A Unification of Intuitionistic Fuzzy Calculus Theories Based on Subtraction Derivatives and Division Derivatives

Qian Lei, Zeshui Xu, *Senior Member*, *IEEE*

***Abstract*—Intuitionistic fuzzy calculus replaces real numbers in classical calculus with intuitionistic fuzzy numbers (IFNs) that are the basic elements of Atanassov's intuitionistic fuzzy sets. Intuitionistic fuzzy calculus consists of two parallel parts, which are respectively developed based on the subtraction derivatives and the division derivatives. This paper focuses on building the relationships between the two independent intuitionistic fuzzy calculus theories, and unifying them into a whole theory.**

***Index Terms*—Atanassov's intuitionistic fuzzy set; intuitionistic fuzzy number; intuitionistic fuzzy calculus; intuitionistic fuzzy function; subtraction derivative; division derivative**

I. INTRODUCTION

Since Zadeh [1] proposed the fuzzy set in 1965, the concept of "fuzzy" has combined rapidly with different disciplines in order to solve a multitude of application problems, which has sufficiently shown the validity and significance of the fuzzy theory. Over the last decades, lots of generalizations of Zadeh's fuzzy set have also been proposed. Atanassov's intuitionistic fuzzy set (A-IFS) [2] (In order to avoid confusion of some technical terms [3,4], it is noted that the word "intuitionistic" should be "Atanassov's intuitionistic" throughout the paper, but for brevity, we omit "Atanassov" before the name "intuitionistic" hereafter) is just one of these generalizations, which extends the fuzzy set by utilizing the membership function and the non-membership function, and can overcome the disadvantage of the classical fuzzy set. Because the A-IFS can actually depict the vagueness and uncertainty of things more exquisitely and comprehensively, it has received great attention, and a large number of research results on A-IFSs have also been achieved in many fields [5-13].

These early research results on A-IFSs have developed the fuzzy theory from various angles, such as aggregation techniques [7,12,14-17] and clustering algorithms [12,18,19]. As a newer emerging research direction, the differential and integral under intuitionistic fuzzy environment have also been studied recently [16-17,20-26], which develop the intuitionistic fuzzy calculus from different points of view. For example, Xu et al. [16,17] investigated Choquet integrals with respect to intuitionistic fuzzy numbers (IFNs), and Ban et al. [20,21,23] defined integrals based on a special measure under intuitionistic fuzzy environment. However, as we pointed out in Ref. [26] that there is still no systematic work building a whole intuitionistic fuzzy calculus, these early intuitionistic fuzzy calculus theories still exist deficiencies and need to be improved. Recently, we have studied this meaningful issue [24-26], which is how to build more systematically the intuitionistic fuzzy differential calculus and integral calculus, by replacing real numbers in the classical calculus with IFNs that are essentially the basic elements of A-IFSs. In Ref. [24], we first proposed the concept of intuitionistic fuzzy function (IFF), which is the main research basis of intuitionistic fuzzy calculus, and then gave the criterion of differentiability of IFFs and the derivative formula. Based on the derivatives of IFFs, we made further research on the differentials of IFFs, and pointed out that the increment of an IFF is approximately equal to its differential. After studying the differential calculus of IFFs, in Ref. [25], we went a step further to consider some issues about the integrals of IFFs. First of all, we defined the concepts of primitive functions and indefinite integrals of IFFs from the definitions of derivatives. Then we defined independently the definite integrals of IFFs, which means that the research process of the definite integrals of IFFs is totally irrelevant to these studies about the differential calculus of IFFs. However, by building the definite integrals of IFFs with the variable upper limits, there exists an interesting and significant connection between the definite integrals and the indefinite integrals of IFFs.

We have researched systematically the differential and integral calculus of IFNs in the literature [24-26]. But there exist two totally parallel calculus theories that are respectively developed based on two different types of derivatives, namely, the subtraction derivatives and the division derivatives. Almost every theorem or conclusion based on the subtraction derivative exists a similar statement based on the division derivative, which makes us think about the problem whether or not there exists some effective techniques unifying the two intuitionistic fuzzy calculus theories into a whole one. As such, intuitionistic fuzzy calculus will be simpler, more elegant and easily understandable, which is just the aim of this paper. The remainder of the paper is set out as follows: Section II mainly introduces some basic knowledges of IFNs, then discusses and analyzes the four fundamental operations of arithmetic (addition, subtraction, multiplication and division) between IFNs and the interrelations among them. Section III shows

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Q. Lei is with the College of Sciences, PLA University of Science and Technology, Nanjing 211101, China (e-mail: leiqian_lq@qq.com).

Z. S. Xu is with the Business School, Sichuan University, Chengdu 610064, China, and also with the College of Sciences, PLA University of Science and Technology, Nanjing 211101, China (Corresponding author: e-mail: xuzeshui @263.net).

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some concepts of intuitionistic fuzzy differential calculus, derivatives and differentials of IFFs, reveals how to build the relationships between these two different types of derivatives (subtraction derivatives and division derivatives) and differentials (subtraction differentials and division differentials). Section IV studies intuitionistic fuzzy integral calculus and explores some connections between those different types of intuitionistic fuzzy integrals in the parallel calculus theories based on the subtraction derivatives and the division derivatives respectively. The paper ends with some concluding remarks in Section V.

II. Basic Knowledges Related to IFNs

We first introduce some basic knowledges related to intuitionistic fuzzy numbers (IFNs) [14], which can be seen as the basic components of Atanassov's intuitionistic fuzzy set (A-IFS) [2]. Let G be a fixed non-empty set, then an A-IFS has the form: $A=\{<x,\mu_A(x),\nu_A(x)>|\, x\in G\}$, each element of which is characterized by a membership function $\mu_A: G\to[0,1]$ and a non-membership function $\nu_A: G\to[0,1]$ with the conditions $0\le \mu_A(x)+\nu_A(x)\le 1$ for all $x\in G$. Moreover, $\mu_A(x)$ and $\nu_A(x)$ represent the membership degree and the non-membership degree of x in A, respectively.

Inspired by the A-IFSs and their operational laws [2,30], Xu and Yager [14,15] defined the concept of intuitionistic fuzzy number (IFN), which is essentially a pair of nonnegative real numbers (μ,ν) for which $\mu+\nu\le 1$. Although the technical term "intuitionistic fuzzy numbers" had been given for the extension of fuzzy numbers [27-29], the paper still uses "intuitionistic fuzzy numbers" to represent the concept of Atanassov's intuitionistic fuzzy values because this paper's study of the calculus in intuitionistic fuzzy environment just considers Atanassov's intuitionistic fuzzy values as basic elements, which is like the real and complex numbers in the traditional mathematical analysis. Also it has been widely used in the existing literature [7,12,16,18-19,24-26,30-33], and hence, we choose to use the technical term "intuitionistic fuzzy numbers" rather than "intuitionistic fuzzy values".

In what follows, we will give some operations of IFNs [14,15]. Let $\alpha=(\mu_\alpha,\nu_\alpha)$ and $\beta=(\mu_\beta,\nu_\beta)$ be two IFNs, then we have

(**Addition**) $\alpha\oplus\beta=(\mu_\alpha+\mu_\beta-\mu_\alpha\mu_\beta,\nu_\alpha\nu_\beta)$;

(**Multiplication**) $\alpha\otimes\beta=(\mu_\alpha\mu_\beta,\nu_\alpha+\nu_\beta-\nu_\alpha\nu_\beta)$.

According to the addition and multiplication between IFNs, we can get $\alpha\oplus\alpha=(1-(1-\mu_\alpha)^2,\nu_\alpha^2)$, $\alpha\otimes\alpha=(\mu_\alpha^2,1-(1-\nu_\alpha)^2)$, $\alpha\oplus\alpha\oplus\alpha=(1-(1-\mu_\alpha)^3,\nu_\alpha^3)$, $\alpha\otimes\alpha\otimes\alpha=(\mu_\alpha^3,1-(1-\nu_\alpha)^3)\cdots\cdots$. Hence, the scalar-multiplication and the power operation of IFNs can be defined as the following forms [14,15]:

(**Scalar-multiplication**) $\lambda\alpha=(1-(1-\mu_\alpha)^\lambda,\nu_\alpha{}^\lambda)$, $\lambda>0$;

(**Power operation**) $\alpha^\lambda=(\mu_\alpha{}^\lambda,1-(1-\nu_\alpha)^\lambda)$, $\lambda>0$.

In order to analyze the above operations, we introduce the complement of an IFN α [31], denoted by $\bar{\alpha}$, i.e.,

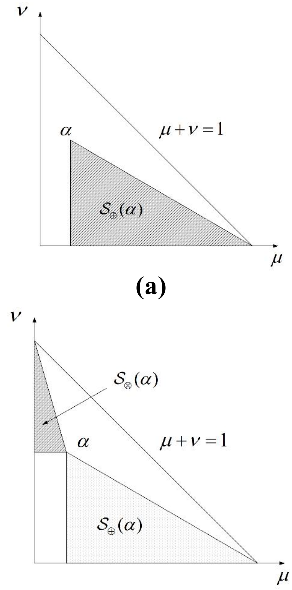
(**Complement**) $\bar{\alpha}=\overline{(\mu_\alpha,\nu_\alpha)}=(\nu_\alpha,\mu_\alpha)$.

Based on which we have the properties [31]: (1) $\overline{\alpha\oplus\beta}=\bar{\alpha}\otimes\bar{\beta}$; and (2) $\overline{\lambda\alpha}=\bar{\alpha}^\lambda$, $\lambda>0$.

Additionally, in Ref. [24], we also discussed the basic addition and multiplication of IFNs in depth, and pointed out two facts:

(1) Let α be an IFN, then $\alpha\oplus\varepsilon$ must fall in the region $\mathcal{S}_\oplus(\alpha)$ for any IFN ε, which is shown in Figure 2.1 (a). It means $\mathcal{S}_\oplus(\alpha)=\{\beta\,|\,\beta=\alpha\oplus\varepsilon,\varepsilon\in IFNS\}$, where $IFNS$ is the set that consists of all IFNs.

(2) $\alpha\otimes\varepsilon$ must fall in the region $\mathcal{S}_\otimes(\alpha)$ for any IFN ε, which is shown in Figure 2.1 (b). It shows $\mathcal{S}_\otimes(\alpha)=\{\beta\,|\,\beta=\alpha\otimes\varepsilon,\varepsilon\in IFNS\}$.



(b)

Figure 2.1. The regions $\mathcal{S}_\oplus(\alpha)$ and $\mathcal{S}_\otimes(\alpha)$

Based on the regions $\mathcal{S}_\oplus(\alpha)$ and $\mathcal{S}_\otimes(\alpha)$ of the IFN α, we get the following result:

Theorem 2.1.

$$\overline{\mathcal{S}_\oplus(\alpha)}=\mathcal{S}_\otimes(\bar{\alpha})$$

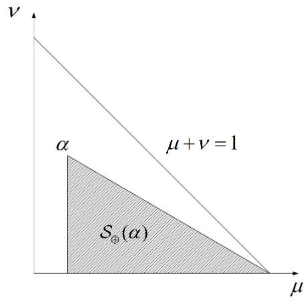
where $\overline{\mathcal{S}_\oplus(\alpha)}$ is the set $\{\bar{\beta}:\beta\in S_\oplus(\alpha)\}$, but does not represent $\{\beta:\beta\notin S_\oplus(\alpha),\beta\in\Theta\}$ ($S_\oplus(\alpha)\subseteq\Theta$), which is actually the complement of a set in the usual sense of set theory.

Proof. Now we give a brief proof for it. Firstly, suppose that any IFN β belongs to the set $\overline{\mathcal{S}_\oplus(\alpha)}$, which means that there must exist an IFN ε satisfying $\alpha\oplus\varepsilon=\bar{\beta}$. Hence,

$$\alpha\oplus\varepsilon=\bar{\beta}\Rightarrow\beta=\bar{\bar{\beta}}=\overline{\alpha\oplus\varepsilon}=\bar{\alpha}\otimes\bar{\varepsilon}\Rightarrow\beta\in\mathcal{S}_\otimes(\bar{\alpha})$$

So we get $\overline{\mathcal{S}_{\oplus}(\alpha)} \subseteq \mathcal{S}_{\otimes}(\overline{\alpha})$. In the same way, we can get $\overline{\mathcal{S}_{\oplus}(\alpha)} \supseteq \mathcal{S}_{\otimes}(\overline{\alpha})$. Therefore, $\overline{\mathcal{S}_{\oplus}(\alpha)} = \mathcal{S}_{\otimes}(\overline{\alpha})$ can be easily deducible.

In fact, this conclusion can also be proven according to Figure 2.2.



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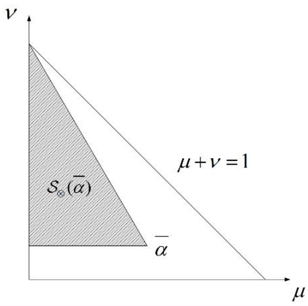


(b)

Figure 2.2. The regions $\mathcal{S}_{\oplus}(\alpha)$ and $\mathcal{S}_{\otimes}(\overline{\alpha})$

According to Figure 2.2, the images of $\mathcal{S}_{\oplus}(\alpha)$ and $\mathcal{S}_{\otimes}(\overline{\alpha})$ are exactly symmetrical when we consider $\mu=\nu$ as the symmetry axis. Hence, $(\nu,\mu)\in\mathcal{S}_{\otimes}(\overline{\alpha})$, for any $(\mu,\nu)\in\mathcal{S}_{\oplus}(\alpha)$, and vice versa. Then, we can conclude that $\overline{\mathcal{S}_{\oplus}(\alpha)} = \mathcal{S}_{\otimes}(\overline{\alpha})$ holds. ■

With the addition and multiplication between IFNs, we will introduce the definitions of their inverse operations (i.e., subtraction and division of IFNs):

According to the subtraction and the division of A-IFSs, in Ref. [20], we proposed the corresponding operations between IFNs as follows:

(Subtraction)

$$\beta \ominus \alpha = \begin{cases} \left(\dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \dfrac{\nu_\beta}{\nu_\alpha}\right), & if\ \ 0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le \dfrac{1-\mu_\beta}{1-\mu_\alpha} \le 1; \\ (0,1), & otherwise, \end{cases};$$

(Division)

$$\beta \oslash \alpha = \begin{cases} \left(\dfrac{\mu_\beta}{\mu_\alpha}, \dfrac{\nu_\beta-\nu_\alpha}{1-\nu_\alpha}\right), & if\ \ 0 \le \dfrac{\mu_\beta}{\mu_\alpha} \le \dfrac{1-\nu_\beta}{1-\nu_\alpha} \le 1, \\ (0,1), & otherwise, \end{cases}.$$

Obviously, the subtraction and the division are respectively the inverse operations of the addition and the multiplication of IFNs. We can calculate the difference $\beta \ominus \alpha$ between two IFNs by utilizing the formula $\left(\dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \dfrac{\nu_\beta}{\nu_\alpha}\right)$ if only $0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le \dfrac{1-\mu_\beta}{1-\mu_\alpha} \le 1$. If the condition cannot be satisfied, then $\left(\dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \dfrac{\nu_\beta}{\nu_\alpha}\right)$ must not be an IFN, which means that at least one of the three inequalities ($0 \le \dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha} \le 1$, $0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le 1$ and $0 \le \dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha} + \dfrac{\nu_\beta}{\nu_\alpha} \le 1$) is false. In order to meet the closure of subtraction, let $\beta \ominus \alpha = (0,1)$ when these inequalities are untenable. Similarly, the division can be analyzed in the same way. In addition, we can find the fact that if β and α meet the condition:

$$0 \le \frac{\nu_\beta}{\nu_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$$

and $\beta \ominus \alpha = \left(\dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}, \dfrac{\nu_\beta}{\nu_\alpha}\right)$ is an IFN, then it is obvious that $\overline{\beta} \oslash \overline{\alpha} = \left(\dfrac{\nu_\beta}{\nu_\alpha}, \dfrac{\mu_\beta-\mu_\alpha}{1-\mu_\alpha}\right)$ is still an IFN.

Since " $\oplus$ " and " $\otimes$ " are connected by the complement operation, i.e., $\overline{\alpha \oplus \beta} = \overline{\alpha} \otimes \overline{\beta}$, it is natural to consider the relationship between " $\ominus$ " and " $\oslash$ ". Then we guess that it can be shown as:

$$\overline{\beta \ominus \alpha} = \overline{\beta} \oslash \overline{\alpha}$$

Clearly, the result is right when $0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le \dfrac{1-\mu_\beta}{1-\mu_\alpha} \le 1$. However, we know that $\beta \ominus \alpha$ will be $(0,1)$ when $0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le \dfrac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ does not hold, which means $\overline{\beta \ominus \alpha} = (1,0)$. But the right-hand of the equation $\overline{\beta} \oslash \overline{\alpha}$ will be $(0,1)$ because $0 \le \dfrac{\nu_\beta}{\nu_\alpha} \le \dfrac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ cannot be satisfied. Hence, we have

$$\overline{\beta \ominus \alpha} = (1,0) \ne (0,1) = \overline{\beta} \oslash \overline{\alpha}$$

In fact, to define the quotient $\beta \oslash \alpha$ between two IFNs β and α as $(0,1)$ only makes the division meet the closure when these inequalities are untenable, hence, we can redefine it as any IFN, such as $(0.2, 0.3)$ and $(0.3, 0.4)$, which definitely includes $(1,0)$. In order to ensure that $\overline{\beta \ominus \alpha} =$

$\bar{\beta} \oslash \bar{\alpha}$ still holds even though $0 \le \frac{\nu_\beta}{\nu_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ cannot be satisfied, we redefine the division between IFNs as follows:

$$\beta \oslash \alpha = \begin{cases} \left(\frac{\mu_\beta}{\mu_\alpha}, \frac{\nu_\beta - \nu_\alpha}{1-\nu_\alpha} \right), & if \ \ 0 \le \frac{\mu_\beta}{\mu_\alpha} \le \frac{1-\nu_\beta}{1-\nu_\alpha} \le 1, \\ (1,0), & otherwise, \end{cases}$$

According to the new definition of division, we can get the following result:

Theorem 2.2.

$$\overline{\beta \ominus \alpha} = \bar{\beta} \oslash \bar{\alpha}$$

Proof. According to the above analysis, we get

(1) When $0 \le \frac{\nu_\beta}{\nu_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ holds, it yields

$$\beta \ominus \alpha = \left(\frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha}, \frac{\nu_\beta}{\nu_\alpha} \right) \text{ and } \bar{\beta} \oslash \bar{\alpha} = \left(\frac{\nu_\beta}{\nu_\alpha}, \frac{\mu_\beta - \mu_\alpha}{1-\mu_\alpha} \right)$$

Hence, $\overline{\beta \ominus \alpha} = \bar{\beta} \oslash \bar{\alpha}$ holds.

(2) When $0 \le \frac{\nu_\beta}{\nu_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ does not hold, we have

$$\beta \ominus \alpha = (0,1) \text{ and } \bar{\beta} \oslash \bar{\alpha} = (1,0)$$

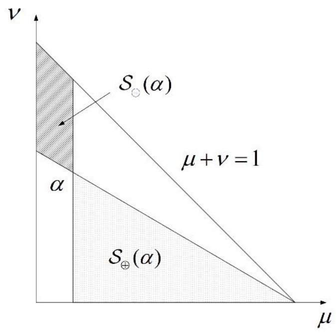
So $\overline{\beta \ominus \alpha} = \bar{\beta} \oslash \bar{\alpha}$ still holds.

Based on (1) and (2), we get that $\overline{\beta \ominus \alpha} = \bar{\beta} \oslash \bar{\alpha}$ is right whether the condition $0 \le \frac{\nu_\beta}{\nu_\alpha} \le \frac{1-\mu_\beta}{1-\mu_\alpha} \le 1$ holds or not. ■

Below we introduce the concepts of "$\mathcal{S}_\ominus(\alpha)$" and "$\mathcal{S}_\oslash(\alpha)$", which were defined in Ref. [20]:

(1) $\alpha \ominus \varepsilon$ must fall in the region $\mathcal{S}_\ominus(\alpha)$ for any IFN ε, which is shown in Figure 2.3 (a). It shows $\mathcal{S}_\ominus(\alpha) = \{\beta \mid \beta = \alpha \ominus \varepsilon, \varepsilon \in IFNS\}$.

(2) $\alpha \oslash \varepsilon$ must fall in the region $\mathcal{S}_\oslash(\alpha)$ for any IFN ε, which is shown in Figure 2.3 (b). $\mathcal{S}_\oslash(\alpha)$ is actually the set $\{\beta \mid \beta = \alpha \oslash \varepsilon, \varepsilon \in IFNS\}$.



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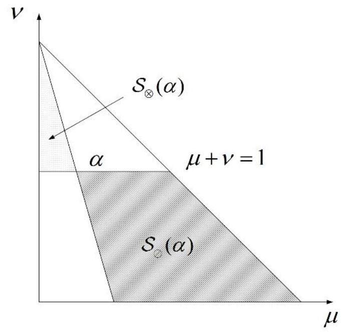


(b)

Figure 2.3. The regions $\mathcal{S}_\ominus(\alpha)$ and $\mathcal{S}_\oslash(\alpha)$

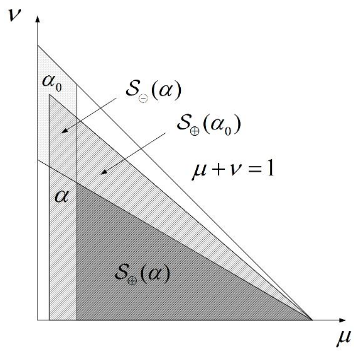
From Figure 2.3, it is clear to get the following result:

Theorem 2.3.

(1) For any given $\alpha_0 \in \mathcal{S}_\ominus(\alpha)$, there must be $\alpha \in \mathcal{S}_\oplus(\alpha_0)$ and $\mathcal{S}_\oplus(\alpha) \subseteq \mathcal{S}_\oplus(\alpha_0)$;

(2) For any given $\alpha_0 \in \mathcal{S}_\oslash(\alpha)$, there must be $\alpha \in \mathcal{S}_\otimes(\alpha_0)$ and $\mathcal{S}_\otimes(\alpha) \subseteq \mathcal{S}_\otimes(\alpha_0)$.

Proof. These results can be proven according to Figure 2.4:



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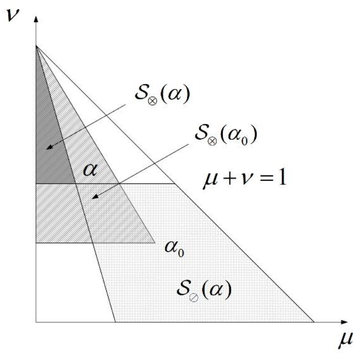


(b)

Figure 2.4. The proof method based on images

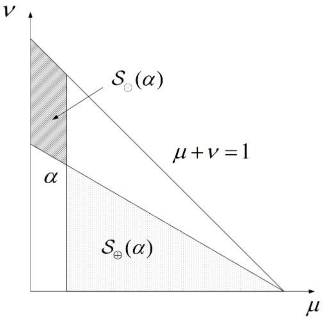
Of course, we can also prove the above statements by simple logical arguments. We first prove the result (1): For any $\alpha_0 \in \mathcal{S}_\ominus(\alpha)$, there exists an IFN ε meeting $\alpha \ominus \varepsilon = \alpha_0$. Then because $\alpha = \varepsilon \oplus \alpha_0$, we get $\alpha \in \mathcal{S}_\oplus(\alpha_0)$. Moreover, for any $\beta \in \mathcal{S}_\oplus(\alpha)$, we have $\beta = \alpha \oplus (\beta \ominus \alpha)$, which can be represented as $\beta = \alpha_0 \oplus \varepsilon \oplus (\beta \ominus \alpha)$, so we know $\beta \in \mathcal{S}_\oplus(\alpha_0)$, which means $\mathcal{S}_{\oplus}(\alpha) \subseteq \mathcal{S}_{\oplus}(\alpha_0)$. However, it is clear that $\alpha_0 = \alpha \ominus \varepsilon \notin \mathcal{S}_{\oplus}(\alpha)$ and $\alpha_0 \in \mathcal{S}_{\oplus}(\alpha_0)$, which means $\mathcal{S}_{\oplus}(\alpha_0) \not\subseteq \mathcal{S}_{\oplus}(\alpha)$. The result (2) can be proven in a similar way. ■

Now, we obtain another result about $\mathcal{S}_{\ominus}(\alpha)$ and $\mathcal{S}_{\oslash}(\alpha)$:

Theorem 2.4.

$$\overline{\mathcal{S}_{\ominus}(\alpha)} = \mathcal{S}_{\oslash}(\overline{\alpha})$$

Proof. The theorem can be proven by the following two methods: (1) For any $\alpha_0 \in \overline{\mathcal{S}_{\ominus}(\alpha)}$, there is an IFN ε meeting $\overline{\alpha}_0 = \alpha \ominus \varepsilon$. Then $\alpha_0 = \overline{\overline{\alpha}}_0 = \overline{\alpha \ominus \varepsilon} = \overline{\alpha} \oslash \overline{\varepsilon}$, and hence, $\alpha_0 \in \mathcal{S}_{\oslash}(\overline{\alpha})$. We know $\overline{\mathcal{S}_{\ominus}(\alpha)} \subseteq \mathcal{S}_{\oslash}(\overline{\alpha})$. In a similar way, we also get $\overline{\mathcal{S}_{\ominus}(\alpha)} \supseteq \mathcal{S}_{\oslash}(\overline{\alpha})$. So $\overline{\mathcal{S}_{\ominus}(\alpha)} = \mathcal{S}_{\oslash}(\overline{\alpha})$. (2) It is easier to test the result by the images of "$\mathcal{S}_{\ominus}(\alpha)$" and "$\mathcal{S}_{\oslash}(\overline{\alpha})$", which are shown below:



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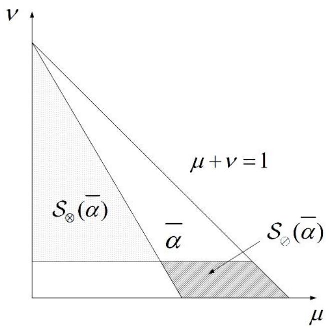


(b)

Figure 2.5. The regions $\mathcal{S}_{\ominus}(\alpha)$ and $\mathcal{S}_{\oslash}(\overline{\alpha})$

According to (1) and (2) of the proof, we can conclude that Theorem 2.4 holds. ■

In what follows, we attempt to present the images of the scalar-multiplication $\lambda\alpha$ and the power operation α^{λ} of IFNs:

We first introduce the symbol $\mathcal{S}_{\lambda\alpha}$, which is the set $\{\beta \mid \beta = \lambda\alpha, \lambda \in (0,\infty)\}$, and the symbol $\mathcal{S}_{\alpha^{\lambda}}$, which is the set $\{\beta \mid \beta = \alpha^{\lambda}, \lambda \in (0,\infty)\}$.

For any given IFN $\alpha_0 = (\mu_0, \nu_0)$, after analyzing the mathematical expression of $\lambda\alpha_0$, we get

(1) The value of $\lambda\alpha_0$ will change when λ changes from zero to infinity;

(2) When $\lambda\alpha_0 = (\mu, \nu)$, we can get the parameter λ if only $\mu_0 \neq 0$, $\mu_0 \neq 1$, $\nu_0 \neq 0$ and $\nu_0 \neq 1$;

(3) The image of $\lambda\alpha_0$ with respect to λ can be depicted with a function $\nu(\mu)$ with respect to μ, whose specific mathematical expression is

$$\nu(\mu) = \nu_0^{\frac{\ln(1-\mu)}{\ln(1-\mu_0)}}$$

(4) The image of $\lambda\alpha_0$ is actually one of the function $\mu(\nu)$ with respect to ν which has the following form:

$$\mu(\nu) = 1 - (1-\mu_0)^{\frac{\ln\nu}{\ln\nu_0}}$$

Below we analyze the function $\nu(\mu)$ in detail, and the discussion about $\mu(\nu)$ is similar:

(1) There is $\nu(\mu_0) = \nu_0$, which shows the situation when $\lambda = 1$;

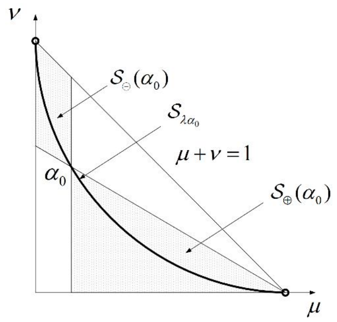
(2) $\nu(1) = 0$ reveals that $\lambda\alpha_0 \to (1,0)$ when $\lambda \to +\infty$;

(3) $\nu(0) = 1$ corresponds to the situation $\lambda\alpha_0 \to (0,1)$ when $\lambda \to 0$;

(4) When $\lambda > 1$, there will be $\lambda\alpha_0 \in \mathcal{S}_{\oplus}(\alpha_0)$ because $\lambda\alpha_0 = \alpha_0 \oplus (\lambda-1)\alpha_0$;

(5) When $0 < \lambda < 1$, there is $\lambda\alpha_0 \in \mathcal{S}_{\ominus}(\alpha_0)$ due to that $\lambda\alpha_0 = \alpha_0 \ominus (1-\lambda)\alpha_0$.

We can also analyze the power operation α^{λ} of IFNs in the aforementioned way. Then, the images of the scalar-multiplication $\lambda\alpha$ and the power operation α^{λ} can be represented as follows:



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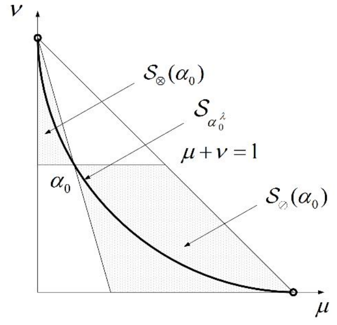

(b)

Figure 2.6. The images of $\mathcal{S}_{\lambda\alpha_0}$ and $\mathcal{S}_{\alpha_0^{\lambda}}$

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It is obvious that we can get a result about $\mathcal{S}_{\lambda\alpha}$ and $\mathcal{S}_{\alpha^{\lambda}}$ as follows:

Theorem 2.5.

$$\overline{\mathcal{S}_{\lambda\alpha}} = \mathcal{S}_{\overline{\alpha}^{\lambda}}$$

The proof of Theorem 2.5 is similar to those of Theorem 2.1 and 2.4, and hence, it is omitted here.

In order to compare and rank IFNs, many methods have been proposed for solving the problem [34-35]. At the end of this section, we introduce a kind of order relations between IFNs based on "$\oplus$", "$\ominus$", "$\otimes$" and "$\oslash$". In Ref. [25], we utilized the four arithmetic operations between IFNs to define a kind of order relations as follows:

(1) If there is an IFN ε, which satisfies $\alpha \oplus \varepsilon = \beta$, then α is less than or equal to β, denoted by $\alpha \trianglelefteq_{\oplus} \beta$. If there is an IFN ε, such that $\alpha \oplus \varepsilon = \beta$ and $\varepsilon \neq (0,1)$, then α is less than β, denoted by $\alpha \triangleleft_{\oplus} \beta$;

(2) If there is an IFN ε, such that $\alpha \otimes \varepsilon = \beta$, then β is less than or equal to α, denoted by $\beta \trianglelefteq_{\otimes} \alpha$. If there is an IFN ε, such that $\alpha \otimes \varepsilon = \beta$ and $\gamma \neq (1,0)$, then β is less than α, denoted by $\beta \triangleleft_{\otimes} \alpha$.

Obviously, "$\trianglelefteq_{\oplus}$" and "$\trianglelefteq_{\otimes}$" satisfy the three basic properties of an order, which are respectively reflexivity, antisymmetry and transitivity. Ref. [33] introduced them as follows:

(1) (Reflexivity) $\alpha \trianglelefteq_{\oplus} \alpha$ because $\alpha \oplus (0,1) = \alpha$;

(2) (Antisymmetry) If $\alpha \trianglelefteq_{\oplus} \beta$ and $\beta \trianglelefteq_{\oplus} \alpha$, then $\alpha = \beta$. In fact, if $\alpha \trianglelefteq_{\oplus} \beta$ and $\beta \trianglelefteq_{\oplus} \alpha$, there exist two IFNs ε_1 and ε_2 meeting $\alpha \oplus \varepsilon_1 = \beta$ and $\beta \oplus \varepsilon_2 = \alpha$, then

$$\alpha \oplus \varepsilon_1 \oplus \varepsilon_2 = \alpha \Rightarrow \varepsilon_1 \oplus \varepsilon_2 = (0,1) \Rightarrow \varepsilon_1 = \varepsilon_2 = (0,1) \Rightarrow \alpha = \beta$$

(3) (Transitivity) If $\alpha \trianglelefteq_{\oplus} \beta$ and $\beta \trianglelefteq_{\oplus} \eta$, then $\alpha \trianglelefteq_{\oplus} \eta$. In fact, if $\alpha \trianglelefteq_{\oplus} \beta$ and $\beta \trianglelefteq_{\oplus} \eta$, there exist two IFNs ε_1 and ε_2 meeting $\alpha \oplus \varepsilon_1 = \beta$ and $\beta \oplus \varepsilon_2 = \eta$, then

$$\alpha \oplus \varepsilon_1 \oplus \varepsilon_2 = \beta \oplus \varepsilon_2 = \eta$$

which indicates $\alpha \trianglelefteq_{\oplus} \eta$.

The above-mentioned (1)-(3) show that "$\trianglelefteq_{\oplus}$" is an order. Similarly, "$\trianglelefteq_{\otimes}$" can be proven in the same way. Moreover, introducing "$\trianglelefteq_{\oplus}$" and "$\trianglelefteq_{\otimes}$" is important for the further research of the calculus in intuitionistic fuzzy environment, because they are very useful in simplifying the expressions, for example, $\mathcal{S}_{\oplus}(\alpha)$ and $\mathcal{S}_{\ominus}(\alpha)$ can be denoted as $\{\varepsilon \mid \alpha \trianglelefteq_{\oplus} \varepsilon\}$ and $\{\varepsilon \mid \varepsilon \trianglelefteq_{\oplus} \alpha\}$, respectively. $\mathcal{S}_{\otimes}(\alpha)$ and $\mathcal{S}_{\oslash}(\alpha)$ are respectively the sets $\{\varepsilon \mid \varepsilon \trianglelefteq_{\otimes} \alpha\}$ and $\{\varepsilon \mid \alpha \trianglelefteq_{\otimes} \varepsilon\}$. Moreover, we can easily get the following result:

Theorem 2.6. If $\alpha \trianglelefteq_{\oplus} \beta$, then $\overline{\beta} \trianglelefteq_{\otimes} \overline{\alpha}$, and vice versa.

Proof. According to the definition of the order "$\trianglelefteq_{\oplus}$", if $\alpha \trianglelefteq_{\oplus} \beta$, then there exists an IFN ε, which satisfies $\alpha \oplus \varepsilon = \beta$, and thus, we get $\overline{\varepsilon}$ such that $\overline{\alpha} \otimes \overline{\varepsilon} = \overline{\beta}$. Therefore, $\overline{\beta} \trianglelefteq_{\otimes} \overline{\alpha}$. In the same way, we can get $\alpha \trianglelefteq_{\oplus} \beta$ if $\overline{\beta} \trianglelefteq_{\otimes} \overline{\alpha}$. So Theorem 2.6 holds. ■

In this section, we have mainly introduced some basic concepts of IFNs and some of their important relationships. As we know, intuitionistic fuzzy calculus consists of two parallel parts, which are respectively developed based on different operations under intuitionistic fuzzy environment. Hence, we first have to unify these basic operations before considering the two parallel intuitionistic fuzzy calculus theories. In conclusion, we have acquired some useful results in this section:

(1) $\overline{\alpha \oplus \beta} = \overline{\alpha} \otimes \overline{\beta}$, $\overline{\lambda\alpha} = \overline{\alpha}^{\lambda}$, $\overline{\beta \ominus \alpha} = \overline{\beta} \oslash \overline{\alpha}$;

(2) $\overline{\mathcal{S}_{\oplus}(\alpha)} = \mathcal{S}_{\otimes}(\overline{\alpha})$, $\overline{\mathcal{S}_{\ominus}(\alpha)} = \mathcal{S}_{\oslash}(\overline{\alpha})$, $\overline{\mathcal{S}_{\lambda\alpha}} = \mathcal{S}_{\overline{\alpha}^{\lambda}}$;

(3) If $\alpha \trianglelefteq_{\oplus} \beta$ (or $\alpha \triangleleft_{\oplus} \beta$), then $\overline{\beta} \trianglelefteq_{\otimes} \overline{\alpha}$ (or $\overline{\beta} \triangleleft_{\otimes} \overline{\alpha}$), and vice versa.

III. Intuitionistic Fuzzy Functions, Derivatives and Differentials

In this section, we discuss the differential calculus of IFNs. We first introduce the intuitionistic fuzzy function (IFF) [20]. The concept of IFF is the research prerequisite of intuitionistic fuzzy calculus that considers the IFNs as the basic elements, which is similar to the real numbers and the complex numbers in the classical mathematical analysis.

Let $\mathbb{D}$ be a non-empty subset of $IFNS$, which is the set that consists of all IFNs, then we call $\varphi : \mathbb{D} \to IFNS$ an intuitionistic fuzzy function (IFF) defined in $\mathbb{D}$, which can be denoted by

$$Y = \varphi(X),\ X \in \mathbb{D}$$

where the set $\mathbb{D}$ is the domain of φ, X is the independent variable, and Y is the dependent variable.

According to the definition of IFF, we can assume that an IFF φ consists of two real functions f and g, i.e.,

$$\varphi = (f(\mu,\nu), g(\mu,\nu)),\ (\mu,\nu) \in \mathbb{D}$$

where f and g satisfy the conditions: $0 \le f(\mu,\nu) \le 1$, $0 \le g(\mu,\nu) \le 1$ and $0 \le f(\mu,\nu) + g(\mu,\nu) \le 1$, for any $(\mu,\nu) \in \mathbb{D}$.

When reviewing the theory of the complex functions, we know that the well-known "C-R condition" is proposed to judge whether or not a complex function has the derivative at one point. "C-R condition" actually confines the objects of study within a smaller range. Similarly, Ref. [24] also pointed out a fact that if the IFF φ is derivable, then

$$\frac{\partial f(\mu,\nu)}{\partial \nu} = \frac{\partial g(\mu,\nu)}{\partial \mu} = 0$$

which means that $\varphi = (f(\mu), g(\nu))$, $(\mu,\nu) \in \mathbb{D}$. Hence, intuitionistic fuzzy calculus mainly concerns about $(f(\mu), g(\nu))$. In fact, the IFFs $\varphi(X) = X \oplus \alpha_0$ and $\varphi(X) = \lambda X$ ($X \in IFNS$) are both this kind of IFFs.

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In the following, we study an issue about the IFF $\varphi(X)$ and its complement $\overline{\varphi(X)}$, that is, whether or not $\overline{\varphi(X)}=\overline{\varphi}(X)$? Obviously, we have $\overline{\varphi(X)}=\overline{(f(\mu),g(\nu))}=(g(\nu),f(\mu))$. However, $\overline{\varphi}(X)$ should be equal to $(g(\mu),f(\nu))$ rather than $(g(\nu),f(\mu))$. Hence, $\overline{\varphi(X)}$ is not equal to $\overline{\varphi}(X)$. After analyzing the expression, we can get the following result:

Theorem 3.1.

$$\overline{\varphi(X)}=\overline{\varphi}(\overline{X})$$

Specially, for a given compound IFFs $(\varphi\circ\psi)(t)$ (or $\varphi(\psi(t))$), which means that φ and ψ are both IFFs and t is independent variable, then

$$\overline{\varphi(\psi(t))}=\overline{\varphi}\left(\overline{\psi(t)}\right)=\overline{\varphi}\left(\overline{\psi}(\overline{t})\right)$$

That is, $\overline{(\varphi\circ\psi)(t)}=\overline{(\varphi\circ\psi)}(\overline{t})=(\overline{\varphi}\circ\overline{\psi})(\overline{t})$.

Proof. Let $\varphi=(f,g)$ be an IFF of the variable $X=(\mu,\nu)$, i.e., $\varphi(X)=(f(\mu),g(\nu))$, then $\overline{\varphi(X)}=\overline{(f(\mu),g(\nu))}=(g(\nu),f(\mu))$, which is actually the IFF $\overline{\varphi}=(g,f)$ with respect to $\overline{X}=(\nu,\mu)$, and thus, $\overline{\varphi(X)}=\overline{\varphi}(\overline{X})$. Therefore, we can easily get the result about the compound IFFs. ■

Now we introduce a special kind of IFFs called monotone increasing IFFs. If $\alpha\trianglelefteq_{\oplus}\beta$, then $\varphi(\alpha)\trianglelefteq_{\oplus}\varphi(\beta)$, in this case, we call the IFF φ a monotone increasing IFF [26]. Obviously, the concept of monotone increasing IFFs is based on the order "$\trianglelefteq_{\oplus}$". Similarly, we can also define the monotone increasing IFFs based on "$\trianglelefteq_{\otimes}$". If $\beta\trianglelefteq_{\otimes}\alpha$, then we get $\varphi(\beta)\trianglelefteq_{\otimes}\varphi(\alpha)$. Thus, we call φ a second monotone increasing IFF. In fact, intuitionistic fuzzy calculus is mainly on the basis of the monotone increasing IFFs.

Theorem 3.2. If φ is a monotone increasing IFF with respect to X, then $\overline{\varphi}$ must be a second monotone increasing IFF with respect to $\overline{X}$.

Proof. Actually, if φ is a monotone increasing IFF, which means $\varphi(Y)\trianglelefteq_{\oplus}\varphi(Z)$ if $Y\trianglelefteq_{\oplus}Z$, then we get $\overline{\varphi(Z)}\trianglelefteq_{\otimes}\overline{\varphi(Y)}$ (or $\overline{\varphi}(\overline{Z})\trianglelefteq_{\otimes}\overline{\varphi}(\overline{Y})$) if $\overline{Z}\trianglelefteq_{\otimes}\overline{Y}$ according to the basic knowledge in Section 2. Hence, the IFF $\overline{\varphi}$ must be a second monotone increasing IFF. ■

For convenience, we denote $\beta\oslash\alpha$ as $\frac{\beta}{\alpha}$. Below we recall the derivatives of IFFs [24], which is the theoretical basis of the differential calculus. Let $\varphi=(f,g)$ be an IFF of X and $O=(0,1)$, then

$$\frac{d\varphi(X)}{dX}=\lim_{\Delta X\to O}\frac{\varphi(X\oplus\Delta X)\ominus\varphi(X)}{\Delta X}$$
$$=\left(\frac{1-\mu}{1-f(\mu)}\frac{df(\mu)}{d\mu},1-\frac{\nu}{g(\nu)}\frac{dg(\nu)}{d\nu}\right)$$

is called the subtraction derivative function of φ. Based on which we introduce the subtraction derivative functions of several simple IFFs below:

(1) If $\varphi(X)=\alpha_0$, then $\frac{d\varphi(X)}{dX}=O$;

(2) If $\varphi(X)=\lambda X$, then $\frac{d\varphi(X)}{dX}=(\lambda,1-\lambda)$.

Let $\varphi(X)=(f(\mu),g(\nu))$ be an IFF of X and $E=(1,0)$, then

$$\frac{l\varphi(X)}{lX}=\lim_{\nabla X\to E}\left(\frac{\varphi(X\otimes\nabla X)}{\varphi(X)}\ominus\nabla X\right)$$
$$=\left(1-\frac{\mu}{f(\mu)}\frac{df(\mu)}{d\mu},\frac{1-\nu}{1-g(\nu)}\frac{dg(\nu)}{d\nu}\right)$$

is called the division derivative function of φ.

The division derivative functions of the following simple IFFs were also developed in Ref. [24]:

(1) If $\varphi(X)=\alpha_0$, then $\frac{l\varphi(X)}{lX}=E$;

(2) If $\varphi(X)=X^{\lambda}$, then $\frac{l\varphi(X)}{lX}=(1-\lambda,\lambda)$.

In fact, we can notice that there should be a closed connection between the subtraction derivative and the division derivative by thinking about the above-mentioned examples, which can be shown in the following theorem:

Theorem 3.3.

$$\overline{\frac{d\varphi(X)}{dX}}=\frac{l\overline{\varphi}(\overline{X})}{l\overline{X}}$$

Proof. The theorem can be proven in two different ways: Firstly, we prove it by using the definition of the intuitionistic fuzzy derivative:

$$\overline{\frac{d\varphi(X)}{dX}}=\overline{\lim_{\Delta X\to O}\frac{\varphi(X\oplus\Delta X)\ominus\varphi(X)}{\Delta X}}$$
$$=\lim_{\overline{\Delta X}\to E}\left(\overline{\varphi(X\oplus\Delta X)\ominus\varphi(X)}\ominus\overline{\Delta X}\right)$$
$$=\lim_{\overline{\Delta X}\to E}\left(\frac{\overline{\varphi(X\oplus\Delta X)}}{\overline{\varphi(X)}}\ominus\overline{\Delta X}\right)=\lim_{\overline{\Delta X}\to E}\left(\frac{\overline{\varphi}(\overline{X}\otimes\overline{\Delta X})}{\overline{\varphi}(\overline{X})}\ominus\overline{\Delta X}\right)$$

Since $\Delta X\to O$, then we get that $\overline{\Delta X}$ approaches to $\overline{O}$. Hence,

$$\overline{\frac{d\varphi(X)}{dX}}=\lim_{\overline{\Delta X}\to E}\left(\frac{\overline{\varphi}(\overline{X}\otimes\overline{\Delta X})}{\overline{\varphi}(\overline{X})}\ominus\overline{\Delta X}\right)=\frac{l\overline{\varphi}(\overline{X})}{l\overline{X}}$$

Moreover, we can also give the proof by utilizing the formulas of derivatives of IFFs. Since $\overline{\varphi}(\overline{X})=(g(\nu),f(\mu))$, then we get

$$\frac{l\overline{\varphi}(\overline{X})}{l\overline{X}}=\left(1-\frac{\nu}{g(\nu)}\frac{dg(\nu)}{d\nu},\frac{1-\mu}{1-f(\mu)}\frac{df(\mu)}{d\mu}\right)$$

and thus,

$$\overline{\frac{d\varphi(X)}{dX}}=\overline{\left(\frac{1-\mu}{1-f(\mu)}\frac{df(\mu)}{d\mu},1-\frac{\nu}{g(\nu)}\frac{dg(\nu)}{d\nu}\right)}=\frac{l\overline{\varphi}(\overline{X})}{l\overline{X}}$$

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Hence, $\overline{\dfrac{d\varphi(X)}{dX}} = \dfrac{l\overline{\varphi}(\overline{X})}{l\overline{X}}$ holds. ■

Therefore, we can analyze the derivation rule of the compound IFFs (the chain rule of derivatives) in intuitionistic fuzzy calculus. As we pointed out in Ref. [26], for a compound IFF $\varphi(\psi(t))$, if all of their derivatives (φ and ψ) exist, then the subtraction derivative of the composite function $\varphi(\psi(t))$ also exists and satisfies $\dfrac{d\varphi(\psi(t))}{dt} = \dfrac{d\varphi(\psi(t))}{d\psi(t)} \otimes \dfrac{d\psi(t)}{dt}$. Although it is actually the chain rule in intuitionistic fuzzy calculus based on the subtraction derivatives, we can derive the corresponding result in the calculus based on the division derivatives according to the aforementioned chain rule. Since

$$\overline{\frac{d\varphi(\psi(t))}{dt}} = \overline{\frac{d\varphi(\psi(t))}{d\psi(t)} \otimes \frac{d\psi(t)}{dt}}$$

then from the left-hand side of the equality, we get $\overline{\dfrac{d\varphi(\psi(t))}{dt}} = \dfrac{l\overline{\varphi(\psi(t))}}{l\overline{t}} = \dfrac{l\overline{\varphi}(\overline{\psi}(\overline{t}))}{l\overline{t}}$; while from the right-hand side of the equality, we have

$$\overline{\frac{d\varphi(\psi(t))}{d\psi(t)} \otimes \frac{d\psi(t)}{dt}} = \frac{l\overline{\varphi}(\overline{\psi}(\overline{t}))}{l\overline{\psi}(\overline{t})} \oplus \frac{l\overline{\psi}(\overline{t})}{l\overline{t}}$$

Hence, it can be obtained that

$$\frac{l\overline{\varphi}(\overline{\psi}(\overline{t}))}{l\overline{t}} = \frac{l\overline{\varphi}(\overline{\psi}(\overline{t}))}{l\overline{\psi}(\overline{t})} \oplus \frac{l\overline{\psi}(\overline{t})}{l\overline{t}}$$

If we let $\overline{\varphi} = \phi$, $\overline{\psi} = Y$ and $\overline{t} = k$, then we know

$$\frac{l\phi(Y(k))}{lk} = \frac{l\phi(Y(k))}{lY(k)} \oplus \frac{lY(k)}{lk}$$

which is just the chain rule of derivatives in intuitionistic fuzzy calculus based on the division derivatives [26].

Now we consider the differential of IFFs [24]: Let φ be an IFF that exists the subtraction derivative function $\dfrac{d\varphi(X)}{dX}$. If noting $\Delta X = X' \ominus X$, then $d\varphi(X) = \dfrac{d\varphi(X)}{dX} \otimes \Delta X$ is called the subtraction differential of φ. Since $dX = (1,0) \otimes \Delta X = \Delta X$, then the subtraction differential can be represented as:

$$d\varphi(X) = \frac{d\varphi(X)}{dX} \otimes dX$$

Based on the subtraction differential of IFFs above, the following result will establish the relationship between the function change value ($\Delta\varphi$) and the differential ($d\varphi$):

Let φ be a monotone increasing IFF of X. If φ exists the subtraction derivative function, then we can get

$$\varphi(Y) \ominus \varphi(X) \approx \frac{d\varphi(X)}{dX} \otimes (Y \ominus X)$$

which is the subtraction differential formula under intuitionistic fuzzy environment [24].

Meanwhile, if φ is an IFF that exists the division derivative function $\dfrac{l\varphi(X)}{lX}$, and we note $\nabla X = X' \oslash X$, then

$$l\varphi(X) = \frac{l\varphi(X)}{lX} \oplus \nabla X$$

is called the division differential of φ. Since $lX = (0,1) \oplus \nabla X = \nabla X$, then the division differential can be represented as:

$$l\varphi(X) = \frac{l\varphi(X)}{lX} \oplus lX$$

Let φ be a second monotone increasing IFF of X. If φ exists the division derivative function, then we can get

$$\varphi(Y) \oslash \varphi(X) \approx \frac{l\varphi(X)}{lX} \oplus (Y \oslash X)$$

which is the division differential formula under intuitionistic fuzzy environment [24].

Based on the two kinds of differentials of IFFs, we can get the following relationship between them:

Theorem 3.4.

$$\overline{d\varphi(X)} = l\overline{\varphi}(\overline{X})$$

Proof. It can be proven as follows:

$$\overline{d\varphi(X)} = \overline{\frac{d\varphi(X)}{dX} \otimes \Delta X} = \frac{l\overline{\varphi}(\overline{X})}{l\overline{X}} \oplus \nabla\overline{X} = l\overline{\varphi}(\overline{X})$$

and thus, $\overline{d\varphi(X)} = l\overline{\varphi}(\overline{X})$ holds. ■

In addition, according to the subtraction differential formula, we can easily get the corresponding one about the division differential. Since

$$\overline{\varphi(Y) \ominus \varphi(X)} \approx \overline{\frac{d\varphi(X)}{dX} \otimes (Y \ominus X)}$$

where $\overline{\varphi(Y) \ominus \varphi(X)} = \overline{\varphi}(\overline{Y}) \oslash \overline{\varphi}(\overline{X})$, and $\overline{\dfrac{d\varphi(X)}{dX} \otimes (Y \ominus X)} = \dfrac{l\overline{\varphi}(\overline{X})}{l\overline{X}} \oplus (\overline{Y} \oslash \overline{X})$, then we get

$$\overline{\varphi}(\overline{Y}) \oslash \overline{\varphi}(\overline{X}) \approx \frac{l\overline{\varphi}(\overline{X})}{l\overline{X}} \oplus (\overline{Y} \oslash \overline{X})$$

which is just the division differential formula. With the chain rule of derivatives and the differential, we have shown the form invariance of intuitionistic fuzzy differential [24].

If $Y = \varphi(\psi(t))$ is a compound IFF, which consists of two IFFs $Y = \varphi(X)$ and $X = \psi(t)$, $\dfrac{dY}{dX}$ and $\dfrac{dX}{dt}$ exist, then

$$\frac{dY}{dt} = \frac{dY}{dX} \otimes \frac{dX}{dt}$$

If considering X as the variable of Y, then there will be a differential form:

$$dY = \frac{dY}{dX} \otimes dX$$

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But if we think the variable is not X but t, then we will get another expression:

$$dY = \frac{dY}{dt} \otimes dt$$

However, if we notice the fact that $\frac{dY}{dt} = \frac{dY}{dX} \otimes \frac{dX}{dt}$ and $\frac{dX}{dt} \otimes dt$ is actually dX, then

$$dY = \frac{dY}{dt} \otimes dt = \frac{dY}{dX} \otimes \frac{dX}{dt} \otimes dt = \frac{dY}{dX} \otimes dX$$

which is just the form invariance of intuitionistic fuzzy differential. On the basis of the form invariance of the subtraction differential, we can get one of the division differentials as follows:

There are two IFFs $\overline{Y} = \overline{\varphi}(\overline{X})$ and $\overline{X} = \overline{\psi}(\overline{t})$. For the compound IFF $\overline{Y} = \overline{\varphi}(\overline{\psi}(\overline{t}))$, which consists of the two IFFs, we can get

(1) According to $\overline{dY} = \overline{\frac{dY}{dX} \otimes dX}$ and $\overline{dY} = \overline{\frac{dY}{dt} \otimes dt}$, we know

$$l\overline{Y} = \frac{l\overline{Y}}{l\overline{X}} \oplus l\overline{X} \text{ and } l\overline{Y} = \frac{l\overline{Y}}{l\overline{t}} \oplus l\overline{t}$$

(2) With $\overline{\frac{dY}{dt}} = \overline{\frac{dY}{dX} \otimes \frac{dX}{dt}}$ and $\overline{dX} = \overline{\frac{dX}{dt} \otimes dt}$, we get

$$\frac{l\overline{Y}}{l\overline{t}} = \frac{l\overline{Y}}{l\overline{X}} \oplus \frac{l\overline{X}}{l\overline{t}} \text{ and } l\overline{X} = \frac{l\overline{X}}{l\overline{t}} \oplus l\overline{t}$$

and then

$$l\overline{Y} = \frac{l\overline{Y}}{l\overline{t}} \oplus l\overline{t} = \frac{l\overline{Y}}{l\overline{X}} \oplus \frac{l\overline{X}}{l\overline{t}} \oplus l\overline{t} = \frac{l\overline{Y}}{l\overline{X}} \oplus l\overline{X}$$

which indicates the form invariance of the division differential given by Ref. [26].

In this section, we have mainly discussed the relationships between the two differential calculus theories based on the subtraction derivatives and the division derivatives respectively, and derived several important results:

(1) $\overline{\varphi(X)} = \overline{\varphi}(\overline{X})$;

(2) $\overline{\frac{d\varphi(X)}{dX}} = \frac{l\overline{\varphi}(\overline{X})}{l\overline{X}}$;

(3) $\overline{d\varphi(X)} = l\overline{\varphi}(\overline{X})$.

In the following section, we will establish some interesting connections among intuitionistic fuzzy integrals of IFFs.

IV. Indefinite Integrals and Definite Integrals of IFFs

The main work of this section is to reveal the relationships between two parallel parts in intuitionistic fuzzy integral calculus of IFFs, which are respectively developed based on the subtraction derivatives and the division derivatives. After getting the derivatives of IFFs, it is a significant and meaningful problem to probe their inverse operations, which means that we need to get the antiderivatives of IFFs. Hence, we should introduce the concept of indefinite integrals of IFFs [25]:

For an IFF $\varphi(X) = (f(\mu), g(\nu))$, we need to solve two ordinary differential equations to acquire the antiderivative $\Phi(X)$ of $\varphi(X)$, which meets $\frac{d\Phi(X)}{dX} = \varphi(X)$. If we suppose $\Phi(X) = (F(\mu), G(\nu))$, then

$$\begin{cases} \frac{1-\mu}{1-F(\mu)}\frac{dF(\mu)}{d\mu} = f(\mu) \\ 1 - \frac{\nu}{G(\nu)}\frac{dG(\nu)}{d\nu} = g(\nu) \end{cases} \Rightarrow \begin{cases} F(\mu) = 1 - c_1 \exp\left\{-\int \frac{f(\mu)}{1-\mu} d\mu\right\} \\ G(\nu) = c_2 \exp\left\{\int \frac{1-g(\nu)}{\nu} d\nu\right\} \end{cases}$$

where c_1 and c_2 are two integral constants, which are real numbers such that $\Phi(X)$ is an IFF.

After solving the ordinary differential equations, we get that the antiderivative of φ has the following form:

$$\Phi(X) = \left(1 - c_1 \exp\left\{-\int \frac{f(\mu)}{1-\mu} d\mu\right\},\ c_2 \exp\left\{\int \frac{1-g(\nu)}{\nu} d\nu\right\}\right) \quad (1)$$

which can be denoted as $\int \varphi(X) dX$.

We can test the correctness of the above-mentioned analysis, which is actually to prove that the derivative function of $\Phi(X)$ is certainly $\varphi(X)$, by the following derivation process:

$$\frac{d\Phi(X)}{dX} = \frac{d\left(1 - c_1 \exp\left\{-\int \frac{f(\mu)}{1-\mu} d\mu\right\},\ c_2 \exp\left\{\int \frac{1-g(\nu)}{\nu} d\nu\right\}\right)}{dX}$$

$$= \left(\frac{(1-\mu) c_1 \exp\left\{-\int \frac{f(\mu)}{1-\mu} d\mu\right\}}{c_1 \exp\left\{-\int \frac{f(\mu)}{1-\mu} d\mu\right\}} \frac{f(\mu)}{1-\mu},\ 1 - \frac{\nu c_2 \exp\left\{\int \frac{1-g(\nu)}{\nu} d\nu\right\}}{c_2 \exp\left\{\int \frac{1-g(\nu)}{\nu} d\nu\right\}} \frac{1-g(\nu)}{\nu} \right)$$

$$= \varphi(X)$$

Obviously, these aforementioned derivation processes are about the subtraction derivatives. Meanwhile, there are also the indefinite integrals of IFFs based on the division derivatives [25]. If there is an IFF $\Phi(X)$, which meets $\frac{l\Phi(X)}{lX} = \varphi(X) = (f(\mu), g(\nu))$, then it should have the following form:

$$\Phi(X) = \left(c_1 \exp\left\{\int \frac{1-f(\mu)}{\mu} d\mu\right\},\ 1 - c_2 \exp\left\{-\int \frac{g(\nu)}{1-\nu} d\nu\right\}\right) \quad (2)$$

which is denoted as $\wr \varphi(X) lX$. Then we can get a conclusion as follows:

Theorem 4.1.

$$\overline{\int \varphi(X) dX} = \wr\, \overline{\varphi}(\overline{X})\, l\overline{X}$$

Proof. Below we will prove it in two different ways, one of which is shown below: For any IFF $\Phi(X) \in \overline{\int \varphi(X) dX}$, we have $\frac{d\overline{\Phi(X)}}{dX}=\varphi(X)$. With $\frac{d\overline{\Phi(X)}}{dX}=\overline{\varphi(X)}$, we get $\frac{l\Phi(X)}{l\bar{X}}=\bar{\varphi}(\bar{X})$, which means $\Phi(X)\in\wr\bar{\varphi}(\bar{X})l\bar{X}$. Hence, $\overline{\int\varphi(X)dX}\subseteq\wr\bar{\varphi}(\bar{X})l\bar{X}$. In the same way, we can also know $\overline{\int\varphi(X)dX}\supseteq\wr\bar{\varphi}(\bar{X})l\bar{X}$. As a result, we get $\overline{\int\varphi(X)dX}=\wr\bar{\varphi}(\bar{X})l\bar{X}$.

Another proof method is to utilize the formulas of indefinite integrals of IFFs. For any antiderivative $\Phi(X)$, which meets $\frac{d\Phi(X)}{dX}=\varphi(X)$, then it must have the form of the equality (1), where c_1 and c_2 are two integral constants, which are real numbers such that $\Phi(X)$ is an IFF. Then, we denote A as the following set:

$$A=\left\{\langle c_1,c_2\rangle \mid \Phi(X)=\left(1-c_1\exp\left\{-\int\frac{f(\mu)}{1-\mu}d\mu\right\},\ c_2\exp\left\{\int\frac{1-g(\nu)}{\nu}d\nu\right\}\right) \text{ is an IFF}\right\}$$

where $\langle c_1,c_2\rangle$ is a two dimensional vector, but not an IFN.

Moreover, for any antiderivative $\Psi(\bar{X})$ satisfying $\frac{l\Psi(\bar{X})}{l\bar{X}}=\bar{\varphi}(\bar{X})$, it is just like:

$$\Psi(\bar{X})=\left(c_3\exp\left\{\int\frac{1-g(\nu)}{\nu}d\nu\right\},\ 1-c_4\exp\left\{-\int\frac{f(\mu)}{1-\mu}d\mu\right\}\right) \quad (2)$$

where c_3 and c_4 are two integral constants, which are real numbers such that $\Psi(\bar{X})$ is an IFF. Then, we let

$$B=\left\{\langle c_3,c_4\rangle \mid \Psi(\bar{X})=\left(c_3\exp\left\{\int\frac{1-g(\nu)}{\nu}d\nu\right\},\ 1-c_4\exp\left\{-\int\frac{f(\mu)}{1-\mu}d\mu\right\}\right) \text{ is an IFF}\right\}$$

where $\langle c_3,c_4\rangle$ is also a two dimensional vector, but not an IFN.

Hence, we can notice that for any given $\Phi(X)$ having the form of the equality (1), it yields

$$\overline{\Phi(X)}=\left(c_2\exp\left\{\int\frac{1-g(\nu)}{\nu}d\nu\right\},\ 1-c_1\exp\left\{-\int\frac{f(\mu)}{1-\mu}d\mu\right\}\right)$$

Obviously, we can get $\langle c_2,c_1\rangle\in B$ according to the definitions of the sets A and B, and hence, $\frac{l\overline{\Phi(X)}}{l\bar{X}}=\bar{\varphi}(\bar{X})$, which means $\overline{\int\varphi(X)dX}\subseteq\wr\bar{\varphi}(\bar{X})l\bar{X}$.

On the other hand, any IFF $\Psi(\bar{X})$ satisfying $\frac{l\Psi(\bar{X})}{l\bar{X}}=\bar{\varphi}(\bar{X})$ has the form of the equality (2), then

$$\overline{\Psi(\bar{X})}=\left(1-c_4\exp\left\{-\int\frac{f(\mu)}{1-\mu}d\mu\right\},\ c_3\exp\left\{\int\frac{1-g(\nu)}{\nu}d\nu\right\}\right)$$

Since $\langle c_4,c_3\rangle\in A$, then we also get $\frac{d\overline{\Psi(\bar{X})}}{dX}=\varphi(X)$, that is, $\wr\bar{\varphi}(\bar{X})l\bar{X}\subseteq\overline{\int\varphi(X)dX}$. Thus, $\wr\bar{\varphi}(\bar{X})l\bar{X}\subseteq\overline{\int\varphi(X)dX}$, and hence, we get $\overline{\int\varphi(X)dX}=\wr\bar{\varphi}(\bar{X})l\bar{X}$. ■

In addition, we investigated the substitution rule for indefinite integrals of IFFs in Ref. [26], as represented below:

(**The substitution rule for indefinite integrals**). If $\Phi(X)=\int\varphi(X)dX$, then $\int\varphi\big(X(t)\big)X'(t)dt=\Phi(X(t))$, where $\varphi\big(X(t)\big)\otimes X'(t)$ is abbreviated as $\varphi\big(X(t)\big)X'(t)$.

The substitution rule can be proven by using the chain rule of derivative. Since

$$\frac{d\Phi\big(X(t)\big)}{dt}=\frac{d\Phi\big(X(t)\big)}{dX(t)}\otimes\frac{dX(t)}{dt}$$

then we can get the following derivation process:

$$\frac{d\Phi\big(X(t)\big)}{dt}=\frac{dX(t)}{dt}\otimes\frac{d}{dX}\int\varphi(X)dX$$

$$=\varphi(X)\otimes\frac{dX(t)}{dt}=\varphi(X)\otimes X'(t)$$

from which we know that $\Phi\big(X(t)\big)$ is certainly the antiderivative of $\varphi(X)\otimes X'(t)$, which means that $\int\varphi\big(X(t)\big)X'(t)dt=\Phi(X(t))$. Based on which, we can get the following results:

$$\overline{\int\varphi\big(X(t)\big)X'(t)dt}=\overline{\int\varphi\big(X(t)\big)\otimes\frac{dX(t)}{dt}dt}=\overline{\Phi(X(t))}$$

$$\Rightarrow\wr\bar{\varphi}\big(\bar{X}(\bar{t})\big)\oplus\frac{l\bar{X}(\bar{t})}{l\bar{t}}l\bar{t}=\bar{\Phi}\big(\bar{X}(\bar{t})\big)$$

where $\bar{\Phi}(\bar{X})=\overline{\int\varphi(X)dX}=\wr\bar{\varphi}(\bar{X})l\bar{X}$.

As we know that the developed formula $\wr\bar{\varphi}\big(\bar{X}(\bar{t})\big)\oplus\frac{l\bar{X}(\bar{t})}{l\bar{t}}l\bar{t}=\bar{\Phi}\big(\bar{X}(\bar{t})\big)$ is just the substitution rule for indefinite integrals based on the division derivatives [26].

In what follows, before introducing the definite integrals of IFFs [25], we first recall the integral of a complex function:

Let C be a simple curve, which means that the curve is smooth or piecewise smooth, in the complex plane $\mathbb{C}$, and let $f(z)=u(x,y)+iv(x,y)$ be a continuous function in C, where $u(x,y)$ and $v(x,y)$ are the real part and imaginary part of $f(z)$. Then we need the following construction process in order to get its integral:

(1) **Dividing the simple curve.** By interpolating some break points z_0, z_1, z_2, …, z_{n-1}, $z_n=z$, we can divide the simple curve C into many smaller arcs $\widehat{z_0z_1}$, $\widehat{z_1z_2}$, …,

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$\widehat{z_{n-1}z}$, and these points z_k ($k=0,1,...,n$) are arranged from z_0 to z, which is shown in Figure 4.2 (a);

(2) **Making the product.** From every small arc $\widehat{z_k z_{k+1}}$, we take a value $\zeta_k=(\xi_k,\eta_k)$ to get the value $f(\zeta_k)(z_{k+1}-z_k)$, which can be represented as:

$$[u(\xi_k,\eta_k)+iv(\xi_k,\eta_k)][(x_{k+1}-x_k)+i(y_{k+1}-y_k)]$$

(3) **Calculating the sum.** We add all $f(\zeta_k)(z_{k+1}-z_k)$ ($k=0,1,...,n-1$) to acquire the sum $\sum_{i=1}^{n-1} f(\zeta_k)(z_{k+1}-z_k)$, that is,

$$\sum_{i=1}^{n-1}[u(\xi_k,\eta_k)+iv(\xi_k,\eta_k)][(x_{k+1}-x_k)+i(y_{k+1}-y_k)]$$

and the sum is:

$$\sum_{i=1}^{n-1}u(\xi_k,\eta_k)(x_{k+1}-x_k)-\sum_{i=1}^{n-1}v(\xi_k,\eta_k)(y_{k+1}-y_k)+ i\left[\sum_{i=1}^{n-1}v(\xi_k,\eta_k)(x_{k+1}-x_k)+\sum_{i=1}^{n-1}u(\xi_k,\eta_k)(y_{k+1}-y_k)\right]$$

(4) **Taking the limit.** When the number of the break points z_k increases infinitely, and meets the condition $\max_i\left\{|z_{k+1}-z_k|=\sqrt{(x_{k+1}-x_k)^2+(y_{k+1}-y_k)^2}\right\}\to 0$, Also the sums $\sum_{i=1}^{n-1}u(\xi_k,\eta_k)(x_{k+1}-x_k)$, $\sum_{i=1}^{n-1}v(\xi_k,\eta_k)(y_{k+1}-y_k)$, $\sum_{i=1}^{n-1}v(\xi_k,\eta_k)(x_{k+1}-x_k)$ and $\sum_{i=1}^{n-1}u(\xi_k,\eta_k)(y_{k+1}-y_k)$ exist the limit values $\int_C u(x,y)dx$, $\int_C v(x,y)dy$, $\int_C v(x,y)dx$ and $\int_C u(x,y)dy$, respectively, then we call that the limit of the sum in Step 3 have the limit value:

$$\int_C u(x,y)dx-v(x,y)dy+i\int_C v(x,y)dx+u(x,y)dy$$

and we define it as the integral of $f(z)$ along the curve C, which is denoted by $\int_C f(z)dz$.

However, researching the integral of IFFs needs a special kind of curves named as intuitionistic fuzzy curves (IFICs) [25]. In what follows, we introduce the concept of IFICs:

There is a curve I linking between α and β that can be written as a bijective mapping $\Im:[0,L]\to I$, where L is the length from α to β, and this mapping satisfies that $\Im(0)=\alpha$ and $\Im(L)=\beta$. If $\Im(t_1)\trianglelefteq_{\oplus}\Im(t_2)$ for $0\le t_1\le t_2\le L$, then we call I an intuitionistic fuzzy integral curve (IFIC). Figure 4.1 shows several IFICs:

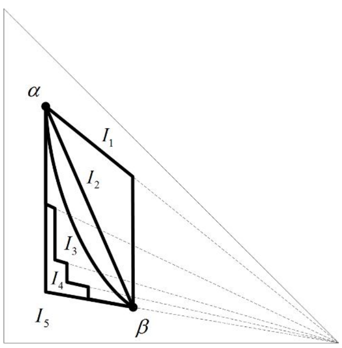


Figure 4.1. IFICs

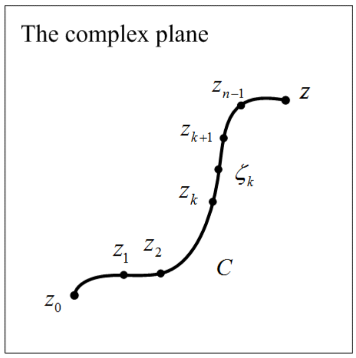
According to the concept of admissible order defined by Ref. [34] and Ref. [35], we can get that the order "$\trianglelefteq_{\oplus}$" on any IFIC I is actually an admissible order because it satisfies the following conditions of the admissible order:

(1) "$\trianglelefteq_{\oplus}$" is a linear order on any IFIC I because $\alpha\trianglelefteq_{\oplus}\beta$ or $\beta\trianglelefteq_{\oplus}\alpha$, for any $\alpha\in I$ and $\beta\in I$;

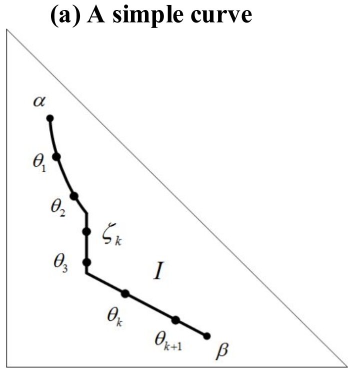
(2) $\mu_\alpha\le\mu_\beta$ and $\nu_\alpha\ge\nu_\beta$ if $\alpha\trianglelefteq_{\oplus}\beta$, for any $\alpha=(\mu_\alpha,\nu_\alpha)\in I$ and $\beta=(\mu_\beta,\nu_\beta)\in I$.

With the IFICs, below we will introduce the integrals of IFFs. Distinguishing from the fact that the integral of the complex functions is based on a simple curve, defining the integral of IFFs needs an IFIC as follows:

(1) **Dividing the IFIC.** By interpolating some break points $\alpha=\theta_0$, θ_1, θ_2, …, θ_{n-1}, $\theta_n=\beta$, we can divide the IFIC I into many smaller arcs $\widehat{\alpha\theta_1}$, $\widehat{\theta_1\theta_2}$, …, $\widehat{\theta_{n-1}\beta}$, and these points θ_k ($k=0,1,...,n$) are arranged from α to β, which is shown in Figure 4.2 (b):



(a) A simple curve



(b) An IFIC

Figure 4.2. A simple curve and an IFIC

(2) **Making the product.** From every small arc $\widehat{\theta_k\theta_{k+1}}$, we take an IFN $\zeta_k=(\xi_k,\eta_k)$ to get the value $\varphi(\zeta_k)\otimes(\theta_{k+1}\ominus\theta_k)$, which can be represented as $\left(f(\mu_{\xi_i}),g(\nu_{\xi_i})\right)\otimes\left(\frac{\mu_{i+1}-\mu_i}{1-\mu_i},\frac{\nu_{i+1}}{\nu_i}\right)$;

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(3) **Calculating the sum.** Add all $\varphi(\zeta_k)\otimes(\theta_{k+1}\ominus\theta_k)$ ($k=0,1,...,n-1$) to get the sum $\bigoplus_{i=1}^{n-1}\varphi(\zeta_k)\otimes(\theta_{k+1}\ominus\theta_k)$, that is, $\bigoplus_{i=1}^{n-1}\left(f(\mu_{\xi_i}),g(\nu_{\xi_i})\right)\otimes\left(\dfrac{\mu_{i+1}-\mu_i}{1-\mu_i},\dfrac{\nu_{i+1}}{\nu_i}\right)$;

(4) **Taking the limit.** If the number of the break points θ_k ($k=0,1,...,n-1$) increases infinitely, and meets the condition $\theta_{k+1}\ominus\theta_k\to O$ ($k=0,1,...,n-1$), then the limits of the membership and non-membership parts of $\bigoplus_{i=1}^{n-1}\varphi(\zeta_k)\otimes(\theta_{k+1}\ominus\theta_k)$ are equal to U and V, respectively, and (U,V) is an IFN. In this case, we call (U,V) the limit of the sum in Step 3, and define it as the integral of $\varphi(X)$ along the IFIC I, which is denoted by $\int_I\varphi(X)dX$.

Since the value of the integral of a complex function $\int_C f(z)dz$ only depends on the starting point and the end point of the curve C, which means that $\int_{C_1}f(z)dz=\int_{C_2}f(z)dz$ ($C_1\neq C_2$) if only the starting point and the end point of C_1 are the same as the ones of C_2. Hence, the integral of a complex function $\int_C f(z)dz$ can be denoted as $\int_{z_0}^{z}f(z)dz$, where z_0 and z are the starting point and the end point of the curve, respectively. Similarly, we have proven that the value of the integral of an IFF $\int_I\varphi(X)dX$ is also only related to the starting point and the end point of the IFIC I in Ref. [21]. So $\int_I\varphi(X)dX$ can be written as $\int_\alpha^\beta\varphi(X)dX$, where α and β are the starting point and the end point of the IFIC I, respectively, and the calculation formula of $\int_I\varphi(X)dX$ can also be given as [25]:

$$\int_\alpha^\beta\varphi(X)\,dX=\left(1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{f(\mu)}{1-\mu}d\mu\right\},\ \exp\left\{\int_{\nu_\alpha}^{\nu_\beta}\frac{1-g(\nu)}{\nu}d\nu\right\}\right)$$

and specially, $\int_\alpha^\alpha\varphi(X)\,dX=(0,1)$.

In the following, we introduce several interesting properties of the definite integral [25]:

(1) $\int_\alpha^\beta(\lambda,1-\lambda)\otimes\varphi(X)\,dX=\lambda\int_\alpha^\beta\varphi(X)\,dX$. Specially, we can get $\int_O^\beta(\lambda,1-\lambda)dX=\lambda\beta$ when $\varphi(X)=(1,0)$ and $\alpha=O$, which means that "$\lambda\beta$" can be developed by "$\oplus$" and "$\otimes$" of IFNs when $0\le\lambda\le1$;

(2) $\int_\alpha^\beta\left(\sum_{i=1}^n f_i(\mu),1-\sum_{i=1}^n(1-g_i(\nu))\right)dX=\bigoplus_{i=1}^n\int_\alpha^\beta\left(f_i(\mu),g_i(\nu)\right)dX$ if $\left(\sum_{i=1}^n f_i(\mu),1-\sum_{i=1}^n(1-g_i(\nu))\right)$ is still an IFF;

(3) $\int_\alpha^\beta\varphi(X)dX=\int_a^b\varphi(X(t))X'(t)dt$ if the IFF $X(t)$ is derivable, and satisfies $X(a)=\alpha$ and $X(b)=\beta$, which is actually the substitution rule for definite integrals of IFFs;

(4) $\int_\alpha^\beta\varphi(X)\,dX\oplus\int_\beta^\gamma\varphi(X)\,dX=\int_\alpha^\gamma\varphi(X)\,dX$, where $\alpha\trianglelefteq_\oplus\beta\trianglelefteq_\oplus\gamma$;

(5) $\int_\alpha^\beta\varphi(X)dX=\Psi(\beta)\ominus\Psi(\alpha)$, where $\Psi(X)$ is an antiderivative of $\varphi(X)$. The result is just the "Newton-Leibniz formula" under intuitionistic fuzzy environment.

It is clear that the "IFICs" is based on the order "$\trianglelefteq_\oplus$". Hence, we can also define a similar kind of curves according to "$\trianglelefteq_\otimes$" as follows:

Let J be a curve linking between α and β that can be written as a bijective mapping $\mathfrak{J}:[0,L]\to I$, where L is the length from α to β. This mapping satisfies $\mathfrak{J}(0)=\alpha$ and $\mathfrak{J}(L)=\beta$. If $\mathfrak{J}(s_2)\trianglelefteq_\otimes\mathfrak{J}(s_1)$ for $0\le s_1\le s_2\le L$, then we call J a second intuitionistic fuzzy integral curve (II-IFIC). Figure 4.3 shows several II-IFICs:

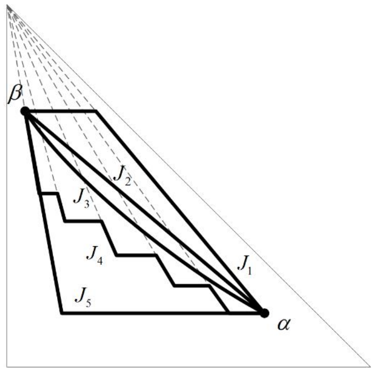

Figure 4.3. II-IFICs

According to the definitions of IFICs and II-IFICs, we can obtain that $\bar{I}$ is an II-IFIC linking $\bar{\alpha}$ and $\bar{\beta}$ if I is an II-IFIC linking α and β, which is shown in Figure 4.4.

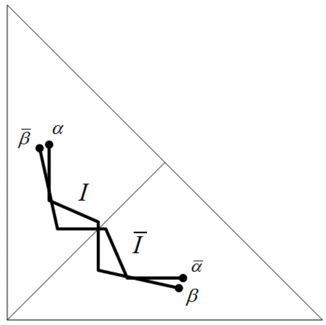

Figure 4.4. The complement of an IFIC $\bar{I}$

By utilizing the concept of "II-IFICs", we can define another integral of IFFs below:

(1) Dividing the II-IFIC. By interpolating some break points $\alpha=\theta_0$, θ_1, θ_2, ..., θ_{n-1}, $\theta_n=\beta$, we can divide the II-IFIC J into several smaller arcs $\widehat{\alpha\theta_1}$, $\widehat{\theta_1\theta_2}$, ..., $\widehat{\theta_{n-1}\beta}$,

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and these points θ_k ($k=0,1,...,n$) are arranged from α to β;

(2) Making the sum. From every small arc $\widehat{\theta_k\theta_{k+1}}$, we take an IFN $\zeta_k=(\xi_k,\eta_k)$ to get the value $\varphi(\zeta_k)\oplus(\theta_{k+1}\oslash\theta_k)$, which can be represented as $\left(f(\mu_{\xi}),g(\nu_{\xi})\right)\oplus\left(\frac{\mu_{i+1}}{\mu_i},\frac{\nu_{i+1}-\nu_i}{1-\nu_i}\right)$;

(3) Calculating the product. We combine all $\varphi(\zeta_k)\oplus(\theta_{k+1}\oslash\theta_k)$ ($k=0,1,...,n-1$) by using multiplication to get the product $\bigotimes_{i=1}^{n-1}\left(\varphi(\zeta_k)\oplus(\theta_{k+1}\oslash\theta_k)\right)$;

(4) Taking the limit. If the number of the break points θ_k ($k=0,1,...,n-1$) increases infinitely, and meets $\theta_{k+1}\oslash\theta_k\to E$ ($k=0,1,...,n-1$), then the limits of the membership and non-membership parts of $\bigotimes_{i=1}^{n-1}\left(\varphi(\zeta_k)\oplus(\theta_{k+1}\oslash\theta_k)\right)$ are equal to U and V, respectively, and (U,V) is an IFN. In this case, we call (U,V) the limit of the sum in Step 3, and define it as the integral of $\varphi(X)$ along the II-IFIC J, denoted by $\wr_J\varphi(X)lX$, which only depends on the two endpoints of the II-IFIC J. Hence, $\wr_J\varphi(X)lX$ is denoted as $\wr_\alpha^\beta\varphi(X)lX$, where $\wr_\alpha^\beta\varphi(X)lX=\left(\exp\left\{\int_{\mu_\alpha}^{\mu_\beta}\frac{1-f(\mu)}{\mu}d\mu\right\},\right.$ $\left.1-\exp\left\{-\int_{\nu_\alpha}^{\nu_\beta}\frac{g(\nu)}{1-\nu}d\nu\right\}\right)$. Specially, $\wr_\alpha^\alpha\varphi(X)lX=(1,0)$.

Clearly, there is a closed connection between two aforementioned kinds of definite integrals of IFFs, which can be shown in the following theorem:

Theorem 4.2.

$$\overline{\int_\alpha^\beta\varphi(X)dX}=\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X$$

Proof. It can be proven by the following two ways: Firstly, according to the definitions of the two definite integrals of IFFs, we get

$$\overline{\int_\alpha^\beta\varphi(X)dX}=\overline{\lim_{d\to O}\bigoplus_{i=1}^k\left(\varphi(\xi_i)\otimes\Delta\delta_i\right)}$$

$$=\lim_{d\to E}\bigotimes_{i=1}^k\overline{\left(\varphi(\xi_i)\otimes\Delta\delta_i\right)}=\lim_{d\to E}\bigotimes_{i=1}^k\left(\bar\varphi(\bar\xi_i)\oplus\nabla\bar\delta_i\right)=\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X$$

Secondly, we can also prove it by using another way: About the left-hand side of the equality, we have

$$\overline{\int_\alpha^\beta\varphi(X)dX}=\left(\exp\left\{\int_{\nu_\alpha}^{\nu_\beta}\frac{1-g(\nu)}{\nu}d\nu\right\},\ 1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{f(\mu)}{1-\mu}d\mu\right\}\right)$$

and for the right-hand side of the equality, we can get

$$\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X=\left(\exp\left\{\int_{\nu_\alpha}^{\nu_\beta}\frac{1-g(\nu)}{\nu}d\nu\right\},\ 1-\exp\left\{-\int_{\mu_\alpha}^{\mu_\beta}\frac{f(\mu)}{1-\mu}d\mu\right\}\right)$$

and thus, $\overline{\int_\alpha^\beta\varphi(X)dX}=\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X$ holds. ■

In addition, based on the connection between two kinds of definite integrals, we can derive the following results on $\int_\alpha^\beta\varphi(X)dX$ in order to acquire some corresponding ones of $\wr_\alpha^\beta\varphi(X)lX$:

(1) Since $\overline{\int_\alpha^\beta(\lambda,1-\lambda)\otimes\varphi(X)\,dX}=\overline{\lambda\int_\alpha^\beta\varphi(X)\,dX}$, then we get

$$\wr_{\bar\alpha}^{\bar\beta}(1-\lambda,\lambda)\oplus\bar\varphi(\bar X)l\bar X=\left(\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X\right)^\lambda$$

Specially, $\wr_E^{\bar\beta}(1-\lambda,\lambda)l\bar X=\bar\beta^\lambda$ when $\bar\alpha=E$ and $\bar\varphi(\bar X)=(0,1)$, which mean $\alpha=O$ and $\varphi(X)=(1,0)$, respectively. The result shows that "β^λ" can also be developed by "$\oplus$" and "$\otimes$" of IFNs when $0\le\lambda\le1$;

(2) By using $\overline{\int_\alpha^\beta\left(\sum_{i=1}^n f_i(\mu),1-\sum_{i=1}^n(1-g_i(\nu))\right)dX}=\overline{\bigoplus_{i=1}^n\int_\alpha^\beta\left(f_i(\mu),g_i(\nu)\right)dX}$, we have

$$\wr_{\bar\alpha}^{\bar\beta}\left(1-\sum_{i=1}^n(1-g_i(\nu)),\sum_{i=1}^n f_i(\mu)\right)l\bar X=\bigotimes_{i=1}^n\wr_{\bar\alpha}^{\bar\beta}\left(g_i(\nu),f_i(\mu)\right)l\bar X$$

(3) According to $\overline{\int_\alpha^\beta\varphi(X)dX}=\overline{\int_a^b\varphi(X(t))X'(t)dt}$, we get

$$\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X=\wr_{\bar a}^{\bar b}\bar\varphi\left(\bar X(\bar t)\right)\oplus\frac{l\bar X(\bar t)}{l\bar t}l\bar t$$

(4) It follows from $\int_\alpha^\beta\varphi(X)\,dX\oplus\int_\beta^\gamma\varphi(X)\,dX=\int_\alpha^\gamma\varphi(X)\,dX$ that

$$\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X\otimes\wr_{\bar\beta}^{\bar\gamma}\bar\varphi(\bar X)l\bar X=\wr_{\bar\alpha}^{\bar\gamma}\bar\varphi(\bar X)l\bar X$$

where $\bar\gamma\trianglelefteq_\otimes\bar\beta\trianglelefteq_\otimes\bar\alpha$;

(5) Based on $\overline{\int_\alpha^\beta\varphi(X)\,dX}=\overline{\Psi(\beta)\ominus\Psi(\alpha)}$, we have

$$\wr_{\bar\alpha}^{\bar\beta}\bar\varphi(\bar X)l\bar X=\bar\Psi(\bar\beta)\oslash\bar\Psi(\bar\alpha)$$

After introducing the definite integrals of IFFs, we will show the methods aggregating the IFNs spreading all over an area, which solve the issues how to aggregate the continuous intuitionistic fuzzy information. Figure 4.2 shows the intuitionistic fuzzy discrete information and the continuous information, respectively. Clearly, the discrete information will be located in several two-dimensional points, however, the continuous information will spread all over an area like D shown in Figure 4.5. Some common aggregation operators only focus on the situations with the discrete intuitionistic fuzzy information. For example, the widely-used IFWA and IFWG operators are only able to aggregate a limit number of IFNs. In fact, we have pointed out that we not only deal with the discrete

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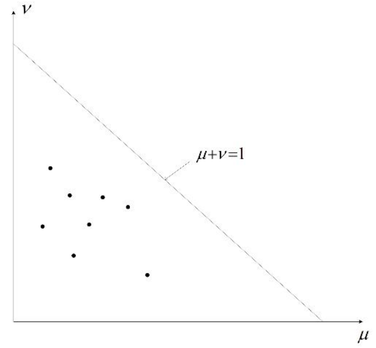
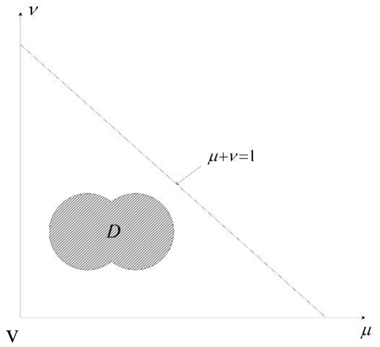
IFNs, but also need to solve a lot of problems with the continuous intuitionistic fuzzy information, which is like that studying the discrete-type random variables and the continuous-type random variables are both significant and necessary for the probability theory and the mathematical statistics [32].



(a) Discrete information



(b) Continuous information

Figure 4.5. Intuitionistic fuzzy discrete information and continuous information

For the discrete intuitionistic fuzzy information or data, there are two common aggregation techniques, namely the IFWA and IFWG operators [14,15]. When $\sum_{i=1}^{n}\omega_i=1$, we have

(The IFWA operator)

$$IFWA_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\mathop{\oplus}_{i=1}^{n}\omega_i\alpha_i=\left(1-\prod_{i=1}^{n}(1-\mu_{\alpha_i})^{\omega_i},\ \prod_{i=1}^{n}\nu_{\alpha_i}^{\omega_i}\right)$$

(The IFWG operator)

$$IFWG_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n)=\mathop{\otimes}_{i=1}^{n}\alpha_i^{\omega_i}=\left(\prod_{i=1}^{n}\mu_{\alpha_i}^{\omega_i},\ 1-\prod_{i=1}^{n}(1-\nu_{\alpha_i})^{\omega_i}\right)$$

Obviously, we can derive the following relationship between the IFWA and IFWG operators:

$$\overline{IFWA_{\omega}(\alpha_1,\alpha_2,\cdots,\alpha_n)}=IFWG_{\omega}(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$$

When considering how to deal with the problems with the continuous intuitionistic fuzzy information, we gave the following method in Ref. [32]:

Firstly, we suppose that there is a real function $P:\mathbb{R}^2\to\mathbb{R}$ defined in the region D, which satisfies $P(\mu,\nu)\geq 0$ for any $(\mu,\nu)\in D$ and $\iint_D P(\mu,\nu)\,d\delta=1$, then we call it a weight density function. The real function P is similar to the probability density function in the probability theory. Then we have the following steps:

Step 1. Dividing the region D into k sub-regions, which are δ_1, δ_2, …, δ_k, respectively;

Step 2. Choosing randomly an IFN (ξ_i,η_i) from the sub-region δ_i ($1\leq i\leq k$), and making the products $P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$ ($1\leq i\leq k$), where $\Delta\delta_i$ is the area of the i-th region;

Step 3. Calculating the sum of $P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$ ($1\leq i\leq k$), that is, $\mathop{\oplus}_{i=1}^{k}P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$;

Step 4. Taking the limit $\lim_{d\to 0}\mathop{\oplus}_{i=1}^{k}P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$, where $d\to 0$ means that

$$d=\max_{0\leq i\leq k}\left\{\sup\left\{d(x,y):x,y\in\delta_i\right\}\right\}\to 0$$

where the distance $d(x,y)$ is an ordinary one of points located in two-dimensional plane, and satisfies three axioms of distances, namely, non-negativity, symmetry and triangle inequality.

If the limit $\lim_{d\to 0}\mathop{\oplus}_{i=1}^{k}P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$ is an IFN, then we denote it as $\iint_D P(\mu,\nu)(\mu,\nu)\,d\delta$, whose calculating formula is

$$\iint_D P(\mu,\nu)(\mu,\nu)\,d\delta$$
$$=\left(1-\exp\left\{\iint_D P(\mu,\nu)\ln(1-\mu)\,d\delta\right\},\ \exp\left\{\iint_D P(\mu,\nu)\ln\nu\,d\delta\right\}\right)$$

Meanwhile, in Ref. [32], we pointed out the fact that $\iint_D P(\mu,\nu)(\mu,\nu)\,d\delta$ is actually a continuous form of the IFWA operator. In addition, we also gave another method to aggregate in the intuitionistic fuzzy continuous information:

Step 1. Dividing the region D into k sub-regions, which are $\delta_i(i=1,2,...,k)$, respectively;

Step 2. Choosing randomly an IFN (ξ_i,η_i) from the sub-region δ_i ($1\leq i\leq k$), and making the produces $(\xi_i,\eta_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$ ($1\leq i\leq k$), where $\Delta\delta_i$ is the area of the i-th region;

Step 3. Calculating the sum of $(\xi_i,\eta_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$ ($1\leq i\leq k$), that is $\mathop{\otimes}_{i=1}^{k}(\xi_i,\eta_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$;

Step 4. Taking the limit $\lim_{d\to 0}\mathop{\otimes}_{i=1}^{k}(\xi_i,\eta_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$.

We denote this limit $\lim_{d\to 0}\mathop{\otimes}_{i=1}^{k}(\xi_i,\eta_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$ as $\wr\wr_D(\mu,\nu)^{P(\mu,\nu)d\delta}$, which is actually a continuous form of the IFWG operator [32], and it has the following form:

$$\wr\wr_D(\mu,\nu)^{P(\mu,\nu)d\delta}$$

$$=\left(\exp\left\{\iint_D P(\mu,\nu)\ln\mu d\delta\right\},1-\exp\left\{\iint_D P(\mu,\nu)\ln(1-\nu)d\delta\right\}\right)$$

Before getting the relationship between $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\wr_D(\mu,\nu)^{P(\mu,\nu)d\delta}$, we first analyze the IFWA and IFWG operators below:

Based on $\overline{IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)}=IFWG_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$, that is

$$\overline{IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)}=\overline{\bigoplus_{i=1}^{n}\omega_i\alpha_i}=\bigotimes_{i=1}^{n}\bar{\alpha}_i^{\omega_i}=IFWG_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$$

and we need the following steps in order to get the complement of $IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$:

(1) Finding the basic components of $IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$. Because $IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)=\bigoplus_{i=1}^{n}\omega_i\alpha_i$, we can know that its basic elements are $\omega_i\alpha_i$ ($1\le i\le n$). Then the IFWA operator combines these $\omega_i\alpha_i$ ($1\le i\le n$) with the operation "$\oplus$";

(2) Getting the complements of the basic components. In order to derive the complement of $IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$, we first acquire these complements of the basic elements $\omega_i\alpha_i$ ($1\le i\le n$), which are actually $\bar{\alpha}_i^{\omega_i}$ ($1\le i\le n$). Significantly, α_i and $\bar{\alpha}_i$ have the same weight value ω_i;

(3) Combining these complements of the basic components with the multiplication "$\otimes$". After getting all complements of the basic elements, namely $\bar{\alpha}_i^{\omega_i}$ ($1\le i\le n$), we assemble them with "$\otimes$" to get $\bigotimes_{i=1}^{n}\bar{\alpha}_i^{\omega_i}$, which is just $IFWG_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$.

We can imitate the above-mentioned steps to get the relationship between $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\wr_D(\mu,\nu)^{P(\mu,\nu)d\delta}$:

(1) Finding the basic components of $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$. Since $\iint_D P(\mu,\nu)(\mu,\nu)d\delta=\lim_{d\to 0}\bigoplus_{i=1}^{k}P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$, then we can easily get its basic component $P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$;

(2) Getting the complements of the basic components $P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$. The complement of $P(\xi_i,\eta_i)(\xi_i,\eta_i)\Delta\delta_i$ is $\overline{(\xi_i,\eta_i)}^{P(\xi_i,\eta_i)\Delta\delta_i}$, which is $(\eta_i,\xi_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$;

(3) Combining these complements of the basic components with the multiplication "$\otimes$". After getting all $(\eta_i,\xi_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$, we can acquire the limit $\lim_{d\to 0}\bigotimes_{i=1}^{k}(\eta_i,\xi_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$.

Based on these steps, it is clear that $\lim_{d\to 0}\bigotimes_{i=1}^{k}(\eta_i,\xi_i)^{P(\xi_i,\eta_i)\Delta\delta_i}$ can be expressed as $\wr\wr_D\overline{(\mu,\nu)}^{P(\mu,\nu)d\delta}$. and then there is the following theorem:

Theorem 4.3.

$$\overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta}=\wr\wr_D\overline{(\mu,\nu)}^{P(\mu,\nu)d\delta}$$

which is just the relationship between "$\iint_D\bullet$" and "$\wr\wr_D\bullet$" that we want.

Based on the above analysis, Theorem 4.3 can be proven easily. On the other hand, we can also define two new concepts, namely the complement of a region D and the complement of a weight density function $P(\mu,\nu)$ in order to get the relationship between the two different integrals: Firstly, the complement of a region D is defined as:

$$\bar{D}=\{\alpha\mid\bar{\alpha}\in D\}\text{ or }\bar{D}=\{\bar{\alpha}\mid\alpha\in D\}$$

which can be represented in the following figure:

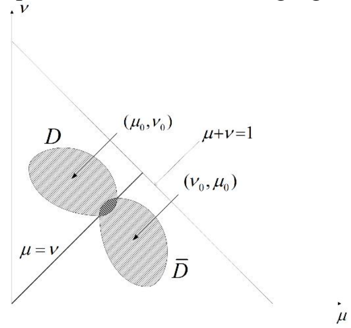


Figure 4.6. The complement of a region D

To get the complement of $IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$, we need to find every complement $\bar{\alpha}_i$ of α_i ($1\le i\le n$). Similarly, we need to get the complement $\bar{D}$ of D when dealing with the continuous intuitionistic fuzzy information. Then we define the complement $\bar{P}(\mu,\nu)$ of a weight density function $P(\mu,\nu)$ as $\bar{P}(\mu,\nu)=P(\nu,\mu)$, and we know that the real function $\bar{P}(\mu,\nu)$ is defined on the region $\bar{D}$ if $P(\mu,\nu)$ is defined on D, which is shown in Figure 4.7:

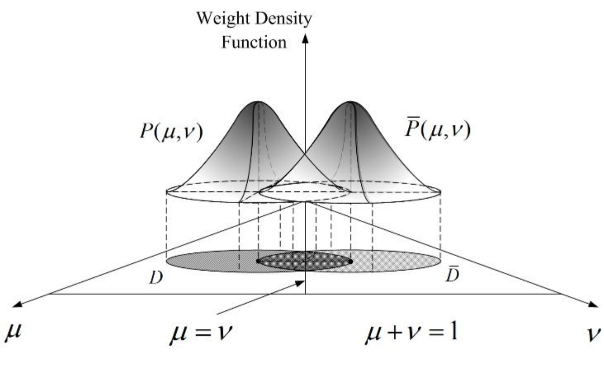


Figure 4.7. The complement of a weight density function $P(\mu,\nu)$

By using the complement $\bar{P}(\mu,\nu)$, we can easily get the weight value of (μ,ν) for any $(\mu,\nu)\in\bar{D}$. By using $\bar{P}(\mu,\nu)$, the weight value of (μ,ν) ($(\mu,\nu)\in\bar{D}$) is actually equal to $P(\nu,\mu)$ according to the definition of $\bar{P}(\mu,\nu)$, which shows that the weight value $\bar{P}(\mu,\nu)$ of (μ,ν) in $\bar{D}$ is just equal to the weight value $P(\nu,\mu)$ of (ν,μ) in D. It is like that α_i and $\bar{\alpha}_i$ have the same weight value ω_i when analyzing $\overline{IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)} = IFWG_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$.

By virtue of $\bar{D}$ and $\bar{P}(\mu,\nu)$, we get $\lim_{d\to 0}\bigotimes_{i=1}^{k}(\eta_i,\xi_i)^{P(\xi_i,\eta_i)\Delta\delta_i} = \wr\wr_{\bar{D}}(\mu,\nu)^{\bar{P}(\mu,\nu)d\delta}$, and hence, we have the following relationship between "$\iint_D \bullet$" and "$\wr\wr_D \bullet$":

Theorem 4.4.

$$\wr\wr_D \overline{(\mu,\nu)}^{P(\mu,\nu)d\delta} = \overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta} = \wr\wr_{\bar{D}}(\mu,\nu)^{\bar{P}(\mu,\nu)d\delta}$$

Proof. With the fore-mentioned analysis, we have $\wr\wr_{\bar{D}}(\mu,\nu)^{\bar{P}(\mu,\nu)d\delta} = \wr\wr_D \overline{(\mu,\nu)}^{P(\mu,\nu)d\delta}$. Clearly, the equality holds because $\wr\wr_D \overline{(\mu,\nu)}^{P(\mu,\nu)d\delta} = \overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta}$.

Moreover, we can also prove it by using another method:

$$\begin{aligned}
&\wr\wr_{\bar{D}}(\mu,\nu)^{\bar{P}(\mu,\nu)d\delta}\\
&=\left(\exp\left\{\iint_D \bar{P}(\nu,\mu)\ln\nu d\delta\right\}, 1-\exp\left\{\iint_D \bar{P}(\nu,\mu)\ln(1-\mu)d\delta\right\}\right)\\
&=\left(\exp\left\{\iint_D P(\mu,\nu)\ln\nu d\delta\right\}, 1-\exp\left\{\iint_D P(\mu,\nu)\ln(1-\mu)d\delta\right\}\right)\\
&=\overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta}
\end{aligned}$$

Considering the right-hand of the equality, we have

$$\begin{aligned}
&\wr\wr_D \overline{(\mu,\nu)}^{P(\mu,\nu)d\delta}\\
&=\left(\exp\left\{\iint_D P(\mu,\nu)\ln\nu d\delta\right\}, 1-\exp\left\{\iint_D P(\mu,\nu)\ln(1-\mu)d\delta\right\}\right)\\
&=\overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta}
\end{aligned}$$

Hence, we get $\wr\wr_{\bar{D}}(\mu,\nu)^{\bar{P}(\mu,\nu)d\delta} = \overline{\iint_D P(\mu,\nu)(\mu,\nu)d\delta} = \wr\wr_D \overline{(\mu,\nu)}^{P(\mu,\nu)d\delta}$. ■

On the other hand, because of the symmetry between "$\iint_D \bullet$" and "$\wr\wr_D \bullet$", we can also get a similar result:

Theorem 4.5.

$$\iint_{\bar{D}} \bar{P}(\mu,\nu)(\mu,\nu)d\delta = \overline{\wr\wr_D (\mu,\nu)^{P(\mu,\nu)d\delta}} = \iint_D P(\mu,\nu)\overline{(\mu,\nu)}d\delta$$

The proof of Theorem 4.5 is omitted here.

In Ref. [33], we studied the IFWA and IFWG operators, $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\wr_D (\mu,\nu)^{P(\mu,\nu)d\delta}$ by utilizing the integrals of IFFs, namely "$\int_\alpha^\beta \varphi(X)dX$" and "$\wr_\alpha^\beta \varphi(X)lX$". We found the fact that the IFWA and IFWG operators, $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\wr_D (\mu,\nu)^{P(\mu,\nu)d\delta}$ are actually the definite integrals of special IFFs "$L(X)$" and "$\mathcal{L}(X)$". Below we introduce them as follows:

If there are n IFNs, which are respectively $\alpha_i=(\mu_i,\nu_i)$ ($i=1,2,\cdots,n$) and meet the condition $\alpha_i \neq \alpha_j$ when $i \neq j$, and their weights are respectively ω_i ($i=1,2,\cdots,n$) and $\sum_{i=1}^n \omega_i = 1$, then we can define the following real functions:

$$R(\mu)=\sum_{\mu_i>\mu}\omega_i \text{ and } T(\nu)=\sum_{\nu_i\geq\nu}\omega_i$$

According to the definitions of R and T, we can get $R:[0,1]\to[0,1]$ and $T:[0,1]\to[0,1]$. Moreover, the real value $R(\mu)$ is actually equal to the sum of all weight values ω_i of α_i, the membership degree μ_i that is greater than or equal to the given real number μ. $T(\nu)$ is actually equal to the sum of all weight values ω_i of α_i, the non-membership degree ν_i that is greater than or equal to ν. For example, if there are two IFNs $\alpha_1=(0.1,0.2)$ and $\alpha_2=(0.1,0.5)$, and their weights are 0.2 and 0.8, respectively, then

$$R(\mu)=\begin{cases}1, & 0\leq\mu<0.1;\\ 0, & 0.1\leq\mu<1;\end{cases} \text{ and } T(\nu)=\begin{cases}1, & 0\leq\nu\leq 0.2;\\ 0.8, & 0.2<\nu\leq 0.5;\\ 0, & 0.5<\nu\leq 1.\end{cases}$$

Then, we build an IFF $(R(\mu),T(\nu))$, which is denoted as $L(X)$, by utilizing the piecewise continuous functions $R(\mu)$ and $T(\nu)$. In fact, the IFF $L(X)$ does not always meet these conditions: $0\leq R(\mu),T(\nu)\leq 1$ and $0\leq R(\mu)+T(\nu)\leq 1$, which are given to define the concept of IFFs in Section 3. It means that these conditions are not always right for any IFN (μ,ν). However, in Refs. [25] and [33], we investigated the situation where we deal with this kind of general IFFs, which does not always meet those conditions that an IFF must need in Section 3. Meanwhile, we also studied the integrals of this kind of general IFFs, and got that their integrals are not different from those defined in Section 3 if only some conditions can be satisfied [25,33]. In addition, the calculation formulas of integrals of general IFFs are also the same as those introduced in Section 3. Thus, we have the following results [33]:

(1) $\int_O^\beta L(X)dX = IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$, where $O=(0,1)$ and $\beta=\left(\max_i\{\mu_i\},\min_i\{\nu_i\}\right)$;

(2) $\wr_E^\gamma L(X)lX = IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$, where $E=(1,0)$ and $\gamma=\left(\min_i\{\mu_i\},\max_i\{\nu_i\}\right)$.

Based on $\overline{\int_\alpha^\beta \varphi(X)dX} = \wr_{\bar{\alpha}}^{\bar{\beta}} \bar{\varphi}(\bar{X})l\bar{X}$ and $\overline{IFWA_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)} = IFWG_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)$, we first get that the right-hand of (2), that is $IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$, can be transformed as:

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$$IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)=\overline{IFWA_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)}$$

Based on $\int_O^\beta L(X)dX = IFWA_\omega\left(\alpha_1,\alpha_2,\cdots,\alpha_n\right)$, we get

$$\int_O^\beta L(\bar{X})d\bar{X} = IFWA_\omega\left(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n\right)$$

and then

$$\wr_E^\gamma \bar{L}(X)lX = \overline{\int_O^\beta L(\bar{X})d\bar{X}}$$
$$= \overline{IFWA_\omega(\bar{\alpha}_1,\bar{\alpha}_2,\cdots,\bar{\alpha}_n)} = IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

which contradicts (2), i.e., $\wr_E^\gamma L(X)lX = IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$. Now, there is a question: Which one of these is right? In what follows, we will show that they are both correct:

After restudying the special IFF $L(X)=\left(R(\mu),T(\nu)\right)$, we get

$$L(X)=\left(R(\mu),T(\nu)\right)=\left(\sum_{\mu_i>\mu}\omega_i,\sum_{\nu_i\ge\nu}\omega_i\right)$$

Then there is $\overline{L(X)}=\overline{\left(\sum_{\mu_i>\mu}\omega_i,\sum_{\nu_i\ge\nu}\omega_i\right)}=\left(\sum_{\nu_i\ge\nu}\omega_i,\sum_{\mu_i>\mu}\omega_i\right)=L(\bar{X})$.

In addition, we know $\overline{L(X)}=\bar{L}(\bar{X})$. Hence, $L(\bar{X})=\bar{L}(\bar{X})$ which means $L=\bar{L}$, and therefore,

$$\wr_E^\gamma \bar{L}(X)lX = IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

and

$$\wr_E^\gamma L(X)lX = IFWG_\omega(\alpha_1,\alpha_2,\cdots,\alpha_n)$$

are both correct.

After expressing the IFWA and IFWG operators as the integrals of the IFF $L(X)$, we analyze $\iint_D P(\mu,\nu)\left(\mu,\nu\right)d\delta$ and $\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta}$ in a similar way: As we know, $\iint_D P(\mu,\nu)\left(\mu,\nu\right)d\delta$ and $\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta}$ are the continuous forms of the IFWA and IFWG operators, respectively. If we want to find the relationships among "$\iint_D\bullet$", "$\wr\wr_D\bullet$", "$\int_\alpha^\beta\bullet$" and "$\wr_\alpha^\beta\bullet$", we need to get the continuous form of $L(X)$. In fact, we studied it in Ref. [33]:

Let $P(\rho,\sigma)$ be a weight density function defined in the region D, then we get the following two subsets D_μ and D_ν of D:

$$D_\mu=\left\{\left(\rho,\sigma\right)\mid\left(\rho,\sigma\right)\in D \; and \; \rho\ge\mu\right\}$$

Based on the definition of "D_μ", we get that D_μ is actually a subset of D, the elements of which are on the right side of $\rho=\mu$ and belong to the region D. Then we express D_ν as:

$$D_\nu=\left\{\left(\rho,\sigma\right)\mid\left(\rho,\sigma\right)\in D \; and \; \sigma\ge\nu\right\}$$

which shows that D_ν is also a subset of D, and all of its elements are on top of $\sigma=\nu$, and belong to the region D. By utilizing D_μ and D_ν, we can define two real functions $\mathcal{R}(\mu)$ and $\mathcal{T}(\nu)$ as:

$$\mathcal{R}(\mu)=\iint_{D_\mu}P(\rho,\sigma)d\rho d\sigma \text{ and } \mathcal{T}(\nu)=\iint_{D_\nu}P(\rho,\sigma)d\rho d\sigma$$

which are the double integrals of $P(\rho,\sigma)$ located in the regions D_μ and D_ν, respectively. Obviously, we can also get $\mathcal{R}:[0,1]\to[0,1]$ and $\mathcal{T}:[0,1]\to[0,1]$, which are very similar to the above functions R and T.

Based on $\mathcal{R}(\mu)$ and $\mathcal{T}(\nu)$, we can construct an IFF $\left(\mathcal{R}(\mu),\mathcal{T}(\nu)\right)$, which is denoted as $\mathcal{L}(X)$. With the IFF $\mathcal{L}(X)$, we showed the following results in Ref. [33]:

(1) $\iint_D P(\mu,\nu)\left(\mu,\nu\right)d\delta = \int_O^\beta \mathcal{L}(X)dX$;

(2) $\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta} = \wr_E^\gamma \mathcal{L}(X)lX$.

According to $\overline{\int_\alpha^\beta \varphi(X)dX} = \wr_{\bar{\alpha}}^{\bar{\beta}}\bar{\varphi}(\bar{X})l\bar{X}$ and $\overline{\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta}} = \iint_{\bar{D}}\bar{P}(\mu,\nu)\left(\mu,\nu\right)d\delta$, we can get

$$\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta} = \overline{\iint_{\bar{D}}\bar{P}(\mu,\nu)\left(\mu,\nu\right)d\delta}$$

Since $\iint_D P(\mu,\nu)\left(\mu,\nu\right)d\delta = \int_O^\beta \mathcal{L}(X)dX$, then there must be

$$\iint_{\bar{D}}\bar{P}(\mu,\nu)\left(\mu,\nu\right)d\delta = \int_O^\beta \mathcal{L}(\bar{X})d\bar{X}$$

Also, we can know

$$\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta} = \overline{\iint_{\bar{D}}\bar{P}(\mu,\nu)\left(\mu,\nu\right)d\delta} = \overline{\int_O^\beta \mathcal{L}(\bar{X})d\bar{X}} = \wr_E^\gamma \bar{\mathcal{L}}(X)lX$$

which contradicts (2), that is $\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta} = \wr_E^\gamma \mathcal{L}(X)lX$. However, both the two results are correct because $\bar{\mathcal{L}}=\mathcal{L}$.

In this section, we have shown the relationships among the integrals of IFFs, which can be summarized as:

(1) $\overline{\int\varphi(X)dX} = \wr\bar{\varphi}(\bar{X})l\bar{X}$;

(2) $\overline{\int_\alpha^\beta\varphi(X)dX} = \wr_{\bar{\alpha}}^{\bar{\beta}}\bar{\varphi}(\bar{X})l\bar{X}$;

(3) $\overline{\iint_D P(\mu,\nu)\left(\mu,\nu\right)d\delta} = \wr\wr_{\bar{D}}\left(\mu,\nu\right)^{\bar{P}(\mu,\nu)d\delta} = \wr\wr_D\overline{\left(\mu,\nu\right)}^{P(\mu,\nu)d\delta}$;

(4) $\overline{\wr\wr_D\left(\mu,\nu\right)^{P(\mu,\nu)d\delta}} = \iint_{\bar{D}}\bar{P}(\mu,\nu)\left(\mu,\nu\right)d\delta = \iint_D P(\mu,\nu)\overline{\left(\mu,\nu\right)}d\delta$;

(5) $\bar{L}=L$ and $\bar{\mathcal{L}}=\mathcal{L}$.

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V. Concluding Remarks

In this paper, we have shown in detail the relationships among subtraction derivatives and division derivatives under intuitionistic fuzzy environment. We have also proven the fact there are some closed connections between the two intuitionistic fuzzy calculus theories based on the subtraction derivatives and the division derivatives respectively, and realized the essential unification between them. The specific contents of the unification involve the derivatives, differentials, indefinite integrals and definite integrals in the intuitionistic fuzzy calculus. Specially, this paper has successfully unified the two fundamental theorems of the calculus that are respectively located in the core of two kinds of intuitionistic fuzzy calculus theories, which represents the realization of the

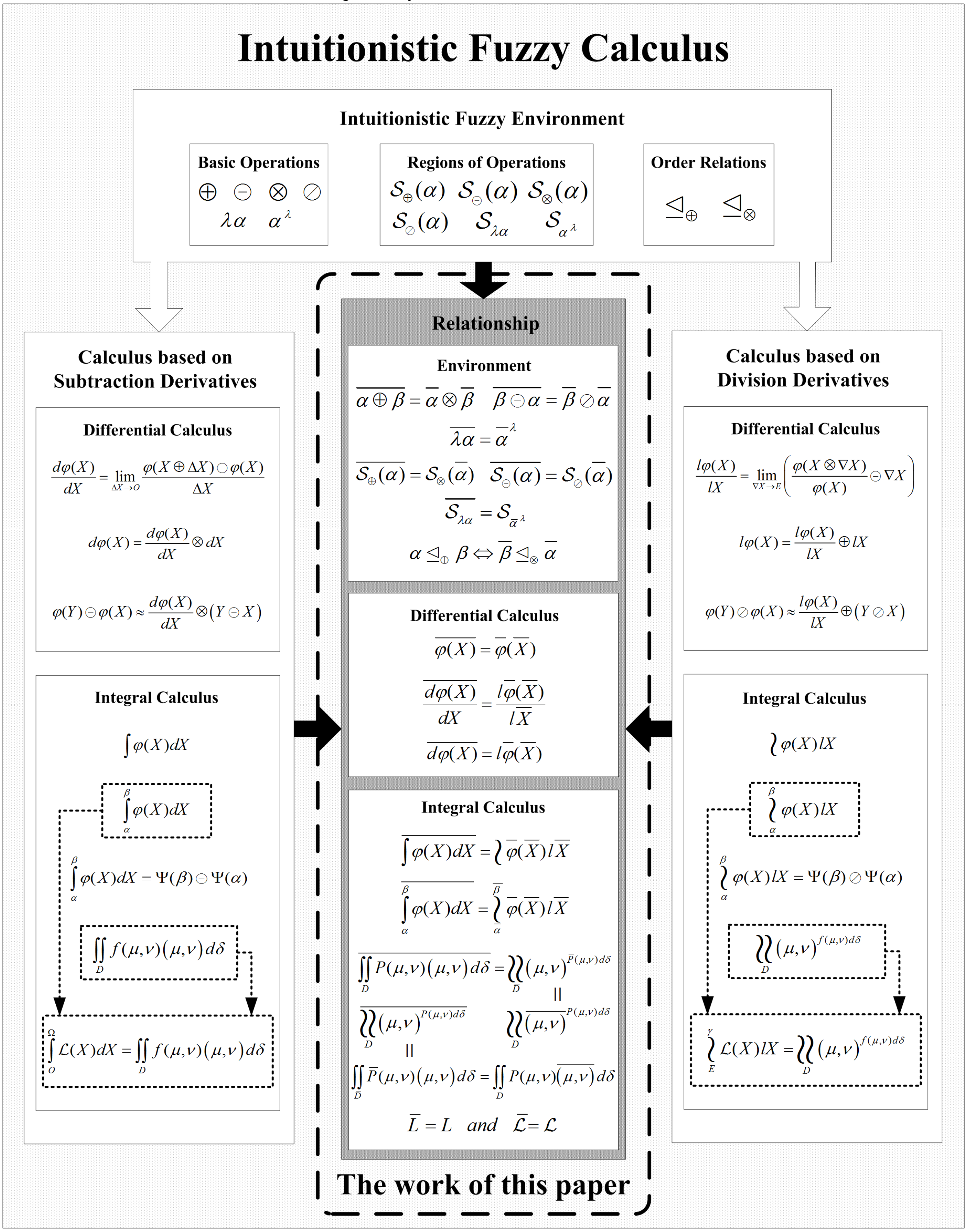

Figure 5.1. The organizational structure of this paper

unification work. In addition, this paper has also studied the relationships among a few aggregation operators of IFNs, such as IFWA, IFWG, $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\!\wr_D (\mu,\nu)^{P(\mu,\nu)d\delta}$. Finally, we have also proven an important fact that $\bar{L}=L$ and $\bar{\mathcal{L}}=\mathcal{L}$. In a word, we attempt to reveal the fact that any statement or conclusion in the intuitionistic fuzzy calculus based on the subtraction derivatives must have a counterpart in the calculus based on the division derivatives. As we know that the definite integrals of intuitionistic fuzzy calculus are usually used to build aggregation operations to deal with continuous or a large number of intuitionistic fuzzy numbers (or information). However, according to these relationships revealed in this paper, we can discover a fact that it is unnecessary to utilize two related operations, like $\iint_D P(\mu,\nu)(\mu,\nu)d\delta$ and $\wr\!\wr_D (\mu,\nu)^{P(\mu,\nu)d\delta}$, to aggregate information because we can utilize one of the two operations and the complement of IFNs to fully realize the work of another aggregation operation.

In order to show the content of this paper more clearly, we draw the diagram of this paper's organizational structure in Fig. 5.1.

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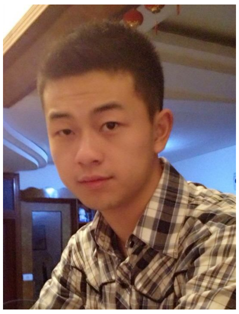

Qian Lei is currently a master student at the College of Sciences, PLA University of Science and Technology, Nanjing, China.

His current research interests include aggregation operators, and fuzzy sets.

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Zeshui Xu (M'08-SM'09) received the Ph.D. degree in management science and engineering from Southeast University, Nanjing, China, in 2003. From April 2003 to May 2005, he was a Postdoctoral Researcher with School of Economics and Management, Southeast University. From October 2005 to December 2007, he was a Postdoctoral Researcher with School of Economics and Management, Tsinghua University, Beijing, China. He is a Distinguished Young Scholar of the National Natural Science Foundation of China and the Chang Jiang Scholars of the Ministry of Education of China. He is currently a Professor with the Business School, Sichuan University, Chengdu, and also with the College of Sciences, PLA University of Science and Technology, Nanjing. He has been selected as a Thomson Reuters Highly Cited Researcher (in the fields of Computer Science (2014, 2015) and Engineering (2014), respectively), included in The World's Most Influential Scientific Minds 2014, and also the Most Cited Chinese Researchers (Ranked first in Computer Science, 2014, Released by Elsevier). His h-index is 92, and has authored the following books: *Uncertain Multiple Attribute Decision Making: Methods and Applications* (Springer, 2015), *Intuitionistic Fuzzy Information Aggregation: Theory and Applications* (Science Press and Springer, 2012), *Linguistic Decision Making: Theory and Methods* (Science Press and Springer, 2012), *Intuitionistic Fuzzy Preference Modeling and Interactive Decision Making* (Springer, 2013), *Intuitionistic Fuzzy Aggregation and Clustering* (Springer, 2013), and *Hesitant Fuzzy Sets Theory* (Springer, 2014). He has contributed more than 480 journal articles to professional journals. His current research interests include information fusion, group decision making, computing with words, and aggregation operators.