

2020-2021 Fall Semester

Linear Algebra & Applications

Solutions for Homework # 3

MAT281E Linear Algebra and Applications HW 3

Instructions: Scan or take photos of clearly handwritten solutions, combine them in one doc, docx or pdf file and submit by 16:00 December 14, 2020. Late homeworks are not accepted. 4-5 problems will be checked in detail which will contribute 80% to the final mark. The rest will be checked for completeness which will contribute 20% to the final mark.

Q1)

let $u \times v$ be equal to \vec{w} . \vec{w} is orthogonal to both \vec{u} and \vec{v}

$$\vec{u} \cdot \vec{w} = \|u\| \cdot \|w\| \cdot \cos 90^\circ = 0$$

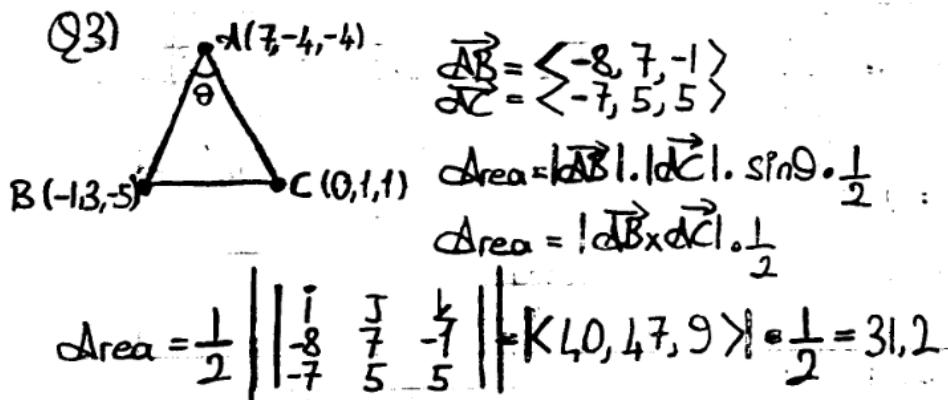
Q2)

$$u = \langle 1, 1, 1 \rangle \quad v = \langle 0, -2, 3 \rangle$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -2 & 3 \end{vmatrix} = \langle 1, -3, -2 \rangle$$

This vector is orthogonal
to both \vec{u} and \vec{v}

Q3)



$$Q4) \quad \begin{aligned} \vec{AB} &= \langle -8, 7, -1 \rangle \\ \vec{AC} &= \langle -7, 5, 5 \rangle \\ \vec{AD} &= \langle -7, 4, 4 \rangle \end{aligned} \quad \left. \begin{aligned} V &\neq 0, (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 56 \end{aligned} \right\} \quad \begin{aligned} &V \neq 0, (\vec{AB} \times \vec{AC}) \cdot \vec{AD} = 56 \\ &= 56 \end{aligned}$$

Since $V \neq 0$, these vectors are NOT coplanar

Q5)

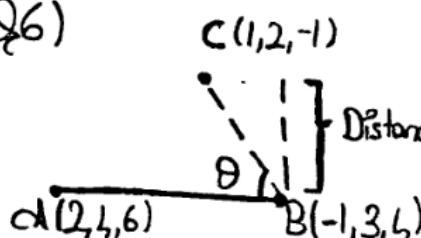
$$\vec{N} = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 2 & -1 & 3 \end{vmatrix} = \langle 1, 2, -2 \rangle \quad \left. \begin{aligned} &\text{Normal vector of the} \\ &\text{determined plane} \end{aligned} \right\}$$

$$\text{let } \vec{p} = \langle a, b, c \rangle, \vec{p} \perp \vec{N} \Rightarrow \vec{p} \cdot \vec{N} = 0, \vec{p} \perp \vec{w} \Rightarrow \vec{p} \cdot \vec{w} = 0$$

$$\begin{cases} 4a + 2b - 2c = 0 \\ 5a + 7b - 4c = 0 \end{cases} \quad \begin{cases} a = b \\ c = 3a \end{cases} \Rightarrow \vec{p}_1 = \langle a, a, 3a \rangle, \vec{p}_2 = \langle -a, -a, -3a \rangle$$

$$\text{Convert } \vec{p} \text{ into unit vector} \Rightarrow \vec{p}_1' = \left\langle \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{3}{\sqrt{11}} \right\rangle, \vec{p}_2' = \left\langle \frac{-1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{-3}{\sqrt{11}} \right\rangle$$

Q6)



$$\text{Distance} = |\vec{BC}| \cdot \sin \theta$$

$$\begin{aligned} \vec{AB} &= \langle -3, -1, -2 \rangle \Rightarrow |\vec{AB}| = \sqrt{14} \\ \vec{BC} &= \langle 2, -1, -5 \rangle \Rightarrow |\vec{BC}| = \sqrt{30} \end{aligned}$$

$$|\vec{AB} \times \vec{BC}| = |\vec{AB}| \cdot |\vec{BC}| \cdot \sin \theta$$

$$\vec{AB} \times \vec{BC} = \begin{vmatrix} i & j & k \\ -3 & -1 & -2 \\ 2 & -1 & -5 \end{vmatrix} = \langle 3, -19, 5 \rangle, |\vec{AB} \times \vec{BC}| = \sqrt{395}$$

$$\sqrt{395} = \sqrt{14} \cdot \sqrt{30} \cdot \sin \theta \Rightarrow \sin \theta = \sqrt{0.94}$$

$$\text{Distance} = |\vec{BC}| \cdot \sin \theta = \sqrt{30} \cdot \sqrt{0.94} = 5.31$$

Q7)

a) $N_1 = \langle 1, 2, -1 \rangle$ and $N_2 = \langle 2, -1, 1 \rangle$

$$\vec{N}_1 \cdot \vec{N}_2 = |N_1| \cdot |N_2| \cdot \cos \theta = \sqrt{6} \cdot \sqrt{6} \cdot \cos \theta = -1 \quad \left. \begin{array}{l} \cos \theta = -\frac{1}{6} \end{array} \right\}$$

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} = \langle 1, -3, -5 \rangle, \quad |N_1 \times N_2| = \sqrt{35} = |N_1| \cdot |N_2| \cdot \sin \theta$$

$$\sqrt{35} = \sqrt{6} \cdot \sqrt{6} \cdot \sin \theta \quad \left. \begin{array}{l} \sin \theta = \frac{\sqrt{35}}{6} \end{array} \right\}$$

! The planes are neither parallel nor perpendicular.

b) $N_1 = \langle 3, 1, -1 \rangle$ and $N_2 = \langle 1, 2, 5 \rangle$

$$\vec{N}_1 \cdot \vec{N}_2 = |N_1| \cdot |N_2| \cdot \cos \theta = \sqrt{11} \cdot \sqrt{30} \cdot \cos \theta = 0 \quad \left. \begin{array}{l} \cos \theta = 0 \\ \theta = 90^\circ \end{array} \right\}$$

! The planes are perpendicular.

c) $N_1 = \langle 1, 1, -3 \rangle$ and $N_2 = \langle 2, 2, -6 \rangle$

$$\vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -3 \\ 2 & 2 & -6 \end{vmatrix} = \langle 0, 0, 0 \rangle, \quad |\vec{N}_1 \times \vec{N}_2| = 0$$

$$|\vec{N}_1 \times \vec{N}_2| = |N_1| \cdot |N_2| \cdot \sin \theta = \sqrt{11} \cdot \sqrt{55} \cdot \sin \theta = 0 \quad \left. \begin{array}{l} \sin \theta = 0 \\ \theta = 0^\circ \end{array} \right\}$$

! The planes are parallel.

Q8)

$$P = (1, 0, -1) \text{ and } M = (x, y, z)$$

$$\vec{PM} = \langle x-1, y, z+1 \rangle \text{ and } \vec{n} \perp \vec{PM}$$

$$\vec{PM} \cdot \vec{n} = |\vec{PM}| \cdot |\vec{n}| \cos 90^\circ = 0$$

$$5(x-1) + 1(y) - 2(z+1) = 0 \quad \text{point normal form of the plane}$$

$$5x - 5 + y - 2z - 2 = 0 \Rightarrow 5x + y - 2z - 7 = 0 \quad \text{General eq. of the plane}$$

$$\vec{OP} = \langle 1, 0, -1 \rangle \text{ and } \vec{OM} = \langle x, y, z \rangle \Rightarrow \vec{OM} - \vec{OP} = \vec{PM} = \langle x-1, y, z+1 \rangle$$

$$\vec{n} \cdot (\vec{OM} - \vec{OP}) = 0 \Rightarrow \langle 5, 1, -2 \rangle \cdot \langle x-1, y, z+1 \rangle = 0 \quad \text{Vector eq. of the plane}$$

Q9) Find the position vector by calculating \vec{QP}

$$\vec{QP} = \langle 0, 2, 3 \rangle \rightarrow \text{Position vector} \quad \left. \begin{array}{l} \text{Find the} \\ \text{parametric} \end{array} \right\} \text{equation}$$

The point $P = (0, 2, 1)$ is on the plane

$$\left. \begin{array}{l} x = 0 + 0 \cdot t \\ y = 2 + 2 \cdot t \\ z = 1 + 3 \cdot t \end{array} \right\} \text{line eq. : } (0, 2+2t, 1+3t)$$

Q10)

let the plane equation be : $ax + by + cz - d = 0$

$$\underline{ax + by + cz - d = 0} \parallel \underline{x + 2y + 5z - 3 = 0}$$

$$\vec{N}_1 = \langle a, b, c \rangle$$

$$\vec{N}_2 = \langle 1, 2, 5 \rangle$$

Since these lines are parallel,

$$\frac{a}{1} = \frac{b}{2} = \frac{c}{5} = k \Rightarrow \text{plane: } kx + 2ky + 5kz - d = 0$$

$$P \text{ is on the plane} \Rightarrow 5k - 4k + 5k - d = 6k - d = 0$$

! With every $k \in \mathbb{R}$ we get a different plane

$$\text{For } k=1, d=6. \text{ and plane: } x + 2y + 5z - 6 = 0$$

Q11) Let the plane equation be : $ax + by + cz + d = 0$

Normal of the plane is $\vec{N} = \langle a, b, c \rangle$

$$\text{Line equation} \Rightarrow (1+3t, -1-2t, 3-t) = P_0 + \vec{V} \cdot t \quad \begin{cases} P_0 = (1, -1, 3) \\ \vec{V} = \langle 3, -2, -1 \rangle \end{cases}$$

$$\textcircled{1} \quad \vec{N} \perp \vec{V} \Rightarrow \vec{N} \cdot \vec{V} = 3a - 2b - c = 0$$

$$\textcircled{2} \quad P_0 \text{ is on the plane} \Rightarrow a - b + 3c + d = 0$$

The plane is orthogonal to the plane given by $x + y - z = 0$
those normal vector is $\vec{N}_1 = \langle 1, 1, -1 \rangle$

$$\textcircled{3} \quad \vec{N} \perp \vec{N}_1 = \vec{N} \cdot \vec{N}_1 = a + b - c = 0$$

Using the equations $\textcircled{1}$, $\textcircled{2}$ and $\textcircled{3}$:

$$2a = 3b, c = a + b$$

$$8b + d = 0$$

for $b = 2$:

$$a - b + 3(a + b) + d = 0$$

* Since we do not have enough data,
there is an infinite amount of planes that
can be written

The plane equation is:

$$4a + 2b + d = 0$$

$$3x + 2y + 5z - 16 = 0$$

Q12)

let the equation of the plane be $ax+by+cz+d=0$

Normal vector of the plane is $\vec{N} = \langle a, b, c \rangle$

$$\overrightarrow{P_2P_1} = \langle 1, 0, 4 \rangle, \overrightarrow{P_2P_1} \perp \vec{N} \Rightarrow \overrightarrow{P_2P_1} \cdot \vec{N} = 0$$

$$\textcircled{1} \quad a + 4c = 0$$

The plane is perpendicular to the plane $2x+2y+3z=1$ whose normal vector is $\vec{N}_1 = \langle 2, 2, 3 \rangle$

$$\vec{N} \perp \vec{N}_1 \Rightarrow \vec{N} \cdot \vec{N}_1 = 0$$

$$\textcircled{2} \quad 2a + 2b + 3c = 0$$

The points P_1 and P_2 are on the plane

$$\textcircled{3} \quad a + 2b + 3c + d = 0$$

$$\textcircled{4} \quad 2b - c + d = 0$$

Using $\textcircled{1}$ and $\textcircled{2} \Rightarrow 2b = 5c \dots \textcircled{5}$

Using $\textcircled{5}$ and $\textcircled{4} \Rightarrow 4c + d = 0$

Since we do not have enough data, there is an infinite amount of planes that can be written.

For $c = -2 \Rightarrow$ The plane eq: $8x - 5y - 2z + 8 = 0$

Q13)

$$\text{The line described } \Rightarrow (1-2t, 1+t, 5-2t) = P_0 + \vec{v} \cdot t \quad \begin{cases} P_0 = (1, 1, 5) \\ \vec{v} = \langle -2, 1, -2 \rangle \end{cases}$$

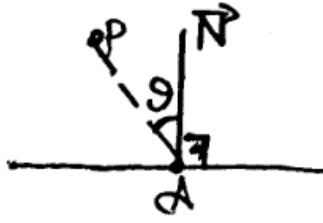
The plane given has the normal vector of $\vec{N} = \langle 1, 2, 0 \rangle$

$$\vec{v} \cdot \vec{N} = |\vec{v}| \cdot |\vec{N}| \cdot \cos \theta = 3\sqrt{3} \cdot \cos \theta = 0 \Rightarrow \cos \theta = 0 \quad \theta = 90^\circ$$

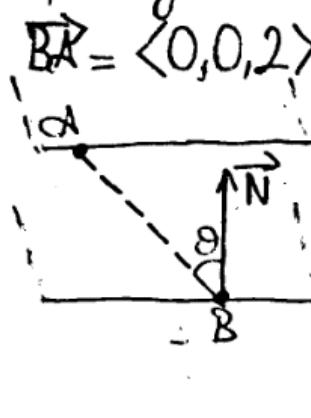
The line and the normal are \Rightarrow The line and the plane are perpendicular //

Q14) let $A = (0, 0, -3)$ be a point on the plane $\Rightarrow \vec{AP} = \langle 1, 3, 3 \rangle$

The normal vector of the plane is $\vec{N} = \langle 5, 4, -1 \rangle$


$$\vec{AP} \cdot \vec{N} = |\vec{AP}| \cdot |\vec{N}| \cdot \cos \theta = 14$$
$$\cos \theta = \frac{14}{\sqrt{13} \cdot \sqrt{42}}, \text{ distance is } |\vec{AP}| \cdot \cos \theta = 2,16$$

Q15) let $A = (0, 0, -3)$ and $B = (0, 0, -5)$ be points on the planes respectively.


$$\vec{BA} = \langle 0, 0, 2 \rangle \quad \text{and the normal vectors for the planes } \vec{N} = \langle 5, 4, -1 \rangle$$
$$\vec{BA} \cdot \vec{N} = |\vec{BA}| \cdot |\vec{N}| \cdot \cos \theta = -2 \quad \cos \theta = \frac{-2}{2 \cdot \sqrt{42}}$$
$$\text{distance} = |\vec{AB}| \cdot \cos \theta$$
$$= \left| 2 \cdot \frac{-2}{2 \cdot \sqrt{42}} \right|$$
$$= 0,31$$

Q16) let $P = (a, b, c)$

$$\begin{array}{l} \text{① } a - 4b - c = -3 \\ \text{② } 2a - 3b + 2c = 0 \end{array} \quad \left. \begin{array}{l} \text{1) for } c = -1 \\ \text{2) for } c = -6 \end{array} \right\} \begin{array}{l} P_1 = (4, 2, -1) \\ P_2 = (15, 6, -6) \end{array}$$

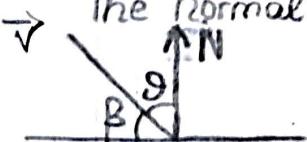
$$P_1 P_2 = \langle 11, 4, -5 \rangle \quad \text{Location vector}$$

The equations for the line are: $\begin{aligned} x &= 4 + 11t \\ y &= 2 + 4t \\ z &= -1 - 5t \end{aligned}$

Q17)

The location vector of the given line is $\vec{V} = \langle -2, 1, -2 \rangle$

The normal vector of the given plane is $\vec{N} = \langle 1, 1, -1 \rangle$



$$\theta + \beta = 90^\circ \quad \vec{V} \cdot \vec{N} = |\vec{V}| \cdot |\vec{N}| \cdot \cos \theta = 1 \Rightarrow \cos \theta = \frac{1}{\sqrt{3}}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 79^\circ \Rightarrow \beta \approx 90 - 79 = 11^\circ$$

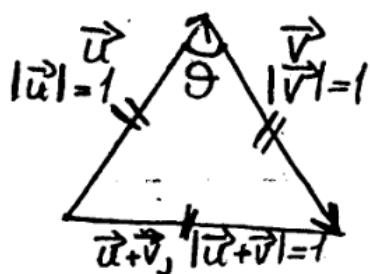
18) If the lines intersect, they must have a common point

$$\begin{aligned} 1-2t &= 3s \\ 1+2t &= 1-s \\ 5-2t &= 1+2s \end{aligned} \Rightarrow \begin{cases} 3s+2t=1 \\ s+2t=0 \\ 2s+2t=4 \end{cases} \text{ Solve for } s, t \quad \left[\begin{array}{cc|c} 3 & 2 & 1 \\ 1 & 2 & 0 \\ 2 & 2 & 4 \end{array} \right] \cdot \left[\begin{array}{c} s \\ t \\ 1 \end{array} \right] = \left[\begin{array}{c} 1 \\ 0 \\ 4 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 2 & 2 & 4 & 4 \end{array} \right] \xrightarrow[E_{2,1}(-3)]{} \left[\begin{array}{ccc|c} 0 & -4 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & -2 & 4 & 4 \end{array} \right] \xrightarrow[E_{3,2}(-2)]{} \left[\begin{array}{ccc|c} 0 & -4 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{array} \right]$$

$$E_{3,1}(4) \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & -7 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 \end{array} \right] \quad \begin{cases} \text{We got an inconsistent system} \\ \rightarrow \text{The lines don't intersect} \end{cases}$$

Q19)



When we add the two vectors, we get a triangle those sides are equal.

Therefore all of the interior angles of the triangle is θ , and since $3\theta = 180^\circ$, θ is 60°

$$|\vec{v} \times \vec{u}| = |\vec{v}| \cdot |\vec{u}| \cdot \sin \theta = 1 \cdot 1 \cdot \sin 60^\circ = \frac{\sqrt{3}}{2}$$

Q20) Gauss-Schwarz Inequality : $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \cdot \|\vec{v}\|$

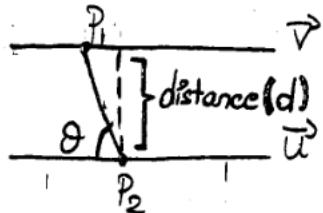
Verify the inequality using $\vec{u} = \langle 2, 0, -1, 1 \rangle$ and $\vec{v} = \langle 0, 2, 1, 1 \rangle$

$$|\vec{u} \cdot \vec{v}| = 2 \cdot 0 + 0 \cdot 2 + (-1) \cdot 1 + 1 \cdot 1 = 3 = \sqrt{9}$$

$$|\vec{u}| = \sqrt{21}, |\vec{v}| = \sqrt{6} \Rightarrow |\vec{u}| \cdot |\vec{v}| = \sqrt{126}$$

As shown above, $|\vec{u} \cdot \vec{v}| = \sqrt{9} < \sqrt{126} = |\vec{u}| \cdot |\vec{v}|$

- Q21) location vector for line $\ell = \vec{v} = \langle -2, 3, -2 \rangle$ and $P_1 = (1, 1, 5)$
 location vector for line $2 = \vec{u} = \langle 3, -1, 2 \rangle$ and $P_2 = (0, 1, 1)$



$$\overrightarrow{P_2P_1} = \langle 1, 0, 4 \rangle, \vec{u} \cdot \overrightarrow{P_2P_1} = |\vec{u}| \cdot |\overrightarrow{P_2P_1}| \cos \theta$$

$$\sqrt{17} \cdot \sqrt{17} \cdot \cos \theta = 11 \Rightarrow \cos \theta = \frac{11}{\sqrt{17} \cdot \sqrt{17}} \Rightarrow \sin \theta = 0.7$$

$$d = |\overrightarrow{P_2P_1}| \sin \theta = 2.89,$$

- Q22) * I have assumed you meant to write:

$$c_1(1, 0, 2, 0) + c_2(1, -2, -2, 0) + c_3(0, 1, 1, -2) = (2, -1, 3, 1)$$

$$\begin{array}{l} c_1 + c_2 = 2 \\ -2c_2 + c_3 = -1 \\ 2c_1 - 2c_2 + c_3 = 3 \\ -2c_3 = 1 \end{array} \left. \begin{array}{c} \\ \\ \\ \end{array} \right\} \left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 2 & -2 & 1 & 3 \\ 0 & 0 & -2 & 1 \end{array} \right] \xrightarrow{E_1(-\frac{1}{2})} \left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

$$\begin{array}{l} E_3(\frac{1}{2}) \\ E_{4,2}(-1) \end{array} \rightarrow \left[\begin{array}{cccc} 1 & 1 & 0 & 2 \\ 0 & -2 & 0 & 1 - \frac{1}{2} \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \xrightarrow{E_{3,1}(-1)} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right] \xrightarrow{E_{4,2}(-1)} \left[\begin{array}{cccc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \end{array} \right]$$

The system is inconsistent, therefore there are no c_1, c_2, c_3 that meet the requirements.

Q23)

The x variables differ between $x_1 \dots x_3$ } Domain is \mathbb{R}^3

The w results differ between $w_1 \dots w_3$ } Codomain is \mathbb{R}^3

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & -1 \\ 5 & 3 & -3 \end{bmatrix}}_{\text{Standard Matrix}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

To find the range, find the constraint on w_i when they are in reduced echelon form.

$$\begin{bmatrix} 2 & 3 & -1 & | & w_1 \\ 1 & -3 & -1 & | & w_2 \\ 5 & 3 & -3 & | & w_3 \end{bmatrix} \xrightarrow{E_{2,1}(-2)} \begin{bmatrix} 0 & 9 & 1 & | & w_1 - 2w_2 \\ 1 & -3 & -1 & | & w_2 \\ 0 & 18 & 2 & | & w_3 - 5w_2 \end{bmatrix} \xrightarrow{E_{3,1}-2} \begin{bmatrix} 0 & 9 & 1 & | & w_1 - 2w_2 \\ 1 & -3 & -1 & | & w_2 \\ 0 & 0 & 0 & | & w_3 - 11w_2 \end{bmatrix}$$

$$E(1/9) \rightarrow \begin{bmatrix} 0 & 1 & 1/9 & | & w_1 - 2w_2/9 \\ 1 & -3 & -1 & | & w_2 \\ 0 & 0 & 0 & | & w_3 - w_2 - 2w_1 \end{bmatrix} \xrightarrow{E_{1,2}(3)} \begin{bmatrix} 0 & 1 & 1/9 & | & (w_1 - 2w_2)/9 \\ 0 & 0 & -2/3 & | & (w_1 + w_2)/3 \\ 0 & 0 & 0 & | & w_3 - w_2 - 2w_1 \end{bmatrix}$$

$$E_{1,2} \rightarrow \begin{bmatrix} 1 & 0 & -2/3 & | & (w_1 + w_2)/9 \\ 0 & 1 & 1/9 & | & (w_1 - 2w_2)/3 \\ 0 & 0 & 0 & | & w_3 - w_2 - 2w_1 \end{bmatrix} \quad \left. \begin{array}{l} w_3 - w_2 - 2w_1 = 0 \\ \hline \end{array} \right\} \text{Constraint}$$

Q24)

a) Standard matrix for the equation is : $\begin{bmatrix} 1 & 0 & 8 \\ 0 & -1 & 8 \\ 0 & 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & -1 & 8 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \quad \left. \begin{array}{l} \text{Result is } (2, -1, 3) \\ \hline \end{array} \right\}$$

b) Standard matrix for the equation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} \right] \text{ Result is } (1, 0, 5)$$

c) Standard matrix for the operation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right] \text{ Result is } (1, 0, 0)$$

d) The standard matrix for the operation is : $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix}$

$$\left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3} - \frac{3}{2} \\ -1 + \frac{3\sqrt{3}}{2} \end{bmatrix} \right] \text{ Result: } \left(1, -\sqrt{3} - \frac{3}{2}, -1 + \frac{3\sqrt{3}}{2} \right)$$

e)

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 \\ 0 & \sin 30 & \cos 30 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}}_{\text{Reflection or. } x=z \quad \text{Dilation by } 2 \quad \text{Rotation by } 30^\circ \text{ around x axis}}$$

Standard matrix for the operation = $\begin{bmatrix} 0 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & -1 \\ 2 & 0 & 0 \end{bmatrix}$

The result : $\begin{bmatrix} 0 & 1 & \sqrt{3} \\ 0 & \sqrt{3} & -1 \\ 2 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3\sqrt{3}-2 \\ -3-2\sqrt{3} \\ 2 \end{bmatrix}$

f)

$$= \underbrace{\begin{bmatrix} \cos 90 & 0 & \sin 90 \\ 0 & 1 & 0 \\ -\sin 90 & 0 & \cos 90 \end{bmatrix}}_{\text{Rotation around } y \text{ axis by } 90^\circ} \cdot \underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 8 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}}_{\text{Contr. with factor 2}} \cdot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Orthogonal pr. onto } x\text{-axis}} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Rotation around y
axis by 90°

Contr. with
factor 2

Orthogonal pr.
onto x-axis

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \rightarrow \text{Standard matrix for the operation is} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \end{bmatrix}$$

Q25)

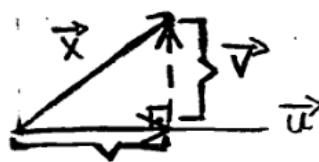
$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_1 \cdot T_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } T_2 \cdot T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

! $T_1 \cdot T_2$ is firstly projects to x-z plane, then projects to x-y plane
 $T_2 \cdot T_1$ is firstly projects to x-y plane, then projects to x-z plane

Since those two operations give the same results, $T_1 \cdot T_2 = T_2 \cdot T_1$.

Q26)



$$\vec{v} \perp \vec{u} \text{ and } T(x) \parallel \vec{u} \Rightarrow \vec{v} \perp T(x) \quad ①$$

$$T(x) + \vec{v} = x, \text{ proj}_u x + \vec{v} = x$$

$$v = x - \text{proj}_u x, \text{ and according to } ①$$

$$(x - \text{proj}_u x) \perp \text{proj}_u x$$

Q27)

a) $\left[\begin{array}{ccc} 1 & 3 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{array} \right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \right]$

for the $T(x): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be one-to-one,
must be invertible.
Therefore $|A| \neq 0$

$$|A| = \left| \begin{array}{ccc} 1 & 3 & -1 \\ 2 & -3 & -1 \\ 5 & 6 & -4 \end{array} \right| \xrightarrow[E_{1,2}(-2)]{E_{1,3}(-5)} \left| \begin{array}{ccc} 1 & 3 & -1 \\ 0 & -9 & -1 \\ 0 & -9 & 1 \end{array} \right| \xrightarrow[E_{2,3}(-1)]{} \left| \begin{array}{ccc} 1 & 3 & -1 \\ 0 & -9 & -1 \\ 0 & 0 & 0 \end{array} \right| = 0,$$

A is not invertible. Therefore, the transformation is not one-to-one.

a₂) Find the range

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 2 & -3 & -1 & w_2 \\ 5 & 6 & -4 & w_3 \end{array} \right] \xrightarrow{E_{1,2}(-2)} \left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 0 & -9 & -1 & w_2 \\ 5 & 6 & -4 & w_3 \end{array} \right] \xrightarrow{E_{1,3}(-5)} \left[\begin{array}{ccc|c} 1 & 3 & -1 & w_1 \\ 0 & -9 & 1 & w_2 - 2w_1 \\ 0 & 0 & 0 & w_3 - 3w_1 - w_2 \end{array} \right]$$

For the system to be consistent, $w_3 - w_2 - 3w_1$ must be 0

The constraint for the range is $w_1 - w_2 - 3w_1 = 0$

There is not an inverse transformation for this transformation

b) $\left[\begin{array}{ccc} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -3 \end{array} \right] \cdot \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} \right] = \left[\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \right]$ } For the $T(x) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ to be one-to-one
 A must be invertible
 Therefore $|A| \neq 0$

$$|A| = \left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -3 \end{array} \right| \xrightarrow{\text{E}_{1,2(+1)}, \text{E}_{1,3(-2)}} \left| \begin{array}{ccc} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -5 \end{array} \right| = 1 \cdot \left| \begin{array}{cc} -2 & -2 \\ -2 & -5 \end{array} \right| = 6 \neq 0$$

A is invertible, therefore the transformation is one-to-one
 Range of the transformation is \mathbb{R}^3

b2) To find the inverse transformation, first find A^{-1}

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{6} \cdot \left[\begin{array}{ccc} 3 & 3 & 0 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{array} \right] = \left[\begin{array}{ccc} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{array} \right]$$

Therefore the inverse transformation $T_{A^{-1}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is:

$$w_1 = \frac{x_1}{2} + \frac{x_2}{2} \quad w_2 = \frac{x_1}{6} - \frac{5x_2}{6} + \frac{x_3}{3} \quad w_3 = \frac{x_1}{3} + \frac{x_2}{3} - \frac{x_3}{3}$$

Q28) We can note the given transformation as:

$$\left. \begin{array}{l} w_1 = 2x_1 + 3x_2 \\ w_2 = 1 \end{array} \right\}$$

We can not define a matrix A where
 $A \cdot x = w$, so the transformation is
 not linear

w_2 doesn't get effected
 as x_1, x_2 changes

Q29) let the base vectors be: $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

① Rotation around z axis by 45°

$$e_1' = \begin{bmatrix} \cos 45^\circ \\ \sin 45^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad e_2' = \begin{bmatrix} -\sin 45^\circ \\ \cos 45^\circ \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} \quad e_3' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

② Orthogonal projection onto the x-axis

$$e_1'' = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix} \quad e_2'' = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ 0 \\ 0 \end{bmatrix} \quad e_3'' = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

To find the standard matrix, write the e_i'' together

$$\text{St. Matrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q30) Let the standard matrices for the transformations be A, B respectively

$T_1 \rightarrow$ one-to-one $\Rightarrow |A| \neq 0$] Let the st. matrix for T_1, T_2 be C
 $T_2 \rightarrow$ Not one-to-one $\Rightarrow |B| = 0$] Let the st. matrix for T_2, T_1 be D

$|C| = |A \cdot B| = |A| \cdot |B| = 0 \Rightarrow T_1, T_2$ is not one-to-one

$|D| = |B \cdot A| = |B| \cdot |A| = 0 \Rightarrow T_2, T_1$ is not one-to-one

Q31)

a) $V: \mathbb{R}^2 \rightarrow \{(2x, x)\}$ } Test the axioms one by one

✓ 1- let $u = (2a, a)$ and $v = (2b, b) \Rightarrow u+v = (2(a+b), (a+b)) \in V$

✓ 2- let $u = (2a, a), v = (2b, b)$ and $w = (2c, c)$

$$\begin{aligned} u + (v+w) &= (2a, a) + (2(b+c), (b+c)) = (2(a+b+c), (a+b+c)) \\ (u+v)+w &= (2(a+b), (a+b)) + (2c, c) = (2(a+b+c), (a+b+c)) \end{aligned} \quad] \text{equal}$$

✓ 3- $u+v = (2(a+b), a+b), v+u = (2(b+a), b+a) \quad] \text{equal}$

✓ 4- $u = (2a, a)$ and $0 = (0, 0) \in V \Rightarrow u+0 = 0+u = (2a, a) = u$

✓ 5- $u = (2a, a)$ and $-u = (-2a, -a) \in V \Rightarrow u+(-u) = (0, 0) = 0$

✓ 6- $u = (2a, a)$ and $ku = (2ak, ak) \in V$

$$\begin{aligned} u+v &= (2(a+b), a+b), k(u+v) = (2k(a+b), k(a+b)) \\ k \cdot u &= (2ka, ka), k \cdot v = (2kb, kb), ku + kv = (2k(a+b), k(a+b)) \end{aligned} \quad] \text{equal}$$

✓ 8- $(k+m)u = (2(k+m)a, (k+m)a), ku = (2ka, ka), mu = (2ma, ma)$

$$ku + mu = (2(k+m)a, (k+m)a) = (k+m)u$$

✓ 9- $k(mu) = k(2ma, ma) = (2kma, kma) \quad] \text{equal}$

$$m(ku) = m(2ka, ka) = (2mka, mka) \quad] \text{equal}$$

✓ 10- $1 \cdot u = (1 \cdot 2a, 1 \cdot a) = (2a, a) = u$

! Axioms are true for every $u \in V \Rightarrow V$ is a vector space

b) $V: \mathbb{R}^2 \rightarrow \{(2x+1, x)\}$ } Test axioms one by one

X 1- $u = (2a+1, a), v = (2b+1, b) \Rightarrow u+v = (2(a+b+1), (a+b)) \notin V$

! $2(a+b+1) = 2a+2b+2 \neq 2(a+b)+1$. Since the axiom is not true for V , it is not a vector space

c) $V: \mathbb{R}^2 \rightarrow \{(x,y), x \geq 0\}$ Test axioms one by one

$\checkmark 1 - u = (a,b), v = (d,e)$ and $a \geq 0, d \geq 0$

$$u+v = (a+d, b+e) \in V$$

$\checkmark 2 - u+v = (a+d, b+e)$ and $v+u = (d+a, e+b)$ } equal

$\checkmark 3 - u = (a,b), v = (d,e), w = (f,g)$ and $a \geq 0, d \geq 0, f \geq 0$

$$\begin{aligned} u + (v+w) &= (a,b) + (d+f, e+g) = (a+d+f, b+e+g) \\ (u+v)+w &= (a+d, b+e) + (f, g) = (a+d+f, b+e+g) \end{aligned} \quad \left. \begin{array}{l} \text{equal} \\ \text{equal} \end{array} \right\}$$

$\checkmark 4 - u = (a,b)$ and $0 = (0,0) \in V$

$$u+0 = 0+u = (a,b) = u$$

$\times 5 - u = (a,b), -u = (-a,-b) \notin V$ (if $a \neq 0$)

∇ for any $a > 0$ the axiom is not met.

V is not a vector space

d) $V: \mathbb{R}^2 \rightarrow \{(1,y)\}$ Test axioms one by one

$\times 1 - u = (1,b), v = (1,e)$

$$u+v = (2, b+e) \notin V$$

∇ Since $2 \neq 1$ the axiom is not met

V is not a vector space

Q32) $V: M_{2x2} \rightarrow \begin{bmatrix} x & 2x \\ 3x & 4x \end{bmatrix}, x \in \mathbb{R} \}$ Test the axioms

$\checkmark 1 - u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}, v = \begin{bmatrix} b & 2b \\ 3b & 4b \end{bmatrix} \text{ and } u+v = \begin{bmatrix} (a+b) & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} \in V$

$\checkmark 2 - u+v = \begin{bmatrix} (a+b) & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix}, v+u = \begin{bmatrix} (b+a) & 2(b+a) \\ 3(b+a) & 4(b+a) \end{bmatrix} \} \text{ equal}$

$\checkmark 3 - u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix}, v = \begin{bmatrix} b & 2b \\ 3b & 4b \end{bmatrix}, w = \begin{bmatrix} c & 2c \\ 3c & 4c \end{bmatrix}$

$$u+(v+w) = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} b+c & 2(b+c) \\ 3(b+c) & 4(b+c) \end{bmatrix} = \begin{bmatrix} a+b+c & 2(a+b+c) \\ 3(a+b+c) & 4(a+b+c) \end{bmatrix} \} \text{ equal}$$

$$(u+v)+w = \begin{bmatrix} a+b & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} + \begin{bmatrix} c & 2c \\ 3c & 4c \end{bmatrix} = \begin{bmatrix} a+b+c & 2(a+b+c) \\ 3(a+b+c) & 4(a+b+c) \end{bmatrix} \} \text{ equal}$$

$\checkmark 4 - 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V \text{ and } u+0=0+u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = u$

$\checkmark 5 - u = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} \text{ and } -u = \begin{bmatrix} -a & -2a \\ -3a & -4a \end{bmatrix} \in V \quad u+(-u) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\checkmark 6 - k \cdot u = \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} \in V$

$\checkmark 7 - k \cdot (u+v) = k \cdot \begin{bmatrix} a+b & 2(a+b) \\ 3(a+b) & 4(a+b) \end{bmatrix} = \begin{bmatrix} k(a+b) & 2k(a+b) \\ 3k(a+b) & 4k(a+b) \end{bmatrix} \} \text{ equal}$

$$ku + kv = \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} + \begin{bmatrix} kb & 2kb \\ 3kb & 4kb \end{bmatrix} = \begin{bmatrix} k(a+b) & 2k(a+b) \\ 3k(a+b) & 4k(a+b) \end{bmatrix} \} \text{ equal}$$

$\checkmark 8 - (k+m) \cdot u = (k+m) \cdot \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = \begin{bmatrix} (k+m)a & 2(k+m)a \\ 3(k+m)a & 4(k+m)a \end{bmatrix} \} \text{ equal}$

$$ku + mu = \begin{bmatrix} ka & 2ka \\ 3ka & 4ka \end{bmatrix} + \begin{bmatrix} ma & 2ma \\ 3ma & 4ma \end{bmatrix} = \begin{bmatrix} (k+m)a & 2(k+m)a \\ 3(k+m)a & 4(k+m)a \end{bmatrix} \} \text{ equal}$$

$\checkmark 9 - k(m \cdot u) = k \cdot \begin{bmatrix} ma & 2ma \\ 3ma & 4ma \end{bmatrix} = \begin{bmatrix} kma & 2kma \\ 3kma & 4kma \end{bmatrix} \} \text{ equal}$

$$(km) \cdot (u) = \begin{bmatrix} kma & 2kma \\ 3kma & 4kma \end{bmatrix} \} \text{ equal}$$

$\checkmark 10 - 1 \cdot u = 1 \cdot \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3a & 4a \end{bmatrix} = u$

∴ All axioms are met for every $u \in V$

V is a vector space

Q33) Test the axioms

✓ 1- $u = a, v = b$ and $a, b \in \mathbb{R} \Rightarrow u+v = \sqrt{a^2+b^2} \in V$

✓ 2- $u+v = \sqrt{a^2+b^2}, v+u = \sqrt{b^2+a^2} \Rightarrow u+v = v+u$

✓ 3- $u=a, v=b, w=c \Rightarrow u+(v+w) = \boxed{\frac{a^2 + (\sqrt{b^2+c^2})^2}{(u+v)+w}} = \boxed{\frac{a^2+b^2+c^2}{\sqrt{(a^2+b^2)^2+c^2}}} \quad \left. \begin{array}{l} \text{equal} \\ (u+v)+w = \sqrt{(a^2+b^2)^2+c^2} = \sqrt{a^2+b^2+c^2} \end{array} \right.$

✗ 4- $\vec{0} = 0 \in V$ and $u+0 = \sqrt{a^2+0^2} = \sqrt{a^2} = |a| \neq a$

The axiom is not met when $a < 0$

Ex: $u = -1 \Rightarrow u+0 = \sqrt{(-1)^2+0^2} = \sqrt{1} = 1 \neq -1$

∴ Since all axioms are not met the set is not a vector space

Q34) a) $W = M_{n \times n}, |M| = 0$

let $u = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}_{n \times n}$ and $v = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n \times n}$

$|u| = 1, 0, \dots, 0 = 0$ and $|v| = 0, 1, \dots, 1 = 0$

$u+v = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}_{n \times n}$ and $|u+v| = 1, \dots, 1 = 1 \neq 0$.

Since $u+v \notin W$, W is not a subspace

b) $W = M_{n \times n}, |W| \neq 0$

let $u = \begin{bmatrix} u_{11} & \dots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nn} \end{bmatrix}$ and $|u| \neq 0$

let $k = 0$

$$k \cdot u = 0, \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix} = \vec{0}$$

$$|k \cdot u| = 0, 0 \dots 0 \cdot 0 = 0$$

Since $|k \cdot u| \notin W, W$ is not a subspace

Q35)

a) $W = M_{n \times m}, a_{11} = a_{12} = \dots = a_{1m} = 0$

let $u = \begin{bmatrix} 0 & \dots & 0 \\ u_{21} & \dots & u_{2n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \dots & u_{mn} \end{bmatrix}$ and $v = \begin{bmatrix} 0 & \dots & 0 \\ v_{11} & \dots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \dots & v_{mn} \end{bmatrix}$

Since W is closed under addition and mult, W is a subspace

$$u+v = \begin{bmatrix} 0 & \dots & 0 \\ u_{21}+v_{21} & \dots & u_{2n}+v_{2n} \\ \vdots & \ddots & \vdots \\ u_{m1}+v_{m1} & \dots & u_{mn}+v_{mn} \end{bmatrix} \in W$$

$$k \cdot u = k \cdot \begin{bmatrix} 0 & \dots & 0 \\ u_{21} & \dots & u_{2n} \\ \vdots & \ddots & \vdots \\ u_{m1} & \dots & u_{mn} \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 \\ k \cdot u_{21} & \dots & k \cdot u_{2n} \\ \vdots & \ddots & \vdots \\ k \cdot u_{m1} & \dots & k \cdot u_{mn} \end{bmatrix} \in W$$

b) $W = H_{n \times m} : a_{11} \dots a_{1m} \neq 0$

$$\text{let } \vec{u} = \begin{bmatrix} 1 & \dots & 1 \\ u_{21} & \dots & u_{2m} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nm} \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} -1 & \dots & -1 \\ v_{21} & \dots & v_{2m} \\ \vdots & \ddots & \vdots \\ v_{n1} & \dots & v_{nm} \end{bmatrix}$$

Test the axioms:

$$u + v = \begin{bmatrix} 0 & \dots & 0 \\ u_{21} + v_{21} & \dots & u_{2m} + v_{2m} \\ \vdots & \ddots & \vdots \\ u_{n1} + v_{n1} & \dots & u_{nm} + v_{nm} \end{bmatrix} \notin W \quad \left\{ \begin{array}{l} \text{Since } W \text{ is not} \\ \text{closed under} \\ \text{addition, it is} \\ \text{not a subspace} \end{array} \right.$$

Q36)

$$a) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & \frac{1}{3} \\ 1 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 2 & 1 & \frac{1}{3} & | & 0 \\ 1 & -1 & 3 & | & 0 \end{bmatrix} \xrightarrow[E_{1,2}(-2)]{E_{1,3}(-1)} \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & -1 & 3 & | & 0 \end{bmatrix} \xrightarrow[E_{2,3}(1)]{} \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 13 & | & 0 \end{bmatrix}$$

$$E_3(\frac{1}{13}) \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \xrightarrow[E_{3,2}(-8)]{E_{3,1}(2)} \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \left\{ \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{array} \right.$$

Solution Space for the given part is $W_{(x_1, x_2, x_3)} : x_1 = x_2 = x_3 = 0$

$$W = \langle (0, 0, 0) \rangle$$

$$b) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 2 & 1 & 4 & | & 0 \end{bmatrix} - E_{1,2(-2)} \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 8 & | & 0 \end{bmatrix}$$

$$\left. \begin{array}{l} x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \\ x_2 + 8x_3 = 0 \Rightarrow x_2 = -8x_3 \end{array} \right\} \text{let } x_3 = t \Rightarrow \begin{array}{l} x_1 = 2t \\ x_2 = -8t \\ x_3 = t \end{array} \quad \begin{array}{l} \text{Infinite} \\ \text{solutions} \\ \text{for } t \end{array}$$

Solution space for the given part is $W(x_1, x_2, x_3) : x_1 = 2t, x_2 = -8t, x_3 = t, t \in \mathbb{R}$

$$\text{Ex: for } t=1, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$$

$$c) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \\ 0 & 1 & 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 2 & 1 & 4 & | & 0 \\ 0 & 1 & 8 & | & 0 \end{bmatrix} - E_{1,2(-2)} \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 8 & | & 0 \\ 0 & 1 & 8 & | & 0 \end{bmatrix} - E_{2,3(-1)} \rightarrow \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 8 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\left. \begin{array}{l} x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3 \\ x_2 + 8x_3 = 0 \Rightarrow x_2 = -8x_3 \end{array} \right\} \text{let } x_3 = t \Rightarrow \begin{array}{l} x_1 = 2t \\ x_2 = -8t \\ x_3 = t \end{array} \quad \begin{array}{l} \text{Infinite} \\ \text{solutions} \\ \text{for } t \end{array}$$

Solution space for the given part is $W(x_1, x_2, x_3) : x_1 = 2t, x_2 = -8t, x_3 = t, t \in \mathbb{R}$

$$\text{Ex: for } t=2, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -16 \\ 2 \end{bmatrix}$$

$$d) \begin{bmatrix} 1 & 0 & -2 \\ 2 & 8 & -2 \\ -1 & 8 & 2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 2 & 8 & -2 & | & 0 \\ -1 & 8 & 2 & | & 0 \end{bmatrix} \xrightarrow[E_{1,2}(1-2)]{E_{1,3}(1)} \begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 8 & 8 & | & 0 \\ 0 & 8 & 0 & | & 0 \end{bmatrix} \quad \text{Infinite solutions}$$

$$x_1 - 2x_3 = 0 \Rightarrow x_1 = 2x_3$$

The solution space for the given part is $W_{(x_1, x_2, x_3)} : x_1 = 2x_3$

$$\text{Ex: for } x_3 = 1 \text{ and } x_3 = 2, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$