2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 4

Q1)

a) let $K = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ and $K = v_1.c_1 + v_2.c_2 + v_3.c_3$

$$\left.\begin{matrix} 4c_1 & -2c_2 & & = a \\ 3c_1 & & +3c_3 & = b \\ 2c_1 & -c_2 & & = c \\ c_1 & & +c_3 & = d \end{matrix}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & a \\ 3 & 0 & 3 & b \\ 2 & -1 & 0 & c \\ 1 & 0 & 1 & d \end{array}\right] \xrightarrow{\substack{E_{1,4}(-4) \\ E_{2,4}(-3) \\ E_{3,4}(-2)}} \left[\begin{array}{ccc|c} 0 & -2 & -4 & a-4d \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

$$E_{1,3}(-2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a-2c \\ 0 & 0 & 0 & b-3d \\ 0 & -1 & -2 & c-2d \\ 1 & 0 & 1 & d \end{array}\right]$$

For the system to be consistent,
$a - 2c = 0 \Rightarrow a = 2c$
$b - 3d = 0 \quad b = 3d$

Given vectors span a subspace consisting of vector K, that

$$K = \begin{bmatrix} 2c \\ 3d \\ c \\ d \end{bmatrix} = -c. \underbrace{\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \end{bmatrix}}_{v_2} + d. \underbrace{\begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

b) The vectors are linearly independent if and only if the eq. $v_1.c_1 + v_2.c_2 + v_3.c_3 = 0$ only has the trivial solution

$$\left.\begin{matrix} 4c_1 & -2c_2 & & = 0 \\ 3c_1 & & +3c_3 & = 0 \\ 2c_1 & -c_2 & & = 0 \\ c_1 & & +c_3 & = 0 \end{matrix}\right\} \begin{bmatrix} 4 & -2 & 0 \\ 3 & 0 & 3 \\ 2 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & -2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\text{Same operations used in the section a}} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right]$$

since there are an infinite amount of solutions, the vectors are linearly dependent

c) Basis of the space spanned by these vectors is : $\left\{\begin{bmatrix}-2\\0\\-1\\0\end{bmatrix},\begin{bmatrix}0\\3\\0\\1\end{bmatrix}\right\}$

$\mathbb{R}^4$'s basis has 4 vectors, therefore, we need to find 2 additional vectors that are not in our initial span

let $V_4 = (0,0,0,1)$ and $V_5 = (1,0,0,0)$ $\exists$ V_4 and V_5 are lin. ind. and not in the span

The basis we get as a result is $= \left\{ \underbrace{\begin{bmatrix}-2\\0\\-1\\0\end{bmatrix}}_{V_2}, \underbrace{\begin{bmatrix}0\\3\\0\\1\end{bmatrix}}_{V_3}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right.$

Q2) ∇ $\cos^2x - \sin^2x = \cos 2x \Rightarrow f_1 = \cos 2x$ and $f_2 = \cos 2x$

Applying Wronskian :

$$\begin{vmatrix}\cos 2x & \cos 2x\\ (\cos 2x)' & (\cos 2x)'\end{vmatrix} = \begin{vmatrix}\cos 2x & \cos 2x\\ -2\sin 2x & -2\sin 2x\end{vmatrix} = -2\sin 2x\cos 2x + 2\sin 2x \cos 2x = 0$$

Since the wronskian is equal to 0, these vectors are linearly dependent.

Q3) Dimension of $\mathbb{R}^3$ is 3, therefore basis for $\mathbb{R}^3$ consists of 3 vectors

Choose $u_1, u_3, u_4 \Rightarrow$ Test for linear independence :

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = 0$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 &= 0\\ -2c_1 &= 0\\ -c_3 + 7c_4 &= 0\end{aligned}\right\} \begin{bmatrix}1&1&0\\-2&0&0\\0&-1&7\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&0&0\\-2&0&0&0\\0&-1&7&0\end{array}\right] \xrightarrow{E_{2,1}(2)} \left[\begin{array}{ccc|c}1&1&0&0\\0&2&0&0\\0&-1&7&0\end{array}\right] \xrightarrow{E_2(1/2)} \left[\begin{array}{ccc|c}1&1&0&0\\0&1&0&0\\0&-1&7&0\end{array}\right]$$

$$\xrightarrow[E_{3,2}(1)]{E_{1,2}(-1)} \left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&7&0\end{array}\right] \left.\begin{aligned}c_1&=0\\c_3&=0\\c_4&=0\end{aligned}\right\} \Rightarrow \text{Vectors are linear independent}$$

Since our number of vectors is equal to the dimention of the plane, we can say u_1, u_3, u_4 form a basis without testing for their span.

Base $= \{(1,-2,0),(1,0,-1),(0,0,7)\}$

∇ Express the remaining vector, u_2, by the base vectors.

$u_1 . c_1 + u_3 . c_3 + u_4 . c_4 = u_2$, find c_1, c_3, c_4

$$\left.\begin{aligned} c_1 + c_3 \qquad &= 1 \\ -2c_1 \qquad\qquad &= 2 \\ -c_3 + 7c_4 &= 4 \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ -2 & 0 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_2(-1/2)} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{1,2}(-1)} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 7 & 4 \end{array}\right]$$

$$\xrightarrow{E_{1,2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & -1 & 7 & 4 \end{array}\right] \xrightarrow{E_{3,2}(1)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 7 & 6 \end{array}\right] \xrightarrow{E_3(1/7)} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 6/7 \end{array}\right]$$

$$\left.\begin{aligned} c_1 &= -1 \\ c_3 &= 2 \\ c_4 &= 6/7 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}}_{u_2} = -1 \cdot \underbrace{\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}_{u_1} + 2 \cdot \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_3} + \frac{6}{7} \cdot \underbrace{\begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}}_{u_4}$$

The dimension of the subspace spanned by these vectors is 4

Q4) If we remove a vector from a set of linearly independent vectors, the remaining ones will also be linearly independent. But since the remaining vectors are less than the dimension, they won't be a basis for the space

Since those 4 vectors are the basis for the space, any other vector can be expressed as a linear combination of the basis vectors. Therefore, if we add another vector to the set, our new set of vectors will be linearly dependent.

Q5) Let M be a subspace of R^4 that is spanned by the given vectors. If we find a vector v such $v \notin M$, then our new set of vectors which additionally has v in, is a larger linearly independent set.

~~Q6) The vectors u and v, are the basis for the uxv plane.~~

~~The vector resulting from $proj_{uxv}\, w$ lies in the uxv plane, therefore it can be expressed as a linear combination of the basis vectors u and v.~~

~~Therefore u, v and $proj_{uxv}\, w$ are not linearly independent~~

Q7) The basis for 2x2 matrices has 4 vectors. If we are given 5 2x2 matrices, only 4 of them can be linearly independent in total. So, in the worst case scenario, at least one of the five vectors is linearly dependent and therefore can be expressed as a linear combination of the remaining vectors.

The basis for 3x3 matrices consists of 9 vectors. If we are given 5 3x3 matrices, every one of them can be linearly independent. In such a case, the given matrices can not be expressed as a linear combination of the remaining ones.

Q8)

a) The dimension of R^6 is 6, so the basis for R^6 has 6 vectors.
Therefore any amount of vectors that is less than 6, cannot form a basis

b) let $v=(-x,y)$ and $m=(x,-y)$

Since $v=-1\cdot m$, these vectors are linearly dependent and therefore, cannot form a basis

c) We're given 3 vectors which is the same amount of vectors in the basis of R^3, so we only need to show whether these vectors are linearly independent or not.

$v_1=(1,2,3) \quad v_2=(1,2,0) \quad v_3=(-1,2,6)$

$c_1v_1+c_2v_2+c_3v_3=0$

$$\left.\begin{aligned} c_1+c_2-c_3&=0\\ 2c_1+2c_2+2c_3&=0\\ 3c_1+\quad\ 6c_3&=0\end{aligned}\right\}\begin{bmatrix}1&1&-1\\2&2&2\\3&0&6\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&1&-1&0\\2&2&2&0\\3&0&6&0\end{array}\right]\xrightarrow[E_{3,1}(-3)]{E_{2,1}(-2)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&6&0\\0&-3&9&0\end{array}\right]\xrightarrow[E_3(-1/3)]{E_2(1/6)}\left[\begin{array}{ccc|c}1&1&-1&0\\0&0&1&0\\0&1&-1&0\end{array}\right]$$

$$\xrightarrow[E_{1,2}(1)]{E_{3,2}(1)}\left[\begin{array}{ccc|c}1&1&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\xrightarrow{E_{1,3}(-1)}\left[\begin{array}{ccc|c}1&0&0&0\\0&0&1&0\\0&1&0&0\end{array}\right]\Big\}\ c_1=c_2=c_3=0$$

$\therefore$ The vectors are linearly independent, therefore they are a basis for R^3

d) Any set of vectors that include the zero vector is linearly dependent. Therefore they can't form a basis.

Q9)

$V_1\cdot c_1+V_2\cdot c_2+V_3\cdot c_3=w$

$2c_1 \quad +c_3=1 \ / \ -2c_1+c_2+5c_3=1 \ / \ -c_2+4c_3=1$

$$\begin{bmatrix}2&0&1\\-2&1&5\\0&-1&4\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\1\end{bmatrix}\Rightarrow\left[\begin{array}{ccc|c}2&0&1&1\\-2&1&5&1\\0&-1&4&1\end{array}\right]\xrightarrow{E_{2,1}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&-1&4&1\end{array}\right]$$

$$\xrightarrow{E_{3,2}(1)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&10&3\end{array}\right]\xrightarrow{E_3(1/10)}\left[\begin{array}{ccc|c}2&0&1&1\\0&1&6&2\\0&0&1&3/10\end{array}\right]\xrightarrow[E_{1,3}(-1)]{E_{2,3}(-6)}\left[\begin{array}{ccc|c}2&0&0&0{,}7\\0&1&0&0{,}2\\0&0&1&0{,}3\end{array}\right]$$

$$-E_1(\tfrac{1}{2}) \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0{,}35 \\ 0 & 1 & 0 & 0{,}20 \\ 0 & 0 & 1 & 0{,}30 \end{array}\right] \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \begin{matrix} c_1 = 0{,}35 \\ c_2 = 0{,}20 \\ c_3 = 0{,}30 \end{matrix}$$

$w = 0{,}35 \cdot v_1 + 0{,}20 \cdot v_2 + 0{,}30 \cdot v_3 \Rightarrow$ Coordinates of w relative to the basis is

$(0{,}35, 0{,}20, 0{,}30)$

Q10)

a) R^3 has a dimension of '3', because of that, the maximum amount of linearly independent vectors we can express is 3. Therefore, the given vectors can't be linearly independent

b)

1) Find the space spanned by v_1, v_2, v_3, v_4

let $u = (a, b, c)$

$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = u$

$$\left.\begin{aligned} 2c_1 \quad\quad + c_3 + c_4 &= a \\ -c_1 + c_2 - c_3 \quad\quad &= b \\ -c_1 + 3c_2 - 2c_3 + c_4 &= c \end{aligned}\right\} \begin{bmatrix} 2 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ -1 & 3 & -2 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 1 & 1 & a \\ -1 & 1 & -1 & 0 & b \\ -1 & 3 & -2 & 1 & c \end{array}\right] \xrightarrow[E_{1,3}(2)]{E_{2,3}(-1)} \left[\begin{array}{cccc|c} 0 & 6 & -3 & 3 & a+2c \\ 0 & -2 & 1 & -1 & b-c \\ -1 & 3 & -2 & 1 & c \end{array}\right]$$

$$\begin{matrix} E_1(1/6) \\ E_3(-1) \end{matrix} \rightarrow \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & -2 & 1 & -1 & b-c \\ 1 & -3 & 2 & -1 & -c \end{array}\right] \xrightarrow[E_{3,1}(3)]{E_{2,1}(2)} \left[\begin{array}{cccc|c} 0 & 1 & -1/2 & 1/2 & (a+2c)/6 \\ 0 & 0 & 0 & 0 & (a+3b-c)/3 \\ 1 & 0 & 1/2 & -1/2 & a/2 \end{array}\right] !$$

$$\left.\frac{a+3b-c}{3} = 0 \Rightarrow a+3b-c = 0 \right\} \text{let } a = t,\ b = k$$

The given vectors span a subspace of R^3 in which $v_i = \begin{bmatrix} t \\ k \\ t+3k \end{bmatrix}$

$v_i = t \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Rightarrow$ Basis for the subspace is $= (\underbrace{\langle 1,0,1 \rangle}_{v_4}, \underbrace{\langle 0,1,3 \rangle}_{v_2})$

c) The dimension of the space spanned by those vectors is 2, since it has 2 basis vectors.

d) * I already calculated the span in section "b"

e) Test if we can express w as a linear combination of the basis vectors

$c_1 \cdot v_4 + c_2 \cdot v_2 = w$

$$\left.\begin{aligned} c_1 + 0 \cdot c_2 &= 2 \\ 0 \cdot c_1 + c_2 &= 2 \\ c_1 + 3c_2 &= 0 \end{aligned}\right\} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 3 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-3)]{E_{3,1}(-1)} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{array}\right]$$

System is inconsistent, w is not in the span

Q11) let $x=t$ and $y=k$ then we can say that our plane consists of Vectors with the rule of:

$$V=\begin{bmatrix} t \\ k \\ 2+k-t \end{bmatrix} = t.\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

The basis for this plane and ONLY for this plane is:

$\{(1,0,-1),(0,1,1),(0,0,2)\}$

Q12) $\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & -1 & 0 \end{array}\right] - E_{2,1}(-2) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & -3 & 4 & -3 & 0 \end{array}\right] \rightarrow E_2(-1/3) - \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$

$E_{1,2}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -4/3 & 1 & 0 \end{array}\right]$ $\quad w+\frac{y}{3}=0 \quad x-\frac{4y}{3}+z=0$ let $x=k$, $y=3t$

$w=-t,\ x=k,\ y=3t,\ z=4t-k$

Vectors in this subspace can be expressed as

$$\begin{bmatrix} -t \\ k \\ 3t \\ 4t-k \end{bmatrix} = t.\underbrace{\begin{bmatrix} -1 \\ 0 \\ 3 \\ 4 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}}_{\text{Basis for the Subspace}}$$

Q13) $\frac{x-1}{3}=t \quad y+\frac{1}{3}=t \quad \frac{1-z}{3}=t, \quad \frac{2x-2}{3}=2t$

$\frac{2x-2}{3}-y-\frac{1}{3}+\frac{z}{3}-\frac{1}{3}=2t-t-t=0$

$\frac{2x}{3}-y+z=\frac{4}{3} \Rightarrow 2x-3y+3z=4$, let $x=3t$ and $y=2k \Rightarrow z=2k-2t+\frac{4}{3}$

The subspace spanned by this equation consist of vectors with the rule of:

$$V=\begin{bmatrix} 3t \\ 2k \\ 2k-2t+4/3 \end{bmatrix} = t.\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + k.\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4/3 \end{bmatrix}$$

The basis for this line and only this line is:

$\{(3,0,-2),(0,2,2),(0,0,4/3)\}$

Q14) let $p(x) = ax^2 + bx + d$

$c_1 . p_1(x) + c_2 . p_2(x) + c_3 p_3(x) + c_4 p_4 = p(x)$

$c_1 . x^2 + c_1 + c_2 . x^2 - 2c_2 x + 3c_2 + c_3 x - c_3 + 3c_4 x^2 + 2c_4 x + c_4 = ax^2 + bx + d$

$x^2(c_1 + c_2 + 3c_4) + x(-2c_2 + c_3 + 2c_4) + (c_1 + 3c_2 - c_3) = ax^2 + bx + d$

$$\left.\begin{aligned} c_1 + c_2 \quad + 3c_4 &= a \\ -2c_2 + c_3 + 2c_4 &= b \\ c_1 + 3c_2 - c_3 + c_4 &= d \end{aligned}\right\} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 1 & 3 & -1 & 1 \end{bmatrix} . \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ d \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 1 & 3 & -1 & 1 & d \end{array}\right] - E_{3,1}(-1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 2 & -1 & -2 & d-a \end{array}\right]$$

$$E_{2,3}(1) \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 3 & a \\ 0 & -2 & 1 & 2 & b \\ 0 & 0 & 0 & 0 & d-a+b \end{array}\right] \Big\} \; d - a + b = 0 \quad \text{let } a = k, b = t \Rightarrow d = k - t$$

These vectors span a subspace of P^2 with the rule of:

$$P(x) = \begin{bmatrix} k \\ t \\ k-t \end{bmatrix} = k . \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t . \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \text{Basis is} : \langle (1,0,1), (0,1,-1) \rangle$$

∇ Dimension of this subspace is 2 (2 vectors in basis)

∇ Since we have $2 < 3$ vectors in our basis, it does not span the space of second order polynomials.

Let $P_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, P_0 is not in the span of our vectors.

By the add-minus theorem, if we add this vector to our basis, the new basis will span all P^2 (3 basis vectors)

Q15) Column vectors of A are $= \Big\{ \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}}_{v_3} \Big\}$, let $v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$c_1 . v_1 + c_2 . v_2 + c_3 v_3 = v$

$$\left.\begin{aligned} 2c_1 - c_2 \quad &= a \\ c_1 + 2c_2 + 5c_3 &= b \\ c_1 + c_2 + 3c_3 &= c \end{aligned}\right\} \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix} . \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & a \\ 1 & 2 & 5 & b \\ 1 & 1 & 3 & c \end{array}\right] \xrightarrow[E_{2,3}(-1)]{E_{1,3}(-2)} \left[\begin{array}{ccc|c} 0 & -3 & -6 & a-2c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right] - E_{1,2}(2) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 0 & a+2b-4c \\ 0 & 1 & 2 & b-c \\ 1 & 1 & 3 & c \end{array}\right]$$

$a + 2b - 4c = 0$, let $a = 4t$, $b = 2k \Rightarrow c = t + k$

These vectors span a subspace of $\mathbb{R}^3$ with the rule:

$$V=\begin{bmatrix}4t\\2k\\t+k\end{bmatrix}=t\cdot\begin{bmatrix}4\\0\\1\end{bmatrix}+k\begin{bmatrix}0\\2\\1\end{bmatrix} \Bigg\} \text{Basis is } \left\{\begin{bmatrix}4\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\1\end{bmatrix}\right\}$$

∇ The dimension of the columnspace is 2

∇ For $Ax=b$ to be consistent for every b, A must be invertible

$$|A|=\begin{vmatrix}2&-1&0\\1&2&5\\1&1&3\end{vmatrix} \xrightarrow[E_{1,3}(-2)]{E_{2,3}(-1)} \begin{vmatrix}0&-3&-6\\0&1&2\\1&-1&3\end{vmatrix}=1\cdot\begin{vmatrix}-3&-6\\1&2\end{vmatrix}=0$$

Since $|A|=0$, A is not invertible. Therefore, $Ax=b$ is <u>not</u> consistent for every b

Q16)

a) columnvectors of A are : $\left\{\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{V_1},\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{V_2},\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{V_3}\right\}$, $b=\begin{bmatrix}-1\\1\\1\end{bmatrix}$

$c_1V_1+c_2V_2+c_3V_3=b$

$$\left.\begin{aligned}c_1-2c_2+c_3&=-1\\c_1+2c_2+2c_3&=1\\2c_1-2c_2+2c_3&=1\end{aligned}\right]\begin{bmatrix}1&-2&1\\1&2&2\\2&-1&2\end{bmatrix}\cdot\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\left[\begin{array}{ccc|c}1&-2&1&-1\\1&2&2&1\\2&-2&2&1\end{array}\right]\xrightarrow[E_{3,1}(-2)]{E_{2,1}(-1)}\left[\begin{array}{ccc|c}1&-2&1&-1\\0&4&1&2\\0&2&0&3\end{array}\right]\xrightarrow[E_{2,3}(-2)]{E_{1,3}(1)}\left[\begin{array}{ccc|c}1&0&1&2\\0&0&1&-4\\0&2&0&3\end{array}\right]$$

$$\xrightarrow[E_{1,2}(-1)]{E_3(1/2)}\left[\begin{array}{ccc|c}1&0&0&6\\0&0&1&-4\\0&1&0&3/2\end{array}\right]\Bigg\}\begin{aligned}c_1&=6\\c_2&=3/2\\c_3&=-4\end{aligned}\Rightarrow\underbrace{\begin{bmatrix}-1\\1\\1\end{bmatrix}}_{b}=6\cdot\underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{V_1}+\frac{3}{2}\cdot\underbrace{\begin{bmatrix}-2\\2\\-2\end{bmatrix}}_{V_2}-4\underbrace{\begin{bmatrix}1\\2\\2\end{bmatrix}}_{V_3}$$

b) $$\begin{bmatrix}1&-2&1\\1&2&2\\2&-1&2\end{bmatrix}\xrightarrow[\text{as in section "a"}]{\text{Same operations}}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}\rightarrow E_{2,3}\rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Rank = no. of pivots in row echolon form = 3

c) Nullity = col. no − rank = 3 − 3 = 0

d)

$\Rightarrow$ $Ax = 0$ } Find the homogeneous solution:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & -2 & 2 & 0 \end{array}\right] \xrightarrow{\text{Same operations as in section "a"}} \left.\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]\right\} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $Ax = b$ } Find the general solution

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & -1 \\ 1 & 2 & 2 & 1 \\ 2 & -2 & 2 & 1 \end{array}\right] \xrightarrow{\text{Same operations as in Section "a"}} \left.\left[\begin{array}{ccc|c} 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 1 & 0 & 3/2 \end{array}\right]\right\} X = \begin{bmatrix} 6 \\ 3/2 \\ -4 \end{bmatrix}$$

Q17)

a) $Ax = 0$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 2 & 4 & -2 & 2 & 0 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 4 & -4 & -4 & 0 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$E_1(1/2) \Rightarrow \left.\left[\begin{array}{cccc|c} 0 & 1 & -1 & -1 & 0 \\ 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\right\} \begin{array}{l} x_2 - x_3 - x_4 = 0 \\ x_1 + x_3 + 3x_4 = 0 \end{array} \Rightarrow \begin{array}{l} \text{let } x_3 = t \text{ and } x_4 = k \\ \star\ x_2 = t+k,\ x_1 = -t-3k \end{array}$$

The space spanned by these vectors is the nullspace

$$NS(A) = \begin{bmatrix} -t-3k \\ t+k \\ t \\ k \end{bmatrix} = t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

b) Nullity = Dimension of the nullspace = 2

c) Rank = No. of non-zero rows in reduced echelon form = 2

d) $\mathcal{A}x = b$

$$\begin{bmatrix} 2 & 2 & 0 & 4 \\ 1 & 0 & 1 & 3 \\ 2 & 4 & -2 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 2 & 2 & 0 & 4 & 5 \\ 1 & 0 & 1 & 3 & 3 \\ 2 & 4 & -2 & 2 & 4 \end{array}\right] \xrightarrow[E_{3,2}(-2)]{E_{1,2}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 4 & -4 & -4 & -2 \end{array}\right] \xrightarrow{E_{3,1}(-2)} \left[\begin{array}{cccc|c} 0 & 2 & -2 & -2 & -1 \\ 1 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{array}{l} 2x_2 - 2x_3 - 2x_4 = -1 \\ x_1 + x_3 + 3x_4 = 3 \end{array}\right\} \begin{array}{l} \text{Let } x_3 = t \text{ and } x_4 = k \\ x_1 = 3 - t - 3k \text{ and } x_2 = -1/2 + t + k \end{array}$$

$$\text{Solution vector } x = \begin{bmatrix} 3 - t - 3k \\ -1/2 + t + k \\ t \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} 3 \\ -1/2 \\ 0 \\ 0 \end{bmatrix} + \underbrace{t \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + k \cdot \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{Homogeneous solution}}}_{\text{General Solution}}$$

e) For $\mathcal{A}x = b$ to be consistent for every b, $\mathcal{A}$ must be invertible. Since $\mathcal{A}$ is not a square matrix, it is not invertible.

Q18)

$$\begin{bmatrix} 1 & 0 & 1 & -4 \\ 2 & 0 & 3 & -1 \\ 2 & 0 & 4 & 6 \end{bmatrix} \xrightarrow[E_{3,1}(-2)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 2 & 14 \end{bmatrix} \xrightarrow{E_{3,2}(-2)} \begin{array}{cc} \begin{bmatrix} 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{l} \to v_1 \\ \to v_2 \\ \end{array} \\ \begin{array}{cccc} \downarrow & & \downarrow & \downarrow \\ v_3 & & v_4 & v_5 \end{array} & \end{array}$$

Basis for row space = {(1,0,1,-4), (0,0,1,7)}

Basis for col. space = {(1,0,0), (1,1,0), (-4,7,0)}

Q19)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 0 & -2 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow[E_{3,1}(-1)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 2 & 5 \end{bmatrix} \xrightarrow{E_{2,3}(1)} \begin{array}{c} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{array} \end{array}$$

Basis for col. space = {(1,0,0,0), (2,0,-2,0), (3,0,-5,0)}

Q20)

$A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \xrightarrow{E_{2,1}(-1)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{2,1}(-1)$ can be expressed as $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = A'$

↳ Elementary matrix

A' can be expressed as $E.A$

① Find the col. space of A

$c_1 . \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 . \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \left.\begin{matrix} c_1 + c_2 = a \\ -c_1 - c_2 = b \end{matrix}\right\} \Rightarrow$ let $c_1 = t$, $c_2 = k$; $a = t+k$ / $b = -t-k$

Col. space of A consists from vectors with the rule: $\begin{bmatrix} t+k \\ -t-k \end{bmatrix} = t . \begin{bmatrix} 1 \\ -1 \end{bmatrix} + k . \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

② Find the col. space of $A' = E.A$

$c_1 . \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 . \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \quad \left.\begin{matrix} c_1 - c_2 = a \\ c_2 = b \end{matrix}\right\} \Rightarrow$ let $c_1 = t$, $c_2 = k$; $a = t-k$ / $b = k$

Col. space of A consists from vectors with the rule: $\begin{bmatrix} t-k \\ k \end{bmatrix} = t . \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k . \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

∇ The rules are different, so A and EA don't have the same columnspace.

Q21)

$A^T = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & -4 & 0 & 1 \\ 4 & 0 & -2 & 3 \end{bmatrix} \xrightarrow[E_{3,1}(-4)]{E_{2,1}(-2)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & -8 & -6 & 3 \end{bmatrix} \xrightarrow{E_{3,2}(-1)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -8 & -2 & 1 \\ 0 & 0 & -4 & 2 \end{bmatrix}$

$\xrightarrow[E_3(-1/4)]{E_2(-1/8)} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow{E_{1,2}(-2)} \begin{bmatrix} 1 & 0 & 1/2 & 1/4 \\ 0 & 1 & 1/4 & -1/8 \\ 0 & 0 & 1 & -1/2 \end{bmatrix} \xrightarrow[E_{1,3}(-1/2)]{E_{2,3}(-1/4)} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Lin. Ind. Col.

A basis for rowspace A is: $\{(1,2,4), (2,-4,0)\}$,