Half-spherical twists on derived categories of coherent sheaves

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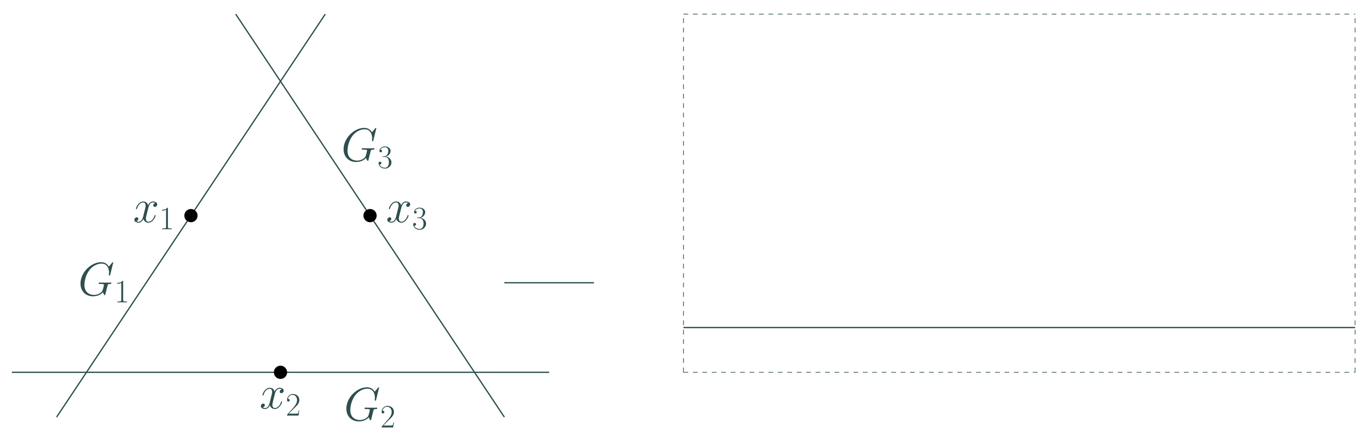
1 Introduction

1.1 Mirror symmetry for singular fibers of type I_n of elliptic surfaces

Let $\pi\colon S \to C$ be a relatively minimal, smooth projective elliptic surface. The possible singular fibers of π are classified by Kodaira and Néron. Among them, the singular fiber of type I_n is the cyclic configuration of n smooth rational curves.

Lekili and Polishchuk [LP17] established mirror symmetry between type I_n singular fiber Y_n and the n-punctured torus T_n, i.e. they showed that the derived category $D^b(Y_n)$ of coherent sheaves on Y_n and the wrapped Fukaya category $D^\pi(\mathcal{W}(T_n))$ of T_n are equivalent.

For example, the equivalence includes the following correspondence of objects:



Combining this with the theory of topological Fukaya categories of surfaces by Haiden, Katzarkov, and Kontsevich [HKK17], Opper [Opp20] described the autoequivalence group of $D^b(Y_n)$ with the following exact sequence:

$$1 \to (\mathbb{C}^\times)^n \times \mathbb{Z}[1] \times \mathrm{Pic}^0(Y_n) \to \mathrm{Auteq}\, D^b(Y_n) \xrightarrow{\Upsilon} \mathrm{MCG}(T_n) \to 1. \quad (1)$$

Here $\mathrm{MCG}(T_n)$ denotes the mapping class group of T_n and the morphism Υ is induced by the equivalence $D^b(C_n) \simeq D^\pi(\mathcal{W}(T_n))$.

1.2 Autoequivalences of elliptic surfaces

Uehara [Ueh16] gave the following description of the autoequivalence group $\mathrm{Auteq}\, D^b(S)$ of $D^b(S)$:

Theorem 1.1.

- *S has non-zero Kodaira dimension*
- *all singular fibers of π are non-multiple and of type I_n, $n \geq 2$*
- *$B = \langle T_{\mathcal{O}_G(a)} \mid G \subset S$: an irreducible component of a singular fiber, $a \in \mathbb{Z}\rangle$: the subgroup of $\mathrm{Auteq}\, D^b(S)$ generated by twist functors $T_{\mathcal{O}_G(a)}$, where $\mathcal{O}_G(a)$ is the line bundle of degree a on $G \simeq \mathbb{P}^1$*

Then there is the exact sequence

$$1 \to \langle B, (-) \otimes \mathcal{O}_S(D) \mid D.F = 0, F \text{ is a fiber } \rangle \rtimes \mathrm{Aut}\, S \times \mathbb{Z}[2] \quad (2)$$
$$\to \mathrm{Auteq}\, D^b(S) \to \mathrm{SL}(2, \mathbb{Z}). \quad (3)$$

1.3 Main results

(1) In a general setting, we construct a "restriction" morphism $B \to \mathrm{Auteq}\, D^b(F)$ for each fiber F, which is nontrivial if F is reducible.

(2) Combining with mirror symmetry for the singular fibers of type $\mathbf{I}_n$, we described the group B in terms of the mapping class group of the n-punctured torus. Then there exists the exact sequence

$$1 \to \langle (-) \otimes \mathcal{O}_S(Y_{n_j}) \mid j = 1, \ldots, m \rangle \to B \xrightarrow{r} \prod_{j=1}^{m} \mathrm{MCG}(T_{n_j}), \quad (4)$$

where Y_{n_j} is the singular fiber of type I_{n_j}.

(3) For $G \subset Y_{n_j}$, $a \in \mathbb{Z}$, and the curve $\gamma_{\mathcal{O}_G(a)}$ on T_{n_j} corresponding to $\mathcal{O}_G(a)$ under the equivalence $D^b(C_n) \simeq D^\pi(\mathcal{W}(T_n))$, the twist functor $T_{\mathcal{O}_G(a)}$ is mapped to the half twist along $\gamma_{\mathcal{O}_G(a)}$.

(4) The image of r is generated by the half twists along the finite number of curves $\{\gamma_{\mathcal{O}_G}, \gamma_{\mathcal{O}_G(-1)} \mid G \subset Y_{n_j}$: an irreducible component, $1 \leq j \leq m\}$.

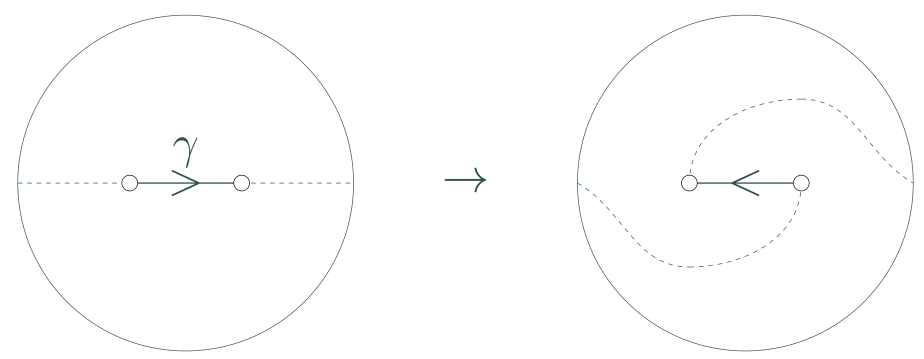


Figure 1: The half twist along the curve γ.

2 Sketch of the proof

2.1 Result (1)

The result (1) is a consequence of the following theorem which generalizes the result of [HT06].

Theorem 2.1.

- *$\pi\colon X \to T$: a flat morphism between smooth quasi-projective varieties*
- *$i\colon Y = \pi^{-1}(0) \hookrightarrow X$: the fiber at $0 \in T$*
- *$E \in D^b(Y)$: an object such that $i_*E \in D^b(X)$ is a spherical object*
- *$T_{i_*E} \in \mathrm{Auteq}\, D^b(X)$: the corresponding twist functor*

Then there is a unique $H_E \in \mathrm{Auteq}\, D^b(Y)$ which makes the following diagram commutative:

$$\begin{tikzcd} D^b(Y) \arrow[r, "i_*"] \arrow[d, "H_E"'] & D^b(X) \arrow[d, "T_{i_*E}"] \\ D^b(Y) \arrow[r, "i_*"] & D^b(X). \end{tikzcd} \quad (5)$$

2.2 Result (3)

(a) There are correspondences between indecomposable objects of $D^b(Y_n)$ and homotopy classes of curves on T_n (+ additional data), and between dimensions of Hom-spaces in $D^b(Y_n)$ and intersection numbers of curves on T_n.

(b) An element of $\mathrm{MCG}(T_n)$ is determined by its action on $\pi_1(T_n)$ (Dehn–Nielsen–Baer theorem).

References

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