An Optimization Algorithm for Imprecise Multi-Objective Problem Functions

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Real world objective functions often produce two types of uncertain output: Noise and imprecision. While there is a distinct difference between both types, most optimization algorithms treat them the same. This paper introduces an alternative way to handle imprecise, interval-valued objective functions, namely imprecision-propagating MOEAs. Hypervolume metrics and imprecision measures are extended to imprecise Pareto sets. The performance of the new approach is experimentally compared to a standard distribution-assuming MOEA.

1 Introduction and motivation

The traditional way to define optimization problems is to create a model of the system and state it to be exact and deterministic. Clearly defined decision values are mapped to likewise clearly defined, non-varying objective values. Respecting the fact that nature doesn't adhere to determinism, noisy optimization problems and their evolutionary solution methods emerged and gained importance [1]. Yet, the main part of this approaches still abide to the certainty of observed objectives. High sampling rates of a given decision value could simply reveal the underlying distribution of the random processes modelled by the system. Thus, models of real systems are built without perfect knowledge of the system simulated. Often the objective values stay highly uncertain even if the real variance (noise) is minimal. This may be due to a fundamental lack of information about environmental factors or the system itself (e.g. measurement errors with unknown distributions, vague input data). In this case stochastic sampling won't help as the distribution to sample from is unknown. This so-called imprecision must not be ignored in the optimization process. Indeed, there is a trend in reliability science and other application areas to formulate models that incorporate and propagate imprecision through the simulation rather than generating sharp values. The outputs are often only given as intervals or imprecise probability distributions, thus the necessity of algorithms capable to handle these types of data is evident.

This work is structured as follows: The remaining section discusses different types of uncertainties, their background and representation while section 2 defines imprecise objective functions and gives an overview of existing approaches. Section 3 presents a new optimization algorithm which is capable to handle imprecise objective functions. Section 4 tries to solve the matter of algorithmic performance assessment under imprecision by proposing metrics and test functions for imprecise objectives. In section 5 the modified algorithm is compared to a distribution-assuming algorithm with promising results. Finally in section 6 some conclusions are made and further research ideas are presented.

1.1 Imprecision and Noise

There are at least two different types of uncertainty that are important for real-world modeling. Well-known and extensively researched is noise, also referred to as aleatory uncertainty. Noise is an inherent property of the system modelled (or is introduced into the model to simulate this behavior) and therefore can't be reduced. [2] defines aleatory uncertainty as the "inherent variation associated with the physical system or the environment under consideration".

On the contrary, imprecision, also known as epistemic uncertainty describes not uncertainty due to system variance but the uncertainty of the outcome due to "any lack of knowledge or information in any phase or activity of the modeling process"[2]. Measurement errors, uncertainty of input parameters and of the correctness of a model are examples of epistemic uncertainty. Imprecision occurs when it is either impossible or too expensive to get more information about the system, so the current state is roughly determined. If more information might be obtained, the imprecision could be reduced (see Fig. 1).

A simple example from reliability science shows the difference: The failure probability of an electronic component $p([0,t])$ can be modelled as seen in Fig. 1. Failures are random events that strike a technical component following a distribution. This distribution is the noisy part of the uncertainty in the model. As reliability prediction is the attempt to predict the future behavior of a component and thus predicting these distributions, there are also uncertainties resulting from imprecision in the model. Uncertain load profiles, unknown production difficulties and vague expert estimates may cause this imprecision. This may result in imprecise distribution parameters or uncertainty about distribution types (e.g. Weibull or log-normal). Propagating this imprecision through the model leads to imprecise objective function formulations. Imprecision propagation instead of generating sharp values is applied to model the system outcome in an “honest” way with all the existing information gaps. Trends in reliability science tend to separate both noise and imprecision. The given example is a result generated by mechatronic reliability prognosis software developed at the University of Duisburg-Essen. The presented algorithms emerged from the practical need for optimization algorithms for design selection [3, 4].

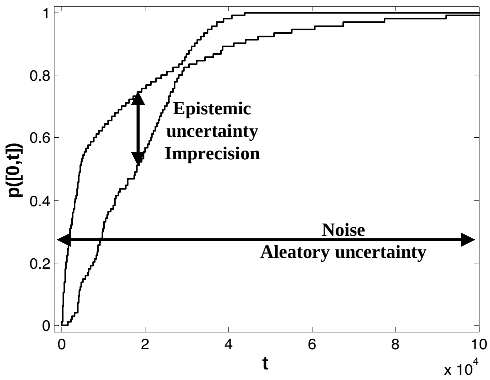

Fig. 1. Exemplary failure probability with imprecision and noise

1.2 The necessity to cope with intervals

There are many extensions to standard probability theory that allow a distinct modeling of both imprecision and noise. A good survey can be found in [5]. Dempster-Shafer models [6], fuzzy sets [7] and sets of probability measures [8] are perhaps the most popular ones. Their commonality is that they display noise through distributions and imprecision through sets or intervals. Statistical properties as the expected value or variance, which average over the noise are also represented as intervals or sets. Therefore a need for special optimization methods capable to handle intervals is evident.

2 Problem description and review of existing approaches

Classical multiple-objective optimization problem formulations are mappings from the decision space X to an objective space $Y \subseteq \mathbb{R}^n$ where the goal is to find a vector $\boldsymbol{x}_{opt} \in X$ that maximizes the objective function

$$f : X \to Y \subseteq \mathbb{R}^n \tag{1}$$

$$f(\boldsymbol{x}) = \boldsymbol{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \tag{2}$$

respective to a (partial) order relation R over Y (Fig. 2, a). If $n = 1$, then we deal with a single-objective optimization problem and R is a total order relation (e.g. the ”bigger than” relation $>$). If $n > 1$, then a relation over vectors has to be chosen. The Pareto dominance $>_P$ is often used in this case:

$$\boldsymbol{y} >_P \boldsymbol{y}' \Leftrightarrow \begin{matrix} \forall i \in 1 \ldots n : y_i \geq y'_i \\ \exists i \in 1 \ldots n : y_i > y'_i \end{matrix} \tag{3}$$

Clearly the above relationship is suitable for exact values, but what if the results of f are imprecise values, represented as multidimensional intervals (Fig. 2, b)? Normal Pareto dominance concepts cease to work and extensions to handle intervals have to be introduced. An interval-valued multiple-objective function can be defined as:

$$\mathsf{f} : X \to \mathsf{Y} \subseteq (\mathbb{R} \times \mathbb{R})^n$$

$$\mathsf{f}(\boldsymbol{x}) = \mathsf{y} = \begin{pmatrix} \mathsf{y}_1 = [\underline{y_1}, \overline{y_1}] \\ \vdots \\ \mathsf{y}_n = [\underline{y_n}, \overline{y_n}] \end{pmatrix} \tag{4}$$

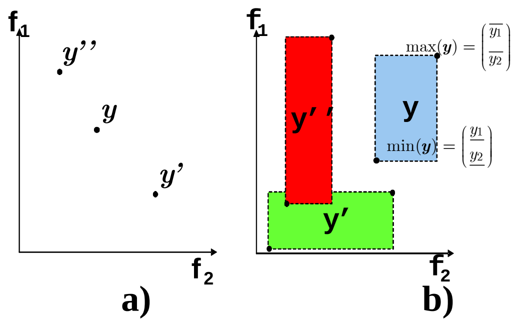

Fig. 2. Normal (a) and imprecise (b) objective values

A literature review showed that not enough work has been devoted to imprecise multiple-objective optimization. Different approaches are described in [9-13]. Where imprecision arises, it is most often simply treated as noise. The existing approaches can be roughly separated into two different types: Distribution assuming (DA-MOEAs) and imprecision-propagating (IP-MOEAs) ones.

DA-MOEAs cope with imprecision like they would do with noisy problems. They assume a distribution (most often Gaussian, uniform) inside the imprecise multiple-objective solution. Then, statistical properties like the expectation value may be used for optimization and the imprecision is erased. Formally, there is a mapping *u* from the imprecise objective function to a precise one:

$$u : \mathsf{Y} \subseteq (\mathbb{R} \times \mathbb{R})^n \to Z \subseteq (\mathbb{R})^n$$

$$u(\mathsf{y}) = \boldsymbol{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \tag{5}$$

This approach has some drawbacks: If the objective function uses sophisticated methods to separate noise with known distribution from imprecision without known distribution, how can the optimizer simply state that the imprecision is Gaussian/uniform/gamma distributed? Yet, it may be feasible for simplicity or other practical reasons as long as the algorithmic performance is not suffering. This question will be analyzed and discussed in section 5. DA-MOEA approaches are for example [9-11].

[9] defines a Gaussian distribution inside the intervals to compute the probability of dominance between two individuals. Hughes [10, 11] uses similar approaches for ranking purposes.

IP-MOEAs do not assume distributions to the objective function. Instead, they handle the objective function as-is and apply the optimization method directly on the interval values. This leads to partial orders and incomparable individuals as described in section 3. In [12] a general framework for a MOEA with a partially ordered fitness function is presented. [13] defines and applies a framework for IP-MOEA optimization.

3 An algorithm for imprecise objective functions

The algorithm proposed is an extended combination of NSGA-II [14] and SMS-EMOA [15], but the modifications are general and could be applied to other MOEAs. The NSGA-II main loop may be described as given in Fig. 3. Modifications are necessary in those steps where imprecise solutions are compared. Thus, steps 4 (selection) and 6 (nondominated sorting) are the crucial ones that must be adapted to deal with imprecision. Step 4 chooses and compares the individuals for the mating pool using tournament selection. Step 6 implicitly uses a comparison criterion to calculate the nondominated fronts. In the following subsections a dominance relation for interval-valued multiple-objective functions is defined, which will be used to compare individuals in step 4 and 6. Then, the SMS-EMOA crowding operator based on the hypervolume metric is extended to handle imprecise objective functions.

```
1  Initialize population P0
2  sort(P0)
3  While (not terminate)
 4 Use selection, mutation and
   crossover to generate children Qi
 5 Pi=Pi∪Qi
 6 sort(Pi)
 7 Pi=Pi[1…popsize]
 8 Pi+1=Pi
 9 i++
10 End While
```

Fig. 3. NSGA-II main loop.

3.1 Dominance relations on multiple-objective interval functions

To be able to optimize multiple-objective interval-valued functions, two relations are necessary. First of all $>_{IN}$, a partial order relation which compares intervals inside a single objective dimension. Second, applying $>_{IN}$, a relation $>_{IP}$ extending the standard Pareto relation may then be developed for the multiple-objective case.

In order to derive $>_{IN}$, it is very valuable to look at possible decision cases between two intervals y, y' (Fig. 4, a-c).

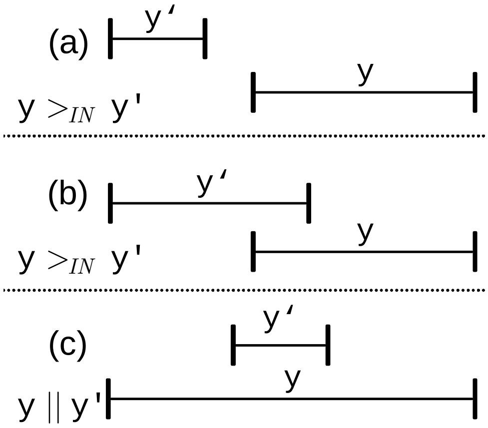

Fig. 4. Different cases of interval comparison

Type (a) represents the case in which it can be decided for sure that y is greater than y'. Thus, $y >_{IN} y'$ must hold. Type (b) is a much more critical case. If there is a distribution inside y and y', it needs not necessarily be the same. The only fact that is known is that the real values are somewhere inside the intervals. So it might well be that y' is greater than y. Yet, most rational decision makers would prefer y to y'. There are two possibilities to treat this case: $y \parallel y'$ (y and y' are incomparable using $>_{IN}$) and $y >_{IN} y'$. Both choices represent a trade-off between the number of indifferent and possibly wrong decisions. $y \parallel y'$ is error free but leads to a very high number of indifferent decisions in case of high imprecision. Thus it can degenerate the optimization to a random search. $y >_{IN} y'$ may contain wrong decisions, but does not suffer this problem. The presented IP-MOEA therefore utilizes the latter. Type (c) is the case of incomparability. One interval encloses the other. Without extra knowledge about the underlying distributions and the decision maker's preferences (risk-avoiding or risk-loving), $y \parallel y'$ should hold.

Regarding this requirements, $>_{IN}$ can be defined as:

$$y >_{IN} y' \Leftrightarrow \underline{y} \geq \underline{y'} \wedge \overline{y} \geq \overline{y'} \wedge y \neq y' \qquad (6)$$

Extending Pareto dominance to the multiple-objective interval-valued case, we face the problem of incomparability inside the single dimensions ($y \parallel y'$ for some i). The Pareto dominance relation therefore has to be ex-

tended to an imprecise Pareto relation $>_{IP}$. Another example might be helpful for this.
Four critical decision cases between solutions $\mathbf{y}$ and $\mathbf{y}'$ in a two-dimensional objective space are plotted in (Fig. 5). Type (a) is the case where $\mathbf{y}$ is greater than $\mathbf{y}'$ in all objectives using $>_{IN}$. Thus, a rational decision maker would prefer $\mathbf{y}$, and thus $\mathbf{y} >_{IP} \mathbf{y}'$. Likewise Type (b) is a case of incomparability between $\mathbf{y}$ and $\mathbf{y}'$. This can be interpreted as ignorance or imprecise knowledge about the decision-makers preferences or indifference of the decision-maker itself.

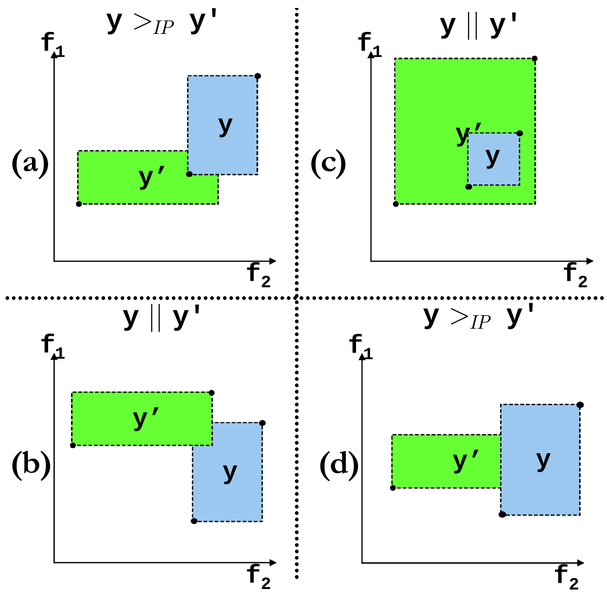

Fig. 5. Four cases of multi-dimensional interval comparison. Which solution is dominant?

Type (c) is also a case of incomparability ($\mathbf{y} \,||\, \mathbf{y}'$), yet for another reason. The incomparability is a result from imprecision of the objective function. Type (d) is the critical case. Using $>_{IN}$, solution $\mathbf{y}$ is greater than $\mathbf{y}'$ in some objectives but incomparable in other. Could a decision between these two cases been made? This depends also heavily on the decision maker. [13] proposes different approaches for both the external repository and the population. The proposed IP-MOEA uses a relation that decides Type (d) as $\mathbf{y} >_{IP} \mathbf{y}'$. Thus, $>_{IP}$ can be derived as:

$$\mathbf{y} >_{IP} \mathbf{y}' \Leftrightarrow \begin{array}{l} \forall i \in 1...n : \mathrm{y}_i >_{IN} \mathrm{y}'_i \vee \mathrm{y}_i \,||\, \mathrm{y}'_i \\ \exists i \in 1...n : \mathrm{y}_i >_{IN} \mathrm{y}'_i \end{array} \tag{7}$$

3.2 Crowding operator

As [15] have shown for the non-interval case, the hypervolume metric $S(M)$ of a solution set M serves not only as an excellent metric but may also be used as a crowding operator. As it can be extended to the imprecise case (see section 4.2), this is a good choice for an IP-MOEA. For each individual $\mathbf{y}$ assign the crowding operator $C(\mathbf{y}, M)$, which has also interval form:

$$\overline{C(\mathbf{y}, M)} := \overline{S(M)} - \overline{S(M \setminus \{\mathbf{y}\})} \tag{8}$$

$$\underline{C(\mathbf{y}, M)} := \underline{S(M)} - \underline{S(M \setminus \{\mathbf{y}\})} \tag{9}$$

On the other hand, the contribution to either certain dominated (worst case) and plausible dominated (best case) hypervolume can be use as a selection criterion. Between two nondominated solutions, one can discern which one should be selected according to its contribution to hypervolume, whereas the one with the smaller contribution is the best candidate to be discarded when pruning the set. The hypervolume measure may therefore be used as a second comparison criterion when the $>_{IP}$ relation stays incomparable.

3.3 Dominance in NSGA-II

To adapt NSGA-II to interval-valued functions, it is necessary to modify both: the selection, which fills the mating pool, and the nondominated sorting algorithm. The tournament selection decision criteria (in order) are:

Tournament selection decision criteria

1. Compare individuals using $>_{IP}$
2. Compare crowding measures $C(\mathbf{y}, M)$ using $>_{IN}$
3. Choose at random

The nondominated sorting criterion is similarly modified. The population is first of all sorted by the number of the front or rank they are in. To calculate the fronts, $>_{IP}$ is used. A second order criterion inside the fronts is the hypervolume measure $C(\mathbf{y}, M)$, which is also sorted according to nondominating $S(M)$ values. This is necessary, as $S(M)$ is not a single value but an interval.

Nondominated sorting decision criteria

1. Compare individuals using $>_{IP}$
2. Compare crowding measures $C(\mathbf{y}, M)$ using nondominated sorting
3. Choose at random

4 Performance assessment for imprecise functions

Two serious problems arise, when the performance of a MOEA should be assessed in the imprecise case. Until now, no generally accepted test function set (like DTLZ, ZDT) exists. And perhaps even more important, no performance metrics have been applied to the imprecision case. Both problems will be tackled in the next subsections

4.1 Test functions

To construct an imprecise test function set, it seems feasible to choose some of the most popular precise test functions in the MOEA community. So we adapted the test functions ZDT 1, ZDT2, ZDT 4 and ZDT 6 [16] with modifications given in [17] to imprecise functions ZDT_I {1,2,4,6} by introducing an imprecision factor ε. These functions serve as the benchmarks for the following comparisons.

$$\varepsilon = \begin{pmatrix} \sin\left(10\pi \sum_i x_i\right)/2 \\ \sin\left(20\pi \sum_i x_i\right)/2 \end{pmatrix} \tag{10}$$

$$\underline{ZDT_I(\boldsymbol{x})} = \min(ZDT_I(\boldsymbol{x}), ZDT_I(\boldsymbol{x}) + \varepsilon) \tag{11}$$

$$\overline{ZDT_I(\boldsymbol{x})} = \max(ZDT_I(\boldsymbol{x}), ZDT_I(\boldsymbol{x}) + \varepsilon) \tag{12}$$

$$ZDT_I(\boldsymbol{x}) = -\left[\underline{ZDT_I(\boldsymbol{x})}, \overline{ZDT_I(\boldsymbol{x})}\right] + 1 \tag{13}$$

For technical reasons, all functions were mapped to maximization problems (eq. 12).

4.2 Measuring algorithmic performance

In this section we introduce a metric that is an extension of the hypervolume metric originally proposed in [18]. The hypervolume or S metric $S(M)$ can be defined as a Lebesgue measure Λ applying

$$S(M) := \Lambda\left(\bigcup_{\boldsymbol{y}\in M} \{\boldsymbol{x} \mid \boldsymbol{y_{ref}} R_P \boldsymbol{x} R_P \boldsymbol{y}\}\right) \tag{14}$$

to a standard nondominated set M. It measures the volume that M dominates limited by a reference point $\boldsymbol{y_{ref}}$. The hypervolume is a well-established and accepted measure for the quality of a nondominated set [19]. So it seems wise to extend it to the imprecise case. Using interval arithmetic and the R_{IP} relation, it can be stated as:

$$S(M) = [\underline{S(M)}, \overline{S(M)}] := \Lambda\left(\bigcup_{\mathbf{y}\in M} \{\boldsymbol{x} \in \mathbb{R} \mid \boldsymbol{y_{ref}} R_{IP} \boldsymbol{x} R_{IP} \mathbf{y}\}\right) \tag{15}$$

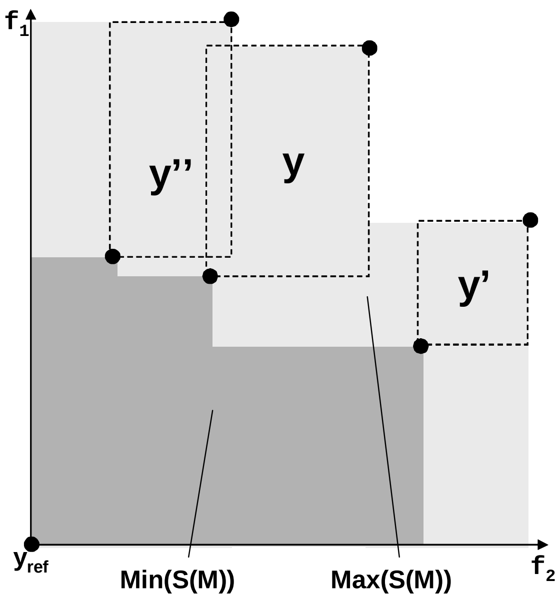

Fig. 6. Hypervolume measure

$S(M)$ has also interval form (Fig. 6). Hypervolume is also susceptible of a convenient semantic interpretation for the decision maker. When coping with intervals, the hypervolume calculated with the lower bounds of the intervals offers a measure of the region of the objective space certain dominated, whereas the hypervolume calculated with the upper bounds gives an idea of the maximal possible gain if the imprecision is overcome.

4.3 Measuring imprecision

Another interesting quantity is the amount of uncertainty in a population. This might be measured as the added volume of all solutions in the front:

$$I(M) := \sum_{\boldsymbol{y}\in M} \Lambda\{\boldsymbol{x} \mid \underline{\boldsymbol{y}} R_P \boldsymbol{x} R_P \overline{\boldsymbol{y}}\} \tag{16}$$

To show, why measuring imprecision might be interesting, see the example in Fig. 7. Both nondominated sets cover the same hypervolume (best case and worst case), but imprecision differs strongly. Therefore, the hypervolume measure does not allow conclusions about the imprecision inside the population.

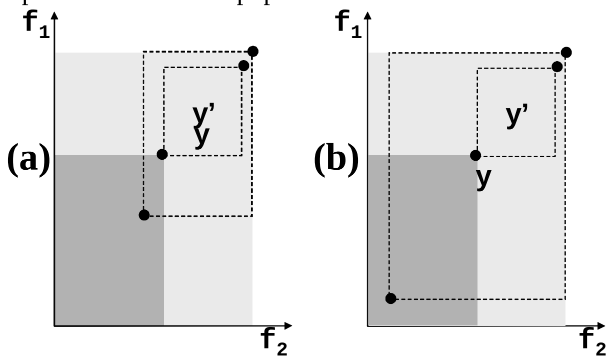

Fig. 7: Solutions with different grades of imprecision and identical hypervolume

5 Results

The most interesting question regarding imprecision preserving MOEAs is: Does an imprecision preserving MOEA perform better than the standard distribution assuming approach? To investigate this crucial point, we chose two test sets.

IP-MOEA: The Algorithm as proposed in the former sections.

DA-MOEA: An identical algorithm, but assuming a uniform distribution inside the solution vector. The expected value of the objective vector is then used for optimization. Thus, all intervals are mapped to points and the algorithm becomes a normal MOEA.

Both algorithms were run for 100 generations with a population size of 20. Crossover probability was set to 0.9 and mutation rate to 0.1. The number of decision variables was fixed on 30 for all test problems. Results were compared using the hypervolume measure (section 4.2). The reference point was fixed to the minimal objective values of all solution sets. Objective values were scaled to [0,1]. For verification purposes, all reference points are listed in Table 1. In Table 2, the worst-case and best-case hypervolume measure and imprecision after 100 generations are listed. Beside the median and interquartile range (IQR), the probability of the null hypothesis, p(0) that both medians are equal is given. Significance testing was done by a Kruskal-Wallis test on the equality of medians. The results are surprisingly clear and support the thesis that imprecision propagation leads to better algorithm performance. Indeed, on all four test functions, both best and worst case hypervolume ($\underline{S(M)}, \overline{S(M)}$) are higher and the null hypothesis can be safely rejected ($p(0)<0.001$ in all tests). Fig. 8 shows a sample population plot of IP-MOEA, where it can be seen that the population is quite diverse and form a broad R_{IP}-optimal set. Detailed plots of $S(M)$ over the optimization run (Fig. 9-Fig. 12) indicate that IP-MOEA seems to have better convergence properties than DA-MOEA. We also investigated the imprecision and its development over the optimization process. Medians of the imprecision measure I(M) can be seen in Fig. 13-Fig. 16. There seems to be a problem-dependent amount of imprecision on the way to the optimal solution which is more or less similar in both algorithms.

Table 1. Reference points and scales for hypervolume results.

	Reference point y_{ref}		Scale	
	y_1	y_2	y_1	y_2
ZDT_I 1	-1.3655	-10.3329	2.8645	12.0196
ZDT_I 2	-1.3994	-10.3480	2.8988	12.3078
ZDT_I 4	-1.0336	-355.1911	2.5334	307.1266
ZDT_I 6	-0.4997	-26.9331	1.6113	14.7861

Table 2. Hypervolume (100 generations), best and worst case.

	IP-MOEA		**DA-MOEA**		
ZDT_I 1	Median	IQR	Median	IQR	p(0)
Worst Case	0.7139	1.15E-02	0.7064	1.57E-02	9.60E-07
Best Case	0.8991	1.59E-02	0.8899	1.97E-02	3.23E-05
Imprecision	2.0721	8.03E-02	2.2136	9.84E-02	2.10E-04
ZDT_I 2	Median	IQR	Median	IQR	p(0)
Worst Case	0.6528	1.91E-02	0.6324	2.32E-02	7.25E-12
Best Case	0.8219	2.50E-02	0.7954	2.74E-02	1.04E-11
Imprecision	2.146	8.74E-02	2.1853	1.08E-01	4.35E-01
ZDT_I 4	Median	IQR	Median	IQR	p(0)
Worst Case	0.7311	3.96E-02	0.7017	4.45E-02	1.04E-09
Best Case	0.9062	4.37E-02	0.8728	5.58E-02	1.74E-09
Imprecision	2.0976	9.44E-02	2.1731	6.03E-02	6.53E-02
ZDT_I 6	Median	IQR	Median	IQR	p(0)
Worst Case	0.4759	3.87E-02	0.4531	3.99E-02	7.12E-07
Best Case	0.7591	5.26E-02	0.7248	4.88E-02	2.92E-06
Imprecision	2.0086	7.08E-02	2.1069	9.50E-02	1.87E-02

6 Conclusions & Outlook

Problems incorporating epistemic uncertainty are of great practical importance as parametric and modeling knowledge is never perfect. Actually, this is a recent trend in reliability engineering and reliability optimization [2, 20, 21], so that the demand for IP-MOEAs will grow in the near future.

The initial approach proposed shows that multiple-objective evolutionary optimization is possible, even if the objective function is disrupted by uncertainty resulting from a lack of knowledge. With adapting a NSGA-II-like MOEA to imprecise objective functions, it is shown, that most popular MOEAs could be applied to imprecise problems.

The results are a remarkably clear indication that imprecision-propagating MOEAs (IP-MOEAs) are an interesting alternative to distribution assuming ones (DA-MOEAs). As they did at least perform equal on all test problems, it might be auspicious both for practical and scientific purposes to intensify the research in this area.

Much more interesting issues remain, perhaps the most crucial one in the development of metrics that incorporate the dimension of imprecision and the integration of such indicators into the evolutionary process.

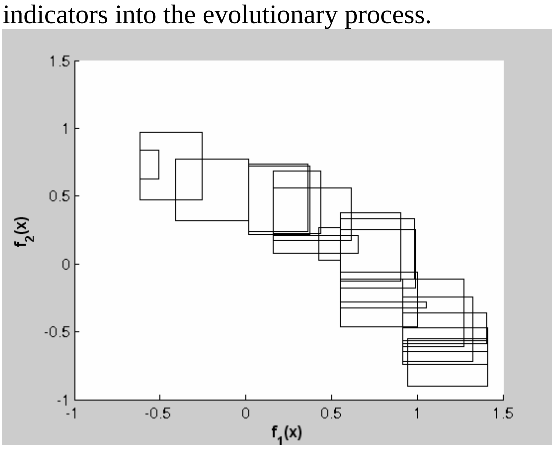

Fig. 8. Sample population plot after 100 generations, ZDT_I 6, IP-MOEA.

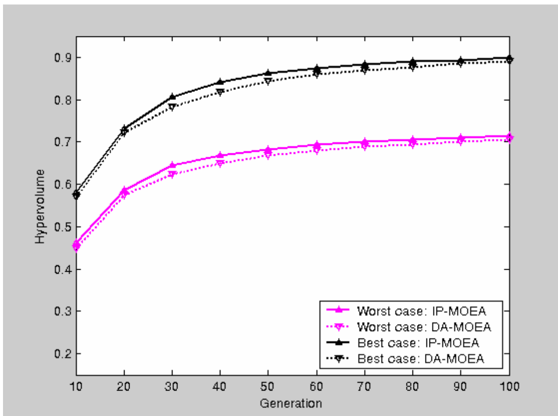

Fig. 9. Hypervolume (Median), ZDT_I 1.

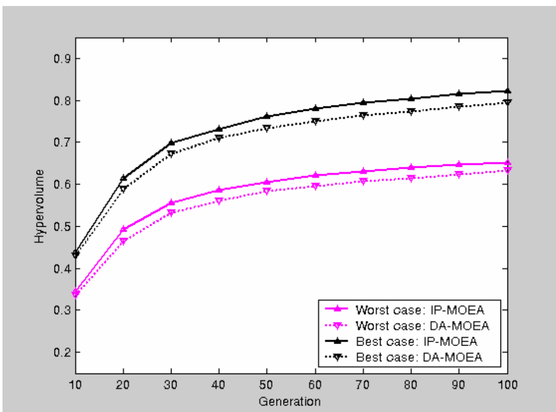

Fig. 10. Hypervolume (Median), ZDT_I 2.

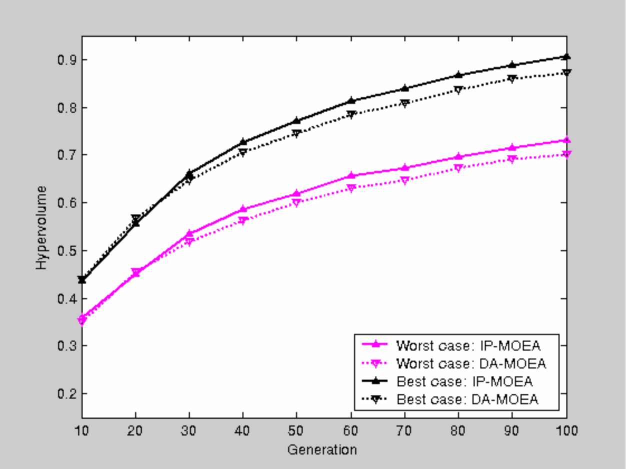

Fig. 11. Hypervolume (Median), ZDT_I 4.

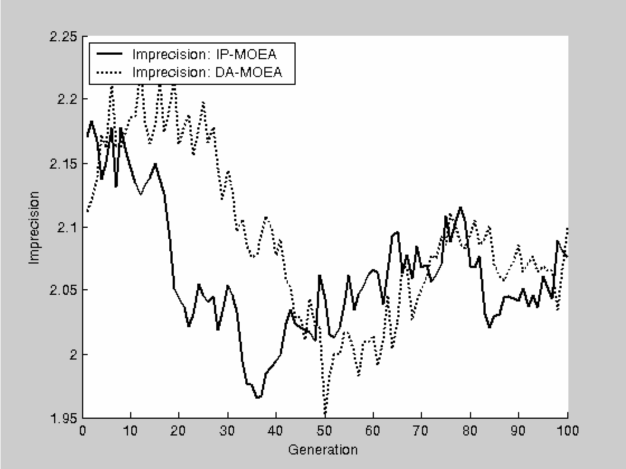

Fig. 14. Imprecision (Median), ZDT_I 2.

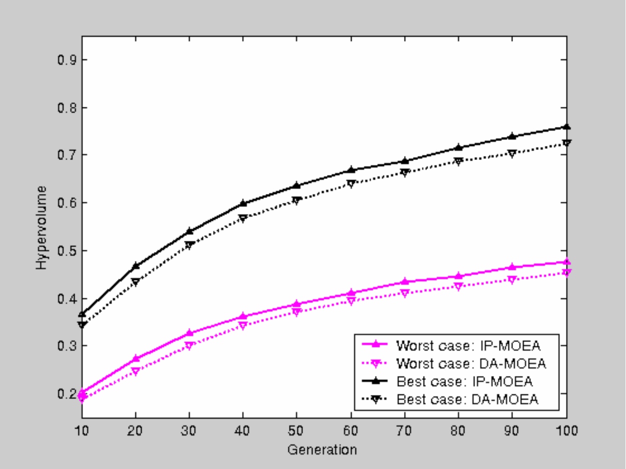

Fig. 12. Hypervolume (Median), ZDT_I 6.

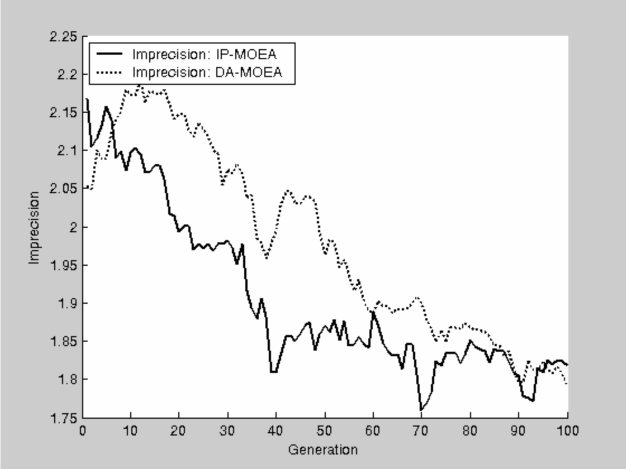

Fig. 15. Imprecision (Median), ZDT_I 4.

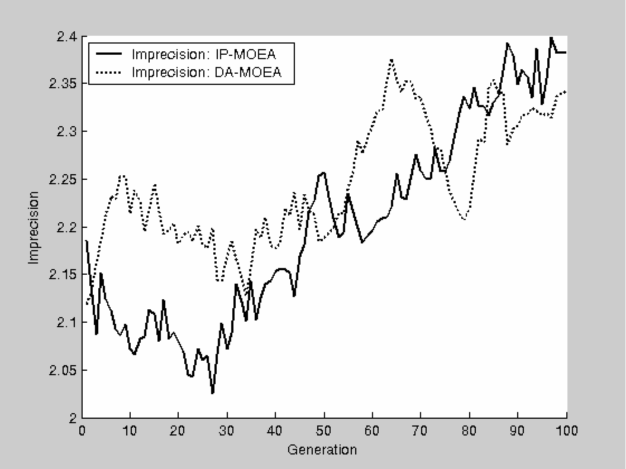

Fig. 13. Imprecision (Median), ZDT_I 1.

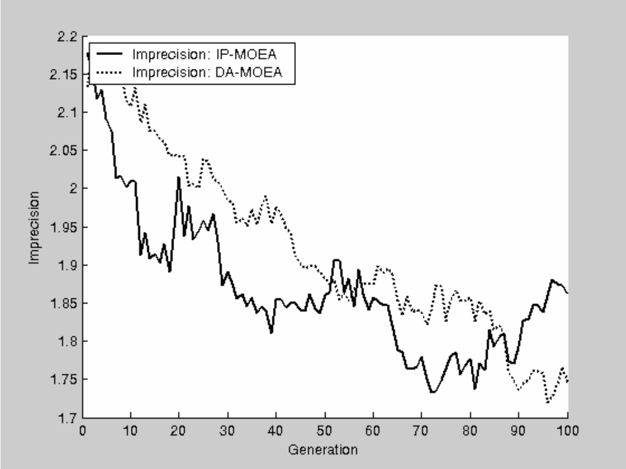

Fig. 16. Imprecision (Median), ZDT_I 6.

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