2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 2

1.

$$\det(A) = \cancel{a_{14}\cdot(-M_{14})} + a_{24}\cdot M_{24} + \cancel{a_{34}\cdot(-M_{34})} + a_{44}\cdot M_{44}$$

$$\quad \hookrightarrow 0 \qquad \hookrightarrow 8 \qquad \hookrightarrow 0 \qquad \hookrightarrow -1$$

$$M_{24} = \begin{vmatrix} 2 & -2 & 4 \\ 1 & 0 & 6 \\ 2 & 3 & 0 \end{vmatrix} = 2\cdot\begin{vmatrix} 0 & 6 \\ 3 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} + 2\cdot\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}$$

$$= -48$$

$$M_{44} = \begin{vmatrix} 2 & -2 & 4 \\ 4 & 1 & 0 \\ 1 & 0 & 6 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} - 4\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix} + 1\cdot\begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix}$$

$$= 56$$

$$\det(A) = 8\cdot(-48) + (-1)\cdot 56$$

$$= -440$$

* To calculate the determinant, the last column is used because it had two zeroes which made the calculations easier.

2.

8.2)

$$\begin{bmatrix} 2 & -2 & 4 & 0 \\ 4 & 1 & 0 & 8 \\ 7 & 0 & 6 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 0 & 4 & 16 \\ 4 & 1 & 0 & 8 \\ 7 & 0 & 6 & 0 \\ -10 & 0 & 0 & -25 \end{bmatrix} \left.\vphantom{\begin{matrix}a\\b\\c\\d\end{matrix}}\right\} \begin{matrix} E_{2,1}(2) \\ E_{2,4}(-3) \end{matrix}$$

$$\det(A) = \underbrace{\cancel{a_{12} \cdot (-M_{12})}}_{\hookrightarrow 0} + \underbrace{a_{22}}_{\hookrightarrow 1} \cdot M_{22} + \underbrace{\cancel{a_{32} \cdot (-M_{32})}}_{\hookrightarrow 0} + \underbrace{\cancel{a_{42} \cdot M_{42}}}_{\hookrightarrow 0}$$

$$M_{22} = \begin{vmatrix} 10 & 4 & 16 \\ 1 & 6 & 0 \\ -10 & 0 & -25 \end{vmatrix} = 10 \cdot \begin{vmatrix} 6 & 0 \\ 0 & -25 \end{vmatrix} - 1 \begin{vmatrix} 4 & 16 \\ 0 & -25 \end{vmatrix} - 10 \begin{vmatrix} 4 & 16 \\ 6 & 0 \end{vmatrix}$$

$$= -440$$

$$\det(A) = 1 \cdot (-440) = -440$$

3.

i)

Q3)

i) $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -6 \\ 8 & 2 & 6 \\ 4 & 1 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 2 & 1 \\ -6 & 6 & 6 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & -1 \\ -6 & 2 \end{vmatrix} = -2$ $\quad C_{21} = 8 \quad C_{31} = 4$

$C_{12} = -2 \quad C_{22} = 2 \quad C_{32} = 1$

$C_{13} = -6 \quad C_{23} = 6 \quad C_{33} = 6$

$\det(A) = \begin{vmatrix} 1 & -4 & 0 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{vmatrix} = 1.(-2) + (-1)(-8) = 6$

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} -1/3 & 4/3 & 2/3 \\ -1/3 & 1/3 & 1/6 \\ -1 & 1 & 1 \end{bmatrix}$

ii)

ii) $A = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{bmatrix} \xrightarrow[E_{12}(-1)]{} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 6 & -2 \\ 0 & -6 & 2 \end{bmatrix} \xrightarrow[E_{23}(1)]{} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$, $\det(A) = 0$ (row of zeroes)

* The inverse of the matrix A doesn't exist since $\det(A) = 0$ Which means such inverse can't be calculated by method of adjoints.

4.

Q4)

$$\begin{bmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 4 & -1 & -5 \\ 1 & 4 & 4 & 1 & -1 \\ 2 & -1 & 4 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} \quad \text{and} \quad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 4 & -1 & -5 \\ 1 & 4 & 4 & 1 & -1 \\ 2 & -1 & 4 & 4 & 0 \end{vmatrix} = \begin{matrix} E_{2,1}(-2) \\ E_{2,3}(-3) \\ E_{2,4}(-1) \\ E_{2,5}(-2) \end{matrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 0 & 4 & 2 & 1 & -7 \\ 0 & -1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ -2 & -2 & -1 & -23 \\ 4 & 2 & 1 & -7 \\ -1 & 0 & 4 & -12 \end{vmatrix}$$

$$\begin{matrix} E_{3,2}(-2) \\ E_{4,3}(1) \end{matrix} \Rightarrow -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 0 & -2 & -9 & 1 \\ 0 & 2 & 17 & -55 \\ -1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 6 & -12 \\ -2 & -9 & 1 \\ 2 & 17 & -55 \end{vmatrix}$$

$$= \left(2 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -55 \end{vmatrix} + 2 \cdot \begin{vmatrix} 6 & -12 \\ -9 & 1 \end{vmatrix}\right) \cdot -1$$

$$= 456$$

$$\det(A_1) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 2 & -2 & 4 & -1 & -5 \\ 2 & 4 & 4 & 1 & -1 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix} \rightarrow \begin{matrix} E_{2,1}(-2) \\ E_{2,3}(-2) \rightarrow \\ E_{2,4}(-2) \end{matrix} \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & 0 & -1 & -17 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix}$$

$$\begin{matrix} E_{5,4}(4) \\ E_{5,3}(-2) \end{matrix} \rightarrow \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & 0 & -8 & -9 & -17 \\ 0 & 0 & 16 & 17 & -13 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 0 & -8 & -9 & -17 \\ 0 & 16 & 17 & -13 \\ -1 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 6 & -12 \\ -8 & -9 & -17 \\ 16 & 17 & -13 \end{vmatrix}$$

$$= \left(8 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -13 \end{vmatrix} + 16 \cdot \begin{vmatrix} 6 & -12 \\ -9 & -17 \end{vmatrix}\right) \cdot -1$$

$$= 2352$$

$$\det(A_3) = \begin{vmatrix} 2 & 0 & 2 & 6 & 0 \\ 1 & 0 & 1 & 0 & 6 \\ 3 & -2 & 2 & -1 & -5 \\ 1 & 4 & 2 & 1 & -1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \to 2 \cdot \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 & 6 \\ 3 & -2 & 2 & -1 & -5 \\ 1 & 4 & 2 & 1 & -1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{\substack{E_{1,2}(-1) \\ E_{1,3}(-3) \\ E_{1,4}(-1) \\ E_{1,5}(-2)}} 2 \cdot \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & -2 & -1 & -10 & -5 \\ 0 & 4 & 1 & -2 & -1 \\ 0 & -1 & -2 & -2 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -3 & 6 \\ -2 & -1 & -10 & -5 \\ 4 & 1 & -2 & -1 \\ -1 & -2 & -2 & 0 \end{vmatrix} \xrightarrow{\substack{E_{4,3}(4) \\ E_{4,2}(-2)}} 2 \begin{vmatrix} 0 & 0 & -3 & 6 \\ 0 & 3 & -6 & -5 \\ 0 & -7 & -10 & -1 \\ -1 & -2 & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 0 & -3 & 6 \\ 3 & -6 & -5 \\ -7 & -10 & -1 \end{vmatrix} = 2 \cdot \left(-3 \cdot \begin{vmatrix} -3 & 6 \\ -10 & -1 \end{vmatrix} - 7 \cdot \begin{vmatrix} -3 & 6 \\ -6 & -5 \end{vmatrix} \right)$$

$$= -1092$$

$$x_1 = \frac{|A_1|}{|A|} = \frac{2352}{456} \approx 5,16 \quad \text{and} \quad x_3 = \frac{|A_3|}{|A|} = \frac{-1092}{456} \approx -2,39$$

5.

Q5)

* For the given matrix to be non-invertible, it's determinant must be equal to zero.

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ x & 1 & -2 \end{vmatrix} = E_{2,3}(-x) \to \begin{vmatrix} 0 & 0 & 1 \\ \cancel{1} & \cancel{x} & \cancel{5} \\ 0 & 1-x^2 & -2-5x \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 1 \\ 1-x^2 & -2-5x \end{vmatrix} = 1-x^2$$

$$1-x^2 = 0 \Rightarrow x = -1 \text{ or } x = 1$$

6.

Q6)

$A^{-1} = \frac{adj(A)}{det(A)}$, Since A is an upper-triangular matrix, determinant equals to the diagonal product

$* \; det(A) = 2 \cdot 1 \cdot 6 \cdot (-1) = -12 \; *$

$$adj(A) = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}^T$$

Since A is an upper-triangular matrix, cofactors which are placed above the diagonal are all equal to zero

$$C_{11} = \begin{vmatrix} 1 & 2 & 8 \\ 0 & 6 & 5 \\ 0 & 0 & -1 \end{vmatrix} = -6$$

$C_{21} = 12$ $\quad C_{22} = -12$ $\quad C_{31} = 8$ $\quad C_{32} = 4$ $\quad C_{33} = -2$

$C_{41} = -56$ $\quad C_{42} = -76$ $\quad C_{43} = -10$ $\quad C_{44} = 12$

$$adj(A) = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 12 & -12 & 0 & 0 \\ 8 & 4 & -2 & 0 \\ -56 & -76 & -10 & 12 \end{bmatrix}^T = \begin{bmatrix} -6 & 12 & 8 & -56 \\ 0 & 12 & 4 & -76 \\ 0 & 0 & -2 & -10 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$$A^{-1} = \frac{adj(A)}{det(A)} = \begin{bmatrix} 1/2 & -1 & -2/3 & 14/3 \\ 0 & 1 & -1/3 & 19/3 \\ 0 & 0 & 1/6 & 5/6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

7.

Q7)

$$= \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 2 & 12 & -1 \end{bmatrix} \xrightarrow{\substack{E_{1,3}(-1) \\ E_{1,4}(-3)}} \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{2,1}(2) \\ E_{2,3}(-2) \\ E_{2,4}(-8)}} 2 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -8 & -9 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & -8 & -9 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{3,2}(-1) \\ E_{3,1}(-6) \\ E_{3,4}(8)}} -10 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & -69/5 \end{vmatrix} = 138 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 138$$

8.

Q8)

$$= \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 0 & 8 & 12 \end{vmatrix} \xrightarrow{\substack{E_{1,3}(-1) \\ E_{1,4}(-3)}} = \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 6 & -4 & 12 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 5 \\ 0 & 6 & -4 & 12 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{2,1}(2) \\ E_{2,3}(-2) \\ E_{2,4}(-6)}} = 2 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -10 & 6 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & -10 & 6 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{3,2}(-1) \\ E_{3,1}(-6) \\ E_{3,4}(10)}} = -10 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = -10 \cdot 0 = 0$$

9.

Q9)

$$A^T = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{vmatrix} = E_{3,1}(-2) \begin{vmatrix} 0 & -9 & 2 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{vmatrix} = -23$$

$$\det((A^T)^6) = \det(\underbrace{A^T \cdot A^T \cdot A^T \cdot A^T \cdot A^T \cdot A^T}_{\text{Square matrices of the same size}}) = \det A^T \cdot \det A^T \ldots \det A^T = (\det A^T)^6$$

$$\det((A^T)^6) = (\det(A^T))^6 = (-23)^6 = 148035889$$

10.

Q10)

$$\det(A) = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 8 & 12 \end{vmatrix} = E_{1,2}(1) \rightarrow \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 8 & 12 \end{vmatrix}$$

$$E_{2,3}(-1) \rightarrow = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 8 & 12 \end{vmatrix} = 0$$

Since $\det(A) = 0$, we can surely say that A^{-1} doesn't exist.

11.

Q 11)

$$\det(B^{-2}A^{-3}) = \det(B^{-2}) \cdot \det(A^{-3})$$
$$= (3)^{-2} \cdot (2)^{-3}$$
$$= \frac{1}{9} \cdot \frac{1}{8}$$
$$= \frac{1}{72}$$

12.

Q 12)

let $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$, $w = \langle w_1, w_2, w_3 \rangle$

a) $u \cdot (v \times w)$

$$= u \cdot \left(\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

$$= u \cdot \left[i \cdot (v_2 w_3 - v_3 w_2) - j \cdot (v_1 w_3 - v_3 w_1) + \hat{k} (v_1 w_2 - v_2 w_1) \right]$$

$$= \underline{u_1 \cdot v_2 \cdot w_3} - \underline{u_1 \cdot v_3 \cdot w_2} - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1$$

b) $v \cdot (u \times w)$

$$= v \cdot \left(\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

$$= v \cdot \left[i (u_2 \cdot w_3 - u_3 w_1) - j (u_1 w_3 - u_3 w_1) + k (u_1 w_2 - u_2 w_1) \right]$$

$$= v_1 u_2 w_3 - v_1 u_3 w_1 - \underline{v_2 u_1 w_3} + v_2 u_3 w_1 + \underline{v_3 u_1 w_2} - v_3 u_2 w_1$$
$$= -(u_1 v_2 w_3 - u_1 \cdot v_3 \cdot w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_3 - u_3 v_2 w_1)$$

c) $w \cdot (u \times v)$

$$= w \cdot \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right)$$

$$= w \cdot \left[\hat{i}(u_2 v_3 - u_3 v_2) - \hat{j}(u_1 v_3 - u_3 v_1) + \hat{k}(u_1 v_2 - u_2 v_1) \right]$$

$$= w_1 u_2 v_3 - w_1 u_3 v_2 - \underline{w_2 u_1 v_3} + w_2 u_3 v_1 + \underline{w_3 u_1 v_2} - w_3 u_2 v_1$$

$$= (u_1 v_2 w_3 - u_1 u_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_3 - u_3 v_3 w_1)$$

★ a, b, c have the same magnitude

13.

Q13)

$$\begin{vmatrix} a+b & e-f \\ c-d & g+h \end{vmatrix} = ag + ah + bg + bh - ec + ed + fc - fd$$

$$\begin{vmatrix} a & c \\ e & g \end{vmatrix} = a \cdot g - ce \qquad \begin{vmatrix} b & -d \\ e & g \end{vmatrix} = b \cdot g + d \cdot e$$

$$\begin{vmatrix} a & c \\ -f & h \end{vmatrix} = a \cdot h + c \cdot f \qquad \begin{vmatrix} b & -d \\ -f & h \end{vmatrix} = bh - df$$

$$★ \begin{vmatrix} a+b & e-f \\ c-d & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & -d \\ e & g \end{vmatrix} + \begin{vmatrix} a & c \\ -f & h \end{vmatrix} + \begin{vmatrix} b & -d \\ -f & h \end{vmatrix}$$

14.

Q14)

$$\begin{vmatrix} k+1 & -4 \\ -2 & k-1 \end{vmatrix} = (k+1)(k-1) - 8$$

$$= k^2 - 1 - 8 = k^2 - 9$$

For the matrix to be non-invertable, $k^2 - 9$ must be 0

$k^2 - 9 = 0 \Rightarrow k^2 = 9 \Rightarrow k = 3 \quad k = -3$

15.

Q15)

let $P_1 = (x_1, y_1, z_1)$ and $\overrightarrow{OP_1} = \langle x_1, y_1, z_1 \rangle$

let $P_2 = (x_2, y_2, z_2)$ and $\overrightarrow{OP_2} = \langle x_2, y_2, z_2 \rangle$

$Q = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ and $\overrightarrow{OQ} = \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right\rangle$

Find the volume of the parallelpipe by $\overrightarrow{OP_1} \cdot (\overrightarrow{OP_2} \times \overrightarrow{OQ})$

$$V = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} & \frac{z_1+z_2}{2} \end{vmatrix} = x_1 \cdot \left(\frac{y_2 z_1 + y_2 z_2}{2} - \frac{z_2 y_1 - z_2 y_2}{2} \right) - x_2 \cdot \left(\frac{y_1 z_1 + y_1 z_2}{2} - \frac{z_1 y_1 - z_1 y_2}{2} \right) + \frac{x_1+x_2}{2} \cdot (y_1 z_2 - z_1 y_2) = 0$$

Since V=0, the vectors are coplanar

16.

$x = (4,0,-2)$ $y = (2,3,-1)$ $z = (0,0,1)$ plane: $ax+by+cz+d=0$

$4a-2c+d=0$ $2a+3b-c+d=0$ $c+d=0$ } Solve the equations

$$\left[\begin{array}{cccc|c} 4 & 0 & -2 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-2)} \left[\begin{array}{cccc|c} 0 & -6 & 0 & -1 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$E_2(\tfrac{1}{2}) \rightarrow \left[\begin{array}{cccc|c} 0 & -6 & 0 & -1 & 0 \\ 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{E_1(-\frac{1}{6})} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1/6 & 0 \\ 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$E_{3,2}(\tfrac{1}{2}) \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1/6 & 0 \\ 1 & 3/2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \Bigg\} \quad b + \frac{d}{6} = 0, \quad c+d=0, \quad a + \frac{3}{2}b + d = 0$$

let $d = 12$

$d=12$, $c=-12$, $b=-2$, $a=-9$

plane: $-9x - 2y - 12z + 12 = 0$

17.

Q17) let $A = \langle 0,1 \rangle$ and $B = \langle 2,0 \rangle$

$A+B = \langle 2,1 \rangle$

$|A| = 1$, $|B| = 2$ and $|A+B| = \sqrt{5}$

$|A+B| < |A| + |B|$

let $C = \langle 0,0 \rangle$ and $D = \langle 0,0 \rangle$

$C+D = \langle 0,0 \rangle$

$|C| = 0$, $|D| = 0$ and $|C+D| = 0$

$|C+D| = |C| + |D|$

18.

Q 18)

let $A = \langle 0,2 \rangle$ and $B = \langle 0,1 \rangle$

$A-B = \langle 0,1 \rangle$

$|A-B| = |A| - |B|$

let $C = \langle 0,3 \rangle$ and $D = \langle 3,0 \rangle$

$C-D = \langle -3,3 \rangle$

$|C-D| > |C| - |D|$

19.

$A = (0,1,0)$, $B = (0,0,-1)$, $C = (2,-1,1)$ are on the plane

$\vec{AB} = \langle 0,-1,-1 \rangle$ $\quad \vec{AC} = \langle 2,-2,1 \rangle$

i)

ii) $\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & -1 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \underbrace{\langle -3,-2,2 \rangle}_{\vec{N} = \text{normal vector}}$

$u_a = k \cdot \vec{N}$, $\vec{N} \cdot (u - u_a) = 0$

$k = \dfrac{\vec{u} \cdot \vec{N}}{|N|^2} = -\dfrac{7}{17}$

$u_a = \left\langle \dfrac{21}{17}, \dfrac{14}{17}, \dfrac{-14}{17} \right\rangle$

$u - u_a = \left\langle \dfrac{-4}{17}, \dfrac{20}{17}, \dfrac{14}{17} \right\rangle$

1

ii)

i) $\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & -1 & -1 \\ 2 & -2 & 1 \end{vmatrix} = \underbrace{\langle -3,-2,2 \rangle}_{\vec{N}}$

$u_a = k \cdot \vec{N}$, $(u - u_a) \cdot \vec{N} = 0$

$u = k \cdot \vec{N} + (u - u_a)$

$u \cdot \vec{N} = k |N|^2 + 0$

$k = \dfrac{u \cdot \vec{N}}{|N|^2} = -\dfrac{7}{17}$

$u_a = \left\langle \dfrac{21}{17}, \dfrac{14}{17}, \dfrac{-14}{17} \right\rangle$ Projection onto plane normal vector

20.

let $P = (x,y,z)$ and $\vec{P_oP} = \langle x, y-1, z-2 \rangle$

$\vec{V} \cdot \vec{P_oP}$ should be equal to 0 since they are orthagonal

$x - (y-1) + (2z-4) = 0$

$x - y + 2z = 3$ => This is the equation of the plane those normal vector is V

21.

i)

Q21)

i) $\vec{A} = \langle x, y, z \rangle$

$\vec{A} \cdot \langle 1,0,-1 \rangle = x - z = 0$ (orthagonality)

$x - z = 0 \Rightarrow x = z$

$\vec{A} = \langle x, y, x \rangle$ and $|A| = \sqrt{2x^2+y^2}$

For all values of x, y; the vector $\left\langle \frac{x}{\sqrt{2x^2+y^2}}, \frac{y}{\sqrt{2x^2+y^2}}, \frac{x}{\sqrt{2x^2+y^2}} \right\rangle$ meets the requirements

ii)

ii) The product of two vectors is orthagonal to both of them

$$\langle 1,0,-1 \rangle \times \langle 1,1,-1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1,0,1 \rangle$$

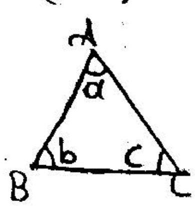
Since $\langle 1,0,1 \rangle$ is orthagonal, $-1 \cdot \langle 1,0,1 \rangle = \langle -1,0,-1 \rangle$ is also orthagonal

Convert the vectors into unit norm;

$\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ and $\left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$

22.

Q22)



$\vec{AB} \times \vec{AC} = |AB| \cdot |AC| \cdot \sin(a) \Rightarrow \frac{|\vec{AB} \times \vec{AC}|}{|AB| \cdot |AC|} = \sin a$

$\vec{CA} \times \vec{CB} = |CA| \cdot |CB| \cdot \sin(c) \Rightarrow \frac{|\vec{CA} \times \vec{CB}|}{|CA| \cdot |CB|} = \sin c$

$\vec{BA} \times \vec{BC} = |BA| \cdot |BC| \cdot \sin(b) \Rightarrow \frac{|\vec{BA} \times \vec{BC}|}{|BA| \cdot |BC|} = \sin b$

Since the value of $\sin\theta$ gets higher as θ raises, and since $\theta < 180$ in a triangle, we can use the calculations above to compare the sin values of the angles. The angle with the largest sine value is the largest angle of the triangle

23.

Q23)

$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 0 & -2 \end{vmatrix} = \langle -4, 0, -4 \rangle$ } Orthogonal to both u and v

If $\langle -4, 0, -4 \rangle$ is orthogonal to the vectors above, we can say that $-\langle -4, 0, -4 \rangle = \langle 4, 0, 4 \rangle$ is also orthogonal.

- Convert $\langle -4, 0, -4 \rangle$ into unit norm

$|\langle -4, 0, -4 \rangle| = 4\sqrt{2}$

Result $= \langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$

- Convert $\langle 4, 0, 4 \rangle$ into unit norm

$|\langle 4, 0, 4 \rangle| = 4\sqrt{2}$

Result $= \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

24.

Q24)

let $\alpha = (1,0)$ be a point on the line $x+2y=1$
The distance between the point α and the line $x+2y=3$ gives us also the distance between those lines

$$D = \frac{|1.1 + 2.0 - 3|}{\sqrt{1^2+2^2}} = \frac{2}{\sqrt{5}}$$

25.

Q25) $A = (0,1,-4)$

- let $B = (x_0, y_0, z_0)$ be a point on the plane

The normal vector of the plane $\vec{N} = \langle 1,0,1 \rangle$

$\vec{AB} = \langle -x_0, 1-y_0, -4-z_0 \rangle$, $\vec{AB} \parallel \vec{N}$.

$\vec{AB}$'s projection over $\vec{N}$ is equal to $\vec{AB}$ because $\vec{AB} \parallel \vec{N}$

$\vec{AB}_a = k.\vec{N}$

$k = \dfrac{\vec{AB}.\vec{N}}{|N|^2}$

$k = \dfrac{-x_0-4-z_0}{2}$

$k = \dfrac{-\overbrace{(x_0+z_0)}^{-1}-4}{2}$

$k = \dfrac{-3}{2}$

$\vec{AB}_a = k.\vec{N}$

$\vec{AB}_a = \langle -\frac{3}{2}, 0, \frac{3}{2} \rangle$

$|\vec{AB}_a| = \dfrac{3}{\sqrt{2}}$

26.

Q26) let $u = \langle u_1, u_2, \dots, u_n \rangle$ and $v = \langle v_1, v_2, \dots, v_n \rangle$

$|\vec{u}.\vec{v}| = |\vec{u}|.|\vec{v}|.\cos 90$
$= 0$

$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}|. \sin 90$
$= |\vec{u}|.|\vec{v}|$

$|\vec{u}+\vec{v}| = |(u_1+v_1), (u_2+v_2), \dots, (u_n+v_n)| = \sqrt{(u_1+v_1)^2+(u_2+v_2)^2+\dots+(u_n+v_n)^2}$

$= \sqrt{\underbrace{u_1^2+u_2^2+\dots+u_n^2}_{|\vec{u}|^2} + \underbrace{v_1^2+v_2^2+\dots+v_n^2}_{|\vec{v}|^2} + \underbrace{\cancel{2(u_1v_1+u_2v_2+\dots+u_nv_n)}}_{2.\vec{u}.\vec{v} = 2.|\vec{u}|.|\vec{v}|.\cos 90 = 0}}$

$= \sqrt{|\vec{u}|^2+|\vec{v}|^2}$

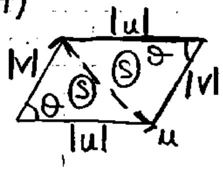
$|\vec{u}-\vec{v}| = |(u_1-v_1), (u_2-v_2), \dots, (u_n-v_n)| = \sqrt{(u_1-v_1)^2+(u_2-v_2)\dots+(u_n-v_n)^2}$

$= \sqrt{\underbrace{u_1^2+u_2^2+\dots+u_n^2}_{|\vec{u}|^2} + \underbrace{v_1^2+v_2^2+\dots+v_n^2}_{|\vec{v}|^2} \underbrace{\cancel{-2(u_1v_1+u_2v_2+\dots+u_nv_n)}}_{-2.\vec{u}.\vec{v} = -2.|\vec{u}|.|\vec{v}|.\cos 90 = 0}}$

$= \sqrt{|\vec{u}|^2+|\vec{v}|^2}$

27.

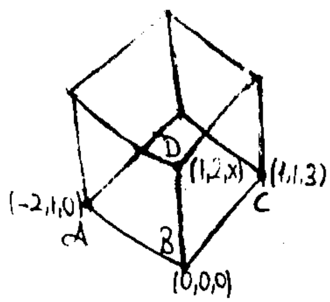
27)



$S = |v| . |u| . \sin\theta . \frac{1}{2}$

Area $= 2S = |v| . |u| . \sin\theta = |v \times u|$

28.



$\vec{BA} = \langle -2, 1, 0 \rangle$

$\vec{BC} = \langle 1, 1, 3 \rangle$

$\vec{BD} = \langle 1, 2, x \rangle$

$V = \vec{BD} \cdot (\vec{BA} \times \vec{BC})$

$$V = \begin{vmatrix} 1 & 2 & x \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 3 - 2(-6) + x(-3) = 15 - 3x$$

$V = 0 = 15 - 3x \Rightarrow x = 5$

29.

Q29) If A has more than one solutions, |A| must be equal to 0, because if it wasn't we would only have one unique solution.

Because of that reason, there is NOT a solution for b in AX=b

30.

If A is a square matrix which has more than one solutions for $AX = b$ just like in the question 29, then it is NOT possible for det(A) to be 9 because det(A) must be 0 for the equation above to be satisfied.

But if A is just a regular square matrix, and if det(A) is 9, then:

$$\cancel{A^{-1}}\,\cancel{A}X = A^{-1}b \quad , \quad A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{\operatorname{adj}(A)}{9}$$

$$X = A^{-1}.b \Rightarrow X = b.\frac{\operatorname{adj}(A)}{9}$$

In this case, the equation has a solution for every b

31.

Q31)

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 2 & 5 & -5 & b_3 \end{array}\right] \rightarrow E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_3-2b_1 \end{array}\right]$$

$$E_{2,3}(-1) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & b_1-2b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3-2b_1-b_2 \end{array}\right]$$

For the system to be consistent, the last row should be zero

$b_3 - 2b_1 - b_2 = 0$