Singular Value Decomposition

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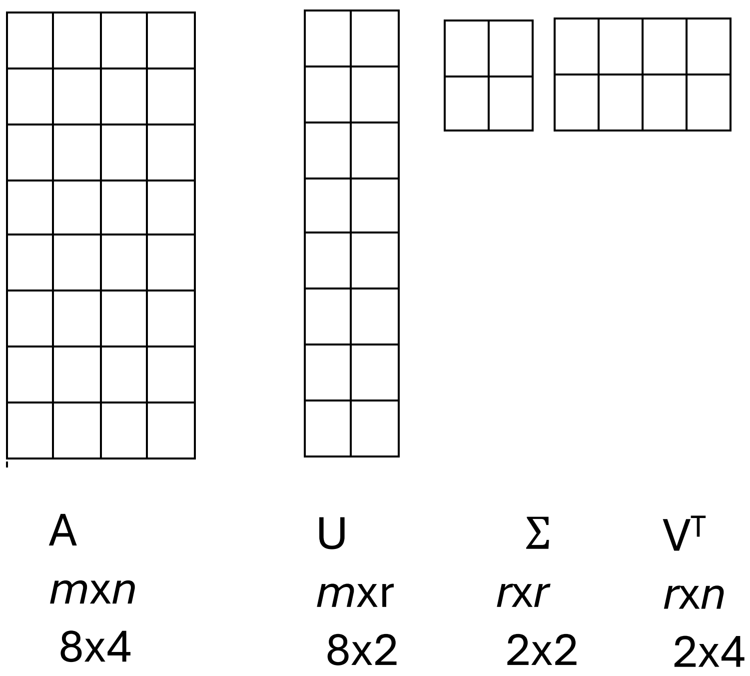
Singular Value Decomposition (SVD)

- Let A be an m × n matrix. Then singular value decomposition of A is **A = U ∑ V** ^{T}
- In the "thin/economy version" of SVD:
  - U is an mxr **orthogonal** matrix and is called the **left singular vectors**.
  - ∑ is rxr and **diagonal** and contains **singular values** of A in descending order, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$
  - V is an nxr **orthogonal** matrix and is called the **right singular vectors**.

Economy SVD

$$A = [\, U_r \quad U_{m-r} \,] \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_r^T \\ V_{n-r}^T \end{bmatrix} = U_r D V_r^T$$



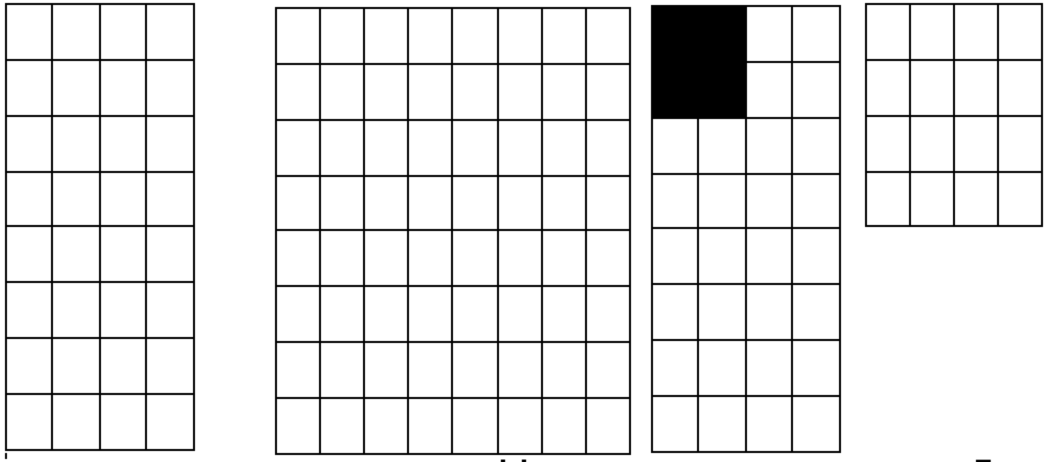
Economy Version of SVD

When ∑ contains rows or columns of zeros, a more compact decomposition of A is possible. Partition the matrices as follow

$$U = [\, U_r \quad U_{m-r} \,], \quad \text{where } U_r = [\, \mathbf{u}_1 \quad \cdots \quad \mathbf{u}_r \,]$$
$$V = [\, V_r \quad V_{n-r} \,], \quad \text{where } V_r = [\, \mathbf{v}_1 \quad \cdots \quad \mathbf{v}_r \,]$$

$$A = [\, U_r \quad U_{m-r} \,]\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_r^T \\ V_{n-r}^T \end{bmatrix} = U_r D V_r^T$$

Regular SVD



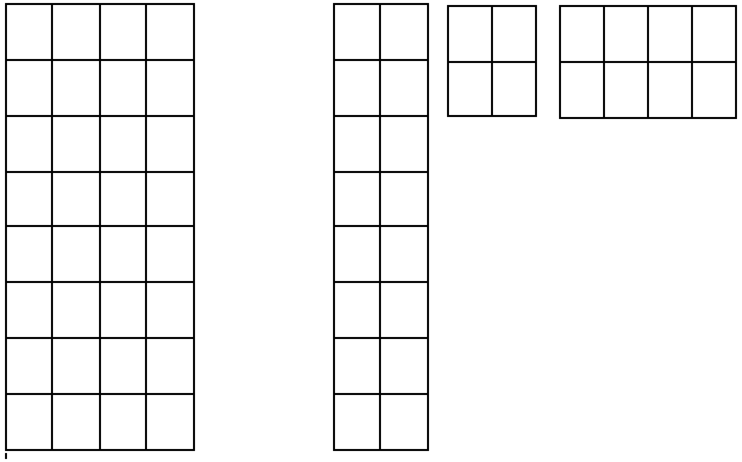
A
mxn
8x4

U
mxm
8x8

Σ
mxn
8x4

V^T
nxn
4x4

Economy SVD



A
mxn
8x4

U
*mx*r
8x2

Σ
rxr
2x2

V^T
rxn
2x4

A Conceptual Example

- An audiobook company likes to make recommendations to users based on the their ratings of other books.
- A: mxn where m rows denote the users and n columns denote the book titles.
- The first 3 columns include titles in historical drama genre and the last 2 column include title in Sci-Fi genre.
- A can be thought of as “user to book” mapping.

	Poisonwood Bible	The Revenant	Ines of my Soul	The Martian	Lord of the Rings
Susan	0	1	0	2	2
Vu	0	0	0	5	5
Joe	0	2	0	4	4
Rafa	5	5	5	0	0
Jose	4	4	4	0	0
Ana	3	3	3	0	0
Mike	1	1	1	0	0

	Poisonwood Bible	The Revenant	Ines of my Soul	The Martian	Lord of the Rings
Susan	0	1	0	2	2
Vu	0	0	0	5	5
Joe	0	2	0	4	4
Rafa	5	5	5	0	0
Jose	4	4	4	0	0
Ana	3	3	3	0	0
Mike	1	1	1	0	0

Historical Drama

Sci-Fi

$$A = U\Sigma V^T$$

U: rows: Users- cols: Concepts

V: rows: Books-cols: Concepts

V^T: rows: concepts-cols: Books

$$= \begin{bmatrix} 0.07 & -0.29 & 0.33 \\ 0.07 & -0.73 & -0.68 \\ 0.15 & -0.60 & 0.65 \\ 0.68 & 0.12 & -0.05 \\ 0.55 & 0.09 & -0.04 \\ 0.41 & 0.07 & -0.03 \\ 0.14 & 0.02 & -0.01 \end{bmatrix} \begin{bmatrix} 12.5 & 0 & 0 \\ 0 & 9.5 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \begin{bmatrix} 0.56 & 0.59 & 0.56 & 0.09 & 0.09 \\ 0.13 & -0.03 & 0.13 & -0.69 & -0.69 \\ 0.41 & -0.80 & 0.41 & 0.09 & 0.09 \end{bmatrix}$$

User to Concept Mapping

Concept Strength

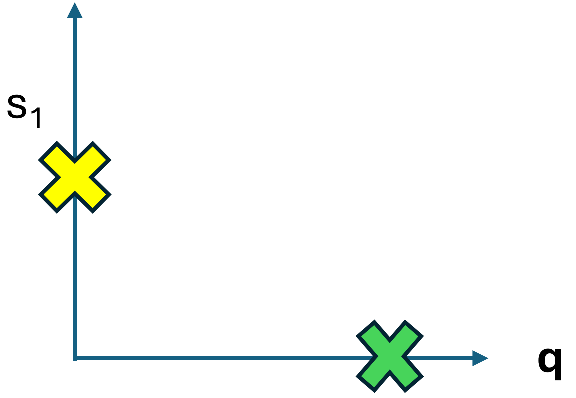
Concept to Book Mapping

We factored the matrix, so what?!

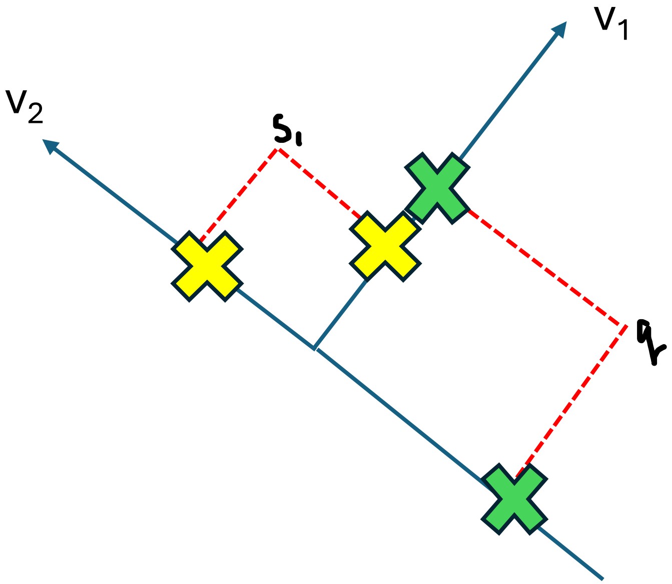
- The goal is to make audiobook recommendations. (say to user S_1)
- e.g. Recommend the Poisonwood Bible (**q**=[5 0 0 0 0]) to people who like this concept/genre (i.e. have had high rating for The Reverent and Ines of My Soul), but haven't read this book yet ($\boldsymbol{S_1} = [0\ 3\ 5\ 0\ 0\ 0]$)
- We need to see how similar S1 is to the query point q. But comparing the two vectors as they are have zero similarity.
- However, if we use the concept space things will be more meaningful. Therefore, we will map these vectors to the concept space spanned by columns of V.

Example

- Map (project) q to the concept space.



- $q * v = [5\ 0\ 0\ 0\ 0] \begin{bmatrix} 0.56 & 0.13 \\ 0.59 & -0.03 \\ 0.56 & 0.13 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix}$=[-2.8 -0.63]



- $s_1 * v = [0\ 3\ 5\ 0\ 0] \begin{bmatrix} 0.56 & 0.13 \\ 0.59 & -0.03 \\ 0.56 & 0.13 \\ 0.09 & -0.69 \\ 0.09 & -0.69 \end{bmatrix}$=[-4.9 -0.55]