2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = V_1 . c_1 + V_2 . c_2 + V_3 . c_3$

$$\left.\begin{aligned} 4c_1 - 2c_2 \quad &= a \\ 3c_1 \quad + 3c_3 &= b \\ 2c_1 - c_2 \quad &= c \\ c_1 \quad + c_3 &= d \end{aligned}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 3 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow{\substack{E_{1,4}(-4) \\ E_{2,4}(-3) \\ E_{3,4}(-2)}} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$\xrightarrow{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

For the system to be consistent, $a-2c=0$, $b-3d=0$ $\Rightarrow$ $a=2c$, $b=3d$

Given vectors span a subspace consisting of vectors K, that

$$K = \begin{bmatrix} 2c \\ 3d \\ c \\ d \end{bmatrix} ; \quad c, d \in \mathbb{R}$$

b) The vectors are linearly independent if and only if the eq. $V_1 . c_1 + V_2 . c_2 + V_3 . c_3 = 0$ only has the trivial solution

$$\left.\begin{aligned} 4c_1 - 2c_2 \quad &= 0 \\ 3c_1 \quad + 3c_3 &= 0 \\ 2c_1 - c_2 \quad &= 0 \\ c_1 \quad + c_3 &= 0 \end{aligned}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Same operations used in the section a}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

since there are an infinite amount of solutions, the vectors are linearly dependent

c) The dimension for R^4 is 4, therefore any linearly ind. 4 vectors form a basis. We already have 3 vectors which are linearly independent, but they don't span R^4, if we can find a fourth vector "v" which is not in the span, we can write the basis for R^4 (add-minus theorem)

We found that subspace spanned of the given vectors consists of vectors such:

$$K = \begin{bmatrix} 2c \\ b \\ c \\ d \end{bmatrix};\ b,c,d \in \mathbb{R} \quad \text{therefore} \quad v = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \text{ is not in the span}$$

A basis for R is $= \{(4,3,2,1), (-2,0,-1,0), (0,3,0,1), (1,2,3,4)\}$

Q2) ∇ $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow f_1 = \cos 2x$ and $f_2 = \cos 2x$

Applying Wronskian:

$$\begin{vmatrix} \cos 2x & \cos 2x \\ (\cos 2x)' & (\cos 2x)' \end{vmatrix} = \begin{vmatrix} \cos 2x & \cos 2x \\ -2\sin 2x & -2\sin 2x \end{vmatrix} = -2\sin 2x \cos 2x + 2\sin 2x \cos 2x = 0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of R^3 is 3, therefore basis for R^3 consists of 3 vectors

Choose u_1, u_3, u_4 $\Rightarrow$ Test for linear independence:

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = 0$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 0 \\ -2c_1 &= 0 \\ -c_3 + 7c_4 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{E_{2,1}(2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{E_2(1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right]$$

$$\xrightarrow[E_{3,2}(1)]{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \left.\begin{aligned} c_1 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}\right\} \Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1, u_3, u_4 form a basis without testing for their span.

Base $= \{(1,-2,0), (1,0,-1), (0,0,7)\}$

∇ Express the remaining vector, u_2, by the base vectors

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = u_2$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 1 \\ -2c_1 &= 2 \\ -c_3 + 7c_4 &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{aligned} c_1 &= -1 \\ c_3 &= 2 \\ c_4 &= 6/7 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

The dimension of the subspace spanned by these vectors is 4

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^4 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

Q6) The vectors u and v, are the basis for the $u \times v$ plane.

The vector resulting from $\text{proj}_{u \times v}\, w$ lies in the $u \times v$ plane, therefore it can be expressed as a linear combination of the basis vectors u and v.

Therefore u, v and $\text{proj}_{u \times v}\, w$ are not linearly independent

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.