A Notion of Robustness in Complex Networks

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***Abstract*—We consider a graph-theoretic property known as r-robustness which plays a key role in a class of consensus (or opinion) dynamics where each node ignores its most extreme neighbors when updating its state. Previous work has shown that if the graph is r-robust for sufficiently large r, then such dynamics will lead to consensus even when some nodes behave in an adversarial manner. The property of r-robustness also guarantees that the network will remain connected even if a certain number of nodes are removed from the neighborhood of every node in the network and thus it is a stronger indicator of structural robustness than the traditional metric of graph connectivity. In this paper, we study this notion of robustness in common random graph models for complex networks; we show that the properties of robustness and connectivity share the same threshold function in Erdös-Rényi graphs, and have the same values in 1-D geometric graphs and certain preferential attachment networks. This provides new insights into the structure of such networks, and shows that they will be conducive to the types of dynamics described before. Although the aforementioned random graphs are inherently robust, we also show that it is coNP-complete to determine whether any given graph is robust to a specified extent.**

***Index Terms*—Complex networks, dynamics on networks, matching cut, random graphs, resilient consensus, robustness.**

I. Introduction

COMPLEX networks abound in both the natural world (e.g., ecological, biological, and social systems), and in engineered applications (e.g., the Internet, the power grid, and large-scale sensor networks). Due to their prevalence, a topic of interest has been the robustness of such networks to disruptions, both in the structure and in the dynamics that are occurring on the network. Studies of *structural robustness* characterize the ability of networks to remain connected despite the loss of nodes and edges, either due to targeted removal [2]–[4], or as the outcome of a dynamical process (e.g., cascading failures) [5]. On the other hand, studies of *dynamical robustness* investigate how global dynamics are affected by structural changes (such as edge removal) [6], or by perturbations in local dynamics where some nodes actively deviate from expected behavior (e.g., due to failures or attacks) [7]–[10]. As one might expect, there is a close coupling between the topology of the underlying network and the ability of dynamics to tolerate deviations in local behavior; in particular, different classes of dynamics and models for deviation will require different conditions on the network topology in order to be robust.

A classical metric of structural robustness to node removal is *node connectivity*. Specifically, a network is r-connected if the network remains connected when any arbitrary set of $r-1$ (or fewer) nodes is removed [11]. The concept of node connectivity also has implications for the robustness of certain dynamics on networks. For instance, if the network is $(2F+1)$-connected (for some non-negative integer F), then there are certain information diffusion dynamics (or algorithms) that allow information to spread reliably in the network, even when there are up to F *malicious* nodes (in total) that deviate from the prescribed dynamics in arbitrary ways [7]–[9], [12].

In this paper, we study a graph property known as r-*robustness*, which was introduced in [13] and [14] in the context of a certain class of resilient consensus dynamics on networks. As we will describe more formally in the next section, one of the consequences of a network being r-robust is that it remains connected even when up to $r-1$ nodes are removed from the neighborhood of *every* remaining node. Thus, r-robustness is generally a much stronger certificate of structural robustness than r-connectivity and, in fact, one can construct graphs that have very high connectivity but very low robustness. Just as r-connectivity has implications for the robustness of certain dynamics, so too does r-robustness: if the network is $(2F+1)$-robust (for some non-negative integer F), then there are certain dynamics that allow the nodes in the network to reach consensus even when there are up to F malicious nodes in the neighborhood of every correctly behaving node [14].

Given the strong nature of the r-robustness property, the contributions of this paper are to provide answers to the following two questions. First, how do the metrics of connectivity and robustness compare in various mathematical models for complex networks? Second, what is the complexity of determining the extent of robustness of any given network? To answer the first question, we study three random graph models (Erdös-Rényi, 1-D geometric, and Barabási-Albert preferential attachment graphs) for complex networks. Our analysis reveals that the notions of robustness and connectivity *coincide* on these random graph models, meaning that random graphs with high connectivity also tend to have high robustness. This is perhaps surprising, given the existence of pathological graphs where these metrics are far apart (as described in the next section), and yields new insights into the structure of certain models for complex networks (namely that such networks

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inherently possess strong robustness properties that go beyond the traditional metric of connectivity). These results also have implications for the study of certain consensus (or opinion) dynamics on complex networks, showing that consensus can be reached even if the nodes ignore a certain number of their most extreme neighbors when they update their values. While these results show that one can efficiently determine the extent of robustness of certain specific classes of networks by checking the connectivity of those networks, in the second half of this paper, we answer the second question posed above and show that this is not likely to be true in general; specifically, we show that the problem of determining the extent of robustness of general networks is **coNP**-complete.

II. r-Robustness of Networks

An undirected network (or graph) is given by a pair $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges in the network. An edge $(i, j) \in \mathcal{E}$ indicates that nodes i and j can communicate with each other. The set of *neighbors* of node i is defined as $\mathcal{V}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, the *degree* of node i is denoted by $d_i = |\mathcal{V}_i|$, and the *minimum degree* of the network is $\min_{i\in\mathcal{V}} d_i$. For a given non-negative integer r, a set $\mathcal{S} \subset \mathcal{V}$ is said to be r*-local* if $|\mathcal{V}_i \cap \mathcal{S}| \leq r$ for all $i \in \mathcal{V} \setminus \mathcal{S}$. The *(node-)connectivity* of a graph is the smallest number of nodes that have to be removed in order to disconnect the graph; such a disconnecting set of nodes is called a *vertex cut*. A graph is r-connected if its connectivity is at least r.

As mentioned in the Introduction, we will be focusing on a graph property known as r*-robustness* in this paper, given by the following two definitions from [13] and [14].

Definition 1 (r-Reachable Set): For a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ and a subset of nodes $\mathcal{S} \subset \mathcal{V}$, $\mathcal{S}$ is an r*-reachable set* if $\exists i \in \mathcal{S}$ such that $|\mathcal{V}_i \setminus \mathcal{S}| \geq r$, where $r \in \mathbb{Z}_{\geq 0}$. In words, a set $\mathcal{S}$ is r-reachable if it contains a node that has at least r neighbors outside that set. □

Definition 2 (r-Robust Graph): A graph $\mathcal{G}$ is r*-robust* if for every pair of nonempty, disjoint subsets of $\mathcal{V}$, at least one of the subsets is r-reachable, where $r \in \mathbb{Z}_{\geq 0}$. □

The following result shows why r-robustness is an indicator of structural robustness.

Theorem 1: Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be an r-robust graph, where $r \in \mathbb{Z}_{\geq 1}$. Let $\mathcal{S} \subset \mathcal{V}$ be an $(r-1)$-local set, and let $\mathcal{G}' = \{\mathcal{V} \setminus \mathcal{S}, \mathcal{E}'\}$ be the graph obtained by removing the nodes in $\mathcal{S}$ and their incident edges from $\mathcal{G}$. Then, $\mathcal{G}'$ is connected. □

Proof: We prove by contradiction. Suppose that $\mathcal{G}'$ is not connected. Pick any two of the components in $\mathcal{G}'$, and let the nodes in those components be denoted by the sets $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Since $\mathcal{G}$ is r-robust, at least one of $\mathcal{S}_1$ or $\mathcal{S}_2$ is r-reachable in $\mathcal{G}$. Assume without loss of generality that $\mathcal{S}_1$ is r-reachable in $\mathcal{G}$ and let $v \in \mathcal{S}_1$ be the node that has r neighbors outside $\mathcal{S}_1$ in $\mathcal{G}$. Since $\mathcal{S}$ is an $(r-1)$-local set, at most $r-1$ of v's neighbors were removed when forming $\mathcal{G}'$. Thus, v has at least one neighbor outside $\mathcal{S}_1$ in $\mathcal{G}'$, contradicting the fact that $\mathcal{S}_1$ is a component. Thus, $\mathcal{G}'$ is connected. ■

Since r-robustness guarantees connectedness of the network even after the removal of any $(r-1)$-local set (which could contain significantly more than $r-1$ nodes), it is a much stronger property than r-connectivity in general. The following result from [14] formalizes this notion.

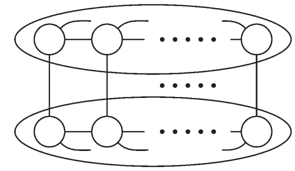

Fig. 1. Example of a graph that has minimum degree $n/2$ and connectivity $n/2$, but that is only 1-robust. Sets $\mathcal{S}_1$ and $\mathcal{S}_2$ induce complete graphs on $n/2$ nodes, and each node in $\mathcal{S}_1$ has exactly one neighbor in $\mathcal{S}_2$ and vice-versa.

Lemma 1 [14]: For any $r \in \mathbb{Z}_{\geq 0}$, if $\mathcal{G}$ is r-robust, then $\mathcal{G}$ is at least r-connected and has a minimum degree of at least r. Furthermore, $\mathcal{G}$ is 1-robust if and only if it is 1-connected.

Thus, the set of r-robust graphs is a subset of the set of r-connected graphs, which itself is a subset of the set of graphs with minimum degree r. Indeed, just as one can construct graphs that have large minimum degree but low connectivity [11], one can construct graphs that have large connectivity but low robustness. For example, consider the network shown in Fig. 1. The sets $\mathcal{S}_1$ and $\mathcal{S}_2$ have $n/2$ nodes (suppose n is even), and each node in each set is connected to all other nodes in its set. Each node has exactly one neighbor from the other set. This network has connectivity $n/2$ and minimum degree $n/2$, but is only 1-robust since $\mathcal{S}_1$ and $\mathcal{S}_2$ are only 1-reachable (i.e., no node in either of those sets has more than 1 neighbor outside its set).

A. Role of r-Robustness in Consensus Dynamics

Consider a setting where each node i in the network holds some private information $x_i[0]$ (an opinion, a measurement, etc.), modeled as a real number. The network operates synchronously, and at each time-step, each normally operating node uses some prescribed rule to update its value (information) based on the values of its neighbors; the value held by node i at time-step k is denoted by $x_i[k]$. In particular, consider the following **Weighted-Mean-Subsequence-Reduced (W-MSR)** dynamics[1]: for some non-negative integer F, at each time-step, each node disregards the largest and smallest F values in its neighborhood (breaking ties arbitrarily) and updates its state to be a weighted average of the remaining values. Mathematically, this is represented as

$$x_i[k+1] = w_{ii}[k]x_i[k] + \sum_{j\in\mathcal{R}_i[k]} w_{ij}[k]x_j[k]$$

where $\mathcal{R}_i[k]$ is the set of nodes whose values were adopted by normal node i at time-step k, and $w_{ii}[k]$ and $\{w_{ij}[k]\}$ are the weights at time-step k. The weights are assumed to satisfy the following conditions:

- $\exists \alpha \in \mathbb{R}_{>0}$ such that $w_{ij}[k] > \alpha,\ \forall j \in \mathcal{R}_i[k] \cup \{i\}, i \in \mathcal{V}, k \in \mathbb{Z}_{\geq 0}$;
- $\sum_{j\in\mathcal{R}_i[k]\cup\{i\}} w_{ij}[k] = 1,\ \forall i \in \mathcal{V}, k \in \mathbb{Z}_{\geq 0}$.

[1]We refer to [13]–[18] for a more complete description of these dynamics, along with proofs of convergence.

Suppose the network contains a set of malicious nodes $\mathcal{M} \subset \mathcal{V}$ which do not necessarily follow the aforementioned dynamics, but instead update their values at each time-step in an arbitrary (potentially worst-case) manner. Denote the set of normal nodes by $\mathcal{N} = \mathcal{V} \setminus \mathcal{M}$. As in [14], we say that the aforementioned dynamics facilitate *resilient asymptotic consensus* if there exists a constant L in the convex hull of the initial values of the normal nodes such that $\lim_{k\to\infty} x_i[k] = L$ for all $i \in \mathcal{N}$. In other words, resilient asymptotic consensus is reached if the malicious nodes cannot prevent the normal nodes from reaching consensus, and furthermore, cannot bias the consensus value excessively (captured by the constraint placed on the consensus value).

To understand the topological conditions required to facilitate consensus under W-MSR dynamics, consider the network shown in Fig. 1. Suppose that nodes in $\mathcal{S}_1$ and $\mathcal{S}_2$ have initial values a and b, respectively, with $a \neq b$. Under the W-MSR dynamics with $F \geq 1$, each node will disregard the value of its neighbor from the opposite set at each time-step and, thus, its own value will remain unchanged, even when there are no misbehaving nodes. Thus, consensus will not be reached in this network, indicating that even networks with high connectivity are not sufficient to guarantee consensus under such dynamics.

Examining Fig. 1, we see that the reason for the failure of consensus in this graph is that it contains two insular communities, where no node in either community receives enough information from *outside* its own community. However, if a graph is r-robust (for sufficiently large r), new information will penetrate at least one out of any two subsets of nodes and pull it toward the other set, preventing stalemates of the aforementioned form. This is formalized in the following result, showing the role that r-robustness plays in the ability of W-MSR dynamics to tolerate arbitrary behavior by a subset of the nodes.

Theorem 2 [14]: Suppose the malicious nodes form an F-local set. Then, resilient asymptotic consensus is reached under W-MSR dynamics if the network is $(2F+1)$-robust. □

Remark 1: Outside settings with misbehaving nodes, W-MSR dynamics can also be viewed in the context of opinion dynamics in social networks. For example, in *DeGroot* opinion dynamics, each node repeatedly updates its opinion as a weighted average of *all* of its neighbors' opinions [19], [20]; W-MSR dynamics generalize this by allowing each node to ignore its neighbors that have the most extreme opinions. In *Hegselmann-Krause (HK)* opinion dynamics, each node removes all values that are sufficiently different from its own opinion at each time-step before averaging the rest [21], [22]; the difference in W-MSR dynamics is that nodes remove values based on *absolute* size (as opposed to *relative* size in HK dynamics). In the opinion dynamics setting with no malicious nodes and where all nodes follow the W-MSR dynamics at each time-step, the proof in [14] directly applies to show that consensus is guaranteed if and only if the network is $(F+1)$-robust. □

Remark 2: The notion of reachable sets also plays a role in the study of *bootstrap percolation* dynamics on networks, where each node maintains a binary state, and changes its state to 1 if a certain number of its neighbors are in state 1 [23]. Bootstrap percolation, reachable sets, and r-robustness are further related to the so-called *Certified Propagation Algorithm* (CPA) for resilient information broadcast in networks, where a single-source node wishes to disseminate its value reliably to all other nodes, even if a certain number of malicious nodes spread misinformation about that value [13], [24]–[26]. For example, in [25], a *t-local pair cut* was defined as a pair of t-local subsets of vertices C_1 and C_2 such that $C_1 \cup C_2$ forms a vertex cut. Such cuts (and their variant defined in [26]) were highlighted as being impediments to reliable information broadcast when the network contains a t-local set of malicious nodes. Since a t-local pair cut forms a $(2t)$-local vertex cut, Theorem 1 indicates that a $(2t+1)$-robust network will not have a t-local pair cut. We refer to [1] and [13] for further discussions on the relationships between these different dynamics. □

Given the strong nature of the r-robustness property and its role in W-MSR (and other) dynamics, it is natural to ask how this property compares to the property of connectivity in commonly studied networks. In the next few sections, we will answer this question by exploring the robustness of three common random graph models for complex networks. We will then analyze the computational complexity of determining the extent to which any given graph is robust. Since all graphs are trivially 0-robust, we will primarily focus on the cases where $r \geq 1$ in the remainder of this paper.

III. Robustness of Erdös-Rényi Random Graphs

Erdös-Rényi random graphs [27], [28] are one of the most common mathematical models for large networks. The version we study here is denoted as $\mathcal{G}_{n,p}$: it consists of n nodes and each possible (undirected) edge between two nodes is present independently with probability p (which may be a function of n), and absent with probability $q = 1 - p$. Let the probability of an event be denoted by $\mathbb{P}(\cdot)$. A *graph property* can be regarded as a class of graphs that is closed under isomorphism.

Definition 3: Assume $\mathcal{P}$ is a graph property and $p = p(n)$ is a function of n. We say that *almost all* $G \in \mathcal{G}_{n,p}$ have property $\mathcal{P}$ if $\mathbb{P}(\mathcal{G}_{n,p} \in \mathcal{P}) \to 1$ as $n \to \infty$, and *almost no* $G \in \mathcal{G}_{n,p}$ has property $\mathcal{P}$ if $\mathbb{P}(\mathcal{G}_{n,p} \in \mathcal{P}) \to 0$ as $n \to \infty$. □

An important feature of $\mathcal{G}_{n,p}$ is that it exhibits phase transitions at certain thresholds for the probability p, defined as follows.

Definition 4: Consider a function $t(n) = g(n)/n$ where $g(n) \to \infty$ as $n \to \infty$, and a function $x = o(g(n))$ satisfying $x \to \infty$ as $n \to \infty$. We say $t(n)$ is a *threshold function* for a graph property $\mathcal{P}$ if $p(n) = (g(n) + x)/n$ implies that almost all $G \in \mathcal{G}_{n,p}$ have property $\mathcal{P}$ and $p(n) = (g(n) - x)/n$ implies that almost no $G \in \mathcal{G}_{n,p}$ has property $\mathcal{P}$. □

Loosely speaking, if the probability of adding an edge is "larger" than $t(n)$ in the sense indicated by Definition 4, then almost all $G \in \mathcal{G}_{n,p}$ will have property $\mathcal{P}$, and if the probability is "smaller" than $t(n)$, almost no $G \in \mathcal{G}_{n,p}$ will have this property.

Definition 5: For $G \in \mathcal{G}_{n,p}$ and constant $r \in \mathbb{Z}_{\geq 1}$, define the properties of *being r-connected*, *r-robust*, and *having minimum degree r* by K_r, R_r, and D_r, respectively. □

Lemma 2 [27]: For any constant $r \in \mathbb{Z}_{\geq 1}$, $t(n) = (\ln n + (r-1)\ln\ln n)/n$ is a threshold function for property K_r. It is also a threshold function for property D_r.

Lemma 2 (by Erdös and Rényi) indicates that K_r and D_r share the same threshold function in $\mathcal{G}_{n,p}$, even though being r-connected is a stronger property than having minimum degree r. The following theorem is one of our main results: it establishes that the aforementioned threshold function for r-connectivity (and minimum degree r) is *also* a threshold function for the stronger property of r-robustness in Erdös-Rényi graphs.

Theorem 3: For any constant $r \in \mathbb{Z}_{\geq 1}$, $t(n) = (\ln n + (r-1)\ln\ln n)/n$ is a threshold function for property R_r. □

Proof: From Lemma 1 and Lemma 2, the result is true for $r = 1$ since 1-connectedness and 1-robustness are equivalent. Thus, we focus on the case where $r \geq 2$.

For the first part of the proof, we show that for any constant $r \in \mathbb{Z}_{\geq 1}$, if $p(n) = (\ln n + (r-1)\ln\ln n + x)/n$, where $x = x(n)$ is some function satisfying $x = o(\ln\ln n)$ and $x \to \infty$ as $n \to \infty$, then almost all $G \in \mathcal{G}_{n,p}$ are r-robust. By the definition of robustness, it is sufficient to show that for almost all $G \in \mathcal{G}_{n,p}$, every subset of $\mathcal{V}$ with size up to $\lfloor n/2 \rfloor$ is r-reachable. Here, we prove a stronger result: if $p(n) = (\ln n + (r-1)\ln\ln n + x)/n$, then almost all $G \in \mathcal{G}_{n,p}$ have the property that every subset of $\mathcal{V}$ with size up to $\lfloor (1-\alpha)n \rfloor$ is r-reachable, where $\alpha = \alpha(n)$ is a positive function satisfying $\sup_n \alpha(n) < 1$ and $\ln\ln n = o(\alpha \ln n)$. Clearly, $\alpha = 1/2$ is included as a special case.

Let $\mathbb{P}_0$ be the probability that some set of cardinality up to $n_c \triangleq \lfloor (1-\alpha)n \rfloor$ is not r-reachable. We need to prove that $\mathbb{P}_0 = o(1)$ when $p(n) = (\ln n + (r-1)\ln\ln n + x)/n$. Denote the probability that some set $\mathcal{S} \subset \mathcal{V}$ with cardinality $k \in \mathbb{Z}_{\geq 1}$ (i.e., $|\mathcal{S}| = k$) is not r-reachable as $\mathbb{P}_k$. By the union bound, we know that $\mathbb{P}_0 \leq \sum_{k=1}^{n_c} \mathbb{P}_k$. For fixed $\mathcal{S}$ of cardinality k, the probability that a node $v \in \mathcal{S}$ has less than r neighbors outside is $\sum_{i=0}^{r-1} \binom{n-k}{i} q^{n-k-i} p^i$, and the probability that $\mathcal{S}$ is not r-reachable is $\left(\sum_{i=0}^{r-1} \binom{n-k}{i} q^{n-k-i} p^i\right)^k$, where $q = 1-p$. Since there are $\binom{n}{k}$ such sets $\mathcal{S}$, we know that $\mathbb{P}_k \leq \binom{n}{k}\left(\sum_{i=0}^{r-1} \binom{n-k}{i} q^{n-k-i} p^i\right)^k$. In the remainder of this proof, we focus on the cases where $k \leq n_c$. Using the fact that $\binom{n}{k} \leq (en/k)^k$ and $\binom{n}{k} \leq n^k$, we obtain

$$\begin{aligned}
\mathbb{P}_k &\leq \binom{n}{k}\left(\sum_{i=0}^{r-1}\binom{n-k}{i} q^{n-k-i} p^i\right)^k \\
&\leq \left(\frac{en}{k}\sum_{i=0}^{r-1}(np)^i(1-p)^{n-k-i}\right)^k \\
&\leq \left(\frac{en}{k}(1-p)^{n-k} r \left(\frac{np}{1-p}\right)^{r-1}\right)^k \\
&\leq \left(\frac{c_1 n (np)^{r-1}}{k}(1-p)^{n-k}\right)^k.
\end{aligned}$$

In the last step above, c_1 is some constant upper bound for $er/(1-p)^{r-1}$ satisfying $0 < c_1 < 2er$ for *sufficiently large* n. The notion of "for sufficiently large n" will be implicitly used throughout the proof. Noting that $1 - p \leq e^{-p}$ and $p(n) = (\ln n + (r-1)\ln\ln n + x)/n$

$$\begin{aligned}
\mathbb{P}_k &\leq \left(\frac{c_1 n (np)^{r-1}}{k} e^{-(n-k)p}\right)^k \\
&= \left(\frac{c_1 n (np)^{r-1}}{k} e^{-\ln n - (r-1)\ln\ln n - x + kp}\right)^k \\
&= \left(c_1 \left(\frac{\ln n + (r-1)\ln\ln n + x}{\ln n}\right)^{r-1} \frac{e^{kp-x}}{k}\right)^k \\
&\leq \left(\frac{c_2 e^{kp-x}}{k}\right)^k.
\end{aligned}$$

Note that $(\ln n + (r-1)\ln\ln n + x)/\ln n < 2$ for sufficiently large n and thus $0 < c_2 < c_1 2^{r-1}$.

Let $f(k) = e^{kp}/k$ be a function of k, where $k \in \mathbb{R}_{>0}$. Since $df/dk < 0$ if $k < 1/p$ and $df/dk > 0$ if $k > 1/p$, $f(k) \leq \max\{f(1), f(n_c)\}$ for $k \in \{1, 2, \ldots, n_c\}$. We know that $f(n_c) = \exp\{n_c p\}/n_c \leq (\exp\{(1-\alpha)np\}/(1-\alpha)n) = (1/(1-\alpha))\exp\{(1-\alpha)np - \ln n\} = (1/(1-\alpha))\exp\{-\alpha\ln n + (1-\alpha)(r-1)\ln\ln n + (1-\alpha)x\}$. Since $\alpha(n)$ is positive, strictly bounded below 1 and $\ln\ln n = o(\alpha \ln n)$, we know that $f(n_c) = o(1)$. Further note that $f(1) = e^p > 1$. Thus, for sufficiently large n, $f(k) \leq f(1) < e$ and $\mathbb{P}_k \leq (c_2 e^{1-x})^k$. We now have

$$\mathbb{P}_0 \leq \sum_{k=1}^{n_c} \mathbb{P}_k \leq \sum_{k=1}^{\infty} (c_2 e^{1-x})^k = \frac{c_2 e^{1-x}}{1 - c_2 e^{1-x}} = o(1)$$

since $x \to \infty$ as $n \to \infty$, completing the first part of the proof. The second part of the proof (showing a lack of r-robustness below the threshold) is obtained by combining Lemmas 1 and 2. ■

Remark 3: The above theorem shows that Erdös-Rényi graphs gain more structure at the threshold $t(n) = (\ln n + (r-1)\ln\ln n)/n$ than simply being r-connected. Whereas r-connectedness implies that given any two disjoint (nonempty) sets, the nodes in at least one of the sets collectively have r neighbors outside that set, Theorem 3 shows that there is (at least) one node in one of the sets that *by itself* has r neighbors outside. Thus, with high probability, "worst-case" graphs such as the one in Fig. 1 will not arise. □

Remark 4: A special case of Erdös-Rényi graphs is when $p(n) = 1/2$; in this case, each graph on n nodes occurs with probability $2^{-\binom{n}{2}}$, corresponding to a uniform distribution over the set of all graphs on n nodes. Thus, the quantity $\mathbb{P}(\mathcal{G}_{n,(1/2)} \in \mathcal{P})$ represents the *fraction* of graphs on n nodes that have property $\mathcal{P}$. Using this fact, our result above indicates that for any fixed $r \in \mathbb{Z}_{\geq 1}$, the fraction of graphs on n nodes that are r-robust goes to 1 as n goes to ∞. □

Remark 5: It is of interest to note that an alternate method to show Theorem 3 would be to first relate the notion of reachable sets to the conditions required for bootstrap percolation dynamics on networks, and then to apply results obtained recently in [23] for such dynamics using branching process techniques. However, the proof provided above is more direct and provides greater insight into the relationship between the underlying graph-theoretic properties of connectivity and robustness. □

IV. Robustness of 1-D Geometric Graphs

Another widely used model for large networks is the *geometric random graph*, which captures edges between nodes that are in close (spatial) proximity to each other. We consider the geometric graph $\mathcal{G}^d_{n,\rho,l}=\{\mathcal{V},\mathcal{E}\}$, which is an undirected graph generated by first placing n nodes (according to some mechanism) in a region $\Omega_d=[0,l]^d$, where $d\in\mathbb{Z}_{\geq 1}$. We denote the position of node $i\in\mathcal{V}$ by $x(i)\in\Omega_d$. Nodes $i,j\in\mathcal{V}$ are connected by an edge if and only if $\|x(i)-x(j)\|\leq\rho$ for some threshold ρ, where $\|\cdot\|$ indicates an appropriate norm (often taken to be the standard Euclidean norm). When the node positions are generated randomly (e.g., uniformly and independently) in the region, one obtains a geometric random graph. In the widely studied model $\mathcal{G}^d_{n,\rho}$, the parameter l is fixed and graph properties are typically explored when $n\to\infty$ and $\rho\to 0$, leading to dense random networks [29]. In the more general model $\mathcal{G}^d_{n,\rho,l}$, however, the length l is also allowed to increase and the density n/l^d can converge to some constant, making it suitable for capturing both dense and sparse random networks [30].

In Section III, we showed that the properties of connectivity and robustness have the same threshold function in Erdös-Rényi graphs. In this section, we will prove similar results for 1-D geometric random graphs (i.e., $d=1$). We start by providing a result showing that connectivity and robustness cannot be very different in 1-D geometric graphs and are, in fact, equal when the nodes are sufficiently spread out (regardless of how the node positions are generated and the relationships between ρ, n, and l). In the following theorem, we assume that the nodes are ordered such that if $i,j\in\mathcal{V}$ and $i<j$, then $x(i)\leq x(j)$.

Theorem 4: In $\Omega_1=[0,l]$, if $\mathcal{G}^1_{n,\rho,l}$ is r-connected, then it is at least $\lfloor r/2\rfloor$-robust. Furthermore, if $x(n)-x(1)>3\rho$, then the graph is r-connected if and only if it is r-robust. □

Proof: First, note that if $x(n)-x(1)\leq\rho$, then the graph is complete and, therefore, it is $(n-1)$-connected and $\lceil n/2\rceil$-robust and, thus, the claim holds. In the rest of the proof, we assume that $x(n)-x(1)>\rho$. In this case, if the graph is r-connected, the following two properties hold.

1) Every interval of the form $(a,a+\rho]\subset(x(1),x(n))$ must have at least r nodes, because otherwise, removing the nodes in that interval would disconnect the nodes in the interval $[x(1),a]$ from those in the interval $(a+\rho,x(n)]$. The same is true for every interval of the form $[a,a+\rho)\subset(x(1),x(n))$ and for every closed interval of length ρ contained in $(x(1),x(n))$.
2) Consider any nonempty set $\mathcal{S}\subset\mathcal{V}$. If there exists an interval $[a,a+\rho]\subset(x(1),x(n))$ with no nodes from $\mathcal{S}$, then there must be a node from $\mathcal{S}$ in the interval $[x(1),a)$ or in the interval $(a+\rho,x(n)]$. By symmetry, assume that $\mathcal{S}$ has nodes in $[x(1),a)$ and let i be the node in $\mathcal{S}$ that is closest to a from this interval. Then, the interval $(x(i),x(i)+\rho]$ contains no nodes from $\mathcal{S}$, but contains at least r nodes and, thus, $\mathcal{S}$ is r-reachable.

Now consider any two disjoint and nonempty subsets $\mathcal{S}_1,\mathcal{S}_2\subset\mathcal{V}$, and any interval $[a,a+\rho]\subset(x(1),x(n))$. If $\mathcal{S}_1$ (respectively, $\mathcal{S}_2$) has no nodes in $[a,a+\rho]$, then $\mathcal{S}_1$ (respectively, $\mathcal{S}_2$) is r-reachable. Thus, suppose $\mathcal{S}_1$ and $\mathcal{S}_2$ have nodes in $[a,a+\rho]$. If $\mathcal{S}_1$ is not $\lfloor r/2\rfloor$-reachable, there are fewer than

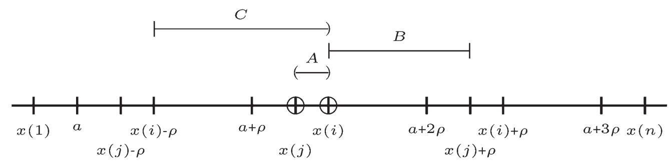

Fig. 2. Illustration of the intervals considered in the proof of Theorem 4. Interval A contains $n_{\mathcal{X}}$ nodes, all from the set $\mathcal{X}$. Interval C contains at least r nodes. Interval $A\cup B$ contains at least r nodes and thus $B\cup C$ has at least $2r-n_{\mathcal{X}}$ nodes.

$\lfloor r/2\rfloor$ nodes from $\mathcal{S}_2$ in $[a,a+\rho]$. Choose any node i from $\mathcal{S}_2$ in the interval. There are at least $r-1$ remaining nodes in the interval and at most $\lfloor r/2\rfloor-1$ of them are in $\mathcal{S}_2$. Thus, i has at least $r-1-\lfloor r/2\rfloor+1\geq\lfloor r/2\rfloor$ neighbors in the interval that are not in $\mathcal{S}_2$. Therefore, for any two disjoint and nonempty subsets $\mathcal{S}_1,\mathcal{S}_2\subset\mathcal{V}$, at least one of them is $\lfloor r/2\rfloor$-reachable. Thus, the graph is at least $\lfloor r/2\rfloor$-robust, proving the first part of the theorem.

For the second part of the theorem, assume that $x(n)-x(1)>3\rho$. Then there exists an interval $[a,a+3\rho]\subset(x(1),x(n))$. Consider any two nonempty and disjoint subsets $\mathcal{S}_1,\mathcal{S}_2\subset\mathcal{V}$ and denote $\mathcal{X}=\mathcal{V}\setminus(\mathcal{S}_1\cup\mathcal{S}_2)$. From the earlier argument, if either $\mathcal{S}_1$ or $\mathcal{S}_2$ do not have any nodes in some closed interval of length ρ within $(x(1),x(n))$, that set will be r-reachable. Thus, suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ have nodes in all closed intervals of length ρ within $(x(1),x(n))$. Pick any node i from $\mathcal{S}_1$ in the interval $[a+\rho,a+2\rho]$, and let $j\in\mathcal{S}_2$ be the node in $[a+\rho,a+2\rho]$ that is closest to i. We assume without loss of generality that $x(j)\leq x(i)$ and that if $x(j)<x(i)$, then there are only nodes from $\mathcal{X}$ between i and j (the latter can always be enforced by redefining i to be the node in $\mathcal{S}_1$ that is closest to j in $[a+\rho,a+2\rho]$).

Suppose that $\mathcal{S}_1$ is not r-reachable. Then, there are fewer than r nodes from $\mathcal{S}_2\cup\mathcal{X}$ in the interval $[x(i)-\rho,x(i)+\rho]$. If $x(j)=x(i)$, then j has at least $2r$ neighbors in $[x(i)-\rho,x(i)+\rho]$ and since fewer than r of them are from $\mathcal{S}_2$, the set $\mathcal{S}_2$ will be r-reachable. Thus, assume that $x(j)<x(i)$. Let the number of nodes from $\mathcal{X}$ strictly between j and i be $n_{\mathcal{X}}$ (see interval A in Fig. 2). The intervals $(x(j),x(j+\rho)]$ ($A\cup B$ in Fig. 2) and $[x(i)-\rho,x(i))$ (interval C in Fig. 2) each contain at least r nodes, and have the interval A in common; thus, the interval $B\cup C$ has at least $2r-n_{\mathcal{X}}$ nodes. Let the number of nodes from $\mathcal{S}_2$ in $B\cup C$ be $n_{\mathcal{S}_2}$. Thus, the number of nodes in $B\cup C$ outside $\mathcal{S}_2$ is at least $2r-n_{\mathcal{S}_2}-n_{\mathcal{X}}$. Since i does not have r neighbors outside $\mathcal{S}_1$, it must be that $n_{\mathcal{S}_2}+n_{\mathcal{X}}<r$ and, thus, there are at least r nodes outside $\mathcal{S}_2$ in $B\cup C$. The set $\mathcal{S}_2$ is then r-reachable (since node j has at least r neighbors outside $\mathcal{S}_2$). Thus, the graph is r-robust. ■

Once again, note that Theorem 4 does not depend on *how* the positions of the nodes are generated. Unfortunately, this strong property does not extend to geometric graphs in higher dimensions. For example, the graph shown in Fig. 1 can be viewed as a geometric graph in two dimensions, where the nodes in each set are all clustered horizontally within a distance ρ, and the two sets are vertically separated by a distance just below ρ so that each node is within a distance ρ of exactly one node in the opposite set. Clearly that graph is only 1-robust, despite having a connectivity of $n/2$. However, as illustrated by our analysis for Erdös-Rényi networks, it may still be possible for robustness and connectivity to coincide in *random* geometric graphs in higher dimensions; an analysis of this for $d \geq 2$ is a ripe avenue for future research. Here, we will present an asymptotic approach to analyzing 1-D random graphs (complementary to the analysis in Theorem 4) to develop scaling laws for r-robustness and r-connectivity. We first define properties for *almost all* graphs in $\mathcal{G}^d_{n,\rho,l}$ as follows, similar to the $\mathcal{G}_{n,p}$ model.

Definition 6: Assume $\mathcal{P}$ is a graph property. We say that *almost all* $G \in \mathcal{G}^d_{n,\rho,l}$ have property $\mathcal{P}$ if $\mathbb{P}(\mathcal{G}^d_{n,\rho,l} \in \mathcal{P}) \to 1$ as $l \to \infty$, and *almost no* $G \in \mathcal{G}^d_{n,\rho,l}$ has property $\mathcal{P}$ if $\mathbb{P}(\mathcal{G}^d_{n,\rho,l} \in \mathcal{P}) \to 0$ as $l \to \infty$. □

Note that we study these properties in $\mathcal{G}^d_{n,\rho,l}$ as $l \to \infty$, and take n and ρ to be functions of l, that is, $n = n(l)$ and $\rho = \rho(l)$. We now present conditions under which the 1-D geometric random graph becomes r-connected and r-robust; the proof of this result builds upon and generalizes the result for 1-D graphs in [30] (which considered scaling laws for connectedness versus disconnectedness of $\mathcal{G}^d_{n,\rho,l}$). Note that if $\rho(l) \geq l$, the graph will be $(n(l)-1)$-connected and $\lceil n(l)/2 \rceil$-robust and thus we focus on the case where $\rho(l) < l$ in the theorem below.

Theorem 5: Assume that $\rho n = kl \ln l$ for some constant $k > 0$.

- If $\rho < l$ and $\rho = \Omega(l)$, then almost all $G \in \mathcal{G}^1_{n,\rho,l}$ are r-connected and r-robust for all $r \in \mathbb{Z}_{\geq 1}$.
- If $\rho = o(l)$ and $\rho l^{(k/(r+1))-1} \to \infty$ for some $r \in \mathbb{Z}_{\geq 1}$, then almost all $G \in \mathcal{G}^1_{n,\rho,l}$ are r-connected and r-robust.
- If $\rho = \Theta(l^{\epsilon})$ and $k \leq (1-\epsilon)$ for some constant $0 < \epsilon < 1$, then almost no $G \in \mathcal{G}^1_{n,\rho,l}$ is r-connected or r-robust.

□

Proof: Fix any $r \in \mathbb{Z}_{\geq 1}$. In order to prove the first two parts, we will show that any interval of length ρ contains at least r nodes; the results will then follow from the arguments in the proof of Theorem 4. Let $\Omega_1 = [0, l]$ be subdivided into nonoverlapping segments of length $h = \rho/(r+1)$. Then, Ω_1 has $c = \lfloor (r+1)l/\rho \rfloor$ whole segments and potentially a fraction of a segment. Any interval of length ρ in Ω_1 will contain at least r whole segments and thus we just need to show that every whole segment contains at least one node.

Let ω be a random variable representing the number of empty whole segments. Since ω is a non-negative integer random variable, by Markov's inequality, we know $\mathbb{P}(\omega > 0) \leq \mathbb{E}(\omega)$, where $\mathbb{E}(\omega) = c(1-(h/l))^n$ is the expected value of ω. Since $1 - x \leq \exp(-x)$, we have

$$\begin{aligned}\mathbb{E}(\omega) &= c\left(1-\frac{h}{l}\right)^n \leq c\exp\left(-\frac{nh}{l}\right)\\ &\leq \frac{(r+1)l}{\rho}\exp\left(-\frac{n\rho}{(r+1)l}\right)\\ &= \frac{(r+1)l}{\rho}\exp\left(-\frac{k}{r+1}\ln l\right)\\ &= \frac{(r+1)}{\rho}l^{1-\frac{k}{r+1}}.\end{aligned}$$

Note that in going from the second line to the third, we replaced n by $(kl \ln l)/\rho$.

Under the conditions in the first part of the theorem, $\rho l^{(k/(r+1))-1} \to \infty$ regardless of the choice of $r \in \mathbb{Z}_{\geq 1}$. Thus, $\mathbb{E}(\omega) \to 0$ and Theorem 4 indicates that almost all graphs will be $\lfloor r/2 \rfloor$-robust for all $r \in \mathbb{Z}_{\geq 1}$ (or, equivalently, r-robust for all $r \in \mathbb{Z}_{\geq 1}$). By Lemma 1, almost all graphs will be r-connected for all $r \in \mathbb{Z}_{\geq 1}$. Similarly, for the second part, $\mathbb{E}(\omega) \to 0$ as $l \to \infty$ if k and r satisfy the given conditions, indicating that the graph will be r-robust and r-connected (again, using Theorem 4). For the third part, Theorem 5 from [30] indicates that almost no $G \in \mathcal{G}^d_{n,\rho,l}$ is connected under the given conditions and thus almost no graph is r-connected or r-robust for any $r \geq 1$. ■

V. Robustness of Barabási-Albert Preferential Attachment Networks

Before discussing the third model for complex networks, we start by reviewing the following construction method for r-robust graphs from [13] and [14].

Theorem 6 [13], [14]: Let $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ be an r-robust graph. Define the graph $\mathcal{G}' = \{\{\mathcal{V}, v_{\text{new}}\}, \{\mathcal{E}, \mathcal{E}_{\text{new}}\}\}$, where v_{new} is a new node added to $\mathcal{G}$ and $\mathcal{E}_{\text{new}}$ is the edge set related to v_{new}. Then $\mathcal{G}'$ is r-robust if $d_{v_{\text{new}}} \geq r$. □

The above theorem indicates that to build an r-robust graph with n nodes (where $n \geq r$), we can start with an r-robust graph of order less than n (such as a complete graph), and continually add new nodes with incoming edges from at least r nodes in the existing graph. The theorem does not specify *which* existing nodes should be chosen as neighbors. When the nodes are selected with a probability proportional to the number of edges that they already have, the aforementioned construction is known as the *Barabási-Albert (BA) preferential-attachment model* and leads to the formation of so-called *scale-free* networks [31].

Theorem 7: In the BA model, when the initial network is r-robust, then the generated (finite) network is r-connected if and only if the network is r-robust. □

Proof: If each new node connects to less than r existing nodes, then the last node added to the network will have a degree less than r, and so the network will be neither r-connected nor r-robust; on the other hand, if all of the new nodes connect to r existing nodes, then by Theorem 6, the network will be r-robust and, thus, r-connected. ■

Note that Theorem 7 relies on the specific construction procedure of the BA model (where each new node connects to the same number of existing nodes); the extension to more general preferential-attachment mechanisms is a venue for future research. To the extent that the BA model is a plausible mechanism for the formation of complex networks, our analysis indicates that these networks will also facilitate dynamics such as W-MSR, provided that r is sufficiently large when the network is forming.

VI. Complexity of Determining the Extent of Robustness of General Graphs

The previous sections showed that for certain classes of graphs (e.g., 1-D geometric and BA preferential attachment graphs generated from a sufficiently robust core), the robustness of the graph can be determined by calculating its connectivity (for which there exist efficient algorithms [32]). Furthermore, as discussed in Remark 4, for any fixed r, the fraction of graphs on n nodes that are r-robust goes to 1 as $n \to \infty$. Despite these facts, we will show in this section that there is unlikely to be an efficient algorithm that determines the extent to which any arbitrary graph is robust. Specifically, we will show that determining whether a given graph is r-robust for any $r \geq 2$ is **coNP**-complete. We start by recalling the following concepts (e.g., see [32]), and defining the r-robustness problem formally.

*Definition 7 (**NP** and **coNP**):* A *decision problem* is a problem whose answer is "Yes" or "No." The set **NP** (respectively, **coNP**) contains those decision problems whose "Yes" (respectively, "No") answers can be verified using a polynomial number of computations. Two problems $\mathcal{P}_1$ and $\mathcal{P}_2$ are *complements* when the output of $\mathcal{P}_1$ to an input instance is "Yes" if and only if the output of $\mathcal{P}_2$ to that instance is "No" and vice-versa. The complement of a problem in **NP** is in **coNP**, and vice-versa. □

*Definition 8 (**NP**-Complete and **coNP**-Complete):* A decision problem $\mathcal{P}_1$ is **NP**-hard if for any problem $\mathcal{P}_2$ in **NP**, there exists a polynomial-time algorithm that transforms any instance of $\mathcal{P}_2$ into an instance of $\mathcal{P}_1$ that has the same answer (i.e., an algorithm for $\mathcal{P}_1$ can be used to solve problem $\mathcal{P}_2$). If $\mathcal{P}_1$ is **NP**-hard and in **NP**, then $\mathcal{P}_1$ is **NP**-complete. The definition of a **coNP**-complete problem is analogous. If a problem is **NP**-hard, then its complement is **coNP**-hard. □

Definition 9 (The r-Robustness Problem): Given a graph $\mathcal{G}$, the *r-robustness problem* is a decision problem that determines whether $\mathcal{G}$ is r-robust for a given $r \in \mathbb{Z}_{\geq 1}$. □

If a graph is not r-robust, then there exist two nonempty and disjoint subsets of nodes $\mathcal{A}, \mathcal{B}$ such that each node in these sets has at most $r-1$ neighbors outside its set. Note that nodes in set $\mathcal{X} = \mathcal{V} \setminus (\mathcal{A} \cup \mathcal{B})$ can have any number of neighbors outside $\mathcal{X}$. There is no apparent way to certify that a graph is r-robust without checking all pairs of disjoint and nonempty subsets of nodes and showing that at least one set out of each pair is r-reachable. This is intractable as the number of such subsets is exponential in the size of the input graph. On the other hand, to certify that a graph is *not* r-robust, one only needs to provide a single pair of disjoint and nonempty subsets of nodes, of which neither set is r-reachable. Therefore, in the r-robustness problem, the "No" instances (input graphs that are not r-robust) have certificates that can be checked in polynomial time, and so the r-robustness problem is in **coNP**. To show that the r-robustness problem is **coNP**-complete, we show the complement of the r-robustness problem, which we call the *relaxed-ρ-degree cut problem*, is **NP**-complete.

Definition 10: For a graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, a partition of $\mathcal{V}$ into two nonempty subsets $\mathcal{A}$ and $\mathcal{B} = \mathcal{V} \setminus \mathcal{A}$ is said to be a *cut*, and is denoted by $\mathcal{C} = (\mathcal{A}, \mathcal{B})$. The *cut-set* of a cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is defined as the subset of the edges of $\mathcal{G}$ with one endpoint in $\mathcal{A}$ and the other in $\mathcal{B}$. A cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ is a *ρ-degree cut* if each node in $\mathcal{A}$ (respectively, $\mathcal{B}$) has at most ρ neighbors outside $\mathcal{A}$ (respectively, $\mathcal{B}$), where $\rho \in \mathbb{Z}_{\geq 0}$. A *relaxed-$\rho$-degree cut* is a pair of nonempty and disjoint subsets of nodes $\mathcal{A}, \mathcal{B} \subset \mathcal{V}$ such that each node in $\mathcal{A}$ (respectively, $\mathcal{B}$) has at most ρ neighbors outside $\mathcal{A}$ (respectively, $\mathcal{B}$), where $\rho \in \mathbb{Z}_{\geq 0}$. The *relaxed-$\rho$-degree cut problem* and the *ρ-degree cut problem* determine whether the graph has a relaxed-ρ-degree cut or a ρ-degree cut, respectively. □

Note that the difference between a ρ-degree cut and a relaxed-ρ-degree cut is that in the former, the two sets need to be nonempty and partition the graph, whereas the latter does not require the two sets to form a partition. The relaxed-ρ-degree cut problem is the complement of the r-robustness problem: a graph has a relaxed-ρ-degree cut if and only if it is *not* $(\rho+1)$-robust. Both the ρ-degree cut problem and the relaxed-ρ-degree cut problem are in complexity class **NP**, because they possess certificates for "Yes" instances that can be checked in polynomial time (i.e., the two sets that comprise the cut).

Our main result in this section is that the ρ-degree cut and the relaxed-ρ-degree cut problems are **NP**-complete for any $\rho \in \mathbb{Z}_{\geq 1}$, from which the **coNP**-completeness of the r-robustness problem follows. The 1-degree cut problem is equivalent to a known **NP**-complete problem called the "matching-cut problem" where the goal is to find whether there exists a cut in the graph such that no two edges in the cut-set share an end-point [33]. This shows that the 1-degree cut problem is **NP**-complete, but does not immediately imply the **NP**-completeness of the ρ-degree cut problem (or relaxed-ρ-degree cut problem) for arbitrary $\rho \geq 1$. We will show these more general results by providing a reduction from the **NP**-complete problem NAE3SAT [34]; this was also the **NP**-complete problem used for reduction in [33], although the details are different.[2] We start by defining NAE3SAT formally; note that a *literal* is a Boolean variable or its complement.

Definition 11 (NAE3SAT): Consider a set of Boolean variables $X = \{x_1, x_2, \ldots, x_t\}$ and a set of clauses $\{C_1, C_2, \ldots, C_m\}$, where each clause is a disjunction of three literals from the set of variables. A formula in *conjunctive normal form* (CNF) is given by $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$. For a given formula in CNF, *NAE3SAT* determines whether there exists a truth assignment of the variables so that each clause contains at least one "True" and one "False" literal [34]. □

We provide a reduction from NAE3SAT to the ρ-degree cut problem by constructing a graph $\mathcal{G}(\phi)$ for any given CNF formula ϕ such that $\mathcal{G}(\phi)$ has a ρ-degree cut if and only if ϕ can be satisfied within the NAE3SAT constraints. The construction of $\mathcal{G}(\phi)$ from a given formula ϕ is as follows.

We first start with two *blocks*, where each block is a complete graph on $4m+t$ nodes (recall that m is the number of clauses in ϕ and t is the number of Boolean variables). The upper and lower blocks are labeled the *True-block* and *False-block*, respectively, as illustrated in Fig. 3. Next, we will add subgraphs (consisting of additional nodes and edges) to represent the variables and clauses of ϕ to these blocks in a carefully chosen way.

The subgraphs to be added to the blocks are of two types: 1) variable gadgets and 2) clause gadgets. A variable gadget is incorporated for each variable $x_i \in X$. This gadget contains

[2] It is also of interest to compare the (relaxed)-ρ-degree cut problem to the *t-local pair cut* problem (and its variants) studied in [25], [26] in the context of the CPA algorithm for reliable broadcast (see Remark 2 for a definition of such a cut). While similar in flavor to a (relaxed)-ρ-degree cut, the fact that the t-local pair cut involves a vertex cut that can be split into two t-local sets allows for a simple reduction from the **NP**-complete *set splitting* problem, thereby proving **NP**-completeness of the t-local pair cut problem [26]. However, it is not apparent whether an equally simple reduction from set splitting applies to the (relaxed)-ρ-degree-cut problem.

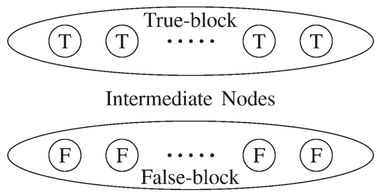

Fig. 3. True-block and False-block in the construction of $\mathcal{G}(\phi)$. Each block is a complete subgraph with $4m + t$ nodes; the edges within each block have been omitted for clarity.

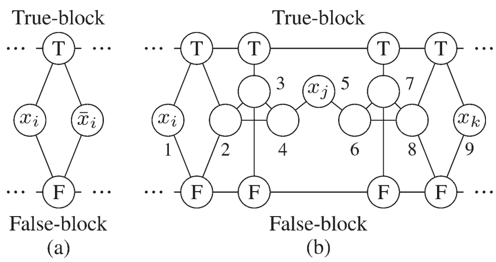

Fig. 4. Figure (a) shows the variable gadget for variable x_i, and (b) shows the clause gadget for clause $x_i \vee x_j \vee x_k$. The intermediate nodes in the clause gadget will be referred to by their numerical labels. The nodes labeled T induce a complete subgraph, as do the nodes labeled F; these edges have been omitted for clarity.

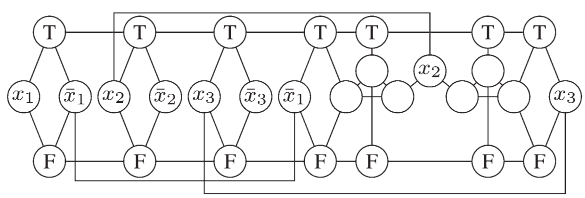

Fig. 5. Graph $\mathcal{G}(\phi)$ corresponding to $\phi = \bar{x}_1 \vee x_2 \vee x_3$ for $\rho = 1$. Each literal node in the clause gadget is connected to its corresponding variable node in the variable gadgets. The nodes labeled "T" induce a complete subgraph, as do the nodes labeled "F"; these edges have been omitted for clarity.

two nodes representing x_i and $\bar{x}_i$ (the binary complement of x_i), each connected to the True and False-blocks as illustrated in Fig. 4(a). Moreover, for each clause in ϕ, a clause-gadget is constructed by connecting three nodes (each representing a literal of the clause) in addition to some extra nodes to the True and False-blocks as depicted in Fig. 4(b). Finally, there are edges, called the *intermediate edges*, connecting each literal node in each clause gadget to its corresponding variable node in the variable gadgets. An example of $\mathcal{G}(\phi)$ for $\phi = \bar{x}_1 \vee x_2 \vee x_3$ is demonstrated in Fig. 5.

The construction of $\mathcal{G}(\phi)$ is now complete for the case where $\rho = 1$. To handle $\rho > 1$, we add additional nodes and edges to the graph as follows. First, for each node v in the True-block (respectively, False-block) of $\mathcal{G}(\phi)$, we add $\rho - 1$ nodes in the False-block (respectively, True-block) and connect them to v. Thus, this step adds a total of $2(\rho - 1)(4m + t)$ nodes to the graph. Second, for each node u in the variable and clause gadgets of $\mathcal{G}(\phi)$ that is not in the True or False-blocks, we add $\rho - 1$ nodes in each of the True and False-blocks and connect u to them. This step adds a total of $2(\rho - 1)(9m + 2t)$ nodes to the graph. Note that the nodes added to the True and False-blocks of $\mathcal{G}(\phi)$ are connected to all other nodes in those blocks and hence the True and False-blocks of $\mathcal{G}(\phi)$ are complete subgraphs (each containing $(4m + t)\rho + (9m + 2t)(\rho - 1)$ nodes). Fig. 6 demonstrates $\mathcal{G}(\phi)$ for the case $\rho = 2$, obtained from the graph shown in Fig. 5 for $\rho = 1$.

If graph $\mathcal{G}(\phi)$ has a ρ-degree cut, then there exists a cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ that partitions the nodes of $\mathcal{G}(\phi)$ into two sets $\mathcal{A}$ and $\mathcal{B} = \mathcal{V} \setminus \mathcal{A}$ such that no node of the graph has more than ρ neighbors outside its set. In the following lemmas, we show that this cut $\mathcal{C}$ satisfies some useful properties (all of these lemmas assume that graph $\mathcal{G}(\phi)$ has a ρ-degree cut and pertains to the cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ just described). Proofs of all results in this section are given in the Appendix.

Lemma 3: Let $\mathcal{T}$ (respectively, $\mathcal{F}$) be the set of all nodes in the True-block (respectively, False-block) of graph $\mathcal{G}(\phi)$. Then, $\mathcal{T} \subseteq \mathcal{A}$ and $\mathcal{F} \subseteq \mathcal{B}$ (or vice-versa).

By Lemma 3, cut $\mathcal{C}$ separates the True and False-blocks of $\mathcal{G}(\phi)$. We assign "True" values to the nodes in the variable and clause gadgets that are in the same set as the True-block in $\mathcal{C}$ and "False" values to the nodes that are in the same set as the False block.

Lemma 4: Cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ has the following two properties:

1) For each variable-gadget, cut $\mathcal{C}$ leaves the variable node and its negation node in opposite sets, that is, they have opposite truth assignments.
2) For each clause gadget, cut $\mathcal{C}$ leaves at least one literal node in set $\mathcal{A}$ and one literal node in $\mathcal{B}$, that is, at least one literal node is assigned "True" and one is assigned "False."

Lemma 5: All literal nodes have the same truth values as their corresponding variable nodes.

Lemmas 3, 4 and 5 connect the ρ-degree cut in $\mathcal{G}(\phi)$ to NAE3SAT, yielding the following result.

Lemma 6: For any $\rho \in \mathbb{Z}_{\geq 1}$, the ρ-degree cut problem is **NP**-complete.

We now show the **NP**-completeness of the more general relaxed-ρ-degree cut problem as follows. We first construct a graph $\mathcal{H}(\phi)$ by taking $2\rho + 1$ copies of $\mathcal{G}(\phi)$ and adding edges to form one complete subgraph on all nodes in the $2\rho + 1$ True-blocks and another complete subgraph on all nodes in the $2\rho + 1$ False-blocks. We refer to each of these copies of $\mathcal{G}(\phi)$ used in building $\mathcal{H}(\phi)$ as a *box*. Fig. 7 illustrates $\mathcal{H}(\phi)$ using the graph $\mathcal{G}(\phi)$ shown in Fig. 5 for $\phi = \bar{x}_1 \vee x_2 \vee x_3$ with $\rho = 1$. Using $\mathcal{H}(\phi)$, we prove the following result.

Theorem 8: For any $\rho \in \mathbb{Z}_{\geq 1}$, the relaxed-$\rho$-degree cut problem is **NP**-complete. □

Knowing that the relaxed-ρ-degree cut problem is **NP**-hard for any $\rho \in \mathbb{Z}_{\geq 1}$, we conclude that its complement problem, that is, the r-robustness problem for any $r \in \mathbb{Z}_{\geq 2}$, is **coNP**-hard. Combining this with the fact that the r-robustness problem is in **coNP** gives the following result.

Corollary 1: For any $r \in \mathbb{Z}_{\geq 2}$, the r-robustness problem is **coNP**-complete.

VII. SUMMARY

We studied a graph property known as r-robustness which provides a metric to measure structural robustness of networks to node removals, and plays a key role in a class of resilient consensus dynamics. While it is **coNP**-complete to determine the extent of robustness in general graphs and one can construct worst-case networks with very large connectivity and low robustness, we showed that the notions of robustness and

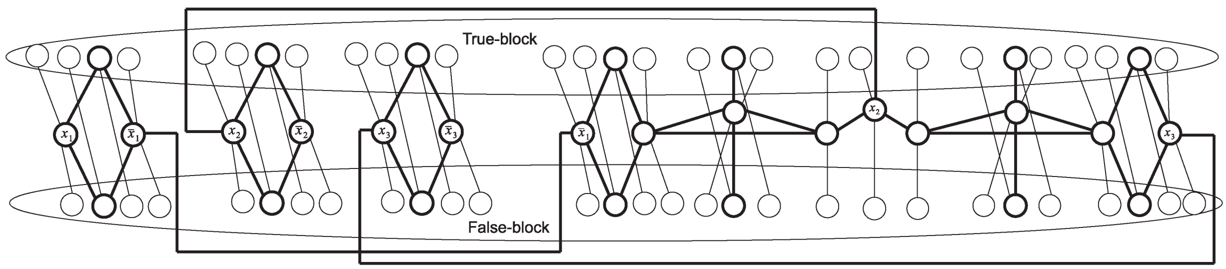

Fig. 6. Graph $\mathcal{G}(\phi)$ for $\rho = 2$ constructed from the graph demonstrated in Fig. 5. The highlighted nodes and edges correspond to the graph for $\rho = 1$.

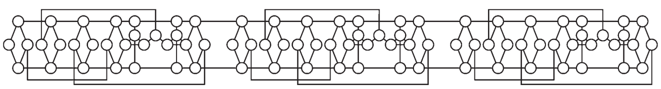

Fig. 7. Graph $\mathcal{H}(\phi)$ constructed by replicating the graph $\mathcal{G}(\phi)$ (depicted in Fig. 5) $2\rho + 1$ times, with $\rho = 1$. The nodes in the top (respectively, bottom) row form the True-block (respectively, False-block) and induce a complete graph. These edges are omitted for clarity.

connectivity coincide in three common models for complex networks. In Erdös-Rényi random graphs, we showed that r-connectivity (and minimum degree) and r-robustness share the same threshold function. In 1-D geometric graphs, we proved that if the nodes are sufficiently spread apart, r-connectedness is equivalent to r-robustness (regardless of how the node locations are generated). In the BA model for preferential attachment networks, we showed that when the initial network is robust, connectivity and robustness are equivalent. Recent work [35] has shown that the properties of connectivity and robustness also share threshold functions in so-called *random intersection graphs*. In total, the aforementioned findings provide new insights into the structural properties of commonly studied models for large-scale networks; investigations of the r-robustness property in other random graph models and extending these results to directed graphs are promising venues for future research.

Appendix

A. Proof of Lemma 3

Since each block is a complete graph with more than $2\rho + 1$ nodes, cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ cannot separate the nodes in the same block; otherwise, there exists a node in the block that has at least $\rho + 1$ neighbors outside its own set.

Now suppose all nodes in both the True and False-blocks are in $\mathcal{A}$ (the case where they are all in $\mathcal{B}$ is handled identically). If there exists a node from a variable gadget in set $\mathcal{B}$, then that node immediately has at least 2ρ neighbors in $\mathcal{A}$, contradicting the definition of cut $\mathcal{C}$ (see Fig. 4(a) for $\rho = 1$). Similarly, it can be argued as follows that no node of any clause gadget can be in set $\mathcal{B}$. Referring to Fig. 4(b), the nodes labeled 1, 2, 3, 7, 8, and 9 cannot be in $\mathcal{B}$ since they would then have at least 2ρ neighbors in $\mathcal{A}$. Since nodes 2, 3, 7, and 8 are in $\mathcal{A}$, nodes 4 and 6 cannot be in $\mathcal{B}$ either. Then node 5 must also be in $\mathcal{A}$. Hence, the only possibility is that all nodes in variable gadgets and clause gadgets are in $\mathcal{A}$. This makes $\mathcal{B}$ empty and violates the definition of cut $\mathcal{C}$. Thus, it must be the case that all nodes in the True-block are in $\mathcal{A}$ and all nodes in the False-block are in $\mathcal{B}$ (or vice-versa). ■

B. Proof of Lemma 4

By Lemma 3, assume without loss of generality that all nodes in the True-block are in $\mathcal{A}$ and all nodes in the False-block are in $\mathcal{B}$. Now, if both nodes of a variable gadget are in the same set (say $\mathcal{A}$), then a node from the False-block in set $\mathcal{B}$ has $\rho + 1$ neighbors in $\mathcal{A}$ (as seen in Fig. 4(a) for $\rho = 1$). This contradicts the definition of cut $\mathcal{C}$. The only possible cuts through variable gadgets for this case are shown in Fig. 8(a), showing the first property of the lemma.

Next, suppose all three literal nodes of a clause-gadget (i.e., nodes 1, 5, and 9 in Fig. 4(b)) are in set $\mathcal{A}$; the case that all three literal-nodes of a clause-gadget are in $\mathcal{B}$ can be handled via identical arguments. Due to the same argument as for variable gadgets, the nodes that share a neighbor in the True and False-blocks with these nodes, that is, the nodes labeled 2 and 8 in Fig. 4(b), should lie in $\mathcal{B}$. Then, nodes labeled 3, 4, 6, and 7 in Fig. 4(b) should also be in $\mathcal{B}$. Now since node 5 in Fig. 4(b) lies in $\mathcal{A}$, it has at least $\rho + 1$ neighbors outside its containing set, contradicting the fact that $\mathcal{C}$ is a ρ-degree cut. Thus, $\mathcal{C}$ cannot leave all literal nodes in a clause gadget in the same set. The only possible cuts through clause gadgets are illustrated in Fig. 8(b) and (c) (only the nodes and edges for the case $\rho = 1$ are shown in that figure). Hence, the second property in the lemma also holds. ■

C. Proof of Lemma 5

First, note that a literal node has the same truth value as its corresponding variable node if and only if they lie on the same side of cut $\mathcal{C} = (\mathcal{A}, \mathcal{B})$. Since a variable node and its negation node in a variable gadget lie in different sets (by Lemma 4), each node is incident with ρ edges in the cutset. Therefore, no other edge connected to these nodes can be excised by cut $\mathcal{C}$. In particular, the intermediate edges connecting literal nodes in clause gadgets to their corresponding variable nodes should be left uncut and thus each literal node in a clause gadget must be in the same set as its corresponding variable node. ■

D. Proof of Lemma 6

We prove the claim by showing that graph $\mathcal{G}(\phi)$ has a ρ-degree cut if and only if ϕ has a solution within the NAE3SAT constraints.

Suppose that $\mathcal{G}(\phi)$ has a ρ-degree cut $\mathcal{C} = (\mathcal{A}, \mathcal{V} \setminus \mathcal{A})$. By the first part of Lemma 4, cut $\mathcal{C}$ has to leave each variable node and its negation node on opposite sides of the cut, thereby specifying their truth assignments. Also, by the second part of Lemma 4, the clause gadgets are cut by $\mathcal{C}$ according to one of

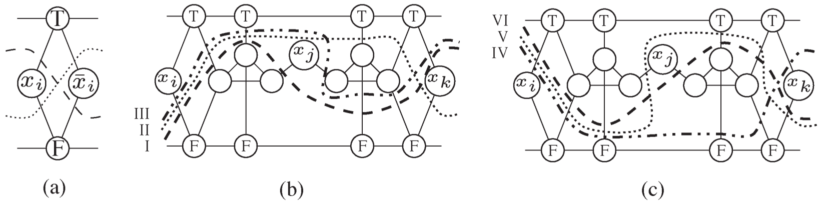

Fig. 8. In the above figures, only the edges and nodes corresponding to the case of $\rho = 1$ are shown, but the analysis holds for any $\rho \geq 1$. Figure (a) shows the two possible cuts through a variable gadget that result in all nodes having at most ρ neighbors on the opposite side of the cut. Figures (b) and (c) show the six allowed cuts through the clause gadget that result in two ρ-reachable but not $(\rho+1)$-reachable sets. Each cut specifies a truth assignment for the literals in the clause (e.g., cut I assigns "True" to x_i and x_j, and "False" to x_k). No cut assigns the same truth values to all literals.

the six cases illustrated in Fig. 8, which results in having at least one "True" and one "False" literal node in each clause gadget. Furthermore, by Lemma 5, all of the literal nodes corresponding to the same variable node are left in the same set as that variable node, and the negated literal nodes are in the other set. Consequently, if $\mathcal{G}(\phi)$ has a ρ-degree cut, then ϕ is satisfiable within the NAE3SAT constraints.

On the other hand, if ϕ has a solution under the NAE3SAT constraints, then a cut $\mathcal{C} = (\mathcal{A}, \mathcal{V} \setminus \mathcal{A})$ can be found in $\mathcal{G}(\phi)$ such that 1) each variable gadget is cut so that the variable node and its negation node are connected to the blocks labeled with their truth values and 2) each clause gadget is cut according to its truth assignment as illustrated in Fig. 8. It can be easily observed that using this cut, no node of graph $\mathcal{G}(\phi)$ is incident with more than ρ edges of the cutset and hence $\mathcal{G}(\phi)$ has a ρ-degree cut. Together with the fact that the ρ-degree cut problem is in **NP**, this shows that the ρ-degree cut problem is **NP**-complete. ■

E. Proof of Theorem 8

Proof: We show that the relaxed-ρ-degree cut problem is **NP**-hard by showing that $\mathcal{H}(\phi)$ has a relaxed-ρ-degree cut if and only if $\mathcal{G}(\phi)$ has a ρ-degree cut for any instance ϕ of NAE3SAT. It can be easily seen that if $\mathcal{G}(\phi)$ has a ρ-degree cut, then $\mathcal{H}(\phi)$ also has a ρ-degree cut (e.g., simply replicate the cut in $\mathcal{G}(\phi)$ for each box in $\mathcal{H}(\phi)$) and thus it has a relaxed-ρ-degree cut. It only remains to show if $\mathcal{H}(\phi)$ has a relaxed-ρ-degree cut then $\mathcal{G}(\phi)$ has a ρ-degree cut. Assume that sets $\mathcal{A}, \mathcal{B}$, and $\mathcal{X}$ partition the nodes of $\mathcal{H}(\phi)$ such that 1) $\mathcal{A}$ and $\mathcal{B}$ are nonempty, and 2) each node in $\mathcal{A}$ and $\mathcal{B}$ has at most ρ neighbors outside its own set (i.e., $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{X}$ specify a relaxed-ρ-degree cut).

First, for any clique in $\mathcal{H}(\phi)$ with at least $2\rho+1$ nodes, the fact that $\mathcal{A}$ and $\mathcal{B}$ are not $(\rho+1)$-reachable implies the following two properties:

1) Set $\mathcal{X}$ can contain up to ρ nodes or all nodes of the clique.
2) If a node of the clique is in $\mathcal{A}$ (respectively, $\mathcal{B}$), then no node of that clique is in $\mathcal{B}$ (respectively, $\mathcal{A}$).

Let $\mathcal{T}$ and $\mathcal{F}$ denote the set of all nodes in the True and False-blocks of graph $\mathcal{H}(\phi)$. Several different scenarios can take place for sets $\mathcal{T}$ and $\mathcal{F}$ with respect to sets $\mathcal{A}, \mathcal{B}$ and $\mathcal{X}$. First, consider the case that $\mathcal{T}$ and $\mathcal{F}$ are subsets of $\mathcal{A}$ (the case that $\mathcal{T}$ and $\mathcal{F}$ are subsets of $\mathcal{B}$ can be analyzed similarly). Since each box of $\mathcal{H}(\phi)$ is isomorphic to $\mathcal{G}(\phi)$, by the same argument as in the proof of Lemma 3, this scenario is not possible as it would leave set $\mathcal{B}$ empty. Now, by property (2) stated above and without loss of generality due to symmetry, assume that $\mathcal{T} \subseteq \mathcal{A} \cup \mathcal{X}$ and $\mathcal{F} \subseteq \mathcal{B} \cup \mathcal{X}$. If $\mathcal{T} \subseteq \mathcal{X}$ or $\mathcal{F} \subseteq \mathcal{X}$, the same argument as above yields that $\mathcal{A}$ or $\mathcal{B}$ would be empty, respectively. Therefore, by property (1) above, $|\mathcal{T} \cap \mathcal{X}| \leq \rho$ and $|\mathcal{F} \cap \mathcal{X}| \leq \rho$. Recall that there are $2\rho+1$ boxes in $\mathcal{H}(\phi)$. Consequently, there exist at least $\rho+1$ boxes in $\mathcal{H}(\phi)$ whose True-blocks are subsets of $\mathcal{A}$, and at least $\rho+1$ boxes whose False-blocks are subsets of $\mathcal{B}$. By the pigeonhole principle, there exists a box in $\mathcal{H}(\phi)$, denoted by $\mathcal{G}'(\phi)$, such that its True-block is a subset of $\mathcal{A}$ and its False-block is a subset of $\mathcal{B}$. We show that no node of $\mathcal{G}'(\phi)$ can be in set $\mathcal{X}$.

Suppose that there exists a node in a variable gadget in $\mathcal{G}'(\phi)$ that lies in $\mathcal{X}$. Now if the other node in that variable gadget lies in $\mathcal{A}$ or $\mathcal{X}$, then the node in the False block connected to both of these variable nodes has $\rho+1$ neighbors outside its set (i.e., the two variable nodes and its $\rho-1$ neighbors in the True block). On the other hand, if the other node in the variable gadget lies in set $\mathcal{B}$, then the node in the True block connected to both of these variable nodes has $\rho+1$ neighbors outside its set. Both cases contradict the fact that we are considering a relaxed-ρ-degree cut and thus no node in the variable gadgets can be in set $\mathcal{X}$.

Now, observe that since the True-block is a subset of $\mathcal{A}$ and the False-block is a subset of $\mathcal{B}$, then each variable node has at least ρ neighbors outside its containing set. Since it was assumed that each node in $\mathcal{A}$ and $\mathcal{B}$ has at most ρ neighbors outside its set, it follows that all other neighbors of a variable node should lie in the same set as that node. Therefore, in $\mathcal{G}'(\phi)$, the endpoints of all intermediate edges (i.e., the edges connecting literal nodes in the clause gadgets to the corresponding variable nodes) lie in the same sets. This, in combination with the fact that none of the variable nodes in $\mathcal{G}'(\phi)$ are in $\mathcal{X}$, shows that no literal node in any clause gadget of $\mathcal{G}'(\phi)$ is in $\mathcal{X}$. It only remains to be seen that the nonliteral nodes in the clause gadgets of $\mathcal{G}'(\phi)$, that is, nodes labeled 2, 3, 4, 6, 7, and 8 in Fig. 4(b), are not in $\mathcal{X}$ either. By the same argument as for variable nodes, nodes labeled 2 and 8 cannot lie in $\mathcal{X}$. Also, note that since the True- and False-blocks of $\mathcal{G}'(\phi)$ are subsets of $\mathcal{A}$ and $\mathcal{B}$, respectively, and node 2 (respectively, node 8) is either in $\mathcal{A}$ or $\mathcal{B}$, then ρ of the edges connecting node 2 (respectively, node 8) to the True and False-blocks are excised. Therefore, all of its other neighbors, that is, nodes 3 and 4 (respectively, nodes 6 and 7), should lie in the same set as node 2 (respectively, node 8). As a result, nodes 3, 4, 6, and 7 cannot be in $\mathcal{X}$. Consequently, no node of $\mathcal{G}'(\phi)$ lies in $\mathcal{X}$.

We have thus shown that if $\mathcal{H}(\phi)$ has a relaxed-ρ-degree cut, then $\mathcal{G}'(\phi)$ has a ρ-degree cut. Since $\mathcal{G}(\phi)$ and $\mathcal{G}'(\phi)$ are isomorphic, graph $\mathcal{G}(\phi)$ also has a ρ-degree cut. Consequently, $\mathcal{H}(\phi)$ has a relaxed-ρ-degree cut if and only if $\mathcal{G}(\phi)$ has a

ρ-degree cut for any NAE3SAT instance ϕ. Hence, the relaxed-ρ-degree cut problem is **NP**-complete. ■

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