Solving Bilevel Multicriterion Optimization Problems With Lower Level Decision Uncertainty

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***Abstract*—Bilevel optimization problems are characterized by a hierarchical leader–follower structure, in which the leader desires to optimize her own strategy taking the response of the follower into account. These problems are referred to as Stackelberg problems in the domain of game theory, and as bilevel problems in the domain of mathematical programming. In a number of practical scenarios, a bilevel problem is solved by a leader who needs to take multiple objectives into account and simultaneously deal with the decision uncertainty involved in modeling the follower's behavior. Such problems are often encountered in strategic product design, homeland security applications, and taxation policy. However, the hierarchical nature makes the problem difficult to solve and they are commonly simplified by assuming a deterministic setup with smooth objective functions. In this paper, we focus our attention on the development of a flexible evolutionary algorithm for solving multicriterion bilevel problems with lower level (follower) decision uncertainty. The performance of the algorithm is evaluated in a comparative study on a number of test problems. In addition to the numerical experiments, we consider two real-world examples from the field of environmental economics and management to illustrate how the framework can be used to obtain optimal strategies.**

***Index Terms*—Bilevel optimization, decision uncertainty, evolutionary algorithms, Stackelberg programming.**

I. INTRODUCTION

THE ORIGINS of hierarchical decision making and optimization problems can be traced to two sources. The first source is in the domain of game theory, in which such problems were first realized by von Stackelberg [45] and came to be known as Stackelberg problems. The second source is in the domain of mathematical programming, in which the problems appeared as bilevel optimization problems containing a nested inner optimization problem as a constraint of an outer optimization problem [7]. The outer optimization problem is often referred to as an upper level (leader's) optimization task and the inner optimization problem is often referred to as a lower level (follower's) optimization task.

Bilevel optimization problems have been widely studied by researchers as well as practitioners. The development has been driven by a number of new applications rising from different fields of operation research. For instance, in the context of homeland security [2], [47], bilevel and even trilevel optimization models have been successfully applied to problems of improving border security, and to defending critical infrastructure (e.g., the pipeline networks, power grids, and supply chains) from a terrorist threat. In these applications, the government commonly assumes the role of a leader (defender) and terrorist (attacker) is modeled as the follower. Recently, bilevel optimization frameworks have come to play an increasingly important role in military applications and public health; e.g., for planning the prepositioning of defensive missile interceptors [9], interdiction of nuclear-weapons projects [10], biometrics; and preparedness and response to bioterror attacks.

Apart from military context, bilevel problems are often encountered in economics and logistics, in which a number of decision-making problems possess a hierarchical structure. For example, consider the study by Benth *et al.* [6] who have applied bilevel optimization to compute optimal recovery policies for financial markets. Cho *et al.* [13] have used a two-level supply chain model to study optimal pricing and rebate strategies. Bilevel frameworks also provide a potent tool for solving theoretical agency problems, where the principal's (leader's) objective is to design an incentive-feasible compensation contract that maximizes its expected payoff while incorporating the reaction of the follower into the optimization process. Solving such problems is challenging and generally necessitates the use of sophisticated numerical methods. For instance, Armstrong *et al.* [3] have recently studied principal-agent models that simultaneously incorporate both adverse selection and moral hazard features. In addition to agency literature, game-theoretic models with a bilevel structure have also been employed to find optimal tax policies [29], [42], [43], model production processes [33], investigate strategic behavior in deregulated markets [27], design networks [8],

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optimize retail channel structures [48], and even to solve management problems for secret projects [34].

The broader adoption of bilevel frameworks has only been limited by computational challenges; optimization problems with a bilevel nature are hard to solve. Having a lower level optimization problem as a constraint to the upper level problem generally "precludes a closed-form solution without relying on strong (and often unrealistic) assumptions" [3]. Most of the classical approaches are based on approximate solution techniques to handle simple bilevel problems with assumptions of smoothness, linearity or convexity [14], [46]. This includes the Karush-Kuhn-Tucker (KKT) approach [26], branch-and-bound techniques [5], and the use of penalty functions [1]. To relax the assumptions and support handling of broader classes of optimization problems, a number of evolutionary studies [32], [44], [50] have been carried out to develop solution procedures for complex single-objective bilevel problems. Substantial efforts have also been devoted to developing test problem suites and test problem generators [11], [40] that help to evaluate the quality of bilevel solution procedures.

The existing research has primarily focused on solving single-objective problems. Apart from a few recent studies, little work has been done in the domain of multiobjective bilevel optimization problems both in the field of classical [22], [23] as well as evolutionary optimization [18], [25], [36]. Most of the studies on multiobjective bilevel programming have aimed at solving the optimization problem without paying much attention to the decision making intricacies involved in realistic problems having a leader and follower. In particular, surprisingly little work has been done to investigate the impacts of decision uncertainty in bilevel frameworks. The discussion has been largely avoided by restricting the studies to a special form of a multiobjective bilevel problem, where the lower level decision maker does not have any actual decision power. When formulating the problem, it has commonly been assumed that the leader is free to choose any Pareto-optimal lower level solution on behalf of the follower that is preferred from her own perspective. Considering the nature of practical problems, this assumption of leader's sovereignty over follower is often unrealistic. Through this paper, we aim to develop a deeper understanding of bilevel multicriteria decision making and optimization aspects.

Our investigation is focused on two aspects: 1) incorporation of multiple objectives at both levels under assumption that the follower has sufficient power to choose a solution from its Pareto-optimal frontier given the leader's decision and 2) handling of lower level decision-uncertainty that arises as the leader can identify the follower's Pareto-frontier for a given upper level decision, but has little idea as to which decision the follower might make on the frontier. In practical decision-making problems, decision makers often have more than one conflicting criterion to consider. Almost by definition, real-world decisions are influenced by a number of stakeholders with their own set of criteria. For instance, the mission of homeland security is to ensure that the country is safe, secure, and resilient against terrorism and other attacks. However, this primary goal needs to be achieved in a cost-effective manner without sacrificing ease of customs operations for citizens, business travelers, and tourists. The discussion gets increasingly interesting when these decision-making aspects are considered within a bilevel framework. When not only the leader but also the follower pursues multiple objectives, the leader faces a substantial lower level decision uncertainty. To optimize her own strategy, the leader has to predict what are the most preferred alternatives for the follower given the leader's own choice. One cannot rely on the assumption that the leader can elicit preference information from the follower. A government responsible for preventing terrorist attacks can only make informed guesses on the actions taken by the attackers. Therefore, to solve the problem, the leader has to model the follower's behavior and bear in mind the uncertainty that arises from not exactly knowing the follower's preferences toward different solutions. Some of the common ways to identify the follower's decision behavior might be to conduct surveys or infer from the past actions of the follower. Such surveys and inferences can only lead to approximate information at best.

The objective of this paper is to present an efficient numerical strategy for handling multiobjective bilevel problems under lower level decision uncertainty. The methodology utilizes a blend of ideas from the domain of decision making and evolutionary computation to tackle the problem. In particular, we leverage tools developed in the context of parametric optimization to guide the evolutionary algorithm. To evaluate the performance, we consider a suite of test problems that represent a number of difficulties encountered in practice. In addition to test problems, we also present two real-world examples in the context of environmental economics and management. Both examples highlight the importance of taking the follower's decision uncertainty into account.

This paper is structured as follows. In the first section, we provide the description of a general multiobjective bilevel optimization problem, which is followed by a review of some recent work on multiobjective bilevel optimization. Thereafter, we introduce the difficulties associated with bilevel multiobjective optimization problems, and develop ideas to efficiently handle these problems. In particular, the focus is on handling the lower level decision uncertainty. Next, we integrate the ideas in an evolutionary algorithm to develop an efficient solution methodology. The algorithm is then evaluated on a number of test problems, and a comparative study is performed with an earlier approach [18]. Finally, we provide a detailed analysis of two real-world problems, and solve them with the proposed evolutionary algorithm.

II. Past Studies on Multiobjective Bilevel Optimization

In the context of single-objective bilevel optimization, it has been shown in [19] that it is possible to write the KKT optimality conditions for such problems and reduce it to a single level problem. However, the presence of many lower level Lagrange multipliers and co-derivative terms makes it difficult to directly implement the idea into practice. Given these difficulties for single-objective bilevel problems, little theoretical attention has been given to multiobjective bilevel optimization problems. Although a substantial body of research exists on single-objective bilevel optimization, relatively few papers have considered bilevel problems with multiple objectives on both levels. Even less research has been done to understand the impacts of decision-uncertainty that arise in multiobjective bilevel problems.

Studies on multiobjective bilevel optimization are mostly directed toward development of techniques for solving optimistic formulation of the problem, where the decision-makers are assumed to co-operate and the leader can freely choose any Pareto-optimal lower level solution. Eichfelder [22], [23] utilized classical techniques to solve simple multiobjective bilevel problems. The lower level problems are handled using a numerical optimization technique, and the upper level problem is handled using an adaptive exhaustive search method. This makes the solution procedure computationally demanding and nonscalable to large-scale problems. The method is close to a nested strategy, where each of the lower level optimization problems is solved to Pareto-optimality. Shi and Xia [36] used the ϵ-constraint method at both levels of a multiobjective bilevel problem to convert the problem into an ϵ-constraint bilevel problem. The ϵ-parameter is elicited from the decision maker, and the problem is solved by replacing the lower level constrained optimization problem with its KKT conditions. The problem is solved for different ϵ-parameters, until a satisfactory solution is found.

With the surge in computation power, a number of nested algorithms have also been proposed, which solve the lower level problem completely for every upper level vector to arrive at the problem optima. One of the first studies, utilizing an evolutionary approach for bilevel multiobjective algorithms was in [50]. The study involved multiple objectives at the upper level, and a single objective at the lower level. The study suggested a nested genetic algorithm, and applied it on a transportation planning and management problem. Later, Halter and Mostaghim [25] used a particle swarm optimization (PSO)-based nested strategy to solve a multicomponent chemical system. The lower level problem in their application problem was linear for which they used a specialized linear multiobjective PSO approach. Recently, a hybrid bilevel evolutionary multiobjective optimization algorithm approach coupled with local search was proposed in [18]. In this paper, the authors handled nonlinear as well as discrete bilevel problems with a relatively large number of variables. The study also provided a suite of test problems for bilevel multiobjective optimization. An extension to this paper [37] attempted to solve bilevel multiobjective optimization with fewer function evaluations by interacting with the leader.

Until now, the focus has been primarily on algorithms for handling deterministic problems. Less emphasis has been paid to the decision-making intricacies that arise in practical multiobjective bilevel problems. The first concern is the reliance on the assumption that transfers decision-making power to the leader by allowing her to freely choose any Pareto-optimal solution from the lower level optimal frontier. In practical problems, the preferences of the lower level decision maker may not be aligned with the leader. Although a leader can anticipate the follower's actions and optimize her strategy accordingly, it is unrealistic to assume that she can decide which tradeoff the follower should choose. To solve hierarchical problems with conflicting decision-makers, a few studies have proposed a line of interactive fuzzy programming models [30], [35]. The methods have been successfully used to handle decentralized bilevel problems that have more than one lower level decision maker [12]. However, the assumption of mutual co-operation and repeated interactions between decision-makers is not necessarily feasible; e.g., in homeland security applications and competitive business decisions. The second concern is the decision-uncertainty. The strategy chosen by the follower may well deviate from what is expected by the leader, which thus gives rise to uncertainty about the realized outcome. It is worthwhile to note that the notion of decision-uncertainty that emanates from not knowing the follower's preferences exactly is different from the uncertainty that follows from nonpreference related factors such as stochastic model parameters. In this paper, we focus only on lower level decision-uncertainty and leave the other sources of uncertainty for future studies.

III. Multiobjective Bilevel Optimization and Decision Making

In this section, we provide three different formulations for a multiobjective bilevel optimization problem. First, we consider the standard formulation, where there is no decision making involved at the lower level and all the lower level Pareto-optimal solutions are considered at the upper level. Second, we consider a formulation, where the decision maker acts at the lower level and chooses a solution to her liking. This becomes a possible feasible solution at the upper level. Finally, we discuss a problem, where the lower level decision maker's preferences are not known with certainty and the upper level decision maker needs to take this decision-uncertainty into account when choosing her optimal strategy. A summary of the notations used in this paper is given in Table I along with references to the sections where they are first introduced.

A. *Multiobjective Bilevel Optimization: The Optimistic Formulation*

Bilevel multiobjective optimization is a nested optimization problem involving two levels of multiobjective optimization tasks. The structure of a bilevel multiobjective problem demands that only the Pareto-optimal solutions to the lower level optimization problem may be considered as feasible solutions for the upper level optimization problem. There are two classes of variables in a bilevel optimization problem; namely, the upper level variables $x_u \in X_U \subset \mathbb{R}^n$, and the lower level variables $x_l \in X_L \subset \mathbb{R}^m$. The lower level multiobjective problem is solved with respect to the lower level variables, x_l, and the upper level variables, x_u act as parameters to the optimization problem. Each x_u corresponds to a different lower level optimization problem, leading to a different Pareto-optimal front. The upper level problem is optimized with respect to both classes of variables, $x = (x_u, x_l)$.

TABLE I
SUMMARY OF CENTRAL NOTATIONS

Category	Notation(s)	Description	Section
Decision variables	$x_u \in X_U$	Leader's (upper level) decision variable and decision space.	3.1.
	$x_l \in X_L$	Follower's (lower level) decision variable and decision space.	3.1.
Objectives	$F = (F_1, \dots, F_p)$	Leader's (upper level) objective functions.	3.1.
	$f = (f_1, \dots, f_q)$	Follower's (lower level) objective functions.	3.1.
Constraints	$G_k,\ k = 1, \dots, K$	Leader's (upper level) constraint functions.	3.1.
	$g_j,\ j = 1, \dots, J$	Follower's (lower level) constraint functions.	3.1.
Preferences	$\xi \in \Xi$	Vector of uncertain parameters and parameter space describing the preferences of the follower.	3.2.
	$\mathcal{D}_\xi$	Distribution that characterizes the uncertainty about follower's parameters.	3.2.
	$\sigma : X_U \times \Xi \to X_L$	Function describing the follower's preferred action given x_u and preferences ξ.	3.2.
	$V(f, \xi)$	Uncertain value function for modeling the follower's preferences.	4.0.
	$\Xi_\alpha \subset \Xi$	Confidence region that contains the follower's preferences with probability greater than $1 - \alpha$.	4.2.
Set-valued mappings	$\Psi : X_U \rightrightarrows X_L$	$\Psi(x_u)$ represents the set of follower's Pareto-optimal solutions corresponding to the given leader's decision x_u.	3.1.
	$\bar{\sigma} : X_U \rightrightarrows X_L$	Gives the set of follower's solutions under expected preferences. In most cases, $\bar{\sigma}$ is an ordinary single-valued function.	4.1.
	$C_\alpha : X_U \rightrightarrows \mathbb{R}^p$	Leader's confidence region in objective space	4.2.

Definition 1: For the upper level objective function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and lower level objective function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, the bilevel problem is given by

$$\begin{aligned} \underset{x_u \in X_U, x_l \in X_L}{\text{minimize}} \quad & F(x_u, x_l) = \big(F_1(x_u, x_l), \dots, F_p(x_u, x_l)\big) \\ \text{subject to} \quad & x_l \in \underset{x_l}{\operatorname{argmin}} \big\{ f(x_u, x_l) = \big(f_1(x_u, x_l), \dots, f_q(x_u, x_l)\big) : \\ & \qquad g_j(x_u, x_l) \le 0, \quad j = 1, \dots, J \big\} \\ & G_k(x_u, x_l) \le 0, \quad k = 1, \dots, K \end{aligned}$$

where $G_k : X_U \times X_L \to \mathbb{R}$, $k = 1, \dots, K$ denote the upper level constraints, and $g_j : X_U \times X_L \to \mathbb{R}$ represent the lower level constraints, respectively. Equality constraints may also exist that have been avoided for brevity.

An equivalent formulation of the above problem can be stated in terms of set-valued mappings as follows.

Definition 2: Let $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping

$$\begin{aligned} \Psi(x_u) = \underset{x_l}{\operatorname{argmin}} \big\{ & f(x_u, x_l) = \big(f_1(x_u, x_l), \dots, f_2(x_u, x_l)\big) : \\ & g_j(x_u, x_l) \le 0, j = 1, \dots, J \big\} \end{aligned}$$

which represents the constraint defined by the lower level optimization problem, i.e., $\Psi(x_u) \subset X_L$ for every $x_u \in X_U$. Then the bilevel multiobjective optimization problem can be expressed as a constrained multiobjective optimization problem

$$\begin{aligned} \underset{x_u \in X_U, x_l \in X_L}{\text{minimize}} \quad & F(x_u, x_l) = \big(F_1(x_u, x_l), \dots, F_p(x_u, x_l)\big) \\ \text{subject to} \quad & x_l \in \Psi(x_u) \\ & G_k(x_u, x_l) \le 0, k = 1, \dots, K \end{aligned}$$

where Ψ can be interpreted as a parameterized range-constraint for the lower level decision vector x_l.

Remark 1: Bilevel optimal solution may not exist for all problems. Therefore, additional regularity and compactness conditions are needed to ensure existence of a solution. This is currently an active area of research. One of the most recent results has been presented by Gadhi and Dempe [24], who established necessary optimality conditions for optimistic multiobjective bilevel problems with the help of Hiriart-Urruty scalarization function. In the case of Definition 1, we would need to start by assuming that all functions are lower semicontinuous and locally Lipschitz. If the problem is also regular enough to allow the set $\bar{S} := \{(x_u, x_l) \in \mathbb{R}^n \times \mathbb{R}^m : G_k(x_u, x_l) \le 0\ \forall\ k \in \{1, \dots, K\}$, and $x_l \in \Psi(x_u)\}$ to be nonempty and compact, then the formulation has at least one optimal solution. For a further look at the existence and necessary conditions the readers may refer to [4], [24], and [49].

In the above formulation, the lower level decision maker is assumed to cooperate with the upper level decision maker, such that she provides all Pareto-optimal points to the upper level decision maker who then chooses the best point according to the upper level objectives. The assumption effectively reduces the influence of the follower and transfers the decision-making power to the leader. Alternatively, one can say that the lower level decision maker is assumed to be indifferent to all lower level Pareto-optimal solutions. Though this formulation has been studied in the past [18], [22], it is a highly unrealistic formulation where decision making aspects at the lower level are not taken into account.

B. Multiobjective Bilevel Optimization With Decision Making at Lower Level

Considering the decision-making situations that arise in practice, a departure from the assumption of an indifferent lower level decision maker is necessary. Rather than providing all Pareto-optimal points to the leader, the follower is likely to act according to her own interests and choose the most preferred lower level solution herself. As a result, the allowance of lower level decision making has a substantial impact on the formulation of multiobjective bilevel optimization problems. First, the lower level problem can no longer be viewed

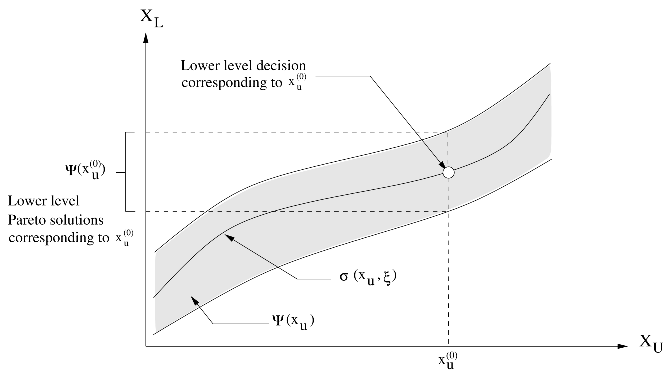

Fig. 1. Decision making under certainty.

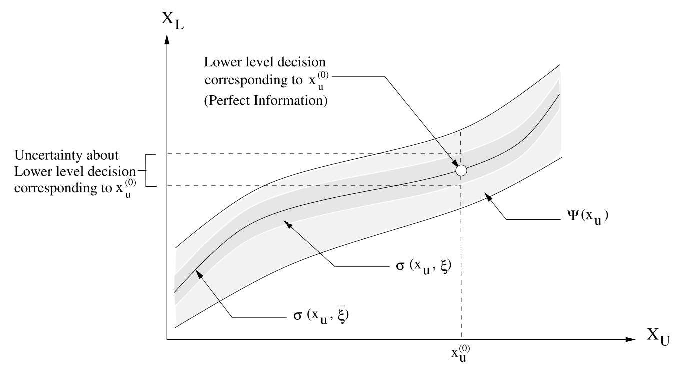

Fig. 2. Decision making under uncertainty.

as a range-constraint that depends only on lower level objectives. Instead it is better interpreted as a selection function that maps a given upper level decision to a corresponding Pareto-optimal lower level solution that it is most preferred by the follower. Second, in order to solve the bilevel problem, the upper level decision maker now needs to model the follower's behavior by anticipating her preferences toward different objectives. Naturally, these changes lead to a number of intricacies that were not encountered in the previous formulations. For simplicity, however, we will begin the discussion under assumption of perfect information.

Definition 3: Let $\xi \in \Xi$ denote a vector of parameters describing the follower's preferences. If the upper level decision maker has complete knowledge of the follower's preferences, the follower's actions can then be modeled via selection mapping

$$\sigma : X_U \times \Xi \to X_L, \quad \sigma(x_u, \xi) \in \Psi(x_u) \tag{1}$$

where Ψ is the set-valued mapping given by Definition 2. The resulting bilevel problem can be rewritten as

$$\begin{aligned} \underset{x_u \in X_U}{\text{minimize}} \quad & F(x_u, x_l) = \big(F_1(x_u, x_l), \ldots, F_p(x_u, x_l)\big) \\ \text{subject to} \quad & x_l = \sigma(x_u, \xi) \in \Psi(x_u) \\ & G_k(x_u, x_l) \leq 0, \quad k = 1, \ldots, K. \end{aligned}$$

Remark 2: In general, $\sigma(\cdot, \xi)$ may not be a continuous function. To guarantee this, stronger assumptions are necessary. For example, one result that leads to continuity of selection mappings is known as Michael representation theorem. If the set $P = \{(x_u, x_l) \in X_U \times X_L : G_k(x_u, x_l) \leq 0 \ \forall k\}$ is compact, and the mapping Ψ is closed convex-valued and lower (inner) semicontinuous at all $x_u \in P$, then Ψ can be represented in terms of continuous selection functions. Provided that the set of feasible solutions is nonempty, the conditions combined with lower semicontinuity of upper level objective functions would allow existence of bilevel optimum. For further discussions, refer to [15] and [31].

To illustrate the definition, consider Fig. 1, where the shaded region

$$\text{gph}\,\Psi = \{(x_u, x_l) : x_l \in \Psi(x_u)\} \tag{2}$$

represents the follower's Pareto-optimal solutions $\Psi(x_u)$ for any given leader's decision x_u. These are the rational reactions, which the follower may choose depending on her preferences. If the leader is aware of the follower's objectives, she will be able to identify the shaded region completely by solving the follower's multiobjective optimization problem for all x_u. However, if the follower is able to act according to her own preferences, she will choose only one preferred solution $\sigma(x_u, \xi)$ for every upper level decision x_u. When the preferences of the follower are perfectly known, the leader can identify $\sigma(\cdot, \xi)$ that characterizes follower's rational reactions for different x_u, and solve the hierarchical optimization task completely.

C. Multiobjective Bilevel Optimization With Lower Level Decision Uncertainty

The assumption that the follower's preferences are perfectly known to the leader is inaccurate as a description of real life scenarios. Most practitioners would find it hard to accept this even when constructing approximations. A natural path toward a more realistic framework would be to relax the axiom of perfect information by assuming that the leader is only partially aware of the follower's preferences. This lack of information leads to the notion of lower level decision uncertainty that is experienced by the leader while solving the bilevel optimization task.

For illustration, consider Fig. 2, where the expected behavior of the follower is shown as the graph of the selection mapping $\sigma(\cdot, \bar{\xi})$, where $\bar{\xi}$ represents the expected preference unknown to the leader. The narrow dark shaded band shows the region of uncertainty in which the follower makes her decisions. For different preferences ξ, $\sigma(\cdot, \xi)$ represents the corresponding decisions of the follower. If the leader is aware of the follower's objectives, the uncertainty region identified by a random ξ is always bounded by gph Ψ because $\sigma(x_u, \xi) \in \Psi(x_u)$ for all $x_u \in X_U$ and $\xi \in \Xi$. However, it is noteworthy that this band is not directly available to the leader but needs to be modeled. In a situation, where the leader cannot elicit follower's preferences by interacting with the follower, a feasible strategy is to utilize the prior information she has about the follower and incorporate it in a tractable stochastic model that characterizes the follower's behavior.

To accommodate the decision uncertainty, we assume that the follower's preferences are described by a random variable $\xi \sim \mathcal{D}_\xi$, which takes values in a set Ξ of $\mathbb{R}^q$. The probability

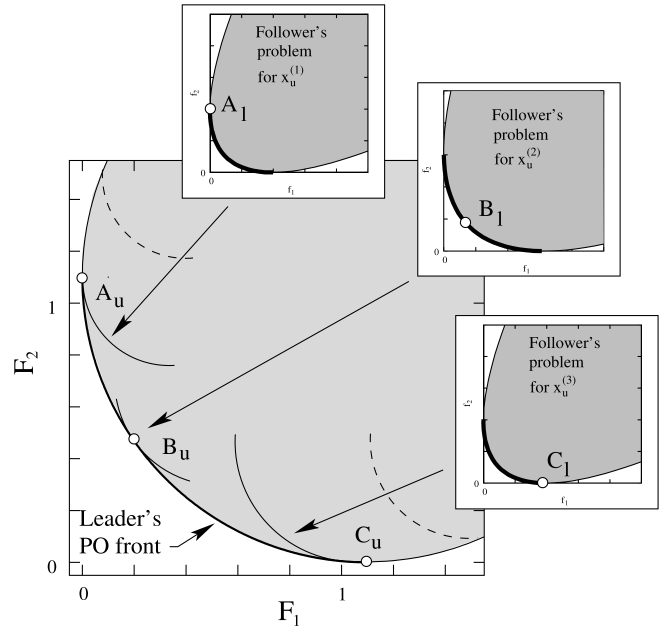

Fig. 3. Insets: follower's problem for different x_u. A_l, B_l, and C_l represent the follower's decisions for $x_u^{(1)}$, $x_u^{(2)}$, and $x_u^{(3)}$, respectively. A_u, B_u, and C_u are the corresponding points for the leader in the leader's objective space.

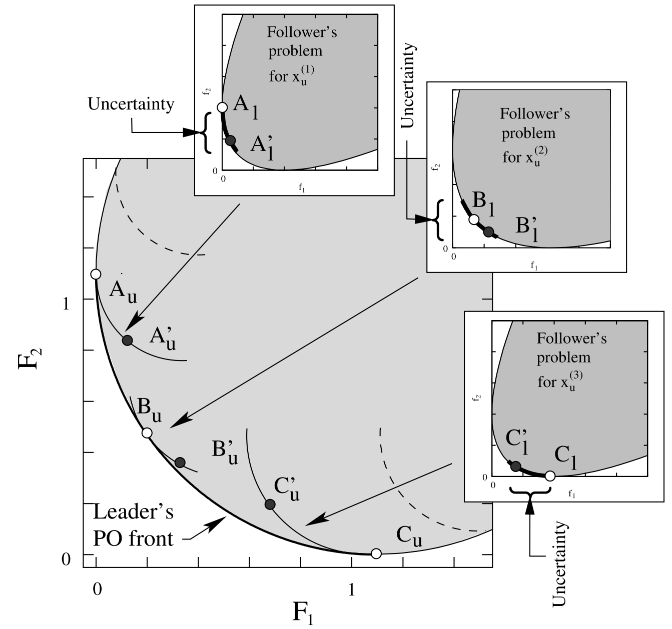

Fig. 4. Insets: follower's problem for different x_u. A_l, B_l, and C_l are the expected decisions of the follower. $A_l^{'}$, $B_l^{'}$, and $C_l^{'}$ are the actual decisions that the follower takes. The corresponding points for the leader are shown in the leader's objective space.

distribution $\mathcal{D}_\xi$ reflects the leader's uncertainty and prior information about follower's expected behavior. In this framework, the assumption of preference uncertainty is equivalent to saying that the lower level decision is a random variable with a distribution that is parametrized by a given upper level decision x_u, i.e., $x_l \sim \mathcal{D}_\sigma(x_u)$. This means that the lower level decision uncertainty experienced by the leader will vary pointwise depending on the follower's objectives and the leader's own decision.

For demonstration of the uncertainty aspects in the objective spaces of the leader and follower, consider Figs. 3 and 4 that show two different scenarios. In the first scenario, we assume a deterministic situation where the follower's preferences and actions are known with certainty. Both leader and follower are assumed to have two objectives, i.e., $p = q = 2$. In this case, the leader solves the bilevel problem in Definition 3 under perfect information. Therefore, each point on the leader's Pareto-frontier corresponds to one of the points on the follower's Pareto-frontier. If $\bar{\xi}$ is the given vector of follower's preferences, then for any leader's choice $x_u^{(i)}$ the corresponding lower level decision is given by $x_l^{(i)} = \sigma(x_u^{(i)}, \bar{\xi})$. This is shown in Fig. 3, where the upper level points $A_u = F(x_u^{(1)}, \sigma(x_u^{(1)}, \bar{\xi}))$, $B_u = F(x_u^{(2)}, \sigma(x_u^{(2)}, \bar{\xi}))$, and $C_u = F(x_u^{(3)}, \sigma(x_u^{(3)}, \bar{\xi}))$ are paired with the points $A_l = f(x_u^{(1)}, \sigma(x_u^{(1)}, \bar{\xi}))$, $B_l = f(x_u^{(2)}, \sigma(x_u^{(2)}, \bar{\xi}))$, and $C_l = f(x_u^{(3)}, \sigma(x_u^{(3)}, \bar{\xi}))$ that lie on the follower's Pareto-front for $x_u^{(1)}$, $x_u^{(2)}$, and $x_u^{(3)}$, respectively.

The situation can be contrasted from another scenario shown in Fig. 4, where the follower's preferences are uncertain. The leader is still assumed to be fully aware of the form of σ, but she no longer knows the true value of ξ. By assuming a prior information $\xi \sim \mathcal{D}_\xi$, the leader can attempt to solve the bilevel problem based on the expected preferences of the follower, that is

$$\begin{aligned} \underset{x_u \in X_U}{\text{minimize}} \quad & F(x_u, \bar{x}_l) \\ \text{subject to} \quad & \bar{x}_l = \sigma(x_u, E[\xi]) \in \Psi(x_u), \quad \xi \sim \mathcal{D}_\xi \\ & G_k(x_u, \bar{x}_l) \leq 0, \quad k = 1, \ldots, K. \end{aligned} \tag{3}$$

For convenience of the example, we assume that the expected actions are the same as the actions in Fig. 3, i.e., $\sigma(x_u, E[\xi])] = \sigma(x_u, \bar{\xi})$ for all x_u. As a result, the leader obtains a Pareto-optimal front corresponding to the follower's expected value function (POF-EVF). However, when she begins to implement the given strategies, the follower's realized actions may deviate from the expected strategies obtained by solving (3). Since ξ is uncertain from the leader's perspective, the follower's true preferences ξ can differ from $\bar{\xi}$ that was expected based on prior information. As shown in the figure, for any strategy $x_u^{(1)}$, $x_u^{(2)}$, or $x_u^{(3)}$ chosen by the leader, the follower may prefer to choose $A_l^{'}$, $B_l^{'}$, or $C_l^{'}$ instead of A_l, B_l, or C_l expected by the leader. It is found that because of the follower's deviation from the expected actions, the leader no longer operates on the POF-EVF. In the objective space, the uncertainty experienced by the leader is reflected in the probability and size of deviations away from the POF-EVF. The follower, on the other hand, does not experience similar uncertainty, because she can always observe the action taken by the leader before making her own decision.

Depending on the problem, uncertainty of the lower level decision maker's preferences may lead to significant losses at the upper level. Therefore, the leader would like to solve the bilevel problem taking the uncertainties into account. While making a decision, the leader might prefer those regions on its frontier, which are less sensitive to lower level uncertainties and at the same time offer an acceptable tradeoff between the objectives. For instance, in the context of the above example, we observe that the expected variation in the objective space is considerably less at the region corresponding to $x_u^{(2)}$ than at $x_u^{(1)}$ or $x_u^{(3)}$. If the leader chooses this point, she knows that the realized upper level objective values are only little affected by the actions of the lower level decision maker. From the perspective of practical decision making, it is valuable for the leader to be aware of the level of uncertainty associated with different strategies.

IV. Solving Multiobjective Bilevel Problems With Lower Level Decision Uncertainty

To solve a bilevel problem with lower level uncertainty, the leader needs to construct a model for the follower's decision making. The purpose of the behavioral model is to help the leader evaluate how the upper level objective values are affected by variations in the follower's preferences. Invoking the value function framework, we can assume that the follower's preferences are characterized by a function $V : \mathbb{R}^q \times \Xi \to \mathbb{R}$ that is parametrized by the preference vector ξ. This allows us to write σ as a selection mapping for a value function optimization problem with x_u and ξ as parameters

$$\sigma(x_u, \xi) \in \underset{x_l \in X_L}{\operatorname{argmin}} \big\{ V(f(x_u, x_l), \xi) : \quad g_j(x_u, x_l) \le 0, \\ j = 1, \dots, J \big\}. \tag{4}$$

For most purposes, it is sufficient to assume that V is a linear form where ξ acts as a stochastic weight vector for the different lower level objectives

$$V(f(x_u, x_l), \xi) = \sum_{i=1}^{q} f_i(x_u, x_l)\xi_i. \tag{5}$$

As observed in an empirical study [28], linear value functions lead to good approximations of actual choice behavior with minimal complexity. This should hold, in particular, when the number of objectives is large.

When the follower's preferences are uncertain, i.e., $\xi \sim \mathcal{D}_\xi$, the value function parametrized by ξ is itself a random mapping. To handle the uncertainty, the leader may use a two-step approach to solve the bilevel problem: 1) the leader can use her expectation of follower's preferences to obtain information about the location of the Pareto-optimal front by solving the bilevel problem with fixed parameters and value function V (see Section IV-A) and 2) the leader can examine the extent of uncertainty by estimating a confidence region around the POF-EVF (see Section IV-B). Based on the joint evaluation of the expected solutions and the uncertainty observed at different parts of the frontier, the leader can make a better tradeoff between her objectives while being aware of the probability of realizing a desired solution. Given the computational complexity of bilevel problems, carrying out these steps requires careful design. In this section, we outline the definitions and theory used within the suggested approach. The details needed for implementation of the algorithm are summarized in Section V.

A. Finding the Expected Pareto-Optimal Frontier

To obtain a set of reference solutions, the leader can solve the following problem to obtain the POF-EVF.

Definition 4: For the upper level objective function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ and lower level objective function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^q$, let $\bar{\sigma} : X_U \rightrightarrows X_L$ denote the set of solutions to the lower level problem under expected follower's preferences, that is

$$\bar{\sigma}(x_u) = \underset{x_l \in \Psi(x_u)}{\operatorname{argmin}} \{ V(f(x_u, x_l), E[\xi]) \}$$

where $\xi \sim \mathcal{D}_\xi$ is a prior distribution over the follower's preferences. Then the bilevel problem with expected follower's preferences can be rewritten as a deterministic problem

$$\begin{aligned} \underset{x_u \in X_U}{\text{minimize}} \quad & F(x_u, \bar{x}_l) \\ \text{subject to} \quad & \bar{x}_l \in \bar{\sigma}(x_u) \\ & G_k(x_u, \bar{x}_l) \le 0, k = 1, \dots, K \end{aligned}$$

where G_k, $k = 1, \dots, K$ denote the upper level constraints.

As suggested before, the lower level decision preferences can be modeled using a linear value function V and a reasonable approximation for POF-EVF can be obtained. For convenience, we also assume that the leader's uncertainty about the follower's preferences can be modeled by choosing a normal prior for the random parameters of the value function, i.e., $\xi \sim N(\mu_\xi, \Sigma_\xi)$.

When the expected value function can be computed analytically, the POF-EVF optimization task can be converted into a deterministic bilevel problem with only one objective at the lower level. To handle the problem, a number of techniques can be considered. In this paper, we propose an evolutionary algorithm that uses sequential approximations of $\bar{\sigma}$ to reduce the computational burden that follows from evaluating the lower level problem. A detailed description of the algorithm and the underlying theoretical results are given in Section V.

B. Analyzing the Degree of Uncertainty

The POF-EVF frontier, which is obtained as a solution to the problem in Definition 4, provides information only about the expected upper level frontier and not about the degree of uncertainty. Therefore, a two-step approach is needed, where the objective of the second step is to analyze the degree of uncertainty at different parts of the expected frontier. Since the uncertainty is a consequence of unknown follower's preferences, a natural approach to examine the effects of preference variations is to draw samples from the assumed prior distribution and see how the leader's objectives are influenced by the preference-induced changes in follower's decisions.

Definition 5 (Leader's Confidence Region in Objective Space): Let $\Xi_\alpha \subset \Xi$ denote the subset of preference vectors that contains the follower's true preferences with probability $\mathbb{P}(\xi \in \Xi_\alpha) \ge 1 - \alpha$, where $\alpha > 0$ denotes a given risk level. Assuming a normal prior distribution, $\xi \sim N(\mu_\xi, \Sigma_\xi)$, the set Ξ_α can be defined as the ellipsoid confidence region

$$\Xi_\alpha = \Big\{ \xi \in \Xi \mid \big(\xi - \mu_\xi\big)' \Sigma_\xi^{-1} \big(\xi - \mu_\xi\big) \le \chi^2(\alpha) \Big\}.$$

If $x_u^\star$ is a solution for the POF-EVF problem in Definition 4, the confidence band at $x_u^\star$ is given by the set of points $C_\alpha(x_u^\star)$

TABLE II
BI-OBJECTIVE PROBLEM WITH LOWER LEVEL UNCERTAINTY

Example 1	Level	Formulation
Variables	Upper level Lower level	x_u $x_l = (x_{l,1}, x_{l,2})$
Objectives	Upper level	$F_1(x_u, x_{l,1}, x_{l,2}) = x_{l,1} - x_u$ $F_2(x_u, x_{l,1}, x_{l,2}) = x_{l,2}$
	Lower level	$f_1(x_u, x_{l,1}, x_{l,2}) = x_{l,1}$ $f_2(x_u, x_{l,1}, x_{l,2}) = x_{l,2}$
Constraints	Upper level	$G_1(x_{l,1}, x_{l,2}) = 1 + x_{l,1} + x_{l,2} \geq 0$ $-1 \leq x_{l,1}, \quad x_{l,2} \leq 1, \quad 0 \leq x_u \leq 1$
	Lower level	$g_1(x_u, x_{l,1}, x_{l,2}) = x_u^2 - x_{l,1}^2 - x_{l,2}^2 \geq 0$
Preference uncertainty	Lower level	$V(f_1, f_2) = \xi_1 x_u^2 f_1 + \xi_2 f_2,$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (5, 1), \quad \Sigma_\xi = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$

in the upper level objective space, where $C_\alpha : X_U \rightrightarrows \mathbb{R}^p$ is the confidence mapping

$$C_\alpha\left(x_u^\star\right) = \left\{F\left(x_u^\star, x_l\right) \in \mathbb{R}^p | x_l \in \sigma\left(x_u^\star, \xi\right) \text{ for some } \xi \in \Xi_\alpha\right\}.$$

The point-wise confidence region $C_\alpha(x_u^\star)$ can be interpreted as the range of leader's objective values that contains the realized objective value with the given probability.

Generally, the confidence mapping C_α cannot be evaluated exactly. Therefore, sampling techniques are necessary to approximate the confidence region at different parts of the expected frontier. Given a set of upper level decisions x_u representing the relevant regions of POF-EVF frontier and a sample of preference vectors from Ξ_α, we solve multiple lower level optimization problems (or a single multiobjective optimization problem) corresponding to the sampled value functions. The optimal (x_u, x_l)-pairs corresponding to different sampled value functions are then mapped to the upper level objective space. This translates the lower level uncertainties to the upper level objective space and leads to an approximation for the confidence region $C_\alpha(x_u)$ at different x_u. Depending on the problem being solved, the confidence region may exhibit substantial variations across the expected upper level frontier. With all the points analyzed together, the upper level decision maker might prefer a point x_u, which offers an acceptable tradeoff between upper level objectives and is less susceptible to lower level decision uncertainty. Therefore, it is important for the leader to be aware of the degree of uncertainty associated with different decision candidates. An efficient way to compute the confidence band is described in Section V.

C. Examples of Bilevel Problems With Lower Level Uncertainty

1) Example 1: For illustration, consider the multiobjective bilevel optimization problem defined in Table II. The problem has two objectives at each level and three decision variables. While modeling the follower's behavior, the leader assumes that the uncertainty about follower's preferences is characterized by a simple bivariate normal distribution with diagonal covariance matrix.

For any fixed value of the upper level decision x_u, the feasible region of the lower level problem is the area inside a circle with a center at origin ($x_{l,1} = x_{l,2} = 0$) and radius equal to x_u. For a given x_u, the Pareto-optimal set for the lower level optimization task is the south-west quarter of the circle, that is

$$\Psi(x_u) = \Big\{\left(x_{l,1}, x_{l,2}\right) \in \mathbb{R}^2 \;\Big|\; x_{l,1}^2 + x_{l,2}^2 = x_u^2, \\ x_{l,1} \leq 0, x_{l,2} \leq 0\Big\}.$$

Before considering the lower level decision making aspects, let us first examine the problem in the sense of Definition 1. The solution to this problem gives the best possible frontier that can be attained at the upper level, when the preferences of the follower are perfectly aligned with the leader. In this ideal scenario, the leader is free to choose any suitable point from the lower level frontier. For the above example, the upper level Pareto-optimal set is given by

$$(x_u, x_{l,1}, y_{l,2})^* = \left\{\left(x_u, x_{l,1}, x_{l,2}\right) \in \mathbb{R}^3 \;\middle|\; x_u \in \left[\frac{1}{\sqrt{2}}, 1\right], \\ x_{l,1} = -1 - x_{l,2}, x_{l,2} = -\frac{1}{2} \pm \frac{1}{4}\sqrt{8x_u^2 - 4}\right\}.$$

The corresponding Pareto-optimal frontier is shown in Fig. 5. It is clear that at most two members from the lower level frontiers corresponding to $x_u \in [\sqrt{0.5}, 1]$ participate in the upper level front. Note that, for $x_u = 0.9$ points B and C are Pareto-optimal at the upper level and point A is infeasible because of the upper level constraint.

Now, let us view the problem in terms of Definitions 4 and 5, where the lower level decision maker has complete power to choose a point from her Pareto-front. In this example, the leader assumes that the follower's preferences can be described by an uncertain value function $V(f_1, f_2) = \xi_1 x_u^2 f_1 + \xi_2 f_2$. To obtain a set of reference solutions, the leader can solve the

TABLE III
BI-OBJECTIVE PROBLEM WITH SCALABLE NUMBER OF LOWER LEVEL VARIABLES

Example 2	Level	Formulation
Variables	Upper level Lower level	x_u $x_l = (x_{l,1}, \ldots, x_{l,K})$
Objectives	Upper level	$F_1(x_u, x_l) = (x_{l,1} - 1)^2 + \sum_{i=2}^{K} x_{l,i}^2 + x_u^2$ $F_2(x_u, x_l) = (x_{l,1} - 1)^2 + \sum_{i=2}^{K} x_{l,i}^2 + (x_u - 1)^2$
	Lower level	$f_1(x_u, x_l) = x_{l,1}^2 + \sum_{i=2}^{K} x_{l,i}^2$ $f_2(x_u, x_l) = \lvert x_u \rvert (x_{l,1} - x_u)^2 + \sum_{i=2}^{K} x_{l,i}^2$
Constraints	Upper level	$-1 \leq (x_u, x_{l,1}, \ldots, x_{l,K}) \leq 2$
Preference uncertainty	Lower level	$V(f_1, f_2) = \xi_1 f_1 + \xi_2 f_2,$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (1, 2), \quad \Sigma_\xi = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}$

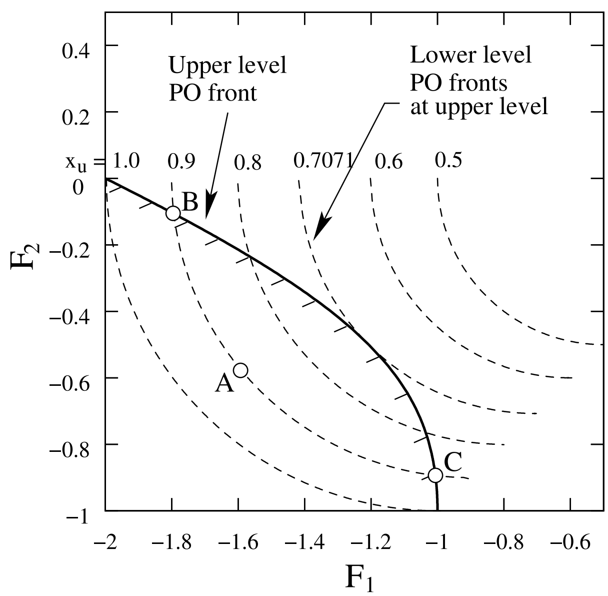

Fig. 5. Upper level Pareto-optimal front (without lower level decision making) and few representative lower level Pareto-optimal fronts in upper level objective space for Example 1.

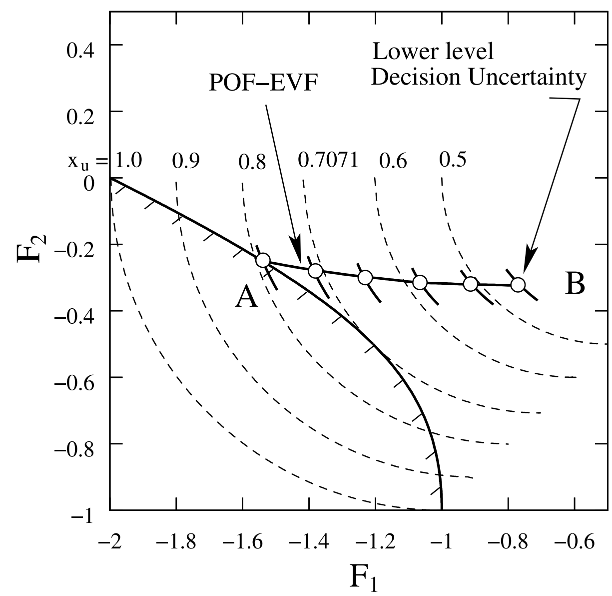

Fig. 6. Expected Pareto-optimal front at upper level (POF-EVF) and lower level decision uncertainty $C_\alpha(x_u)$ at different x_u for Example 1.

problem in Definition 4 to obtain the Pareto-optimal frontier (see Fig. 6) that corresponds to the follower's expected value function $V(f_1, f_2) = 5x_u^2 f_1 + f_2$. It is noteworthy that the assumed value function also contains x_u. This kind of dependency may not always exist, but has been considered here to show that the lower level value function may take any possible form. For a few Pareto-optimal x_u, the lower level Pareto-optimal solutions corresponding to decision uncertainty $C_{0.01}(x_u)$ are also shown by bold lines. On the frontier AB, while moving from B to A, the leader gains substantially in terms of F_1 while losing only little in terms of F_2. Therefore, the leader might be interested in choosing an x_u corresponding to A. However, if the leader chooses x_u corresponding to point A, then she might end up with an infeasible point subject to the decision taken by the follower. Therefore, the leader should choose an x_u, which is close to A but still feasible with a high probability. From leader's perspective both feasibility as well as the extent of uncertainty are important considerations while making a decision.

2) Example 2: The second example presents a bi-objective problem that is scalable in terms of lower level variables; see Table III. In this paper, we choose $K = 14$, such that the problem contains 15 variables in total: one upper level variable and 14 lower level variables. The follower's preferences are again assumed to follow a linear value function with a bivariate normal distribution for the weights.

For any x_u, the Pareto-optimal solutions of the lower level optimization problem are given by

$$\Psi(x_u) = \left\{ x_l \in \mathbb{R}^K \;\middle|\; x_{l,1} \in [0, x_u],\; x_{l,i} = 0 \text{ for } i = 2, \ldots, K \right\}.$$

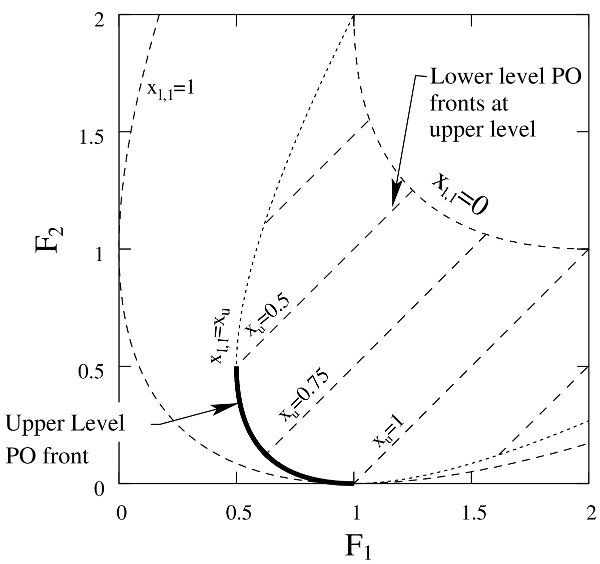

Fig. 7. Upper level Pareto-optimal front (without lower level decision making) and few representative lower level Pareto-optimal fronts in upper level objective space for Example 2.

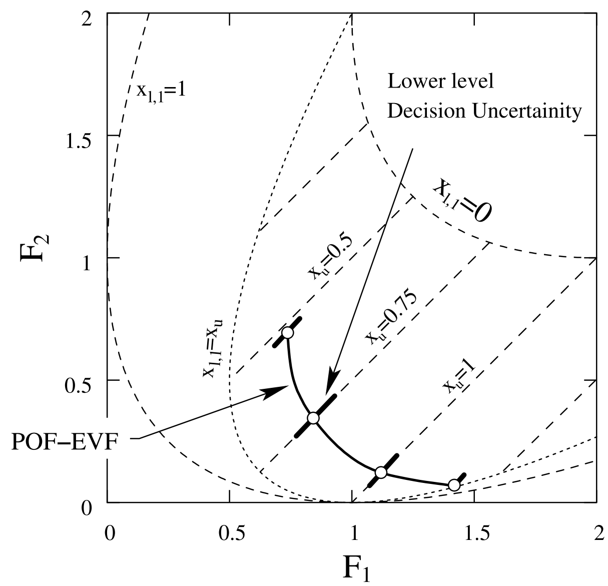

Fig. 8. Expected Pareto-optimal front at upper level and lower level decision uncertainty for Example 2.

The best possible frontier at the upper level may be obtained when the lower level decision maker does not have any real decision-making power; see Fig. 7. In this example, the leader's ideal frontier, that is obtained by solving the problem in Definition 1, is given by

$$\Big\{(x_u, x_l) \in \mathbb{R}^{K+1} \big| x_u \in [0.5, 1.0], x_{l,1} = x_u, x_{l,i} = 0 \text{ for } i = 2, \ldots, K\Big\}.$$

Now, let us consider the full example with a lower level decision-maker, whose preferences are assumed to follow $V(f_1, f_2) = \xi_1 f_1 + \xi_2 f_2$. The upper level front corresponding to the expected value function is obtained by solving the POF-EVF problem in Definition 4. The outcome is shown in Fig. 8, where the follower's influence on the bilevel solution is shown as shift of the expected frontier away from the leader's optimal frontier. The extent of decision uncertainty is again described using the bold lines around the POF-EVF front. Each line corresponds to the leader's confidence region $C_\alpha(x_u)$ with $\alpha = 0.01$ at different x_u. When examining the confidence regions at different parts of the frontier, substantial variations can be observed.

V. Algorithm

In this section, we provide a detailed description of an evolutionary algorithm for handling multiobjective bilevel problems with lower level decision uncertainty. The main steps can be summarized as follows.

1) *Specification of the Preference Model:* The leader defines a model for the follower's behavior by choosing a value function V and a distribution $\xi \sim \mathcal{D}_\xi$ that reflect her prior information and expectations about the follower's preferences.
2) *Finding the Expected Frontier:* Given the assumptions on follower's behavior, the leader obtains a set of reference solutions by solving the expected Pareto-optimal frontier in Definition 4. To find the frontier, we propose multiobjective bilevel evolutionary algorithm based on quadratic (m-BLEAQ) algorithm:[1] a multiobjective bilevel evolutionary method that utilizes quadratic approximations for the lower level decisions.
3) *Finding the Confidence Regions:* After observing the expected frontier, the leader needs to examine the confidence regions at different parts of the frontier to understand the impact of lower level decision uncertainty. To approximate the regions, we propose a multiobjective optimizer that can solve the problem in Definition 5 efficiently without the need for repeated runs.

A. Bilevel Evolutionary Algorithm for Finding the Expected Frontier

The bilevel problem in Definition 4, that leads to the expected frontier (POF-EVF), can be solved using any algorithm which admits multiple objectives at the upper level and a single objective at the lower level. However, given the complexity encountered in practical problems, computational efficiency is an important concern. Therefore, brute force methods such as nested algorithms are generally not appealing. In this section, we describe m-BLEAQ algorithm, which is a model-based evolutionary method that relies on the insights drawn from parametric optimization theory. The core idea is quite simple: instead of solving the lower level problem every time from scratch, the algorithm can learn to approximate the follower's optimal decision mapping $\bar{\sigma}$. When viewing the lower level problem as a parametrized constraint, we observe that the complexity of the bilevel problem can be

[1]Earlier versions of the algorithm for solving single objective bilevel optimization problems can be found in [38] and [39], and for solving multiobjective bilevel optimization problems can be found in [41].

TABLE IV
STEP-BY-STEP PROCEDURE FOR M-BLEAQ

Step	Description
1	**Generate an initial population $\mathcal{P}$ of size** N: (a) Draw a random sample of upper level vectors $x_u^{(1)},\ldots,x_u^{(N)}$; (b) For each $x_u^{(j)}$, find a corresponding optimal lower level solution $x_l^{(j)} \in \bar{\sigma}(x_u^{(j)})$ by solving the lower level problem (see Section V-A2 for outline of lower-level algorithm). Set $\mathcal{P} = \{(x_u^{(j)}, x_l^{(j)}), j = 1,\ldots,N\}$.
2	**Lower level quality check**: Tag all vectors $x^{(j)} \in \mathcal{P}$ for which a lower level optimization has been successfully performed as members of set $\mathcal{P}_1$, and others as members of set $\mathcal{P}_0$.
3	**Parent selection**: Choose randomly $2(\mu-1)$ number of members from the population $\mathcal{P}$, and perform a tournament selection based on the upper level fitness. This produces $\mu-1$ number of parents, denoted by $\mathcal{P}_{\text{par}}$.
4	**Offspring generation**: Create λ offspring, denoted by $\mathcal{P}_{\text{off}}$, from the set of parents $\mathcal{P}_{\text{par}}$ using genetic operators. For each offspring $x^{(j)} = (x_u^{(j)}, x_l^{(j)}) \in \mathcal{P}_{\text{off}}$, update the lower level decision $x_l^{(j)}$ to reflect $x_u^{(j)}$ using the following strategy: *Case 1. [Optimization]*: If the size of set $\mathcal{P}_1$ is less than $\frac{1}{2}[(dim(x_u)+1)(dim(x_u)+2)] + dim(x_u)$, perform lower level optimizations to ensure that $x_l^{(j)} \in \bar{\sigma}(x_u^{(j)})$ (see Section V-A2 for outline of lower-level algorithm). *Case 2. [Approximation]*: If the size of set $\mathcal{P}_1$ is more than $\frac{1}{2}[(dim(x_u)+1)(dim(x_u)+2)] + dim(x_u)$, use the points in $\mathcal{P}_1$ to construct a quadratic approximation q for the mapping $\bar{\sigma}$. For each offspring $x^{(j)} = (x_u^{(j)}, x_l^{(j)}) \in \mathcal{P}_{\text{off}}$, update the lower level decision associated with the upper level $x_u^{(j)}$ by setting $x_l^{(j)} = q(x_u^{(j)})$.
5	**Lower level quality check**: If successful lower level optimizations were performed for the offspring (in above Case 1), then add them to set $\mathcal{P}_1$; otherwise to set $\mathcal{P}_0$. If quadratic approximations were performed (in above Case 2) and the mean squared error, e_{mse}, in approximation is less than e_0, then tag the offspring as members of $\mathcal{P}_1$; otherwise as members of $\mathcal{P}_0$.
6	**Population update**: Form a pool of worst r members from the parent population $\mathcal{P}_{\text{par}}$ and offspring $\mathcal{P}_{\text{off}}$. The best r members from the pool replace the chosen members from the population.
7	**Termination check:** Perform a termination check. If false, proceed to the next generation (Step 3).

substantially reduced when the decision mapping $\bar{\sigma}$ is available. Therefore, approximation of the decision mapping with single-valued functions can be an effective technique to avoid frequent optimization of the lower level problem. Even when the approximation is poor, it can provide an educated guess that can help the algorithm to quickly converge to the true lower level optimal solution.

1) Recipe and Parameters: A step-by-step procedure for m-BLEAQ is provided in Table IV. Although the overall structure of the algorithm resembles a standard evolutionary procedure, crucial differences are observed in the strategy used for generating offspring and the subsequent need for lower level quality checks. The challenge in bilevel programming is that the solution candidates for the upper level problem are always required to be optimal with respect to the lower level problem. However, performing repeated lower level optimizations is costly. To reduce the computational burden, we adopt two complementary strategies for offspring generation: a) creation of offspring with fully optimized lower level decisions and b) generation of offspring with approximately optimal lower level decisions. Of course, model-based approximations can only be used to speed up the algorithm between iterations, but they can never be given as final solutions. Furthermore, incorporation of approximate solutions requires a continuous quality monitoring mechanism to eliminate the influence of bad intermediate approximations on subsequent generations. To complement the steps shown in Table IV, additional details on the lower level optimization, implementation of the quadratic approximations, and termination check is given below in Sections V-A2–V-A5. A summary of the parameters is given in Table V.

TABLE V
PARAMETERS IN M-BLEAQ

Parameter	Description	Value
N	Population size	50
μ	Number of parents	3
λ	Number of offspring from μ parents	2
r	Solutions replaced at population update	2
e_0	Approximation error threshold	0.001
p_c	Crossover probability	0.9
p_m	Mutation probability	0.1
α_u^{stop}	Upper level termination threshold	1e-5
α_l^{stop}	Lower level termination threshold	1e-5

2) Lower Level Optimization: To solve the lower level problem (at steps 2 and 4 in Table IV), another evolutionary algorithm is used. The steps are shown in Table VI. The optimization task is considered to terminate successfully, if the termination criteria discussed in Section V-A5 are met. The fitness assignment at this level is performed based on lower level value function and constraints (see Section V-A4). If the lower level problem can be handled using a classical optimization algorithm, then it should be preferred over an evolutionary algorithm.

3) Approximation of $\bar{\sigma}$: As briefly discussed above, the complexity of the POF-EVF problem can be substantially reduced if the mapping $\bar{\sigma}$ is readily available. Knowing $\bar{\sigma}$ would reduce the bilevel problem to single level, which can be handled using a suitable optimizer. Although the mapping is rarely available in an analytical form, it may still be possible to approximate it locally. In fact, based on [20, Th. 1] or [21, Th. 2G.9], we can expect the mapping $\bar{\sigma}$ to behave almost

TABLE VI
STEP-BY-STEP PROCEDURE FOR LOWER LEVEL OPTIMIZATION IN m-BLEAQ

Step	Description
1	**Initialization:** Use all the lower level members from the upper level as initial solutions. If the lower level members are not available, then initialize them randomly. The upper level variable for which the optimization is being performed is kept fixed. Assign fitness to the members (Refer to Sub-section V-A4) based on lower level objective functions and constraints.
2	**Parent selection:** Randomly choose 2μ members from the population, and perform a tournament selection. This gives μ parents for crossover.
3	**Offspring generation:** Create λ offspring from the parents using genetic operators for the lower level variables only.
4	**Population update:** Randomly choose r random members from the population, and pool them with λ offspring. The best r members from the pool replace the chosen r members from the population.
5	**Termination:** Perform a termination check. Proceed to next generation (Step 2), if the termination criterion is not satisfied.

like a single-valued smooth function the closer one is to the optimal point. After recognizing the connection with parametric optimization and sensitivity analysis, it is intuitive, that under appropriate regularity conditions, a small change in the upper level variable would usually lead to a minor change in the lower level optimization problem.

Therefore, to reduce the number of lower level optimizations, it may be possible to construct a strategy for approximating $\bar{\sigma}$ with a suitable function. Any approximation requires a set of sample points, and an evolutionary framework that maintains a population of members at every generation, readily provides such a sample that can be used to approximate the mapping. In the case of m-BLEAQ, one approach is to use the set $\mathcal{P}_1$ containing the members that have passed lower level quality check. If we want to find an estimate of $\bar{\sigma}$ around a particular solution $(x_u, x_l) : x_l \in \bar{\sigma}(x_u)$ in the population, where

$$\left\{\left(x_u^{(i)}, x_l^{(i)}\right) \in X_U \times X_L \mid x_l^{(i)} \in \bar{\sigma}\left(x_u^{(i)}\right), i \in \mathcal{I}\right\} \subset \mathcal{P}_1$$

are a sample of population members in the neighborhood of (x_u, x_l), then the task of finding a good approximation can be viewed as an ordinary supervised learning problem.

Definition 6 (Learning of $\bar{\sigma}$): Let $\mathcal{H}$ be the hypothesis space, i.e., the set of functions that can be used to predict the optimal lower level decision from the given upper level decision. Given the sample from $\mathcal{P}_1$, our goal is to choose a model $q \in \mathcal{H}$ such that it minimizes the empirical error on the sample dataset, that is

$$q = \operatorname*{argmin}_{h \in \mathcal{H}} \sum_{i \in \mathcal{I}} \left\| x_l^{(i)} - h\left(x_u^{(i)}\right) \right\|_L^2 \tag{6}$$

where $||\cdot||_L$ is the norm in the lower level decision space X_L. In this paper, we have chosen $\mathcal{H}$ to be the set of quadratic functions.

4) Constraint Handling and Fitness Assignment: The constraint handling scheme defined in [16] is used at both levels. It should be noted that the lower level optimization task is a constraint for the upper level problem. However, we compute the constraint violations at the upper and lower levels only with respect to the upper and lower level inequality constraints, respectively. The overall constraint violation for any solution at a level is the summation of the violations of all the inequality constraints at the level. The fitness assignment at both upper and lower level is done in a manner that automatically incorporates the constraint handling mechanism.

a) Upper level fitness: For the infeasible upper level solutions, the fitness value is defined as the negative of the sum of upper level constraint violations. For feasible upper level solutions, the fitness is assigned based on the nondominated rank and crowding distance for a given point using

$$F_u(x) = \frac{1}{\mathrm{NR}(x) + e^{-\mathrm{CD}(x)}}$$

where $\mathrm{NR}(x)$ denotes the nondominated rank for a solution x in the objective space, and $\mathrm{CD}(x)$ denotes the crowding distance for a solution x in the objective space.

b) Lower level fitness: For feasible lower level solutions, the fitness is assigned based on the lower level value function. For infeasible lower level solutions, the fitness is assigned by subtracting the sum of lower level constraint violations from the fitness value of the least fit feasible lower level solution.

5) Operators and Termination: Below we have described the genetic operators and the termination criterion used in the algorithm.

a) Crossover and mutation: A parent centric crossover (PCX) and a polynomial mutation are performed to generate new parents at step 3 in Table IV. The crossover operator used in step 4 is a modified PCX operator, where three parents are required to create an offspring

$$c = x^{(p)} + \omega_\xi d + \omega_\eta \frac{p^{(2)} - p^{(1)}}{2}$$

where $x^{(p)}$ is the index parent, $d = x^{(p)} - g$ with g denoting the mean of all parents, $p^{(1)}$ and $p^{(2)}$ are the other two parents, $\omega_\xi = 0.1$, and $\omega_\eta = \dim(x^{(p)})/||d||$. The same crossover and mutation operators are used at both upper as well as lower level.

b) Termination: To check convergence, we use an expected improvement-based termination criterion at both levels. When we observe that the expected improvement in hypervolume at the upper level over N function evaluations is less than α_u^{stop}, then we terminate the algorithm at the upper level. At the lower level, when we observe that the expected improvement in fitness over N function evaluations is less than α_l^{stop}, then we terminate the algorithm at the lower level. The termination conditions for both levels are as

$$\frac{H^{\max} - H^{\min}}{H^{\max} + H^{\min}} \le \alpha_u^{\mathrm{stop}}, \qquad \frac{F^{\max} - F^{\min}}{F^{\max} + F^{\min}} \le \alpha_l^{\mathrm{stop}} \tag{7}$$

TABLE VII
STEP-BY-STEP PROCEDURE FOR APPROXIMATION OF LEADER'S CONFIDENCE REGIONS

Step	Description
1	**Preference sampling:** Given a fixed $\alpha > 0$, draw a sample of N preference vectors $\xi^{(n)}$, $n = 1, \ldots, N$ from the prior distribution such that $\xi^{(n)} \in \Xi_\alpha$, where $$\Xi_\alpha = \left\{\xi \in \Xi \mid (\xi - \mu_\xi)' \Sigma_\xi^{-1} (\xi - \mu_\xi) \leq \chi^2(\alpha)\right\}.$$
2	**POF-EVF sampling:** Choose a sample $P \subset X_U$ of upper level points $x_u^\star$ that are solutions to the POF-EVF problem. The points should be chosen such that they are evenly spaced across the leader's expected Pareto optimal frontier or represent a region preferred by the leader. In this paper we suggest the use of crowding distance to perform the selection.
3	**Evaluation of confidence regions:** For every $x_u^\star \in P$, compute an approximation of the confidence set by using $$\widehat{C}_\alpha(x_u^\star) = \{F(x_u^\star, x_l) \mid x_l \in \underset{x_l \in \Psi(x_u^\star)}{\operatorname{argmin}} \{V(f(x_u, x_l), \xi) \text{ for some } \xi \in \widehat{\Xi}_\alpha\},$$ where $\widehat{\Xi}_\alpha = \{\xi^{(n)} \in \Xi_\alpha, n = 1, \ldots, N\}$ is the sample of preference vectors constructed at Step 1. An efficient approach for computation of $\widehat{C}_\alpha(x_u^\star)$ is described in Section V-B2.

TABLE VIII
MULTIOBJECTIVE APPROACH FOR COMPUTING $\widehat{C}_\alpha(x_u^\star)$

Step	Description
1	For a given $x_u^\star$, consider the problem to be optimized as $$\Psi(x_u^\star) = \underset{x_l}{\operatorname{argmin}} \{f(x_u, x_l) = (f_1(x_u, x_l), \ldots, f_q(x_u, x_l)) : g_j(x_u, x_l) \leq 0, j = 1, \ldots, J\}.$$
2	Choose a multi-objective evolutionary algorithm, say NSGA-II or SPEA-II, to solve the problem.
3	Modify the dominance criteria such that, at any generation of the algorithm, points that are best for at least one of the N value functions, $\{V(\cdot, \xi) \ : \ \xi \in \widehat{\Xi}_\alpha\}$, are considered to be non-dominated and the rest are considered to be dominated.
4	Execute the algorithm until termination. The points obtained as a solution are considered as representatives of set $\widehat{C}_\alpha(x_u^\star)$.

where H denotes hypervolume at upper level and F denotes fitness at lower level. The maximum and minimum values are calculated over any N consecutive function evaluations.

B. Multiobjective Method for Efficient Approximation of the Confidence Regions

The m-BLEAQ algorithm described in Section V-A leads to the estimate of the expected upper level frontier (POF-EVF). The next step is to examine the impact of lower level decision uncertainty at the upper level.

1) Step-by-Step Procedure: Following our earlier discussion in Section IV-B, we consider a sampling mechanism to approximate the leader's confidence regions at different parts of the POF-EVF frontier. A stepwise description of the procedure is given in Table VII. Although it appears straightforward, it is computationally heavy due to the optimization tasks incorporated in step 3. If approached using brute force techniques, estimation of the confidence regions would require solving N optimization problems for every $x_u^\star$ that is chosen to represent the POF-EVF frontier. Therefore, we suggest an improved mechanism for evaluating $\widehat{C}_\alpha(x_u^\star)$ that requires only solving a single optimization problem instead of N.

2) Improved Approach for Computing $\widehat{C}_\alpha(x_u^\star)$: To avoid the computational burden of solving N different single-objective tasks to evaluate $\widehat{C}_\alpha(x_u^\star)$, we suggest using a multiobjective optimization approach to achieve the optimal strategies corresponding to the different value functions in a single run. The key steps are summarized in Table VIII. The approach is flexible and allows the use of almost any multiobjective evolutionary algorithm to handle the task. The only requirement is the incorporation of a modified dominance criterion. In our experiments, we have used $N = 50$ as the number of different parameter vectors sampled from Ξ_α. For optimization, we used NSGA-II with population size of 50 and 200 generations for all the problems. The points obtained as a solution can then be plotted in the upper level objective space along with the POF-EVF to get an estimate of the confidence region $C_\alpha(x_u^\star)$.

VI. EXPERIMENTS AND RESULTS

In this section, we evaluate the two-step algorithm described in Section V on four numerical test problems and two practical problems. To establish baseline results, we compare the performance of m-BLEAQ against an earlier algorithm hybrid bilevel evolutionary multiobjective optimization (H-BLEMO) [18] in terms of accuracy as well as computational efficiency.

A. Test Problems

In addition to the two example problems introduced in Sections IV-C1 and IV-C2, we use modified DS [17], [18] test problems (10 and 20 variable instances) in this paper.

TABLE IX
DS TEST PROBLEMS WITH DECISION UNCERTAINTY

DS 1	Level	Formulation
Variables	Upper level Lower level	$x = (x_1, \ldots, x_K) \in \mathbb{R}^K$ $y = (y_1, \ldots, y_K) \in \mathbb{R}^K$
Objectives	Upper level	$F_1(x,y) = (1 + r - \cos(\alpha\pi x_1)) + \sum_{j=2}^{K}(x_j - \frac{j-1}{2})^2 + \tau\sum_{i=2}^{K}(y_i - x_i)^2 - r\cos\left(\gamma\frac{\pi}{2}\frac{y_1}{x_1}\right)$ $F_2(x,y) = (1 + r - \sin(\alpha\pi x_1)) + \sum_{j=2}^{K}(x_j - \frac{j-1}{2})^2 + \tau\sum_{i=2}^{K}(y_i - x_i)^2 - r\sin\left(\gamma\frac{\pi}{2}\frac{y_1}{x_1}\right)$
	Lower level	$f_1(x,y) = y_1^2 + \sum_{i=2}^{K}(y_i - x_i)^2 + \sum_{i=2}^{K}10(1 - \cos(\frac{\pi}{K}(y_i - x_i)))$ $f_2(x,y) = \sum_{i=1}^{K}(y_i - x_i)^2 + \sum_{i=2}^{K}10\lvert\sin(\frac{\pi}{K}(y_i - x_i)\rvert$
Constraints	Upper level	$y_i \in [-K, K],\ i = 1, \ldots, K,$ $x_1 \in [1, 4],\ x_j \in [-K, K],\ j = 2, \ldots, K$
Preference uncertainty	Lower level	$V(f_1, f_2) = \xi_1 f_1 + \xi_2 f_2,$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (5.0, 1.5), \quad \Sigma_\xi = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.16 \end{bmatrix}$

DS 2	Level	Formulation
Variables	Upper level Lower level	$x = (x_1, \ldots, x_K) \in \mathbb{R}^K$ $y = (y_1, \ldots, y_K) \in \mathbb{R}^K$
Objectives	Upper level	$F_1(x,y) = v_1(x_1) + \sum_{j=2}^{K}\left[x_j^2 + 10(1 - \cos(\frac{\pi}{K}x_i))\right] + \tau\sum_{i=2}^{K}(y_i - x_i)^2 - r\cos\left(\gamma\frac{\pi}{2}\frac{y_1}{x_1}\right)$ $F_2(x,y) = v_2(x_1) + \sum_{j=2}^{K}\left[x_j^2 + 10(1 - \cos(\frac{\pi}{K}x_i))\right] + \tau\sum_{i=2}^{K}(y_i - x_i)^2 - r\sin\left(\gamma\frac{\pi}{2}\frac{y_1}{x_1}\right)$ $v_1(x_1) = \begin{cases} \cos(0.2\pi)x_1 + \sin(0.2\pi)\sqrt{\lvert 0.02\sin(5\pi x_1)\rvert}, & \text{for } 0 \le x_1 \le 1 \\ x_1 - (1 - \cos(0.2\pi)), & x_1 > 1. \end{cases}$ $v_2(x_1) = \begin{cases} -\sin(0.2\pi)x_1 + \cos(0.2\pi)\sqrt{\lvert 0.02\sin(5\pi x_1)\rvert}, & \text{for } 0 \le x_1 \le 1 \\ 0.1(x_1 - 1) - \sin(0.2\pi), & \text{for } x_1 > 1. \end{cases}$
	Lower level	$f_1(x,y) = y_1^2 + \sum_{i=2}^{K}(y_i - x_i)^2, \quad f_2(x,y) = \sum_{i=1}^{K}i(y_i - x_i)^2$
Constraints	Upper level	$y_i \in [-K, K],\ i = 1, \ldots, K,$ $x_1 \in [0.001, K],\ x_j \in [-K, K],\ j = 2, \ldots, K$
Preference uncertainty	Lower level	$V(f_1, f_2) = \xi_1 f_1 + \xi_2 f_2,$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (6, 1), \quad \Sigma_\xi = \begin{bmatrix} 0.09 & 0 \\ 0 & 0.09 \end{bmatrix}$

A summary of these test problems is given in Table IX. All test problems have been chosen such that it is possible to generate an analytical Pareto-optimal front corresponding to the expected value function. Also, the lower level Pareto-optimal members corresponding to 99% decision uncertainty can be theoretically determined for any x_u. The advantage of choosing these test problems is that they offer various difficulties found in multiobjective bilevel problems, while at the same time their analytical solutions are known.

The results for all test problems are summarized in Tables X and XI. First, we evaluate m-BLEAQ in terms of its ability to find the expected frontier correctly (POF-EVF). As the inverted generalization distance (IGD) metric is able to evaluate the performance of the algorithm both in terms of convergence and diversity, we decide to use it in this paper. The IGD values achieved by m-BLEAQ and the baseline algorithm H-BLEMO are reported in Table X, where smaller values denote better results. The results look promising; the IGD values obtained by m-BLEAQ are clearly better than those for H-BLEMO on all test problems.

The next step is to consider the computational expense required by the two methods. As proxies for the

TABLE X
MINIMUM, MEDIAN, AND MAXIMUM IGD VALUES OBTAINED FROM 21 RUNS OF m-BLEAQ AND H-BLEMO

Prob.	No of Vars.	IGD (m-BLEAQ)			IGD (H-BLEMO)		
		Min	Med	Max	Min	Med	Max
Ex1	3	**0.0011**	**0.0015**	**0.0018**	0.0034	0.0049	0.0064
Ex2	10	**0.0010**	**0.0013**	**0.0015**	0.0029	0.0046	0.0058
DS1	10	**0.0047**	**0.0069**	**0.0116**	0.0088	0.0121	0.0198
DS2	10	**0.0068**	**0.0079**	**0.0189**	0.0097	0.0134	0.0236
DS1	20	**0.0219**	**0.0435**	**0.1154**	0.0808	0.1106	0.1457
DS2	20	**0.0326**	**0.0623**	**0.1874**	0.0601	0.1321	0.2719

TABLE XI
MINIMUM, MEDIAN, AND MAXIMUM UPPER LEVEL FUNCTION EVALUATIONS (ULFEs) AND LLFEs FROM 21 RUNS OF m-BLEAQ AND H-BLEMO

Prob.	ULFE (m-BLEAQ)			LLFE (m-BLEAQ)			Savings: $\frac{\text{H-BLEMO (Med)}}{\text{m-BLEAQ (Med)}}$	
	Min	Med	Max	Min	Med	Max	LLFE	ULFE
Ex1	4,410	5,035	5,964	77,546	91,344	112,443	**2.57**	**5.66**
Ex2	5,124	6,464	7,345	59,453	77,653	91,343	**2.75**	**6.16**
DS1 (10)	20,352	22,223	27,753	248,343	345,345	446,574	**2.42**	**4.77**
DS2 (10)	22,543	25,364	32,548	312,235	411,445	523,356	**2.61**	**5.24**
DS1 (20)	29,499	34,110	43,585	323,781	475,374	632,288	**3.21**	**5.61**
DS2 (20)	32,331	36,439	46,043	438,728	643,833	688,239	**3.36**	**5.93**

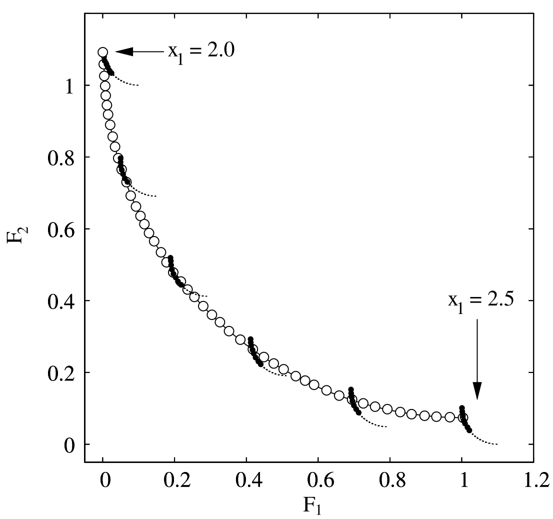

Fig. 9. DS1: POF-EVF obtained using m-BLEAQ and lower level decision uncertainity corresponding to a few x_u vectors.

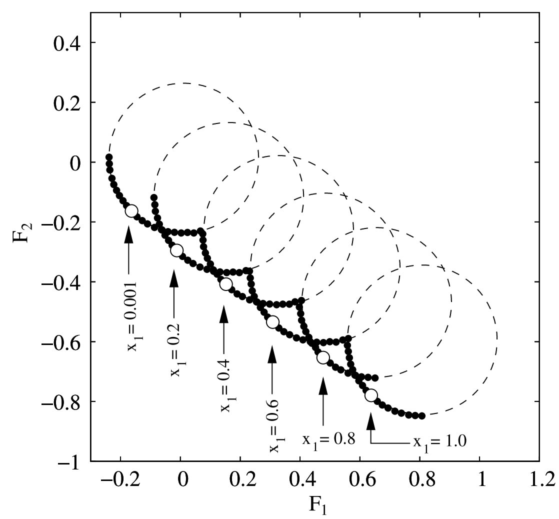

Fig. 10. DS2: POF-EVF obtained using m-BLEAQ and lower level decision uncertainity corresponding to a few x_u vectors.

computational burden, we consider the number of upper as well as lower level function evaluations (LLFEs). The results are shown in Table XI. By judging the outcome, it is clear that m-BLEAQ is able to achieve similar IGD values in less function evaluations. Once the POF-EVF is obtained, the lower level uncertainty is determined by executing the algorithm described in Section V-B. The results for ten variable instances of DS1 and DS2 are displayed in Figs. 9 and 10. The corresponding results for Examples 1 and 2 match the theoretical frontiers already shown in Figs. 6 and 8.

B. Practical Problem 1: Gold Mining in Kuusamo

In the northern part of Finland lies the Kuusamo region, which is a popular tourist destination known for its natural beauty. Recently, there has been a lot of interest in this region, as it is considered to be a "highly prospective Paleoproterozoic Kuusamo Schist Belt" containing large amounts of gold deposits. An Australia-based company, primarily operating in the Nordic region, has been performing drill tests to evaluate the mining prospects in the region. The expected gold content in the ore is around 4.9 g per ton, which is worth millions of euros considering the overall deposits present in the area. The mining project could lead to a large amount of gold resources and also generate a number of jobs, but it is being opposed for fear of harming the environment. There are three primary reasons for the opposition against the gold mining operations in Kuusamo: 1) the river Kitkajoki is located in Kuusamo, and the environmentalists fear that the run-off water generated from the gold mining operations might pollute the river water; 2) the ore contains uranium, which if mined, would blemish the reputation of the tourist resort; and 3) the visible open-pit mines located next to the Ruka slopes will be a big turn-off for skiing and hiking enthusiasts.

TABLE XII
GOLD MINING IN KUUSAMO

Category	Level	Formulation
Variables	Upper level Lower level	τ [per unit tax imposed on the mine] q [amount of metal extracted by the mine]
Objectives	Upper level	$F_1(\tau, q) = E(\tau, q) = \tau q$ [tax revenue] $F_2(\tau, q) = -D(q) = -kq$ [environmental damage]
	Lower level	$f_1(\tau, q) = \pi(\tau, q) = (\alpha - \beta q)q - (\delta q^2 + \gamma q + \phi) - \tau q$ [profit] $f_2(\tau, q) = R(q) = -\eta q$ [reputation]
Constraints	Upper level Lower level	$q \geq 0,\ \tau \geq 0$ $\pi(\tau, q) \geq 0$
Preference uncertainty	Lower level	$V(f_1, f_2) = \xi_1 f_1 + \xi_2 f_2,$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (1, 1), \quad \Sigma_\xi = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$
Parameters in example		$k = 1,\ \eta = 1$ $\alpha = 100,\ \beta = 1,\ \delta = 1,\ \gamma = 1,\ \phi = 0$

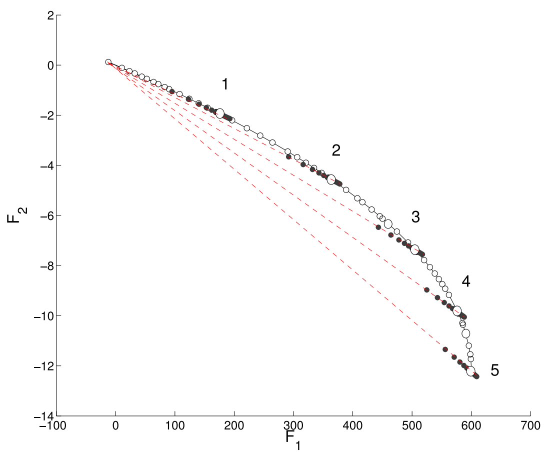

Fig. 11. POF-EVF and lower level frontiers in upper level objective space.

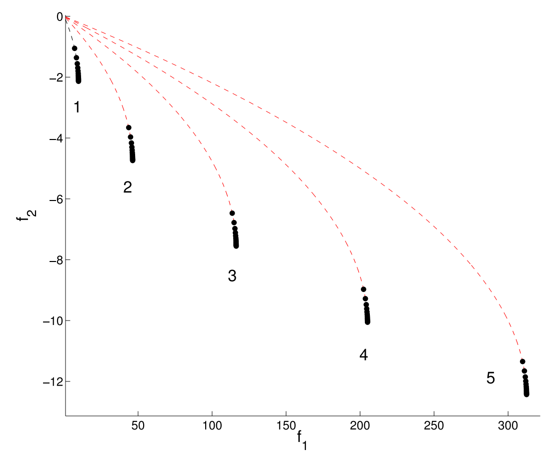

Fig. 12. Lower level frontiers corresponding to five locations in upper level objective space.

Given the situation, the government needs to make a decision whether to allow mining and to what extent. The government has primarily two conflicting objectives: the first objective is maximization of revenues generated by the project (e.g., taxes and additional jobs); the second objective is to minimize the harm done to the environment as a result of mining. The mining company, on the other hand, aims to maximize its profits as well as its reputation to attract future contracts and other concessions. Clearly also these objectives are conflicting, as large profits may be generated at the cost of the environment that might lead to a poor public image. Aware of the problem structure, the government would like to have a taxation policy that maximizes revenues while restraining the mining company from causing extensive environmental damage. To optimize its taxation strategy, the government as a leader needs to model how the mining company would respond to any given tax structure. Before starting to solve the problem, the government needs to have the knowledge of the mining company's value function. However, it is difficult to determine an exact value function for the company. Therefore, we assume that only an uncertain value function is available. Simplified illustration of the problem is given in Table XII.

Fig. 11 shows the POF-EVF in white circles obtained using the m-BLEAQ approach. For five different locations on the POF-EVF we provide the lower level frontiers in upper level space with the help of broken lines. If the upper level decision maker chooses a point 4 from the POF-EVF then the lower level decision uncertainty may cause the realized point to lie on the band represented by black circles in the figure with 99% probability. It is interesting to note that there is a good chance that choosing a point from the right side of the POF-EVF might make the realized point significantly worse in terms of both objectives than the expected point on POF-EVF. Fig. 12 shows the lower level frontiers corresponding to locations $1-5$ on the upper level objective space. This plot also gives an idea to the leader that how the choice of a particular point on the POF-EVF will impact the profit and reputation of the follower. Information obtained from the two figures might be useful for the leader while making a decision.

C. Practical Problem 2: Decision Making in Company

Generally, in manufacturing business, the objectives of the chief executive officer (CEO) are to maximize the profit and

TABLE XIII
HIERARCHICAL DECISION MAKING IN A COMPANY

Category	Level	Formulation
Variables	Upper level Lower level	$x = (x_1, x_2)$ $y = (y_1, y_2, y_3)$
Objectives	Upper level	$F_1(x, y) = 3.38x_1 + 7.78x_2 + 8.54y_1 - 2.35y_2 + 4.97y_3$ $F_2(x, y) = 6.64x_1 + 4.26x_2 + 4.67y_1 + 4.59y_2 + 3.73y_3$
	Lower level	$f_1(x, y) = 4.47x_1 + 5.46x_2 - 6.23y_1 - 4.78y_2 + 7.34y_3$ $f_2(x, y) = 5.34x_1 + 3.74x_2 + 9.45y_1 + 6.37y_2 + 5.45y_3$
Constraints	Upper level	$G_1(x, y) = 4.55x_1 + 7.35x_2 + 9.65y_1 + 6.23y_2 + 4.24y_3 \leq 987,$ $G_2(x, y) = -5.33x_1 - 1.35x_2 + 2.67y_1 - 4.22y_2 + 1.75y_3 \leq 135,$ $G_3(x, y) = -2.11x_1 + 2.67x_2 + 4.34y_1 + 9.26y_2 + 8.33y_3 \leq 830,$ $G_4(x, y) = 2.42x_1 + 7.43x_2 + 4.51y_1 - 3.56y_2 + 1.46y_3 \leq 565,$ $x_1, x_2 \geq 0.$
	Lower level	$g_1(x, y) = 3.67x_1 - 7.84x_2 - 6.78y_1 - 5.87y_2 + 1.26y_3 \leq 105,$ $g_2(x, y) = 4.34x_1 + 9.26x_2 + 8.33y_1 - 2.11y_2 - 2.67y_3 \leq 830,$ $g_3(x, y) = 4.51x_1 - 3.56x_2 + 1.46y_1 + 2.42y_2 + 7.43y_3 \leq 565,$ $g_4(x, y) = 4.47x_1 + 5.46x_2 - 6.23y_1 - 4.78y_2 + 7.34y_3 \geq 0,$ $g_5(x, y) = 5.34x_1 + 3.74x_2 + 9.45y_1 + 6.37y_2 + 5.45y_3 \geq 0,$ $y_1, y_2, y_3 \geq 0.$
Preference uncertainty	Lower level	$V(f_1, f_2) = f_1^{\xi_1} f_2^{\xi_2},$ $\xi \sim N_2(\mu_\xi, \Sigma_\xi), \quad \mu_\xi = (1, 1), \quad \Sigma_\xi = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$

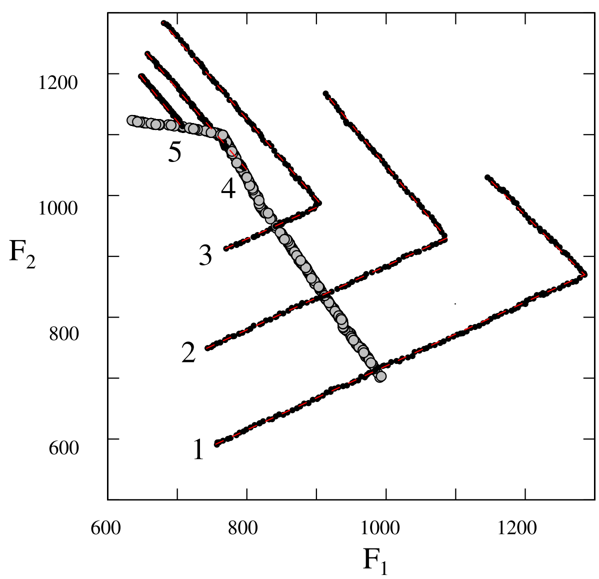

Fig. 13. POF-EVF and lower level frontiers in upper level objective space.

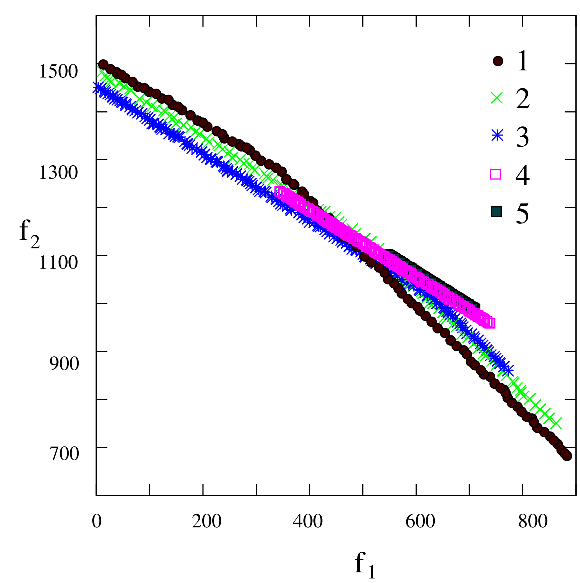

Fig. 14. Lower level frontiers corresponding to five locations in upper level objective space.

the quality of the products. Often there are branch heads working under the CEO. Their responsibility includes manufacturing one or more constituents of the final product. The heads are faced with objectives that include maximizing their branch profits and at the same time maximizing the worker satisfaction. A fuzzy version of this problem is solved in [51] and the deterministic version is discussed in [18]. Both studies attempt to find the best solution for the CEO without considering the decisions of the head. In this paper, we take into account the decisions of the head, while solving the problem. Once again we consider that the exact decision of the head is not known to the CEO beforehand.

The multiobjective bilevel formulation of the problem is given in Table XIII. The first objective for the leader is profit maximization and the second objective for the leader is quality maximization. The first objective for the follower is maximizing worker satisfaction and the second objective is maximizing branch profits. In this problem, the leader is assumed to be aware of the expected value function for the follower. However, she considers the lower level decisions to be highly uncertain. Therefore, she is rather interested in evaluating the complete lower level frontiers corresponding to the points on POF-EVF in the upper level objective space. The expected value function for the follower is given as $V(f_1, f_2) = f_1 f_2$.

It is not possible to solve this problem analytically, so we proceed with the m-BLEAQ approach to find the POF-EVF and then produce the lower level frontier in the upper level space at five different locations to help the leader make a decision.

Fig. 13 provides the POF-EVF points (shown in gray) obtained by the m-BLEAQ approach along with the lower level frontier in upper level objective space at five different locations. The lower level frontiers have been shown in the lower level objective space in Fig. 14. It could be observed from Fig. 13 that choosing a point from the right end of POF-EVF could make the realized decisions very different from the planned decisions because of the uncertainty from the branch heads. The realized decisions could be significantly better or worse than the planned decisions. On the other hand choosing a decision from the left extreme of the POF-EVF offers comparatively little variation because of uncertainty. Frontiers 4 and 5 do not get significantly worse than POF-EVF, and if the CEO is satisfied with the tradeoff between company profit and quality then she might like to make a choice in this region.

VII. Conclusion

In this paper, we have provided a detailed discussion on bilevel multicriterion optimization problems involving lower level decision uncertainty. The discussion is followed by the proposal of an evolutionary methodology to handle such problems. A number of example problems with known solutions have been used to evaluate the proposed methodology. A couple of real-life examples of bilevel multicriterion problems are also discussed and solved. The proposed method successfully handled all the problems considered in this paper. Given the practical relevance of bilevel problems with multiple objectives, the proposed strategy should be a useful optimization and decision tool when solving problems from the perspective of the leader.

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