3 V_3 = V$

$$\left.\begin{aligned} 2c_1 - c_2 \quad\quad &= a \\ c_1 + 2c_2 + 5c_3 &= b \\ c_1 + c_2 + 3c_3 &= c \end{aligned}\right\} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & a \\ 1 & 2 & 5 & b \\ 1 & 1 & 3 & c \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 0 & -3 & -6 & a-2c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right] - E_{1,2}(2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a+2b-4c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right]$$

$a + 2b - 4c = 0$, let $a = 4t$, $b = 2k \Rightarrow c = t + k$

These vectors span a subspace of $\mathbb{R}^3$ with the rule:

$$V=\begin{bmatrix}4t\\2k\\t+k\end{bmatrix}=t\cdot\begin{bmatrix}4\\0\\1\end{bmatrix}+k\begin{bmatrix}0\\2\\1\end{bmatrix}\;\Bigg\}\text{ Basis is }\left\{\begin{bmatrix}4\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}$$

▽ The dimension of the columnspace is 2

▽ For $Ax=b$ to be consistent for every b, A must be invertible

$$|A|=\begin{vmatrix}2&-1&0\\1&2&5\\1&1&3\end{vmatrix}\xrightarrow[E_{1,3}(-2)]{E_{2,3}(-1)}\begin{vmatrix}0&-3&-6\\0&1&2\\1&-1&3\end{vmatrix}=1\cdot\begin{vmatrix}-3&-6\\1&2\end{vmatrix}=0$$

Since $|A|=0$, A is not invertible. Therefore, $Ax=b$ is <u>not</u> consistent for every b.

Q16)

a) columnvectors of A are:

$$\Bigg\{\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{v_1},\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{v_2},\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{v_3}\Bigg\},\quad b=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$c_1v_1+c_2v_2+c_3v_3=b$$

$$\left.\begin{aligned}c_1-2c_2+c_3&=-1\\c_1+2c_2+2c_3&=1\\2c_1-2c_2+2c_3&=1\end{aligned}\right\}\begin{bmatrix}1&-2&1\\1&2&2\\2&-1&2\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&-2&1&-1\\1&2&2&1\\2&-2&2&1\end{array}\right]\xrightarrow[E_{3,1}(-2)]{E_{2,1}(-1)}\left[\begin{array}{ccc|c}1&-2&1&-1\\0&4&1&2\\0&2&0&3\end{array}\right]\xrightarrow[E_{2,3}(-2)]{E_{1,3}(1)}\left[\begin{array}{ccc|c}1&0&1&2\\0&0&1&-4\\0&2&0&3\end{array}\right]$$

$$\xrightarrow[E_{1,2}(-1)]{E_3(1/2)}\left[\begin{array}{ccc|c}1&0&0&6\\0&0&1&-4\\0&1&0&3/2\end{array}\right]\;\left.\begin{aligned}c_1&=6\\c_2&=3/2\\c_3&=-4\end{aligned}\right\}\Rightarrow\underbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}}_{b}=6\cdot\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{v_1}+\frac{3}{2}\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{v_2}-4\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{v_3}$$

b)

$$\begin{bmatrix}1&-2&1\\1&2&2\\2&-1&2\end{bmatrix}\xrightarrow{\text{Same operations as in section "a"}}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\xrightarrow{E_{2,3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Rank = no. of pivots in row echelon form = 3

c) Nullity = col. no − rank = 3 − 3 = 0

d)

$\rightarrow$ $Ax = 0$ } Find the homogeneous solution :

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] - \text{Same operations as in Section "a"} - \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Big\} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ $Ax = b$ } Find the general solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] - \text{Same operations as in Section "a"} - \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3/2 \end{array}\right] \Big\} X = \begin{bmatrix} 6 \\ 3/2 \\ -4 \end{bmatrix}$$

Q17)

a) $Ax = 0$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 4 & -4 & -4 & 0 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_1(1/2) \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Big\} \begin{array}{l} x_2 - x_3 - x_4 = 0 \\ x_1 + x_3 + 3x_4 = 0 \end{array} \Rightarrow \begin{array}{l} \text{let } x_3 = t \text{ and } x_4 = k \\ \star\ x_2 = t + k,\ x_1 = -t - 3k \end{array}$$

The space spanned by these vectors is the nullspace

$$NS(A) = \begin{bmatrix} -t-3k \\ t+k \\ t \\ k \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b) Nullity = Dimension of the nullspace = 2

c) Rank = No. of non-zero rows in reduced echelon form $=2$,

d) $\mathcal{A}x = b$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 4 & -4 & -4 & -2 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} 2x_2 - 2x_3 - 2x_4 = -1 \\ x_1 + x_3 + 3x_4 = 3 \end{array}\right\} \begin{array}{l} \text{Let } x_3 = t \text{ and } x_4 = k \\ x_1 = 3 - t - 3k \text{ and } x_2 = -1/2 + t + k \end{array}$$

$$\text{Solution vector } x = \begin{bmatrix} 3 - t - 3k \\ -1/2 + t + k \\ t \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Homogeneous solution}}}_{\text{General Solution}}$$

e) For $\mathcal{A}x = b$ to be consistent for every b, $\mathcal{A}$ must be invertible. Since $\mathcal{A}$ is not a square matrix, it is not invertible.

Q18)

$$\mathcal{A} = \begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix} \xrightarrow[E_{3,1}(-2)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{bmatrix} \xrightarrow{E_{3,2}(-2)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_{1,2}(-1) \rightarrow \begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots

Base for the columnspace: $\{(1,2,2), (1,3,4)\}$

$$\mathcal{A}^T = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \\ -4 & -1 & 6 \end{bmatrix} \xrightarrow[E_{4,1}(4)]{E_{3,1}(-1)} \begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 7 & 14 \end{bmatrix} \xrightarrow[E_{4,3}(-7)]{E_{1,3}(-2)} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Pivots

Base for the rowspace: $\{(1,0,1,-4), (2,0,3,-1)\}$

Q19)

$$A=\begin{bmatrix}1&2&3\\2&4&6\\1&0&-2\\0&2&5\end{bmatrix}\xrightarrow[E_{3,1}(-1)]{E_{2,1}(-2)}\begin{bmatrix}1&2&3\\0&0&0\\0&-2&-5\\0&2&5\end{bmatrix}\xrightarrow{E_{2,3}(1)}\begin{bmatrix}1&2&3\\0&0&0\\0&-2&-5\\0&0&0\end{bmatrix}\xrightarrow{E_{1,3}(1)}\begin{bmatrix}1&0&-2\\0&0&0\\0&-2&-5\\0&0&0\end{bmatrix}$$

$$\xrightarrow{E_2(-1/2)}\underbrace{\begin{bmatrix}1&0&-2\\0&0&0\\0&1&5/2\\0&0&0\end{bmatrix}}_{\text{Pivots}}$$

Basis for columnspace: $\{(1,2,1,0),(2,4,0,2)\}$

$$A^T=\begin{bmatrix}1&2&1&0\\2&4&0&2\\3&6&-2&5\end{bmatrix}\xrightarrow[E_{3,1}(-3)]{E_{2,1}(-2)}\begin{bmatrix}1&2&1&0\\0&0&-2&2\\0&0&-5&5\end{bmatrix}\xrightarrow[E_{1,2}(1/2)]{E_{3,2}(-5/2)}\begin{bmatrix}1&2&0&1\\0&0&-2&2\\0&0&0&0\end{bmatrix}\xrightarrow{E_2(-1/2)}\underbrace{\begin{bmatrix}1&2&0&1\\0&0&1&-1\\0&0&0&0\end{bmatrix}}_{\text{Pivots}}$$

Basis for rowspace:
$\{(1,2,3),(1,0,-2)\}$

Q20)

$A=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\xrightarrow{E_{2,1}(-1)}\begin{bmatrix}1&1\\0&0\end{bmatrix}$, $E_{2,1}(-1)$ can be expressed as $\begin{bmatrix}1&-1\\0&1\end{bmatrix}=A'$
↳ Elementary matrix

A' can be expressed as $E.A$

① Find the col. space of A

$$c_1.\begin{bmatrix}1\\-1\end{bmatrix}+c_2.\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}\ \left.\begin{matrix}c_1+c_2=a\\-c_1-c_2=b\end{matrix}\right\}\Rightarrow \text{let } c_1=t,\ c_2=k \quad a=t+k\ /\ b=-t-k$$

Col. space of A consists from vectors with the rule: $\begin{bmatrix}t+k\\-t-k\end{bmatrix}=t.\begin{bmatrix}1\\-1\end{bmatrix}+k.\begin{bmatrix}1\\-1\end{bmatrix}$

② Find the col. space of $A'=E.A$

$$c_1.\begin{bmatrix}1\\0\end{bmatrix}+c_2.\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}a\\b\end{bmatrix}\ \left.\begin{matrix}c_1-c_2=a\\c_2=b\end{matrix}\right\}\Rightarrow \text{let } c_1=t,\ c_2=k \quad a=t-k\ /\ b=k$$

Col. space of A consists from vectors with the rule: $\begin{bmatrix}t-k\\k\end{bmatrix}=t.\begin{bmatrix}1\\0\end{bmatrix}+k.\begin{bmatrix}-1\\1\end{bmatrix}$

∇ The rules are different, so A and EA don't have the same columnspace.

Q21)

$$A^T = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix} \xrightarrow[E_{3,1}(-4)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & -8 & -6 & 3 \end{bmatrix} \xrightarrow{E_{3,2}(-1)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & 0 & -4 & 2 \end{bmatrix}$$

$$\xrightarrow[E_3(-1/4)]{E_2(-1/8)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow{E_{1,2}(-2)} \begin{bmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow[E_{1,3}(-1/2)]{E_{2,3}(-1/4)} \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$$

Lin. Ind. Col.

A basis for rowspace A is : $\{(1,2,4),(2,-4,0),(1,0,-2)\}$, Q22) $\frac{x}{2}=t$, $y=t$, $2z=t$

① $\frac{x}{2}=y \Rightarrow x-2y=0$ ② $y=2z \Rightarrow y-2z=0$ ③ $\frac{x}{2}=2z \Rightarrow x-4z=0$

Our matrix can be expressed as:

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -4 \end{bmatrix} \xrightarrow{E_{3,1}(-1)} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{E_{3,2}(-2)} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{E_{1,2}(2)} \left.\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}\right\} \text{Reduced row echolon form} \Rightarrow \text{Rank}=2$$

∇ All 3x3 matrices that can be reduced to the matrix above, meet the condition.

$$A = E.\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$, E is a matrix which is the product of elementary matrices specific for A

Q23)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 2 & -1 & 3 & b_2 \\ -1 & 3 & 1 & b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right] \xrightarrow[E_{3,1}(1)]{E_{2,1}(-2)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -3 & 5 & b_2-2b_1 \\ 0 & 4 & 0 & b_1+b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right] \xrightarrow{E_3(1/4)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -3 & 5 & b_2-2b_1 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 2 & -1 & b_4 \end{array}\right]$$

$$\xrightarrow[E_{4,3}(-2)]{E_{2,3}(3)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 0 & 5 & (-5b_1+4b_2+3b_3)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right] \xrightarrow{E_{2,4}(5)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 0 & 0 & (-15b_1+4b_2-7b_3+20b_4)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right] ★$$

$$\xrightarrow[E_{1,4}(-1)]{E_{1,3}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (5b_1+b_3-4b_3)/4 \\ 0 & 0 & 0 & (-15b_1+4b_2-7b_3+20b_4)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right]$$

★ → For the system to be consistent,

$$-15b_1+4b_2-7b_3+20b_4=0$$

★ In the case the system is consistent, there is a unique solution which:

$x_1 = \frac{5b_1+b_3-b_4}{4}$ $x_2 = \frac{b_1+b_3}{4}$ $x_3 = \frac{b_1+b_3-2b_4}{2}$

Q24) We have 6 variables but only 3 equations. Therefore, there is $6-3=3$ free variables for the system $Ax=0$

Nullity = row number − rank = $3-3=0$

Since there is not a row of zeroes in the reduced echolon form, Q25) Assume we have a matrix shown as:

$$A_{4\times3} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A \cdot X = 0 \Rightarrow X = \begin{bmatrix} -ax_3 \\ -bx_3 \\ x_3 \end{bmatrix}$$

① Nullspace of A, defines a line which $x = -a.t$; $y = -b.t$, $z = t$
And for $t = 0$, we can see the line passes through origin.

② The columnspace of A is the subspace of R^4 spanned by the pivots.

$$\text{Col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$ } This subspace defines a plane (Not a line)

! Therefore, the nullspace, column space and the row space can not all be a line through origin

Assume we have a matrix shown as:

$$A_{2\times4} = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} \quad A \cdot X = 0 \Rightarrow X = \begin{bmatrix} (-x_4 - x_3)/a \\ (-x_4 - x_3)/b \\ x_3 \\ x_4 \end{bmatrix} = x_3 \cdot \begin{bmatrix} -1/a \\ -1/b \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/a \\ -1/b \\ 0 \\ 1 \end{bmatrix}$$

① Nullspace of A defines a plane. And for $x_3 = x_4 = 0$, we can see it passes through origin

② The columnspace for A is the subspace of R^2 spanned by the pivots

$$\text{col}(A) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$ } This subspace spans a plane passing through origin

③ The row space of A is the subspace of R^4 spanned by the non-zero rows

$$\text{row}(A) = \text{span}\{(1,0,a,c), (0,1,b,d)\}$$ } This subspace spans a plane passing through origin

! Therefore, the nullspace, column space and the row space can all be a plane through origin

★ For the nullspace to define a line, there must be exactly one free variable. And it can only be achieved by a matrix which can be reduced to:

★ For the nullspace to define a plane there must be two free variables And it can only be achieved by a matrix which can be reduced to:

Q261

① $\begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & 1/2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow[E_{3,1}(-3)]{E_{2,1}(-t)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1-t/2 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow{E_{2,3}(1/2)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/4 \\ 0 & t-3/2 \end{bmatrix}$

$\xrightarrow{E_2(4)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow[E_{3,2}(3/2-t)]{E_{1,2}(-1/2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ } Since the remaining matrix does not have an element with t, the column space is the same for every t

② $A^T = \begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \Rightarrow$ Find the col. space for A^T (row sp. for A)

$\begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_{2,1}(-2)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 0 & 1-t/2 & t-3/2 \end{bmatrix}$ } Can't be reduced further

∇ Since the remaining matrix still has elements with t, the row space for A is not the same for every t.

(Note: We didn't use $E(1/t)$ because since $t \in \mathbb{R}$, it can be 0)

③ Let b be in the col. space of A:

$$b = c_1 \cdot \begin{bmatrix} 4 \\ t \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2 \\ 1 \\ t \end{bmatrix} = \begin{bmatrix} 4c_1+2c_2 \\ c_1 \cdot t + c_2 \\ 3c_1 + c_2 \cdot t \end{bmatrix}$$

Find the solution space for $Ax=b$

$\left[\begin{array}{cc|c} 4 & 2 & 4c_1+2c_2 \\ t & 1 & c_1 t + c_2 \\ 3 & t & 3c_1 + c_2 t \end{array}\right] \xrightarrow{E_1(1/4)} \left[\begin{array}{cc|c} 1 & 1/2 & c_1 + c_2/2 \\ t & 1 & c_1 t + c_2 \\ 3 & t & 3c_1 + c_2 t \end{array}\right] \xrightarrow[E_{3,1}(-3)]{E_{2,1}(-t)} \left[\begin{array}{cc|c} 1 & 1/2 & c_1 + c_2/2 \\ 0 & 1-t/2 & c_2(1-t/2) \\ 0 & t-3/2 & c_2(t-3/2) \end{array}\right]$

$\xrightarrow{E_{2,3}(1/2)} \left[\begin{array}{cc|c} 1 & 1/2 & c_1 + c_2/2 \\ 0 & 1/4 & c_2/4 \\ 0 & t-3/2 & c_2(t-3/2) \end{array}\right] \xrightarrow{E_2(4)} \left[\begin{array}{cc|c} 1 & 1/2 & c_1 + c_2/2 \\ 0 & 1 & c_2 \\ 0 & t-3/2 & c_2(t-3/2) \end{array}\right]$

$\xrightarrow[E_{3,2}(3/2-t)]{E_{1,2}(-1/2)} \left[\begin{array}{cc|c} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \end{array}\right]$ } $X = \begin{bmatrix} c_1 \\ c_2 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

∇ Since our solution matrix does not have an element with "t", our solution space is the same for all t

Q.271

b) $A.x = \lambda.x \Rightarrow (A-\lambda I).x = 0 \Rightarrow |A-\lambda I| = 0$

$$= \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} \xrightarrow{E_{3,2}(1)} \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix} \xrightarrow{E_{1,2}(-\lambda)} \begin{vmatrix} 0 & -\lambda+\lambda^2 & 1+\lambda \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix}$$

$$= \cancel{-1.(-1)^3}. \begin{vmatrix} -\lambda+\lambda^2 & 1+\lambda \\ -\lambda & -1-\lambda \end{vmatrix}$$

$$= \lambda(\lambda-1).-1.(\lambda+1) + \lambda^2+\lambda = -\lambda^3+\lambda^2+2\lambda = \lambda(-\lambda^2+\lambda-2) = 0 \Big\}\ \text{Characteristic equation}$$

Solutions for the char. eq. $\Rightarrow \lambda = 0_1,\ \lambda = 2,\ \lambda = -1$

$\lambda=0$/ $(A-\lambda I).X = A.x = 0$ } Find the eigenvector "$\vec{X}$"

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \left\{\begin{array}{l} X_3 = 0 \\ x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \\ \text{Let } x_1 = x_2 = t \end{array}\right.$$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Basis for the eigenspace $= \{(1,1,0)\}$

$$\lambda I - A = -A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{E_{3,2}(1)} \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = 2$ / $A - \lambda I = A - 2I = 0$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(1)]{E_{1,3}(2)} \left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right]$$

$$E_2(-1/2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & 0 & -1/2 & 0 \end{array}\right] \left\} \begin{array}{l} x_1 - 1/2\, x_3 = 0 \\ x_2 + 3/2\, x_3 = 0 \\ \text{Let } x_3 = 2t \end{array} \right.$$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$$

Basis for the eigenspace $= \{(1,-3,2)\}$

$$\lambda I - A = 2I - A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -3/2 \\ -1 & 0 & 1/2 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = -1$ / $A - \lambda I = A + I = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[E_{3,1}(-1)]{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(2)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \left\} \begin{array}{l} x_1 + x_3 = 0 \\ x_2 = 0 \\ \text{let } x_1 = t \end{array} \right.$$

$$\text{Eigenvectors} = X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Basis for the eigenspace $= \{(1,0,-1)\}$

$$\lambda I - A = -I - A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{as above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

a) $A \cdot X = X \cdot \lambda \Rightarrow (A - \lambda I)X = 0 \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 9 = \underbrace{\boxed{\lambda^2 - 2\lambda - 8 = 0}}_{\text{Characteristic eq.}} \Big\} \underbrace{\boxed{\begin{matrix}\lambda_1 = 4 \\ \lambda_2 = -2\end{matrix}}}_{\text{Eigenvalues}}$$

$\lambda = 4$ // $(A - 4I) \cdot X = 0$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Big\} \begin{matrix} -3x_1 + 3x_2 = 0 \\ \text{let } x_1 = t \end{matrix}$$

Eigenvectors $= X = \begin{bmatrix} t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(1,1)\}$

$$\lambda I - A = 4I - A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \xrightarrow{E_{2,1}(+1)} \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

$\lambda = -2$ // $(A + 2I) \cdot X = 0$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-1)} \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{matrix} 3x_1 + 3x_2 = 0 \\ \text{let } x_1 = t \end{matrix}$$

Eigenvectors $= X = \begin{bmatrix} t \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda^2 - 3\lambda + 2 - 1$

$-1 + (2 - 2\lambda - \lambda + \lambda^2)$

Basis for the eigenspace $= \{(1,-1)\}$

$$\lambda I - A = -2I - A = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

c) $A \cdot X = X \cdot \lambda \Rightarrow (A - \lambda I) \cdot \lambda = 0 \Rightarrow |A - \lambda I| = 0$

$$= \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ 1-2\lambda & -\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(-2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$\xrightarrow{E_{1,2}(1-\lambda)} \begin{vmatrix} 0 & -1+(2-\lambda)(1-\lambda) & 3-5\lambda \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} = -1(-1)^3 \cdot \begin{vmatrix} \lambda^2 - 3\lambda + 1 & 3-5\lambda \\ 1 & 3-\lambda \end{vmatrix}$$

$= -\lambda^3 + 6\lambda^2 - 5\lambda = 0$ } Characteristic equation

$-\lambda(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 5,\ \lambda_3 = 1$ } Eigenvalues

$\lambda=0$ / $(A-\lambda I)X = A.X = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 2 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{1,2}(+1)]{E_{3,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + 3x_3 = 0 \end{array}\right\}$ let $x_3 = t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ -3t \\ t \end{bmatrix} = t.\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1,-3,1)\}$

$$\lambda I - A = -A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & -5 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

$\lambda=5$ / $(A-\lambda I)X = (A-5I).X = 0$

$$\left[\begin{array}{ccc|c} -4 & -1 & -2 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-4)} \left[\begin{array}{ccc|c} 0 & 11 & -22 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(3)]{E_{1,3}(-11)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$\left.\begin{array}{l} -x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array}\right\}$ let $x_3 = t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t.\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1,2,1)\}$

$$\lambda I - A = 5I - A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

$\lambda=1$ / $(A-\lambda I)X = (A-I).X = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ -1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$ $\left.\begin{array}{l} -x_1 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}\right\}$ Let $x_3 = t$

Eigenvectors $= X = \begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} = t.\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(3,-2,1)\}$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{as above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & -1 & -2 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

Q28) $A^5 . x = A.(A.(A.(A.(\underbrace{A.x}_{x.\lambda})))) = \ldots = \lambda^5 . x$

Eigenvalues for A^5 are: $\lambda_1' = 0^5 = 0$, $\lambda_2' = 2^5 = 32$, $\lambda_3' = (-1)^5 = -1$

Eigenvectors for A^5 are the same:

★ For $\lambda_1' = 0$: $X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ Basis $= \{(1,1,0)\}$, Dimension: 1

★ For $\lambda_2' = 32$: $X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t.\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \Rightarrow$ Basis $= \{(1,-3,2)\}$, Dimension: 1

★ For $\lambda_3' = -1$: $X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t.\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow$ Basis $= \{(1,0,-1)\}$, Dimension: 1

3 eigenspaces

Q29)

a) Since we have a 2nd degree polynomial, the dimension of the matrix is 2x2 (it has to be a square matrix for it to have a ch. pol.)

b) The characteristic polynomial is equal to det $(\lambda I - A)$ for $\lambda = 0$;

det $(-A) = (-1)^2$. det $(A) = \cancel{0^2} + \cancel{6.0} + 9 = 9$

c) $A^{-1}/A.x = A^{-1}/\lambda.x$

$I.x = A^{-1}.x.\lambda$

$I.\frac{1}{\lambda}.x = A^{-1}.x$

For the characteristic polynomial, $\lambda' = \frac{1}{\lambda}$

$\frac{1}{\lambda^2} + \frac{6}{\lambda} + 9 = p'(\lambda)$

d) The characteristic polynomial is calculated by det $(\lambda I - A) = 0$. If we take A^T instead of A, the ch. pol. won't change for the following reasons:

1) With $\lambda I - A^T$, we substract λ from values in the main diagonal of A^T. And by definition, taking the transpoze of a matrix doesn't change it's main diagonal.

2) det(A) = det(A^T)

Q30) Find the eigenvalues of A $\Rightarrow |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-1 & 1 & 4 \\ 0 & \lambda-2 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\cdot\begin{vmatrix} \lambda-2 & -4 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1).(\lambda-1).(\lambda-2)$$

$\lambda_{1,2} = 1$, $\lambda_3 = 2$

Find the eigenvectors :

$\lambda=1$ /

$$\left[\begin{array}{ccc|c} 0 & -1 & -4 & 0 \\ 0 & -1 & +4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-1)} \left[\begin{array}{ccc|c} 0 & -1 & -4 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{1,2}(1/2)} \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_2=0 \\ x_3=0 \\ \text{Let } x_1=t \end{array}$$

$$\text{Eigenvectors} = \left.\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t.\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \text{Basis} = \{(1,0,0)$$

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda=2$ /

$$\left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 0 & +4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[E_{1,3}(+4)]{E_{2,3}(-4)} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \left\{\begin{array}{l} x_1-x_2=0 \\ x_3=0 \\ \text{Let } x_1=t \end{array}\right.$$

$$\text{Eigenvectors} = \left.\begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} \text{Basis} = \{(1,1,0)\}$$

$$P_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

! Since there are $2 < 3$ basis vectors in total A is <u>NOT</u> diagonalizable

! Since there isn't a P which $P^{-1}.A.P = D$, A cannot be written as a linear combination of rank 1 matrices formed by its eigenvectors

Q31) Since A is a symmetric matrix, it's orthagonally diagonalizable

Find the eigenvalues of A such $|\lambda I - A| = 0$

$$0 = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda+1 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix} \xrightarrow{E_{2,3}(-1)} \begin{vmatrix} \lambda-2 & 1 & 1 \\ 0 & \lambda+3 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix} \xrightarrow{E_{1,3}(2-\lambda)} \begin{vmatrix} 0 & 2\lambda-3 & -\lambda^2+\lambda+3 \\ 0 & \lambda+3 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix}$$

$$0 = \underbrace{\lambda^3 - 9\lambda + 8}_{\lambda_1 = 1} = (\lambda-1).\underbrace{(\lambda^2+\lambda-8)}_{\Delta=33}, \quad \lambda_2 = \frac{-1+\sqrt{33}}{2}, \quad \lambda_3 = \frac{-1-\sqrt{33}}{2}$$

Find eigenvectors:

$\lambda = 1$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & 2 & 2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[E_{1,2}(-1)]{E_{3,2}(-1)} \left[\begin{array}{ccc|c} 0 & -3 & -3 & 0 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(2/3)} \left[\begin{array}{ccc|c} 0 & -3 & -3 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_2 + x_3 = 0 \\ x_1 = 0 \end{array}\right\} \text{let } x_2 = t \Rightarrow \text{Eigenvectors} = \left.\begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\} \text{Basis} = \{(0,1,-1)\}$$

- Apply Gram-Schmidt process to find the orthonormal vector:

$$V_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda_2 = \frac{1-\sqrt{33}}{2}$

$$\begin{bmatrix} (-5-\sqrt{33})/2 & -1 & -1 \\ -1 & (1+\sqrt{33})/2 & 2 \\ -1 & 2 & (1+\sqrt{33})/2 \end{bmatrix} - \text{* For the sake of my mental health, I'll be skipping the steps *} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{33}-5)/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 + \frac{\sqrt{33}-5}{2} x_2 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \text{let } x_2 = t \Rightarrow \text{Ei. V.} = \left.\begin{bmatrix} (5-\sqrt{33})/2 \cdot t \\ t \\ t \end{bmatrix}\right\} \text{Basis} = \left\{\left(\frac{5-\sqrt{33}}{2}, 1, 1\right)\right\}$$

! For the love of God, assume $\frac{5-\sqrt{33}}{2} \approx -0{,}37$

- Apply Gram-Schmidt process to find the orthonormal vector:

$$V_2 = \begin{bmatrix} -0{,}25 \\ 0{,}68 \\ 0{,}68 \end{bmatrix}$$

$\lambda_3 = \frac{1+\sqrt{33}}{2}$

$$\left[\begin{array}{ccc|c} (-5+\sqrt{33})/2 & -1 & -1 & 0 \\ -1 & (1+\sqrt{33})/2 & 2 & 0 \\ -1 & 2 & (1+\sqrt{33})/2 & 0 \end{array}\right] - \cdots \to \left[\begin{array}{ccc|c} 1 & 0 & (-\sqrt{33}-5)/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 + \left(\frac{-\sqrt{33}-5}{2}\right) \cdot x_2 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \text{Let } x_2 = t \Rightarrow \text{Ei. V.} = \left.\begin{bmatrix} (+5+\sqrt{33})/2 \cdot t \\ t \\ t \end{bmatrix}\right\} \text{Basis} = \left\{\left(\frac{+5+\sqrt{33}}{2}, 1, 1\right)\right\}$$

! Assume $\frac{+5+\sqrt{33}}{2} \approx +5{,}37$

- Apply Gram-Schmidt process to find the orthonormal vector

$$V_3 = \begin{bmatrix} 0{,}97 \\ 0{,}18 \\ 0{,}18 \end{bmatrix}$$

• Form matrix P from vectors V_1, V_2, V_3

$$P=\begin{bmatrix} 0 & -0{,}25 & 0{,}97 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \Big\} \text{Matrix } P, \text{ orthagonally diagonalizes } A$$

$$P^T \cdot A \cdot P = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{33})/2 & 0 \\ 0 & 0 & (-1+\sqrt{33})/2 \end{bmatrix}$$

• Can we write A in diagonal form as a linear combination formed from its eigenvectors?

$$P/P^{-1} \cdot A \cdot P = P/D \Rightarrow A \cdot P \setminus P^{-1} = P \cdot D \setminus P^{-1} \Rightarrow A = P \cdot D \cdot P^{-1}$$

Since P is orthagonal, $P^{-1} = P^T$ and $A = P \cdot D \cdot P^T$

$$A=\begin{bmatrix} 0 & -0{,}25 & 0{,}97 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{33})/2 & 0 \\ 0 & 0 & (-1+\sqrt{33})/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -0{,}25 & 0{,}68 & 0{,}68 \\ 0{,}97 & 0{,}18 & 0{,}18 \end{bmatrix}$$

$$A = P \cdot \left\{ \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ -0{,}25 & 0{,}68 & 0{,}68 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0{,}97 & 0{,}18 & 0{,}18 \end{bmatrix} \cdot \lambda_3 \right\}$$

∇ Since both P and the row matrices for $D \cdot P^T$ have rank=1, the product will also be rank=1

$$A=\begin{bmatrix} 0 & -0{,}25 & 0{,}97 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_1 + \dots$$

$$A = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0{,}5 & -0{,}5 \\ 0 & -0{,}5 & 0{,}5 \end{bmatrix}}_{\text{Rank}=1} \cdot \lambda_1 + \underbrace{\begin{bmatrix} 0{,}0625 & -0{,}17 & -0{,}17 \\ -0{,}17 & 0{,}4624 & 0{,}4624 \\ -0{,}17 & 0{,}4624 & 0{,}4624 \end{bmatrix}}_{\text{Rank}=1} \lambda_2 + \underbrace{\begin{bmatrix} 0{,}9409 & 0{,}1746 & 0{,}1746 \\ 0{,}1746 & 0{,}0324 & 0{,}0324 \\ 0{,}1746 & 0{,}0324 & 0{,}0324 \end{bmatrix}}_{\text{Rank}=1} \lambda_3$$

∇ Yes, we can write A in diagonal form as a linear combination of rank 1 matrices formed from it's eigenvectors

Q32) $\lambda(x) = \lambda(a\lambda^2 + b\lambda + c) \Rightarrow \lambda_1 = 0,\ \lambda_2 = \lambda_2,\ \lambda_3 = \lambda_3$

Let P be a matrix that diagonalizes A

$$P^{-1}/P \cdot A \cdot P^{-1} = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow A \cdot P^{-1} \setminus P = P^{-1} \cdot D \setminus P \Rightarrow A = P^{-1} \cdot D \cdot P$$

$$A = P^{-1} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot P = P^{-1} \cdot (P_1 \cdot 0 + P_2 \cdot \lambda_2 + P_3 \cdot \lambda_3)$$

$$A = \cancel{P^{-1} \cdot P_1 \cdot 0} + P^{-1} \cdot P_2 \cdot \lambda_2 + P^{-1} \cdot P_3 \cdot \lambda_3 = \underbrace{P^{-1} \cdot P_2 \cdot \lambda_2 + P^{-1} \cdot P_3 \cdot \lambda_3}_{\text{rank}=1}$$

∇ Since A can be expressed as a linear combination of 2, rank=1 matrices, we can say that A has a reduced row echolon form which can be shown as:

$$\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \end{bmatrix} \Big\} A \text{ has the rank of "2"}$$

Nullity = col num. − rank = 3 − 2 = 1

∇ A has a row of zeroes when it's reduced to row echolon form. Therefore $\det(A) = 0$ which means A is not invertible

∇ Characteristic eq. of a matrix calculated by: $P(\lambda) = |\lambda I - A|$. For $\lambda = 0$, $P(\lambda) = \det(-A) = (-1)^n \cdot \det(A)$, for the matrix to be invertible $\det(A) \neq 0$ and therefore $P(0) \neq 0$