Example

- Write the following set of equations in the matrix form:

$$\begin{cases} x_1 + x_2 = 3 \\ 0x_1 + x_2 = 1 \end{cases}$$

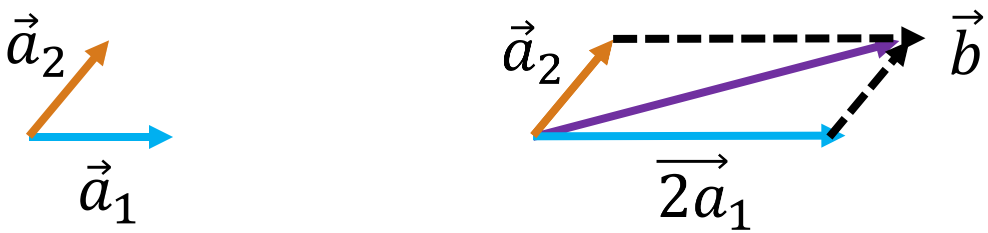
- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$
- $x = A^{-1}b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- Let's look at the geometrical interpretation of this…

Geometry of Least Square

- Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and vector

$\vec{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

- Is it possible to write $\vec{b}$ as a linear combinations $\vec{a}_1$ and $\vec{a}_2$, where $\vec{a}_1$ and $\vec{a}_2$ are columns of matrix A?

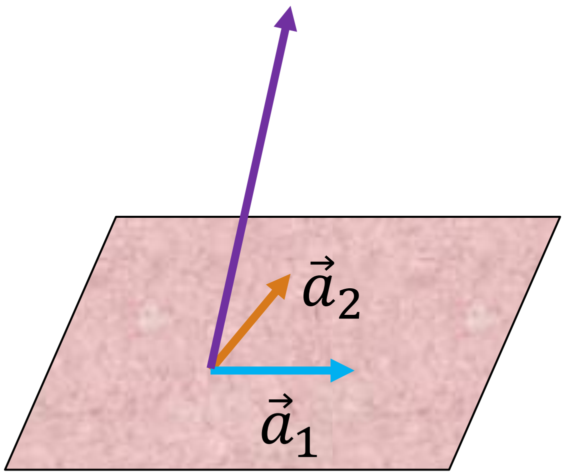


- Yes! Because $\vec{b}$ on the plane that is spanned by $\vec{a}_1$ and $\vec{a}_2$. So you can find some value of x_1 and x_2 for which $\boldsymbol{x_1\vec{a}_1 + x_1\vec{a}_2}= \boldsymbol{\vec{b}}$.

- In this case $2\boldsymbol{\vec{a}_1} + \boldsymbol{1\vec{a}_2}= \boldsymbol{\vec{b}}$.

Geometry of Least Square

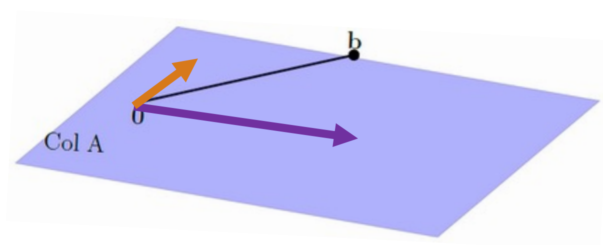
- How about this case?

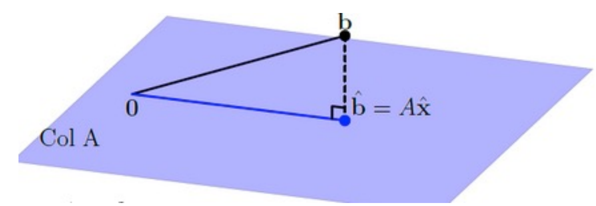


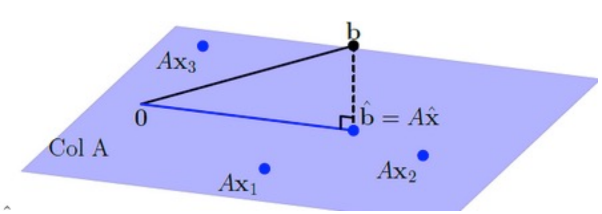
- No! Because $\vec{b}$ does not lie on the plane that is spanned by $\vec{a}_1$ and $\vec{a}_2$. In other words, $\vec{b}$ does not lie in the column space of A.

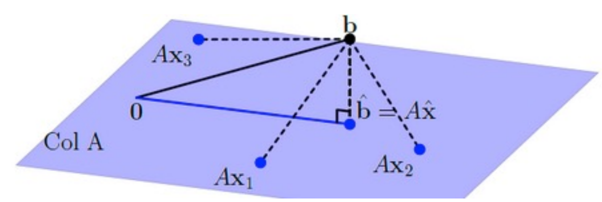
Least Square Problem

- Consider Ax=b where the systems is inconsistent. i.e. there doesn’t exist a solution x that satisfies this matrix equation.
- When a solution is demanded and none exists, the best one can do is to find an x that makes **Ax as close as possible to b.**
- Think of Ax as an approximation to b. The smaller the distance between b and Ax given by ||b-Ax||, the better the approximation.



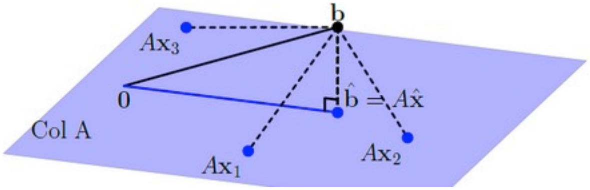




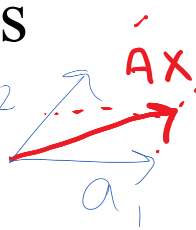


LEAST-SQUARES PROBLEMS

- **Definition:** If A is $m \times n$ and **b** is in $\mathbb{R}^m$, a **least-squares solution** of $A\mathrm{x} = \mathrm{b}$ is an $\hat{\mathrm{x}}$ in $\mathbb{R}^n$ such that
  $$\|\mathrm{b} - A\hat{\mathrm{x}}\| \le \|\mathrm{b} - A\mathrm{x}\|$$
  for all **x** in $\mathbb{R}^n$.



- Note that vector Ax is a linear combinations of columns of A. (Recall the 2nd version of matrix-vector multiplication definition).
- This implies that no matter what x we select, the vector Ax will be on the plane spanned by columns of A (column space of A)

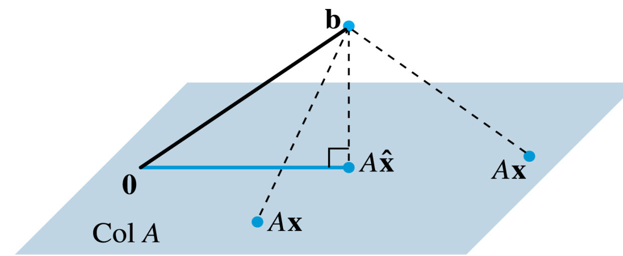


$A = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad Ax = x_1 a_1 + x_2 a_2$

Shortest Distance

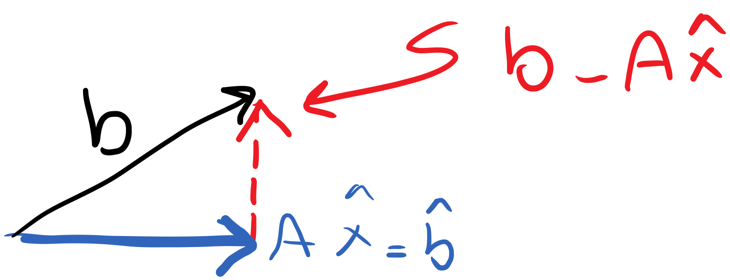
- So we seek an x that makes Ax the closest point in Col A to b in $A\mathrm{x} = \mathrm{b}$
- Recall that the shortest distance is

the orthogonal distance.



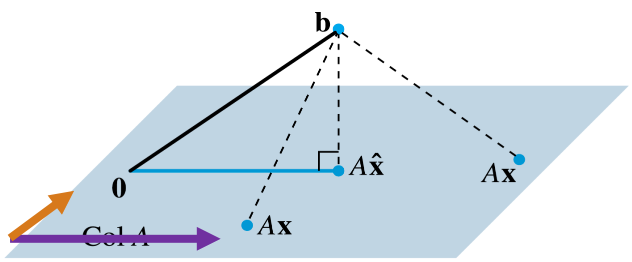
The vector **b** is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other **x**.

- Therefore, the dotted line that is perpendicular to the plane will result in smallest distance between b and Ax.
- What's the expression for the dotted line?



Shortest Distance

- If a vector is orthogonal to a plane it'll be orthogonal to any vector on that plane, including columns of matrix A, namely a_1 and a_2.
- So $b - A\hat{x}$ is orthogonal to each column of A.
- If $\mathbf{a}_j$ is any column of A, then $a_j \cdot (b - A\hat{x}) = 0$, and $a_j^T(b - A\hat{x}) = 0$.



The vector **b** is closer to $A\hat{\mathbf{x}}$ than to $A\mathbf{x}$ for other **x**.

SOLUTION OF THE GENREAL LEAST-SQUARES PROBLEM

- Since each a_j^T is a row of A^T,

$$A^T(\mathrm{b} - A\hat{\mathrm{x}}) = 0 \qquad (2)$$

- Thus

$$A^T\mathrm{b} - A^T A\hat{\mathrm{x}} = 0$$

$$A^T A\hat{\mathrm{x}} = A^T\mathrm{b}$$

- These calculations show that each least-squares solution of $A\mathrm{x} = \mathrm{b}$ satisfies the equation

$$A^T A\mathrm{x} = A^T\mathrm{b} \qquad (3)$$

- The matrix equation (3) represents a system of equations called the **normal equations** for $A\mathrm{x} = \mathrm{b}$.
- A solution of (3) is often denoted by $\hat{\mathrm{x}}$.

$$\hat{\mathrm{x}} = (A^T A)^{-1} A^T\mathrm{b}$$

Example

- **Example 1:** Find a least-squares solution of the inconsistent system $A\mathrm{x} = \mathrm{b}$ and the error.

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \mathrm{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Solution in Python

```
A = np.array([[4,0],[0,2],[1,1]])
b = np.array([[2],[0],[11]])
print(A)
print(b)
```

```
[[4 0]
 [0 2]
 [1 1]]
[[ 2]
 [ 0]
 [11]]
```

```
from numpy.linalg import inv
x = inv(A.T@A)@A.T@b
print(x)
```

```
[[1.]
 [2.]]
```

OR

```
from numpy.linalg import pinv
x = pinv(A)@b
print(x)
```

```
[[1.]
 [2.]]
```

```
In [68]: err = norm(A@x - b)
         print(err)
```

```
9.16515138991168
```