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June Exam
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Question 1: Coins

In a box there are $n+1$ coins. Coin i shows heads with probability $\frac{i}{n}$, $i = 0, \ldots, n$. You choose a coin at random and flip it.

Let A_i be the event of choosing coin i and K be the event of getting heads. It is given that $\mathbb{P}(K \mid A_i) = \frac{i}{n}$ and $\mathbb{P}(A_i) = \frac{1}{n+1}$.

a) [10] Show that the probability of getting heads is $\frac{1}{2}$.

Since the events A_i are disjoint and their union gives us the sample space, by the law of total probability we have,

$$\mathbb{P}(K) = \sum_{i=0}^{n} \mathbb{P}(K \mid A_i)\mathbb{P}(A_i) = \sum_{i=0}^{n} \frac{i}{n}\frac{1}{n+1} = \frac{1}{n(n+1)}\frac{n(n+1)}{2} = \frac{1}{2}\,. \tag{1}$$

b) [10] Given that heads came up, calculate the probability that you chose coin i.

By Bayes' theorem,

$$\mathbb{P}(A_i \mid K) = \frac{\mathbb{P}(K \mid A_i)\,\mathbb{P}(A_i)}{\mathbb{P}(K)} = \frac{\frac{i}{n}\frac{1}{n+1}}{\frac{1}{2}} = \frac{2i}{n(n+1)}\,. \tag{2}$$

(20 points)

Question 2: Uniform Distribution [20]

Let X_1, X_2, X_3 be independent random variables that follow the uniform distribution on the interval $[0, 1]$. The probability density function of the uniform distribution equals 1 for values of X in $[0, 1]$ and 0 elsewhere. Calculate the probability that the largest of the three is smaller than the sum of the other two.

The event whose probability we want to calculate can be written as,

$$\begin{aligned} A = &\{X_1 < X_2 + X_3\,,\ X_1 > X_2\,,\ X_1 > X_3\} \cup \\ &\{X_2 < X_1 + X_3\,,\ X_2 > X_1\,,\ X_2 > X_3\} \cup \\ &\{X_3 < X_1 + X_2\,,\ X_3 > X_1\,,\ X_3 > X_2\} \\ = &\ A_1 \cup A_2 \cup A_3\,. \end{aligned} \tag{3}$$

Since the sets are disjoint, we have

$$\mathbb{P}(A) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + \mathbb{P}(A_3) = 3\mathbb{P}(A_1)\,, \tag{4}$$

where the last equality holds due to the symmetry of the problem. We will calculate the probability of event A_1.

For each of the random variables, we have that the largest of all, X_1 here, can belong anywhere in $[0, 1]$. X_2 is certainly smaller than X_1 and greater than 0, so it belongs to the interval $[0, X_1]$. For X_3 we have that $X_3 > X_1 - X_2$ and $X_3 < X_1$, so it belongs to the interval $[X_1 - X_2, X_1]$. Therefore,

$$\mathbb{P}(X_1 < X_2 + X_3) = \int_0^1 \int_0^{x_1} \int_{x_1 - x_2}^{x_1} 1\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 = \ldots = \frac{1}{6}\,. \tag{5}$$

Therefore, $\mathbb{P}(A) = \frac{1}{2}$.

Information

- A total of 40 points are given. The maximum score is 40.
- The duration of the exam is 150 minutes.
- Justify your answers clearly. Answers without justification are not considered correct.
- Solve the problems on draft paper and present the final solutions neatly written. Answers with scribbles will not be graded.
- Give consolidated answers for each exercise. If you proceed to the next without having completed the previous one, leave sufficient blank space in case you want to return.