2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 2

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MAT 281E Linear Algebra&Applications – HW#2 – Due date November 27, 2020.

Submission: Scan or take photos of clearly handwritten solutions, combine them in **<u>one</u>** doc, docx or pdf file and submit by 16:00 November 27, 2020. Late homeworks are not accepted!

1.Evaluate the determinant of the following matrix by cofactor expansion rows or columns of your choice. Explain your reasoning for choosing the rows or columns. $A = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 4 & 1 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$

$$\det(A) = \cancel{a_{14}\cdot(-M_{14})} + a_{24}\cdot M_{24} + \cancel{a_{34}\cdot(-M_{34})} + a_{44}\cdot M_{44}$$

$$a_{14} \to 0 \quad a_{24} \to 8 \quad a_{34} \to 0 \quad a_{44} \to -1$$

$$M_{24} = \begin{vmatrix} 2 & -2 & 4 \\ 1 & 0 & 6 \\ 2 & 3 & 0 \end{vmatrix} = 2\cdot\begin{vmatrix} 0 & 6 \\ 3 & 0 \end{vmatrix} - 1\cdot\begin{vmatrix} -2 & 4 \\ 3 & 0 \end{vmatrix} + 2\cdot\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix}$$
$$= -48$$

$$M_{44} = \begin{vmatrix} 2 & -2 & 4 \\ 4 & 1 & 0 \\ 1 & 0 & 6 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & 0 \\ 0 & 6 \end{vmatrix} - 4\begin{vmatrix} -2 & 4 \\ 0 & 6 \end{vmatrix} + 1\cdot\begin{vmatrix} -2 & 4 \\ 1 & 0 \end{vmatrix}$$
$$= 56$$

$$\det(A) = 8\cdot(-48) + (-1)\cdot 56$$
$$= -440$$

* To calculate the determinant, the last column is used because it had two zeroes which made the calculations easier.

2. Apply two multiply-add type elementary row operations followed by cofactor expansion to compute the determinant of

$$A = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 4 & 1 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix}$$

Q2)

$$\begin{bmatrix} 2 & -2 & 4 & 0 \\ 4 & 1 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ 2 & 3 & 0 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 10 & 0 & 4 & 16 \\ 4 & 1 & 0 & 8 \\ 1 & 0 & 6 & 0 \\ -10 & 0 & 0 & -25 \end{bmatrix} \left.\right\} \begin{matrix} E_{2,1}(2) \\ E_{2,4}(-3) \end{matrix}$$

$$\det(A) = \cancel{\underbrace{a_{12}}_{\hookrightarrow 0} \cdot (-M_{12})} + \underbrace{a_{22}}_{\hookrightarrow 1} \cdot M_{22} + \cancel{\underbrace{a_{32}}_{\hookrightarrow 0} \cdot (-M_{32})} + \cancel{\underbrace{a_{42}}_{\hookrightarrow 0} \cdot M_{42}}$$

$$M_{22} = \begin{vmatrix} 10 & 4 & 16 \\ 1 & 6 & 0 \\ -10 & 0 & -25 \end{vmatrix} = 10 \cdot \begin{vmatrix} 6 & 0 \\ 0 & -25 \end{vmatrix} - 1 \begin{vmatrix} 4 & 16 \\ 0 & -25 \end{vmatrix} - 10 \begin{vmatrix} 4 & 16 \\ 6 & 0 \end{vmatrix}$$

$$= -440$$

$$\det(A) = 1 \cdot (-440) = -440$$

3. Can you evaluate $\underline{\underline{A}}^{-1}$ by the method of adjoints where

i) $A = \begin{bmatrix} 1 & -4 & 0 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{bmatrix}$

Q3)

i) $\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} -2 & -2 & -6 \\ 8 & 2 & 6 \\ 4 & 1 & 6 \end{bmatrix}^T = \begin{bmatrix} -2 & 8 & 4 \\ -2 & 2 & 1 \\ -6 & 6 & 6 \end{bmatrix}$

$C_{11} = (-1)^{1+1} \cdot \begin{vmatrix} 2 & -1 \\ -6 & 2 \end{vmatrix} = -2$ $\quad C_{21} = 8 \quad C_{31} = 4$

$C_{12} = -2 \quad C_{22} = 2 \quad C_{32} = 1$

$C_{13} = -6 \quad C_{23} = 6 \quad C_{33} = 6$

$\det(A) = \begin{vmatrix} 1 & -4 & 0 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{vmatrix} = 1.(-2) + (-1)(-8) = 6$

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} -1/3 & 4/3 & 2/3 \\ -1/3 & 1/3 & 1/6 \\ -1 & 1 & 1 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{bmatrix}$ (Subtract the first row from the second, then add the second row to the third)

ii) $A = \begin{bmatrix} 1 & -4 & 1 \\ 1 & 2 & -1 \\ 0 & -6 & 2 \end{bmatrix} \xrightarrow{E_{12}(-1)} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 6 & -2 \\ 0 & -6 & 2 \end{bmatrix} \xrightarrow{E_{23}(1)} \begin{bmatrix} 1 & -4 & 1 \\ 0 & 6 & -2 \\ 0 & 0 & 0 \end{bmatrix}$

$A^{-1} = \dfrac{\operatorname{adj}(A)}{\det(A)}$, $\det(A) = 0$ (row of zeroes)

* The inverse of the matrix A doesn't exist since det(A) = 0 Which means such inverse can't be calculated by method of adjoints.

4.Let

$$2x_1 + 4x_3 + 6x_4 = 2$$
$$x_1 + 2x_3 + 6x_5 = 1$$
$$3x_1 + -2x_2 + 4x_3 - x_4 - 5x_5 = 2$$
$$x_1 + 4x_2 + 4x_3 + x_4 - x_5 = 2$$
$$2x_1 - x_2 + 4x_3 + 4x_4 = 0$$

Determine only x_1, x_3 by Cramer's method.

Q4)

$$\begin{bmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 4 & -1 & -5 \\ 1 & 4 & 4 & 1 & -1 \\ 2 & -1 & 4 & 4 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix}$$

$$x_1 = \frac{\det(A_1)}{\det(A)} \quad \text{and} \quad x_3 = \frac{\det(A_3)}{\det(A)}$$

$$\det(A) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 3 & -2 & 4 & -1 & -5 \\ 1 & 4 & 4 & 1 & -1 \\ 2 & -1 & 4 & 4 & 0 \end{vmatrix} \begin{matrix} E_{2,1}(-2) \\ E_{2,3}(-3) \\ E_{2,4}(-1) \\ E_{2,5}(-2) \end{matrix} \Rightarrow \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & -2 & -1 & -23 \\ 0 & 4 & 2 & 1 & -7 \\ 0 & -1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ -2 & -2 & -1 & -23 \\ 4 & 2 & 1 & -7 \\ -1 & 0 & 4 & -12 \end{vmatrix}$$

$$\begin{matrix} E_{3,2}(-2) \\ E_{4,3}(1) \end{matrix} \Rightarrow -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 0 & -2 & -9 & 1 \\ 0 & 2 & 17 & -55 \\ -1 & 0 & 4 & -12 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 6 & -12 \\ -2 & -9 & 1 \\ 2 & 17 & -55 \end{vmatrix}$$

$$= \left(2 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -55 \end{vmatrix} + 2 \cdot \begin{vmatrix} 6 & -12 \\ -9 & 1 \end{vmatrix} \right) \cdot -1$$

$$= 456$$

$$\det(A_1) = \begin{vmatrix} 2 & 0 & 4 & 6 & 0 \\ 1 & 0 & 2 & 0 & 6 \\ 2 & -2 & 4 & -1 & -5 \\ 2 & 4 & 4 & 1 & -1 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix} \rightarrow \begin{matrix} E_{2,1}(-2) \\ E_{2,3}(-2) \\ E_{2,4}(-2) \end{matrix} \rightarrow \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & -2 & 0 & -1 & -17 \\ 0 & 4 & 0 & 1 & -13 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix}$$

$$\begin{matrix} E_{5,4}(4) \\ E_{5,3}(-2) \end{matrix} \rightarrow \begin{vmatrix} 0 & 0 & 0 & 6 & -12 \\ 1 & 0 & 2 & 0 & 6 \\ 0 & 0 & -8 & -9 & -17 \\ 0 & 0 & 16 & 17 & -13 \\ 0 & -1 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 0 & 6 & -12 \\ 0 & -8 & -9 & -17 \\ 0 & 16 & 17 & -13 \\ -1 & 4 & 4 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 6 & -12 \\ -8 & -9 & -17 \\ 16 & 17 & -13 \end{vmatrix}$$

$$= \left(8 \cdot \begin{vmatrix} 6 & -12 \\ 17 & -13 \end{vmatrix} + 16 \cdot \begin{vmatrix} 6 & -12 \\ -9 & -17 \end{vmatrix} \right) \cdot -1$$

$$= 2352$$

$$\det(A_3) = \begin{vmatrix} 2 & 0 & 2 & 6 & 0 \\ 1 & 0 & 1 & 0 & 6 \\ 3 & -2 & 2 & -1 & -5 \\ 1 & 4 & 2 & 1 & -1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \rightarrow 2 \cdot \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 & 6 \\ 3 & -2 & 2 & -1 & -5 \\ 1 & 4 & 2 & 1 & -1 \\ 2 & -1 & 0 & 4 & 0 \end{vmatrix} \xrightarrow{\substack{E_{1,2}(-1) \\ E_{1,3}(-3) \\ E_{1,4}(-1) \\ E_{1,5}(-2)}} 2 \cdot \begin{vmatrix} 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & -3 & 6 \\ 0 & -2 & -1 & -10 & -5 \\ 0 & 4 & 1 & -2 & -1 \\ 0 & -1 & -2 & -2 & 0 \end{vmatrix}$$

$$= 2 \begin{vmatrix} 0 & 0 & -3 & 6 \\ -2 & -1 & -10 & -5 \\ 4 & 1 & -2 & -1 \\ -1 & -2 & -2 & 0 \end{vmatrix} \xrightarrow{\substack{E_{4,3}(4) \\ E_{4,2}(-2)}} 2 \begin{vmatrix} 0 & 0 & -3 & 6 \\ 0 & 3 & -6 & -5 \\ 0 & -7 & -10 & -1 \\ -1 & -2 & -2 & 0 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 0 & -3 & 6 \\ 3 & -6 & -5 \\ -7 & -10 & -1 \end{vmatrix} = 2 \cdot \left(-3 \cdot \begin{vmatrix} -3 & 6 \\ -10 & -1 \end{vmatrix} - 7 \cdot \begin{vmatrix} -3 & 6 \\ -6 & -5 \end{vmatrix} \right)$$

$$= -1092$$

$x_1 = \frac{|A_1|}{|A|} = \frac{2352}{456} \approx 5{,}16$ and $x_3 = \frac{|A_3|}{|A|} = \frac{-1092}{456} \approx -2{,}39$

5. For what value(s) of x is the following matrix noninvertible?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ x & 1 & -2 \end{bmatrix}$$

Q5)

* For the given matrix to be non-invertible, it's determinant must be equal to zero.

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ x & 1 & -2 \end{vmatrix} = E_{2,3}(-x) \rightarrow \begin{vmatrix} 0 & 0 & 1 \\ 1 & x & 5 \\ 0 & 1-x^2 & -2-5x \end{vmatrix} = -1 \cdot \begin{vmatrix} 0 & 1 \\ 1-x^2 & -2-5x \end{vmatrix} = 1-x^2$$

$1-x^2 = 0 \Rightarrow x=-1$ or $x=1$

6. Let $A = \begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Determine $\underline{\underline{A}}^{-1}$ without computing all elements. Which ones do you need not compute?

Q6)

$A^{-1} = \frac{adj(A)}{det(A)}$, Since A is an upper-triangular matrix, determinant equals to the diagonal product

$* \; det(A) = 2 \cdot 1 \cdot 6 \cdot (-1) = -12 \; *$

$adj(A) = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ C_{21} & C_{22} & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}^T$ Since A is an upper-triangular matrix, cofactors which are placed above the diagonal are all equal to zero

$C_{11} = \begin{vmatrix} 1 & 2 & 8 \\ 0 & 6 & 5 \\ 0 & 0 & -1 \end{vmatrix} = -6$

$C_{21} = 12 \quad C_{22} = -12 \quad C_{31} = 8 \quad C_{32} = 4 \quad C_{33} = -2$

$C_{41} = -56 \quad C_{42} = -76 \quad C_{43} = -10 \quad C_{44} = 12$

$adj(A) = \begin{bmatrix} -6 & 0 & 0 & 0 \\ 12 & -12 & 0 & 0 \\ 8 & 4 & -2 & 0 \\ -56 & -76 & -10 & 12 \end{bmatrix}^T = \begin{bmatrix} -6 & 12 & 8 & -56 \\ 0 & 12 & 4 & -76 \\ 0 & 0 & -2 & -10 \\ 0 & 0 & 0 & 12 \end{bmatrix}$

$A^{-1} = \frac{adj(A)}{det(A)} = \begin{bmatrix} 1/2 & -1 & -2/3 & 14/3 \\ 0 & 1 & -1/3 & 19/3 \\ 0 & 0 & 1/6 & 5/6 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

7. Reduce the matrix to row echelon form by row reduction (elementary row operations) to evaluate the determinant of $A = \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 2 & 12 & -1 \end{bmatrix}$.

Q7)

$$= \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 2 & 12 & -1 \end{bmatrix} \xrightarrow{\substack{E_{1,3}(-1) \\ E_{1,4}(-3)}} \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 5 \\ 0 & 8 & 0 & -1 \end{vmatrix}$$

$$= \xrightarrow{\substack{E_{2,1}(2) \\ E_{2,3}(-2) \\ E_{2,4}(-8)}} 2 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -8 & -9 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & -8 & -9 \end{vmatrix}$$

$$= \xrightarrow{\substack{E_{3,2}(-1) \\ E_{3,1}(-6) \\ E_{3,4}(8)}} -10 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & -69/5 \end{vmatrix} = 138 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 138$$

8. Reduce the matrix to row echelon form by row reduction (elementary row operations) to evaluate the determinant of $A = \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 0 & 8 & 12 \end{bmatrix}$.

Q8)

$$= \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 1 & 0 & 1 & 5 \\ 3 & 0 & 8 & 12 \end{vmatrix} \xrightarrow{\substack{E_{1,3}(-1) \\ E_{1,4}(-3)}} = \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 6 & -4 & 12 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 5 \\ 0 & 6 & -4 & 12 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{2,1}(2) \\ E_{2,3}(-2) \\ E_{2,4}(-6)}} = 2 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 3 \\ 0 & 0 & -10 & 6 \end{vmatrix} = -10 \cdot \begin{vmatrix} 1 & 0 & 6 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & -10 & 6 \end{vmatrix}$$

$$\xrightarrow{\substack{E_{3,2}(-1) \\ E_{3,1}(-6) \\ E_{3,4}(10)}} = -10 \cdot \begin{vmatrix} 1 & 0 & 0 & 28/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & -3/5 \\ 0 & 0 & 0 & 0 \end{vmatrix} = -10 \cdot 0 = 0$$

9. Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & -5 \\ 0 & -3 & 1 \end{bmatrix}$. Determine $det(\left(\underline{A^T}\right)^6)$.

Q9)

$$A^T = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{bmatrix} \Rightarrow \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{vmatrix} = E_{3,1}(2) \begin{vmatrix} 0 & -9 & 2 \\ 0 & 2 & -3 \\ -1 & -5 & 1 \end{vmatrix} = -23$$

$$\det((A^T)^6) = \det(\underbrace{A^T.A^T.A^T.A^T.A^T.A^T}_{\text{Square matrices of the same size}}) = \det A^T.\det A^T \ldots \det A^T = (\det A^T)^6$$

$$\det((A^T)^6) = (\det(A^T))^6 = (-23)^6 = 148035889$$

10. By inspection explain why the following matrix is not invertible. Do not try to compute the inverse!

$$A = \begin{bmatrix} -1 & 1 & 2 & 2 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 8 & 12 \end{bmatrix}$$

Q10)

$$\det(A) = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 1 & -1 & -2 & 0 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 8 & 12 \end{vmatrix} = E_{1,2}(1) \rightarrow \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \\ 3 & 0 & 8 & 12 \end{vmatrix}$$

$$E_{2,3}(-1) \rightarrow = \begin{vmatrix} -1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 3 & 0 & 8 & 12 \end{vmatrix} = 0$$

Since $\det(A) = 0$, we can surely say that A^{-1} doesn't exist.

11. If $\det(\underline{\underline{A}}) = 2$, $\det(\underline{\underline{B}}) = 3$ what is $det(\underline{B}^{-2}\underline{A}^{-3})$?

Q 11)

$$\det(B^{-2}A^{-3}) = \det(B^{-2}) . \det(A^{-3})$$
$$= (3)^{-2} . (2)^{-3}$$
$$= \frac{1}{9} . \frac{1}{8}$$
$$= \frac{1}{72}$$

12. Show that **u** · (**v** × **w**), **v** · (**u** × **w**), **w** · (**u** × **v**) have the same magnitude. What is their magnitude?

Q 12)

let $u = \langle u_1, u_2, u_3 \rangle$, $v = \langle v_1, v_2, v_3 \rangle$, $w = \langle w_1, w_2, w_3 \rangle$

a) $u . (v \times w)$

$$= u . \left(\begin{vmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

$$= u . [i . (v_2 w_3 - v_3 w_2) - j . (v_1 w_3 - v_3 w_1) + \hat{k} (v_1 w_2 - v_2 w_1)]$$

$$= \underline{u_1 . v_2 . w_3} - \underline{u_1 . v_3 . w_2} - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_3 v_2 w_1$$

b) $v . (u \times w)$

$$= v . \left(\begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \right)$$

$$= v . [i (u_2 . w_3 - u_3 w_1) - j (u_1 w_3 - u_3 w_1) + k (u_1 w_2 - u_2 w_1)]$$

$$= v_1 u_2 w_3 - v_1 u_3 w_1 - \underline{v_2 u_1 w_3} + v_2 u_3 w_1 + \underline{v_3 u_1 w_2} - v_3 u_2 w_1$$
$$= - (u_1 v_2 w_3 - u_1 . v_3 . w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_3 - u_3 v_2 w_1)$$

c) $w \cdot (u \times v)$

$$= w \cdot \left(\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \right)$$

$$= w \cdot \left[\hat{i}(u_2 v_3 - u_3 v_2) - \hat{j}(u_1 v_3 - u_3 v_1) + \hat{k}(u_1 v_2 - u_2 v_1) \right]$$

$$= w_1 u_2 v_3 - w_1 u_3 v_2 - \underline{w_2 u_1 v_3} + w_2 u_3 v_1 + \underline{w_3 u_1 v_2} - w_3 u_2 v_1$$

$$= -(u_1 v_2 w_3 - u_1 u_3 w_2 - u_2 v_1 w_3 + u_2 v_3 w_1 + u_3 v_1 w_3 - u_3 v_3 w_1)$$

★ a, b, c have the same magnitude

13. Determine $\begin{vmatrix} a+b & \text{e-f} \\ c-d & \text{g+h} \end{vmatrix}$ in terms of the determinants of $\begin{bmatrix} a & c \\ e & g \end{bmatrix}$, $\begin{bmatrix} b & -d \\ e & g \end{bmatrix}$, $\begin{bmatrix} a & c \\ -f & h \end{bmatrix}$ and $\begin{bmatrix} b & -d \\ -f & h \end{bmatrix}$.

Q13)

$$\begin{vmatrix} a+b & e-d \\ c-d & g+h \end{vmatrix} = ag + ah + bg + bh - ec + ed + fc - fd$$

$$\begin{vmatrix} a & c \\ e & g \end{vmatrix} = a.g - ce \qquad \begin{vmatrix} b & -d \\ e & g \end{vmatrix} = b.g + d.e$$

$$\begin{vmatrix} a & c \\ -f & h \end{vmatrix} = a.h + c.f \qquad \begin{vmatrix} b & -d \\ -f & h \end{vmatrix} = bh - df$$

$$★ \begin{vmatrix} a+b & e-d \\ c-d & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & -d \\ e & g \end{vmatrix} + \begin{vmatrix} a & c \\ -f & h \end{vmatrix} + \begin{vmatrix} b & -d \\ -f & h \end{vmatrix}$$

14. For what values of k is $\begin{bmatrix} k+1 & -4 \\ -2 & k-1 \end{bmatrix}$ noninvertible?

Q14)

$$\begin{vmatrix} k+1 & -4 \\ -2 & k-1 \end{vmatrix} = (k+1)(k-1) - 8$$

$$= k^2 - 1 - 8 = k^2 - 9$$

For the matrix to be non-invertable, k^2-9 must be 0

$k^2 - 9 = 0 \Rightarrow k^2 = 9 \Rightarrow k = 3 \quad k = -3$

15. Let $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ be two vectors from the origin to points P_1 and P_2. Let point Q be the midpoint of the line between P_1 and P_2. Show that $\overline{OQ}$, $\overline{OP}_1$ and $\overline{OP}_2$ are coplanar. (You can try to either i)show that the projection of $\overrightarrow{OQ}$ onto the plane defined by $\overrightarrow{OP_1}$ and $\overline{OP}_2$is the same as $\overline{OQ}$ or ii)try to solve $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ for a, b, c by letting $\overline{OP}_1 = (x_1 - x_0, y_1 - y_0, z_1 - z_0)$, $\overrightarrow{OP_2} = (x_2 - x_0, y_2 - y_0, z_2 - z_0)$ or iii)argue that the volume of the parallelpiped formed by $\overrightarrow{OQ}$, $\overline{OP}_1$ and $\overline{OP}_2$ as edges is 0 by employing determinant properties.)

Q15)

let $P_1 = (x_1, y_1, z_1)$ and $\overrightarrow{OP_1} = \langle x_1, y_1, z_1 \rangle$

let $P_2 = (x_2, y_2, z_2)$ and $\overrightarrow{OP_2} = \langle x_2, y_2, z_2 \rangle$

$Q = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ and $\overrightarrow{OQ} = \left\langle \frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2} \right\rangle$

Find the volume of the parallelpipe by $\overrightarrow{OP_1} \cdot (\overrightarrow{OP_2} \times \overrightarrow{OQ})$

$$V = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} & \frac{z_1+z_2}{2} \end{vmatrix} = x_1 \cdot \left(\frac{y_2z_1+y_2z_2}{2} - \frac{z_2y_1-z_2y_2}{2}\right) - x_2 \cdot \left(\frac{y_1z_1+y_1z_2}{2} - \frac{z_1y_1-z_1y_2}{2}\right) + \frac{x_1+x_2}{2} \cdot (y_1z_2 - z_1y_2) = 0$$

Since $V=0$, the vector are coplanar

16. Setup a linear system and solve to get the equation of a plane that contains points $\mathbf{x} = (4,0,-2)$, $\mathbf{y} = (2,3,-1)$ and $\mathbf{z} = (0,0,1)$.

$x = (4,0,-2)$ $y = (2,3,-1)$ $z = (0,0,1)$ plane: $ax+by+cz+d=0$

$4a-2c+d=0$ $2a+3b-c+d=0$ $c+d=0$ } Solve the equations

$$\left[\begin{array}{cccc|c} 4 & 0 & -2 & 1 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{E_{2,1}(-2)} \left[\begin{array}{cccc|c} 0 & -6 & 0 & -1 & 0 \\ 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$E_2(\tfrac{1}{2}) \rightarrow \left[\begin{array}{cccc|c} 0 & -6 & 0 & -1 & 0 \\ 1 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{E_1(-\frac{1}{6})} \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1/6 & 0 \\ 1 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$E_{3,2}(-\tfrac{1}{2}) \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & 1/6 & 0 \\ 1 & 3/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \Big\}$$

$b + \frac{d}{6} = 0$ $c+d=0$

$a + \frac{3}{2}b = 0$ *let $d = 12$*

$d=12$, $c=-12$, $b=-2$, $a=3$

plane : $3x - 2y - 12z + 12 = 0$

17. Construct two examples in $\Re^2$ to demonstrate $\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$ with equality and inequality?

Q17) let $A = \langle 0,1 \rangle$ and $B = \langle 2,0 \rangle$

$A+B = \langle 2,1 \rangle$

$|A| = 1$, $|B| = 2$ and $|A+B| = \sqrt{5}$

$|A+B| < |A| + |B|$

let $C = \langle 0,0 \rangle$ and $D = \langle 0,0 \rangle$

$C+D = \langle 0,0 \rangle$

$|C| = 0$, $|D| = 0$ and $|C+D| = 0$

$|C+D| = |C| + |D|$

18. Construct two examples in $\Re^2$ to demonstrate $|\,\|\mathbf{u}\|-\|\mathbf{v}\|\,| \leq \|\mathbf{u}-\mathbf{v}\|$ with equality and inequality?

Q 18)

let $A = \langle 0,2 \rangle$ and $B = \langle 0,1 \rangle$

$A-B = \langle 0,1 \rangle$

$|A-B| = |A| - |B|$

let $C = \langle 0,3 \rangle$ and $D = \langle 3,0 \rangle$

$C-D = \langle -3,3 \rangle$

$|C-D| > |C| - |D|$

19. Find the orthogonal projection of vector $\underline{u} = (1,2,0)$ onto

$A=(0,1,0)$, $B=(0,0,-1)$, $C=(2,-1,1)$ are on the plane

$\vec{AB} = \langle 0,-1,-1\rangle$ $\vec{AC} = \langle 2,-2,1\rangle$

i) the plane described by equation $-3x-2y+2z=-2$.

i) $\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 2 & -1 \end{vmatrix} = \underbrace{\langle -3,-2,2\rangle}_{\vec{N}}$

$u_a = k.\vec{N}$, $(u-u_a).\vec{N}=0$

$u = k.\vec{N} + (u-u_a)$

$u.\vec{N} = k|N|^2 + 0$

$k = \frac{u.\vec{N}}{|N|^2} = -\frac{7}{17}$

$u_a = \langle \frac{21}{17}, \frac{14}{17}, \frac{-14}{17}\rangle$ } Projection onto plane normal vector

Projection onto the normal vector is the orthagonal projection onto the plane

ii) a line along the normal vector of the plane in i).

ii) $\vec{AB}\times\vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 2 & -1 \end{vmatrix} = \underbrace{\langle -3,-2,2\rangle}_{\vec{N}=\text{normal vector}}$

$u_a = k.\vec{N}$, $\vec{N}.(u-u_a)=0$

$k = \frac{\vec{u}.\vec{N}}{|N|^2} = -\frac{7}{17}$

$u_a = \langle \frac{21}{17}, \frac{14}{17}, \frac{-14}{17}\rangle$

$u-u_a = \langle \frac{-4}{17}, \frac{20}{17}, \frac{14}{17}\rangle$

20. Find equation of all points $P \in \Re^3$ such that $\overline{P_0P}$ is orthogonal to vector $\underline{v} = (1,-1,2)$ where $P_0 = (0,1,2)$. What is this the equation of?

let $P=(x,y,z)$ and $\vec{P_0P} = \langle -x, 1-y, 2-z\rangle$

$\vec{V}.\vec{P_0P}$ should be equal to 0 since they are orthagonal

$-x + (y-1) + (4-2z) = 0$

$-x + y - 2z = -3$ => This is the equation of the plane whose normal vector is V

21. Find all vectors that can be described as (x, y, z) that are

Q21)

i) $\vec{A} = \langle x, y, z \rangle$

$\vec{A} \cdot \langle 1,0,-1 \rangle = x - z = 0$ (orthagonality)

$x - z = 0 \Rightarrow x = z$

$\vec{A} = \langle x, y, x \rangle$ and $|A| = \sqrt{2x^2+y^2}$

For all values of x, y ; the vector $\left\langle \frac{x}{\sqrt{2x^2+y^2}}, \frac{y}{\sqrt{2x^2+y^2}}, \frac{x}{\sqrt{2x^2+y^2}} \right\rangle$ meets the requirements.

ii) orthogonal to both $(1,0,-1)$ and $(1,1,-1)$ and have unit norm.

ii) The product of two vectors is orthagonal to both of them

$$\langle 1,0,-1 \rangle \times \langle 1,1,-1 \rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \langle 1,0,1 \rangle$$

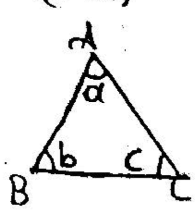
Since $\langle 1,0,1 \rangle$ is orthagonal, $-1 . \langle 1,0,1 \rangle = \langle -1,0,-1 \rangle$ is also orthagonal

Convert the vectors into unit norm;

$\left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$ and $\left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right\rangle$

22. Describe a method to determine the largest interior angle of a triangle.

Q22)



$\vec{AB} \times \vec{AC} = |AB| . |AC| . \sin(a) \Rightarrow \frac{|\vec{AB} \times \vec{AC}|}{|AB| . |AC|} = \sin a$

$\vec{CA} \times \vec{CB} = |CA| . |CB| . \sin(c) \Rightarrow \frac{|\vec{CA} \times \vec{CB}|}{|CA| . |CB|} = \sin c$

$\vec{BA} \times \vec{BC} = |BA| . |BC| . \sin(b) \Rightarrow \frac{|\vec{BA} \times \vec{BC}|}{|BA| . |BC|} = \sin b$

Since the value of $\sin\theta$ gets higher as θ rises, and since $\theta < 180$ in a triangle, we can use the calculations above to compare the sin values of the angles. The angle with the largest sine value is the largest angle of the triangle

23. Find two unit vectors (vectors of unit norm) orthogonal to vectors **u** =(1,2,-1) and **v** = (2,0,-2).

Q23)

$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 0 & -2 \end{vmatrix} = \langle -4, 0, -4 \rangle$ } Orthogonal to both u and v

If $\langle -4, 0, -4 \rangle$ is orthogonal to the vectors above, we can say that $-\langle -4, 0, -4 \rangle = \langle 4, 0, 4 \rangle$ is also orthogonal.

- Convert $\langle -4, 0, -4 \rangle$ into unit norm

$|\langle -4, 0, -4 \rangle| = 4\sqrt{2}$

Result $= \langle \frac{-1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$

- Convert $\langle 4, 0, 4 \rangle$ into unit norm

$|\langle 4, 0, 4 \rangle| = 4\sqrt{2}$

Result $= \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle$

24. Determine the distance between the lines x+2y=1 and x+2y=3 in $\Re^2$ using the method of orthogonal projections.

Q24)

let $\alpha = (1,0)^T$ be a point on the line $x+2y=1$

The distance between the point α and the line $x+2y=3$ gives us also the distance between those lines

$$D = \frac{|1.1 + 2.0 - 3|}{\sqrt{1^2+2^2}} = \frac{2}{\sqrt{5}}$$

25. Determine the distance between the point (0,1,-4) and the plane x+z=-1 in $\Re^3$ using the method of orthogonal projections.

Q25) $A = (0,1,-4)$

- let $B = (x_0, y_0, z_0)$ be a point on the plane

The normal vector of the plane $\vec{N} = \langle 1, 0, 1 \rangle$

$\vec{AB} = \langle -x_0, 1-y_0, -4-z_0 \rangle$, $\vec{AB} \parallel \vec{N}$.

- $\vec{AB}$'s projection over $\vec{N}$ is equal to $\vec{AB}$ because $\vec{AB} \parallel \vec{N}$

$\vec{AB_a} = k.\vec{N}$

$k = \dfrac{\vec{AB}.\vec{N}}{|N|^2}$

$k = \dfrac{-x_0-4-z_0}{2}$

$k = \dfrac{-\overbrace{(x_0+z_0)}^{-1}-4}{2}$

$k = \dfrac{-3}{2}$

$\vec{AB_a} = k.\vec{N}$

$\vec{AB_a} = \langle -\frac{3}{2}, 0, \frac{3}{2} \rangle$

$|\vec{AB_a}| = \dfrac{3}{\sqrt{2}}$

26. If **u** ⊥ **v** what can you say about the norms of the vectors **u** · **v**, **u** × **v**, **u** + **v** and **u** − **v** in terms of the norms of **u** and **v** ?Assume that the vector dimension is arbitrary.

Q26) let $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$

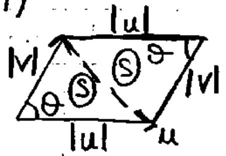
$\vec{u}.\vec{v} = |u|.|v|.\cos 90$ $\quad |\vec{u} \times \vec{v}| = |u|.|v|.\sin 90$

$\vec{u}.\vec{v} = |u|.|v|.0 = 0$ $\quad |\vec{u} \times \vec{v}| = |u|.|v|.1 = |u|.|v|$

$|\vec{u}+\vec{v}| = \sqrt{(u_1+v_1)^2 + (u_2+v_2)^2}$ $\quad |\vec{u}-\vec{v}| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2}$

27. Let **u** and **v** be two vectors that form two sides of a parallelogram. Use the determinant to determine the area of the parallelogram in terms of **u**, **v** and norms of **u** and **v**.

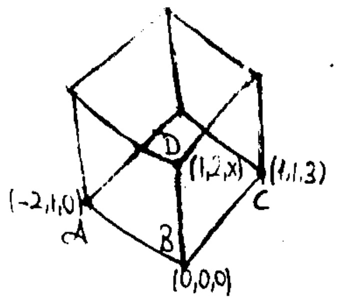
27)



$S = |v|.|u|.\sin\theta.\frac{1}{2}$

$Area = 2S = |v|.|u|.\sin\theta = |v \times u|$

28. Let points (0,0,0), (1,2,x), (-2,1,0) and (1,1,3) be at four corners of a parallelpiped. Determine the volume of the parallelepiped by using the determinant in terms of x. For what value of x is the volume 0. Interpret this case geometrically.



$\vec{BA} = \langle -2, 1, 0 \rangle$
$\vec{BC} = \langle 1, 1, 3 \rangle$
$\vec{BD} = \langle 1, 2, x \rangle$

$V = \vec{BD} \cdot (\vec{BA} \times \vec{BC})$

$$V = \begin{vmatrix} 1 & 2 & x \\ -2 & 1 & 0 \\ 1 & 1 & 3 \end{vmatrix} = 3 - 2(-6) + x(-3) = 15 - 3x$$

$V = 0 = 15 - 3x \Rightarrow x = 5$

29. A homogenous linear system $A\mathbf{x} = \mathbf{0}$ has more than one solution for $\mathbf{x}$ where A is a square matrix. Is it true that $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$? Explain.

Q29) If A has more than one solutions, |A| must be equal to 0, because if it wasn't we would only have one unique solution.

Because of that reason, there is NOT a solution for b in $AX = b$

30.Is it possible for A to be the matrix described in Problem 29 where $\det(A) = 9$? Explain.

If A is a square matrix which has more than one solutions for $AX = b$ just like in the question 29, then it is NOT possible for det(A) to be 9 because det(A) must be 0 for the equation above to be satisfied

But if A is just a regular square matrix, and if det(A) is 9, then:

$A^{-1}AX = A^{-1}b$, $A^{-1} = \frac{\text{adj}(A)}{\det(A)} = \frac{\text{adj}(A)}{9}$

$X = A^{-1} \cdot b \Rightarrow X = b \cdot \frac{\text{adj}(A)}{9}$

In this case, the equation has a solution for every b

31.Determine the condition on b_i so that the following linear system has solution(s).

$$
\begin{aligned}
&x + 2y - 3z = b_1 \\
&y + z = b_2 \\
&2x + 5y - 5z = b_3
\end{aligned}
$$

Q31)

$$
\left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 2 & 5 & -5 & b_3 \end{array}\right] \rightarrow E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 1 & 1 & b_3-2b_1 \end{array}\right]
$$

$$
E_{2,3}(-1) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -5 & b_1-2b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & b_3-2b_1-b_2 \end{array}\right]
$$

For the system to be consistent, the last row should be zero

$b_3-2b_1-b_2=0$