2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 1

$$\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ -1 & 1 & 3 \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}_{3\times1}$$ Using Gauss-Jordan Elemination

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 4 & 0 & 1 & 2 \\ -1 & 1 & 3 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -12 & 5 & -2 \\ 0 & 4 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & -12 & 5 & -2 \\ 0 & 1 & 1/2 & 3/4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 1 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & -12 & 5 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -5/4 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & 0 & 11 & 7 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5/2 & -5/4 \\ 0 & 1 & 1/2 & 3/4 \\ 0 & 0 & 1 & 7/11 \end{array}\right]$$

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 15/44 \\ 0 & 1 & 0 & 19/44 \\ 0 & 0 & 1 & 7/11 \end{array}\right]}_{\text{Reduced Echelon Form}} \quad \left.\right\} \begin{aligned} x &= 15/44 \\ y &= 19/44 \\ z &= 7/11 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 1 \\ -1 & 2 & 7 \end{bmatrix}_{3\times3} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}_{3\times1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 3 \\ -1 & 2 & 7 & 5 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 4 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & -5 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{array}\right]$$

! As seen in the last row, we got $0 = -6$ which is not possible. Hence, this system has no solutions.

$$\underset{2\times 4}{\begin{bmatrix} 1 & 3 & -1 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}} \cdot \underset{4\times 1}{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}} = \underset{2\times 1}{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 3 & 0 & 1 & 2 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 0 & -9 & 4 & -1 & -4 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 1 & 1 \\ 0 & 1 & -4/9 & 1/9 & 4/9 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3/9 & 6/9 & -3/9 \\ 0 & 1 & -4/9 & 1/9 & 4/9 \end{array}\right]$$

? Since the number of equations is less then the unknowns the system has infinite solutions with $(4-2=2)$ variables.

- let $z = 9k$ and $w = 9t$

$$\left.\begin{aligned} x + 3k + 6t &= -3/9 \\ y - 4k + t &= 4/9 \end{aligned}\right\} \quad \begin{aligned} x &= -3/9 - 3k - 6t \\ y &= 4/9 + 4k - t \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -3 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -1/3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & -2 \\ 0 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 2/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & -2/3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $\Rightarrow$ Row Echelon Form

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ $\Rightarrow$ Reduced Row Echelon Form

The given matrix is in "row echelon form" for the following reasons:

- The first non-zero number on each row is a one
- In each row, the leading one is further right than the upper one.

The given matrix is ALSO "reduced" row echelon form because:

- There are columns which have leading ones also have zeros everywhere else.

Let's divide the given expression into three parts:

$$(C^T - AB).B^T$$

C^T $\quad$ AB $\quad$ B^T

a) Since C is a 4×2 matrix, C^T has a dimension of 2×4

b) $\underbrace{A_{2x3} \cdot B_{3x4}}$ $\Rightarrow M_{2x4}$ (We get a $2x4$ matrix)

c) Since B is a $3x4$ matrix, B^T has a dimension of 4×3

$$C^T_{2x4} - AB_{2x4} = K_{2x4} \Rightarrow \underbrace{K_{2x4} \cdot B^T_{4x3}} = L_{2x3}$$

The size of the matrix resulting from the given expression is 2×3

• If L is a lower triangular matrix, we can asume that L^{-1} is a lower triangular matrix as well.

• If U is an upper triangular matrix, then U^T is a lower triangular matrix

$L^{-1} \times U^T \Rightarrow$ If we multiply two lower triangular $n \times n$ matrices, we get another lower triangular $n \times n$ matrix as result.

Q6)

$$A \cdot B = [A_1 \mid A_2] \cdot \left[\frac{B_1}{B_2}\right] = [A_1 B_1 + A_2 B_2]$$

$$A_1 \cdot B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_2 \cdot B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A \cdot B = A_1 B_1 + A_2 \cdot B_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 2 \\ 2 & 2 \end{bmatrix}$$ //

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

* $a_{11} = 0$, $|1-1| < 2$
* $a_{12} = 0$, $|1-2| < 2$
* $a_{21} = 0$, $|2-1| < 2$
* $a_{22} = 0$, $|2-2| < 2$
* $a_{23} = 0$, $|2-3| < 2$
* $a_{32} = 0$, $|3-2| < 2$
* $a_{33} = 0$, $|3-3| < 2$
* $a_{34} = 0$, $|3-4| < 2$
* $a_{43} = 0$, $|4-3| < 2$
* $a_{44} = 0$, $|4-4| < 2$

! We can fill other indices with "1" since they don't affect the given rule.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow A.A^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\left.\begin{matrix} a.e + b.g = 1 & c.e + d.g = 0 \\ a.f + b.h = 0 & c.f + d.h = 1 \end{matrix}\right\} \text{Solve the equations}$$

$\frac{-c}{ad}$

$$\begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix} \cdot \begin{bmatrix} e \\ f \\ g \\ h \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} a & 0 & b & 0 & 1 \\ 0 & a & 0 & b & 0 \\ c & 0 & d & 0 & 0 \\ 0 & c & 0 & d & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & b/a & 0 & 1/a \\ 0 & 1 & 0 & b/a & 0 \\ c/d & 0 & 1 & 0 & 0 \\ 0 & c/d & 0 & 1 & 1/d \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} \\ 0 & 1 & 0 & \frac{b}{a} & 0 \\ 0 & 0 & \frac{ad-cb}{ad} & 0 & \frac{-c}{ad} \\ 0 & 0 & 0 & \frac{ad-cb}{ad} & \frac{1}{d} \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & \frac{b}{a} & 0 & \frac{1}{a} \\ 0 & 1 & 0 & \frac{b}{a} & 0 \\ 0 & 0 & 1 & 0 & \frac{-c}{ad-cb} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-cb} \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{d}{(ad-cb)} \\ 0 & 1 & 0 & 0 & \frac{-b}{(ad-cb)} \\ 0 & 0 & 1 & 0 & \frac{-c}{ad-cb} \\ 0 & 0 & 0 & 1 & \frac{a}{ad-cb} \end{array}\right] \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \quad \begin{aligned} e &= \frac{d}{(ad-cb)} \\ f &= \frac{-b}{(ad-cb)} \\ g &= \frac{-c}{(ad-cb)} \\ h &= \frac{a}{(ad-cb)} \end{aligned}$$

$$A^{-1} = \begin{bmatrix} \frac{d}{(ad-bc)} & \frac{-b}{(ad-cb)} \\ \frac{-c}{(ad-bc)} & \frac{a}{(ad-cb)} \end{bmatrix} = \frac{1}{(ad-bc)} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^3 - 2A^2 + 3A = 0 \quad \Rightarrow A^3 \cdot A^{-1} - 2A^2 \cdot A^{-1} + 3 \cdot A \cdot A^{-1} = 0$$

$$A^2 - 2A + 3I = 0 \quad \Rightarrow A^2 \cdot A^{-1} - 2A \cdot A^{-1} + 3I \cdot A^{-1} = 0$$

$$A - 2I + 3A^{-1} = 0 \quad \Rightarrow A^{-1} = \frac{1}{3}\left(2I - A\right)$$

* let $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$A . A^{-1} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$ $a.e + 0.g = 1$ $b.e + 0.g = 0$
$a.f + 0h = 0$ $bf + 0.h = 1$

$a.e + \cancel{0.g} = 1$ $\quad$ $b.e + \cancel{0.g} = 0$ $\quad$ $\cancel{bf} + \cancel{0.h} = 1$

$a.e = 1$ $\quad$ $b.e = 0$ $\quad$ * $0 + 0 = 1$ *

$e \neq 0$ $\quad$ $e \neq 0 \Rightarrow b = 0$ $\quad$ impossible

Therefore, if a matrix has a column of zeroes, it's inverse doesn't exist

* let $A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ which doesn't have a row of zeroes.

By using row operations, we get:

$\begin{bmatrix} a & b \\ a & b \end{bmatrix} \Rightarrow \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ } Even if we don't have a row of zeroes in the beginning, we get a row of zeroes by reducing the matrix.

Therefore, a matrix which doesn't have a row of zeroes can be non-invertible.

$$(A+B)^2 = (A+B).(A+B) = A.A + \overbrace{A.B + B.A}^{\text{Equal}} + B.B = A^2 + 2AB + B^2$$

* If AB is symmetric, we can say that $(AB) = (AB)^T = B^T.A^T$
* Since A and B are symmetric as well, $B^T = B$ and $A^T = A$

$\Big\}$ $AB = (AB)^T = B^T.A^T = BA$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

No, such a thing is not possible. If a matrix has a whole column of zeroes after elementary operations, it means that matrix does not have an inverse.

By that logic, if a matrix is invertible it is not able to be turned into a matrix with an all zeroes column by elementary operations.

a) $$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1/2 & 0 & 1 \end{array}\right]$$

The matrix is non-invertible since it has a row of zeroes

b) $$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & -1/2 & 1 \end{array}\right] \Rightarrow \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I} \middle| \underbrace{\begin{matrix} 0 & -1/2 & 1 \\ 0 & 1/2 & 0 \\ -1 & -1/2 & 1 \end{matrix}}_{A^{-1}}\right]$$

c) $$\left[\begin{array}{ccc|ccc} 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \\ c & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} c & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & a & 1 & 0 & 0 \\ 0 & 0 & b & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/c \\ 0 & 0 & 1 & 1/a & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/b & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1/c \\ 0 & 0 & 1 & 1/a & 0 & 0 \\ 0 & 0 & 0 & -1/a & 1/b & 0 \end{array}\right]$$

Matrix is non-invertible (Row of zeroes)

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_1 \text{ (Interchange Rows)}} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{E_2 \text{ (Interchange columns)}} = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an invertible matrix

* Using Gauss elemination it is possible to turn A into I since A^{-1} exists.

$$E_1 \left\{ \begin{bmatrix} \frac{1}{a} & 0 \\ -\frac{c}{a} & 1 \end{bmatrix} \right. \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix}$$

* Note that E_1 is lower triangular

$$E_2 \left\{ \begin{bmatrix} 1 & \frac{-b}{da-bc} \\ 0 & \frac{a}{ad-bc} \end{bmatrix} \right. \cdot \begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

* Note that E_2 is upper triangular

* Find the inverses for E_1 and E_2

$$[E_1 \mid I] = \left[\begin{array}{cc|cc} 1/a & 0 & 1 & 0 \\ -c/a & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & a & 0 \\ -c/a & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & a & 0 \\ 0 & 1 & c & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} a & 0 \\ c & 1 \end{matrix}}_{E_1^{-1}}$$

$$[E_2 \mid I] = \left[\begin{array}{cc|cc} 1 & \frac{-b}{da-bc} & 1 & 0 \\ 0 & \frac{a}{da-bc} & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|cc} 1 & \frac{-b}{da-bc} & 1 & 0 \\ 0 & 1 & 0 & \frac{da-bc}{a} \end{array}\right] \longrightarrow \ldots$$

$$\ldots \left[\begin{array}{cc|cc} 1 & 0 & 1 & \frac{b}{a} \\ 0 & 1 & 0 & \frac{da-bc}{a} \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{matrix}}_{E_2^{-1}}$$

* Since E_1 is a lower triangular matrix, E_1^{-1} is also lower triangular

Since E_2 is an upper triangular matrix, E_2^{-1} is also upper triangular

* If we multiply $E_1^{-1} \cdot E_2^{-1}$:

$$\underbrace{\begin{bmatrix} a & 0 \\ c & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da-bc}{a} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{A}$$

$$\left[\begin{array}{cccc|cccc} a & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1/a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/b & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1/c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1/d \end{array}\right]$$

$$\underbrace{}_{I_4} \qquad \underbrace{}_{D^{-1}}$$

$$\begin{bmatrix} 2 & 5 & 4 & 9 & -1 \\ 0 & 0 & -3 & 8 & 2 \\ 0 & 0 & 6 & 1 & -4 \\ 0 & 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 2 & 9/2 & -1/2 \\ 0 & 0 & 1 & -8/3 & 2/3 \\ 0 & 0 & 1 & 1/6 & -2/3 \\ 0 & 0 & 0 & 1 & -2/3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 2 & 9/2 & 0 \\ 0 & 0 & 1 & -8/3 & 0 \\ 0 & 0 & 1 & 1/6 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5/2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 5/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix is non-invertible since we got a row of zeroes

a) If AB is invertible, then A and B should also be invertible

$* \ (AB)^{-1} = B^{-1} . A^{-1} \ *$

$\cancel{A^{-1}} AB = \cancel{A^{-1}} AC \Rightarrow \underbrace{A^{-1}A}_{I} B = \underbrace{A^{-1}A}_{I} C$

$IB = IC \Rightarrow B = C$

b) If A can be reduced to a matrix with an all zeroes row, we can say A is non-invertible.

Therefore if $AB = AC$, we can't say $B = C$ directly

Ex

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow$ an all zeroes row.

$AB = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$ but $B \neq C$

$$\left[\begin{array}{ccc|c|c|c} 2 & 3 & -1 & 1 & -1 & 3 \\ 4 & 1 & -1 & 0 & 2 & 3 \\ 1 & 1 & 1 & 2 & 0 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 4 & 1 & -1 & 0 & 2 & 3 \\ 2 & 3 & -1 & 1 & -1 & 3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 0 & -3 & -5 & -8 & 2 & -9 \\ 0 & 1 & -3 & -3 & -1 & -3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 1 & 1 & 2 & 0 & 3 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & -3 & -5 & -8 & 2 & -9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 4 & 5 & 1 & 6 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & 0 & -14 & -17 & -1 & -18 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c|c|c} 1 & 0 & 4 & 5 & 1 & 6 \\ 0 & 1 & -3 & -3 & -1 & -3 \\ 0 & 0 & 1 & 17/14 & 1/14 & 9/7 \end{array}\right]$$

$$\left[\begin{array}{ccc|c|c|c} 1 & 0 & 0 & 1/7 & 5/7 & 6/7 \\ 0 & 1 & 0 & 9/14 & -11/14 & 6/7 \\ 0 & 0 & 1 & 17/14 & 1/14 & 9/7 \end{array}\right]$$

Set 1	Set 2	Set 3
$x = 1/7$	$x = 5/7$	$x = 6/7$
$y = 9/14$	$y = -11/14$	$y = 6/7$
$z = 17/14$	$z = 1/14$	$z = 9/7$