Intuitionistic Fuzzy Relation Equations

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***Abstract*—In this paper, we extend the fuzzy relation equations to intuitionistic fuzzy relation equations. The definition of intuitionistic fuzzy relation equations is proposed on the basis of theory of fuzzy relation equation. Intuitionistic fuzzy relation equation is an L-fuzzy relation equation. The method with special significance is obtained as the lattice is embodied. The solvability criteria for the intuitionistic fuzzy relation equations are given, and the maximum solution of intuitionistic fuzzy relation equation is research.**

Keywords-Fuzzy sets; fuzzy relation equations; intuitionistic fuzzy relation equations

I. INTRODUCTION AND PRELIMINARIES

Intuitionistic fuzzy sets theory (IFS theory) introduced by K. Atanassov in [1-4], is a significant extension of fuzzy set theory by Zadeh. Fuzzy sets can be viewed as intuitionistic fuzzy sets, but the converse is not true. It has been asserted by many authors that there are a large and large number of life problems for which IFS theory is a more suitable tool than fuzzy set theory for searching solution. For example, in decision making problems, particularly in the case of medical diagnosis, sales analysis, new product marketing, financial services, etc. there is a fair chance of the existence of a non-null hesitation part at each moment of evaluation of an unknown object. To be more precise – intuitionistic fuzzy sets let us express e.g., the fact that the temperature of a patient changes, and other symptoms are not quite clear.

The notion of fuzzy relational equations based upon the max-min composition was first investigated by Sanchez. He studied conditions and theoretical methods to resolve fuzzy relations on fuzzy sets defined as mappings from sets to [0,1]. Some theorems for existence and determination of solutions of certain basic fuzzy relation equations were given by him.

Our objective in this paper is to extend fuzzy relation equations to intuitionistic fuzzy relation equations. The definition of Intuitionistic fuzzy relation equations is proposed, the solvability criteria for the intuitionistic fuzzy relation equations are given, and the maximum solution of intuitionistic fuzzy relation equation is research.

Now we are going to remember the concept of intuitionistic fuzzy set in [1].

Let X be a non-empty sets. An intuitionistic fuzzy sets in X is an expression A given by

$$A=\{<x,\mu_A(x),v_A(x)>|\,x\in X\} \tag{1}$$

where

$$\begin{aligned}\mu_A &: X\to[0,1]\\ v_A &: X\to[0,1]\end{aligned} \tag{2}$$

with the condition

$$0\le\mu_A(x)+v_A(x)\le 1 \tag{3}$$

for all x in X.

The numbers $\mu_A(x)$ and $v_A(X)$ denote respectively the degree of membership and the degree of non-membership of the element x in set A. We will represent as *IFS(X)* the set of all the intuitionistic fuzzy sets in X.

Obviously, each fuzzy set corresponds to the following intuitionistic fuzzy set

$$A=\{<x,\mu_A(x),1-\mu_A(x)>|\,x\in X\} \tag{4}$$

For each intuitionistic fuzzy set in X, we will call

$$\pi_A(x)=1-\mu_A(x)-v_A(x) \tag{5}$$

a hesitation margin of $x\in A$, and it is a hesitation degree of whether x belongs to A or not. It is obvious that $0\le\pi_A(x)\le 1$, for each $x\in X$.

For every $A,B\in IFS(x)$ the following expressions are known in [5]:

- $A\le B$ if and only if $\mu_A(x)\le\mu_B(x)$ and $v_A(x)\ge v_A(x)$ for all $x\in X$
- $A\preceq B$ if and only if $\mu_A(x)\le\mu_B(x)$ and $v_A(x)\le v_A(x)$ for all $x\in X$
- $A=B$ if and only if $A\le B$ and $B\le A$
- $A_c=\{<x,v_A(x),\mu_A(x)>|\,x\in X\}$.

Let *X, Y and Z* be ordinary finite non-empty sets. We know that an intuitionistic fuzzy relation in [6] is an intuitionistic fuzzy subset of $X\times Y$, that is an expression R given by

$$R=\{<(x,y),\mu_R(x,y),v_R(x,y)>|\,(x,y)\in X\times Y\} \tag{6}$$

where

$$\begin{aligned}\mu_R &: X\times Y\to[0,1]\\ v_R &: X\times Y\to[0,1]\end{aligned} \tag{7}$$

satisfy the condition $0\le\mu_R(x,y)+v_R(x,y)\le 1$, for every $(x,y)\in X\times Y$.

We will represent as $IFR(X \times Y)$ the set of all the intuitionistic fuzzy subsets in $X \times Y$.

For any $R, P \in IFR(X \times Y)$ the following expressions are known:

- $R \le P$ if and only if $\mu_R(x,y) \le \mu_P(x,y)$ and $\nu_R(x,y) \ge \nu_P(x,y)$ for every $(x,y) \in X \times Y$
- $R \preceq P$ if and only if $\mu_R(x,y) \le \mu_P(x,y)$ and $\nu_R(x,y) \le \nu_P(x,y)$ for every $(x,y) \in X \times Y$
- $R \vee P = \{<(x,y), \mu_R(x,y) \vee \mu_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y) > | (x,y) \in X \times Y\}$
- $R \wedge P = \{<(x,y), \mu_R(x,y) \wedge \mu_P(x,y), \nu_R(x,y) \vee \nu_P(x,y) > | (x,y) \in X \times Y\}$.

Definition 1. Let $R \in IFR(X \times Y)$ and $P \in IFR(X \times Z)$. We will call composed relation $R \circ P \in IFR(X \times Z)$ the one defined by

$$R \circ P = \left\{ \langle (x,z), \mu_{R\circ P}(x,z), \nu_{R \circ P}(x,z) \rangle \mid x \in X, z \in Z \right\}$$

where

$$\begin{aligned} \mu_{R\circ P}(x,z) &= \vee_{y \in Y} \{\mu_R(x,y) \wedge \mu_P(y,z)\} \\ \nu_{R\circ P}(x,z) &= \wedge_{y \in Y} \{\nu_R(x,y) \vee \nu_P(y,z)\}. \end{aligned} \tag{8}$$

II. INTUITIONISTIC FUZZY RELATION EQUATIONS

We propose intuitionistic fuzzy relation equations as follows:

Definition 2. Let U, V and W be ordinary finite non-empty sets, and Let $R \in IFR(V \times W)$ and $S \in IFR(U \times W)$ are given. Then an equation

$$X \circ R = S \tag{9}$$

is called an intuitionistic fuzzy relation equations (*IFRE*), where $X \in IFR(U \times V)$ is unknown.

An intuitionistic fuzzy relation X satisfying (9) is called a solution of the equation. The maximum solution of the equation is a solution $\overline{X}$ such that $X \le \overline{X}$ for any solution X of the equation.

Theorem 1. Let $R \in IFR(V \times W)$ and $S \in IFR(U \times W)$. Equation $X \circ R = S$ has a solution $X \in IFR(U \times V)$ if for each $v \in V$ there exists $(u,w) \in U \times W$ such that either $S(u,w) \le R(v,w)$ or $R(v,w) \le S(u,w)$.

Proof.

We proof the theorem by contradiction.

If for any $v \in V, (u,w) \in U \times W$, both $S(u,w) \le R(v,w)$ and $R(v,w) \le S(u,w)$ do not tenable, then either $S(u,w) \preceq R(v,w)$ or $R(v,w) \preceq S(u,w)$ tenable.

If $R(v,w) \preceq S(u,w)$, then $X(u,v) \wedge R(v,w) \le R(v,w) \preceq S(u,w)$.

If $S(u,w) \preceq R(v,w)$, then either $X(u,v) \wedge R(v,w) \le S(u,w)$ or $S(u,w) \preceq X(u,v) \wedge R(v,w)$.

(1) if $X(u,v) \le S(u,w)$,then $X(u,v) \wedge R(v,w) \le S(u,w)$.

(2) If $S(u,w) \le X(u,v)$, that is $(S_1(u,w), S_2(u,w)) \le (X_1(u,v), X_2(u,v))$, then $S_1(u,w) \le X_1(u,v)$, $S_2(u,w) \ge X_2(u,v)$. From $S(u,w) \preceq R(v,w)$, we know $S_1(u,w) \le R_1(v,w)$, $S_2(u,w) \le R_2(v,w)$, so $X_1(u,v) \wedge R_1(v,w) \ge S_1(u,w)$ and $X_2(u,v) \vee R_2(v,w) \ge S_2(u,w)$, that is $S(u,w) \preceq X(u,v) \wedge R(v,w)$.

(3) If $X(u,v) \preceq S(u,w)$, that is $(X_1(u,v), X_2(u,v)) \preceq (S_1(u,w), S_2(u,w))$, then $X_1(u,v) \le S_1(u,w)$, $X_2(u,v) \le S_2(u,w)$. From $S(u,w) \preceq R(v,w)$, we know $S_1(u,w) \le R_1(v,w), S_2(u,w) \le R_2(v,w)$, so $X_1(u,v) \wedge R_1(v,w) \le S_1(u,w)$, $X_2(u,v) \vee R_2(v,w) \ge S_2(u,w)$, that is $X(u,v) \wedge R(v,w) \le S(u,w)$

(4) If $S(u,w) \preceq X(u,v)$, then $S(u,w) \preceq X(u,v) \wedge R(v,w)$.

On the other hand, the equation has a solution X, that is $\vee_{v \in V} (X(u,v) \wedge R(v,w)) = S(u,w), \forall (u,w)$, then for any $v \in V$, for any $(u,w) \in U \times W$, $X(u,v) \wedge R(v,w) \le S(u,w)$.

The contradiction is come, so the theorem is proofed.

Theorem 2. Let $R \in IFR(V \times W)$ and $S \in IFR(U \times W)$. Equation $X \circ R = S$ is solvability if and only if $\overline{X}$ satisfy the equation, where

$$\overline{X}(u,v) = \wedge_w \left\{ S(u,w) \middle| S(u,w) \le R(v,w), S(u,w) \ne R(v,w) \right\}.$$

is the maximal solution of the equation.

Proof.

"$\Rightarrow$" If he equation has a solution X, that is $\vee_{v \in V} (X(u,v) \wedge R(v,w)) = S(u,w), \forall (u,w)$.

$\Rightarrow \forall (u,w), \forall v, X(u,v) \wedge R(v,w) \le S(u,w)$.

$\Rightarrow \forall (u,w), \forall v$, if $S(u,w) \le R(v,w)$ and $S(u,w) \ne R(v,w)$, then $(0,1) \le X(u,v) \le S(u,w)$; if $R(v,w) \le S(u,w)$, then $(0,1) \le X(u,v) \le (1,0)$.

$\Rightarrow (0,1) \le X(u,v) \le (1,0) \wedge (\wedge_w \{S(u,w) \mid S(u,w) \le R(v,w), S(u,w) \ne R(v,w)\}) = \wedge_w \{S(u,w) \mid S(u,w) \le R(v,w), S(u,w) \ne R(v,w)\} = \overline{X}(u,v)$

$\Rightarrow X \le \overline{X}$

On the other hand, $\forall (u,w), \forall v$

(1) If $S(u,w) \leq R(v,w)$ and $S(u,w) \neq R(v,w)$, then $\overline{X}(u,v) = \underset{w}{\wedge}\{S(u,w) \mid S(u,w) \leq R(v,w), S(u,w) \neq R(v,w)\} \leq S(u,w) \Rightarrow \overline{X}(u,v) \wedge R(v,w) \leq S(u,w)$

(2) If $R(v,w) \leq S(u,w)$, then $\overline{X}(u,v) \wedge R(v,w) \leq R(v,w) \leq S(u,w) \Rightarrow \underset{v}{\vee}(\overline{X}(u,v) \wedge R(v,w)) \leq S(u,w)$,

that is $\overline{X} \circ R \leq S$. Since $S = X \circ R \leq \overline{X} \circ R \leq S$, we know $\overline{X} \circ R = S$, that means $\overline{X}$ is the maximal solution of the equation.

" $\Leftarrow$ " If $\overline{X} \circ R = S$, then $\overline{X}$ is a solution of the equation. So the equation is solvability.

The theorem above gives the necessary and sufficient condition of the solvability of intuitionistic fuzzy relation equations.

III. Example Analysis

There are two examples as follow:

Example 1.

$$(x_1, x_2, x_3) \circ \begin{pmatrix} (0.3,0.5) & (0.2,0.4) & (0.7,0.2) \\ (0.1,0.9) & (0.8,0.1) & (0.4,0.3) \\ (0.6,0.2) & (0.3,0.3) & (0.5,0.1) \end{pmatrix} =$$

$$((0.1,0.6),(0.2,0.4),(0.2,0.4))$$

Solution. According to the theorem 2,

$$\overline{x}_k = \underset{j}{\wedge}\left\{s_j \,\middle|\, s_j \leq r_{kj}, s_j \neq r_{kj}\right\}, (k=1,2,3)$$

and hence

$$\overline{x}_1 = \wedge\{s_1, s_3\} = (0.1,0.6) \wedge (0.2,0.4) = (0.1,0.6)$$

$$\overline{x}_2 = \wedge\{s_2, s_3\} = (0.2,0.4) \wedge (0.2,0.4) = (0.2,0.4)$$

$$\overline{x}_3 = \wedge\{s_1, s_2, s_3\} = (0.1,0.6) \wedge (0.2,04) \wedge (0.2,0.4) = (0.1,0.6)$$

that is

$$\overline{X} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) = ((0.1,0.6),(0.2,0.4),(0.1,0.6)).$$

It is easy to see

$\overline{X} \circ R = ((0.1,0.6),(0.2,04),(0.2,0.4))$ and $\overline{X}$ is the maximal solution of the equation.

Example 2.

$$(x_1, x_2, x_3) \circ \begin{pmatrix} (0.2,0.1) & (0.1,0.3) & (0.4,0.3) \\ (0.5,0.2) & (0.6,0.4) & (0.7,0.1) \\ (0.6,0.1) & (0.1,0.1) & (0.5,0.4) \end{pmatrix} =$$

$$((0.2,0.4),(0.6,0.2),(0.4,0.3))$$

Solution. According to the theorem 2,

$$\overline{x}_k = \underset{j}{\wedge}\left\{s_j \,\middle|\, s_j \leq r_{kj}, s_j \neq r_{kj}\right\}, (k=1,2,3)$$

and hence

$$\overline{x}_1 = \wedge\{s_1\} = (0.2,0.4)$$

$$\overline{x}_2 = \wedge\{s_1, s_3\} = (0.2,0.4) \wedge (0.4,0.3) = (0.2,0.4)$$

$$\overline{x}_3 = \wedge\{s_1, s_3\} = (0.2,0.4) \wedge (0.4,0.3) = (0.2,0.4)$$

that is

$$\overline{X} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) = ((0.2,0.4),(0.2,0.4),(0.2,0.4)).$$

It is easy to see

$\overline{X} \circ R = ((0.2,0.4),(0.2,0.4),(0.2,0.4)) \leq ((0.2,0.4),(0.6,0.2),(0.4,0.3))$ and the equation is not solvability.

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