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Construction and Optimization of Fuzzy Relation Matrices model based-on Semi-tensor Product

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Abstract: On base of semi-tensor product (STP) of matrices, this paper proposes a new and more general framework to construct a matrix-based fuzzy relation structure model for multi-input multi-output (MIMO) fuzzy control systems. The measured sampling data of inputs and outputs are assumed to be obtained from experiments. Instead of building the fuzzy logical rule sets of a fuzzy dynamical controller, the whole designing process is realized via matrix expression and algebraic algorithms. After the input and output data pairs which satisfy a fuzzy dynamic process are described by vectors, the fuzzy relation matrixes of all input and output variables for each data pair are calculated respectively so its fuzzy relation matrix can be got. Final fuzzy relation matrix of all measured data can be achieved by fuzzy disjunction operation and then optimized by the fuzzy α-interception matrix. The new technique gives one general design method to obtain fuzzy relation matrix expression of multiple variables fuzzy control systems. It is particularly suitable to design non-decomposable multi-output fuzzy controllers, which is not solvable directly by the traditional decomposed control design methods.

Key Words: Fuzzy relation matrix, Dual fuzzy structure, Semi-tensor product, α-interception matrix

1 Introduction

The fuzzy logic theory and its applications have developed into a relatively mature theoretical system for more than forty years since Zadeh firstly proposed fuzzy logic theory in 1965 in [1]. As an intelligent control strategy, fuzzy control theory plays more and more important roles in industrial control process. The most remarkable characteristic of fuzzy control is no need of precise mathematical model for objective systems. But it does not mean fuzzy control can not be used for control systems which have well known models [2]. Nowadays more and more efforts have been put on the multi-input multi-output (MIMO) fuzzy control systems [3]. It should be pointed out that most MIMO fuzzy systems are based on T-S fuzzy models which are more suitable for theoretical analysis and design.

Recently a new matrix product, namely, semi-tensor product (STP) of matrices, was proposed [4]. It converts logical systems into standard discrete-time dynamic systems and it may be used to deal with control problems of fuzzy logical systems. It generalizes conventional matrix product to any two matrices and the method has been successfully used for analysis and control synthesis of logical and k-valued logical dynamic systems [5, 6]. The dynamics of a fuzzy controller is a logical system. A major difficulty in dealing with logical systems is that few analysis tools can be used. Paper [7] focused on the designing process of MIMO fuzzy control systems based-on Mamdani model, and we assumed the input-output multiple fuzzy relations were not output decomposable. Then the traditional method is not applicable and new technique has to be developed to deal with such systems. By constructing and applying structure matrices of logical operators, a logical equation can be expressed as an algebraic equation. Based-on this, STP converts complex fuzzy logic reasoning process into solving a set of algebraic equations, which greatly simplifies the process of fuzzy logical reasoning.

But when we apply this novel design method into a real practical system, we need to handle how to get fuzzy structure matrices from practical measured data to construct a general MIMO fuzzy control system. As a result, we build one general matrix method to identify any fuzzy dynamical model as well as any multi-variable fuzzy logical system and then optimized by the fuzzy α-interception matrix. Compared with the existing methods, the most advantage of our method is that only one fuzzy structure matrix needs to be identified.

The rest of this paper is organized as follows. In Section 2 some necessary preliminaries are introduced. Section 3 presents how to convert fuzzy dynamics model based-on language and rules into its algebraic form and realize fuzzy control based-on fuzzy structure relation matrix. In Section 4, we propose the fuzzy α-interception matrix and then it is used to optimize the fuzzy relation matrix. In Section 5, a general algorithm is provided to identify a based-on matrix fuzzy controller through the observed data. Section 6 gives two examples to illustrate concrete design process. Section 7 consists of some concluding remarks.

2 Preliminaries

To begin with, we introduce some notations, which are used throughout this paper. The basic tool in this paper is STP of matrices, which is a generalization of conventional matrix product.

Definition 2.1 [5] Let $A=(a_{i,j})\in D_k^{m\times n}$, $B=(b_{i,j})\in D_k^{p\times q}$, their **semi-tensor product (STP)** is defined as

(i) If $n=pt$, then $A \rhd B = A\times_B (B\otimes_B I_t)\in D_k^{m\times tq}$ (1)

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(ii) If $nt = p$, then $A \triangleright B = \left(A \times_B I_t \right) \times_B B \in D_k^{tm\times q}$ (2)

where $\times_B$ is the Boolean product and $\otimes_B$ is the Kronecker product.

Definition 2.2 [5] Assume the universe of discourse is finite, we express it as $E = [e_1, \cdots e_n]$. Let A be a fuzzy set over E. Then A is conventionally expressed as

$$A = \alpha_1/e_1 + \alpha_2/e_2 + \cdots + \alpha_n/e_n$$

where $\alpha_i = \mu_A(e_i)$ is the membership degree of e_i on A.

The vector expression of one fuzzy set is denoted by υ_A, as

$$\upsilon_A = (\alpha_1, \cdots, \alpha_n)^T \in R^n \quad (3)$$

Then the matrix expression of multiple fuzzy relations is considered. In most literature the fuzzy relations are defined over two universes of discourse. In practice the fuzzy relations over multiple universes of discourse need to be treated. We give the following definition first.

Let $E_i,\ i = 1, \cdots, k$ be k universes of discourse. A multiple fuzzy relation R over $E_i\ (i = 1, \cdots, k)$ is **a fuzzy set on the product space** $\prod_{i=1}^{k} E_i = E_1 \times \cdots \times E_k$. Precisely, for each point $(e_1, \cdots, e_k) \in \prod_{i=1}^{k} E_i$ there is a membership degree $\mu_R(e_1, \cdots, e_k) \in [0,1]$, to present the matrix expression of a multiple fuzzy relation, we introduce a multi-index notation.

Definition 2.3 [5] Let $A = \left\{ a_{i_1, \cdots, i_k} \middle| i_j = 1, \cdots, n_j; j = 1, \cdots, k \right\}$ be a set of data, labeled by k-fold index $i = (i_1, \cdots, i_k)$. Then $\mathrm{Id}(i_1, \cdots, i_k; n_1, \cdots, n_k)$ is an index order, which arranges the data in such an order that $a_{i_1, \cdots, i_k}$ is ahead of $a_{i'_1, \cdots, i'_k}$, iff there is an $1 \le s \le k$, such that $i_j = i'_j,\ \ j < s$ and $i_s < i'_s$.

Now assume $E_i = \left\{ e_1^i, \cdots, e_{n_i}^i \right\},\ i = 1, \cdots, k$. Denote membership function of the fuzzy relation by $r_{j_1, \cdots, j_k} = \mu_R\left(e_{j_1}^1, \cdots, e_{j_k}^k \right),\ j_t = 1, \cdots, n_t,\ \ t = 1, \cdots, k$. Let $\{\alpha_1, \cdots, \alpha_p\}$ and $\{\beta_1, \cdots, \beta_q\}$ be a partition of $p + q = k,\ \ p, q \in \{1, 2, \cdots, k\}$. We can arrange the data $\left\{ r_{j_1, \cdots, j_k} \middle| j_t = 1, \cdots, n_t,\ t = 1, \cdots, k \right\}$ into **a fuzzy relation matrix** $M_R \in M_{(n_{\alpha_1} \times \cdots \times n_{\alpha_p}) \times (n_{\beta_1} \times \cdots \times n_{\beta_q})}$ **in the order of** $\mathrm{Id}\left(j_{\alpha_1}, \cdots, j_{\alpha_p}; n_{\alpha_1}, \cdots, n_{\alpha_p} \right) \times \mathrm{Id}\left(j_{\beta_1}, \cdots, j_{\beta_q}; n_{\beta_1}, \cdots, n_{\beta_q} \right)$, which means that the rows of M_R are arranged in the order of $\mathrm{Id}\left(j_{\alpha_1}, \cdots, j_{\alpha_p}; n_{\alpha_1}, \cdots, n_{\alpha_p} \right)$ and the columns of M_R are arranged in the order of $\mathrm{Id}\left(j_{\beta_1}, \cdots, j_{\beta_q}; n_{\beta_1}, \cdots, n_{\beta_q} \right)$.

3 Fuzzy Controls Based-on STP

In order to realize the design of MIMO fuzzy control systems, in paper [7] a new framework, based on matrix approach, is investigated. There one novel design strategy of fuzzy controller based-on STP of matrices is proposed. This method designs one fuzzy relation structure matrix between inputs and outputs to realize the fuzzy control directly, which turns the fuzzy rule and fuzzy inference into algebraic form and simplifies its complexity. The basic structure of this novel fuzzy controller is shown in Fig. 1. One different unit is the fuzzy relation structure matrix which takes the place of fuzzy rules basis in the traditional fuzzy design.

We firstly introduce some definitions about the dual fuzzy structure.

Definition 3.1 [7] Let E be a universe of discourse, and $\boldsymbol{A} = \{A_1, \cdots, A_k\}$ be a set of fuzzy sets on E. Then $(E, \boldsymbol{A})$ is called **a fuzzy structure**. **The support of** A_i is defined as $\mathrm{Supp}(A_i) = \left\{ e \in E \middle| \mu_{A_i}(e) \ne 0 \right\} \subset E$.

Definition 3.2 [7] Assume E is a well ordered set. $\boldsymbol{A} = \{A_1, \cdots, A_k\}$ is called **a set of degree-based fuzzy sets**, if $\sup\left(\mathrm{Supp}(A_i)\right) < \sup\left(\mathrm{Supp}(A_{i+1})\right)$ and

$$\inf\left(\mathrm{Supp}(A_i)\right) < \inf\left(\mathrm{Supp}(A_{i+1})\right),\ \ i = 1, \cdots, k-1 \quad (4)$$

Definition 3.3 [7] Given a fuzzy structure $(E, \boldsymbol{A})$ as in Definition 3.1. Assume E is a well ordered set and A is a set of degree-based fuzzy sets, i.e., (4) is satisfied. Then we may consider $(\mathbf{A}, E)$ as a fuzzy structure, where $\boldsymbol{A} = \{A_1, \cdots, A_k\}$ is considered as a universe of discourse, each $e \in E$ is a fuzzy set, with

$$\mu_e\left(A_i\right) = \mu_{A_i}(e),\ \ i = 1, \cdots, k \quad (5)$$

This fuzzy structure is called **the dual structure** of $(E, \boldsymbol{A})$. When A is a finite set, the dual structure has a finite universe of discourse.

In fact, fuzzification is basically to find the dual structure. Then we can design a fuzzy controller. Roughly speaking, a fuzzy controller is a fuzzy inference mechanism. Assume the controlled system has m inputs and p outputs; the fuzzy controller has the form as

$$\Sigma \in f\left(Y_1 \times \cdots \times Y_p \times U_1 \times \cdots \times U_m \right) \quad (6)$$

where $Y_i,\ \ i = 1, \cdots, p$ and $U_j,\ \ j = 1, \cdots, m$ have been fuzzificated. Note that for the fuzzy controller $\{Y_i\}$ becomes the input set and $\{U_i\}$ does the output set. In general case a fuzzy controller is mathematically equivalent to a fuzzy relation (6). We describe this as follows. First, we specify degree-based fuzzy sets of Y_i and U_j as

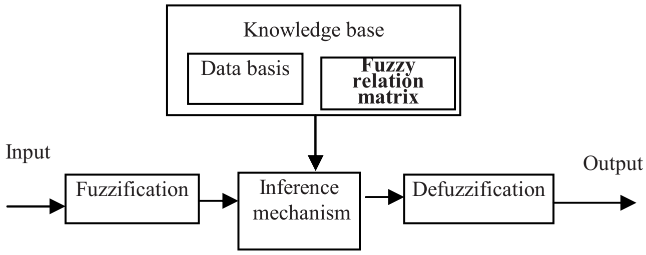

Fig. 1 The basic structure of a novel fuzzy controller

$$E_{Y_i} = \{y_1^i, \cdots, y_{\alpha_i}^i\}, \quad i = 1, \cdots, p;$$
$$E_{U_i} = \{u_1^j, \cdots, u_{\beta_j}^j\}, \quad j = 1, \cdots, m \tag{7}$$

Note that y_k^i, $k = 1, \cdots, \alpha_i$ correspond to "negative big", "negative middle", ..., which are what we mean the degree-based fuzzy sets. We use the dual fuzzy structure. That is, consider E_{Y_i} as the universe of discourse for Y_i, and E_{U_j} as the universe of discourse for U_j; and meanwhile, consider each true value $y_i \in Y_i$ as a fuzzy set over E_{Y_i}, and $u_j \in U_j$ as a fuzzy set over E_{Y_i}.

A fuzzy controller is a fuzzy relation among $\{Y_1, \cdots, Y_P; U_1, \cdots, U_m\}$ and now assume it is known as Σ. That is, Σ is a fuzzy relation on $\prod_{i=1}^{p} Y_i \times \prod_{j=1}^{m} U_j$. Then for each element in this product space we have its membership degree as

$$\mu_\Sigma\left(y_{\varepsilon_1}^1, \cdots, y_{\varepsilon_p}^p, u_{\eta_1}^1, \cdots, u_{\eta_m}^m\right) = \gamma_{\eta_1 \cdots \eta_m}^{\varepsilon_1 \cdots \varepsilon_p},$$
$$\varepsilon_i = 1, \cdots, \alpha_i, \quad i = 1, \cdots, p; \quad \eta_j = 1, \cdots, \beta_j, \quad j = 1, \cdots, m. \tag{8}$$

Arranging $\{\gamma_{\eta_1 \cdots \eta_m}^{\varepsilon_1 \cdots \varepsilon_p}\}$ into a matrix in the order of $\mathrm{Id}(\eta_1, \cdots, \eta_m; \beta_1, \cdots, \beta_m) \times \mathrm{Id}(\varepsilon_1, \cdots, \varepsilon_p; \alpha_1, \cdots, \alpha_p)$, we have

$$M_\Sigma = \begin{bmatrix} \gamma_{1\cdots 11}^{1\cdots 11} & \gamma_{1\cdots 11}^{1\cdots 12} & \cdots & \gamma_{1\cdots 11}^{1\cdots 1\alpha_p} & \cdots & \gamma_{1\cdots 11}^{\alpha_1 \cdots \alpha_p} \\ \gamma_{1\cdots 12}^{1\cdots 11} & \gamma_{1\cdots 12}^{1\cdots 12} & \cdots & \gamma_{1\cdots 12}^{1\cdots 11} & \cdots & \gamma_{1\cdots 12}^{1\cdots 11} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \gamma_{\beta_1 \cdots \beta_m}^{1\cdots 11} & \gamma_{\beta_1 \cdots \beta_m}^{1\cdots 12} & \cdots & \gamma_{\beta_1 \cdots \beta_m}^{1\cdots 1\alpha_p} & \cdots & \gamma_{\beta_1 \cdots \beta_m}^{\alpha_1 \cdots \alpha_p} \end{bmatrix} \tag{9}$$

As aforementioned a fuzzy controller is essentially a fuzzy relation among the outputs and the controls. As the fuzzy relation is expressed by a matrix, we are convinced that a fuzzy controller is essentially a fuzzy matrix, which is expressed as (9).

Now given a set of $y_i \in Y_i$, $i = 1, \cdots, p$, recall that in dual fuzzy structure they are fuzzy sets and fuzzification provides their vector forms as υ_{y_i}, $i = 1, \cdots, p$. Then the fuzzy controller produces the corresponding controls u_j, $j = 1, \cdots, m$ as

$$\triangleright_{j=1}^{m} \upsilon_{u_j} = M_\Sigma \triangleright_{i=1}^{p} \upsilon_{y_i} \tag{10}$$

Finally, through defuzzification, the result obtained from the fuzzy mapping, which are fuzzy sets, can be converted back to control values. That is, the purpose of defuzzification is to provide controls $(u_1, \cdots, u_m)$ from a fuzzy set $u \in F(U_1 \times \cdots \times U_m)$. Reference [7] proposed two methods to deal with defuzzification of multiple control case. One method is **Joined Defuzzification** (JD), the other one is **Separated- Defuzzification** (SD). Note that JD Method is particularly suitable for the case when the controls are coupled, i.e., interacted to each other. While SD Method is particularly suitable for the case the controls are independent to each other. Experience is necessary for judging whether the controls are coupled.

4 Fuzzy α -interception Matrix and Optimization for Fuzzy Structure Matrix

The fuzzy relation structure matrix in (9) is relatively complex, which included much information. In order to apply the fuzzy control based-on relation matrix into the practice systems simply and efficiently, we can optimize the fuzzy structure matrix by α -interception fuzzy matrix, and then the α -interception Boolean matrix is the simplest form. At first we give the definition of α -interception fuzzy matrix and spread the classical α -interception fuzzy sets, then the traditional fuzzy decomposition theory can be spreaded into the more general theories based-on MIMO fuzzy relation matrixes.

Definition 4.1 Consider a m inputs n outputs fuzzy controller, its fuzzy relation structure matrix is $M = (r_{ij})_{p \times q}$, where $p = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_m$, α_i is the number of fuzzy subsets of the *i*-th input variable, $i = 1, 2, \cdots, m$ and $q = \beta_1 \times \beta_2 \times \cdots \times \beta_n$, β_j is the number of fuzzy subsets of the *j*-th output variable, $j = 1, 2, \cdots, n$. $\forall \alpha \in [0,1]$. We define the α **-interception fuzzy matrix of** M as

$$M_\alpha = \begin{cases} r_{ij}, & r_{ij} \ge \alpha, \\ 0, & r_{ij} < \alpha. \end{cases} \tag{11}$$

We define α **-interception Boolean matrix of** M as

$$B_{M_\alpha} = \begin{cases} 1, & r_{ij} \ge \alpha, \\ 0, & r_{ij} < \alpha. \end{cases} \tag{12}$$

Theorem 4.1 Under the assumptions that M_α is the α -interception fuzzy matrix of M, $\alpha \in [0,1]$, M can be equivalent to a decomposition

$$M = \bigcup_{\alpha \in [0,1]} \alpha B_{M_\alpha} \tag{13}$$

or

$$M = \bigcup_{\alpha \in [0,1]} M_\alpha \tag{14}$$

Where αB_{M_α} is the number product operator of matrixes.

Proof:

(1) Set $B = \bigcup_{\alpha \in [0,1]} \alpha B_{M_\alpha} = (b_{ij})_{p \times q}$, $\forall \alpha \in [0,1]$ and $M = (r_{ij})_{p \times q}$, so we have to proof $r_{ij} = b_{ij}$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$.

$$b_{ij} = \max_{\alpha \in [0,1]} \alpha B_{M_\alpha} = \bigvee_{\alpha \in [0,1]} (\alpha \wedge b_{ij})$$
$$= \bigvee_{\alpha \in [0,1]} (\alpha \wedge 1) = \bigvee_{\substack{r_{ij} \ge \alpha \\ \alpha \in [0,1]}} \alpha = r_{ij}$$

So $M = \bigcup_{\alpha \in [0,1]} \alpha B_{M_\alpha}$.

(2) Set $A = \bigcup_{\alpha \in [0,1]} M_\alpha = (a_{ij})_{p \times q}$, $\forall \alpha \in [0,1]$ and $M = (r_{ij})_{p \times q}$, so we have to proof $r_{ij} = a_{ij}$, $i = 1, 2, \cdots, p$, $j = 1, 2, \cdots, q$.

$$a_{ij} = \max_{\alpha\in[0,1]} M_\alpha = \wedge_{\alpha\in[0,1]} (\alpha \vee r_{ij}) = r_{ij}$$

So $M = \bigcup_{\alpha\in[0,1]} M_\alpha$.

It is obvious that Theorem 4.1 is the generalization and matrix expression of conventional fuzzy α -interception sets and the traditional fuzzy decomposition theory. Then we will apply the two new definitions into the construction of optimized fuzzy relation structure matrixes.

Assumed we have gotten the appropriate value of α, then we can use two methods to optimize the end fuzzy relation structure matrix (9). One is to get its α -interception fuzzy matrix M_α and the other is to get its α -interception Boolean matrix B_{M_α} according to the definition 4.1. The control effect of the α -interception fuzzy matrix is more precise but more complex while the α -interception Boolean matrix is more fit for the theatrical study.

5 Model Construction of Novel Fuzzy Controller Based-on Fuzzy Relation Matrix

When this design is applied into real dynamical systems as shown in Fig. 2, model realization is the most important for the design of matrix-based fuzzy control systems proposed in Section 3. So here we will study how to construct a fuzzy system from measured data. If the groups of measured input-output data pairs in real controlled plant can be obtained from experiments, then we will provide a following general approach to identify a MIMO fuzzy relation matrix model of the plant. It is useful for simulation purposes and sometimes for use in a real practical controller plant.

5.1 Determine the Structure of the Fuzzy Controller

In order to realize fuzzy control, for one practical controlled process, we should understand all the composed elements, including all equipments, component, their physical principles, application and relations. Then define all parameters of the closed-loop fuzzy control process, including constants, variables, scalars etc. We need to explicit all the units of the system and their input, output variables and relations. Thus the structure figures of the whole closed-loop system can be derived. The characteristic of system are determined such as continuous or discrete, time-variable or constant, linear or nonlinear etc.

According to the concrete control conditions and objection, every input and output fuzzy language variables of the fuzzy controller and their language values, fuzzy subsets, fuzzy membership functions are defined. All parameters in fuzzification process need to be identified.

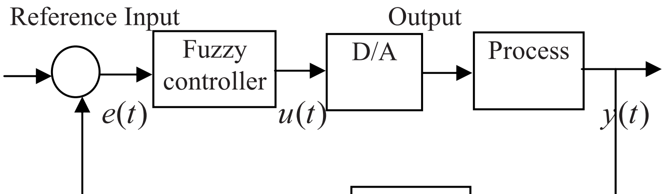

Fig. 2 The principle of fuzzy control systems

5.2 Fuzzification of Every Fuzzy Language Variable

According to sampling data and practical experiences, define the universe of discourse for every fuzzy language variable. Every fuzzy variable is divided some fuzzy subsets, their number is decided by the practical process. On every fuzzy subset the suitable membership function is defined. Every membership function on each fuzzy subset needs to define its shapes and position parameters. Ten all parameters in fuzzification process need to be identified and turn into the dual fuzzy structure.

5.3 Choose Fuzzy Inference Operations

In one fuzzy controller there exist the fuzzy “and”, “or”, “implication” algorithms are three basis kinds of fuzzy inference operations. Because the membership degree of conclusion is determined by the membership degree of both condition and implication operation, the essence of the fuzzy control rules is the fuzzy implication relation in a fuzzy controller. There exist many fuzzy inference operations as fuzzy relation has many forms. In this paper we adopt max-min fuzzy inference.

5.4 Construction of Fuzzy Relation Matrix

In one practical system, we assume there are p input and m output variables named $\{Y_1, \cdots, Y_p; U_1, \cdots, U_m\}$.

(1) Assume we have obtained N groups of sampling data pairs about the input and output variables, that is

$$\left(Y_{1i}^*, Y_{2i}^*, \cdots, Y_{pi}^*; U_{1i}^*, \cdots, U_{mi}^*\right), \quad i = 1, \cdots, N$$

(2) Firstly we specify the degree-based fuzzy sets of Y_i and U_j as

$$E_{Y_i} = \left\{y_1^i, \cdots, y_{\alpha_i}^i\right\}, \quad i = 1, \cdots, p;$$

$$E_{U_j} = \left\{u_1^j, \cdots, u_{\beta_j}^j\right\}, \quad j = 1, \cdots, m. \tag{15}$$

We use the dual fuzzy structure in Section 3.

(3) Through the fuzzification, we convert these data into the fuzzy vector forms:

$$v_{Y_{1i}} = \left(\mu_{y_1^1}\left(Y_{1i}^*\right), \cdots, \mu_{y_{\alpha_1}^1}\left(Y_{1i}^*\right)\right)^T, \cdots, v_{Y_{pi}} = \left(\mu_{y_1^p}\left(Y_{pi}^*\right), \cdots, \mu_{y_{\alpha p}^p}\left(Y_{pi}^*\right)\right)^T \tag{16}$$

$$v_{U_{1i}} = \left(\mu_{u_1^1}\left(U_{1i}^*\right), \cdots, \mu_{u_{\beta_1}^1}\left(U_{1i}^*\right)\right)^T, \cdots, v_{U_{mi}} = \left(\mu_{u_1^m}\left(U_{mi}^*\right), \cdots, \mu_{u_{\beta m}^m}\left(U_{mi}^*\right)\right)^T \tag{17}$$

(4) The fuzzy relation matrix among all input variables of the *i*-th data is

$$R_{Y_{1i}, Y_{2i}, \cdots, Y_{pi}} = v_{Y_{1i}} \triangleright v_{Y_{2i}} \triangleright v_{Y_{pi}} = R_Y^i \tag{18}$$

(5) The fuzzy relation matrix among all output variables of the *i*-th data is

$$R_{U_{1i}, U_{2i}, \cdots, U_{mi}} = v_{U_{1i}} \triangleright v_{U_{2i}} \triangleright v_{U_{mi}} = R_U^i \tag{19}$$

(6) Then the *i*-th fuzzy relation matrix between the input and output variables from the *i*-th data is

$$R^i = R_U^i \triangleright \left(R_Y^i\right)^T, \quad R^i \in M_{\beta_1\beta_2\cdots\beta_m \times \alpha_1\alpha_2\cdots\alpha_p} \tag{20}$$

(7) The end fuzzy relation matrix from all N groups of data is

$$R = \bigcup_{i=1}^{N} R^i \tag{21}$$

5.5 Choose One Method of Defuzzification

In fact all the traditional defuzzification algorithms are fit for this new designed fuzzy controller. The defuzzification algorithm is used to transform the fuzzy subsets of outputs into the concrete output values. Here we choose either JD or SD defuzzification algorithms.

5.6 Optimization of the Fuzzy Controller

In order to simplified real control process, when we get the final fuzzy relation structure matrix R, we can choose suitable α to optimize it. But the most important and key is how to get the appropriate value of α. It have to be determined by the real systems in the practice plant so it is difficult to get a general method.

Finally, after the simulation test and verification we can apply this designed fuzzy controller into practical process plant. And then if it appears some small error and disturbance in the real control process, we need adjust and optimize the fuzzy controller to adapt the practical system performance.

6 Illustrative Examples

Example 6.1 Recall the Example in [8]. For a two inputs X_1, X_2 and one output Y_1 fuzzy controller, the sampling data is $\left(X_{1i}^*, X_{2i}^*, Y_1^*\right)$, $i = 1, \cdots, N$

Assume fuzzy subsets of the input variable X_1 are $A_1^1, \cdots, A_1^7$ and fuzzy subsets of the input variable X_2 are $A_1^1, \cdots, A_1^5$ and fuzzy subsets of the output variable Y_1 are $B_1^1, \cdots, B_1^3$. They are shown in Fig. 3.

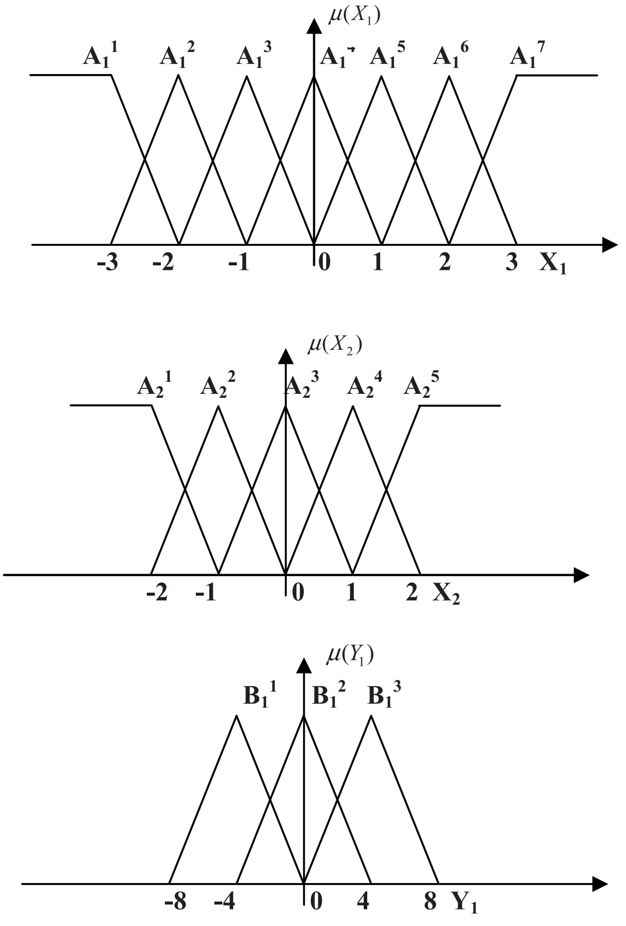

Fig. 3 Membership functions of variables

When $N = 2$, that is, we get two sampling data $\left(X_{11}^*, X_{21}^*, Y_{11}^*\right)$, $\left(X_{12}^*, X_{22}^*, Y_{12}^*\right)$ and their membership functions are

$\nu_{X_{11}} = (1, 0.9, 0.7, 0.5, 0.3, 0.1, 0)^T$, $\nu_{X_{21}} = (0.2, 0.4, 0.6, 0.8, 1)^T$,

$\nu_{Y_{11}} = (0.1, 0.5, 0.8)^T$; $\nu_{X_{12}} = (0, 0.2, 0.4, 0.6, 0.8, 1, 1)^T$,

$\nu_{X_{22}} = (1, 0.7, 0.5, 0.3, 0.1)^T$, $\nu_{Y_{12}} = (0.7, 0.3, 0.1)^T$

Then the fuzzy matrixes of output variables are

$$R_X^1 = R_{X_{11} \times X_{21}} = \nu_{X_{11}} \triangleright \nu_{X_{21}}$$
$$= [0.2\ 0.4\ 0.6\ 0.8\ 1\ 0.2\ 0.4\ 0.6\ 0.8\ 0.9\ 0.2\ 0.4\ 0.6\ 0.7\ 0.7\ 0.2\ 0.4\ 0.5\ 0.5\ 0.5\ 0.2\ 0.3\ 0.3\ 0.3\ 0.3\ 0.1\ 0.1\ 0.1\ 0.1\ 0.1\ 0\ 0\ 0\ 0\ 0]^{\mathrm{T}} \in M_{35\times1}$$

$$R_X^2 = R_{X_{12} \times X_{22}} = \nu_{X_{12}} \triangleright \nu_{X_{22}}$$
$$= [0\ 0\ 0\ 0\ 0\ 0.2\ 0.2\ 0.2\ 0.2\ 0.1\ 0.4\ 0.4\ 0.4\ 0.3\ 0.1\ 0.6\ 0.6\ 0.5\ 0.3\ 0.1\ 0.8\ 0.7\ 0.5\ 0.3\ 0.1\ 1\ 0.7\ 0.5\ 0.3\ 0.1\ 1\ 0.7\ 0.5\ 0.3\ 0.1]^{\mathrm{T}} \in M_{35\times1}$$

The fuzzy matrix of output variable is

$$R_Y^1 = \nu_{Y_{11}} = (0.1, 0.5, 0.8)^T \quad R_Y^2 = \nu_{Y_{12}} = (0.7, 0.3, 0.1)^T$$

The fuzzy relation matrix of each data is

$$R^1 = R_Y^1 \triangleright \left(R_X^1\right)^T$$
$$= \left(\begin{array}{cccccccccccccccccccc} 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0.2&0.4&0.5&0.5&0.5&0.2&0.4&0.5&0.5&0.5&0.2&0.4&0.5&0.5&0.5&0.2&0.4&0.5&0.5&0.5\\ 0.2&0.4&0.6&0.8&0.8&0.2&0.4&0.6&0.8&0.8&0.2&0.4&0.6&0.7&0.7&0.2&0.4&0.5&0.5&0.5 \end{array}\right.$$
$$\left.\begin{array}{ccccccccccccccc} 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0 \end{array}\right) \in M_{3\times35}$$

$$R^2 = R_Y^2 \triangleright \left(R_X^2\right)^T$$
$$= \left(\begin{array}{cccccccccccccccccccc} 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0&0&0&0&0&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{array}\right.$$
$$\left.\begin{array}{ccccccccccccccc} 0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0.3&0.3&0.3&0.3&0.1\\ 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{array}\right) \in M_{3\times35}$$

So the final fuzzy relation matrix from the two data is

$$R = R^1 \cup R^2$$
$$= \left(\begin{array}{cccccccccccccccccccc} 0.1&0.1&0.1&0.1&0.1&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0.2&0.4&0.5&0.5&0.5&0.2&0.4&0.5&0.5&0.5&0.3&0.4&0.5&0.5&0.5&0.3&0.4&0.5&0.5&0.5\\ 0.2&0.4&0.6&0.8&0.8&0.2&0.4&0.6&0.8&0.8&0.2&0.4&0.6&0.7&0.7&0.2&0.4&0.5&0.5&0.5 \end{array}\right.$$
$$\left.\begin{array}{ccccccccccccccc} 0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1\\ 0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{array}\right) \in M_{3\times35}$$

This matrix is the same as result in [8] if it is described by fuzzy matrix. Finally we can optimize this fuzzy relation matrix by α-interception fuzzy matrix and α-interception Boolean matrix.

When $\alpha = 0.5$,

$$R_{0.5}=\left(\begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.6&0.6&0.5&0&0\\ 0&0&0.5&0.5&0.5&0&0&0.5&0.5&0.5&0&0&0.5&0.5&0.5&0&0&0.5&0.5&0.5\\ 0&0&0.6&0.8&0.8&0&0&0.6&0.8&0.8&0&0&0.6&0.7&0.7&0&0&0.5&0.5&0.5 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.7&0.7&0.5&0&0&0.7&0.7&0.5&0&0&0.7&0.7&0.5&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix}\right)\in M_{3\times35}$$

$$B_{R_{0.5}}=\left(\begin{matrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&0&0\\ 0&0&1&1&1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&1\\ 0&0&1&1&1&0&0&1&1&1&0&0&1&1&1&0&0&1&1&1 \end{matrix}\right.$$

$$\left.\begin{matrix} 1&1&1&0&0&1&1&1&0&0&1&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{matrix}\right)\in B_{3\times35}$$

When the fuzzy control system have more than one output variable, the old method (compounded inference) does not work, but through matrix semi-tensor product, the fuzzy relation between multi-input and multi-output variables can become matrix expression. So we can get the fuzzy relation matrix using this new construction method.

Example 6.2 Recall the Examples 6.1, assume to add one output variable Y_2, where other variables takes same values as in Example 6.1. Assume Y_2 have two fuzzy subsets B_2^1, B_2^2 shown in Fig. 4, and its two data are

$Y_{21}=(0.3,0.4), Y_{22}=(0.6,0.8)$

So $\nu_{Y_{21}}=(0.3,0.4)^T, \nu_{Y_{22}}=(0.6,0.8)^T$

Then we need calculate R_Y^1 and R_Y^2 again

$$R_Y^1=R_{Y_{11}\times Y_{21}}=\nu_{Y_{11}}\triangleright\nu_{Y_{21}}=\begin{pmatrix}0.1&0.1&0.3&0.4&0.3&0.4\end{pmatrix}^T$$

$$R_Y^2=R_{Y_{12}\times Y_{22}}=\nu_{Y_{12}}\triangleright\nu_{Y_{22}}=\begin{pmatrix}0.6&0.7&0.3&0.3&0.1&0.1\end{pmatrix}^T$$

As a result,

$$R^1=R_Y^1\triangleright\left(R_X^1\right)^T$$

$$=\left(\begin{matrix} 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0&0&0&0&0 \end{matrix}\right)\in M_{6\times35}$$

$$R^2=R_Y^2\chi\left(R_X^2\right)^T$$

$$=\left(\begin{matrix} 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0&0&0&0&0&0.2&0.2&0.2&0.2&0.1&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0&0&0&0&0&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0&0&0&0&0&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.6&0.6&0.5&0.3&0.1&0.6&0.6&0.5&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1\\ 0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0.2&0.3&0.3&0.3&0.1&0.2&0.3&0.3&0.3&0.1&0.2&0.3&0.3&0.3&0.1\\ 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{matrix}\right)\in M_{6\times35}$$

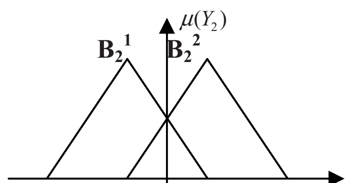

Fig. 4 Membership functions of Y_2 in Ex. 6.2

Then final fuzzy matrix becomes:

$$R=R^1\cup R^2$$

$$=\left(\begin{matrix} 0.1&0.1&0.1&0.1&0.1&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0.1&0.1&0.1&0.1&0.1&0.2&0.2&0.2&0.2&0.1&0.4&0.4&0.4&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4&0.3&0.4&0.4&0.4&0.4&0.3&0.4&0.4&0.4&0.4\\ 0.2&0.3&0.3&0.3&0.3&0.2&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3\\ 0.2&0.4&0.4&0.4&0.4&0.2&0.4&0.4&0.4&0.4&0.3&0.4&0.4&0.4&0.4&0.3&0.4&0.4&0.4&0.4 \end{matrix}\right.$$

$$\left.\begin{matrix} 0.6&0.6&0.5&0.3&0.1&0.6&0.6&0.5&0.3&0.1&0.6&0.6&0.5&0.3&0.1\\ 0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1&0.7&0.7&0.5&0.3&0.1\\ 0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.3&0.1&0.3&0.3&0.3&0.3&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1\\ 0.2&0.3&0.3&0.3&0.3&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1&0.1 \end{matrix}\right)\in M_{6\times35}$$

So from two groups of practical sampling data we get the fuzzy relation matrix about X_1, X_2, Y_1, Y_2 of the TITO fuzzy control system but the classical fuzzy controller is difficult to realize.

7 Conclusion

With the observed datasets, in the general framework of matrix-based multi-variables fuzzy control systems, we have constructed fuzzy relation matrix from measured data based-on STP of matrices. A new designing procedure is proposed for practical fuzzy controlled systems. Then the fuzzy α -interception matrix is given to optimize the fuzzy relation matrix. Some interesting examples are included to demonstrate the result. This new approach is particularly suitable for the control interacted case, where the multiple controls are coupled and the fuzzy relation can not be separated. Conventional design method is not implicated to multiple-outputs fuzzy systems.

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