# Complex Analysis: Important Results

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*MATH 324: Introduction to Complex Analysis*

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**Definition 1.** *(Multiplication in $\mathbb{C}$)* Let $z$, $w \in \mathbb{C}$. The product **zw** is the complex number whose polar coordinates are $r = |z||w|$ and $\theta = \alpha + \beta$, where $\alpha$ and $\beta$ are the polar angles of $z$ and $w$. Equivalently, if $z = x + iy$ and $w = p + iq$, then $zw = xp - yq + i(xq + yp)$.

**Definition 2.** Let $z = x + iy \in \mathbb{C}$. The **modulus**, or **absolute value**, of $z$ is denoted $|z|$, and is defined as follows:

$$|z| = \sqrt{x^2 + y^2}.$$

**Definition 3.** Let $z = x + iy \in \mathbb{C}$. The **complex conjugate** of z is denoted $\overline{z}$, and is defined as follows:

$$\overline{z} = x - iy.$$

**Definition 4.** Let $z \in \mathbb{C}$, $z \neq 0$. The polar angle of $z$ is called the **argument**, and denoted $\arg(z)$. The value of $\arg(z)$ in the range $[-\pi, \pi)$ is called the **principle value**, and is denoted Arg(z).

**Definition 5.** *(Circles)* The circle with center $c \in \mathbb{R}$ and radius $r \in \mathbb{R}$, $r > 0$ is parameterized by the equation $|z - c| = r$, $z \in \mathbb{C}$.

**Definition 6.** *(Lines)* Consider a line $L$. Let $c \in \mathbb{C}$ represent a point on the line, and let $d \in \mathbb{C}, d \neq 0$ represent the direction of the line. The **parametric equation** of $L$ is written as $z(t) = ct + d$, for $t \in \mathbb{R}$. The **intrinsic equation** of $L$ is written as $\Re[(z - c)i\overline{d}] = 0$. For a (nonvertical) straight line $y = mx + b$, $m$ and $b$ real, the equation can be formulated as $\Re[(m + i)z + b] = 0$. Moreover, for $Px + Qy = K$, we get $\Re[z(P - iQ)] = K$.

**Definition 7.** Let $p \in \mathbb{C}$. The **translation** $T_p$ is defined as $T_p(z) = z + p$.

**Definition 8.** Let $u = \cos\theta + i \sin\theta$. The **rotation** $R_\theta$ is defined as $R_\theta(z) = u \cdot z$ (counterclockwise rotation about the origin by angle $\theta$).

**Definition 9.** Let $z \in \mathbb{C}$, $a \in \mathbb{R}$, $a > 0$. The **scaling** $S_a$ is defined as $S_a(z) = a \cdot z$.

**Definition 10.** Let $z \in \mathbb{C}$, $b \in \mathbb{C}$. The **multiplication** operator $M_b$ is defined as $M_b(z) = b \cdot z$.

**Definition 11.** Let $z \in \mathbb{C}$. The **inversion** mapping, $J$, is defined as $J(z) = \frac{1}{z}$.

**Definition 12.** Let $p \in \mathbb{C}$ and $r > 0$. The **open ball** of radius $r$ about $p$ is the set $\{z \mid |z - p| < r\}$.

**Definition 13.** If $p \in S \subset \mathbb{C}$, then $p$ is an **interior point** of $S$ if $S$ contains some open ball about $p$.

**Definition 14.** $S \subset \mathbb{C}$ is an **open set** if every point of $S$ is an interior point of $S$.

**Definition 15.** $S \subset \mathbb{C}$ is **closed** if its complement $\mathbb{C} \setminus S$ is open.

**Definition 16.** $p \in \mathbb{C}$ is a **boundary point** of $S$ (denoted $p \in \partial S$) if it is neither interior to $S$ nor to its complement.

**Definition 17.** $S \subset \mathbb{C}$ is a **neighborhood** of $p \in \mathbb{C}$ if $p$ is an interior point of $S$.

**Definition 18.** $N$ is a **neighborhood of infinity** if $\{0\} \cup \{\frac{1}{z} \mid z \in N\}$.

**Definition 19.** *(Connectedness and Domains)* Consider an open subset $S \subset \mathbb{C}$.

a. $S \subset \mathbb{C}$ is **disconnected** if it is the disjoint union of two nonempty open sets.

b. $S \subset \mathbb{C}$ is **connected** if it is not disconnected.

c. $S$ is **polygonally connected** if for any $z$, $w \in S$ there is a finite sequence of points $p_0$, $p_1$, $...p_n$ with $z = p_0$, $w = p_n$, such that $S$ contains the line segment $p_{j-1}p_j$ for each $1 \leq j \leq n$.

**Definition 20.** A **domain** is an open, connected subset of $\mathbb{C}$.

**Definition 21.** A subset $S \subset \mathbb{C}$ is **convex** if for any $z, w \in S$, the line segment from $z$ to $w$ is in $S$.

**Definition 22.** A set $K$ is **compact** if, whenever there is a collection of open sets whose union contains $K$, then there is a finite sub-collection of the open sets that contains $K$.

**Definition 23.** A **function** $f : X \to Y$ is a rule assigning a single point $f(x) \in Y$ to each point $x \in X$.

**Definition 24.** The **image**, or **range**, of $f$ is the set $\{f(x) \mid x \in X\} \subset Y$.

**Definition 25.** $\lim_{z \to p} f(z) = L$ means that $\forall \epsilon > 0$, $\exists \delta > 0$ such that $0 < |z - p| < \delta \implies |f(z) - L| < \epsilon$.

**Definition 26.** $\lim_{z \to \infty} f(z) = L$ means that $\forall \epsilon > 0$, $\exists r > 0$ such that $|z| > r \implies |f(z) - L| < \epsilon$.

**Definition 27.** A function $f$ is **continuous** at $p \in \mathbb{C}$ if $f$ is defined on a neighborhood of $p$ and if $\lim_{z \to p} f(z) = f(p)$.

**Definition 28.** If $f : X \to Y$ and $S \subset Y$, then the **pre-image** of $S$ under $f$, denoted $f^{-1}(S)$, is the set of all $x \in X$ such that $f(x) \in S$.

**Definition 29.** A **sequence** $f(n) = a(n) : \mathbb{N} \to \mathbb{C}$ converges to $L$ if $\lim_{n\to\infty} f(n) = L$.

**Definition 30.** A **series** in $\mathbb{C}$ is a sequence $s_n = \sum_{j=1}^{n} a_j$ of partial sums of another sequence $a_n$.

**Definition 31.** *(Exponential)* For $z \in \mathbb{C}$, $e^z = e^{x+iy} = e^x e^{iy} = e^x[\cos(y) + i\sin(y)]$.

**Definition 32.** *(Logarithm)* The **logarithm** function is a set-valued function defined on $\mathbb{C}\backslash\{0\}$ that sends a point $w$ to the set of all of its pre-images under the exponential function: $\log(w) = \ln|w| + i\ \arg(w)$.

**Definition 33.** Let $D = \mathbb{C} \setminus \{tc \mid t \geq 0\}$ for fixed $c \neq 0$. Let $\theta = \operatorname{Arg}(z)$. There exists a single-valued **branch** of the logarithm continuous on all of $D$, defined as $\ln|z| + ia(z)$, where $a(z)$ is the value of $\arg(z)$ lying in the range $[\theta, \theta + 2\pi)$.

**Definition 34.** The **principle value** of the logarithm is the single-valued function $\operatorname{Log}(w) = \ln|w| + i\ \operatorname{Arg}(w)$.

**Definition 35.** *(Arbitrary Powers)* For $a,\ z \in \mathbb{C}\backslash\{0\}$, $a^z$ is defined to be the multi-valued function $e^{z\log(a)}$.

**Definition 36.** *(Hyperbolic Functions)* The **hyperbolic cosine** is defined as $\cosh(z) = \frac{e^z+e^{-z}}{2}$. The **hyperbolic sine** is defined as $\sinh(z) = \frac{e^z-e^{-z}}{2}$.

**Definition 37.** *(Sines and Cosines)* For $z \in \mathbb{C}$, $\cos(z) = \frac{e^{iz}+e^{-iz}}{2}$, and $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$

**Definition 38.** A **curve** is a vector-valued function of a single variable; it is **smooth** if it is continuously differentiable. It is **piecewise smooth** if it is continuous and its domain, which is an interval, is a union of finite number of subintervals, and the curve is smooth on each of these.

**Definition 39.** A piecewise smooth simple closed curve is **positively oriented** if the inside of the curve is immediately to the left as you proceed around the curve in the given orientation.

**Definition 40.** If $\gamma : [a, b] \to \mathbb{C}$ is a continuous curve in $\mathbb{C}$, then the *reversed curve*, denoted $-\gamma$, is another curve defined as:

$$(-\gamma)(s) = \gamma(a + b - s).$$

**Definition 41.** *(Concatenation of Curves)* If $\alpha : [a, b] \to \mathbb{C}$ and $\beta : [b, c] \to \mathbb{C}$ are curves with $\alpha(b) = \beta(b)$, the curve $\alpha + \beta : [a, c] \to \mathbb{C}$ is defined by:

$$(\alpha + \beta)(t) = \begin{cases} \alpha(t) & \text{if } a \leq t \leq b \\ \beta(t) & \text{if } b \leq t \leq c \end{cases}$$

**Definition 42.** A **VH chain** is a piecewise smooth curve whose pieces are alternately horizontal and vertical.

**Definition 43.** *(Analytic Functions)* Suppose $D$ is a domain, $p \in D$, and $f : D \to \mathbb{C}$. Then $f$ is **differentiable** at $p$, with **derivative** $f'(p) \in \mathbb{C}$ if:

$$\lim_{z \to p} \frac{f(z) - f(p)}{z - p} = f'(p) \quad \left( = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \right)$$

and this limit exists. If $f$ is differentiable at every $p \in D$, then $f$ is said to be **analytic** on $D$. If $f$ is analytic on all of $\mathbb{C}$, then $f$ is called **entire**.

**Definition 44.** Suppose $U$ is an open subset of $\mathbb{C}$ and $w : U \to \mathbb{R}$. Then $w$ is **harmonic** on $U$ if $\forall z \in U, w_{xx}(z) + w_{yy}(z) = 0$.

**Definition 45.** If $D$ is a domain in $\mathbb{C}$, then two real-valued functions $u$ and $w$ on $D$ with continuous second order partials are **harmonic conjugates** if either is harmonic and they satisfy the Cauchy-Riemann equations.

**Definition 46.** A **power series** in $z \in \mathbb{C}$ is $\sum_{n=0}^{\infty} a_n(z-p)^n$, where $a_n \in \mathbb{C}$. The **radius of convergence** $0 \le r < \infty$ of the power series is such that the series converges absolutely at any $z$ with $|z - p| < r$, and it diverges at $z$ if $|z - p| > r$.

**Definition 47.** A domain $D$ is called **simply connected** if $D$ contains the inside of every simple closed curve in $D$.

**Definition 48.** A **graph** $G$ in $\mathbb{C}$ is a finite collection of points, called the **vertices** of $G$ along with a finite collection of simple curves, called the **edges** of $G$, with the restrictions that both ends of any edge are among the vertices, and that any intersection of two edges occurs at an end of each.

**Definition 49.** A graph $G$ is a **directed graph** if each of its edges is oriented such that it has an **initial vertex** and a **terminal vertex**.

**Definition 50.** The **parity** of a vertex $v$ is the difference between the number of edges ending at $v$ and the number of edges beginning at $v$. A graph $G$ is called a **parity-0 graph** if the parity of every vertex in $G$ is 0.

**Definition 51.** An ordered collection of edges $(E_i)$ of a graph $G$ is a **semisimple closed edge-path** if the $E_j$ are all distinct, the end of $E_j$ is the start of $E_{j+1}$, and the end of $E_n$ is the start of $E_1$. A semisimple closed edge-path is **simple** if the only times that the end of an edge and the start of an edge are the same are those listed above. Removing the edges of a semisimple closed edge path from a graph preserves the parity of the graph.

**Definition 52.** If $X$ is a nonempty subset of $\mathbb{C}$, the **diameter** of $X$ is the supremum of $|z - w|$ for all pairs of points $z, w$ of $X$ (this may be infinite).

**Definition 53.** Suppose $D$ is a domain and $f : D \to \mathbb{C}$ is analytic, but not constant. Let $p \in D$ and $f(p) = 0$. We say that $p$ is a **zero of order k** of $f$ if $k$ is the smallest natural number $n$ such that $f^{(n)}(p) \neq 0$.

**Definition 54.** We call $p$ a **singularity** of a function $f$ if $f$ is not differentiable at $p$. This can happen for two reasons:

a. $f$ is not even defined at $p$

b. $f$ is defined at $p$, but the limit defining the derivative does not exist.

**Definition 55.** $p$ is an **isolated singularity** of a function $f$ if $p$ is a singularity, yet there is some $r > 0$ such that $f$ is analytic on $B_r(p) \backslash \{p\}$ (a **"punctured disk"**). Here, there are three possibilities:

a. $|f(z)| \to \infty$ as $z \to p$. In this case we say that $f$ has a **pole** at p.

b. There is an $L > 0$ such that $|f(z)| < L$ for all $z \neq p$ near p. In this case we say that $f$ has a **removable singularity** at $p$.

c. Neither (a) nor (b); in this case we say that $f$ has an **essential singularity** at $p$.

**Definition 56.** A series of the form $\sum a_n(z-p)^n$ where we allow both positive and negative values of $n$ is called a **Laurent series**.

**Definition 57.** Let $p$ be an isolated singularity of $f$, and let $\gamma$ be a small positively oriented circle centered at $p$. The **residue** of $f$ at $p$ is defined to be

$$\text{Res}(f;p) = \frac{1}{2\pi i}\int_\gamma f(z)dz.$$

**Definition 58.** A rational function $f(z) = \frac{P(z)}{Q(z)}$ is called a **simple rational function**, if the polynomials $P$ and $Q$ satisfy the following 3 conditions:

a. degree of $P <$ degree of $Q$.

b. $P$ and $Q$ have no roots in common.

c. $Q$ has no repeated roots.

**Theorem 1.** *(Properties of Addition and Multiplication)* Let $z$, $w$, $r \in \mathbb{C}$. Then the following hold:

- $z + w = w + z$
- $zw = wz$
- $z + (w + r) = (z + w) + r$
- $z(wr) = (zw)r$
- $r(z + w) = rz + rw$

**Theorem 2.** *(Properties of Conjugation)* Let $z$, $w \in \mathbb{C}$. Let $\arg(z) = \alpha$, and $\arg(\overline{z}) = \alpha'$. Then the following hold:

- $\overline{\overline{z}} = z$
- $\alpha' = -\alpha$
- $z \cdot \overline{z} = |z|^2 \geq 0$
- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$

**Theorem 3.** *(Powers and Roots)* Let $z \in \mathbb{C}$. Then the following hold:

- $|z^n| = |z|^n$
- $\arg(z^n) = n \cdot \arg(z)$

If, additionally, $z$ is an $n^{th}$ root of $w \in \mathbb{C}$, then

- $|z| = \sqrt[n]{|w|}$
- $n \cdot \arg(z) = \arg(w)$
- $\alpha \in \arg(w) \implies \frac{\alpha}{n} + \frac{2\pi k}{n} \in \arg(z)\ \forall k \in \mathbb{Z}$

**Corollary 1.** Any nonzero complex number has exactly $n$ different $n^{th}$ roots.

**Theorem 4.** *(Polar Representation)* Let $z = x + iy \in \mathbb{C}$. Let $\alpha = \arg(z)$. The **polar representation** of $z$ is

$$z = |z|(\cos \alpha + i \sin \alpha).$$

**Theorem 5.** *(De Moivre)* For all $t \in \mathbb{R}$, $n \in \mathbb{N}$,

$$[\cos(t) + i \sin(t)]^n = \cos(nt) + i \sin(nt).$$

**Theorem 6.** Let $z$, $w \in \mathbb{C}$. $z$ is perpendicular to $w$ if and only if $\Re(\overline{z}w) = 0$.

**Theorem 7.** Let $p \in \mathbb{C}$, $a \in \mathbb{R}$, $a > 0$, $u = \cos\theta + i \sin\theta$. Then, $T_p$, $R_\theta$, $S_a$, and $M_p$ preserve angles, map circles to circles and lines to lines, and are invertible. Their inverses are, respectively, $T_{-p}$, $R_{-\theta}$, $S_{\frac{1}{a}}$, and $M_{\frac{1}{p}}$.

**Corollary 2.** Let $b \in \mathbb{C}$, $a = |b|$, $\theta \in \arg(b)$. Then, $M_b = S_a \circ R_\theta$.

**Theorem 8.** The inversion mapping $J$ is invertible, and is its own inverse. However, it need not map circles to circles, or lines to lines.

**Lemma 1.** *(Inversion)* The following holds $\forall z \in \mathbb{C}$:

- $J(M_W(z)) = M_{\frac{1}{w}}(J(z))$
- $J(R_\theta(z)) = R_{-\theta}(J(z))$
- $J(S_a(z)) = S_{\frac{1}{a}}(J(z))$
- $\overline{J(z)} = J(\overline{z})$

**Theorem 9.** The inversion mapping $J$ takes any circle not passing through zero to another circle not passing through zero.

**Theorem 10.** *(More About Circles)* If $b \neq 0$ is in $\mathbb{C}$ and $r > 0$, but $r \neq 1$, then:

a. The set $\{z \in \mathbb{C} \mid r|z| = |z - b|\}$ is a circle.

b. The center of the circle is $\frac{b}{1-r^2}$ and its radius is $\frac{r|b|}{|1-r^2|}$.

c. 0 and b are on opposite sides of this circle.

**Corollary 3.** If $p \neq q$ and $0 < r < 1$, then $r|z-p| = |z-q|$ is a circle. Additionally:

a. Its center is $\frac{q-pr^2}{1-r^2}$.

b. Its radius is $\frac{r|q-p|}{1-r^2}$.

c. $p$ and $q$ are on opposite sides of the circle.

**Corollary 4.** In Corollary 3, if $q > 0$ and $p = -q$, then:

a. The center is at $v = q\frac{1+r^2}{1-r^2} > q > 0$.

b. The radius is $\frac{2r|q|}{1-r^2}$.

c. $q$ is inside the circle, and $-q$ is outside.

**Theorem 11.** *(Open and Closed Sets)*

a. $S$ and $\mathbb{C} \setminus S$ have the same boundary.

b. $S$ is open $\iff$ $S \cap \partial S = \varnothing$.

c. $S$ is closed $\iff$ $\partial S \subset S$.

**Theorem 12.** *(Compactness)* Subsets of $\mathbb{R}^n$ or $\mathbb{C}$ are compact if and only if they are closed and bounded.

**Theorem 13.** $f(z) \to L \iff \Re[f(z)] \to \Re[L]$ and $\Im[f(z)] \to \Im[L]$.

**Theorem 14.** *(Limits and Arithmetic)* Suppose that as $z \to p$ (or $z \to \infty$), both $f(z) \to L$ and $g(z) \to M$. Then:

a. $f(z) + g(z) \to L + M$

b. $f(z)g(z) \to LM$

c. As long as $M \neq 0$, $\frac{f(z)}{g(z)} \to \frac{L}{M}$.

**Theorem 15.** A function $f$ is continuous at $p$ if and only if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $|z-p| < \delta \implies |f(z) - f(p)| < \epsilon$.

**Theorem 16.** Under the appropriate assumptions, continuous functions preserve continuity under addition, multiplication, division, and composition.

**Theorem 17.** $f : \mathbb{C} \to \mathbb{C}$ is continuous if and only if $\forall$ open $U \subset \mathbb{C}$, $f^{-1}(U)$ is also open.

**Theorem 18.** Suppose $X$ is a compact subset of $\mathbb{C}$ and $f : X \to \mathbb{C}$ is continuous. Then $f(X)$ is compact.

**Corollary 5.** If $X \neq \varnothing$ is compact and $f : X \to \mathbb{R}$ is continuous, then $f$ achieves its maximum and minimum values on $X$.

**Theorem 19.** *(Convergence Tests)*

a. $\sum_{n=1}^{\infty} a_n \to L \implies a_n \to 0$.

b. $a_n \nrightarrow 0 \implies \sum a_n$ does not converge.

c. A series of nonnegative terms either converges or diverges to $\pm\infty$

d. For real or complex numbers $x_n$, if $\sum |x_n|$ converges, so does $\sum x_n$.

e. If $|r| < 1$ then $\sum_{n=o}^{\infty} r^n$ converges to $\frac{1}{1-r}$. Otherwise, the series diverges.

**Theorem 20.** *(Properties of the Complex Exponential)*

a. $|e^z| = e^x$

b. $\arg(e^z) = y$

c. $e^0 = 1$

d. $e^z e^w = e^{z+w}$

e. $e^{-z} = \frac{1}{e^z}$

f. $e^z = e^{z+2\pi i}$

g. $e^{\overline{z}} = \overline{e^z}$

h. $e^{i\pi} = -1$

**Theorem 21.** $\log(zw) = \log(z) + \log(w)$, as sets.

**Theorem 22.** *(Properties of Hyperbolic Functions)* For $z \in \mathbb{C}$:

a. $\cosh(-z) = \cosh(z)$

b. $\sinh(-z) = -\sinh(z)$

c. $\cosh^2(z) - \sinh^2(z) = 1$

d. For $t \in \mathbb{R}$, $[\cosh(t)]' = \sinh(t)$ and $[\sinh(t)]' = \cosh(t)$

**Theorem 23.** *(Properties of Sines and Cosines)* For $z \in \mathbb{C}$:

a. $\sin^2(z) + \cos^2(z) = 1$

b. $\sin(-z) = -\sin(z)$

c. $\cos(-z) = \cos(z)$

d. $\cos(\overline{z}) = \overline{\cos(z)}$, $\sin(\overline{z}) = \overline{\sin(z)}$

e. $\cos(z) = \cosh(iz)$

f. $\sin(z) = -i\sinh(iz)$

g. $\cos(iz) = \cosh(z)$

h. $\sin(iz) = i\sinh(z)$

i. cos, sin are unbounded.

**Theorem 24.** *(Trig Identities)* For $z, w \in \mathbb{C}$:

a. $\cos(z) + i\sin(z) = e^{iz}$

b. $\sin(z)\cos(w) + \sin(w)\cos(z) = \sin(z+w)$

c. $\sin(w + \frac{\pi}{2}) = \cos(w)$

d. $\sin(w + \pi) = -\sin(w)$

e. $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

**Theorem 25.** *(Jordan Curve Theorem)* The complement of the image of a simple closed curve in $\mathbb{C}$ is a pair of disjoint domains, and the image of the curve is the boundary of each of these. Of the two domains, one is bounded (the inside of the curve), and the other is unbounded (the outside of the curve).

**Theorem 26.** Let $z = x + iy \in \mathbb{C}$. The principal argument of $z$ is:

$$\mathbf{Arg}(z) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0,\ y \in \mathbb{R} \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0,\ y \geq 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0,\ y < 0 \\ \frac{\pi}{2} & \text{if } x = 0,\ y > 0 \\ -\frac{\pi}{2} & \text{if } x = 0,\ y < 0 \\ \text{undefined} & \text{if } x = 0,\ y = 0 \end{cases}$$

**Theorem 27.** *(Integral along a parameterized curve)* If $\gamma : [a,b] \to \mathbb{C}$ is continuous, then

$$\int_a^b \gamma(t)\,dt = \int_a^b \Re[\gamma(t)]\,dt + i\int_a^b \Im[\gamma(t)]\,dt.$$

**Theorem 28.** *(Integral of a function along a parameterized curve)* If $\gamma : [a,b] \to \mathbb{C}$ is a smooth curve with image $S \subset \mathbb{C}$, and $f : S \subset \mathbb{C} \to \mathbb{C}$ is a continuous function, then the line integral of $f$ over $\gamma$ is

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

**Theorem 29.** *(Linearity Properties of Line Integrals)*

a. $\int_\gamma f + g\,dz = \int_\gamma f\,dz + \int_\gamma g\,dz$

b. $\forall c \in \mathbb{R}, \int_\gamma cf\,dz = c\int_\gamma f\,dz$

**Theorem 30.** *(Line Integral of Reversed Curve)*

$$\int_{-\gamma} f\,dz = -\int_\gamma f\,dz$$

**Theorem 31.** *(Line Integral over Concatenated Curves)*

$$\int_{\alpha+\beta} f = \int_\alpha f + \int_\beta f$$

**Theorem 32.** Suppose $\phi : [a,b] \to \mathbb{C}$ is continuous. Then:

$$\int_a^b |\phi(t)|\,dt \geq \left|\int_a^b \phi(t)\,dt\right|$$

**Corollary 6.** If $f$ is continuous and $\gamma$ is piecewise smooth, then

$$\left| \int_\gamma f\, dz \right| \leq (\max |f(\gamma(t))|)(\text{length of}\gamma).$$

**Lemma 2.** *(Product Rule for Differentiable Curves)* If $f,\ g : [a,b] \to \mathbb{C}$ are both differentiable curves, then so is $fg$, and

$$(fg)' = f'g + fg'.$$

**Corollary 7.** Suppose $g : [a,b] \to \mathbb{C}$ is a differentiable curve. Then:

a. So is $g\overline{g} = |g|^2$.

b. So is $g^m$ for any $m \geq 1$, and $[g^m]' = mg^{m-1}g'$.

**Theorem 33.** *(Quotient Rule for Curves)* If $f : [a,b] \to \mathbb{C}$ is a differentiable curve that is never 0, then $\frac{1}{f}$ is also a differentiable curve, and

$$\left(\frac{1}{f}\right)' = \frac{-f'}{f^2}.$$

**Corollary 8.** Suppose $f, g : [a,b] \to \mathbb{C}$ are differentiable curves and $g$ is never zero.

a. $\frac{f}{g}$ is a differentiable curve, and $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

b. For $m \geq 1$, $\frac{1}{g^m}$ is a differentiable curve, and $(g^{-m})' = -mg^{-m-1}g'$

**Theorem 34.** *(Line integral of $z^m$ over a curve)* Suppose $\gamma : [a,b] \to \mathbb{C}$ is piecewise smooth and $m \neq -1$ is an integer. Then,

$$\int_\gamma z^m dz = \frac{\gamma(b)^{m+1} - \gamma(a)^{m+1}}{m+1}.$$

**Corollary 9.** Suppose $\gamma$ is a piecewise smooth closed curve, and $m \neq -1$ is an integer. Then,

$$\int_\gamma z^m dz = 0.$$

**Theorem 35.** *(Green's Theorem)* Let $\{\gamma_j\}$ be a finite disjoint collection of piecewise smooth simple closed curves. Let each $\gamma_j$ be inside of some curve $\gamma_0$, and any two of the $\gamma_j$'s outside of each other. Let $D$ denote the region inside of $\gamma_0$ and outside all of the other $\gamma_j$'s. Define $\partial D = \cup_{k=0}^n \gamma_k$ with a positive orientation. Define $\int_{\partial D} f dz$ to mean $\sum_{k=0}^n \int_{\gamma_k} f dz$. Let $G$ be an open set containing $D$ and $\partial D$. Let $f : G \to \mathbb{C}$ be continuous and write $f(z) = f(x+iy) = u(x+iy) + iw(x+iy)$. Suppose the partial derivatives $u_x, u_y, w_x, w_y$ exist and are continuous on $G$. Then,

$$\int_{\partial D} f dz = i \int\int_D \left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] dxdy = i\left(\int\int_D \frac{\partial f}{\partial x} dxdy\right) - \int\int_D \frac{\partial f}{\partial y} dxdy.$$

**Theorem 36.** Suppose $\gamma$ is a piecewise smooth, positively oriented, simple closed curve, and $p \in \mathbb{C}$ is not in the image of $\gamma$. Then:

$$\int_\gamma \frac{1}{z-p} dz = \begin{cases} 2\pi i & \text{if } p \in \gamma \\ 0 & \text{otherwise} \end{cases}$$

**Theorem 37.** If $D$ is an open disk containing $p$ and $q$, there there is a VH chain in $D$ from $p$ to $q$.

**Theorem 38.** Suppose $p, q \in \mathbb{C}$ and that $D$ is an open set containing the closed line segment $L$ going from $p$ to $q$. Then there is a VH chain in $D$ going from $p$ to $q$.

**Theorem 39.** *(Characterization of domains)* An open set $D$ is a domain if and only if $\forall\, p, q \in D, \exists$ a VH chain in $D$ going from $p$ to $q$.

**Theorem 40.** *(Characterization of constant functions on domains)* Suppose $D$ is a domain and $u : D \to \mathbb{R}$ is smooth. If both $u_x, u_y$ are identically 0 on $D$, then $u$ is constant on $D$.

**Theorem 41.** *(Sums and constant multiples of differentiable functions)* Suppose that $f'(p)$ and $g'(p)$ exist, and that $c \in \mathbb{C}$ is a constant. Then:

a. $f + g$ is differentiable at $p$, with derivative $f'(p) + g'(p)$

b. $cf$ is differentiable at $p$, with derivative $c \cdot f'(p)$

**Theorem 42.** *(Continuity Estimate)* Suppose $f$ is differentiable at $p$. Let $\epsilon > 0$. Then there is a $\delta > 0$ such that for any $z$ with $|z - p| < \delta$,

$$|f(z) - f(p)| \le (|f'(p)| + \epsilon)|z - p|.$$

It follows that if $f$ is differentiable at $p$, then $f$ is continuous at $p$.

**Theorem 43.** *(Product Rule)* If $f(z)$ and $g(z)$ are both differentiable at $p$, then so is $h(z) = f(z)g(z)$, and

$$h'(p) = f'(p)g(p) + f(p)g'(p).$$

**Theorem 44.** *(Chain Rule)* If $f$ is differentiable at $p$ and $g$ is differentiable at $p$, then $g \circ f$ is differentiable at $p$, and

$$(g \circ f)'(p) = g'(f(p))f'(p).$$

**Theorem 45.** *(Quotient Rule)* Suppose $f$ and $g$ are functions that are both differentiable at $p$, and that $g(p) \neq 0$. Let $h(z) = \frac{f(z)}{g(z)}$. Then $h$ is differentiable at $p$, and

$$h'(p) = \frac{f'(p)g(p) - f(p)g'(p)}{g(p)^2}.$$

**Theorem 46.** *(Cauchy-Riemann Equations)* Suppose $D \subset \mathbb{C}$ is a domain, and that $f : D \to \mathbb{C}$ is analytic. Write

$$f(z) = f(x + iy) = u(x + iy) + iw(x + iy)).$$

Then the partial derivatives of $u$ and $w$ with respect to $x$ and $y$ exist, $u_x = w_y$, and $u_y = -w_x$, so that $u_x u_y + w_x w_y = 0$.

**Theorem 47.** Suppose that $u, w$ are real-valued functions defined on a domain $D$, that the second order partial derivatives of $u$ and $w$ exist and are continuous, and that $u, w$ satisfy the Cauchy-Riemann equations. Then $u$ and $w$ are harmonic on $D$.

**Corollary 10.** If $f = u + iw$ is analytic on a domain $D$, then $u$ and $w$ are harmonic.

**Theorem 48.** *(Cauchy-Riemann implies analyticity)* Suppose $D$ is an open disk centered at $p$, and that $f = u + iw$ is a function from $D$ to $\mathbb{C}$. If the Cauchy-Riemann equations for $u$ and $w$ hold at $p$, then $f$ is analytic at $p$, and $f'(p) = u_x(p) + iw_x(p)$.

**Theorem 49.** *(Harmonic conjugates always exist on disks)* Suppose $D \subset \mathbb{C}$ is a disk, and that $u : D \to \mathbb{R}$ is harmonic. Then there is another function $w : D \to \mathbb{R}$ such that the function $f(z) = u(z) + iw(z)$ satisfies the Cauchy-Riemann equations.

**Theorem 50.** *(Restrictions on Analytic Functions)* Suppose $f = u + iw$ is an analytic function on a domain $D$.

a. If $u$ is constant on $D$, then so is $f$.

b. If $u^2 + w^2$ is constant on $D$, then so is $f$.

**Theorem 51.** If $\sum_{n=0}^{\infty} a_n(z-p)^n$ converges at $z$, then it converges absolutely at $w$ whenever $|w - p| < |z - p|$.

**Corollary 11.** If a power series $\sum_{n=0}^{\infty} a_n(z-p)^n$ diverges at $z$, then it also diverges at any $w$ with $|w - p| > |z - p|$.

**Theorem 52.** *(Ratio and Root Tests)* Consider the power series

$$\sum_{n=0}^{\infty} a_n(z-p)^n$$

.

a. Suppose $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists; call it $L$. If $L > 0$ then the radius of convergence is $\frac{1}{L}$; if $L = 0$ then the radius of convergence is infinite; if the limit is infinite, then the radius of convergence is 0.

b. Suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists; call it $L$. Then the radius of convergence is as above.

**Lemma 3.** Suppose that the radius of convergence of $\sum a_n(z-p)^n$ is $R > 0$. Then the radius of convergence of $\sum na_n(z-p)^{n-1}$ is at least $R$.

**Theorem 53.** *(Derivatives of power series)* If a power series $\sum_{n=0}^{\infty} a_n(z-p)^n$ has radius of convergence $R > 0$ then we can define a function $f$ on the open disk $D$ of radius $R$ centered at $p$ by $f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n$. Then:

a. $f$ is analytic on $D$

b. $\forall z \in D, f'(z)$ is given by the power series $\sum_{n=0}^{\infty} na_n(z-p)^{n-1}$

**Corollary 12.** *(Derivatives of power series)*

a. Every derivative of $f$ exists and is given as a power series with radius of convergence at least $R$.

b. The power series for the derivatives can be found by termwise differentiation.

c.
$$a_n = \frac{f^n(p)}{n!}$$

d. A power series with a positive radius of convergence is just a Taylor series:
$$\sum_{n=0}^{\infty} a_n(z-p)^n = \sum_{n=0}^{\infty} \frac{f^n(p)(z-p)^n}{n!}$$

**Lemma 4.** Suppose that $G$ is a parity-0 graph with at least one edge. Then $G$ contains a simple closed edge-path $L$ with at least one edge.

**Theorem 54.** Suppose that $G$ is a parity-0 graph whose set of edges $E$ is nonempty. The $E$ is the disjoint union of simple closed edge-paths $\{L_j\}$.

**Lemma 5.** If $D$ is a domain containing the closed line segment $K$ joining $p$ and $q$, there there is a positive constant $\epsilon > 0$ such that for any point $z \in K$, the ball of radius $\epsilon$ about $z$ is contained in $D$.

**Theorem 55.** If $D$ is a simply connected domain and $f : D \to \mathbb{C}$ is analytic, then there is an analytic $g : D \to \mathbb{C}$ with $g' = f$.

**Lemma 6.** *(Green analyticity lemma)* Suppose that $D$ is a domain and that $f : D \to \mathbb{C}$ is analytic. Then at any point of $D$,
$$f_x + if_y = 0.$$

**Theorem 56.** *(Cauchy's Formula)* Suppose $D \subset \mathbb{C}$ is a domain, that $f : D \to \mathbb{C}$ is analytic, and that $\gamma$ is a positively-oriented, piecewise smooth, simply connected curve in $D$, such that the inside of $\gamma$ is also in $D$. If $p$ is a point inside of $\gamma$, then:
$$\frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-p} dz = f(p).$$

**Theorem 57.** For a triangle $\mathcal{T} = PQR$ (including its inside), the diameter is the length of its longest side.

**Theorem 58.** Suppose $K_n \subset \mathbb{C}$ are closed nonempty sets, with $K_{n+1} \subset K_n$ for each $n$, and that the diameter of $K_n$ goes to 0 as $n \to \infty$. Then, $\cup K_n$ is a single point.

**Theorem 59.** *(Cauchy-Goursat Theorem)* Suppose $D \subset \mathbb{C}$ is a domain, $f : D \to \mathbb{C}$ is analytic, and that $T$ is a triangular simple closed curve in $D$ whose inside is also in $D$. Then,
$$\int_T f dz = 0.$$

**Theorem 60.** *(General Cauchy Theorem)* Suppose $D \subset \mathbb{C}$ is a simply connected domain, and that $f : D \to \mathbb{C}$ is analytic. Then,

a. There is an analytic function $g : D \to \mathbb{C}$ such that $g' = f$.

b. If $\gamma$ is a piecewise smooth closed curve in $D$, then $\int_\gamma f dz = 0$.

**Theorem 61.** *(Extending Cauchy's Theorem)* Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$, and $f'$ is continuous on $U$. For $p \in U$, choose $r > 0$ so that $B_r(p) \subset U$.

a. There is a power series $\sum a_n(z-p)^n$ with radius of convergence at least $r$ such that $f(z) = \sum a_n(z-p)^n$ for every $z$ with $|z-p| < r$.

b. The coefficients $a_n$ are given by

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-p)^{n+1}} dw$$

where $\gamma$ is any counterclockwise circle centered at $p$ with radius $d < r$.

**Theorem 62.** *(General Cauchy Formula)* Suppose $p \in \mathbb{C}$, $r > 0$, and $f : U = B_r(p) \to \mathbb{C}$ analytic. Let $\gamma$ be a counterclockwise circle centered at p with radius $d < r$. Define

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-p)^{n+1}} dw.$$

Then $f(z) = \sum_{n=0}^{\infty} a_n(z-p)^n$ for every $z$ with $|z-p| < r$, and the radius of convergence of this power series is at least $r$.

**Theorem 63.** *(Liouville)* If $f : \mathbb{C} \to \mathbb{C}$ is bounded and entire, then $f$ is constant.

**Theorem 64.** *(Morera)* Suppose $D$ is a domain, that $f : D \to \mathbb{C}$ is continuous, and that if $T$ is any triangle in $D$ whose inside is also in $D$, then the line integral of $f$ around $T$ is 0. Then $f$ is analytic on $D$.

**Theorem 65.** *(Removable singularities are removable)* If f has a removable singularity at $p$, then it is possible to make $f$ become analytic at $p$ by redefining $f(p)$.

**Theorem 66.** *(Laurent series near poles)* If f has a pole at $p$, then for any large positive integer $m$, $(z-p)^m f(z)$ has a removable singularity at $p$. The smallest such $m$ is called the **order** of the pole.

**Theorem 67.** If $p$ is an essential singularity of $f$, then there exists a Laurent series for $f$ in powers of $(z-p)$, and it must contain arbitrarily high powers of $\frac{1}{(z-p)}$.

**Theorem 68.** *(Residues)* Suppose $p$ is an isolated singularity of $f$. Then for all sufficiently small $r > 0$, the line integral of $f$ over the positively oriented circle of radius $r$ centered at $p$ is independent of $r$.

**Lemma 7.** *(Residues at removable singularities)* If $f$ has a removable singularity at $p$, then $\text{Res}(f;p) = 0$.

**Lemma 8.** *(Residues at poles)* Suppose that $p$ is a pole of order $m$ for $f$. It is known that, for $z$ near $p$, $f(z)$ is given by a Laurent series

$$f(z) = \sum_{j=-m}^{\infty} a_j (z-p)^j$$

. Then Res(f;p) $= a_{-1}$.

**Corollary 13.** If $p$ is a pole of order $m$ for $f$, and $g$ is the function that agrees with $(z-p)^m f(z)$ near $p$ and is analytic on a disc about $p$, then the Laurent coefficient $a_{-1}$ for $f$ is the coefficient of $(z-p)^{m-1}$ in the Taylor series for $g$:

$$a_{-1} = \frac{g^{(m-1)}(p)}{(m-1)!}$$

.

**Theorem 69.** *(Residue Theorem)* Suppose $D \subset \mathbb{C}$ is a simply connected domain and $f : D \to \mathbb{C}$ is analytic except for finitely many isolated singularities $p_j$. Suppose $\gamma$ is a positively oriented, piecewise smooth, simple closed curve in $D$ that does not hit any of the $p_j$. Let $U$ be the inside of $\gamma$. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{p_j \in U} \mathrm{Res}(f; p_j).$$

**Theorem 70.** *(Laurent series on annuli)* Suppose $0 \le r < R$ and that $f$ is analytic on the open annulus $A$ given by $r < |z| < R$. Then there are two analytic functions given by series:

a. $f_1(z) = \sum_{n=0}^{\infty} a_n z^n$, converging on $|z| < R$
b. $f_2(z) = \sum_{n=1}^{\infty} \frac{b_n}{z_n}$

such that $f(z) = f_1(z) + f_2(z)$ for each $z \in A$.

**Corollary 14.** For any $b$ with $r < b < R$, if for each integer $n$ we set

$$a_n = \frac{1}{2\pi i} \int_{|w|=b} \frac{f(w)}{w^{n+1}} dw$$

, then $f_1(z) = \sum_{n \ge 0} a_n z^n$ and $f_2(z) = \sum_{n \le -1} a_n z^n$.

**Lemma 9.** Suppose $D$ is a domain and $f : D \to \mathbb{C}$ is analytic except for finitely many poles $p_j$, of order $m_j$. Then there is an analytic function $g(z)$ on $D$ and for each $p_j$ a polynomial $h_j$ of degree $m_j$, with no constant term, such that on $D$ $\{p_j\}$,

$$f(z) = g(z) + \sum_j h_j \frac{1}{(z-p_j)}$$

**Theorem 71.** *(Partial fractions for simple rational functions)* Suppose $f(z) = \frac{P(z)}{Q(z)}$ is a simple rational function. Let $c_n$ be the $n^{th}$ root of $Q$. Then

$$f(z) = \sum_n \frac{\mathrm{Res}(f; c_n)}{z - c_n}.$$

**Theorem 72.** *(Real integrals using residues)* Suppose $P$ and $Q$ are **real** polynomials, with $\deg(Q) \geq 2+\deg(P)$, and that $Q$ has no real roots. Let $f(x) = \frac{P(x)}{Q(x)}$. Let $U$ be the set of roots of $Q$ that are in the upper half plane in $C$. Then

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{q \in U} \operatorname{Res}(f; q).$$

**Lemma 10.** *(Jordan)* Let $\gamma(t) = re^{it}$ for $0 \leq t \leq \pi$. If $g(z)$ is continuous and $|g(z)| \leq B_r$ for all $z$ on $\gamma$, then $|\int_\gamma e^{iz} g(z)dz| \leq \pi B_r$.

**Lemma 11.** Suppose $D \subset \mathbb{C}$ is a domain containing a point $p$, that $g$ is an analytic function on $D$ with $g(p) \neq 0$, and that $f(z) = (z-p)^k g(z)$ for some integer $k \neq 0$.

a. If $k > 0$ then $f$ has a zero of order $k$ at $p$.

b. If $k < 0$ then $f$ has a pole of order $|k| = -k$ at $p$.

c. In either case, $p$ is a pole of $\frac{f'}{f}$ of order 1.

d. $\operatorname{Res}(\frac{f'}{f}; p) = k$.

**Theorem 73.** Suppose $D$ is a domain, $f : D \to \mathbb{C}$ is non-constant and analytic except at a finite set $P \subset D$ of poles. Let $Z$ be the set of *zeros* of $f$ in $D$. Assume $\gamma : [a, b] \to D$ is a piecewise smooth, positively oriented, simple closed curve in $D$ that does not meet $P$ or $Z$, with the inside of $\gamma$ contained in $D$.

a. The number of zeros of $f$ inside $\gamma$ is finite.

b. $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N_Z - N_P$, where $N_Z$ is the sum of the orders of the zeros of $f$ inside $\gamma$, and $N_p$ is the sum of the orders of the poles of $f$ inside $\gamma$.

**Theorem 74.** *(Argument Principle)* Suppose $D$ is a domain, $f : D \to \mathbb{C}$ is non-constant and analytic except at a finite set $P \subset D$ of poles. Let $Z$ be the set of *zeros* of $f$ in $D$. Assume $\gamma : [a, b] \to D$ is a piecewise smooth, positively oriented, simple closed curve in $D$ that does not meet $P$ or $Z$, with the inside of $\gamma$ contained in $D$. Suppose that $\alpha(t)$ is a continuous branch of $\arg(f(\gamma(t)))$ that is defined on $[a, b]$. Then

$$\frac{\alpha(b) - \alpha(a)}{2\pi} = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = N_Z - N_P.$$

**Theorem 75.** *(Rouché)* Suppose $f$ and $g$ are analytic on a domain $D$, containing a piecewise smooth simple closed curve $\gamma : [a, b] \to D$ and its inside. If $|g(z) - f(z)| < |f(z)|$ for every $z \in \gamma$, then, counted with multiplicity, $f$ and $g$ have the same number of zeros inside $\gamma$.

**Theorem 76.** *(Fundamental Theorem of Algebra)* In $\mathbb{C}$, a polynomial of degree $n \geq 1$ has exactly $n$ roots (counted with multiplicity).

**Theorem 77.** *(Open Mapping Theorem)* Suppose $D \subset \mathbb{C}$ is a domain and that $f : D \to \mathbb{C}$ is analytic. Then either $f(D)$ is a single point, or an open set.

**Theorem 78.** *(Maximum Principle)* Suppose $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is analytic and not constant. Then $|f|$ has no local maxima in $D$.

**Theorem 79.** *(Maximum Principle Version 2)* Suppose $D \subset \mathbb{C}$ is a domain and $f : D \to \mathbb{C}$ is analytic on $D$ and not constant. Suppose also that $D$ is bounded, so that its closure (the union of $D$ and its boundary) is compact, and that $f$ is continuous on the closure of $D$. Then the following are all achieved only on the boundary of $D$:

a. The maximum value of $|f|$.

b. The maximum and minimum values of $\Re[f]$.

c. The maximum and minimum values of $\Im[f]$.