Chapter 8
Volatility Investing with Variance Swaps

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Abstract Traditionally volatility is viewed as a measure of variability, or risk, of an underlying asset. However, recently investors began to look at volatility from a different angle. It happened due to emergence of a market for new derivative instruments - variance swaps. In this chapter, first we introduce the general idea of the volatility trading using variance swaps. Then we describe valuation and hedging methodology for vanilla variance swaps as well as for the third generation volatility derivatives: gamma swaps, corridor variance swaps, conditional variance swaps. Finally, we show the results of the performance investigation of one of the most popular volatility strategies - dispersion trading. The strategy was implemented using variance swaps on DAX and its constituents during the 5-year period from 2004 to 2008.

8.1 Introduction

Traditionally volatility is viewed as a measure of variability, or risk, of an underlying asset. However recently investors have begun to look at volatility from a different angle, variance swaps have been created.

The first variance swap contracts were traded in late 1998, but it was only after the development of the replication argument using a portfolio of vanilla options that

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variance swaps became really popular. In a relatively short period of time these over-the-counter (OTC) derivatives developed from simple contracts on future variance to more sophisticated products. Recently we have been able to observe the emergence of 3G volatility derivatives: gamma swaps, corridor variance swaps, conditional variance swaps and options on realised variance.

Constant development of volatility instruments and improvement in their liquidity allows for volatility trading almost as easily as traditional stocks and bonds. Initially traded OTC, now the number of securities having volatility as underlying are available on exchanges. Thus the variance swaps idea is reflected in volatility indices, also called "fear" indices. These indices are often used as a benchmark of equity market risk and contain option market expectations on future volatility. Among those are VIX – the Chicago Board Options Exchange (CBOE) index on the volatility of S&P500, VSTOXX on Dow Jones EURO STOXX 50 volatility, VDAX – on the volatility of DAX. These volatility indices represent the theoretical prices of one-month variance swaps on the corresponding index. They are calculated daily and on an intraday basis by the exchange from the listed option prices. Also, recently exchanges started offering derivative products, based on these volatility indices – options and futures.

8.2 Volatility Trading with Variance Swaps

Variance swap is a forward contract that at maturity pays the difference between realised variance σ_R^2 (floating leg) and predefined strike K_{var}^2 (fixed leg) multiplied by notional N_{var}.

$$(\sigma_R^2 - K_{\text{var}}^2) \cdot N_{\text{var}} \tag{8.1}$$

When the contract expires the realised variance σ_R^2 can be measured in different ways, since there is no formally defined market convention. Usually variance swap contracts define a formula of a final realised volatility σ_R. It is a square root of annualized variance of daily log-returns of an underlying over a swap's maturity calculated in percentage terms:

$$\sigma_R = \sqrt{\frac{252}{T} \sum_{t=1}^{T} \left(\log \frac{S_t}{S_{t-1}} \right)^2} \cdot 100 \tag{8.2}$$

There are two ways to express the variance swap notional: variance notional and vega notional. Variance notional N_{var} shows the dollar amount of profit (loss) from difference in one point between the realised variance σ_R^2 and the strike K_{var}^2. But since market participants usually think in terms of volatility, vega notional N_{vega} turns out to be a more intuitive measure. It shows the profit or loss from 1% change in volatility. The two measures are interdependent and can substitute each other:

$$N_{\text{vega}} = N_{\text{var}} \cdot 2K_{\text{var}}. \tag{8.3}$$

Example 1. Variance notional $N_{\text{var}} = 2500$. If K_{var} is 20% ($K^2_{\text{var}} = 400$) and the subsequent variance realised over the course of the year is $(15\%)^2$ (quoted as $\sigma^2_R = 225$), the investor will make a loss:

$$\begin{aligned} \text{Loss} &= N_{\text{var}} \cdot (\sigma^2_R - K^2_{\text{var}}) \\ 437500 &= 2500 \cdot (400 - 225). \end{aligned}$$

Marking-to-market of a variance swap is straightforward. If an investor wishes to close a variance swap position at some point t before maturity, he needs to define a value of the swap between inception 0 and maturity T. Here the additivity property of variance is used. The variance at maturity $\sigma^2_{R,(0,T)}$ is just a time-weighted sum of variance realised before the valuation point $\sigma^2_{R,(0,t)}$ and variance still to be realised up to maturity $\sigma^2_{R,(t,T)}$. Since the later is unknown yet, we use its estimate $K^2_{\text{var},(t,T)}$. The value of the variance swap (per unit of variance notional) at time t is therefore:

$$T^{-1}\left\{t\sigma^2_{R,(0,t)} - (T-t)K^2_{\text{var},(t,T)}\right\} - K^2_{\text{var},(0,T)} \tag{8.4}$$

8.3 Replication and Hedging of Variance Swaps

The strike K^2_{var} of a variance swap is determined at inception. The realised variance σ^2_R, on the contrary, is calculated at expiry (8.2). Similar to any forward contract, the future payoff of a variance swap (8.1) has zero initial value, or $K^2_{\text{var}} = E[\sigma^2_R]$. Thus the variance swap pricing problem consists in finding the fair value of K^2_{var} which is the expected future realised variance.

To achieve this, one needs to construct a trading strategy that captures the realised variance over the swap's maturity. The cost of implementing this strategy will be the fair value of the future realised variance.

One of the ways of taking a position in future volatility is trading a delta-hedged option. The P&L from delta-hedging (also called hedging error) generated from buying and holding a vanilla option up to maturity and continuously delta-hedging it, captures the realised volatility over the holding period.

Some assumptions are needed:

- The existence of futures market with delivery dates $T' \geq T$
- The existence of European futures options market, for these options all strikes are available (market is complete)
- Continuous trading is possible
- Zero risk-free interest rate ($r = 0$)
- The price of the underlying futures contract F_t following a diffusion process with no jumps:

$$\frac{\mathrm{d}F_t}{F_t} = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t \tag{8.5}$$

We assume that the investor does not know the volatility process σ_t, but believes that the future volatility equals σ_{imp}, the implied volatility prevailing at that time on the market. He purchases a claim (for example a call option) with σ_{imp}. The terminal value (or payoff) of the claim is a function of F_T. For a call option the payoff is denoted: $f(F_T) = (F_T - K)^+$. The investor can define the value of a claim $V(F_t, t)$ at any time t, given that σ_{imp} is predicted correctly. To delta-hedge the long position in V over $[0, T]$ the investor holds a dynamic short position equal to the option's delta: $\Delta = \partial V / \partial F_t$. If his volatility expectations are correct, then at time t for a delta-neutral portfolio the following relationship holds:

$$\Theta = -\frac{1}{2}\sigma_{\text{imp}}^2 F_t^2 \Gamma \tag{8.6}$$

subject to terminal condition:

$$V(F_T, T) = f(F_T) \tag{8.7}$$

$\Theta = \partial V / \partial t$ is called the option's theta or time decay and $\Gamma = \partial^2 V / \partial F_t^2$ is the option's gamma. Equation (8.6) shows how the option's value decays in time (Θ) depending on convexity (Γ).

Delta-hedging of V generates the terminal wealth:

$$P\&L_\Delta = -V(F_0, 0, \sigma_{\text{imp}}) - \int_0^T \Delta \mathrm{d}F_t + V(F_T, T) \tag{8.8}$$

which consists of the purchase price of the option $V(F_0, 0, \sigma_{\text{imp}})$, P&L from delta-hedging at constant implied volatility σ_{imp} and final pay-off of the option $V(F_T, T)$.

Applying Itô's lemma to some function $f(F_t)$ of the underlying process specified in (8.5) gives:

$$f(F_T) = f(F_0) + \int_0^T \frac{\partial f(F_t)}{\partial F_t}\mathrm{d}F_t + \frac{1}{2}\int_0^T F_t^2\sigma_t^2 \frac{\partial^2 f(F_t)}{\partial F_t^2}\mathrm{d}t + \int_0^T \frac{\partial f(F_t)}{\partial t}\mathrm{d}t \tag{8.9}$$

For $f(F_t) = V(F_t, t, \sigma_t)$ we therefore obtain:

$$V(F_T, T) = V(F_0, 0, \sigma_{\text{imp}}) + \int_0^T \Delta \mathrm{d}F_t + \frac{1}{2}\int_0^T F_t^2 \Gamma \sigma_t^2 \mathrm{d}t + \int_0^T \Theta \mathrm{d}t \tag{8.10}$$

Using relation (8.6) for (8.10) gives:

$$V(F_T, T) - V(F_0, 0, \sigma_{\text{imp}}) = \int_0^T \Delta \mathrm{d}F_t + \frac{1}{2}\int_0^T F_t^2 \Gamma (\sigma_t^2 - \sigma_{\text{imp}}^2)\mathrm{d}t \tag{8.11}$$

Finally substituting (8.11) into (8.8) gives $P\&L_\Delta$ of the delta-hedged option position:

$$P\&L_\Delta = \frac{1}{2}\int_0^T F_t^2 \Gamma(\sigma_t^2 - \sigma_{\text{imp}}^2)\mathrm{d}t \tag{8.12}$$

Thus buying the option and delta-hedging it generates P&L (or hedging error) equal to differences between instantaneous realised and implied variance, accrued over time $[0, T]$ and weighed by $F_t^2\Gamma/2$ (dollar gamma).

However, even though we obtained the volatility exposure, it is path-dependent. To avoid this one needs to construct a portfolio of options with path-independent P&L or in other words with dollar gamma insensitive to F_t changes. Figure 8.1 represents the dollar gammas of three option portfolios with an equal number of vanilla options (puts or calls) and similar strikes lying in a range from 20 to 200. Dollar gammas of individual options are shown with thin lines, the portfolio's dollar gamma is a bold line.

First, one can observe, that for every individual option dollar gamma reaches its maximum when the option is ATM and declines with price going deeper out of the money. One can make a similar observation by looking at the portfolio's dollar gamma when the constituents are weighted equally (first picture). However, when we use the alternative weighting scheme $(1/K)$, the portfolio's dollar gamma becomes flatter (second picture). Finally by weighting options with $1/K^2$ the

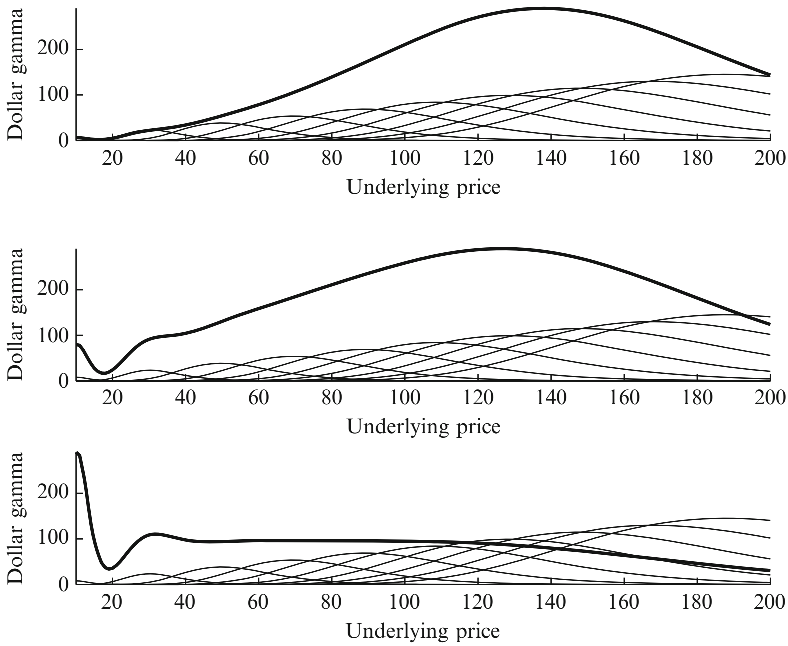

Fig. 8.1 Dollar gamma of option portfolio as a function of stock price. Weights are defined: equally, proportional to $1/K$ and proportional to $1/K^2$

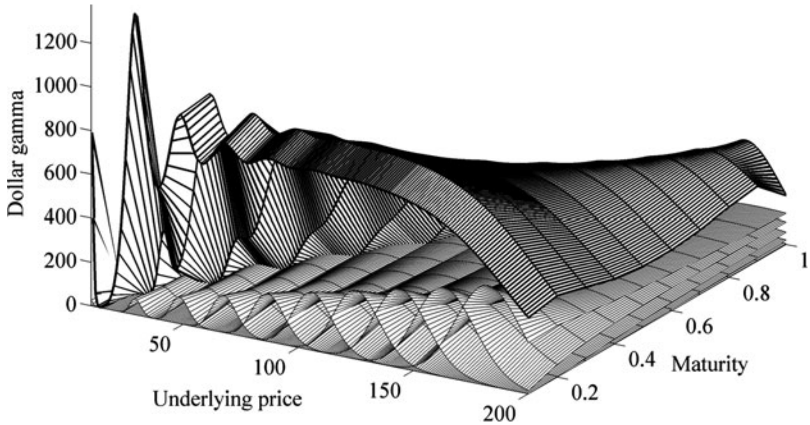

Fig. 8.2 Dollar gamma of option portfolio as a function of stock price and maturity. Weights are defined proportional to $1/K^2$

portfolio's dollar gamma becomes parallel to the vertical axis (at least in 20–140 region), which suggests that the dollar gamma is no longer dependent on the F_t movements.

We have already considered a position in a single option as a bet on volatility. The same can be done with the portfolio of options. However the obtained exposure is path-dependent. We need, however the static, path-independent trading position in future volatility. Figures 8.1 and 8.2 illustrate that by weighting the options' portfolio proportional to $1/K^2$ this position can be achieved. Keeping in mind this intuition we proceed to formal derivations.

Let us consider a payoff function $f(F_t)$:

$$f(F_t) = \frac{2}{T}\left(\log\frac{F_0}{F_t} + \frac{F_t}{F_0} - 1\right) \tag{8.13}$$

This function is twice differentiable with derivatives:

$$f'(F_t) = \frac{2}{T}\left(\frac{1}{F_0} - \frac{1}{F_t}\right) \tag{8.14}$$

$$f''(F_t) = \frac{2}{TF_t^2} \tag{8.15}$$

and

$$f(F_0) = 0 \tag{8.16}$$

One can give a motivation for the choice of the particular payoff function (8.13). The first term, $2\log F_0/TF_t$, is responsible for the second derivative of the payoff $f(F_t)$ w.r.t. F_t, or gamma (8.15). It will cancel out the weighting term in (8.12) and therefore will eliminate path-dependence. The second term $2/T(F_t/F_0 - 1)$ guarantees the payoff $f(F_t)$ and will be non-negative for any positive F_t.

Applying Itô's lemma to (8.13) (substituting (8.13) into (8.9)) gives the expression for the realised variance:

$$\frac{1}{T}\int_0^T \sigma_t^2 \mathrm{d}t = \frac{2}{T}\left(\log\frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right) - \frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right)\mathrm{d}F_t \tag{8.17}$$

Equation (8.17) shows that the value of a realised variance for $t \in [0, T]$ is equal to

- A continuously rebalanced futures position that costs nothing to initiate and is easy to replicate:

$$\frac{2}{T}\int_0^T \left(\frac{1}{F_0} - \frac{1}{F_t}\right)\mathrm{d}F_t \tag{8.18}$$

- A *log contract*, static position of a contract that pays $f(F_T)$ at expiry and has to be replicated:

$$\frac{2}{T}\left(\log\frac{F_0}{F_T} + \frac{F_T}{F_0} - 1\right) \tag{8.19}$$

Carr and Madan (2002) argue that the market structure assumed above allows for the representation of any twice differentiable payoff function $f(F_T)$ in the following way:

$$f(F_T) = f(k) + f'(k)\left[\{(F_T - k)^+ - (k - F_T)^+\}\right] + \tag{8.20}$$

$$+\int_0^k f''(K)(K - F_T)^+\mathrm{d}K + \int_k^\infty f''(K)(F_T - K)^+\mathrm{d}K$$

Applying (8.20) to payoff (8.19) with $k = F_0$ gives:

$$\log\left(\frac{F_0}{F_T}\right) + \frac{F_T}{F_0} - 1 = \int_0^{F_0} \frac{1}{K^2}(K - F_T)^+\mathrm{d}K + \int_{F_0}^\infty \frac{1}{K^2}(F_T - K)^+\mathrm{d}K \tag{8.21}$$

Equation (8.21) represents the payoff of a log contract at maturity $f(F_T)$ as a sum of:

- The portfolio of OTM puts (strikes are lower than forward underlying price F_0), inversely weighted by squared strikes:

$$\int_0^{F_0} \frac{1}{K^2}(K - F_T)^+\mathrm{d}K \tag{8.22}$$

- The portfolio of OTM calls (strikes are higher than forward underlying price F_0), inversely weighted by squared strikes:

$$\int_{F_0}^{\infty} \frac{1}{K^2}(F_T - K)^+ \mathrm{d}K \tag{8.23}$$

Now coming back to equation (8.17) we see that in order to obtain a constant exposure to future realised variance over the period 0 to T the trader should, at inception, buy and hold the portfolio of puts (8.22) and calls (8.23). In addition he has to initiate and roll the futures position (8.18).

We are interested in the costs of implementing the strategy. Since the initiation of futures contract (8.18) costs nothing, the cost of achieving the strategy will be defined solely by the portfolio of options. In order to obtain an expectation of a variance, or strike K_{var}^2 of a variance swap at inception, we take a risk-neutral expectation of a future strategy payoff:

$$K_{\text{var}}^2 = \frac{2}{T} e^{rT} \int_0^{F_0} \frac{1}{K^2} P_0(K) \mathrm{d}K + \frac{2}{T} e^{rT} \int_{F_0}^{\infty} \frac{1}{K^2} C_0(K) \mathrm{d}K \tag{8.24}$$

8.4 Constructing a Replication Portfolio in Practice

Although we have obtained the theoretical expression for the future realised variance, it is still not clear how to make a replication in practice. Firstly, in reality the price process is discrete. Secondly, the range of traded strikes is limited. Because of this the value of the replicating portfolio usually underestimates the true value of a log contract.

One of the solutions is to make a discrete approximation of the payoff (8.19). This approach was introduced by Demeterfi et al. (Summer 1999).

Taking the logarithmic payoff function, whose initial value should be equal to the weighted portfolio of puts and calls (8.21), we make a piecewise linear approximation. This approach helps to define how many options of each strike investor should purchase for the replication portfolio.

Figure 8.3 shows the logarithmic payoff (dashed line) and the payoff of the replicating portfolio (solid line). Each linear segment on the graph represents the payoff of an option with strikes available for calculation. The slope of this linear segment will define the amount of options of this strike to be put in the portfolio.

For example, for the call option with strike K_0 the slope of the segment would be:

$$w(K_0) = \frac{f(K_{1,c}) - f(K_0)}{K_{1,c} - K_0} \tag{8.25}$$

where $K_{1,c}$ is the second closest call strike.

The slope of the next linear segment, between $K_{1,c}$ and $K_{2,c}$, defines the amount of options with strike $K_{1,c}$. It is given by

$$w(K_{1,c}) = \frac{f(K_{2,c}) - f(K_{1,c})}{K_{2,c} - K_{1,c}} - w(K_0) \tag{8.26}$$

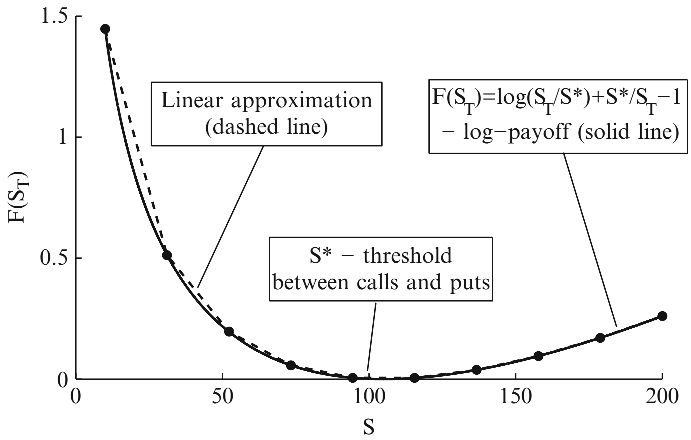

Fig. 8.3 Discrete approximation of a log payoff

Finally for the portfolio of n calls the number of calls with strike $K_{n,c}$:

$$w(K_{n,c}) = \frac{f(K_{n+1,c}) - f(K_{n,c})}{K_{n+1,c} - K_{n,c}} - \sum_{i=0}^{n-1} w(K_{i,c}) \tag{8.27}$$

The left part of the log payoff is replicated by the combination of puts. For the portfolio of m puts the weight of a put with strike $K_{m,p}$ is defined by

$$w(K_{m,p}) = \frac{f(K_{m+1,p}) - f(K_{m,p})}{K_{m,p} - K_{m+1,p}} - \sum_{j=0}^{m-1} w(K_{j,p}) \tag{8.28}$$

Thus constructing the portfolio of European options with the weights defined by (8.27) and (8.28) we replicate the log payoff and obtain value of the future realised variance.

Assuming that the portfolio of options with narrowly spaced strikes can produce a good piecewise linear approximation of a log payoff, there is still the problem of capturing the "tails" of the payoff. Figure 8.3 illustrates the effect of a limited strike range on replication results. Implied volatility is assumed to be constant for all strikes ($\sigma_{\text{imp}} = 25\%$). Strikes are evenly distributed one point apart. The strike range changes from 20 to 1,000. With increasing numbers of options the replicating results approach the "true value" which equals to σ_{imp} in this example. For higher maturities one needs a broader strike range than for lower maturities to obtain the value close to actual implied volatility.

Table 8.1 Replication of a variance swaps strike by portfolio of puts and calls

Strike	IV	BS Price	Type of option	Weight	Share value
200	0.13	0.01	Put	0.0003	0.0000
210	0.14	0.06	Put	0.0002	0.0000
220	0.15	0.23	Put	0.0002	0.0000
230	0.15	0.68	Put	0.0002	0.0001
240	0.16	1.59	Put	0.0002	0.0003
250	0.17	3.16	Put	0.0002	0.0005
260	0.17	5.55	Put	0.0001	0.0008
270	0.18	8.83	Put	0.0001	0.0012
280	0.19	13.02	Put	0.0001	0.0017
290	0.19	18.06	Put	0.0001	0.0021
300	0.20	23.90	Call	0.0000	0.0001
310	0.21	23.52	Call	0.0001	0.0014
320	0.21	20.10	Call	0.0001	0.0021
330	0.22	17.26	Call	0.0001	0.0017
340	0.23	14.91	Call	0.0001	0.0014
350	0.23	12.96	Call	0.0001	0.0011
360	0.24	11.34	Call	0.0001	0.0009
370	0.25	9.99	Call	0.0001	0.0008
380	0.25	8.87	Call	0.0001	0.0006
390	0.26	7.93	Call	0.0001	0.0005
400	0.27	7.14	Call	0.0001	0.0005
				K_{var}	0.1894

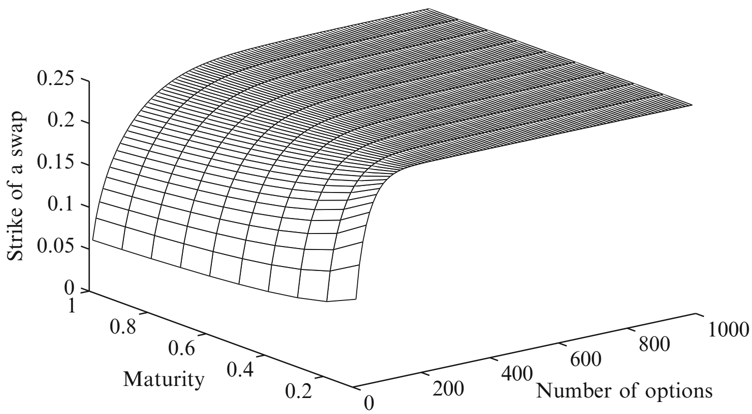

Fig. 8.4 Dependence of replicated realised variance level on the maturity of the swap and the number of options

Table 8.1 shows the example of the variance swap replication. The spot price of $S^* = 300$, riskless interest rate $r = 0$, maturity of the swap is one year $T = 1$, strike range is from 200 to 400 (Fig. 8.4). The implied volatility is 20% ATM and changes linearly with the strike (for simplicity no smile is assumed). The weight of each option is defined by (8.27) and (8.28).

8.5 3G Volatility Products

If we need to capture some particular properties of realised variance, standard variance swaps may not be sufficient. For instance by taking asymmetric bets on variance. Therefore, there are other types of swaps introduced on the market, which constitute the third-generation of volatility products. Among them are: gamma swaps, corridor variance swaps and conditional variance swaps.

By modifying the floating leg of a standard variance swap (8.2) with a weight process w_t we obtain a *generalized variance swap*.

$$\sigma_R^2 = \frac{252}{T} \sum_{t=1}^{T} w_t \left(\log \frac{F_t}{F_{t-1}} \right)^2 \tag{8.29}$$

Now, depending on the chosen w_t we obtain different types of variance swaps: Thus $w_t = 1$ defines a *standard variance swap*.

8.5.1 Corridor and Conditional Variance Swaps

The weight $w_t = w(F_t) = \mathbf{I}_{F_t \in C}$ defines a *corridor variance swap* with corridor C. $\mathbf{I}$ is the indicator function, which is equal to one if the price of the underlying asset F_t is in corridor C and zero otherwise.

If F_t moves sideways, but stays inside C, then the corridor swap's strike is large, because some part of volatility is accrued each day up to maturity. However if the underlying moves outside C, less volatility is accrued resulting the strike to be low. Thus corridor variance swaps on highly volatile assets with narrow corridors have strikes K_C^2 lower than usual variance swap strike K_{var}^2.

Corridor variance swaps admit model-free replication in which the trader holds statically the portfolio of puts and calls with strikes within the corridor C. In this case we consider the payoff function with the underlying F_t in corridor $C = [A, B]$

$$f(F_t) = \frac{2}{T} \left(\log \frac{F_0}{F_t} + \frac{F_t}{F_0} - 1 \right) \mathbf{I}_{F_t \in [A,B]} \tag{8.30}$$

The strike of a corridor variance swap is thus replicated by

$$K_{[A,B]}^2 = \frac{2}{T} e^{rT} \int_A^{F_0} \frac{1}{K^2} P_0(K) \mathrm{d}K + \frac{2}{T} e^{rT} \int_{F_0}^{B} \frac{1}{K^2} C_0(K) \mathrm{d}K \tag{8.31}$$

$C = [0, B]$ gives a *downward variance swap*, $C = [A, \infty]$ – an *upward variance swap*.

Since in practice not all the strikes $K \in (0, \infty)$ are available on the market, corridor variance swaps can arise from the imperfect variance replication, when just strikes $K \in [A, B]$ are taken to the portfolio.

Similarly to the corridor, realised variance of conditional variance swap is accrued only if the price of the underlying asset in the corridor C. However the accrued variance is averaged over the number of days, at which F_t was in the corridor (T) rather than total number of days to expiry T. Thus ceteris paribus the strike of a conditional variance swap $K^2_{C,cond}$ is smaller or equal to the strike of a corridor variance swap K^2_C.

8.5.2 Gamma Swaps

As it is shown in Table 8.2, a standard variance swap has constant dollar gamma and vega. It means that the value of a standard swap is insensitive to F_t changes. However it might be necessary, for instance, to reduce the volatility exposure when the underlying price drops. Or in other words, it might be convenient to have a derivative with variance vega and dollar gamma, that adjust with the price of the underlying.

The weight $w_t = w(F_t) = F_t/F_0$ defines a price-weighted variance swap or *gamma swap*. At maturity the buyer receives the realised variance weighted to each t, proportional to the underlying price F_t. Thus the investor obtains path-dependent exposure to the variance of F_t. One of the common gamma swap applications is equity dispersion trading, where the volatility of a basket is traded against the volatility of basket constituents.

The realised variance paid at expiry of a gamma swap is defined by

$$\sigma_{\text{gamma}} = \sqrt{\frac{252}{T}\sum_{t=1}^{T}\frac{F_t}{F_0}\left(\log\frac{S_t}{S_{t-1}}\right)^2} \cdot 100 \tag{8.32}$$

Table 8.2 Variance swap greeks

Greeks		Call	Put	Standard variance swap	Gamma swap
Delta	$\frac{\partial V}{\partial F_t}$	$\Phi(d_1)$	$\Phi(d_1) - 1$	$\frac{2}{T}(\frac{1}{F_0} - \frac{1}{F_t})$	$\frac{2}{TF_0}\log\frac{F_t}{F_0}$
Gamma	$\frac{\partial^2 V}{\partial F_t^2}$	$\frac{\phi(d_1)}{F_t\sigma\sqrt{\tau}}$	$\frac{\phi(d_1)}{F_t\sigma\sqrt{\tau}}$	$\frac{2}{F_t^2 T}$	$\frac{2}{TF_0F_t}$
Dollar gamma	$\frac{F_t^2\partial^2 V}{2\partial F_t^2}$	$\frac{F_t\phi(d_1)}{2\sigma\sqrt{\tau}}$	$\frac{F_t\phi(d_1)}{2\sigma\sqrt{\tau}}$	$\frac{1}{T}$	$\frac{F_t}{TF_0}$
Vega	$\frac{\partial V}{\partial \sigma_t}$	$\phi(d_1)F_t\sqrt{\tau}$	$\phi(d_1)F_t\sqrt{\tau}$	$\frac{2\sigma\tau}{T}$	$\frac{2\sigma\tau}{T}\frac{F_t}{F_0}$
Variance vega	$\frac{\partial V}{\partial \sigma_t^2}$	$\frac{F_t\phi(d_1)}{2\sigma\sqrt{\tau}}$	$\frac{F_t\phi(d_1)}{2\sigma\sqrt{\tau}}$	$\frac{\tau}{T}$	$\frac{\tau}{T}\frac{F_t}{F_0}$

One can replicate a gamma swap similarly to a standard variance swap, by using the following payoff function:

$$f(F_t) = \frac{2}{T}\left(\frac{F_t}{F_0}\log\frac{F_t}{F_0} - \frac{F_t}{F_0} + 1\right) \tag{8.33}$$

$$f'(F_t) = \frac{2}{TF_0}\log\frac{F_t}{F_0} \tag{8.34}$$

$$f''(F_t) = \frac{2}{TF_0F_t} \tag{8.35}$$

$$f(F_0) = 0 \tag{8.36}$$

Applying Itô's formula (8.9) to (8.33) gives

$$\frac{1}{T}\int_0^T \frac{F_t}{F_0}\sigma_t^2 \mathrm{d}t = \frac{2}{T}\left(\frac{F_T}{F_0}\log\frac{F_T}{F_0} - \frac{F_T}{F_0} + 1\right) - \frac{2}{TF_0}\int_0^T \log\frac{F_t}{F_0}\mathrm{d}F_t \tag{8.37}$$

Equation (8.37) shows that accrued realised variance weighted each t by the value of the underlying is decomposed into payoff (8.33), evaluated at T, and a continuously rebalanced futures position $\frac{2}{TF_0}\int_0^T \log\frac{F_t}{F_0}\mathrm{d}F_t$ with zero value at $t = 0$. Then applying the Carr and Madan argument (8.20) to the payoff (8.33) at T we obtain the $t = 0$ strike of a gamma swap:

$$K^2_{\text{gamma}} = \frac{2}{TF_0}e^{2rT}\int_0^{F_0}\frac{1}{K}P_0(K)\mathrm{d}K + \frac{2}{TF_0}e^{2rT}\int_{F_0}^{\infty}\frac{1}{K}C_0(K)\mathrm{d}K \tag{8.38}$$

Thus gamma swap can be replicated by the portfolio of puts and calls weighted by the inverse of strike $1/K$ and rolling the futures position.

8.6 Equity Correlation (Dispersion) Trading with Variance Swaps

8.6.1 Idea of Dispersion Trading

The risk of the portfolio (or basket of assets) can be measured by the variance (or alternatively standard deviation) of its return. Portfolio variance can be calculated using the following formula:

$$\sigma^2_{\text{Basket}} = \sum_{i=1}^{n} w_i^2\sigma_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} w_i w_j \sigma_i\sigma_j\rho_{ij} \tag{8.39}$$

where σ_i – standard deviation of the return of an i-th constituent (also called volatility), w_i – weight of an i-th constituent in the basket, ρ_{ij} – correlation coefficient between the i-th and the j-th constituent.

Let's take an arbitrary market index. We know the index value historical development as well as price development of each of index constituent. Using this information we can calculate the historical index and constituents' volatility using, for instance, formula (8.2). The constituent weights (market values or current stock prices, depending on the index) are also known to us. The only parameter to be defined are correlation coefficients of every pair of constituents ρ_{ij}. For simplicity assume $\rho_{ij} = const$ for any pair of i, j and call this parameter $\overline{\rho}$ - *average index correlation, or dispersion*. Then having index volatility σ_{index} and volatility of each constituent σ_i, we can express the average index correlation:

$$\overline{\rho} = \frac{\sigma^2_{\text{index}} - \sum_{i=1}^{n} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} w_i w_j \sigma_i \sigma_j} \tag{8.40}$$

Hence it appears the idea of *dispersion trading*, consisting of buying the volatility of index constituents according to their weight in the index and selling the volatility of the index. Corresponding positions in variances can be taken by buying (selling) variance swaps.

By going short index variance and long variance of index constituents we go short dispersion, or enter the direct dispersion strategy.

Why can this strategy be attractive for investors? This is due to the fact that index options appear to be more expensive than their theoretical Black-Scholes prices, in other words investors will pay too much for realised variance on the variance swap contract expiry. However, in the case of single equity options one observes no volatility distortion. This is reflected in the shape of implied volatility smile. There is growing empirical evidence that the index option skew tends to be steeper then the skew of the individual stock option. For instance, this fact has been studied in Bakshi et al. (2003) on example of the S&P500 and Branger and Schlag (2004) for the German stock index DAX.

This empirical observation is used in dispersion trading. The most widespread dispersion strategy, direct strategy, is a long position in constituents' variances and short in variance of the index. This strategy should have, on average, positive payoffs. Hoverer under some market conditions it is profitable to enter the trade in the opposite direction. This will be called – the inverse dispersion strategy.

The payoff of the direct dispersion strategy is a sum of variance swap payoffs of each of i-th constituent

$$(\sigma^2_{R,i} - K^2_{\text{var},i}) \cdot N_i \tag{8.41}$$

and of the short position in index swap

$$(K^2_{\text{var,index}} - \sigma^2_{R,\text{index}}) \cdot N_{\text{index}} \tag{8.42}$$

where

$$N_i = N_{\text{index}} \cdot w_i \tag{8.43}$$

The payoff of the overall strategy is:

$$N_{\text{index}} \cdot \left(\sum_{i=1}^{n} w_i \sigma_{R,i}^2 - \sigma_{R,\text{index}}^2 \right) - \text{ResidualStrike} \tag{8.44}$$

The residual strike

$$\text{ResidualStrike} = N_{\text{index}} \cdot \left(\sum_{i=1}^{n} w_i K_{\text{var},i}^2 - K_{\text{var,index}}^2 \right) \tag{8.45}$$

is defined by using methodology introduced before, by means of replication portfolios of vanilla OTM options on index and all index constituents.

However when implementing this kind of strategy in practice investors can face a number of problems. Firstly, for indices with a large number of constituent stocks (such as S&P500) it would be problematic to initiate a large number of variance swap contracts. This is due to the fact that the market for some variance swaps did not reach the required liquidity. Secondly, there is still the problem of hedging vega-exposure created by these swaps. It means a bank should not only virtually value (use for replication purposes), but also physically acquire and hold the positions in portfolio of replicating options. These options in turn require dynamic delta-hedging. Therefore, a large variance swap trade (as for example in case of S&P500) requires additional human capital from the bank and can be associated with large transaction costs. The remedy would be to make a stock selection and to form the offsetting variance portfolio only from a part of the index constituents.

It has already been mentioned that, sometimes the payoff of the strategy could be negative, in order words sometimes it is more profitable to buy index volatility and sell volatility of constituents. So the procedure which could help in decisions about trade direction may also improve overall profitability.

If we summarize, the success of the volatility dispersion strategy lies in correct determining:

- The direction of the strategy
- The constituents for the offsetting variance basket

The next sections will present the results of implementing the dispersion trading strategy on DAX and DAX constituents' variances. First we implement its classical variant meaning short position in index variance against long positions in variances of all 30 constituents. Then the changes to the basic strategy discussed above are implemented and the profitability of these improvements measured.

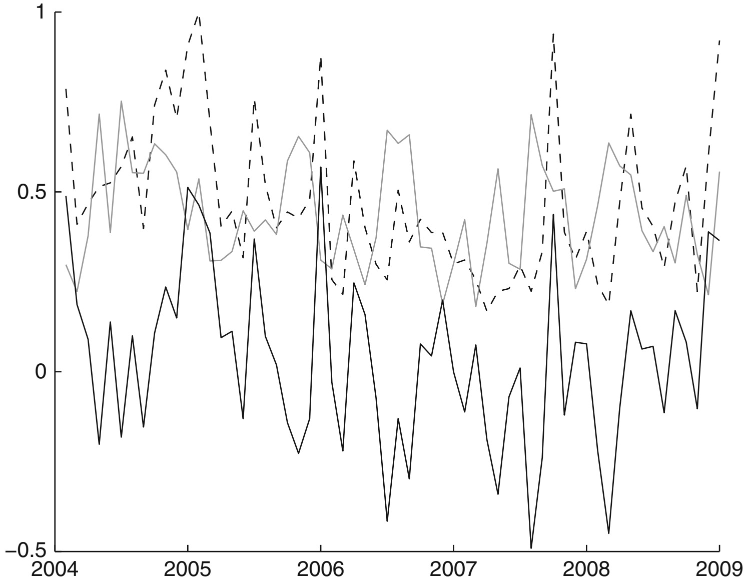

Fig. 8.5 Average implied correlation (*dotted*), average realized correlation (*gray*), payoff of the direct dispersion strategy (*solid black*)

8.6.2 Implementation of the Dispersion Strategy on DAX Index

In this section we investigate the performance of a dispersion trading strategy over the 5 years period from January 2004 to December 2008. The dispersion trade was initiated at the beginning of every moth over the examined period. Each time the 1-month variance swaps on DAX and constituents were traded.

First we implement the basic dispersion strategy, which shows on average positive payoffs over the examined period (Fig. 8.5). Descriptive statistics shows that the average payoff of the strategy is positive, but close to zero. Therefore in the next section several improvements are introduced.

It was discussed already that index options are usually overestimated (which is not the case for single equity options), the future volatility implied by index options will be higher than realized volatility meaning that the direct dispersion strategy is on average profitable. However the reverse scenario may also take place. Therefore it is necessary to define whether to enter a direct dispersion (short index variance, long constituents variance) or reverse dispersion (long index variance and short constituents' variances) strategy.

This can be done by making a forecast of the future volatility with GARCH (1,1) model and multiplying the result by 1.1, which was implemented in the paper of Deng (2008) for S&P500 dispersion strategy. If the variance predicted by GARCH is higher than the variance implied by the option market, one should enter the reverse

Table 8.3 Comparison of basic and improved dispersion strategy payoffs for the period from January 2004 to December 2008

Strategy	Mean	Median	SD	Skewness	Kurtosis	J-B	Probability
Basic	0.032	0.067	0.242	0.157	2.694	0.480	0.786
Improved	0.077	0.096	0.232	−0.188	3.012	0.354	0.838

dispersion trade (long index variance and short constituents variances). After using the GARCH volatility estimate the average payoff increased by 41.7% (Table 8.3).

The second improvement serves to decrease transaction cost and cope with market illiquidity. In order to decrease the number of stocks in the offsetting portfolio the Principal Components Analysis (PCA) can be implemented. Using PCA we select the most "effective" constituent stocks, which help to capture the most of index variance variation. This procedure allowed us to decrease the number of offsetting index constituents from 30 to 10. According to our results, the 1-st PC explains on average 50% of DAX variability. Thereafter each next PC adds only 2–3% to the explained index variability, so it is difficult to distinguish the first several that explain together 90%. If we take stocks, highly correlated only with the 1-st PC, we can significantly increase the offsetting portfolio's variance, because by excluding 20 stocks from the portfolio we make it less diversified, and therefore more risky.

However it was shown that one still can obtain reasonable results after using the PCA procedure. Thus in the paper of Deng (2008) it was successfully applied to S&P500.

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