2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 1

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1. For each of the following i) Express the linear system in matrix form, i.e. as $A\mathbf{x} = \mathbf{b}$. Indicate dimensions of $A, \mathbf{x}, \mathbf{b}$. Determine the solution for $\underline{x}$ (if any) using Gauss-Jordan Elimination.

   a. $x + 3y - z = 1$
   $4x + z = 2$
   $-x + y + 3z = 2$

$$\left.\underbrace{\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}}_{3\times 3} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}_{3\times 1} \right\}$$

Using Gauss-Jordan Elemination

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 4 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -12 & 5 & -2 \\ 0 & 4 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -12 & 5 & -2 \\ 0 & 1 & 1/2 & 3/4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & -12 & 5 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -5/4 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & 0 & 11 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -5/4 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & 0 & 1 & 7/11 \end{array}\right]$$

$$\left.\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 15/44 \\ 0 & 1 & 0 & 19/44 \\ 0 & 0 & 1 & 7/11 \end{array}\right]}_{\text{Reduced Echdon Form}} \right\} \quad \begin{aligned} x &= 15/44 \\ y &= 19/44 \\ z &= 7/11 \end{aligned}$$

   b. $x + 2y - 3z = 1$
   $y + z = 3$
   $-x + 2y + 7z = 5$

$$\underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ -1 & 2 & 7 \end{bmatrix}}_{3\times 3} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}}_{3\times 1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 3 \\ -1 & 2 & 7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 4 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

! As seen in the last-row, we got $0 = -6$ which is not possible. Hence, this system has no solutions.

c. $x + 3y - z + w = 1$
$3x + z + 2w = -1$

$$\underbrace{\begin{bmatrix} 1 & 3 & -1 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}}_{2\times 4} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{4\times 1} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{2\times 1}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 3 & 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 0 & -9 & 4 & -1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 0 & 1 & -4/9 & 1/9 & 4/9 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3/9 & 6/9 & -3/9 \\ 0 & 1 & -4/9 & 1/9 & 4/9 \end{array}\right]$$

? Since the number of equations is less then the unknowns the system has infinite solutions with $(4-2=2)$ variables

• let $z = 9k$ and $w = 9t$

$$\left.\begin{array}{l} x + 3k + 6t = -3/9 \\ y - 4k + t = 4/9 \end{array}\right\} \quad \begin{array}{l} x = -3/9 - 3k - 6t \\ y = 4/9 + 4k - t \end{array}$$

2. Bring the following matrix to row echelon form and then to reduced row echelon form by applying elementary row operations. Specify the elementary matrices with which you premultiply at each step.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Row Echelon Form}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{Reduced Row Echelon Form}$$

3. Consider $A = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Is this matrix in row echelon form, reduced row echelon form or neither? Explain.

The given matrix is in "row echelon form" for the following reasons:

- The first non-zero number on each row is a one
- In each row, the leading one is further right than the upper one.

The given matrix is ALSO "reduced" row echelon form because:

- There are columns which have leading ones also have zeros everywhere else.

4. If A is 2x3, B is 3x4 and C is 4x2 what are the dimensions (size) of the matrix resulting from $(C^T - AB)B^T$?

Let's divide the given expression into three parts:

$$(C^T - AB).B^T$$

$$\swarrow \quad \downarrow \quad \searrow$$

$$C^T \qquad AB \qquad B^T$$

a) Since C is a 4×2 matrix, C^T has a dimension of 2×4

b) $\underbrace{A_{2x3} \cdot B_{3x4}}$ => M_{2x4} (We get a 2x4 matrix)

c) Since B is a 3×4 matrix, B^T has a dimension of 4×3

$$C^T_{2\times4} - AB_{2\times4} = K_{2\times4} \Rightarrow \underbrace{K_{2\times4} \cdot B^T_{4\times3}} = L_{2\times3}$$

The size of the matrix resulting from the given expression is 2×3

5. If L and U are lower triangular and upper triangular matrices, respectively, of size nxn what can you say about the matrix product $L^{-1}U^T$? Explain.

- If L is a lower triangular matrix, we can assume that L^{-1} is a lower triangular matrix as well.
- If U is an upper triangular matrix, then U^T is a lower triangular matrix

$L^{-1} \times U^T \Rightarrow$ If we multiply two lower triangular $n \times n$ matrices, we get another lower triangular $n \times n$ matrix as result.

6. Perform the matrix product by first expressing the result in terms as of the submatrices.

$$A = [A_1|A_2] = \left[\begin{array}{ccc|cc} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&1 \\ 1&0&0&1&0 \end{array}\right] \quad B = \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} = \left[\begin{array}{cc} 1&0 \\ 1&1 \\ 0&1 \\ \hline 1&2 \\ -1&1 \end{array}\right].$$

Q6)

$$A \cdot B = [A_1 \vdots A_2] \cdot \begin{bmatrix} B_1 \\ \hline B_2 \end{bmatrix} = [A_1B_1 + A_2B_2]$$

$$A_1 \cdot B_1 = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{bmatrix} \cdot \begin{bmatrix} 1&0 \\ 1&1 \\ 0&1 \end{bmatrix} = \begin{bmatrix} 1&0 \\ 1&1 \\ 0&1 \\ 1&0 \end{bmatrix}$$

$$A_2 \cdot B_2 = \begin{bmatrix} 0&0 \\ 0&0 \\ 0&1 \\ 1&0 \end{bmatrix} \cdot \begin{bmatrix} 1&2 \\ -1&1 \end{bmatrix} = \begin{bmatrix} 0&0 \\ 0&0 \\ -1&1 \\ 1&2 \end{bmatrix}$$

$$AB = A_1B_1 + A_2 \cdot B_2 = \begin{bmatrix} 1&0 \\ 1&1 \\ -1&2 \\ 2&2 \end{bmatrix} //$$

7. Write down a 4x4 matrix A for which the elements satisfy $a_{ij} = 0$ if $|i-j| < 2$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$*\ a_{11} = 0, \quad |1-1| < 2$
$*\ a_{12} = 0, \quad |1-2| < 2$
$*\ a_{21} = 0, \quad |2-1| < 2$
$*\ a_{22} = 0, \quad |2-2| < 2$
$*\ a_{23} = 0, \quad |2-3| < 2$
$*\ a_{32} = 0, \quad |3-2| < 2$
$*\ a_{33} = 0, \quad |3-3| < 2$
$*\ a_{34} = 0, \quad |3-4| < 2$
$*\ a_{43} = 0, \quad |4-3| < 2$
$*\ a_{44} = 0, \quad |4-4| < 2$

! We can fill other indices with "1" since they don't affect the given rule.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

8. Prove that the inverse of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ by solving a set of 4 linear equations in 4 unknowns.

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow A.A^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$a.e + b.g = 1 \qquad c.e + d.g = 0$
$a.f + b.h = 0 \qquad c.f + d.h = 1$ } Solve the equations

$\frac{-c}{ad}$

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \cdot \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ c/d & 0 & 1 & 0 & 0 \\ 0 & c/d & 0 & 1 & 1/d \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} \\ 0 & 1 & 0 & \frac{b}{a} & 0 \\ 0 & 0 & \frac{ad-cb}{ad} & 0 & \frac{-c}{ad} \\ 0 & 0 & 0 & \frac{ad-cb}{ad} & \frac{1}{d} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} \\ 0 & 1 & 0 & \frac{b}{a} & 0 \\ 0 & 0 & 1 & 0 & \frac{-c}{ad-cb} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-cb} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{d}{(ad-cb)} \\ 0 & 1 & 0 & 0 & \frac{-b}{(ad-cb)} \\ 0 & 0 & 1 & 0 & \frac{-c}{ad-cb} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-cb} \end{array}\right] \quad \left\{ \begin{array}{l} e = \frac{d}{(ad-cb)} \\ f = \frac{-b}{(ad-cb)} \\ g = \frac{-c}{(ad-cb)} \\ h = \frac{a}{(ad-cb)} \end{array} \right.$$

$$A^{-1} = \begin{bmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-cb)} \\ \frac{-c}{(ad-bc)} & \frac{a}{(ad-cb)} \end{bmatrix} = \frac{1}{(ad-bc)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

9. Let $A^3 - 2A^2 + 3A = 0$ Write down a formula for A^{-1} in terms of A.

$$A^3 - 2A^2 + 3A = 0 \Rightarrow A^3 . A^{-1} - 2A^2 . A^{-1} + 3.A.A^{-1} = 0$$
$$A^2 - 2A + 3I = 0 \Rightarrow A^2 . A^{-1} - 2A.A^{-1} + 3I.A^{-1} = 0$$
$$A - 2I + 3A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{3}(2I - A)$$

10. Let a matrix have a column of zeroes. Does its inverse exist? Explain by using $AA^{-1} = I$. If a matrix does not have a column or row of zeroes can it be noninvertible? Explain by using row reduction operations.

* let $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$A.A^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ $a.e + 0.g = 1$ $\quad b.e + 0.g = 0$
$a.f + 0h = 0$ $\quad bf + 0.h = 1$

$a.e + \cancel{0.g} = 1$ $\qquad b.e + \cancel{0.g} = 0$ $\qquad \cancel{bf} + \cancel{0.h} = 1$

$a.e = 1$ $\qquad b.e = 0$ $\qquad$ * $0 + 0 = 1$ *

$e \neq 0$ $\qquad e \neq 0 \Rightarrow b = 0$ $\qquad$ impossible

Therefore, if a matrix has a column of zeroes, it's inverse <u>doesn't exist</u>

* let $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ which doesn't have a row of zeroes.

By using row operations, we get:

$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ } Even if we don't have a row of zeroes in the beginning, we get a row of zeroes by reducing the matrix.

Therefore, a matrix which doesn't have a row of zeroes <u>can</u> be non-invertible.

11. Can you write the following where A, B, AB are all symmetric? Explain.

$$(A+B)^2 \stackrel{?}{=} A^2 + 2AB + B^2$$

$$(A+B)^2 = (A+B).(A+B) = A.A + \overbrace{A.B + B.A}^{\text{Equal}} + B.B = A^2 + 2AB + B^2$$

$*$ If AB is symmetric, we can say that $(AB) = (AB)^T = B^T A^T$

$*$ Since A and B are symmetric as well, $B^T = B$ and $A^T = A$

$AB = (AB)^T = B^T A^T = BA$

12. Find three row operations that will turn $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ into $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Write down the elementary matrices for each.

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

13. Is it possible to apply elementary row operations to turn an invertible matrix into a matrix with an all zeroes column?

No, such a thing is not possible. If a matrix has a whole column of zeroes after elementary operations, it means that matrix does not have an inverse.

By that logic, if a matrix is invertible it is not able to be turned into a matrix with an all zeroes column by elementary operations.

14. Try to find the inverse of the following matrices by Gauss Jordan elimination on augmented matrices of the form $[A \mid I]$

a) $A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ b) $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ c) $A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix}$

a)

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1/2 & 0 & 1 \end{array}\right]$$

The matrix is non-invertible since it has a row of zeroes

b)

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right] \Rightarrow \left[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \middle| \underbrace{\begin{array}{ccc} 0 & -1/2 & 1 \\ 0 & 1/2 & 0 \\ -1 & -1/2 & 1 \end{array}}_{A^{-1}}\right]$$

c)

$$\left[\begin{array}{ccc|ccc} 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} c & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/c \\ 0 & 0 & 1 & 1/a & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/b & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/c \\ 0 & 0 & 1 & 1/a & 0 & 0 \\ 0 & 0 & 0 & -1/a & 1/b & 0 \end{array}\right]$$

Matrix is non-invertible (Row of zeroes)

15. State the elementary row/column operations needed to turn $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ into $B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$. (Hint: Specify elementary row and column operations that interchange rows and interchange columns)

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_1 \text{ (Interchange Rows)}} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_2 \text{ (Interchange columns)}} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

16. LU decomposition: Prove that an invertible matrix can be expressed as $A = LU$ where L is lower triangular and U is upper triangular. (Hint: Product of lower triangular elementary matrices is lower triangular. What about the row echelon form?)

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix

* Using Gauss elemination it is posible to turn A into I since A^{-1} exists.

$$E_1 \left\{ \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{c}{a} & 1 \end{bmatrix} \right. \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix}$$

* Note that E_1 is lower triangular

$$E_2 \left\{ \begin{bmatrix} 1 & \frac{-b}{da-bc} \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \right. \cdot \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Note that E_2 is upper triangular

* Find the inverses for E_1 and E_2

$$[E_1 \mid I] = \left[\begin{array}{cc|cc} 1/a & 0 & 1 & 0 \\ -c/a & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & a & 0 \\ -c/a & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & a & 0 \\ 0 & 1 & c & 1 \end{array}\right]$$

$$E_1^{-1} = \begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}$$

$$[E_2 \mid I] = \left[\begin{array}{cc|cc} 1 & \frac{-b}{da-bc} & 1 & 0 \\ 0 & \frac{a}{da-bc} & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-b}{da-bc} & 1 & 0 \\ 0 & 1 & 0 & \frac{da-bc}{a} \end{array}\right] \rightarrow \dots$$

$$\dots \left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{b}{a} \\ 0 & 1 & 0 & \frac{da-bc}{a} \end{array}\right]$$

$$E_2^{-1} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix}$$

* Since E_1 is a lower triangular matrix, E_1^{-1} is also lower triangular

Since E_2 is an upper triangular matrix, E_2^{-1} is also upper triangular

* If we multiply $E_1^{-1} \cdot E_2^{-1}$:

$$\underbrace{\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}$$

17. Determine the inverse of the diagonal matrix $D = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} a & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{I_4} \quad \underbrace{\begin{matrix} 1/a & 0 & 0 & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1/d \end{matrix}}_{D^{-1}}$$

18. Is the given triangular matrix invertible? Explain by saying what happens when the matrix is reduced to a row echelon or reduced row echelon form. Do you get a row of zeroes or not? (Do not try to compute the inverse.)

$$A_{5x5} = \begin{bmatrix} 2 & 5 & 4 & 9 & -1 \\ 0 & 0 & -3 & 8 & 2 \\ 0 & 0 & 6 & 1 & -4 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & 4 & 9 & -1 \\ 0 & 0 & -3 & 8 & 2 \\ 0 & 0 & 6 & 1 & -4 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 2 & 9/2 & -1/2 \\ 0 & 0 & 1 & -8/3 & 2/3 \\ 0 & 0 & 1 & 1/6 & -2/3 \\ 0 & 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 2 & 9/2 & 0 \\ 0 & 0 & 1 & -8/3 & 0 \\ 0 & 0 & 1 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix is non-invertible since we got a row of zeroes

19. If matrices satisfy $AB = AC$ is it true that $B = C$ when

a) AB is invertible? (if yes explain, if no, give an example for which $B \neq C$.)

a) If AB is invertible, then A and B should also be invertible

$* \ (AB)^{-1} = B^{-1} \cdot A^{-1} \ *$

$A^{-1}/AB = A^{-1}/AC \Rightarrow \underbrace{A^{-1}A}_{I}B = \underbrace{A^{-1}A}_{I}C$

$IB = IC \Rightarrow B = C$

b) A can be reduced to a matrix with all zeroes row? (if yes explain, if no, give an example for which $B \neq C$.)

b) If A can be reduced to a matrix with an all zeroes row, we can say A is non-invertible.

Therefore if $AB = AC$, we can't say $B = C$ directly

Ex

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ an all zeroes row.

$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ but $B \neq C$

20. Simultaneously solve the following linear systems of equations by reducing one 3x6 matrix:

$$2x + 3y - z = 1$$
$$4x + y - z = 0$$
$$x + y + z = 2$$

$$2x + 3y - z = -1$$
$$4x + y - z = 2$$
$$x + y + z = 0$$

$$2x + 3y - z = 3$$
$$4x + y - z = 3$$
$$x + y + z = 3$$

$$\left[\begin{array}{ccc|c|c|c} 2 & 3 & -1 & 1 & -1 & 3 \\ 4 & 1 & -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 4 & 1 & -1 & 0 & 2 & 3 \\ 2 & 3 & -1 & 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 0 & -3 & -5 & -8 & 2 & -9 \\ 0 & 1 & -3 & -3 & -1 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & -3 & -5 & -8 & 2 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 4 & 5 & 1 & 6 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & 0 & -14 & -17 & -1 & -18 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 4 & 5 & 1 & 6 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & 0 & 1 & 17/14 & 1/14 & 9/7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1/7 & 5/7 & 6/7 \\ 0 & 1 & 0 & 9/14 & -11/14 & 6/7 \\ 0 & 0 & 1 & 17/14 & 1/14 & 9/7 \end{array}\right]$$

Set 1	Set 2	Set 3
$x = 1/7$	$x = 5/7$	$x = 6/7$
$y = 9/14$	$y = -11/14$	$y = 6/7$
$z = 17/14$	$z = 1/14$	$z = 9/7$