Learning Objectives

- Characteristic Equation
- Find eigenvalues from the characteristic equation
- Find eigenvalues of a triangular matrix

Finding Eigenvalues: Example

- Find the eigenvalues of A=$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$
- Let's go to the definition of eigenvalues and eigenvectors:$Ax = \lambda x$ where λ is a scalar and $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - 6x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + 3x_2 - \lambda x_1 \\ 3x_1 - 6x_2 - \lambda x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} (2-\lambda)x_1 + 3x_2 \\ 3x_1 + (-6-\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example – cont.

- Let's think about this system of equations geometrically. Each equation represents a line in 2D that passes through the origin.
- The only way for the two lines to have a nontrivial solution(s) is for the two lines to overlap.
- This requires the ratio of the coefficients of the two lines to be the same which means that the determinant of $\begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$ has to be zero. (This matrix has to be **non-invertible**).
- Recall $\det\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$
- Therefore $(2-\lambda)\,(-6-\lambda) - (3)(3) = 0$ => $\lambda^2 + 4\lambda - 21 = 0$
- The above equation is called **Characteristic Equation.**
- Solving for lambda results in values of 3 and -7.

Characteristic Equation

- Given a square matrix A, one can obtain the eigenvalues of A from the Characteristic Equation:

$$\mathbf{det}(\boldsymbol{A} - \boldsymbol{\lambda I}) = \mathbf{0}$$

- Note that the characteristic equation transforms the matrix equation $(A - \lambda I)x = 0$ which involves two unknowns (λ and x) into the scalar equation which involves only one unknown (λ).

Finding Eigenvectors Example

- Find Eigenvectors of A=$\begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.
- Recall that eigenvalues of A were λ=-7 and λ=3.
- Eigenvectors can be found by plugging in the eigenvalues in either of two equations:
- $\begin{bmatrix} (2-\lambda)x_1 + 3x_2 \\ 3x_1 + (-6-\lambda)x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- $\boldsymbol{\lambda}$**=3**$\overset{\textbf{top eq.}}{\Longrightarrow} -x_1 + 3x_2 = 0$
- $x_1 = 3x_2$
- $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $x_2 = 1 \Rightarrow x_1 = 3$
- X=$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or any of its multiples
- $\begin{bmatrix} 0.9487 \\ 0.3162 \end{bmatrix}$ normal eigenvector corresponding to λ=3

- $\boldsymbol{\lambda}$**=-7**$\overset{\textbf{top eq.}}{\Longrightarrow} 9x_1 + 3x_2 = 0$
- $x_1 = -\frac{x_2}{3}$
- $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$
- $x_2 = 1 \Rightarrow x_1 = -\frac{1}{3}$
- X=$\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$ or any of its multiples
- Normalizing the above vector will result in:

$$\begin{bmatrix} \frac{1}{3\sqrt{(1+\frac{1}{9})}} \\ \frac{1}{\sqrt{(1+\frac{1}{9})}} \end{bmatrix} = \begin{bmatrix} -0.3162 \\ 0.9487 \end{bmatrix}$$ normal eigenvector corresponding to λ=-7

Finding Eigenvalues and Eigenvectors in Python

```
from numpy import linalg as LA
import numpy as np
```

```
A = np.array([[2,3],[3,-6]])
print(A)
```

```
[[ 2  3]
 [ 3 -6]]
```

```
l, ev = LA.eig(A)
print('Eigen values are:', l)
print('Eigen vectors are:' , ev[:,0], 'and', ev[:,1])
```

```
Eigen values are: [ 3. -7.]
Eigen vectors are: [0.9486833  0.31622777] and [-0.31622777  0.9486833 ]
```

Eigenvalue of a Triangular Matrix

- **Theorem 1:** The eigenvalues of a triangular matrix are the entries on its main diagonal.

- Example: What are the eigen $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$.

- The eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 and 1.