Axler LADR Exercise Solutions Chapter 1

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Some notes and disclaimers: Not all problems might be included, especially those I feel are too niche and could be skipped without harming understanding. I also use bold notation for vectors and column vector notation as they are more familiar in linear algebra. A ($\star$) indicates the problem is interesting and tricky, while ($\star\star$) indicates a problem is genuinely challenging.

1A: $\mathbb{R}^n$ and $\mathbb{C}^n$

1. Let $\alpha = a + bi$ and $\beta = c + di$. Using the definition (1.1), we can write

$$\begin{aligned}\alpha + \beta &= (a + bi) + (c + di)\\ &= (a + c) + (b + d)i\\ &= (c + a) + (d + b)i\\ &= (c + di) + (a + bi)\\ &= \beta + \alpha.\end{aligned}$$

   Note: A key assumption that is made here is that the real numbers commute (in the third =). It is already assumed in the book.

2. Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = s + ti$, for $a, b, c, d, s, t \in \mathbb{R}$. Then

$$\begin{aligned}(\alpha + \beta) + \lambda &= ((a + bi) + (c + di)) + (s + ti)\\ &= ((a + c) + (b + d)i) + (s + ti)\\ &= ((a + c) + s) + ((b + d) + t)i\\ &= (a + (c + s)) + (b + (d + t))i\\ &= (a + bi) + ((c + s) + (d + t))i\\ &= (a + bi) + ((c + di) + (s + ti))\\ &= \alpha + (\beta + \lambda).\end{aligned}$$

3. Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = s + ti$ for $a, b, c, d, s, t \in \mathbb{R}$. Then

$$\begin{aligned}
(\alpha\beta)\lambda &= [(a+bi)(c+di)](s+ti) \\
&= [(ac-bd)+(ad+bc)i](s+ti) \\
&= (acs-bds-adt-bct)+(act-bdt+ads+bcs)i \\
&= (a(cs-dt)-b(ds+ct))+(a(ct+ds)+b(cs-dt))i \\
&= (a+bi)[(cs-dt)+(ct+ds)i] \\
&= (a+bi)[(c+di)(s+ti)] \\
&= \alpha(\beta\lambda).
\end{aligned}$$

4. Let $\alpha = a + bi$, $\beta = c + di$, and $\lambda = s + ti$ for $a, b, c, d, s, t \in \mathbb{R}$. Then

$$\begin{aligned}
\lambda(\alpha+\beta) &= (s+ti)[(a+bi)+(c+di)] \\
&= (s+ti)[(a+c)+(b+d)i] \\
&= (s(a+c)-t(b+d))+(s(b+d)+t(a+c))i \\
&= (sa+sc-tb-td)+(sb+sd+ta+tc)i \\
&= [(sa-tb)+(sb+ta)i]+[(sc-td)+(sd+tc)i] \\
&= [(s+ti)(a+bi)]+[(s+ti)(c+di)] \\
&= \lambda\alpha+\lambda\beta.
\end{aligned}$$

5. Let $\alpha = a + bi$, where $a, b \in \mathbb{R}$. We know that for any real number $x \in \mathbb{R}$, there exists an additive inverse $-x \in \mathbb{R}$. Then let $\beta = -a - bi$. Then

$$\begin{aligned}
\alpha+\beta &= (a+bi)+(-a+(-b)i) \\
&= (a+(-a))+(b+(-b))i \\
&= 0+0i = 0.
\end{aligned}$$

Now we show uniqueness. Suppose there exist two additive inverses β_1, β_2 of α. Then we can use the additive identity as follows:

$$\begin{aligned}
\beta_1 &= \beta_1 + 0 \\
&= \beta_1 + (\alpha + \beta_2) \\
&= (\beta_1 + \alpha) + \beta_2 \\
&= 0 + \beta_2 \\
&= \beta_2.
\end{aligned}$$

Thus the additive inverse is unique. (This can also be shown by just saying that the additive inverses of real numbers are unique, but this is more informative.)

6. Let $\alpha \in \mathbb{C}$ with $\alpha \neq 0$. Then $\alpha = a + bi$ with $a, b \in \mathbb{R}$, with at least one of a and b nonzero. Define

$$\beta = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i.$$

Note that since at least one of a, b is nonzero, $a^2 + b^2 \neq 0$ and β is well-defined. Then

$$\begin{aligned}
\alpha\beta &= (a + bi)\left(\frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i\right) \\
&= \left(\frac{a^2}{a^2+b^2} - \frac{-b^2}{a^2+b^2}\right) + \left(\frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right)i \\
&= 1 + 0i = 1.
\end{aligned}$$

To prove uniqueness of β, follow the exact same procedure as in exercise 5, except instead of addition we use multiplication, and instead of 0 we use 1.

7. Let

$$\lambda = \frac{-1 + \sqrt{3}i}{2}.$$

Then

$$\begin{aligned}
\lambda^2 &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \left(\frac{1}{4} - \frac{3}{4}\right) + \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}\right) \\
&= -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

Now

$$\begin{aligned}
\lambda^3 = \lambda^2 \cdot \lambda &= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\
&= \left(\frac{1}{4} - \left(-\frac{3}{4}\right)\right) + \left(-\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)i \\
&= 1 + 0i = 1.
\end{aligned}$$

Thus $\lambda^3 = 1$ and λ is a cube root of 1.

8. Let $\alpha = a + bi$, where $a, b \in \mathbb{R}$. We want $\alpha^2 = i$. Then

$$\begin{aligned}
\alpha^2 &= (a + bi)(a + bi) \\
&= (a^2 - b^2) + (2ab)i.
\end{aligned}$$

Thus we have $(a^2 - b^2) + (2ab)i = i$. Matching coefficients, we see that we must have

$$\begin{aligned}
a^2 - b^2 &= 0 \\
2ab &= 1.
\end{aligned}$$

There are many ways of solving this system. Here is one way. Since $a^2 - b^2 = (a+b)(a-b) = 0$, we know that $a + b = 0$ or $a - b = 0$. If $a + b = 0$, then $a = -b$. Substituting into the second equation gets us

$$-2b^2 = 1 \implies a^2 = -\frac{1}{2},$$

which is not possible (recall a, b are real). Then $a - b = 0$ and thus $a = b$. As a result, *weget*

$$2b^2 = 1 \implies b^2 = \frac{1}{2},$$

which gives us

$$b = \pm\frac{1}{\sqrt{2}}.$$

This gives us two (and only two) square roots of i:

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

9. We are asked to find $\mathbf{x} \in \mathbb{R}^4$ such that

$$\begin{bmatrix} 4 \\ -3 \\ 1 \\ 7 \end{bmatrix} + 2\mathbf{x} = \begin{bmatrix} 5 \\ 9 \\ -6 \\ 8 \end{bmatrix}$$

This is similar to solving an equation. We subtract from the left hand side and multiply by 1/2:

$$\begin{aligned} 2\mathbf{x} &= \begin{bmatrix} 1 \\ 12 \\ -7 \\ 1 \end{bmatrix} \\ \mathbf{x} &= \begin{bmatrix} 1/2 \\ 6 \\ -7/2 \\ 1/2 \end{bmatrix}. \end{aligned}$$

10. Suppose there does exist some $\lambda \in \mathbb{C}$ such that

$$\lambda \begin{bmatrix} 2-3i \\ 5+4i \\ -6+7i \end{bmatrix} = \begin{bmatrix} 12-5i \\ 7+22i \\ -32-9i \end{bmatrix}.$$

Then we must have

$$\begin{aligned} \lambda(2-3i) &= 12-5i \\ \lambda(5+4i) &= 7+22i \\ \lambda(-6+7i) &= -32-9i. \end{aligned}$$

The first equation implies that

$$\lambda = 3 + 2i.$$

However, $(2+3i)(5+4i) = -2+23i \neq 7+22i$. So such a λ cannot exist.

11. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}^n$. Then $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n)$, and $\mathbf{z} = (z_1, \ldots, z_n)$. Then

$$\begin{aligned}
(\mathbf{x}+\mathbf{y})+\mathbf{z} &= \left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \\
&= \begin{bmatrix} (x_1 + y_1) + z_1 \\ \vdots \\ (x_n + y_n) + z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 + (y_1 + z_1) \\ \vdots \\ x_n + (y_n + z_n) \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 + z_1 \\ \vdots \\ y_n + z_n \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \left(\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\right) \\
&= \mathbf{x} + (\mathbf{y} + \mathbf{z}).
\end{aligned}$$

12. Let $\mathbf{x} \in \mathbb{F}^n$ and $a, b \in \mathbb{F}$. Then $\mathbf{x} = (x_1, \ldots, x_n)$, and

$$\begin{aligned}
(ab)\mathbf{x} &= (ab)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} abx_1 \\ \vdots \\ abx_n \end{bmatrix} \\
&= a\begin{bmatrix} bx_1 \\ \vdots \\ bx_n \end{bmatrix} \\
&= a(b\mathbf{x}).
\end{aligned}$$

13. This is trivial. Use the fact that 1 is the multiplicative identity.

14. Let $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$ and $\lambda \in \mathbb{F}$. Then

$$\begin{aligned}
\lambda(\mathbf{x}+\mathbf{y}) &= \lambda\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}\right) \\
&= \lambda \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda(x_1 + y_1) \\ \vdots \\ \lambda(x_n + y_n) \end{bmatrix} = \begin{bmatrix} \lambda x_1 + \lambda y_1 \\ \vdots \\ \lambda x_n + \lambda y_n \end{bmatrix} \\
&= \begin{bmatrix} \lambda x_1 \\ \vdots \\ \lambda x_n \end{bmatrix} + \begin{bmatrix} \lambda y_1 \\ \vdots \\ \lambda y_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \lambda \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \\
&= \lambda\mathbf{x} + \lambda\mathbf{y}.
\end{aligned}$$

15. Let $\mathbf{x} \in \mathbb{F}^n$ and let $a, b \in \mathbb{F}$. Then $\mathbf{x} = (x_1, \ldots, x_n)$ and

$$\begin{aligned}
(a+b)\mathbf{x} &= (a+b) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= \begin{bmatrix} (a+b)x_1 \\ \vdots \\ (a+b)x_n \end{bmatrix} = \begin{bmatrix} ax_1 + bx_1 \\ \vdots \\ ax_n + bx_n \end{bmatrix} \\
&= \begin{bmatrix} ax_1 \\ \vdots \\ ax_n \end{bmatrix} + \begin{bmatrix} bx_1 \\ \vdots \\ bx_n \end{bmatrix} = a \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + b \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\
&= a\mathbf{x} + b\mathbf{x}.
\end{aligned}$$

1B: Definition of Vector Space

1. We wish to show that $-(-\mathbf{v}) = \mathbf{v}$ for every $\mathbf{v} \in V$. That is, the additive inverse of $-\mathbf{v}$ is $\mathbf{v}$. However, note that

$$\mathbf{v} + (-\mathbf{v}) = -\mathbf{v} + \mathbf{v} = \mathbf{0},$$

so $\mathbf{v}$ is an additive inverse of $-\mathbf{v}$. However, 1.27 says that the additive inverse is unique, so $\mathbf{v}$ is the additive inverse of $-\mathbf{v}$, i.e. $\mathbf{v} = -(-\mathbf{v})$.

2. Suppose $a \in \mathbb{F}$, $\mathbf{v} \in V$, and $a\mathbf{v} = \mathbf{0}$. Suppose $a \neq 0$. Then we show that $\mathbf{v} = \mathbf{0}$. Note that since $a\mathbf{v} = \mathbf{0}$, we can multiply by $1/a$ (which exists since $a \neq 0$), and get

$$\mathbf{0} = \frac{1}{a}\mathbf{0} = \frac{1}{a}(a\mathbf{v}) = 1\mathbf{v} = \mathbf{v}.$$

3. We first show existence. let

$$\mathbf{x} = \frac{1}{3}(\mathbf{w} - \mathbf{v}).$$

Then

$$\begin{aligned}\mathbf{v} + 3\mathbf{x} &= \mathbf{v} + 3 \cdot \frac{1}{3}(\mathbf{w} - \mathbf{v}) \\ &= \mathbf{v} + (\mathbf{w} - \mathbf{v}) \\ &= \mathbf{w}.\end{aligned}$$

To show uniqueness, suppose $\mathbf{x}_1, \mathbf{x}_2$ are two solutions. Then

$$\begin{aligned}\mathbf{u} + 3\mathbf{x}_1 &= \mathbf{w} \\ \mathbf{u} + 3\mathbf{x}_2 &= \mathbf{w}.\end{aligned}$$

Subtracting these two equations, we have

$$3(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}.$$

Then $\mathbf{x_1} - \mathbf{x}_2 = 0$ by exercise 2, and so $\mathbf{x}_1 = \mathbf{x}_2$, proving uniqueness.

4. The empty set is not a vector space because it fails to satisfy the additive identity condition. Specifically, since the empty set contains no elements, it cannot satisfy an existential condition.

5. We will show that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition. To do that, suppose that V is a vector space by the original definition (with additive inverse). Then it is a vector space by the new definition due to 1.30.

   We now prove that if V is a vector space by the new definition, it is a vector space by the original definition. To do so, we prove that the additive inverse condition holds. Let $\mathbf{v} \in V$. Then $1\mathbf{v} = \mathbf{v}$. Now, if we let -1 be the additive inverse of 1 in $\mathbb{F}$, then

$$\begin{aligned}\mathbf{0} &= 0\mathbf{v} \\ &= (1 + (-1))\mathbf{v} \\ &= 1\mathbf{v} + (-1)\mathbf{v} \\ &= \mathbf{v} + (-1)\mathbf{v}.\end{aligned}$$

   Then letting $\mathbf{w} = (-1)\mathbf{v}$ gives us an additive inverse.

6. No, $\mathbb{R} \cup \{-\infty, \infty\}$ is not a vector space, as the distributive property does not hold. To see that, consider
$$(-1+2)\infty = 1\infty = \infty.$$
However,
$$-1\infty + 2\infty = -\infty + \infty = 0,$$
so $(-1+2)\infty) \neq -1\infty + 2\infty$.

7. Let V be a vector space over field $\mathbb{F}$. We will show that V^S is a vector space. Note that $f+g \in V^S$ and $\lambda f \in V^S$ for $\lambda \in \mathbb{F}$. Let $f, g \in V^S$. Then for $x \in S$,
$$(f+g)(x) = f(x) + g(x) = g(x) + f(x) = (g+f)(x),$$
where the second equality is due to commutativity of V. Now, let $f, g, h \in V^S$. Then for $x \in S$,
$$\begin{aligned}((f+g)+h)(x) &= (f+g)(x) + h(x)\\ &= (f(x)+g(x)) + h(x)\\ &= f(x) + (g(x)+h(x))\\ &= f(x) + (g+h)(x)\\ &= (f+(g+h))(x).\end{aligned}$$
As before, the third equality is due to associativity of V. Now, the additive identity is $0(x) = \mathbf{0}$, the zero function. Then for any $f \in V^S$,
$$(f+0)(x) = f(x) + 0(x) = f(x) + \mathbf{0} = f(x).$$
Now, the additive inverse is given by $(-f)(x) = -f(x)$, the additive inverse of $f(x) \in V$. Then
$$(f+(-f))(x) = f(x) + (-f(x)) = \mathbf{0} = 0(x).$$
If $f \in V^S$, then
$$(1f)(x) = 1f(x) = f(x),$$
satisfying the multiplicative identity property. Finally, the distributive property is satisfied if we let $f, g \in V^S$ and $a \in \mathbb{F}$, so that
$$\begin{aligned}(a(f+g))(x) &= a(f+g)(x)\\ &= a(f(x)+g(x))\\ &= af(x) + ag(x)\\ &= (af+ag)(x).\end{aligned}$$
Similarly, if $f \in V^S$ and $a, b \in \mathbb{F}$, then
$$\begin{aligned}((a+b)f)(x) &= (a+b)f(x)\\ &= af(x) + bf(x)\\ &= (af+bf)(x).\end{aligned}$$
Thus we have verified all the properties needed for V^S to be a vector space.

1C: Subspaces

1. (a) This is a subspace. We see that $\mathbf{0} \in U$ since

$$0 + 2 \cdot 0 + 3 \cdot 0 = 0.$$

Also, if $\mathbf{x}, \mathbf{y} \in U$, then

$$(x_1+y_1)+2(x_2+y_2)+3(x_3+y_3) = (x_1+2x_2+3x_3)+(y_1+2y_2+3y_3) = 0+0 = 0,$$

so $\mathbf{x} + \mathbf{y} \in U$. Similarly, if $\mathbf{u} \in U$ and $a \in \mathbb{F}$, then

$$ax_1 + 2ax_2 + 3ax_3 = a(x_1 + 2x_2 + 3x_3) = a \cdot 0 = 0.$$

Then $a\mathbf{u} \in U$, so U is a subspace.

(b) This is not a subspace, since

$$0 + 2 \cdot 0 + 3 \cdot 0 \neq 4.$$

(c) This is not a subspace, since $(0, 1, 1)$ and $(1, 1, 0)$ are in the set, but $(0, 1, 1) + (1, 1, 0) = (1, 2, 1)$ is not: $1 \cdot 2 \cdot 1 = 2 \neq 0$.

(d) This is a subspace.

2. (a) Consider the set

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \, : \, x_3 = 5x_4 + b\}.$$

If $b = 0$, then U is a subspace: $(0, 0, 0, 0) \in U$, and if $\mathbf{u} \in U$ and $\mathbf{v} \in U$, then $\mathbf{u} + \mathbf{v} \in U$, since

$$u_3 + v_3 = 5(u_4 + v_4)$$

as $u_3 = 5u_4$ and $v_3 = 5v_4$. Finally, if $c \in \mathbb{F}$ and $\mathbf{u} \in U$, then $c\mathbf{u} \in U$, as

$$cu_3 = 5cu_4$$

as $u_3 = 5u_4$. This proves U is a subspace.

Now, suppose U is a subspace. Then $\mathbf{0} \in U$, so $0 = 5 \cdot 0 + b$, so $b = 0$.

(b) This is a subspace because the zero function is continuous, the sum of continuous functions are continuous, and constant multiples are continuous.

(c) This is a subspace because the zero function is differentiable, the sum of differentiable functions is differentiable, and the constant multiples of differentiable functions are also differentiable.

(d) If $b = 0$, then the set is clearly a subspace. All that is needed is to verify that if the set is a subspace, then $b = 0$. If the set is a subspace, then it contains the zero function $0(x)$. However, $0(2) = b = 0$, so $b = 0$.

(e) This is a subspace by the properties of sequences.

3. Let U be the set of differentiable functions on $(-4, 4)$ such that $f'(-1) = 3f(2)$. We wish to show that U is a subspace of $\mathbb{R}^{(-4,4)}$. We see that the zero function $Z(x) \in U$, as $Z'(-1) = 0 = 3Z(2)$. Now let $f, g \in U$. Then since $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$,

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f+g)(2),$$

so $f + g \in U$. Finally, let $f \in U$ and $a \in \mathbb{R}$. Then since $f'(-1) = 3f(2)$,

$$(af)'(-1) = af'(-1) = 3af(2),$$

so $af \in U$. Thus U is a subspace.

4. This is trivial.

5. No, $\mathbb{R}^2$ is not a subspace of the complex vector space $\mathbb{C}^2$, since $\mathbb{F} = \mathbb{C}$, and as a result it is not closed under scalar multiplication: $i(1, 1) = (i, i) \notin \mathbb{R}^2$.

6. (a) Yes, this is a subspace of $\mathbb{R}^3$, since $a^3 = b^3$ implies $a = b$ for real numbers. Then the proof that this is a subspace is quite trivial.

   (b) No, this is not a subspace of $\mathbb{C}^3$. For instance, consider

$$\left(1, \frac{-1+\sqrt{3}i}{2}, 0\right), \left(1, \frac{-1-\sqrt{3}i}{2}, 0\right).$$

   The first two values are both cube roots of 1, so $a^3 = b^3$. However, if we add them, we get

$$(2, -1, 0)$$

   and clearly $2^3 \neq (-1)^3$, so the set is not closed under addition.

7. ($\star$) This statement is not true: take $\mathbb{Z}^2$. This is closed under addition and under taking additive inverses, but is not a subspace, because it is not closed under scalar multiplication: $0.5(1, 1) = (0.5, 0.5) \notin \mathbb{Z}^2$.

8. Consider $U = \{(x_1, x_2) \in \mathbb{R}^2 : x_1x_2 \geq 0\}$. This is the first and third quadrants in $\mathbb{R}^2$. Then U is closed under scalar multiplication: If $\mathbf{u} \in U$ and $a \in \mathbb{R}$, then $(ax_1)(ax_2) = a^2x_1x_2 \geq 0$ since $a^2 \geq 0$ and $x_1x_2 \geq 0$. However, this is not a subspace of $\mathbb{R}^2$ as it is not closed under addition: $(1, 5) + (-3, -3) = (-2, 2)$, and $-2 \cdot 2 = -4 < 0$.

9. ($\star$) This is not a subspace, as it is not closed by addition. We see that $f(x) = \sin(x)$ is periodic with period 2π and $g(x) = \sin(\pi x)$ is periodic with period 2. However, $h = f + g$ is not periodic. To do this, note that the derivatives of a periodic function is periodic with the same period (Prove this.). Then examining the second derivative, we see that

$$h''(x) = -\sin(x) - \pi^2 \sin(\pi x).$$

If h was periodic with period T, then $h(0) = h(T)$ and $h''(0) = h''(T)$. Then $h(0) = 0 = h''(0)$, so

$$\sin(T) + \sin(\pi T) = 0$$

and

$$\sin(T) + \pi^2 \sin(\pi T) = 0.$$

These two equations imply $\sin(T) = 0$ and $\sin(\pi T) = 0$. This implies that $\pi T = m\pi$, so $T = m$ for some integer m. But this is not possible if $T = n\pi$ for some integer n. So no such T exists.

10. Suppose V_1 and V_2 are subspaces of V. To prove that $V_1 \cap V_2$ is a subspace of V, note that $\mathbf{0} \in V_1$ and $\mathbf{0} \in V_2$ as V_1 and V_2 are both subspaces. As a result, $\mathbf{0} \in V_1 \cap V_2$. Now let $\mathbf{u}, \mathbf{v} \in V_1 \cap V_2$. Then $\mathbf{u} + \mathbf{v} \in V_1$ and $\mathbf{u} + \mathbf{v} \in V_2$ since V_1 and V_2 are subspaces. Then $\mathbf{u} + \mathbf{v} \in V_1 \cap V_2$. Finally let $\mathbf{u} \in V_1 \cap V_2$ and $c \in \mathbb{F}$. Then $c\mathbf{u} \in V_1$ and $c\mathbf{u} \in V_2$ since V_1 and V_2 are subspaces. Then $c\mathbf{u} \in V_1 \cap V_2$. Thus $V_1 \cap V_2$ is a subspace.

11. Follow the blueprint of exercise 10.

12. ($\star$) Let V_1 and V_2 be subspaces of V. WLOG, suppose $V_1 \subseteq V_2$. Then $V_1 \cup V_2 = V_2$, and clearly $V_1 \cup V_2$ is a subspace.

    Now suppose $V_1 \cup V_2$ is a subspace. Suppose for the sake of contradiction that $V_1 \cup V_2 \neq V_1$ and $V_1 \cup V_2 \neq V_2$. There exist vectors $\mathbf{v}_1 \in V_1$ but not in V_2 and $\mathbf{v}_2 \in V_2$ but not in V_1. Consider the sum $\mathbf{v}_1 + \mathbf{v}_2$. Since $V_1 \cup V_2$ is a subspace, $\mathbf{v}_1 + \mathbf{v}_2 \in V_1 \cup V_2$. If $\mathbf{v}_1 + \mathbf{v}_2 \in V_1$, then we can subtract out $\mathbf{v}_1$ and since V_1 is a subspace, $\mathbf{v}_2 \in V_1$. But this is a contradiction. So $\mathbf{v}_1 + \mathbf{v}_2 \in V_2$. But similarly, since V_2 is a subspace, we can subtract $\mathbf{v}_2$ and get $\mathbf{v}_1 \in V_2$. In either case we have a contradiction.

13. ($\star\star$) Let V_1, V_2, and V_3 be subspaces of V. Suppose WLOG that $V_1 \subseteq V_3$ and $V_2 \subseteq V_3$. Then $V_1 \cup V_2 \cup V_3 = V_3$, and since V_3 is a subspace, $V_1 \cup V_2 \cup V_3$ is clearly a subspace.

    Now suppose $V_1 \cup V_2 \cup V_3$ is a subspace. We first consider the case where V_1 contains V_2 or V_2 contains V_1. WLOG, suppose $V_1 \subseteq V_2$. Then let $W = V_1 \cup V_2$. Since V_1 and V_2 are subspaces, by exercise 12, so is $V_1 \cup V_2 = W$. Now note that $V_1 \cup V_2 \cup V_3 = W \cup V_3$ is a subspace, so again using exercise 2, we have W containing V_3 (so V_2 contains V_1 and V_3), or V_3 containing W (so V_3 contains V_1 and V_2). In any case we have one subspace containing two others.

    Otherwise, V_1 and V_2 do not contain each other. Consider $\mathbf{u} \in V_1 \setminus V_2$ and $\mathbf{v} \in V_2 \setminus V_1$. Consider the sum $\mathbf{u} + \mathbf{v}$. Note that $\mathbf{u} + \mathbf{v} \in V_1 \cup V_2 \cup V_3$ as $V_1 \cup V_2 \cup V_3$ is a subspace, and $\mathbf{u}, \mathbf{v} \in V_1 \cup V_2 \cup V_3$. Now, if $\mathbf{u} + \mathbf{v} \in V_1$, $\mathbf{v} = (\mathbf{u} + \mathbf{v}) - \mathbf{u} \in V_1$, which is a contradiction, so $\mathbf{u} + \mathbf{v} \notin V_1$. Similarly, $\mathbf{u} + \mathbf{v} \notin V_2$. Thus, we must have $\mathbf{u} + \mathbf{v} \in V_3$. We can use similar logic to conclude that $\mathbf{u} - \mathbf{v} \in V_3$. This means that

    $$(\mathbf{u} + \mathbf{v}) + (\mathbf{u} - \mathbf{v}) = (1 + 1)\mathbf{u} \in V_3.$$

Multiplying by $(1+1)^{-1}$, the inverse of $1+1$, gives us $\mathbf{u} \in V_3$. Playing the exact same game with the vectors $\mathbf{v}+\mathbf{u}$ and $\mathbf{v}-\mathbf{u}$, we see that $\mathbf{v} \in V_3$. Thus $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are both contained in V_3.

Finally, we look at $V_1 \cap V_2$. Suppose $\mathbf{u} \in V_1 \cap V_2$. Now let $\mathbf{v} \in V_1 \setminus V_2$ and consider $\mathbf{u}+\mathbf{v}$. If $\mathbf{u}+\mathbf{v} \in V_1 \cap V_2$, then $\mathbf{u}+\mathbf{v} \in V_2$, and subtracting $\mathbf{u}$ leads to a contradiction. But $\mathbf{u}+\mathbf{v} \in V_1$ as V_1 is a subspace and $\mathbf{u}, \mathbf{v} \in V_1$, so $\mathbf{u}+\mathbf{v} \in V_1 \setminus V_2$. But $V_1 \setminus V_2 \subseteq V_3$ from the previous paragraph, so $\mathbf{u}+\mathbf{v} \in V_3$. Since $\mathbf{v} \in V_3$ (again, since $V_1 \setminus V_2 \subseteq V_3$), and V_3 is a subspace, this must imply that

$$\mathbf{u} = (\mathbf{u}+\mathbf{v}) - \mathbf{v} \in V_3.$$

Since $\mathbf{u}$ was arbitrary, $V_1 \cap V_2 \subseteq V_3$.

Since $V_1 \cup V_2 = (V_1 \setminus V_2) \cup (V_1 \cap V_2) \cup (V_2 \setminus V_1)$, and $V_1 \setminus V_2$, $V_2 \setminus V_1$, and $V_1 \cap V_2$ are all contained in V_3, we can conclude that $V_1 \cup V_2 \subseteq V_3$, and V_3 contains both V_1 and V_2.

NOTE: Note that $1+1$ is used because $\mathbb{F}$ is assumed to be arbitrary. Note that $\mathbb{F}$, however, cannot be a two-element field, as that forces $1+1=0$, and we have no way of proving that $V_1 \setminus V_2$ and $V_2 \setminus V_1$ are contained in V_3. Indeed, if we look at the vector space $V = \mathbb{Z}_2^2$ over the finite field $\mathbb{Z}_2 = \{0, 1\}$, and consider the subspaces $V_1 = \{(0,0), (1,0)\}$, $V_2 = \{(0,0), (0,1)\}$, and $V_3 = \{(0,0), (1,1)\}$, we see that $V_1 \cup V_2 \cup V_3 = V$ is a subspace, but none of the subspaces are contained in the other. For more information on finite fields, look at an abstract algebra source.

14. With symbols,

$$U + W = \{(x+y, -x+y, 2x+2y) \in \mathbb{F}^3 \,:\, x, y \in \mathbb{F}\}.$$

Without symbols, $U+W$ is the plane through the origin containing the vectors $(1,-1,2)$ and $(1,1,2)$.

15. Suppose U is a subspace of V. Then we claim $U+U=U$. Clearly, if $\mathbf{v} \in U+U$, then $\mathbf{v}$ can be written as the sum of two vectors in U. But since U is a subspace, the sum of two vectors in U is also in U, so $\mathbf{v} \in U$ and $U+U \subseteq U$. Also, $U \subseteq U+U$ as any $\mathbf{u} \in U$ can be written as $\mathbf{u}+\mathbf{0}$. Thus, $U+U=U$.

16. Yes, the addition of subspaces is commutative. To quickly see this, note that $\mathbf{v} \in U+W$ if and only if $\mathbf{v} = \mathbf{u}+\mathbf{w}$ for $\mathbf{u} \in U, \mathbf{w} \in W$. However, $\mathbf{u}+\mathbf{w} = \mathbf{w}+\mathbf{u}$, so $\mathbf{v} = \mathbf{w}+\mathbf{u} \in W+U$.

17. Yes, the operation of addition is associative. Suppose $\mathbf{v} \in (V_1+V_2)+V_3$. Then there is

$$\mathbf{v} = (\mathbf{v}_1+\mathbf{v}_2)+\mathbf{v}_3,$$

where $\mathbf{v}_1+\mathbf{v}_2 \in V_1+V_2$ and $\mathbf{v}_3 \in V_3$. Then since vector addition is associative,

$$\mathbf{v} = (\mathbf{v}_1+\mathbf{v}_2)+\mathbf{v}_3 = \mathbf{v}_1+(\mathbf{v}_2+\mathbf{v}_3) \in V_1+(V_2+V_3).$$

18. The operation of subspace addition has an additive identity, the zero subspace $\{\mathbf{0}\}$. To see this, let U be any subspace of V. Then $U + \{\mathbf{0}\} = U$ as any vector $\mathbf{u} = \mathbf{u} + \mathbf{0}$.

    The only subspace with an additive inverse is the zero subspace. To see this, suppose U, W be subspaces such that $U + W = \{\mathbf{0}\}$. We show that U and W are both $\{\mathbf{0}\}$. We see that if $\mathbf{u} \in U$, then $\mathbf{u} + \mathbf{0} \in \{\mathbf{0}\}$, so $\mathbf{u} + \mathbf{0} = \mathbf{0}$ and $\mathbf{u} = \mathbf{0}$. In a similar vein any $\mathbf{w} \in W$ can be shown to satisfy $\mathbf{w} = \mathbf{0}$. Thus $U, W = \{\mathbf{0}\}$ and the only subspace with an additive inverse under addition is $\{\mathbf{0}\}$, whose additive inverse is itself.

19. This is not true. Consider $V = \mathbb{F}^2$, and the subspaces $V_1 = \{(x, 0) \in \mathbb{F}^2 : x \in \mathbb{F}\}$, $V_2 = \{(0, x) \in \mathbb{F}^2 : x \in \mathbb{F}\}$, and $U = \{(x, x) \in \mathbb{F}^2 : x \in \mathbb{F}\}$. We now show that $\mathbb{F}^2 = V_1 + U$. Let $(x_1 . x_2) \in \mathbb{F}^2$. Then
    $$(x_1, x_2) = (x_1 - x_2, 0) + (x_2, x_2).$$
    In a similar fashion, $\mathbb{F}^2 = V_2 + U$ as
    $$(x_1, x_2) = (x_1, x_1) + (0, x_2 - x_1).$$
    However, clearly $V_1 \neq V_2$.

20. Let
    $$U = \{(x, x, y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$
    Now consider
    $$W = \{(0, x, y, 0) \in \mathbb{F}^4 : x, y \in \mathbb{F}\}.$$
    We claim $\mathbb{F}^4 = U \oplus W$. We first show that $U + W = \mathbb{F}^4$. CLearly $U + W \subseteq \mathbb{F}^4$, so consider $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{F}^4$. Then
    $$\mathbf{x} = (x_1, x_1, x_4, x_4) + (0, x_2 - x_1, x_3 - x_4, 0),$$
    so $\mathbf{x} \in U + W$ and $\mathbb{F}^4 = U + W$. To show that the sum is direct, suppose $\mathbf{v} \in U \cap W$. Then if $\mathbf{v} = (v_1, v_2, v_3, v_4)$, then since $\mathbf{v} \in U$, we have $v_1 = v_2$ and $v_3 = v_4$. However, since $\mathbf{v} \in W$, this implies $v_1 = v_4 = 0$. This means $v_2 = v_3 = 0$, so $\mathbf{v} = \mathbf{0}$ and $U \cap W = \mathbf{0}$. By 1.46, this implies the sum is direct and $\mathbb{F}^4 = U \oplus W$.

21. ($\star$) Let
    $$U = \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 : x, y \in \mathbb{F}\}.$$
    Now consider
    $$W = \{(x, y, z, 0, 0) \in \mathbb{F}^5 : x, y, z \in \mathbb{F}\}.$$
    We claim $\mathbb{F}^5 = U \oplus W$. We first show $\mathbb{F}^5 = U + W$. Clearly, $U + W \subseteq \mathbb{F}^5$. Now consider $\mathbf{x} \in \mathbb{F}^5$, so that $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$. Then
    $$\mathbf{x} = (a, b, a + b, a - b, 2a) + (r, s, t, 0, 0)$$
    with $a = x_5/2$, $b = -x_4 + x_5/2$, $r = x_1 - x_5/2$, $s = x_2 + x_4 - x_5/2$, and $t = x_3 + x_4 - x_5$. Thus $U + W = \mathbb{F}^5$. To prove that this is a direct sum, consider $\mathbf{v} \in U \cap W$. Then if $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$, then since $\mathbf{v} \in U$, we have $v_3 = v_1 + v_2$, $v_4 = v_1 - v_2$, and $v_5 = 2v_1$. Also, since $\mathbf{v} \in W$, we have $v_4 = v_5 = 0$. Thus $v_1 = 0$. Also, since $v_2 = v_1 - v_4$, $v_2 = 0$. Thus $v_3 = 0$ and $\mathbf{v} = \mathbf{0}$. Thus $\mathbb{F}^5 = U \oplus W$.

22. This exercise deals with the same subspace U as in exercise 21. The only modification we need is to split W into three separate subspaces

$$W_1 = \{(x, 0, 0, 0, 0) \in \mathbb{F}^5 \,:\, x \in \mathbb{F}\}$$
$$W_2 = \{(0, x, 0, 0, 0) \in \mathbb{F}^5 \,:\, x \in \mathbb{F}\}$$
$$W_3 = \{(0, 0, x, 0, 0) \in \mathbb{F}^5 \,:\, x \in \mathbb{F}\}.$$

Then by the same arrangement as before, we see that $\mathbb{F}^5 = U + W_1 + W_2 + W_3$. All we need to show is that this sum is still direct. To do that, suppose $\mathbf{u} + \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 = \mathbf{0}$ for $\mathbf{u} \in U$, $\mathbf{w}_1 \in W_1$, $\mathbf{w}_2 \in W_2$, and $\mathbf{w}_3 \in W_3$.Then

$$(x, y, x + y, x - y, 2x) + (a, 0, 0, 0, 0) + (0, b, 0, 0, 0) + (0, 0, c, 0, 0) = \mathbf{0}.$$

This implies

$$(x + a, y + b, x + y + c, x - y, 2x) = (0, 0, 0, 0, 0)$$

giving us the equations

$$\begin{aligned} x + a &= 0 \\ y + b &= 0 \\ x + y + c &= 0 \\ x - y &= 0 \\ 2x &= 0. \end{aligned}$$

The last equation implies $x = 0$. This along with the fourth equation implies $y = x = 0$. The first three equations are then used to imply $a = b = c = 0$. Thus $\mathbf{u} = \mathbf{w}_1 = \mathbf{w}_2 = \mathbf{w}_3 = \mathbf{0}$, so by 1.45, this is a direct sum, and $\mathbb{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

23. This is not true. The counterexample is the same as that in exercise 19. The only thing we must do is verify that the sums are direct. Suppose $\mathbf{v} \in U \cap V_1$. Then $\mathbf{v} = (v_1, v_2)$. Since $\mathbf{v} \in U$, $v_1 = v_2$. Since $\mathbf{v} \in V_1$, $v_2 = 0$. Thus $v_1 = 0$, and $V \cap U_1 = \{\mathbf{0}\}$. By 1.46, this implies that $V = V_1 \oplus U$. A very similar argument can be used to prove $V = V_2 \oplus U$. However. as previously stated, $V_1 \neq V_2$.

24. ($\star$) We first show that $\mathbb{R}^{\mathbb{R}} = V_e + V_o$. Let $f \in \mathbb{R}^{\mathbb{R}}$, and define the functions

$$f_e(x) = \frac{1}{2}[f(x) + f(-x)]$$

and

$$f_o(x) = \frac{1}{2}[f(x) - f(-x)].$$

We first verify that $f_e \in V_e$. Note that

$$f_e(-x) = \frac{1}{2}[f(-x) + f(-(-x))] = \frac{1}{2}[f(-x) + f(x)] = f_e(x),$$

so $f_e \in V_e$. Next, we check that $f_o \in V_o$. Then

$$f_o(-x) = \frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)] = -f_o(x).$$

Thus $f_o \in V_o$. Now we have

$$f_e(x) + f_o(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)] = \frac{1}{2}f(x) + \frac{1}{2}f(x) = f(x).$$

Since f is arbitrary, $\mathbb{R}^{\mathbb{R}} = V_e + V_o$. Now we verify that the sum is direct. Suppose $f \in V_e \cap V_o$. Since $f \in V_e$, we have $f(-x) = f(x)$. However, since $f \in V_o$, we have $f(-x) = -f(x)$. Thus we have $f(x) = -f(x)$ or $2f(x) = 0$, which implies $f(x) = 0$ and f is the zero function. Thus the sum is direct, and $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.