2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = v_1 \cdot c_1 + v_2 \cdot c_2 + v_3 \cdot c_3$

$$\left.\begin{array}{l} 4c_1 - 2c_2 = a \\ 3c_1 + 3c_3 = b \\ 2c_1 - c_2 = c \\ c_1 + c_3 = d \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 3 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow[E_{3,4}(-2)]{E_{1,4}(-4),\ E_{2,4}(-3)} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

For the system to be consistent,
$a - 2c = 0$, $b - 3d = 0$ $\Rightarrow$ $a = 2c$, $b = 3d$

Given vectors span a subspace consisting of vector K, that

$$K = \begin{bmatrix} 2c \\ 3d \\ c \\ d \end{bmatrix} = c \cdot \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{v_2} + d \cdot \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

b) The vectors are linearly independent if and only if the eq. $v_1 \cdot c_1 + v_2 \cdot c_2 + v_3 \cdot c_3 = 0$ only has the trivial solution

$$\left.\begin{array}{l} 4c_1 - 2c_2 = 0 \\ 3c_1 + 3c_3 = 0 \\ 2c_1 - c_2 = 0 \\ c_1 + c_3 = 0 \end{array}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[\text{section a}]{\text{Same operations used in the}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

since there are an infinite amount of solutions, the vectors are linearly dependent

c) Basis of the space spanned by these vectors is : $\left\{\begin{bmatrix}-2\\0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\3\\0\\1\end{bmatrix}\right\}$

R^4's basis has 4 vectors, therefore, we need to find 2 additional vectors that are not in our initial span

let $v_4 = (0,0,0,1)$ and $v_5 = (1,0,0,0)$ $\exists$ v_4 and v_5 are lin. ind. and not in the span

The basis we get as a result is $= \left\{ \underbrace{\begin{bmatrix}-2\\0\\-1\\0\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}0\\3\\0\\1\end{bmatrix}}_{v_3}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \right.$

Q2) ! $\cos^2 x - \sin^2 x = \cos 2x \Rightarrow f_1 = \cos 2x$ and $f_2 = \cos 2x$

applying Wronskian :

$$\begin{vmatrix} \cos 2x & \cos 2x \\ (\cos 2x)' & (\cos 2x)' \end{vmatrix} = \begin{vmatrix} \cos 2x & \cos 2x \\ -2\sin 2x & -2\sin 2x \end{vmatrix} = -2\sin 2x \cos 2x + 2\sin 2x \cos 2x = 0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of R^3 is 3, therefore basis for R^3 consist of 3 vectors

Choose $u_1, u_3, u_4 \Rightarrow$ Test for linear independence :

$u_1 \cdot c_1 + u_3 \cdot c_3 + u_4 \cdot c_4 = 0$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 0 \\ -2c_1 &= 0 \\ -c_3 + 7c_4 &= 0 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_{2,1(2)}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right] \xrightarrow{-E_2(1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 7 & 0 \end{array}\right]$$

$$\xrightarrow[E_{3,2(1)}]{E_{1,2(-1)}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \Bigg\} \begin{aligned} c_1 &= 0 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned} \Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1, u_3, u_4 form a basis without testing for their span.

Base $= \{(1,-2,0), (1,0,-1), (0,0,7)\}$

! Express the remaining vector, u_2, by the base vectors.

$u_1 \cdot c_1 + u_3 \cdot c_3 + u_4 \cdot c_4 = u_2$, find c_1, c_3, c_4

$$\left.\begin{array}{l} c_1 + c_3 = 1 \\ -2c_1 = 2 \\ -c_3 + 7c_4 = 4 \end{array}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{array}{l} c_1 = -1 \\ c_3 = 2 \\ c_4 = 6/7 \end{array}\right\} \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

The dimension of the subspace spanned by these vectors is 4

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^4 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

Q6) Let's assume that u, v and w are linearly independent vectors in R^n where $n > 3$

In that case, the space R^n has $n > 3$ basis vectors and therefore the maximum amount of linearly ind. vectors we can write is also $n > 3$

Since the product of $u \times v$ is orthagonal to both u and v, it cannot be expressed as a linear combination of u, v. So; u, v and $(u \times v)$ are linearly independent.

For this newly formed vector, there are 2 possibilities:

1) It can be parallel to w, since w is also linearly ind. from u, v

2) It can also be orthagonal to w, since for our vector space R^n, $n > 3$ linearly ind. vectors can be written.

For ①, $proj_{u \times v} w$ is equal to w, since they are parallel. In this case we can say u, v and $proj_{u \times v} w = w$ are linearly independent.

However, for ② $proj_{u \times v}$ is $\vec{0}$, since they are orthogonal. In this case u, v and $proj_{u \times v} = \vec{0}$ are linearly dependent, because a set of vectors which contains the 0 vector is always linearly dependent.

As a conclusion, no, we can't necessarily say that u, v and $proj_{u \times v} w$ are linearly independent.

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.

Q8)

a) The dimension of R^6 is 6, so the basis for R^6 has 6 vectors. Therefore any amount of vectors that is less than 6, cannot form a basis

b) let $v=(-x,y)$ and $m=(x,-y)$

Since $v=-1\cdot m$ these vectors are linearly dependent and therefore, cannot form a basis

c) We're given 3 vectors which is the same amount of vectors in the basis of R^3, so we only need to show whether these vectors are linearly independent or not.

$v_1=(1,2,3)$ $v_2=(1,2,0)$ $v_3=(-1,2,6)$

$c_1v_1+c_2v_2+c_3v_3=0$

$$\left.\begin{aligned} c_1+c_2-c_3&=0\\ 2c_1+2c_2+2c_3&=0\\ 3c_1+\quad 6c_3&=0\end{aligned}\right\}\begin{bmatrix}1&1&-1\\2&2&2\\3&0&6\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\2&2&2&0\\3&0&6&0\end{array}\right]\xrightarrow[E_{3,1}(-3)]{E_{2,1}(-2)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&6&0\\0&-3&9&0\end{array}\right]\xrightarrow[E_3(-1/3)]{E_2(1/6)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&1&0\\0&1&-3&0\end{array}\right]$$

$$\xrightarrow[E_{1,2}(1)]{E_{3,2}(3)}\left[\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\xrightarrow{E_{1,3}(-1)}\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\Bigg\}\ c_1=c_2=c_3=0$$

! The vectors are linearly independent, therefore they are a basis for R^3

d) Any set of vectors that include the zero vector is linearly dependent Therefore they can't form a basis.

Q9)

$v_1\cdot c_1+v_2\cdot c_2+v_3\cdot c_3=w$

$2c_1+c_3=1$ / $-2c_1+c_2+5c_3=1$ / $-c_2+4c_3=1$

$$\begin{bmatrix}2&0&1\\-2&1&5\\0&-1&4\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}2&0&1&1\\-2&1&5&1\\0&-1&4&1\end{array}\right]\xrightarrow{E_{2,1}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&-1&4&1\end{array}\right]$$

$$\xrightarrow{E_{3,2}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&10&3\end{array}\right]\xrightarrow{E_3(1/10)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&1&3/10\end{array}\right]\xrightarrow[E_{1,3}(-1)]{E_{2,3}(-6)}\left[\begin{array}{ccc|c}2&0&0&0{,}7\\0&1&0&0{,}2\\0&0&1&0{,}3\end{array}\right]$$

$$-E_1(\tfrac{1}{2}) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0{,}35 \\ 0 & 1 & 0 & 0{,}20 \\ 0 & 0 & 1 & 0{,}30 \end{array}\right] \Rightarrow \begin{array}{l} c_1 = 0{,}35 \\ c_2 = 0{,}20 \\ c_3 = 0{,}30 \end{array}$$

$w = 0{,}35 \cdot v_1 + 0{,}20 \cdot v_2 + 0{,}30 \cdot v_3 \Rightarrow$ Coordinates of w relative to the basis is

$(0{,}35, 0{,}20, 0{,}30)$

Q10)

a) R^3 has a dimension of 3, because of that, the maximum amount of linearly independent vectors we can express is 3. Therefore, the given vectors can't be linearly independent

b)

① Find the space spanned by v_1, v_2, v_3, v_4

let $u = (a, b, c)$

$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = u$

$$\left.\begin{array}{rrrrl} 2c_1 & & +c_3 & +c_4 & = a \\ -c_1 & +c_2 & -c_3 & & = b \\ -c_1 & +3c_2 & -2c_3 & +c_4 & = c \end{array}\right\} \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & a \\ -1 & 1 & -1 & 0 & b \\ -1 & 3 & -2 & 1 & c \end{array}\right] \xrightarrow[E_{1,3}(2)]{-E_{2,3}(-1)} \left[\begin{array}{cccc|c} 0 & 6 & -3 & 3 & a+2c \\ 0 & -2 & 1 & -1 & b-c \\ -1 & 3 & -2 & 1 & c \end{array}\right]$$

$$\begin{array}{l} E_1(1/6) \\ E_3(-1) \end{array} \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & -2 & 1 & -1 & b-c \\ 1 & -3 & 2 & -1 & -c \end{array}\right] \xrightarrow[E_{3,1}(3)]{-E_{2,1}(2)} \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & 0 & 0 & 0 & (a+3b-c)/3 \\ 1 & 0 & 1/2 & -1/2 & a/2 \end{array}\right] !$$

$$\left.\frac{a+3b-c}{3} = 0 \Rightarrow a+3b-c = 0 \right\} \text{let } a = t,\ b = k$$

The given vectors span a subspace of R^3 in which $v_i = \begin{bmatrix} t \\ k \\ t+3k \end{bmatrix}$

$$v_i = t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Rightarrow \text{Basis for the subspace is} = (\underbrace{\langle 1,0,1 \rangle}_{v_1}, \underbrace{\langle 0,1,3 \rangle}_{v_2})$$

c) The dimension of the space spanned by those vectors is 2, since it has 2 basis vectors.

d) * I already calculated the span in section "b"

e) Test if we can express w as a linear combination of the basis vectors

$c_1 \cdot v_1 + c_2 \cdot v_2 = w$

$$\left.\begin{array}{rrl} c_1 & + 0 \cdot c_2 & = 2 \\ 0 \cdot c_1 & + c_2 & = 2 \\ c_1 & + 3c_2 & = 0 \end{array}\right\} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-3)]{-E_{3,1}(-1)} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{array}\right]$$

System is inconsistent, w is not in the span

Q11) $x-y+z=2 \Rightarrow x=(2+y)-z$, let $2+y=t$ and $z=k$

$$\text{Vectors} = \begin{bmatrix} t-k \\ t-2 \\ k \end{bmatrix} = t.\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + k.\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

Since our last vector $[0,-2,0]$ is a constant, we cannot express vectors on the plane using a scalar multiple of it other than "1".
Therefore, we cannot form a basis.

Q12) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-2)} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & -3 & 0 \end{array}\right] \xrightarrow{E_2(-1/3)} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$

$\xrightarrow{E_{1,2}(-1)} \left[\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$ $\quad w+\frac{y}{3}=0 \quad x-\frac{4y}{3}+z=0$ let $x=k, y=3t$

$w=-t,\ x=k,\ y=3t,\ z=4t-k$

Vectors in this subspace can be expressed as

$$\begin{bmatrix} -t \\ k \\ 3t \\ 4t-k \end{bmatrix} = t.\underbrace{\begin{bmatrix} -1 \\ 0 \\ 3 \\ 4 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\text{Basis for the Subspace}}$$

Dimension of this subspace is 2

Q13) $\frac{x}{3}+\frac{1}{3}=t \quad y+\frac{1}{3}=t \quad \frac{1}{3}-\frac{z}{3}=t$

① $\frac{x}{3}=y$ ② $y=-\frac{z}{3}$ ③ $\frac{x}{3}=-\frac{z}{3}$ $\Big\}$ let $z=3k$, $x=-3k$, $y=-k$

Our vectors have the form of: $\begin{bmatrix} -3k \\ -k \\ 3k \end{bmatrix} = k.\begin{bmatrix} -3 \\ -1 \\ 3 \end{bmatrix}$

Basis for the line is $= \{(-3,-1,3)\}$

Q14) let $p(x) = ax^2 + bx + d$

$$c_1 \cdot p_1(x) + c_2 \cdot p_2(x) + c_3 p_3(x) + c_4 p_4 = p(x)$$

$$c_1 \cdot x^2 + c_1 + c_2 \cdot x^2 - 2c_2 x + 3c_2 + c_3 x - c_3 + 3c_4 x^2 + 2c_4 x + c_4 = ax^2 + bx + d$$

$$x^2(c_1 + c_2 + 3c_4) + x(-2c_2 + c_3 + 2c_4) + (c_1 + 3c_2 - c_3) = ax^2 + bx + d$$

$$\left.\begin{aligned} c_1 + c_2 \quad\quad + 3c_4 &= a \\ -2c_2 + c_3 + 2c_4 &= b \\ c_1 + 3c_2 - c_3 + c_4 &= d \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 1 & 3 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 1 & 3 & -1 & 1 & d \end{array}\right] - E_{3,1}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 2 & -1 & -2 & d-a \end{array}\right]$$

$$E_{2,3}(1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & d-a+b \end{array}\right] \Bigg\} \; d-a+b=0 \quad \text{let } a=k,\ b=t \Rightarrow d=k-t$$

These vectors span a subspace of P^2 with the rule of:

$$P(x) = \begin{bmatrix} k \\ t \\ k-t \end{bmatrix} = k \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \text{Basis is} : \langle (1,0,1), (0,1,-1) \rangle$$

! Dimension of this subspace is 2 (2 vectors in basis)

! Since we have $2 < 3$ vectors in our basis, it does not span the space of second order polynomials.

Let $P_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, P_0 is not in the span of our vectors.

By the add-minus theorem, if we add this vector to our basis, the new basis will span all P^2 (3 basis vectors)

Q15) Columnvectors of A are $= \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}}_{v_3} \right\}$, let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$$c_1 \cdot v_1 + c_2 \cdot v_2 + c_3 v_3 = v$$

$$\left.\begin{aligned} 2c_1 - c_2 \quad\quad &= a \\ c_1 + 2c_2 + 5c_3 &= b \\ c_1 + c_2 + 3c_3 &= c \end{aligned}\right\} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & a \\ 1 & 2 & 5 & b \\ 1 & 1 & 3 & c \end{array}\right] \begin{matrix} -E_{1,3}(-2) \\ E_{2,3}(-1) \end{matrix} \rightarrow \left[\begin{array}{ccc|c} 0 & -3 & -6 & a-2c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right] - E_{1,2}(2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a+2b-4c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right]$$

$a + 2b - 4c = 0$, let $a = 4t$, $b = 2k \Rightarrow c = t + k$

These vectors span a subspace of $\mathbb{R}^3$ with the rule:

$$V=\begin{bmatrix}t+t\\2t\\t+t\end{bmatrix}=t\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}+t\cdot\begin{bmatrix}0\\2\\1\end{bmatrix}\Bigg\}\text{ Basis is }\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}$$

▽ The dimension of the columnspace is 2

▽ For $Ax=b$ to be consistent for every b, A must be invertible

$$|A|=\begin{vmatrix}2&-1&0\\1&2&5\\1&1&3\end{vmatrix}\xrightarrow[E_{1,3}(-2)]{E_{2,3}(-1)}\begin{vmatrix}0&-3&-6\\0&1&2\\1&1&3\end{vmatrix}=1\cdot\begin{vmatrix}-3&-6\\1&2\end{vmatrix}=0$$

Since $|A|=0$, A is not invertible. Therefore, $Ax=b$ is <u>not</u> consistent for every b.

Q16)

a) columnvectors of A are : $\left\{\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{V_1},\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{V_2},\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{V_3}\right\}$, $b=\begin{bmatrix}-1\\1\\1\end{bmatrix}$

$C_1V_1+C_2V_2+C_3V_3=b$

$$\left.\begin{aligned}C_1-2C_2+C_3&=-1\\C_1+2C_2+2C_3&=1\\2C_1-2C_2+2C_3&=1\end{aligned}\right\}\begin{bmatrix}1&-2&1\\1&2&2\\2&-2&2\end{bmatrix}\cdot\begin{bmatrix}C_1\\C_2\\C_3\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&-2&1&-1\\1&2&2&1\\2&-2&2&1\end{array}\right]\xrightarrow[E_{3,1}(-2)]{E_{2,1}(-1)}\left[\begin{array}{ccc|c}1&-2&1&-1\\0&4&1&2\\0&2&0&3\end{array}\right]\xrightarrow[E_{2,3}(-2)]{E_{1,3}(1)}\left[\begin{array}{ccc|c}1&0&1&2\\0&0&1&-4\\0&2&0&3\end{array}\right]$$

$$\xrightarrow[E_{1,2}(-1)]{E_3(1/2)}\left[\begin{array}{ccc|c}1&0&0&6\\0&0&1&-4\\0&1&0&3/2\end{array}\right]\Bigg\}\begin{array}{l}C_1=6\\C_2=3/2\\C_3=-4\end{array}\Rightarrow\underbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}}_{b}=6\cdot\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{V_1}+\frac{3}{2}\cdot\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{V_2}-4\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{V_3}$$

b) $\begin{bmatrix}1&-2&1\\1&2&2\\2&-1&2\end{bmatrix}\xrightarrow[\text{as in section 'a'}]{\text{Same operations}}\begin{bmatrix}1&0&0\\0&9&1\\0&1&0\end{bmatrix}\xrightarrow{E_{2,3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

Rank = no. of pivots in row echelon form = 3

c) Nullity = col. no − rank = 3 − 3 = 0

d)

→ $Ax=0$ } Find the homogeneous solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow[\text{as in section "a"}]{\text{Same operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \Big\} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

→ $Ax=b$ } Find the general solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow[\text{as in Section "a"}]{\text{Same operations}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3/2 \end{array}\right] \Big\} X = \begin{bmatrix} 6 \\ 3/2 \\ -4 \end{bmatrix}$$

Q17)

a) $Ax=0$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 4 & -4 & -4 & 0 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_1(1/2) \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Big\} \begin{array}{l} x_2 - x_3 - x_4 = 0 \\ x_1 + x_3 + 3x_4 = 0 \end{array} \Rightarrow \begin{array}{l} \text{let } x_3 = t \text{ and } x_4 = k \\ \star\ x_2 = t + k,\ x_1 = -t - 3k \end{array}$$

The space spanned by these vectors is the nullspace

$$NS(A) = \begin{bmatrix} -t-3k \\ t+k \\ t \\ k \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b) Nullity = Dimension of the nullspace = 2

c) Rank = No. of non-zero rows in reduced echelon form = 2

d) $Ax = b$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 4 & -4 & -4 & -2 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} 2x_2 - 2x_3 - 2x_4 = -1 \\ x_1 \quad + x_3 + 3x_4 = 3 \end{array}\right\} \text{let } x_3 = t \text{ and } x_4 = k$$

$$x_1 = 3 - t - 3k \text{ and } x_2 = -\tfrac{1}{2} + t + k$$

$$\text{Solution vector } x = \begin{bmatrix} 3-t-3k \\ -\frac{1}{2}+t+k \\ t \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -\frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \underbrace{t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Homogeneous Solution}}}_{\text{General Solution}}$$

e) For $Ax = b$ to be consistent for every b, A must be invertible. Since A is not a square matrix, it is not invertible.

Q18)

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix} \xrightarrow[E_{3,1}(-2)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{bmatrix} \xrightarrow{E_{3,2}(-2)} \begin{array}{c} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \to v_1 \\ \to v_2 \\ \end{array} \\ \begin{array}{cccc} \downarrow & & \downarrow & \downarrow \\ v_3 & & v_4 & v_5 \end{array} \end{array}$$

Basis for row space = $\{(1,0,1,-4), (0,0,1,7)\}$

Basis for col. space = $\{(1,0,0), (1,1,0), (-4,7,0)\}$

Q19)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow[E_{3,1}(-1)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{E_{2,3}(1)} \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array} \end{array}$$

Basis for col. space = $\{(1,0,0,0), (2,0,-2,0), (3,0,-5,0)\}$

Q20)

$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{2,1}(-1)$ can be expressed as $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = A'$

↳ Elementary matrix

A' can be expressed as $E \cdot A$

① Find the col. space of A

$c_1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Big\} \begin{matrix} c_1 + c_2 = a \\ -c_1 - c_2 = b \end{matrix} \Rightarrow$ let $c_1 = t, c_2 = k$, $a = t+k$ / $b = -t-k$

Col. space of A consists from vectors with the rule: $\begin{bmatrix} t+k \\ -t-k \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

② Find the col. space of $A' = E \cdot A$

$c_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \Big\} \begin{matrix} c_1 - c_2 = a \\ c_2 = b \end{matrix} \Rightarrow$ let $c_1 = t$, $c_2 = k$, $a = t-k$ / $b = k$

Col. space of A consists from vectors with the rule: $\begin{bmatrix} t-k \\ k \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

∴ The rules are different, so A and EA don't have the same columnspace.

Q21)

$A^T = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix} \xrightarrow[E_{3,1}(-4)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & -8 & -6 & 3 \end{bmatrix} \xrightarrow{E_{3,2}(-1)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & 0 & -4 & 2 \end{bmatrix}$

$\xrightarrow[E_3(-1/4)]{E_2(-1/8)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow{E_{1,2}(-2)} \begin{bmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow[E_{1,3}(-1/2)]{E_{2,3}(-1/4)} \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{bmatrix}$

Lin. Ind. Col.

A basis for rowspace A is: $\{(1,2,4), (2,-4,0), (1,0,-2)\}$

Q22) $x = 2t,\ y = t,\ z = t/2$

The basis for the solution space $AX=0$ is : $t.\begin{bmatrix}2\\1\\1/2\end{bmatrix}$

The nullity is $= 1$. Rank = row n. − nullity $= 3-1=2$

$x = 2t,\ y = t,\ z = t/2 \Rightarrow x-2y=0,\ y-2z=0,\ x-4z=0$

* Write an augmented matrix for $A.X=0$

$$A = \left[\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 1 & 0 & -4 & 0 \end{array}\right]$$

Q23)

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 2 & -1 & 3 & b_2 \\ -1 & 3 & 1 & b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right] \xrightarrow[E_{3,1}(1)]{E_{2,1}(-2)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -3 & 5 & b_2-2b_1 \\ 0 & 4 & 0 & b_1+b_3 \\ 0 & 2 & -1 & b_4 \end{array}\right] \xrightarrow{E_3(1/4)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & -3 & 5 & b_2-2b_1 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 2 & -1 & b_4 \end{array}\right]$$

$$\xrightarrow[E_{4,3}(-2)]{E_{2,3}(3)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 0 & 5 & (-5b_1+4b_2+3b_3)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right] \xrightarrow{E_{2,4}(5)} \left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 0 & 0 & (-15b_1+4b_2-7b_3+20b_4)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right]$$

$$\xrightarrow[E_{1,4}(-1)]{E_{1,3}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & (5b_1+b_3-4b_3)/4 \\ 0 & 0 & 0 & (-15b_1+4b_2-7b_3+20b_4)/4 \\ 0 & 1 & 0 & (b_1+b_3)/4 \\ 0 & 0 & -1 & (-b_1-b_3+2b_4)/2 \end{array}\right]$$

* → For the system to be consistent, $-15b_1+4b_2-7b_3+20b_4=0$

* In the case the system is consistent, there is a unique solution which:

$x_1 = \dfrac{5b_1+b_3-b_4}{4}$ $\quad x_2 = \dfrac{b_1+b_3}{4}$ $\quad x_3 = \dfrac{b_1+b_3-2b_4}{2}$

Q24) We have 6 variables but only 3 equations. Therefore, there is $6-3=3$ free variables for the system $AX=0$

Nullity = row number − rank $= 3-3=0$

Since there is not a row of zeroes in the reduced echelon form.

Q25) Assume we have a matrix shown as:

$$A_{4\times3} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A.X = 0 \Rightarrow X = \begin{bmatrix} -ax_3 \\ -bx_3 \\ x_3 \end{bmatrix}$$

① Nullspace of A, defines a line which $x = -a.t$; $y = -b.t$, $z = t$

And for $t=0$, we can see the line passes through origin.

② The columnspace of A is the subspace of R^4 spanned by the pivots.

$$Col(A) = span\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}\right)$$ } This subspace defines a <u>plane</u> (Not a line)

! Therefore, the nullspace, column space and the row space cannot all be a line through origin

Assume we have a matrix shown as:

$$A_{2\times4} = \begin{bmatrix} 1 & 0 & a & c \\ 0 & 1 & b & d \end{bmatrix} \quad A.X = 0 \Rightarrow X = \begin{bmatrix} (-x_4-x_3)/a \\ (-x_4-x_3)/b \\ x_3 \\ x_4 \end{bmatrix} = x_3.\begin{bmatrix} -1/a \\ -1/b \\ 1 \\ 0 \end{bmatrix} + x_4.\begin{bmatrix} -1/a \\ -1/b \\ 0 \\ 1 \end{bmatrix}$$

① Nullspace of A defines a plane. And for $x_3 = x_4 = 0$, we can see it passes through origin

② The columnspace for A is the subspace of R^2 spanned by the pivots

$$col(A) = span\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$$ } This subspace spans a plane passing through origin

③ The row space of A is the subspace of R^4 spanned by the non-zero rows

$$row(A) = span\{(1,0,a,c), (0,1,b,d)\}$$ } This subspace spans a plane passing through origin

! Therefore, the nullspace, column space and the row space can all be a plane through origin

★ For the nullspace to define a line, there must be exactly one free variable. And it can only be achieved by a matrix which can be reduced to:

★ For the nullspace to define a plane there must be two free variables And it can only be achieved by a matrix which can be reduced to:

Q26)

① $\begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & 1/2 \\ t & 1 \\ 3 & t \end{bmatrix} \xrightarrow[E_{3,1}(-3)]{E_{2,1}(-t)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1-t/2 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow{E_{2,2}(1/2)} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/4 \\ 0 & t-3/2 \end{bmatrix}$

$E_2(4) \rightarrow \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \\ 0 & t-3/2 \end{bmatrix} \xrightarrow[E_{3,2}(3/2-t)]{E_{1,2}(-1/2)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ } Since the remaining matrix does not have an element with t, the column space is the same for every t

② $A^T = \begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \Rightarrow$ Find the col. space for A^T (row sp. for A)

$\begin{bmatrix} 4 & t & 3 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_1(1/4)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 2 & 1 & t \end{bmatrix} \xrightarrow{E_{2,1}(-2)} \begin{bmatrix} 1 & t/4 & 3/4 \\ 0 & 1-t/2 & t-3/2 \end{bmatrix}$ } Can't be reduced further

! Since the remaining matrix still has elements with t, the row space for A is not the same for every t.

(Note: We didn't use $E(1/t)$ because since $t \in \mathbb{R}$, it can be 0)

③ $Ax = b \Rightarrow \begin{bmatrix} 4 & 2 \\ t & 1 \\ 3 & t \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$\left[\begin{array}{cc|c} 4 & 2 & b_1 \\ t & 1 & b_2 \\ 3 & t & b_3 \end{array}\right] \xrightarrow[\text{in step ①}]{\text{Same op.}} \left[\begin{array}{cc|c} 1 & 0 & b_1' \\ 0 & 1 & b_2' \\ 0 & 0 & b_3' \end{array}\right]$ } ! assume the condition $b_3' = 0$ is met for the system to have a solution.

$x = \begin{bmatrix} b_1' \\ b_2' \end{bmatrix}$ } Our solution space is in $\mathbb{R}^2$ but our column space is in $\mathbb{R}^3$. Therefore the solution space is not in the span of our column vectors.

Q27)

b) $A.x = \lambda.x \Rightarrow (A - \lambda I).x = 0 \Rightarrow |A - \lambda I| = 0$

$= \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} \xrightarrow{E_{3,2}(1)} \begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix} \xrightarrow{E_{1,2}(-\lambda)} \begin{vmatrix} 0 & -\lambda+\lambda^2 & 1+\lambda \\ -1 & 1-\lambda & -1 \\ 0 & -\lambda & -1-\lambda \end{vmatrix}$

$= \cancel{-1.(-1)}^{3} \cdot \begin{vmatrix} -\lambda+\lambda^2 & 1+\lambda \\ -\lambda & -1-\lambda \end{vmatrix}$

$= \lambda(\lambda-1).-1.(\lambda+1) + \lambda^2+\lambda = -\lambda^3+\lambda^2+2\lambda = \lambda(-\lambda^2+\lambda-2) = 0$ } Characteristic equation

Solutions for the char. eq. $\Rightarrow \lambda = 0, \lambda = 2, \lambda = -1$

$\lambda = 0$ / $(A - \lambda I).x = A.x = 0$ } Find the eigenvector "$\vec{x}$"

$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right]$ } $x_3 = 0$, $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$, let $x_1 = x_2 = t$

Eigenvectors $= X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Basis for the eigenspace $= \{(1,1,0)\}$

$$\lambda I - A = -A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{E_{3,2}(1)} \begin{bmatrix} 0 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = 2$ / $A - \lambda I = A - 2I = 0$

$$\left[\begin{array}{ccc|c} -2 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(1)]{E_{1,3}(2)} \left[\begin{array}{ccc|c} 0 & -2 & -3 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right]$$

$$E_2(-1/2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & -1 & -2 & 0 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/2 & 0 \\ 1 & 0 & -1/2 & 0 \end{array}\right] \Bigg\} \begin{array}{l} x_1 - 1/2\, x_2 = 0 \\ x_2 + 3/2\, x_3 = 0 \\ \text{Let } x_3 = 2t \end{array}$$

Eigenvectors $= X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$

Basis for the eigenspace $= \{(1,-3,2)\}$

$$\lambda I - A = 2I - A = \begin{bmatrix} -2 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & -3/2 \\ -1 & 0 & 1/2 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

$\lambda = -1$ / $A - \lambda I = A + I = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[E_{3,1}(-1)]{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,3}(2)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right] \Bigg\} \begin{array}{l} x_1 + x_3 = 0 \\ x_2 = 0 \\ \text{let } x_1 = t \end{array}$$

Eigenvectors $= X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Basis for the eigenspace $= \{(1,0,-1)\}$

$$\lambda I - A = -I - A = \begin{bmatrix} -1 & 0 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "2"

a) $A \cdot x = x \cdot \lambda \Rightarrow (A - \lambda I)x = 0 \Rightarrow |A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 9 = \boxed{\lambda^2 - 2\lambda - 8 = 0} \Big] \boxed{\begin{matrix} \lambda_1 = 4 \\ \lambda_2 = -2 \end{matrix}}$$

★ Characteristic eq. ★ ★ Eigenvalues ★

$\lambda = 4$ / $(A - 4I) \cdot x = 0$

$$\left[\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \Big\} \; -3x_1 + 3x_2 = 0 \quad \text{let } x_1 = t$$

Eigenvectors $= X = \begin{bmatrix} t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(1,1)\}$

$$\lambda I - A = 4I - A = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \xrightarrow{E_{2,1}(+1)} \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

$\lambda = -2$ / $(A + 2I) \cdot x = 0$

$$\left[\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-1)} \left[\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \; 3x_1 + 3x_2 = 0 \quad \text{let } x_1 = t$$

Eigenvectors $= X = \begin{bmatrix} t \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda^2 - 3\lambda + 2 - 1$

$-1 + (2 - 2\lambda - \lambda + \lambda^2)$

Basis for the eigenspace $= \{(1,-1)\}$

$$\lambda I - A = -2I - A = \begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix}$$

Rank for $\lambda I - A$ is "1"

c) $A \cdot x = x \cdot \lambda \Rightarrow (A - \lambda I) \cdot \lambda = 0 \Rightarrow |A - \lambda I| = 0$

$$= \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ 1-2\lambda & -\lambda & 1 \\ 0 & 1 & 3-\lambda \end{vmatrix} \xrightarrow{E_{2,1}(-2)} \begin{vmatrix} 1-\lambda & -1 & -2 \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix}$$

$$\xrightarrow{E_{1,2}(1-\lambda)} \begin{vmatrix} 0 & -1+(2-\lambda)(1-\lambda) & 3-5\lambda \\ -1 & 2-\lambda & 5 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \cancel{-1} \cdot (\cancel{-1})^3 \cdot \begin{vmatrix} \lambda^2 - 3\lambda + 1 & 3-5\lambda \\ 1 & 3-\lambda \end{vmatrix}$$

$= -\lambda^3 + 6\lambda^2 - 5\lambda = 0$ } Characteristic equation

$-\lambda(\lambda^2 - 6\lambda + 5) = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 5,\ \lambda_3 = 1$ } Eigenvalues

$\lambda = 0$ / $(A - \lambda I)X = A \cdot X = 0$

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ -1 & 2 & 5 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow{E_{2,1}(1)} \left[\begin{array}{ccc|c} 1 & -1 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{1,2}(+1)]{E_{3,2}(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\left.\begin{array}{l} x_1 + x_3 = 0 \\ x_2 + 3x_3 = 0 \end{array}\right\}$ let $x_3 = t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ -3t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1, -3, 1)\}$

$$\lambda I - A = -A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & -5 \\ 0 & -1 & -3 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

$\lambda = 5$ / $(A - \lambda I)X = (A - 5I) \cdot X = 0$

$$\left[\begin{array}{ccc|c} -4 & -1 & -2 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow{E_{1,2}(-4)} \left[\begin{array}{ccc|c} 0 & 11 & -22 & 0 \\ -1 & -3 & 5 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(3)]{E_{1,3}(-11)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$\left.\begin{array}{l} -x_1 - x_3 = 0 \\ x_2 - 2x_3 = 0 \end{array}\right\}$ let $x_3 = t \Rightarrow$ Eigenvectors $= X = \begin{bmatrix} -t \\ 2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(-1, 2, 1)\}$

$$\lambda I - A = 5I - A = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & -5 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{above}]{\text{Same op.}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

$\lambda = 1$ / $(A - \lambda I)X = (A - I) \cdot X = 0$

$$\left[\begin{array}{ccc|c} 0 & -1 & -2 & 0 \\ -1 & 1 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(1)} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \left.\begin{array}{l} -x_1 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}\right\} \text{Let } x_3 = t$$

Eigenvectors $= X = \begin{bmatrix} 3t \\ -2t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

Basis for the eigenspace $= \{(3, -2, 1)\}$

$$\lambda I - A = I - A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & -5 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow[\text{as above}]{\text{Same op.}} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & -1 & -2 \end{bmatrix}$$

$\lambda I - A$ has the rank of "2"

Q28) $A^5 \cdot x = A \cdot (A \cdot (A \cdot (A \cdot (A \cdot x)))) = \ldots = \lambda^5 \cdot x$

$x \cdot \lambda$ $\quad$ $x \cdot \lambda$

Eigenvalues for A^5 are: $\lambda_1' = 0^5 = 0$, $\lambda_2' = 2^5 = 32$, $\lambda_3' = (-1)^5 = -1$

Eigenvectors for A^5 are the same:

★ For $\lambda_1' = 0$: $X = \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow$ Basis $= \{(1,1,0)\}$, Dimension: 1

★ For $\lambda_2' = 32$: $X = \begin{bmatrix} t \\ -3t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \Rightarrow$ Basis $= \{(1,-3,2)\}$, Dimension: 1

★ For $\lambda_3' = -1$: $X = \begin{bmatrix} t \\ 0 \\ -t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \Rightarrow$ Basis $= \{(1,0,-1)\}$, Dimension: 1

3 eigenspaces

Q29)

a) Since we have a 2nd degree polynominal, the dimension of the matrix is 2×2 (It has to be a square matrix for it to have a ch. pol.)

b) The characteristic polynominal is equal to $\det(\lambda I - A)$ for $\lambda = 0$;

$\det(-A) = \cancel{(-1)^2} \cdot \det(A) = \cancel{\lambda^2} + \cancel{6 \cdot 0} + 9 = 9$

c) $\lambda^{-1} / \lambda \cdot X = \lambda^{-1} / \lambda \cdot X$

$I \cdot X = \lambda^{-1} \cdot X \cdot \lambda$

$I \cdot \frac{1}{\lambda} \cdot X = \lambda^{-1} \cdot X$

For the characteristic polynomial, $\lambda' = \frac{1}{\lambda}$

$\frac{1}{\lambda^2} + \frac{6}{\lambda} + 9 = p'(\lambda)$

d) The characteristic polynomial is calculated by $\det(\lambda I - A) = 0$. If we take A^T instead of A, the ch. pol. won't change for the following reasons:

1) With $\lambda I - A^T$, we substract λ from values in the main diagonal of A^T. And by definition, taking the transpoze of a matrix doesn't change it's main diagonal.

2) $\det(A) = \det(A^T)$

Q30) Find the eigenvalues of A : $\Rightarrow |\lambda I - A| = 0$

$$\begin{vmatrix} \lambda-1 & 1 & 4 \\ 0 & \lambda-2 & -4 \\ 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\cdot\begin{vmatrix} \lambda-2 & -4 \\ 0 & \lambda-1 \end{vmatrix} = (\lambda-1)\cdot(\lambda-1)\cdot(\lambda-2)$$

$\lambda_{1,2} = 1 \quad , \lambda_3 = 2$

Find the eigenvectors :

$\lambda=1$ /

$$\left[\begin{array}{ccc|c} 0 & -1 & -4 & 0 \\ 0 & -1 & +4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-1)} \left[\begin{array}{ccc|c} 0 & -1 & -4 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{1,2}(1/2)} \left[\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} x_2=0 \\ x_3=0 \\ \text{Let } x_1=t \end{array}$$

$$\text{Eigenvectors} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Big\} \text{ Basis} = \{(1,0,0)$$

$$P_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda=2$ /

$$\left[\begin{array}{ccc|c} 1 & -1 & -4 & 0 \\ 0 & 0 & +4 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[E_{1,3}(+4)]{E_{2,3}(-4)} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \Big\} \begin{array}{l} x_1+x_2=0 \\ x_3=0 \\ \text{Let } x_1=t \end{array}$$

$$\text{Eigenvectors} = \begin{bmatrix} t \\ -t \\ 0 \end{bmatrix} = t\cdot\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \Big\} \text{ Basis} = \{(1,-1,0)\}$$

$$P_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

! Since there are $2 < 3$ basis vectors in total A is <u>NOT</u> diagonalizable

! Since there isn't a P which $P^{-1}\cdot A\cdot P = D$, A cannot be written as a linear combination of rank 1 matrices formed by its eigenvectors

Q31) Since A is a symmetric matrix, it's orthogonally diagonalizable

Find the eigenvalues of A such $|\lambda I - A| = 0$

$$0 = \begin{vmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda+1 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix} \xrightarrow{E_{2,3}(-1)} \begin{vmatrix} \lambda-2 & 1 & 1 \\ 0 & \lambda+3 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix} \xrightarrow{E_{1,3}(2-\lambda)} \begin{vmatrix} 0 & 2\lambda-3 & -\lambda^2+\lambda+3 \\ 0 & \lambda+3 & -2 \\ 1 & -2 & \lambda+1 \end{vmatrix}$$

$$0 = \underbrace{\lambda^3 - 9\lambda + 8}_{\lambda_1=1} = (\lambda-1)\cdot\underbrace{(\lambda^2+\lambda-8)}_{\Delta=33}, \quad \lambda_2 = \frac{-1+\sqrt{33}}{2}, \quad \lambda_3 = \frac{-1-\sqrt{33}}{2}$$

Find eigenvectors:

$\lambda=1$

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & 2 & 2 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[E_{1,2}(-1)]{E_{3,2}(-1)} \left[\begin{array}{ccc|c} 0 & -3 & -3 & 0 \\ -1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{E_{2,1}(2/3)} \left[\begin{array}{ccc|c} 0 & -3 & -3 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_2 + x_3 = 0 \\ x_1 = 0 \end{array}\right\} \text{let } x_2 = t \Rightarrow \text{Eigenvectors} = \begin{bmatrix} 0 \\ t \\ -t \end{bmatrix} = t.\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Bigg\} \text{Basis} = \{(0,1,-1)\}$$

- Apply Gram-Schmidt process to find the orthanormal vector:

$$V_1 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$\lambda_2 = \frac{-1-\sqrt{33}}{2}$

$$\left[\begin{array}{ccc|c} (-5-\sqrt{33})/2 & -1 & -1 & 0 \\ -1 & (1+\sqrt{33})/2 & 2 & 0 \\ -1 & 2 & (1+\sqrt{33})/2 & 0 \end{array}\right] \xrightarrow{\text{* For the sake of my mental health, I'll be skipping the steps *}} \left[\begin{array}{ccc|c} 1 & 0 & (\sqrt{33}-5)/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 + \frac{\sqrt{33}-5}{2} x_2 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \text{let } x_2 = t \Rightarrow \text{Ei. V.} = \begin{bmatrix} (5-\sqrt{33})/2 \cdot t \\ t \\ t \end{bmatrix} \Bigg\} \text{Basis} = \left\{\left(\frac{5-\sqrt{33}}{2}, 1, 1\right)\right\}$$

! For the love of God, assume $\frac{5-\sqrt{33}}{2} \approx -0{,}4$

- Apply Gram-Schmidt process to find the orthanormal vector:

$$V_2 = \begin{bmatrix} -0{,}27 \\ 0{,}68 \\ 0{,}68 \end{bmatrix}$$

$\lambda_3 = \frac{-1+\sqrt{33}}{2}$

$$\left[\begin{array}{ccc|c} (-5+\sqrt{33})/2 & -1 & -1 & 0 \\ -1 & (1+\sqrt{33})/2 & 2 & 0 \\ -1 & 2 & (1+\sqrt{33})/2 & 0 \end{array}\right] \text{---} \cdots \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & (-\sqrt{33}-5)/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} x_1 + \left(\frac{-\sqrt{33}-5}{2}\right) \cdot x_2 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} \text{let } x_2 = t \Rightarrow \text{Ei. v.} = \begin{bmatrix} (-5-\sqrt{33})/2 \cdot t \\ t \\ t \end{bmatrix} \Bigg\} \text{Basis} = \left\{\left(\frac{-5-\sqrt{33}}{2}, 1, 1\right)\right\}$$

! Assume $\frac{-5-\sqrt{33}}{2} \approx -5{,}4$

- Apply Gram-Schmidt process to find the orthanormal vector

$$V_3 = \begin{bmatrix} -0{,}97 \\ 0{,}18 \\ 0{,}18 \end{bmatrix}$$

• Form matrix P from vectors v_1, v_2, v_3

$$P = \begin{bmatrix} 0 & -0{,}27 & -0{,}97 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \Big\} \text{Matrix } P \text{, orthagonally diagonalizes } A$$

$$P^T \cdot A \cdot P = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{33})/2 & 0 \\ 0 & 0 & (-1+\sqrt{33})/2 \end{bmatrix}$$

• Can we write A in diagonal form as a linear combination formed from its eigenvectors?

$$P/P^{-1} \cdot A \cdot P = P/D \Rightarrow A \cdot P \backslash P^{-1} = P \cdot D \backslash P^{-1} \Rightarrow A = P \cdot D \cdot P^{-1}$$

Since P is orthagonal, $P^{-1} = P^T$ and $A = P \cdot D \cdot P^T$

$$A = \begin{bmatrix} 0 & -0{,}27 & -0{,}97 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1-\sqrt{33})/2 & 0 \\ 0 & 0 & (-1+\sqrt{33})/2 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -0{,}27 & 0{,}68 & 0{,}68 \\ -0{,}97 & 0{,}18 & 0{,}18 \end{bmatrix}$$

$$A = \left\{ \begin{bmatrix} 0 & -0{,}27 & -0{,}97 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_1 \cdot I + \begin{bmatrix} 0 & 0 & 0 \\ 1/\sqrt{2} & 0{,}68 & 0{,}18 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_2 \cdot I + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1/\sqrt{2} & 0{,}68 & 0{,}18 \end{bmatrix} \cdot \lambda_3 I \right\} \cdot P^{-1}$$

$$A = \begin{bmatrix} 0 & -0{,}27 & -0{,}97 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -0{,}27 & 0{,}68 & 0{,}68 \\ -0{,}97 & 0{,}18 & 0{,}18 \end{bmatrix} \cdot \lambda_1 + \cdots$$

$$A = \begin{bmatrix} 1{,}01 & -0{,}36 & -0{,}36 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ -0{,}36 & 1 & -0{,}005 \\ 0 & 0 & 0 \end{bmatrix} \cdot \lambda_2 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0{,}36 & -0{,}005 & 1 \end{bmatrix} \cdot \lambda_3$$

! Yes, we can write A in diagonal form as a linear combination of rank 1 matrices formed from it's eigenvectors

Q3.2) Eigenvalues for $A \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

Let P be a matrix which diagonalizes A

$$P^{-1}/P \cdot A \cdot P^{-1} = P^{-1}/D \Rightarrow A \cdot P^{-1} \backslash P = P^{-1} \cdot D \backslash P \Rightarrow A = P \cdot D \cdot P^{-1}$$

$$A = P^{-1} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \cdot P = P^{-1} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

• Nullity of A is "0"

• Since $\det(A) = 0$, the matrix is not invertible

• Characteristic eq. of a matrix is calculated as: $P(\lambda) = |\lambda I - A|$
For $\lambda = 0$, $P(\lambda)$ is equal to $-\det(A)$. For the matrix to be invertible $\det(A) \neq 0$ and therefore $P(0) \neq 0$