Half-spherical twists on derived categories of coherent sheaves

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1 Introduction

1.1 Mirror symmetry for singular fibers of type I_n of elliptic surfaces

Let $\pi\colon S \to C$ be a relatively minimal, smooth projective elliptic surface. The possible singular fibers of π are classified by Kodaira and Néron. Among them, *the singular fiber of type* I_n is the cyclic configuration of n smooth rational curves.

Lekili and Polishchuk [LP17] established mirror symmetry between the type I_n singular fiber Y_n and the n-punctured torus T_n, i.e. they showed that the derived category $D^b(Y_n)$ of coherent sheaves on Y_n and the wrapped Fukaya category $D^\pi(\mathcal{W}(T_n))$ of T_n are equivalent.

For example, the equivalence includes the following correspondence of objects, where G_i's are irreducible components and γ_F denotes the corresponding curve to the object $F \in D^b(Y_n)$:

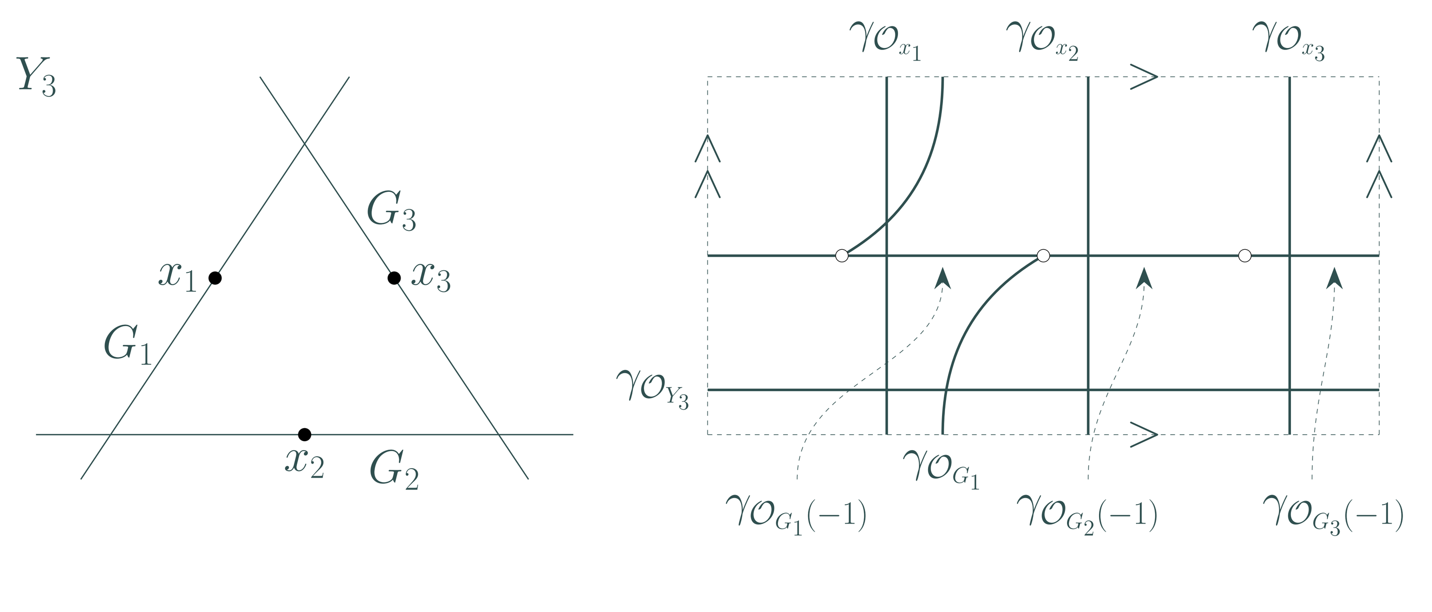

Figure 1: The correspondence of objects via $D^b(Y_n) \simeq D^\pi(\mathcal{W}(T_n))$, $n = 3$.

Combining this with the theory of topological Fukaya categories of surfaces developed by Haiden, Katzarkov, and Kontsevich [HKK17], Opper [Opp20] described the autoequivalence group of $D^b(Y_n)$ with the following exact sequence:

$$1 \to (\mathbb{C}^\times)^n \times \mathbb{Z}[1] \times \mathrm{Pic}^0(Y_n) \to \mathrm{Auteq}\, D^b(Y_n) \xrightarrow{\Upsilon} \mathrm{MCG}(T_n) \to 1.$$

Here $\mathrm{MCG}(T_n)$ denotes the mapping class group of T_n and the morphism Υ is induced by the equivalence $D^b(Y_n) \simeq D^\pi(\mathcal{W}(T_n))$.

1.2 Autoequivalences of elliptic surfaces

Uehara [Ueh16] gave the following description of the autoequivalence group $\mathrm{Auteq}\, D^b(S)$. This result implies that the study of the structure of $\mathrm{Auteq}\, D^b(S)$ reduces to that of B.

Theorem 1.1.

- *S has non-zero Kodaira dimension*
- *all singular fibers of π are non-multiple and of type $\mathrm{I_n}$, $n \geq 2$*
- *$B = \langle T_{\mathcal{O}_G(a)} \mid G \subset S$: an irreducible component of a singular fiber, $a \in \mathbb{Z}\rangle$: the subgroup of $\mathrm{Auteq}\, D^b(S)$ generated by twist functors $T_{\mathcal{O}_G(a)}$, where $\mathcal{O}_G(a)$ is the line bundle of degree a on $G \simeq \mathbb{P}^1$*

Then there is the exact sequence

$$1 \to \langle B, (-) \otimes \mathcal{O}_S(D) \mid D.F = 0, F \textit{ is a fiber} \rangle \rtimes \mathrm{Aut}\, S \times \mathbb{Z}[2] \to \mathrm{Auteq}\, D^b(S) \to \mathrm{SL}(2, \mathbb{Z}).$$

1.3 Main results

We study the group B in terms of the mapping class group of the n-punctured torus so that we reveal the whole structure of $\mathrm{Auteq}\, D^b(S)$.

(1) We construct a natural "restriction" morphism $\mathrm{res}\colon B \to \mathrm{Auteq}\, D^b(F)$ for each fiber F, which is nontrivial if F is a reducible fiber.

(2) Combining with mirror symmetry for the singular fibers of type I_n, we have the exact sequences

$$1 \to \langle (-) \otimes \mathcal{O}_S(Y_{n_j}) \mid 1 \leq j \leq m \rangle \to B \xrightarrow{\Upsilon \circ \mathrm{res}} \prod_{j=1}^{m} \mathrm{MCG}(T_{n_j}),$$

where Y_{n_j} is the singular fiber of type I_{n_j} and $\{Y_{n_j}\}_{j=1}^m$ is the set of all singular fibers of S.

(3) For $G \subset Y_n$, $a \in \mathbb{Z}$, and the curve $\gamma_{\mathcal{O}_G(a)}$ on T_n corresponding to $\mathcal{O}_G(a)$ under the equivalence $D^b(Y_n) \simeq D^\pi(\mathcal{W}(T_n))$, the twist functor $T_{\mathcal{O}_G(a)} \in B$ is mapped to *the half twist along* $\gamma_{\mathcal{O}_G(a)}$ in $\mathrm{MCG}(T_n) \subset \prod_{j=1}^m \mathrm{MCG}(T_{n_j})$.

(4) The image of $\Upsilon \circ \mathrm{res}$ is generated by the half twists along the finite number of curves $\{\gamma_{\mathcal{O}_G}, \gamma_{\mathcal{O}_G(-1)} \mid G \subset Y_{n_j}$: an irreducible component, $1 \leq j \leq m\}$.

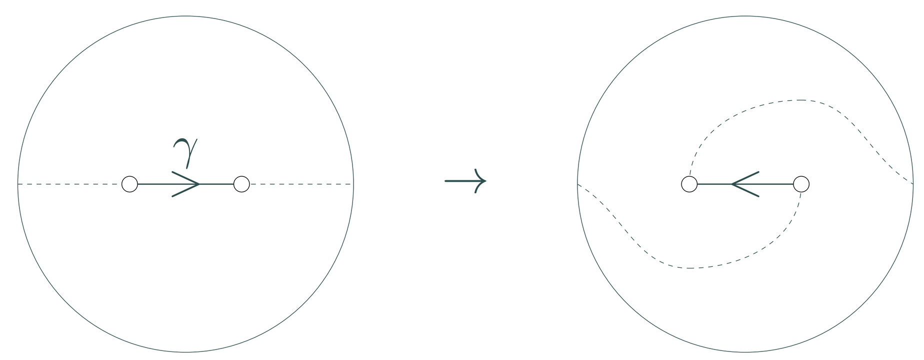

Figure 2: The half twist along the curve γ.

2 Sketch of the proof

We provide a sketch of the proofs of the result (1) and (3).

2.1 Result (1)

We prepare the following theorem to prove the result (1). This is a generalization of the relation between twist functors and $\mathbb{P}$-twists by Huybrechts and Thomas.

Theorem 2.1.

- *$\pi\colon X \to T$: a flat morphism between smooth quasi-projective varieties*
- *$i\colon Y = \pi^{-1}(0) \hookrightarrow X$: the fiber at $0 \in T$*
- *$E \in D^b(Y)$: an object such that $i_* E \in D^b(X)$ is a spherical object*
- *$T_{i_*E} \in \mathrm{Auteq}\, D^b(X)$: the corresponding twist functor*

Then there is a unique $H_E \in \mathrm{Auteq}\, D^b(Y)$ which makes the following diagram commutative:

$$\begin{tikzcd} D^b(Y) \arrow[r, "i_*"] \arrow[d, "H_E"'] & D^b(X) \arrow[d, "T_{i_*E}"] \\ D^b(Y) \arrow[r, "i_*"] & D^b(X). \end{tikzcd} \tag{1}$$

Proof of the result (1). The uniqueness and the construction of H_E ensure that we have a well-defined morphism $\mathrm{res}\colon B \to \mathrm{Auteq}\, D^b(F)$ such that $\mathrm{res}(T_{\mathcal{O}_G(a)}) = H_{\mathcal{O}_G(a)}$ if $G \subset F$ and $\mathrm{res}(T_{\mathcal{O}_G(a)}) = \mathrm{id}$ otherwise. □

2.2 Result (3)

For the proof of the result (3), we use the following three facts.

- The group $\pi_1(T_n)$ is generated by the curves $\gamma_{\mathcal{O}_{Y_n}}, \gamma_{\mathcal{O}_{x_1}}, \ldots, \gamma_{\mathcal{O}_{x_n}}$ (see Figure 1).
- An element of $\mathrm{MCG}(T_n)$ is determined by its action on $\pi_1(T_n)$ (Dehn–Nielsen–Baer theorem).
- [HKK17, Opp20] There are correspondences between indecomposable objects of $D^b(Y_n)$ and homotopy classes of curves on T_n (+ some data), and between dimensions of Hom-spaces in $D^b(Y_n)$ and intersection numbers of curves on T_n.

Proof of the result (3). For simplicity, we assume that $x_1 \in G$ and $a = 0$. We denote the image under the morphism $\Upsilon \circ \mathrm{res}$ of the twist functor $T_{\mathcal{O}_G}$ by H, and the half twist along $\gamma_{\mathcal{O}_G}$ by H'.

Given the first and second facts, we only need to check the identities $H(\gamma_{\mathcal{O}_{Y_n}}) = H'(\gamma_{\mathcal{O}_{Y_n}}), H(\gamma_{\mathcal{O}_{x_1}}) = H'(\gamma_{\mathcal{O}_{x_1}}), \ldots, H(\gamma_{\mathcal{O}_{x_n}}) = H'(\gamma_{\mathcal{O}_{x_n}})$ of curves.

Next, due to the construction of Υ and res, the curve $H(\gamma_{\mathcal{O}_{Y_n}})$ on T_n (resp. $H(\gamma_{\mathcal{O}_{x_i}})$) corresponds to the object $H_{\mathcal{O}_G}(\mathcal{O}_{Y_n}) \in D^b(Y_n)$ (resp. $H_{\mathcal{O}_G}(\mathcal{O}_{x_i})$) under the third fact.

Finally, these correspondences and the third fact enable us to compute some intersection numbers between $H(\gamma_{\mathcal{O}_{Y_n}})$ (or $H(\gamma_{\mathcal{O}_{x_i}})$) and other curves using basic computations in homological algebra. We can collect enough intersection numbers to determine the curves $H(\gamma_{\mathcal{O}_{Y_n}})$ and $H(\gamma_{\mathcal{O}_{x_i}})$, and this finishes the proof. □

References

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