Affine varieties:

1) K - will denote a fixed algebraically closed ground field ($\mathbb{C}$)

2) Rings are always assumed to be commutative with unity. If J is an ideal in a ring R we will write this as $J \trianglelefteq R$ and denote radical of J, by $\sqrt{J} := \{ f \in R \mid f^k \in J$ for some $k \in \mathbb{N} \}$

The ideal generated by a subset S of a ring will be written as $\langle S \rangle$.

3) Degree of a polynomial $f \neq 0$ is always meant be its total degree i.e the biggest integer $d \in \mathbb{N}$ s.t. f contains a non-zero monomial $x_1^{i_1} \cdots x_n^{i_n}$ with $i_1 + \cdots + i_n = d$.

Defⁿ 1.1 (Affine ~~varieties~~ algebraic sets)

(i) We call $\mathbb{A}^n := \mathbb{A}^n_K := \{ (a_1, \ldots, a_n) \mid a_i \in K \ \forall i \}$ the affine n-space over K.

(ii) For a subset $S \subset K[x_1, \ldots, x_n]$ of polynomials we call $V(S) := \{ x \in \mathbb{A}^n \mid f(x) = 0 \ \forall f \in S \} \subset \mathbb{A}^n$ the zero locus of S. Subsets of this form are called affine ~~varieties~~ algebraic set. If $S = \{ f_1, \ldots, f_k \}$ is a finite set we will write $V(S) = V(\{f_1, \ldots, f_k\}) = V(f_1, \ldots, f_k)$

Lemma: I.2

(a) For any $S_1 \subset S_2 \subset K[x_1, \dots, x_n]$ we have $V(S_1) \supset V(S_2)$

(b) For any $S_1, S_2 \subset K[x_1, \dots, x_n]$ we have $V(S_1) \cup V(S_2)$ $= V(S_1 S_2)$, where as usual we set $S_1 S_2 := \{fg : f \in S_1, g \in S_2\}$.

(c) If J is any index set and $S_i \subset K[X_1, \dots, X_n]$ for all $i \in J$ then $\bigcap_{i \in J} V(S_i) = V(\bigcup_{i \in J} S_i)$.

In particular, finite unions and arbitrary intersections of affine algebraic sets are again affine algebraic set.

Pf: (a) If $x \in V(S_2) \Rightarrow f(x) = 0 \ \forall f \in S_2 \Rightarrow f(x) = 0$ $\forall f \in S_1 \Rightarrow x \in V(S_1)$.

(b) $\supset$ If $x \in V(S_1) \cup V(S_2) \Rightarrow f(x) = 0 \ \forall f \in S_1$ or $g(x) = 0$ $\forall g \in S_2$. In any case this means that $(fg)(x) = 0$ $\forall f \in S_1$ and $g \in S_2 \Rightarrow x \in V(S_1 S_2)$.

$\subset$ If $x \notin V(S_1) \cup V(S_2) \Rightarrow x \notin V(S_1)$ and $x \notin V(S_2)$. Then $\exists f, g$ s.t. $f(x) \neq 0$ & $g(x) \neq 0$. Then $\underset{\in S_1}{f}, \underset{\in S_2}{g}$ $(fg)(x) \neq 0$, and hence $x \notin V(S_1 S_2)$.

(c) We have $x \in \bigcap_{i \in J} V(S_i) \Leftrightarrow f(x) = 0 \ \forall f \in S_i \ \forall i \in J$.

$\Leftrightarrow x \in V(\bigcup_{i \in J} S_i)$.

* **Examples: 1.3.**

(1) The Affine n-space $\mathbb{A}^n = V(0)$. Similarly the empty set $\emptyset = V(1)$ is also a.a.s.

(2) Any pt. $(a_1, \dots, a_n) \in \mathbb{A}^n$ is an a.a.s.
$\{a\} = V(x_1 - a_1, \dots, x_n - a_n)$.
By the previous lemma, finite subsets of $\mathbb{A}^n$ are then a.a.s as well.

(3) Linear subspaces of $\mathbb{A}^n$ are a.a.s. (why?)

(4) If $X \subset \mathbb{A}^n$ & $Y \subset \mathbb{A}^m$ are a.a.s. Then so is the product $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m = \mathbb{A}^{n+m}$.

Remark 1.4 let f & g be polynomials that vanish on a certain subset $X \subset \mathbb{A}^n$. Then $f+g$ & hf vanish on X as well for any polynomial h clearly.
This means that the set $S \subset K[x_1, \dots, x_n]$ defining an affine variety $X = V(S)$ is certainly not unique. For any $f, g \in S$ and any polynomial h we can add $f+g$ & hf to S without changing its zero locus, so that we always have $V(\langle S \rangle) = V(S)$. In particular any affine a.a.s. ~~variety~~ in $\mathbb{A}^n$ can be written as the zero locus of an ideal in $K[x_1, \dots, x_n]$

As any ideal in $K[x_1, \ldots, x_n]$ is f.g. by [Hilbert's Basis Theorem] This means ~~that~~ moreover that any affine algebraic set can be written as the zero locus of finitely many polynomials.

* Prop of $V(\cdot)$: For any $J, J_1, J_2 \trianglelefteq K[x_1, \ldots, x_n]$. we have.

(a) $V(\sqrt{J}) = V(J)$,

(b) $V(J_1) \cup V(J_2) = V(J_1 J_2) = V(J_1 \cap J_2)$

(c) $V(J_1) \cap V(J_2) = V(J_1 + J_2)$.

Pf: Checking.

* Remark 1.4. is important since it is in some sense the basis of AG. It relates geometric objects (a.a.s.) to algebraic objects (ideals): We have already assigned a.a.s. to ideals in Def^n 1.1. and Remark (1.4), so let's now introduce an operation which does the opposite job.

Def^n (1.6): (Ideal of a subset of $\mathbb{A}^n$):

Let $X \subset \mathbb{A}^n$ be any subset, Then (Ideal of X) $\Rightarrow$

$$I(X) := \{ f \in K[x_1, \ldots, x_n] \mid f(x) = 0 \ \forall \ x \in X \}.$$

This is indeed an ideal (check).

Remark: (1.7):

(a) In analogy to Lemma (1.2) it is obvious that the ideal of a subset reverses inclusions as well:

If $X_1 \subset X_2 \subset \mathbb{A}^n$ then $I(X_1) \supset I(X_2)$.

(b) Note that $I(X)$ is always a radical ideal: If $f^k \in I(X)$ for some $f \in K[x_1, \dots, x_n]$ and $k \in \mathbb{N}$ then $f^k(x) = 0 \ \forall x \in X$ and hence

$f(x) = 0 \ \forall x \in X \Rightarrow f \in I(X)$.

Hence the operation $I(\cdot) \Longleftrightarrow V(\cdot)$
$\downarrow$ a.a.s. $\downarrow$ radical ideals.

By a ~~certain~~ result of C.A. that this is actually a bijection. (Hilbert Nullstellensatz).

Prop. 1.10.

(a) For any a.a.s. $X \subset \mathbb{A}^n$ we have $V(I(X)) = X$

(b) For any ideal $J \trianglelefteq K[x_1, \dots, x_n]$ we have $I(V(J)) = \sqrt{J}$.

In particular there is an "inclusion-reversing" bijection

$$\{\text{affine a.s. in } \mathbb{A}^n\} \xleftrightarrow{1:1} \{\text{radical ideals in } K[x_1, \dots, x_n]\}.$$

$$X \longmapsto I(X)$$

$$V(J) \longleftarrow J.$$

Pf:- Three of the four inclusions of (a) & (b) are actually easy:

(a) "$\supset$": If $x \in X$ then $f(x) = 0 \; \forall \; f \in I(X)$ $\Rightarrow$ $x \in V(I(X))$.

(b) "$\supset$": If $f \in \sqrt{J}$, then $f^K \in J$ for some $K \in \mathbb{N}$. It follows that $f^K(x) = 0 \; \forall \; x \in V(J)$, hence also $f(x) = 0 \; \forall \; x \in V(J)$, & thus $f \in I(V(J))$.

(a) "$\subset$": As X is a.a.s we know by Remark 1.9 that $X = V(J)$ for some ideal J. Then $I(V(J)) \supset \sqrt{J} \supset J$ by (b) "$\supset$", so taking the zero locus we obtain $V(I(J)) \subset V(J)$ by Lemma 1.9 (a). This means exactly that $V(I(X)) \subset X$.

Only the inclusions $I(V(J)) \subset \sqrt{J}$ of (b) "$\subset$" is hard to prove and it uses commutative algebra ~~and~~ with the convention K is algebraically closed.

The additional bijection statement now follows directly from (a) & (b), together with the observations $\sqrt{J} = J$ since J is radical, $I(X)$ is actually a radical ideal by Remark ( ), and both operations ~~reverse~~ reverse inclusions by Lemma ( ). & Remark ( ).

Examples:

(a) Let $J \trianglelefteq K$ be a non-zero ideal. As $K[x_1]$ is a P.I.D.: we have $J = \langle f \rangle$ for a polynomial $f = (x_1 - a_1)^{k_1} \cdots (x_1 - a_r)^{k_r}$ for some distinct pts. $a_1, \dots, a_r \in \mathbb{A}^1$ & $k_1, \dots, k_r \in \mathbb{N}_{>0}$. The zero locus $V(J) = V(f) = \{a_1, \dots, a_r\} \subset \mathbb{A}^1$ Then contains the data of the zeroes of f, but no longer of the multiplicities $k_1, \dots, k_r$. Consequently, by proposition, $I(V(J)) = \sqrt{J} = \langle (x_1 - a_1) \cdots (x_1 - a_r) \rangle$ is just the ideal of all polynomial vanishing at $a_1, \dots, a_r$. (with any order).

(b) If we had not assumed K to be a.c., the Nullstellensatz would already break down in the simple example (a): The prime (and hence radical) ideal $J = \langle x_1^2 + 1 \rangle \trianglelefteq \mathbb{R}[x_1]$ has empty zero locus in $\mathbb{A}^1_{\mathbb{R}}$, so we would obtain $I(V(J)) = I(\phi) = \mathbb{R}[x_1] \neq J = \sqrt{J}$.

(c) The ideal $J = \langle x_1 - a_1, \dots, x_n - a_n \rangle \trianglelefteq K[x_1, \dots, x_n]$ is maximal, and hence radical. As its zero locus is $V(J) = \{a\}$. $a = (a_1, \dots, a_n)$. We conclude by Prop. that the ideal of the pt. a is $I(\{a\}) = I(V(J))$
$= J = \langle x_1 - a_1, \dots, x_n - a_n \rangle$.

In fact, pts in $\mathbb{A}^n$ are clearly just the minimal non-empty algebraic sets in $\mathbb{A}^n$, so by the inclusion-reversing operations of the Nullstellensatz they correspond exactly to the maximal (proper) ideals in $K[x_1, \ldots, x_n]$. Hence the bijection in prop. restricts to a bijection

$$\{\text{points in } \mathbb{A}^n\} \xleftrightarrow{1:1} \{\text{maximal ideals in } K[x_1, \ldots, x_n]\}.$$

So the maximal ideals considered above are actually the only maximal ideals in the polynomial ring.

Now with the help of nullstellensatz $\rightarrow$ we have

Lemma:

For any affine a.s. $X_1, X_2 \subset \mathbb{A}^n$

(a) $I(X_1 \cup X_2) = I(X_1) \cap I(X_2)$

(b) $I(X_1 \cap X_2) = \sqrt{I(X_1) + I(X_2)}$.

Pf $\Rightarrow$ (a) A polynomial $f \in K[x_1, \ldots, x_n]$ is contained in $I(X_1 \cup X_2) \Leftrightarrow f(x) = 0 \ \forall x \in X_1 \ \& \ \forall x \in X_2 \Leftrightarrow$. $f \in I(X_1) \cap I(X_2)$.

(b) By the nullstellensatz of Prop. we obtain.

$$I(X_1 \cap X_2) \overset{1.10.(a)}{=} I(V(I(X_1)) \cap V(I(X_2))) = I(V(I(X_1) + I(X_2)))$$

$$\overset{1.10(b)}{=} \sqrt{I(X_1) + I(X_2)}.$$

Remark: Recall from Remark 1.9. that ideals of affine varieties are always radical. So in particular

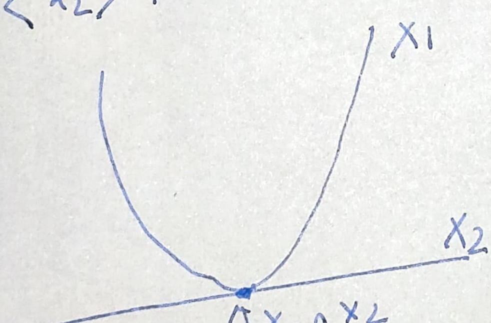
Lemma shows that the intersections of radical ideals are again radical → which could be checked directly →.

In contrast, sums of radical ideals are in general not radical, which is why taking the radical in Lemma . is really necessary.

There's also a geometric interpretation behind this fact:

$X_1, X_2 \subset \mathbb{A}^n$ affine a.s. with $I(X_1) = \langle x_2 - x_1^2 \rangle$. whose real pts

& $I(X_2) = \langle x_2 \rangle$.



$X_1 \cap X_2$ is the just the origin hence with the ideal

$I(X_1 \cap X_2) = I(0) = \langle x_1, x_2 \rangle$

But $I(X_1) + I(X_2) = \langle x_2 - x_1, x_2 \rangle = \langle x_1^2, x_2 \rangle$. is not a radical ideal : only its radical $= I(X_1 \cap X_2) = \langle x_1, x_2 \rangle$.

The geometric meaning of the non-radical ideal $I(X_1)+I(X_2) = \langle x_1^2, x_2\rangle$ is simply that X_1 & X_2 are tangent at their intersection point: In a linear approximation coincide, and both defining eqn's $x_2 = x_1^2$ & $x_2 = 0$ describe the x_1-axis. Hence we could imagine that the intersection $X_1 \cap X_2$ extends from the origin to an infinitesimally small amount in the x_1-direction, as indicated in the picture — so that ~~the~~ x_1-does not quite vanish on the intersections, only x_1^2 does.

more precise way to explain: Intersection multiplicity at the origin, encoding the

$$\dim_{\mathbb{C}} \mathbb{C}[x,y]/\langle x_1^2, x_2\rangle = 2$$

tangential intersection numerically. ~~Hence~~ we will introduce "schemes" shortly where it enlarges the geometric category of varieties to include "objects extending by infinitesimally small amounts in some directions", which will then yield a statement analogus to Proposition (A). That affine subschemes of $\mathbb{A}^n$ are in bijection to arbitrary ideals in $K[x_1, \dots, x_n]$.

In this language of X_1 & X_2 will then be the scheme corresponding to the non-radical ideal $\langle x_1^2, x_2\rangle$.

* **Defn:** (Zariski Topology):

we will give some notions :→

Before introducing we will give some notions :→
$X \subset \mathbb{A}^n$ be an affine algebraic set. A polynomial funcⁿ on X is a map $X \to K$ that is of the form $x \mapsto f(x)$ for some $f \in K[x_1, \dots, x_n]$, by

Remark 1. The ring of all polynomial function X is just the quotient ring

$$A(X) = K[x_1, \dots, x_n] / I(X).$$

→ Used in relative setting.

polynomial funcⁿ on X. Co-ordinate ring of the affine ~~variety~~ a.a.s. X.

It also has a K-vector space and which is K-bilinear structure ⇒ K-algebra structure.

By this Co-ordinate ring we can study subset of polynomial funcⁿs on Y we define $S \subset A(Y)$

(a) the zero locus as $V(S) := V_Y(S) := \{ x \in Y \mid f(x) = 0 \ \forall f \in S \}$.

(b) for a subset $X \subset Y$ the ideal of X in Y is defined to be

$$I(X) := I_Y(X) := \{ f \in A(Y) \mid f(x) = 0 \ \forall x \in X \} \trianglelefteq A(Y).$$

Hence we define the Zariski-topology on X to be the topology whose closed sets are exactly the affine sub-a.a.s. of X. $\Rightarrow$ the subsets of the form $V(S)$ for some $S \subset A(X)$.

By Lemma, & Example. This forms a topology.

Here $V(0) = X$ & $V(c) = \Phi$ | c - any non-zero constant

Also, for just affine algebraic sets. These can be defined (in complex varieties):-

Example: (Topologies in complex varieties)

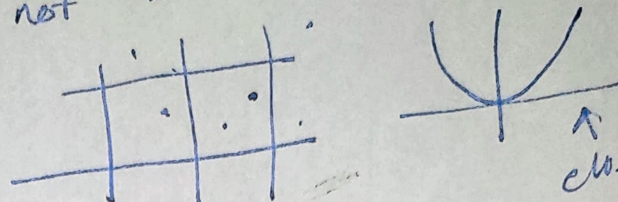
(1) The metric unit ball $A := \{ x \in \mathbb{A}^1_{\mathbb{C}} \mid |a| \leq 1 \}$ in $\mathbb{A}^1_{\mathbb{C}}$ is clearly closed in the classical (euclidean) top but not in Zariski-topology:

In fact the Zariski-closed subsets of $\mathbb{A}^1$ are only the finite sets & $\mathbb{A}^1$ itself. In particular the Zariski-closure of A (a) $= \mathbb{A}^1$. Hence we saw that any Zariski-closed subset is also closed in the classical topology (since it is given by equ's among polynomial func's which are cont. in the classical topology).

(2) $f : \mathbb{A}^1 \to \mathbb{A}^1$ be any injective map $\Rightarrow f$ is automatica cont. in the Zariski-topology.

(3) In general there is a notion of product topology. However the Zariski top of an affine variety $X \times Y$ is not the product top. For example →

$V(x_1 - x_2) = \{(a,a),\ a \in K\} \subseteq \mathbb{A}^2$ is closed in the Zariski top. but not in the product top of $\mathbb{A}^1 \times \mathbb{A}^1$.



$\mathbb{A}^2 \times \mathbb{A}^2$.

closed in $\mathbb{A}^2$ but not in $\mathbb{A}^2 \times \mathbb{A}^2$

~~Distinguished open sets:→~~

~~eg. For $f \in A(X)$. we define the~~

(4). In $\mathbb{A}^2$ there are 3 - building blocks of closed subsets:

$a = \{0\} \rightsquigarrow V(0) = \mathbb{A}^2 \leftarrow$ all of $\mathbb{A}^2$

$a = (f(x,y)) \rightsquigarrow V(a) = \{(x,y) \in \mathbb{A}^2 \mid f(x,y) = 0\}$

↑ a curve

$a = (x-a, y-b) \rightsquigarrow V(a) = \{(a,b)\} \rightarrow$ a point.

Distinguished open sets:→

For $f \in A(X)$, we define the distinguished $D(f)$ as $D(f) = \{x \in X \mid f(x) \neq 0\} = X \setminus V(f)$.

Prop: X be an a.a.s. Then $D(f)$ form a basis.

Proposition: A non-empty affine ~~variety~~ a.s. X is irreducible

$\iff A(X)$ is an I.D.

Pf: As X is non-empty, $A(X) \neq 0$.

" $\Rightarrow$ " S'pose not, then $\exists$ $f_1, f_2 \neq 0 \in A(X)$ s.t. $f_1 f_2 = 0$. $\Rightarrow$ $X_1 = V(f_1)$, $X_2 = V(f_2)$ are not closed, not equal to X, and $X_1 \cup X_2 = V(f_1) \cup V(f_2)$
$= V(f_1 f_2) = V(0) = X$.

hence X is reducible.

" $\Leftarrow$ ". Assume that X is reducible. $X = X_1 \cup X_2$ for $X_1, X_2 \subsetneq X$. By the bijection of the relative Nullstellensatz, $I(X_i) \neq \{0\}$ in $A(X)$ for $i = 1, 2$. So we can choose non-zero $f_i \in I(X_i)$. Then $f_1 f_2$ vanishes on $X_1 \cup X_2 = X$. Hence $f_1 f_2 = 0 \in A(X)$

$\neq$ I.D.

relative Nullstellensatz $\rightarrow$.

$\{$affine sub a.s. of $Y\} \overset{1:1}{\longleftrightarrow} \{$radical ideals in $A(Y)\}$.

Other prop: (1) open subsets of irr. top space are irre and dense. (2) $Y \subset X$ irr. $\Rightarrow \overline{Y}$ also irr.

(3) $f: X \to P$ cont. If X is irr. $\Rightarrow f(X)$ is.

Defn: (Irreducible):
A top space X is irreducible if it can't be written as $X = X_1 \cup X_2$ for closed subsets, $X_1, X_2 \subsetneq X$. We call irr. a.s. affine variety $X \subseteq \mathbb{A}^n$.

In the Zariski top. The algebraic characterization of irreducibility & connectedness is the following:

Prop 2.7
Let X be a disconnected affine ~~variety~~ a.s.; with $X = X_1 \cup X_2$ for two disjoint closed subsets $X_1, X_2 \subsetneq X$.

Then $A(X) \cong A(X_1) \times A(X_2)$.

Pf: As $X_1 \cup X_2 = X$ we obtain in $A(X)$

$I(X_1) \cap I(X_2) = I(X_1 \cup X_2) = I(X) = \{0\}$.

On the other hand, from $X_1 \cap X_2 = \phi$ we have in $A(X)$,

$\sqrt{I(X_1) + I(X_2)} = I(X_1 \cap X_2) = I(\phi) = \langle 1 \rangle$.

Thus also $I(X_1) + I(X_2) = \langle 1 \rangle$, so by C.R.T.

we conclude that

$$A(X) \cong A(X)/I(X_1) \times A(X)/I(X_2)$$

Example:

(1) Let f be an irr. polynomial in $K[x,y]$. Then f generates a prime ideal in $K[x,y]$ since $K[x,y]$ is an UFD, so the zero set $Y = V(f)$ is irreducible. We call it the affine curve of degree d ($\deg f = d$).

More generally if f is irr. in $A = K[x_1, \dots, x_n]$ we obtain a affine variety $Y = V(f)$, which is called a surface if $n = 3$ or a hypersurface if $n > 3$.

Algebra-Geometric Dictionary:

$A = k[x_1, \dots, x_n]$, $k = \bar{k}$.

$X \longmapsto I(X)$

$J \longmapsto V(J)$.

Algebra.	Geometry.
max. ideals of A	points in $\mathbb{A}^n$
prime " " A	affine varieties in $\mathbb{A}^n$
radical " " A	closed subsets of $\mathbb{A}^n$
maximal " of $A/\mathfrak{a}$	pts of $V(\mathfrak{a})$.