Exercise 1.2

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Proof of Equation in Exercise 1.2

$$f'''(x_k) = \frac{f(x_{k+2}) - 2f(x_{k+1}) + 2f(x_{k-1}) - f(x_{k-2})}{2h^3} + \mathcal{O}(h^2)$$

We use Taylor expansions around x_k evaluated at neighboring points:

$$\begin{aligned}
f(x_{k+1}) &= f(x_k) + hf'(x_k) + \frac{h^2}{2}f''(x_k) + \frac{h^3}{6}f'''(x_k) + \mathcal{O}(h^4) \\
f(x_{k-1}) &= f(x_k) - hf'(x_k) + \frac{h^2}{2}f''(x_k) - \frac{h^3}{6}f'''(x_k) + \mathcal{O}(h^4) \\
f(x_{k+2}) &= f(x_k) + 2hf'(x_k) + 2h^2f''(x_k) + \frac{4h^3}{3}f'''(x_k) + \mathcal{O}(h^4) \\
f(x_{k-2}) &= f(x_k) - 2hf'(x_k) + 2h^2f''(x_k) - \frac{4h^3}{3}f'''(x_k) + \mathcal{O}(h^4)
\end{aligned}$$

We consider the following combination (found in the numerator)

$$f(x_{k+2}) - 2f(x_{k+1}) + 2f(x_{k-1}) - f(x_{k-2})$$

Substituting the Taylor expansions:

$$\begin{aligned}
&\left[f(x_k) + 2hf'(x_k) + 2h^2f''(x_k) + \frac{4h^3}{3}f^{(3)}(x_k)\right] \\
&\quad - 2\left[f(x_k) + hf'(x_k) + \frac{h^2}{2}f''(x_k) + \frac{h^3}{6}f^{(3)}(x_k)\right] \\
&\quad + 2\left[f(x_k) - hf'(x_k) + \frac{h^2}{2}f''(x_k) - \frac{h^3}{6}f^{(3)}(x_k)\right] \\
&\quad - \left[f(x_k) - 2hf'(x_k) + 2h^2f''(x_k) - \frac{4h^3}{3}f^{(3)}(x_k)\right]
\end{aligned}$$

We compare the coefficients of each term:

- The $f(x_k)$ terms: $1 - 2 + 2 - 1 = 0$
- The $f'(x_k)$ terms: $2h - 2h - 2h + 2h = 0$
- The $f''(x_k)$ terms: $2h^2 - h^2 - h^2 + 2h^2 = 0$
- The $f^{(3)}(x_k)$ terms:

$$\frac{4h^3}{3} - 2 \cdot \frac{h^3}{6} - 2 \cdot \frac{h^3}{6} + \left(-\left(-\frac{4h^3}{3}\right)\right) = 2h^3$$

and the expression reduces to:

$$f(x_{k+2}) - 2f(x_{k+1}) + 2f(x_{k-1}) - f(x_{k-2}) = 2h^3 f^{(3)}(x_k) + \mathcal{O}(h^5)$$

From this we obtain:

$$f^{(3)}(x_k) = \frac{f(x_{k+2}) - 2f(x_{k+1}) + 2f(x_{k-1}) - f(x_{k-2})}{2h^3} + \mathcal{O}(h^2)$$

$\square$