Axler LADR Exercise Solutions Ch 2

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2A: Span and Linear Independence

1. Let $U = \{(x, y, z) \in \mathbb{F}^3 : x+y+z = 0\}$. We claim that $U = \operatorname{span}((1, 0, -1), (0, 1, -1))$. Suppose $\mathbf{x} \in U$. Then $\mathbf{x} = (x_1, x_2, x_3)$ where $x_1 + x_2 + x_3 = 0$. Then $x_3 = -x_1 - x_2$, and
$$\mathbf{x} = (x_1, x_2, -x_1 - x_2) = x_1 \cdot (1, 0, -1) + x_2 \cdot (0, 1, -1),$$
so $\mathbf{x} \in \operatorname{span}((1, 0, -1), (0, 1, -1))$. Now suppose $\mathbf{x} \in \operatorname{span}((1, 0, -1), (0, 1, -1))$. Then
$$\mathbf{x} = c_1(1, 0, -1) + c_2(0, 1, -1) = (c_1, c_2, -c_1 - c_2).$$
Then $c_1 + c_2 + (-c_1 - c_2) = 0$, so $\mathbf{x} \in U$. Thus $U = \operatorname{span}((1, 0, -1), (0, 1, -1))$. We now just need to take these two vectors plus any two vectors also in U: For example,
$$\{(1, 0, -1), (0, 1, -1), (1, 1, -2), (1, -1, 0)\}.$$

2. This is true. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$ span V. Let $\mathbf{v} \in V$. Then there exists $c_1, c_2, c_3, c_4 \in \mathbb{F}$ such that
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4.$$
However, note that with some algebraic manipulation,
$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+c_4\mathbf{v}_4 = c_1(\mathbf{v}_1-\mathbf{v}_2)+(c_1+c_2)(\mathbf{v}_2-\mathbf{v}_3)+(c_1+c_2+c_3)(\mathbf{v}_3-\mathbf{v}_4)+(c_1+c_2+c_3+c_4)\mathbf{v}_4.$$
Thus $\mathbf{w}$ can be written as a linear combination of $\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{v}_3 - \mathbf{v}_4$, and $\mathbf{v}_4$, and as a result, $\mathbf{v}_1 - \mathbf{v}_2$, $\mathbf{v}_2 - \mathbf{v}_3$, $\mathbf{v}_3 - \mathbf{v}_4$, $\mathbf{v}_4$ span V.

3. ($\star$) Let $\mathbf{v}_1, \dots, \mathbf{v}_m \in V$, and for each $k \in \{1, \dots, m\}$, let $\mathbf{w}_k = \mathbf{v}_1 + \ldots + \mathbf{v}_k$. We wish to prove that $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_m) = \operatorname{span}(\mathbf{w}_1, \dots, \mathbf{w}_m)$. There are many ways to prove this: this is a proof by induction on the number of vectors m. Suppose $m = 1$. Then $\mathbf{w}_1 = \mathbf{v}_1$, so it is clear that $\operatorname{span}(\mathbf{v}_1) = \operatorname{span}(\mathbf{w}_1)$. Now let k be an integer. We wish to prove that $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1}) = \operatorname{span}(\mathbf{w}_1, \dots, \mathbf{w}_{k+1})$. Suppose $\mathbf{u} \in \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_{k+1})$. Then
$$\mathbf{u} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1}.$$

Then by the inductive hypothesis $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, there exist constants $d_1, \ldots, d_k \in \mathbb{F}$ such that

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = d_1\mathbf{w}_1 + \ldots + d_k\mathbf{w}_k.$$

Then

$$\begin{aligned}
\mathbf{u} &= c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v}_{k+1} \\
&= d_1\mathbf{w}_1 + \ldots + d_k\mathbf{w}_k + c_{k+1}\mathbf{v}_{k+1} \\
&= d_1\mathbf{w}_1 + \ldots + d_k\mathbf{w}_k + c_{k+1}(\mathbf{w}_{k+1} - \mathbf{w}_k) \\
&= d_1\mathbf{w}_1 + \ldots + (d_k - c_{k+1})\mathbf{w}_k + c_{k+1}\mathbf{w}_{k+1}.
\end{aligned}$$

Thus $\mathbf{u} \in \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{k+1})$. Now suppose $\mathbf{u} \in \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{k+1})$. Then there exist constants $d_1, \ldots, d_{k+1} \in \mathbb{F}$ such that

$$\mathbf{u} = d_1\mathbf{w}_1 + \ldots + d_{k+1}\mathbf{w}_{k+1}.$$

Now by the inductive hypothesis, there exist $c_1, \ldots, c_k \in \mathbb{F}$ such that

$$d_1\mathbf{w}_1 + \ldots + d_k\mathbf{w}_k = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k.$$

Then

$$\begin{aligned}
\mathbf{u} &= d_1\mathbf{w}_1 + \ldots + d_k\mathbf{w}_k + d_{k+1}\mathbf{w}_{k+1} \\
&= c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k + d_{k+1}\mathbf{w}_{k+1} \\
&= c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k + d_{k+1}(\mathbf{v}_1 + \ldots + \mathbf{v}_{k+1}) \\
&= (c_1 + d_{k+1})\mathbf{v}_1 + \ldots + (c_k + d_{k+1})\mathbf{v}_k + d_{k+1}\mathbf{v}_{k+1},
\end{aligned}$$

so $\mathbf{u} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1})$, so $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}) = \text{span}(\mathbf{w}_1, \ldots, \mathbf{w}_{k+1})$, completing the induction.

4. (a) Suppose $\mathbf{v} \in V$. Suppose $\mathbf{v} \neq \mathbf{0}$. Then if we consider

$$c_1\mathbf{v} = \mathbf{0},$$

and since $\mathbf{v} \neq \mathbf{0}$, then $c_1 = 0$, and the list $\{\mathbf{v}\}$ is linearly independent.

If $\mathbf{v} = \mathbf{0}$, then for any $c \neq 0$,

$$c\mathbf{v} = \mathbf{0},$$

so the list $\{\mathbf{v}\}$ is linearly dependent.

(b) Consider the list $\{\mathbf{v}_1, \mathbf{v}_2\}$. Suppose WLOG that $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$, so $\mathbf{v}_1 = c\mathbf{v}_2$. Then

$$\mathbf{v}_1 - c\mathbf{v}_2 = c\mathbf{v}_2 - c\mathbf{v}_2 = \mathbf{0},$$

so the list is not linearly independent.

Now suppose the list is not linearly independent, so that there are constants $c_1, c_2 \in \mathbb{F}$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Suppose WLOG that $c_1 \neq 0$. Then

$$c_1\mathbf{v}_1 = -c_2\mathbf{v}_2$$

and

$$\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2,$$

so $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$.

5. Consider the list

$$(3,1,4), (2,-3,5), (5,9,t).$$

If the list is not linearly independent in $\mathbb{R}^3$, then there exist constants $c_1, c_2, c_3 \in \mathbb{R}$, not all zero, such that

$$c_1(3,1,4) + c_2(2,-3,5) + c_3(5,9,t) = (0,0,0).$$

Thus we have the equations

$$\begin{aligned} 3c_1 + 2c_2 + 5c_3 &= 0 \\ c_1 - 3c_2 + 9c_3 &= 0 \\ 4c_1 + 5c_2 + tc_3 &= 0. \end{aligned}$$

Rewriting these equations gives us

$$\begin{aligned} c_1 - 3c_2 + 9c_3 &= 0 \\ 3c_1 + 2c_2 + 5c_3 &= 0 \\ 4c_1 + 5c_2 + tc_4 &= 0 \end{aligned}$$

Now subtracting the first equation from the second and third gives us

$$\begin{aligned} c_1 - 3c_2 + 9c_3 &= 0 \\ 11c_2 - 22c_3 &= 0 \\ 17c_2 + (t-36)c_3 &= 0 \end{aligned}$$

Dividing the second equation by 2 gives us

$$\begin{aligned} c_1 - 3c_2 + 9c_3 &= 0 \\ c_2 - 2c_3 &= 0 \\ 17c_2 + (t-36)c_3 &= 0 \end{aligned}$$

and subtracting 17 times the second equation gives us $(t-2)c_3 = 0$. Now if $t-2 \neq 0$, then $c_3 = 0$, which forces $c_2 = c_1 = 0$. Thus $t-2 = 0$ for linear dependence, and $t = 2$. If $t = 2$, then a possible arrangement is

$$-3 \cdot (3,1,4) + 2 \cdot (2,-3,5) + 1 \cdot (5,9,2) = (0,0,0).$$

6. Consider the list $\{(2,3,1),(1,-1,2),(7,3,c)\}$. Suppose they are linearly dependent in $\mathbb{F}^3$. Then there are constants $a_1, a_2, a_3 \in \mathbb{F}$, not all 0, such that

$$a_1(2,3,1)+a_2(1,-1,2)+a_3(7,3,c)=(0,0,0).$$

Then

$$\begin{aligned} 2a_1+a_2+7a_3&=0\\ 3a_1-a_2+3a_3&=0\\ a_1+2a_2+ca_3&=0 \end{aligned}$$

Subtracting (1/2) of equation 1 from equation 3 and (3/2) of equation 1 from equation 2 gives us the following two equations:

$$\begin{aligned} -\frac{5}{2}a_2-\frac{15}{2}a_3&=0\\ \frac{3}{2}a_2+\left(c-\frac{7}{2}\right)a_3&=0 \end{aligned}$$

We can further simplify these equations to read:

$$\begin{aligned} a_2+3a_3&=0\\ 3a_2+(2c-7)a_3&=0 \end{aligned}$$

Now subtracting 3 times the top equation from the bottom equation gives us

$$(2c-16)a_3=0$$

Now, either $2c-16=0$ or $a_3=0$. If $a_3=0$, the other equations imply $a_2=a_1=0$, so we must have $2c-16=0$ or $c=8$.

Now suppose $c=8$. Then the list is linearly dependent:

$$-2\cdot(2,3,1)-3\cdot(1,-1,2)+(7,3,8)=(0,0,0).$$

NOTE: The process used is called row reduction and should be familiar from a first course in linear algebra. The linear dependence constants can be determined by the same equations we solved: since the third equation is $0=0$ if $c=8$ (as shown above), a_3 can be anything, so we let $a_3=1$ and use the equations to back-solve for a_2 and a_1.

7. (a) Suppose $\mathbb{C}$ is a vector space over $\mathbb{R}$. Suppose there are numbers $c_1, c_2 \in \mathbb{R}$ such that

$$c_1(1+i)+c_2(1-i)=0.$$

Then $c_1+c_2=0$ and $c_1-c_2=0$ by equating real and imaginary parts. Then $c_1=c_2$, $2c_1=0$, and $c_1=c_2=0$, so the list is linearly independent.

(b) Suppose $\mathbb{C}$ is a vector space over $\mathbb{C}$. Then the list $\{1+i, 1-i\}$ is linearly dependent:

$$(1+i) - i(1-i) = 0.$$

8. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in V$ are linearly independent. Now suppose $c_1, c_2, c_3, c_4 \in \mathbb{F}$ are constants such that

$$c_1(\mathbf{v}_1 - \mathbf{v}_2) + c_2(\mathbf{v}_2 - \mathbf{v}_3) + c_3(\mathbf{v}_3 - \mathbf{v}_4) + c_4\mathbf{v}_4 = \mathbf{0}.$$

We can now rewrite this expression as

$$c_1\mathbf{v}_1 + (c_2 - c_1)\mathbf{v}_2 + (c_3 - c_2)\mathbf{v}_3 + (c_4 - c_3)\mathbf{v}_4 = \mathbf{0}.$$

From linear independence of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, we know that $c_1 = c_2 - c_1 = c_3 - c_2 = c_4 - c_3 = 0$. Since $c_1 = 0$, we know $c_2 = c_1 = 0$. Similarly, $c_3 = c_2 = 0$ and $c_4 = c_3 = 0$. Thus the vectors $\mathbf{v}_1 - \mathbf{v}_2, \mathbf{v}_2 - \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_4, \mathbf{v}_4$ are linearly independent.

9. This is true. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are linearly independent vectors. Let $c_1, \ldots, c_m \in \mathbb{F}$ such that

$$c_1(5\mathbf{v}_1 - 4\mathbf{v}_2) + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

Then rewriting this gives us

$$5c_1\mathbf{v}_1 + (c_2 - 4c_1)\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

From the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_m$, we see that $5c_1 = 0$, $c_2 - 4c_1 = 0$, and $c_3 = \cdots = c_m = 0$. Since $5c_1 = 0$, $c_1 = 0$. Then $c_2 - 4c_1 = c_2 = 0$. Thus all constants are 0, and the list $5\mathbf{v}_1 - 4\mathbf{v}_2, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

10. This is true. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ form a list of linearly independent vectors, and let $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Now let $c_1, \ldots, c_m \in \mathbb{F}$ be such that

$$c_1(\lambda\mathbf{v}_1) + \ldots + c_m(\lambda\mathbf{v}_m) = \mathbf{0}.$$

Then

$$(\lambda c_1)\mathbf{v}_1 + \ldots + (\lambda c_m)\mathbf{v}_m = \mathbf{0}.$$

From linear independence, $\lambda c_i = 0$ for all $i \in \{1, \ldots, m\}$. Since $\lambda \neq 0$, this implies $c_i = 0$ for all $i \in \{1, \ldots, m\}$, and $\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_m$ are linearly independent.

11. This is false. Consider $V = \mathbb{R}^2$, and suppose $\mathbf{v}_1 = (1,0)$, $\mathbf{v}_2 = (0,1)$, $\mathbf{w}_1 = (-1,0)$, and $\mathbf{w}_2 = (0,-1)$. Then we can see easily that $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{w}_1, \mathbf{w}_2$ are both linearly independent lists, but both $\mathbf{v}_1 + \mathbf{w}_1 = \mathbf{0}$ and $\mathbf{v}_2 + \mathbf{w}_2 = \mathbf{0}$, and clearly $\{\mathbf{0}, \mathbf{0}\}$ is a linearly dependent list.

12. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ are linearly independent. Let $\mathbf{w} \in V$ such that $\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_m + \mathbf{w}$ are linearly dependent. Then there exist constants $c_1, \ldots, c_m \in \mathbb{F}$, not all zero, such that

$$c_1(\mathbf{v}_1 + \mathbf{w}) + \ldots + c_m(\mathbf{v}_m + \mathbf{w}) = \mathbf{0}.$$

Then we can write

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m + (c_1 + \ldots + c_m)\mathbf{w} = \mathbf{0},$$

or

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = -(c_1 + \ldots + c_m)\mathbf{w}.$$

Now suppose $c_1 + \ldots + c_m = 0$. Then we would have

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = -0\mathbf{w} = \mathbf{0}.$$

However, since not all c_i are zero, this would contradict the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. So $c_1 + \ldots + c_m \neq 0$, and thus

$$\mathbf{w} = -\frac{c_1}{\sum c_i}\mathbf{v}_1 + \ldots - \frac{c_m}{\sum c_i}\mathbf{v}_m,$$

and $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

13. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ is linearly independent, and let $\mathbf{w} \in V$. First suppose $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Then there exist constants $c_1, \ldots, c_m \in \mathbb{F}$ such that

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m.$$

Then

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m - \mathbf{w} = \mathbf{0},$$

so $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}$ is linearly dependent. Now suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m.\mathbf{w}$ is linearly dependent. Then there exist constants $c_1, \ldots, c_m, c \in \mathbb{F}$, not all zero, such that

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m + c\mathbf{w} = \mathbf{0}.$$

Now suppose $c = 0$. Then $c\mathbf{w} = \mathbf{0}$, and

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

However, one of the c_i must be nonzero, which contradicts the linear independence of $\mathbf{v}_1, \ldots, \mathbf{v}_m$. So $c \neq 0$, and we can write

$$\mathbf{w} = -\frac{c_1}{c}\mathbf{v}_1 - \ldots - \frac{c_m}{c}\mathbf{v}_m$$

and $\mathbf{w} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$.

14. ($\star$) We will prove this statement by induction on the number of vectors k. If $m = 1$, then the statement is trivial, since $\mathbf{v}_1 = \mathbf{w}_1$. Now suppose the statement is true for $m = k$. We wish to prove that $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ is linearly independent if and only if $\mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$ is linearly independent.

First suppose $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ is linearly independent. Then the sublist $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is linearly independent (the proof of this fact is simple). By the inductive hypothesis, this means $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is linearly independent. We now show that $\mathbf{w}_{k+1} \notin \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, because then exercise 13 implies that $\mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$ is linearly independent. If $\mathbf{w}_{k+1} \in \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$, then by exercise 3, $\mathbf{w}_{k+1} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, and

$$\mathbf{w} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$$

for some $c_1, \ldots, c_k \in \mathbb{F}$. Then

$$\mathbf{v}_1 + \ldots + \mathbf{v}_{k+1} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k,$$

so

$$\mathbf{v}_{k+1} = (c_1 - 1)\mathbf{v}_1 + \ldots + (c_k - 1)\mathbf{v}_k$$

and $\mathbf{v}_{k+1} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. However, using exercise 13, we know that $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ is linearly dependent, contradicting our initial assumption that $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ was linearly independent. So $\mathbf{w}_{k+1} \notin \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ and $\mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$ is linearly independent.

The reverse direction is very similar. If $\mathbf{w}_1, \ldots, \mathbf{w}_{k+1}$ is linearly independent, then so is $\mathbf{w}_1, \ldots, \mathbf{w}_k$. By the inductive hypothesis, this implies $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is also linearly independent. We now show that $\mathbf{v}_{k+1} \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Suppose for the sake of contradiction that $\mathbf{v}_{k+1} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. Then by exercise 3, $\mathbf{v}_{k+1} \in \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$ Then there exists $c_1, \ldots, c_k \in \mathbb{F}$

$$\mathbf{v}_{k+1} = c_1\mathbf{w}_1 + \ldots + c_k\mathbf{w}_k$$

However, since $\mathbf{v}_{k+1} = \mathbf{w}_{k+1} - \mathbf{w}_k$,

$$\mathbf{w}_{k+1} = c_1\mathbf{w}_1 + \ldots + (c_k + 1)\mathbf{w}_k$$

and $\mathbf{w}_{k+1} \in \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. But this implies $\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{w}_{k+1}$ is linearly dependent, a contradiction. Thus $\mathbf{v}_{k+1} \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ is linearly independent, as desired. This completes the induction.

NOTE: A lot of previous exercises were used. However, the benefit of this is that this avoids a lot of nasty expressions involving constants if we had tried to use "brute force" and prove the statement directly.

15. Consider the list $\{1, x, x^2, x^3, x^4\}$ in $\mathcal{P}_4(\mathbb{F})$. Then this list is a spanning list: If $p \in \mathcal{P}_4(\mathbb{F})$, then

$$p = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

and thus is trivially a linear combination of $1, x, x^2, x^3, x^4$. However by 2.22, the length of any linearly independent list cannot be greater than the length of any spanning list, and this spanning list has length 5, so there is no list of six polynomials in $\mathcal{P}_4(\mathbb{F})$ that is linearly independent.

16. Consider the list $\{p_0, p_1, p_2, p_3, p_4\}$ in $\mathcal{P}_4(\mathbb{F})$, where

$$\begin{aligned} p_0 &= x^4 - 10x^3 + 35x^2 - 50x + 24 = (x-1)(x-2)(x-3)(x-4) \\ p_1 &= x^4 - 9x^3 + 26x^2 - 24x = x(x-2)(x-3)(x-4) \\ p_2 &= x^4 - 8x^3 + 19x^2 - 12x = x(x-1)(x-3)(x-4) \\ p_3 &= x^4 - 7x^3 + 14x^2 - 8x = x(x-1)(x-2)(x-4) \\ p_4 &= x^4 - 6x^3 + 11x^2 - 6x = x(x-1)(x-2)(x-3). \end{aligned}$$

We now prove that this list is linearly independent. Suppose there exist constants $c_0, \ldots, c_4 \in \mathbb{F}$ such that

$$c_0 p_0 + \ldots + c_4 p_4 = 0,$$

the zero polynomial. This would mean that

$$c_0 p_0(x) + \ldots + c_4 p_4(x) = 0$$

for all x. However, note that $p_i(i) \neq 0$ and $p_i(j) \neq 0$ for $i \neq j$ (the reason why these polynomials were chosen). This means that for $x = i$ for $i \in \{0, 1, 2, 3, 4\}$, $c_i p_i(i) = 0$. But since $p_i(i) \neq 0$, this implies $c_i = 0$ for all i. Thus these polynomials are linearly independent.

But by 2.22, the length of any spanning must be at least the length of any linearly independent list, and this is a linearly independent list of length 5, so a spanning list must have length at least 5. Thus no list of four polynomials spans $\mathcal{P}_4(\mathbb{F})$.

17. Suppose there exists a sequence $\mathbf{v}_1, \mathbf{v}_2, \ldots$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent for all m. Suppose for the sake of contradiction that V is finite-dimensional. Then there exists a spanning list $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of V. However, $\mathbf{v}_1, \ldots, \mathbf{v}_{n+1}$ is linearly independent by the definition of our sequence, so we have a linearly independent list that is longer in length than a spanning list, which contradicts 2.22. Thus V cannot be finite-dimensional, and V is infinite-dimensional.

Now suppose V is infinite-dimensional. Select $\mathbf{v}_1 \in V$ such that $\mathbf{v}_1 \neq \mathbf{0}$ (this exists, since $\{\mathbf{0}\}$ is finite dimensional). Now, we will construct a sequence according to following rule: given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_i$ in our sequence, select $\mathbf{v}_{i+1} \in V$ such that $\mathbf{v}_{i+1} \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_i)$. Such a vector must exist in V since otherwise $\mathbf{v}_1, \ldots, \mathbf{v}_i$ would form a spanning list for V and V would be finite-dimensional, which is not true. We claim that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent for all m. We can use an inductive argument: since $\mathbf{v}_1 \neq \mathbf{0}$, then $\mathbf{v}_1$ is linearly independent. Now suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is linearly independent. Since $\mathbf{v}_{k+1}$ was chosen such that $\mathbf{v}_{k+1} \notin \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, by the inductive hypothesis and exercise 13, we must have $\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}$ as linearly independent.

18. We can use exercise 17. Consider the sequence defined by

$$\mathbf{v}_i = (0, 0, \ldots, 0, 0, 1, 0, 0, \ldots)$$

where $\mathbf{v}_i$ has a 1 in the i-th position and zeros everywhere else. Then $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent for all m. To see this, suppose

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = \mathbf{0}.$$

Then

$$c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = (c_1, c_2, \ldots, c_m, 0, 0, \ldots) = (0, 0, 0, \ldots)$$

and $c_1 = \cdots = c_m = 0$.

19. ($\star$) We use exercise 17 again. Consider the infinite sequence of continuous functions defined by

$$f_i = x^i$$

for $i \geq 0$ We prove that $f_0, \ldots, f_m$ is linearly independent. To do this, suppose there exist constants $c_0, \ldots, c_m \in \mathbb{R}$, not all zero, such that

$$c_0 f_0 + \ldots + c_m f_m = 0,$$

so

$$c_0 + c_1 x + \ldots + c_m x^m = 0.$$

There are many ways to prove that $c_0 = \cdots = c_m = 0$: this is one way. Suppose that there exists some $c_i \neq 0$. Then we can take i derivatives of both sides. This gives us a polynomial

$$\frac{c_i}{i!} + \ldots + Cx^{m-i} = 0.$$

Now evaluating this expression at $x = 0$ gives us the contradiction

$$\frac{c_i}{i!} = 0,$$

so $c_i = 0$ for all i. Thus $1, x, x^2, \ldots, x^m$ is linearly independent, and by exercise 17, the set of continuous real-valued functions on $[0, 1]$ is infinite-dimensional.

NOTE: Some may be unhappy with the use of calculus, since the space in question is made up of continuous functions. However, the polynomials are also continuous, and there are no other stipulations required on the elements of a sequence. There are other sequences you can use: one I think of at the top of my head are piecewise functions defined such that they are nonzero on $[2^{-m}, 2^{-m+1}]$ and zero everywhere else. But it can get messy, and I don't like messy things.

20. ($\star$) Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_k(2) = 0$. To prove that they are not linearly independent, consider the set $U = \{p \in \mathcal{P}_m(\mathbb{F}) : p(2) = 0\}$. This is a subspace (as you can verify). let $p \in U$. Then

$$p(x) = a_0 + a_1 x + \ldots + a_m x^m$$

and

$$p(2) = a_0 + a_1 \cdot 2 + \ldots + a_m \cdot 2^m = 0$$

Then if we solve for a_0, we notice that

$$a_0 = -a_1 \cdot 2 - \ldots - a_m \cdot 2^m,$$

so substituting into the expression for $p(x)$, we see that

$$p(x) = a_1(x-2) + \ldots + a_m(x^m - 2^m).$$

Thus we claim that the set $\{x-2, x^2-2^2, \ldots, x^m-2^m\}$ spans U. As we shown above, $U \subseteq \text{span}(x-2, \ldots, x^m-2^m)$. Now, if we denote $f_i(x) = x^i - 2^i$, then $f_i(2) = 0$ and each $f_i \in U$. Since U is a subspace, this implies $\text{span}(f_1, \ldots, f_m) \subseteq U$ and thus $U = \text{span}(x-2, \ldots, x^m-2^m)$. Thus a list of m vectors forms a spanning list for U. Now consider $p_0, p_1, \ldots, p_m$, as defined above, Then $p_0, p_1, \ldots, p_m \in U$. Now considering U is a finite-dimensional vector space itself, we can apply 2.22 to U, so any linearly independent list must be less than or equal to m. Thus no linearly independent list of length $m+1$ exists in U, so there exist constants $c_0, \ldots, c_m \in \mathbb{F}$ such that

$$c_0 p_0 + \ldots + c_m p_m = 0.$$

Since the same choice of constants work regardless of the space, the list $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$.

NOTE: Even though this is not an extremely difficult exercise, this is a truly beautiful result, because it is totally nonconstructive: we never construct the constants and only have the seemingly basic information that $p_k(2) = 0$ for all k. The step of constructing a subspace to use as an intermediary is also a tricky one this early on, and is part of what cements this problem as one of my favorites from the entire book.

2B: Bases

1. Note that $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is a basis for V, then so is any nonzero constant multiple of the vectors, i.e. $\lambda\mathbf{v}_1, \ldots, \lambda\mathbf{v}_m$ is also a basis for V for $\lambda \neq 0$. Thus if we have even one nonzero vector $\mathbf{v}$, then we can form infinitely many bases. Thus the only possibility is $\{\mathbf{0}\}$. However, the span of the empty list is defined to be $\{\mathbf{0}\}$, and the empty list is vacuously linearly independent, so $\{\mathbf{0}\}$ is the only vector space with exactly one basis.

   NOTE: This uses the assumption stated throughout the chapter that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. However, if $\mathbb{F}$ is allowed to be a finite field, then there are only finitely many choices of λ. Specifically, letting $V = \mathbb{Z}_2 = \{0, 1\}$ over the field $\mathbb{F} = \mathbb{Z}_2$, with vector addition being defined as addition in $\mathbb{Z}_2$ and scalar multiplication defined as multiplication on $\mathbb{Z}_2$ gives us another vector space with only one basis, namely $\{1\}$. However, finite fields aren't examined in detailed in this book, so for all intensive purposes, these other answers do not really matter.

2. (a) This is trivial to show that it is a basis.

   (b) This is a basis. Note that $(1, 2), (3, 5)$ is linearly independent, as no vector is a constant multiple of another (the proof of this is exercise 2A.4b). It also spans $\mathbb{F}^2$: If $\mathbf{v} = (v_1, v_2) \in \mathbb{F}^2$, then
   $$\mathbf{v} = (-5v_1 + 3v_2) \cdot (1, 2) + (2v_1 - v_2) \cdot (3, 5).$$
   So this list is a basis.

   (c) This list does not span $\mathbb{F}^3$, as it is a two element list and any list spanning $\mathbb{F}^3$ must have length at least 3 (due to the existence of the linearly independent list $(1, 0, 0), (0, 1, 0), (0, 0, 1)$ and 2.22)

   (d) This list is not linearly independent, as it is of length 3, and any linearly independent list of $\mathbb{F}^2$ must be of length at most 2 (due to the existence of the spanning list $(1, 0), (0, 1)$ and 2.22)

   (e) Note that $(1, 1, 0), (0, 0, 1)$ is linearly independent as none is a scalar multiple of the other. Now let $\mathbf{v} \in \{(x, x, y) \in \mathbb{F}^3 : x, y \in \mathbb{F}\}$, so $\mathbf{v} = (v_1, v_1, v_2)$. Then
   $$(v_1, v_1, v_2) = v_1 \cdot (1, 1, 0) + v_2 \cdot (0, 0, 1).$$
   Also, $(1, 1, 0), (0, 0, 1)$ are both in this subspace, so this list spans this subspace. Thus this list forms a basis for this subspace.

   (f) The list $(1, -1, 0), (1, 0, -1)$ is linearly independent as neither is a scalar multiple of the other. Now let $U = \{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$. Then since $(1, -1, 0), (1, 0, -1) \in U$ and U is a subspace, so is $\operatorname{span}((1, -1, 0), (1, 0, -1))$. Now let $\mathbf{v} \in U$, so $\mathbf{v} = (v_1, v_2, v_3)$ such that $v_1 + v_2 + v_3 = 0$. Then $v_1 = -v_2 - v_3$ and $\mathbf{v} = (-v_2 - v_3, v_2, v_3)$, and
   $$\mathbf{v} = -v_3 \cdot (1, 0, -1) - v_2 \cdot (1, -1, 0)$$
   and $(1, 0, -1), (1, -1, 0)$ spans U. Thus it forms a basis for U.

(g) Showing that $1, z, \ldots, z^m$ spans $\mathcal{P}_m(\mathbb{F})$ is trivial. Also, we can prove that $1, x, \ldots, x^m$ is linearly independent in a similar way as shown in exercise 2A.19. Thus it forms a basis for $\mathcal{P}_m(\mathbb{F})$.

3. (a) Let
$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \,:\, x_1 = 3x_2, x_3 = 7x_4\}.$$
We propose that a basis for U is
$$\{(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)\}.$$
To prove that this is indeed a basis for U, we first show linear independence. Note that if
$$c_1 \cdot (3,1,0,0,0) + c_2 \cdot (0,0,7,1,0) + c_3 \cdot (0,0,0,0,1) = (3c_1, c_1, 7c_2, c_2, c_3) = (0,0,0,0,0),$$
then clearly $c_1 = c_2 = c_3 = 0$. Now to show that this list spans U, note that if $(x_1, x_2, x_3, x_4, x_5) \in U$, then $x_1 = 3x_2$ and $x_3 = 7x_4$. Then we can write
$$(x_1, x_2, x_3, x_4, x_5) = (3x_2, x_2, 7x_4, x_4, x_5)$$
and
$$(x_1, x_2, x_3, x_4, x_5) = x_2 \cdot (3,1,0,0,0) + x_4 \cdot (0,0,7,1,0) + x_5 \cdot (0,0,0,0,1).$$
Thus this list is a basis for U.

(b) We can extend this list to a basis of $\mathbb{R}^5$ by appending the standard basis of $\mathbb{R}^5$ to the list:
$$(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0), (0,1,0,0,0), (0,0,1,0,0), (0,0,0,1,0).$$
(Note that the fifth basis vector is already in our original basis, so we don't add it again.) Then we can perform the process described in the proof of 2.30, which uses the linear dependence lemma repeatedly to remove vectors that are in the span in the previous vectors:
$$(0,0,0,1,0) = (0,0,7,1,0) - 7 \cdot (0,0,1,0,0)$$
so this vector is removed. The vector $(0,0,1,0,0)$ cannot be written as a linear combination of the vectors before it, so it stays. However,
$$(0,1,0,0,0) = (3,1,0,0,0) - 3 \cdot (1,0,0,0,0)$$
so it gets removed. Now, $(1,0,0,0,0)$ cannot be written as any of the previous vectors, and the first three vectors are linearly independent. Thus we have our basis for $\mathbb{R}^5$:
$$(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1), (1,0,0,0,0), (0,0,1,0,0).$$

(c) We can use our basis from above to define

$$W = \text{span}((1,0,0,0,0),(0,0,1,0,0)) = \{(x,0,y,0,0) \in \mathbb{R}^5 : x, y \in \mathbb{R}^5\}.$$

Then by 2.33, $\mathbb{R}^5 = U \oplus W$.

4. (a) Let $U = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$ Then a basis for U is given by

$$\{(1,6,0,0,0),(0,0,-2,1,0),(0,0,-3,0,1)\}.$$

To show linear independence, suppose

$$c_1 \cdot (1,6,0,0,0) + c_2 \cdot (0,0,-2,1,0) + c_3 \cdot (0,0,-3,0,1) = (c_1, 6c_1, -2c_2 - 3c_3, c_2, c_3) = (0,0,0,0,0).$$

Note that matching the arguments implies $c_1 = c_2 = c_3 = 0$. To show span, note that if $(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5$ then since $6z_1 = z_2$ and $z_3 + 2z_4 + 3z_5 = 0$, we can replace $z_2 = 6z_1$ and $z_3 = -2z_4 - 3z_5$. Then

$$(z_1, z_2, z_3, z_4, z_5) = z_1 \cdot (1,6,0,0,0) + z_4 \cdot (0,0,-2,1,0) + z_5 \cdot (0,0,-3,0,1).$$

(b) Using a similar process as in exercise 3, we can extend the basis in (a) to the following basis:

$$(1,6,0,0,0),(0,0,-2,1,0),(0,0,-3,0,1),(1,0,0,0,0),(0,0,1,0,0).$$

(c) Define $W = \{(z_1, 0, z_2, 0, 0) \in \mathbb{C}^5 : z_1, z_2 \in \mathbb{C}\}$. Then $W = \text{span}((1,0,0,0,0),(0,0,1,0,0))$ and by 2.33, $\mathbb{C}^5 = V \oplus W$.

5. Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Since U and W are subspaces, there exist bases $\mathbf{u}_1, \ldots, \mathbf{u}_m$ of U and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ of W. Now consider the list $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n$. We prove that this is a spanning list for V. If $\mathbf{v} \in V$, then since $V = U + W$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. Then since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ form a basis for U there exist $c_1, \ldots, c_m \in \mathbb{F}$ such that

$$\mathbf{u} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m.$$

Similarly, since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ form a basis for W, there exist $d_1, \ldots, d_n \in \mathbb{F}$ such that

$$\mathbf{w} = d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n.$$

Then

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m + d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n,$$

so $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n$ form a spanning list for V. Then by 2.30, we can form a basis for V by removing some vectors from this combined list. Thus we have formed a basis of V consisting entirely of vectors either in U or W, or vectors in $U \cup W$.

6. This is false. Consider the polynomials, $1, x, x^3 + x^2, x^3$. This list is linearly independent in $\mathcal{P}_3(\mathbb{F})$. To see this, let $a_0, a_1, a_2, a_3 \in \mathbb{F}$ such that

$$a_0 + a_1 x + a_2(x^3 + x^2) + a_3 x^3 = 0.$$

Then expanding, we can write

$$a_0 + a_1 x + a_2 x^2 + (a_2 + a_3)x^3 = 0.$$

But $\{1, x, x^2, x^3\}$ is linearly independent, so $a_0 = a_1 = a_2 = 0$ and $a_3 = 0$ since $a_2 + a_3 = 0 + a_3 = 0$. It is also a spanning list, since if we consider $p \in \mathcal{P}_3(\mathbb{F})$, then

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

But we can write

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_2 x^3 - a_2 x^3 + a_3 x^3 = a_0 + a_1 x + a_2(x^2 + x^3) + (a_3 - a_2)x^3.$$

so this is a spanning list for $\mathcal{P}_3(\mathbb{F})$. Thus this is a basis for $\mathcal{P}_3(\mathbb{F})$. However, there is no polynomial of degree 2.

7. This is false. Consider $V = \mathbb{R}^4$ and the standard basis $(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)$. Now consider the subspace $U = \operatorname{span}((1,0,0,0), (0,1,0,0), (0,0,1,1))$. Clearly $(1,0,0,0), (0,1,0,0)$ do not form a basis for U, but $(0,0,1,0)$ and $(0,0,0,1)$ are not in U, as the third and fourth elements in any vector in this subspace must be the same.

8. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ and suppose $\mathbf{w}_k = \mathbf{v}_1 + \ldots + \mathbf{v}_k$ for $k \in \{1, \ldots, m\}$. By exercise 2A.3, $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \operatorname{span}(\mathbf{w}_1, \ldots, \mathbf{w}_m)$, so $\mathbf{v}_1, \ldots, \mathbf{v}_m$ spans V if and only if $\mathbf{w}_1, \ldots, \mathbf{w}_m$ spans V. By exercise 2A.14, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ is linearly independent if and only if $\mathbf{w}_1, \ldots, \mathbf{w}_m$ is linearly independent. Thus $\mathbf{v}_1, \ldots, \mathbf{v}_m$ forms a basis in V if and only if $\mathbf{w}_1, \ldots, \mathbf{w}_m$ forms a basis in V.

9. Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is a basis of U and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is a basis of W. To show that $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n$ form a basis in V, we first show linear independence. Suppose there exist $c_1, \ldots, c_m, d_1, \ldots, d_n \in \mathbb{F}$ such that

$$c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m + d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n.$$

Now let $\mathbf{u} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m \in U$ and $\mathbf{w} = d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n \in W$. Since $V = U \oplus W$, If $\mathbf{0} = \mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$, then $\mathbf{u} = \mathbf{0}$ and $\mathbf{w} = \mathbf{0}$. Thus

$$c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m = 0$$
$$d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n = 0.$$

However, since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ is linearly independent, $c_1 = \ldots = c_m = 0$. Similarly, since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ is linearly independent, $d_1 = \ldots = d_n = 0$. Thus all coefficients are 0 and $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent. Now we show they span V. Let $\mathbf{v} \in V$. Since $V = U \oplus W$, there exist vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$.

Since $\mathbf{u}_1, \ldots, \mathbf{u}_m$ span U and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ span W, there exist $c_1, \ldots, c_m \in \mathbb{F}$ and $d_1, \ldots, d_n \in \mathbb{F}$ such that

$$\begin{aligned}\mathbf{u} &= c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m \\ \mathbf{w} &= d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n.\end{aligned}$$

Then

$$\mathbf{v} = \mathbf{u} + \mathbf{w} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m + d_1\mathbf{w}_1 + \ldots + d_n\mathbf{w}_n$$

and $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{w}_1, \ldots, \mathbf{w}_n$ span V. So this list forms a basis for V.

2C: Dimension

1. Let $U \subseteq \mathbb{R}^2$ be a subspace. Then it must have either dimension 0, 1, or 2. If $\dim U = 2$, then $U = \mathbb{R}^2$ by 2.39. If $\dim U = 1$, then U has a single basis vector $\mathbf{v} \in \mathbb{R}^2$. Then every vector (x, y) in U is $t \cdot (v_1, v_2) = (tv_1, tv_2)$. Then $x = cv_1$ and $y = cv_2$, and substituting for c, we get $y = (v_2/v_1)x$, a line through the origin. If $\dim U = 0$, then $U = \{\mathbf{0}\}$. Thus the only subspaces in $\mathbb{R}^2$ are $\{\mathbf{0}\}$, $\mathbb{R}^2$, and lines passing through the origin.

2. Let $U \subseteq \mathbb{R}^3$ be a subspace. Then it must have a dimension of 0, 1, 2, or 3. If $\dim U = 3$, then $U = \mathbb{R}^3$ by 2.39. If $\dim U = 1$, then $U = \operatorname{span}((v_1, v_2, v_3))$ for some $(v_1, v_2, v_3) \in \mathbb{R}^3$ and any vector in U is equal to $(v_1 t, v_2 t, v_3 t)$ for some t. These are the parametric equations of a line through the origin. Now suppose $\dim U = 2$. Then $U = \operatorname{span}((u_1, u_2, u_3), (v_1, v_2, v_3))$. Then every vector in our subspace can be represented as
$$(x, y, z) = s \cdot (u_1, u_2, u_3) + t \cdot (v_1, v_2, v_3) = (su_1 + tv_1, su_2 + tv_2, su_3 + tv_3).$$
Note that this is a parametric representation of a plane in $\mathbb{R}^3$ through the origin. (You can also demonstrate this by using the first two arguments to solve for s and t and obtaining the standard form). Thus the only subspaces of $\mathbb{R}^3$ are $\{\mathbf{0}\}$, lines through the origin, planes through the origin, and $\mathbb{R}^3$.

3. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{F}) : p(6) = 0\}$. Note that $x - 6$, $x(x - 6)$, $x^2(x - 6)$, and $x^3(x - 6)$ satisfy $p(6) = 0$. Note also that they are linearly independent. To see this, suppose
$$a_1(x - 6) + a_2 x(x - 6) + a_3 x^2(x - 6) + a_4 x^3(x - 6) = 0.$$
Then $a_4 = 0$ as there are no x^4 terms on the right hand side. The coefficient of x^3 is $a_3 - 6a_4$, which must be 0. Since $a_4 = 0$, $a_3 = 0$. We can show similarly that $a_2 = 0$, and thus $a_1 = 0$. Thus we have a list of four linearly independent vectors in U. Thus $4 \le \dim U \le \dim \mathcal{P}_4(\mathbb{F}) = 5$. But if $\dim U = 5$, then $U = \mathcal{P}_4(\mathbb{F})$, which is not true as $p(6) \neq 0$ for $p(x) = 1$. Thus $\dim U = 4$ and $x - 6, x(x - 6), x^2(x - 6), x^3(x - 6)$ forms a basis for U by 2.38.
   (b) Consider adding 1. Then the resulting list is linearly independent: if
$$a_0 + a_1(x - 6) + a_2 x(x - 6) + a_3 x^2(x - 6) + a_4 x^3(x - 6) = 0,$$
then $a_1 = a_2 = a_3 = a_4 = 0$ by the same reasoning as before. Then the constant term is $a_0 - 6a_1 = 0$. Since $a_1 = 0$, $a_0 = 0$. So we have a linearly independent list of length 5, so it forms a basis for $\mathcal{P}_4(\mathbb{F})$ by 2.38.
   (c) Let $W = \{p \in \mathcal{P}_4(\mathbb{F}) : p(x) = c, c \neq 0\}$. Then $\mathcal{P}_4(\mathbb{F}) = U \oplus W$ by 2.33.

4. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p''(6) = 0\}$. Then note that $1, x, (x - 6)^3, (x - 6)^4$ all satisfy these conditions. We prove that this
$$a_0 + a_1 x + +a_2(x - 6)^3 + a_3(x - 6)^4.$$

We know that the coefficient of x^4 must be 0 on the right hand side, and the left hand side has a a_3x^4 term, so $a_3 = 0$. Similarly, $a_2 = 0$ as the left hand side has an a_2x^3 term. Thus $a_0 = a_1 = 0$ trivially. Thus we have a linearly independent list of 4 vectors, so $4 \leq \dim U \leq \dim \mathcal{P}_4(\mathbb{R}) = 5$. However, $x^2 \notin U$, as $p''(6) = 2 \neq 0$. So $\dim U = 4$, and this list forms a basis for U.

(b) Consider adding x^2. Then since $x^2 \notin U$, $x^2 \notin \operatorname{span}(1, x, (x-6)^3, (x-6)^4)$ and as a result it maintains linear independence (exercise 2A.13). Thus $1, x, x^2, (x-6)^3, (x-6)^4$ is a linear independent list of length 5. Thus $\{1, x, x^2, (x-6)^3, (x-6)^4\}$ is a basis for $\mathcal{P}_4(\mathbb{R})$.

(c) Let $W = \{p \in \mathcal{P}_4(\mathbb{R}) : p(x) = cx^2, c \in \mathbb{R}\}$. Then $\mathcal{P}_4(\mathbb{R}) = U \oplus W$ by 2.33.

5. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : p(2) = p(5)\}$. Note that $1, (x-2)(x-5), x(x-2)(x-5), x^2(x-2)(x-5)$ satisfy these conditions and form a linearly independent list. So $\dim U \geq 4$. However $\dim U \neq 5$ because that would imply $U = \mathcal{P}_4(\mathbb{R})$ by 2.39, which is not true as $x \notin U$. Thus this list forms a basis for U by 2.38.

   (b) Append x to the list. Then $1, (x-2)(x-5), x(x-2)(x-5), x^2(x-2)(x-5), x$ form a basis for $\mathcal{P}_4(\mathbb{R})$.

   (c) Let $W = \{p \in \mathcal{P}_4(\mathbb{R}) : p(x) = cx, c \in \mathbb{R}\}$. Then by 2.33, we have $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

6. (a) Let $U = \{p \in \mathcal{P}_2(\mathbb{F}) : p(2) = p(5) = p(6)\}$. Then the polynomials $1, (x-2)(x-5)(x-6), x(x-2)(x-5)(x-6)$ are in U. Now note that this list is linearly independent, so $\dim U \geq 3$. However, note that no polynomial of degree 1 or 2 can satisfy this condition, as we can translate it down to obtain a polynomial with three zeros, which is not possible for a polynomial of degree 1 or 2. If we let $W = \operatorname{span}(x, x^2)$, then $U \cap W = \{\mathbf{0}\}$, so

$$\dim(U + W) = \dim U + \dim W - \dim(U \cap W) \leq 5$$

   as $U + W$ is a subspace of $\mathcal{P}_4(\mathbb{F})$. Thus $\dim U + 2 \leq 5$ and $\dim U \leq 3$, so $\dim U = 3$ and this is a basis for U.

   (b) Append x, x^2. Note that $W = \operatorname{span}(x, x^2)$ forms a direct sum with U as $U \cap W = \{\mathbf{0}\}$, and $\dim(U + W) = 5$, so $U + W = \mathcal{P}_4(\mathbb{R})$ and $\mathcal{P}_4(\mathbb{R}) = U \oplus W$. Then the bases of two components of a direct sum can be combined to form a basis for the whole space, as was shown in exercise 2B.9.

   (c) We can use $W = \operatorname{span}(x, x^2) = \{p \in \mathcal{P}_4(\mathbb{R}) : p(x) = ax + bx^2,\ a, b \in \mathbb{R}\}$.

7. (a) Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p = 0\}$. Note that x and x^3 are in U. Also note that $x^4 - \frac{3}{5}x^2$ and $x^2 - \frac{1}{3}$ also satisfy the conditions. Also note that this list is linearly independent, so $\dim U \geq 4$. However, $\dim U \neq 5$ as that would imply that $U = \mathcal{P}_4(\mathbb{R})$, but $p = 1$ is not in this set.

   (b) Append 1 to this set. Note that $1 \notin \operatorname{span}(x, x^3, x^4 - \frac{3}{5}x^2, x^2 - \frac{1}{3})$, as that would imply $1 \in U$. Then $1, x, x^3, x^4 - \frac{3}{5}x^2, x^2 - \frac{1}{3}$ forms a linearly independent list of length 5 and thus it is a basis for $\mathcal{P}_4(\mathbb{R})$.

(c) Let $W = \text{span}(1) = \{p \in \mathcal{P}_4(\mathbb{R}) : p(x) = c, c \in \mathbb{R}\}$. Then by 2.33, $\mathcal{P}_4(\mathbb{R}) = U \oplus W$.

8. ($\star$) Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$ be linearly independent and let $\mathbf{w} \in V$. Let

$$U = \text{span}(\mathbf{v}_1 + \mathbf{w}, \ldots, \mathbf{v}_m + \mathbf{w}).$$

Note that for $k \in \{1, \ldots, m-1\}$

$$\mathbf{v}_k - \mathbf{v}_{k+1} = (\mathbf{v}_k + \mathbf{w}) - (\mathbf{v}_{k+1} + \mathbf{w}),$$

and thus $\mathbf{v}_k - \mathbf{v}_{k+1} \in U$. Now we show that $\mathbf{v}_1 - \mathbf{v}_2, \ldots, \mathbf{v}_{m-1} - \mathbf{v}_m$ are linearly independent. Suppose

$$c_1(\mathbf{v}_1 - \mathbf{v}_2) + \ldots + c_{m-1}(\mathbf{v}_{m-1} - \mathbf{v}_m) = \mathbf{0}.$$

Then we can rewrite this as

$$c_1\mathbf{v}_1 + (c_2 - c_1)\mathbf{v}_2 + \ldots + (c_{m-1} - c_{m-2})\mathbf{v}_{m-1} - c_{m-1}\mathbf{v}_m = \mathbf{0}.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent, all coefficients must be zero. Then $c_1 = 0$. Thus $c_2 = c_1 = 0$, $c_3 = c_2 = 0, \ldots, c_m = c_{m-1} = 0$. Thus the list $\mathbf{v}_1 - \mathbf{v}_2, \ldots, \mathbf{v}_{m-1} - \mathbf{v}_m$ is linearly independent. However, note that this is a list of $m-1$ vectors, and any spanning list (and thus basis) thus must have at least $m-1$ vectors. Thus $\dim U \geq m-1$.

9. Let m be a positive integer and let $p_0, \ldots, p_m \in \mathcal{P}(\mathbb{F})$ be polynomials such that p_k has degree k. We prove that these polynomials are linearly independent, as that will imply that the list is a basis, since we have $m+1$ polynomials and $\dim \mathcal{P}_m(\mathbb{F}) = m+1$. Let $c_0, \ldots, c_m \in \mathbb{F}$ such that

$$c_0p_0 + \ldots + c_mp_m = 0.$$

Expanding c_mp_m would get us $a_mc_mx^m$ on the right hand side, where $a_m \neq 0$ is the coefficient of x^m in p_m. Since no other polynomial to the left of p_m is of degree m, and the right hand side has no x^m term, this implies $a_mc_m = 0$ and thus $c_m = 0$. We can continue this reasoning from right to left to show that $c_{m-1} = 0, c_{m-2} = 0$, and we are eventually left with $c_0 = 0$. This shows the list is linearly independent, as desired.

10. ($\star$) Let m be a positive integer, and define $p_k(x) = x^k(1-x)^{m-k}$ for $k \in \{0, \ldots, m\}$. We show that $p_0, \ldots, p_m$ form a basis form a basis for $\mathcal{P}_m(\mathbb{F})$. We will do so by an inductive argument. Specifically, we will show that for every m, each of $1, x, \ldots, x^m$ can be written as a linear combination of some of the p_k. Then since every $p \in \mathcal{P}_m(\mathbb{F})$ can be written as a linear combination of $1, x, \ldots, x^m$, a simple substitution will show that p can be written as a linear combination of the p_k, which shows that $p_0, \ldots, p_m$ spans $\mathcal{P}_m(\mathbb{F})$. This would imply that $p_0, \ldots, p_m$ form a basis for $\mathcal{P}_m(\mathbb{F})$ since we have $m+1$ polynomials forming a spanning set.

To start, let's note that x^m can be trivially represented by x^m, since $x^m = p_m(x)$. We now perform a backward induction. Suppose x^i can be represented by a linear combination of $p_0, \ldots, p_m$ for $i \in \{k+1, \ldots, m\}$. Consider

$$p_k(x) = x^k(1-x)^{m-k}.$$

Then $(1-x)^{m-k}$ is a polynomial of degree $m-k$. and can be written as $1 + a_1 x + \ldots + a_{m-k}x^{m-k}$. Then multiplying in x^k gives us

$$p_k(x) = x^k + a_1 x^{k+1} + \ldots + a_{m-k}x^m.$$

Now, by our inductive hypothesis, $x^{k+1}, \ldots, x^m$ are all spanned by $p_0, \ldots, p_m$, so we can write

$$p_k(x) = x^k + c_0 p_0(x) + \ldots + c_m p_m(x).$$

Then

$$x^k = p_k(x) - c_0 p_0(x) - \ldots - c_m p_m(x),$$

so x^k is also spanned by $p_0, \ldots, p_m$. Thus by induction, each of $1, x, \ldots, x^m$ is in the span of $p_0, \ldots, p_m$, as desired.

11. Suppose U and W are four-dimensional subspaces of $\mathbb{C}^6$. Then $U + W$ is a subspace of $\mathbb{C}^6$ and $\dim(U+W) \leq 6$. This implies

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 4 + 4 - \dim(U \cap W) \leq 6.$$

This implies $\dim(U \cap W) \geq 2$. Thus there is a basis of $U \cap W$ of length at least 2. Let $\mathbf{v}_1, \mathbf{v}_2$ be two vectors in this basis. Then they are linearly independent, so neither of these vectors is a scalar multiple of the other.

12. Let U and W be subspaces of $\mathbb{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbb{R}^8$. To prove that $\mathbb{R}^8 = U \oplus W$, we need to show that this sum is direct. We can do this by showing that $U \cap W = \{\mathbf{0}\}$. Since $\mathbb{R}^8 = U + W$, $\dim(U+W) = 8$. Then

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 3 + 5 - \dim(U \cap W) = 8.$$

Thus $\dim(U \cap W) = 0$, but the only zero-dimensional subspace is the zero subspace. Thus $U \cap W = \{\mathbf{0}\}$, and $\mathbb{R}^8 = U \oplus W$, as desired.

13. Let U and W be five-dimensional subspaces of $\mathbb{R}^9$. Then $U + W$ is a subspace of $\mathbb{R}^9$, so $\dim(U+W) \leq 9$. Then we can see that

$$\dim(U+W) = \dim U + \dim W - \dim(U \cap W) = 5 + 5 - \dim(U \cap W) \leq 9.$$

Thus $10 - \dim(U \cap W) \leq 9$, and $\dim(U \cap W) \geq 1$, so there is a nonzero vector in $U \cap W$, and $U \cap W \neq \{\mathbf{0}\}$.

14. Suppose V is a ten-dimensional vector space and V_1, V_2, and V_3 are subspaces of V such that $\dim V_1 = \dim V_2 = \dim V_3 = 7$. First note that $V_1 + V_2$ is a subspace of V, so that $\dim(V_1 + V_2) \leq 10$. This means that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) = 7 + 7 - \dim(V_1 \cap V_2) \leq 10.$$

Solving this inequality gives $\dim(V_1 \cap V_2) \geq 4$. Now note that $(V_1 \cap V_2) + V_3$ is another subspace of V since $V_1 \cap V_2$ is a subspace of V. Thus $\dim((V_1 \cap V_2) + V_3) \leq 10$, and we can use the dimension formula again:

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim(V_3) - \dim(V_1 \cap V_2 \cap V_3) \leq 10.$$

Then since $\dim(V_1 \cap V_2) \geq 4$, we can write

$$10 \geq \dim(V_1 \cap V_2) + \dim(V_3) - \dim(V_1 \cap V_2 \cap V_3) \geq 4 + 7 - \dim(V_1 \cap V_2 \cap V_3),$$

so $\dim(V_1 \cap V_2 \cap V_3) \geq 1$. However, this implies there is a nonzero basis vector in $V_1 \cap V_2 \cap V_3$, so $V_1 \cap V_2 \cap V_3 \neq \{\mathbf{0}\}$.

15. ($\star$) Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces such that $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. We can use the same reasoning as the previous exercise. First note that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) \leq \dim V$$

since $V_1 + V_2$ is a subspace of $\dim V$. Then

$$\dim(V_1 \cap V_2) \geq \dim V_1 + \dim V_2 - \dim V.$$

Now, note that $\dim((V_1 \cap V_2) + V_3) \leq \dim V$, so

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3) \leq \dim V.$$

Then

$$\begin{aligned} \dim(V_1 \cap V_2 \cap V_3) &\geq \dim(V_1 \cap V_2) + \dim V_3 - \dim V \\ &\geq (\dim V_1 + \dim V_2 - \dim V) + \dim V_3 - \dim V \\ &= \dim V_1 + \dim V_2 + \dim V_3 - 2 \dim V \\ &> 0. \end{aligned}$$

Since $\dim(V_1 \cap V_2 \cap V_3) > 0$, there exists a nonzero basis vector of $V_1 \cap V_2 \cap V_3$, so $V_1 \cap V_2 \cap V_3 \neq \{\mathbf{0}\}$.

16. Suppose that V is finite-dimensional and U is a subspace of V with $U \neq V$. Also let $n = \dim V$ and $m = \dim U$. First let $\mathbf{u}_1, \ldots, \mathbf{u}_m$. Since this list is linearly independent, we can extend this list to a basis of V, say $\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_{n-m}$. Now let $U_i = \operatorname{span}(\{\mathbf{u}_1, \ldots, \mathbf{u}_m, \mathbf{v}_1, \ldots, \mathbf{v}_{n-m}\} \setminus \{\mathbf{v}_i\})$, which is the subspace spanned by the basis vectors, removing $\mathbf{v}_i$. Since there are $n - m$ $\mathbf{v}$ vectors, there are $n - m$ ways of

removing a single $\mathbf{v}_i$, and thus there are $n-m$ subspaces. Also, since we have only removed a single vector, $\dim U_i = n-1$ for all i.

We now show that $U_1 \cap \cdots \cap U_{m-n} = U$. To do this, note that if $\mathbf{u} \in U$, then $\mathbf{u}$ can be written as the linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_m$. However, $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are in the span of every U_i by definition, so $\mathbf{u} \in U_1 \cap \cdots \cap U_{m-n}$. Now suppose $\mathbf{v} \in U_1 \cap \cdots \cap U_{m-n}$. Suppose

$$\mathbf{v} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m + d_1\mathbf{v}_1 + \ldots + d_{m-n}\mathbf{v}_m.$$

However, since $\mathbf{v} \in U_1$, $d_1 = 0$, for if $d_1 \neq 0$, we could subtract all other terms except $d_1\mathbf{v}_1$ to the left hand side, which would imply $\mathbf{v}_1 \in U_1$, contradicting the definition of U_1. Thus $d_1 = 0$. Using similar reasoning, we find that $d_2 = \ldots = d_{m-n} = 0$, so $\mathbf{v} = c_1\mathbf{u}_1 + \ldots + c_m\mathbf{u}_m$ and $\mathbf{v} \in U$. Thus $U = U_1 \cap \cdots \cap U_{m-n}$.

17. Suppose $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. We will prove the statement by induction on m. Note that if $m = 1$, then $\dim(V_1) \leq \dim(V_1)$ is trivial. Now suppose

$$\dim(V_1 + \ldots + V_k) \leq \dim V_1 + \ldots + \dim V_k$$

for a positive integer k. Then

$$\begin{aligned}
\dim(V_1 + \ldots + V_{k+1}) &= \dim((V_1 + \ldots + V_k) + V_{k+1}) \\
&= \dim(V_1 + \ldots + V_k) + \dim V_{k+1} - \dim((V_1 + \ldots + V_k) \cap V_{k+1}) \\
&\leq \dim(V_1 + \ldots + V_k) + \dim V_{k+1} \\
&\leq \dim V_1 + \ldots + \dim V_k + \dim V_{k+1},
\end{aligned}$$

where the first inequality is due to the fact that $\dim((V_1 + \ldots + V_k) \cap V_{k+1}) \geq 0$ and the second inequality is from the inductive hypothesis. This completes the induction.

18. Suppose V is finite-dimensional and suppose $n = \dim V$ is a positive integer. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of V. Now let $V_i = \operatorname{span}(\mathbf{v}_i)$ for $i \in \{1, \ldots, n\}$. Then note that $\dim V_i = 1$. Now we show that $V = V_1 \oplus \cdots \oplus V_n$. To show this, note that $V = V_1 + \ldots V_n$ since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a spanning list for V and thus every vector in V can be written as the sum of vectors in V_i. To show a direct sum, suppose there are vectors $\mathbf{u}_1 \in V_1, \ldots, \mathbf{u}_n \in V_n$ such that

$$\mathbf{u}_1 + \ldots + \mathbf{u}_n = \mathbf{0}.$$

Then $\mathbf{u}_i = c_i\mathbf{v}_i$ by definition of V_i. However, this would imply

$$c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0},$$

which would imply $c_1 = \cdots = c_n = 0$ and thus $\mathbf{u}_1 = \cdots = \mathbf{u}_n = \mathbf{0}$. Thus $V = V_1 \oplus \cdots \oplus V_n$.

19. The formula

$$\begin{aligned}
\dim(V_1 + V_2 + V_3) = {} & \dim V_1 + \dim V_2 + \dim V_3 \\
& - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) + \dim(V_1 \cap V_2 \cap V_3).
\end{aligned}$$

can be guessed from the inclusion-exclusion formula for the sizes of finite sets:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

However, this is not true in general. To see this, let $V = \mathbb{R}^2$, $V_1 = \operatorname{span}((1,0)), V_2 = \operatorname{span}((1,1)), V_3 = \operatorname{span}((0,1))$. Then $V_1 \cap V_2 = V_1 \cap V_3 = V_2 \cap V_3 = V_1 \cap V_2 \cap V_3 = \{\mathbf{0}\}$. Then since $V_1 + V_2 + V_3 = \mathbb{R}^2$, $\dim(V_1 + V_2 + V_3) = 2$. However, our formula gives us

$$\dim(V_1 + V_2 + V_3) = \dim V_1 + \dim V_2 + \dim V_3 = 3,$$

which is false.

20. We will prove the equivalent formula

$$\begin{aligned} 3\dim(V_1 + V_2 + V_3) = 3\dim V_1 &+ 3\dim V_2 + 3\dim V_3 \\ &- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &- \dim((V_1 + V_2) \cap V_3) - \dim((V_1 + V_3) \cap V_2) - \dim((V_2 + V_3) \cap V_1). \end{aligned}$$

To do this, we realize that

$$\dim(V_1 + V_2 + V_3) = \dim((V_1 + V_2) + V_3) = \dim((V_1 + V_3) + V_2) = \dim((V_2 + V_3) + V_1).$$

Then we can write

$$\begin{aligned} \dim((V_1 + V_2) + V_3) &= \dim(V_1 + V_2) + \dim V_3 - \dim((V_1 + V_2) \cap V_3) \\ \dim((V_1 + V_3) + V_2) &= \dim(V_1 + V_3) + \dim V_2 - \dim((V_1 + V_3) \cap V_2) \\ \dim((V_2 + V_3) + V_1) &= \dim(V_2 + V_3) + \dim V_1 - \dim((V_2 + V_3) \cap V_1). \end{aligned}$$

Then expanding further gives us

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3) \\ \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 \cap V_3) - \dim((V_1 + V_3) \cap V_2) \\ \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_2 \cap V_3) - \dim((V_2 + V_3) \cap V_1). \end{aligned}$$

Now adding these three equations gives us

$$\begin{aligned} 3\dim(V_1 + V_2 + V_3) = 3\dim V_1 &+ 3\dim V_2 + 3\dim V_3 \\ &- \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &- \dim((V_1 + V_2) \cap V_3) - \dim((V_1 + V_3) \cap V_2) - \dim((V_2 + V_3) \cap V_1), \end{aligned}$$

as desired.