Learning Objectives

- Define eigenvalue and eigenvectors.
- Given an eigenvalue, find the parametric form of the eigenvector.
- Define eigenspace and spectrum

Eigenvectors and Eigenvalues

- **Definition:** An **eigenvector** of an $n \times n$ matrix A is a nonzero vector **x** such that $A\mathrm{x} = \lambda\mathrm{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution **x**. Such an **x** is called an *eigenvector corresponding to λ*.

- λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation $A\mathrm{x} = \lambda\mathrm{x}$

$$(A - \lambda I)\mathrm{x} = 0$$

has a nontrivial solution.

$$\text{e.g. } \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x1 \\ x2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

- Show that 7 is an eigenvalue of matrix

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and find the corresponding eigenvectors.

Solution

- **Solution:** The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathrm{x} = 7\mathrm{x} \qquad \text{----(2)}$$

  has a nontrivial solution.

- But (2) is equivalent to $A\mathrm{x} - 7\mathrm{x} = 0$, or

$$(A - 7I)\mathrm{x} = 0 \qquad \text{----(3)}$$

- To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

Solution

- The columns of $A - 7I$ are obviously linearly dependent, so (3) has nontrivial solutions.
- To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} : \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

```
import numpy as np
import sympy as sp
A = np.array([[-6,6,0],[5,-5,0]])
A_rref = sp.Matrix(A).rref()
A_rref[0]
```

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$
- $x_1 - x_2 = 0 \Rightarrow x_1 = x_2$
- The general solution has the parametric of form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
- Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to

  $\lambda = 7$

Eigenspace and Spectrum

- Let A be an nxn matrix and let λ be an eigenvalue of A. This implies that the equation $(A - \lambda I)\mathrm{x} = 0$ has a nontrivial solution.
- The set of all solutions of above equation is a subspace of $\mathbb{R}^n$ and is called the eigenspace of A corresponding to λ.
- Therefore, the collection of all eigenvectors corresponding to λ, together with the zero vector is called the ***eigenspace***.
- The set of all eigenvalues of a matrix A is called the ***spectrum***

Example

- **Example 2:** Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.
- **Solution:** Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathrm{x} = 0$

Solution

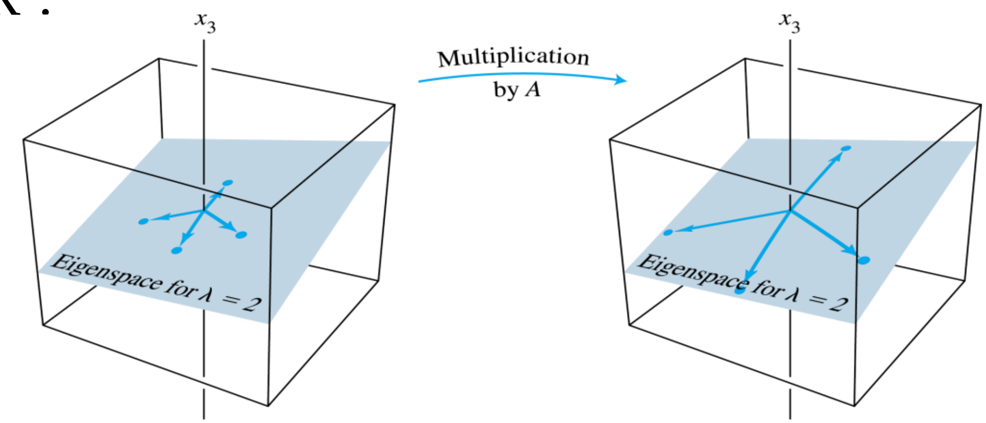
$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} : \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A-2I)\mathrm{x}=0$ has free variables.
- $2x_1$-$x_2 + 6x_3 = 0$ => $x_1 = \frac{1}{2}x_2 - 3x_3$
- The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$, x_2 and x_3 free.

Solution

- The eigenspace, shown in the following figure, is a two-dimensional subspace of R^3.



A acts as a dilation on the eigenspace.

- Any multiple of $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ would be a basis. E.g. $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$