Fault-Tolerate Gathering Algorithms for Autonomous Mobile Robots

N. AGMON AND D. PELEG

Outline

- Why gathering algorithms
- Definition of gathering and fault-tolerance in robotic
- Model parameters
- Impossibility of gathering under $(3,1) - Byzantine$ failures in asynchronous and semi-synchronous systems
- An algorithm for Fully-synchronous gathering with $N \geq 3f + 1$
- The algorithm in practice

Why Cooperative Activities by Robots?

- Perform tasks not possible by a single robot.
- Decreases the cost of operation.
- Various applications
  - Military
  - Space mission e.g. exploration

Types of pattern formation problems

- Gathering and convergence
- Flocking (following a leader)
- Even Distribution
- Partitioning

The Gathering Problem

- N autonomous robots to occupy a single point within a finite number of steps.
- Similar to ***Convergence*** problem in message passing systems.

Fault-Tolerance

- N robots with f faulty ones should reach a point.
- Only simple failures are addressed before:
  - Transient failure (robots get lost)
  - Sensor failure which is known to other robots
- This paper addresses crash and Byzantine failures

Robot operation cycle

- Every Cycle include:
- **Look:** identify the locations of all robots (ids **Unknown**)
- **Compute:** execute an algorithm to choose a goal point P_G
- **Move:** move towards P_G, at least by a distance S

Synchronization models

1. **Semi-Synchronous (SSYNC):** robots use same clock but not necessarily active in all cycles.
2. **Fully- Asynchronous(ASYNC):** every robot acts independently.
3. **Fully-Synchronous (FSYNC):**
   - **All robots are active in all cycles**
   - **A lower and upper bound for maximum movements**

Proposed model assumption

- Robots are assumed *Oblivious*
  - In dynamic environment knowledge is not useful
  - Obliviousness is the worst case scenario
- The only input is the set of positions P of all robots.
- Robots are Transparent

The role of "Adversary"

- An external adversary is assigned that can,
  - Decide the distance a non-faulty robot can travel (no less than S)
  - Define arbitrary action for faulty robots

Gathering Under Byzantine Failure

- It is impossible to perform gathering in SSYNC and ASYNC models
  - If a problem is solvable in ASYNC, it is also in SSYNC
  - If prove no solvable in SSYNC, it also proves not solvable in ASYNC

Definition 1

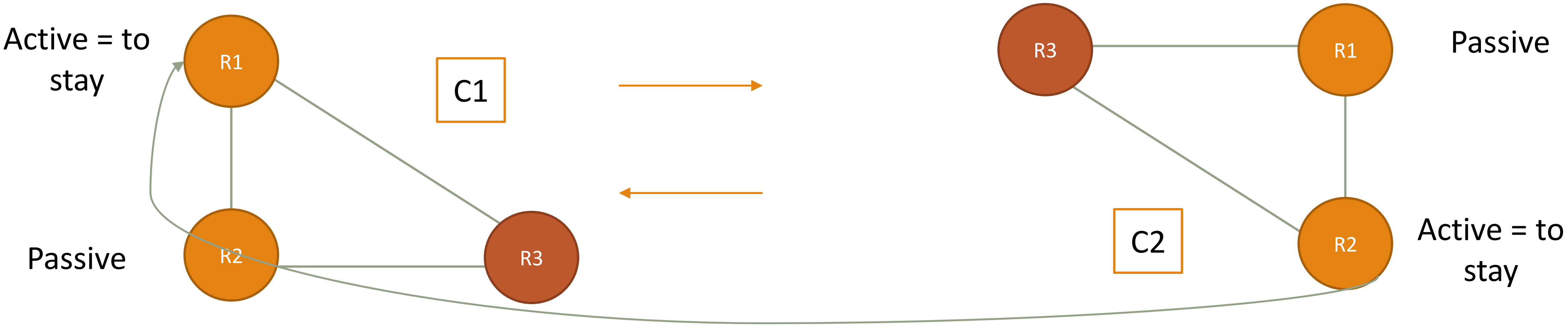
- A gathering algorithm is *hyperactive* if it instructs every robot to make a move in every cycle

Theorem 1

- $N = 3$ *and* $f = 1$, *under the* ***SSYNC*** *model => no non-hyperactive algorithm exists for gathering or convergence*

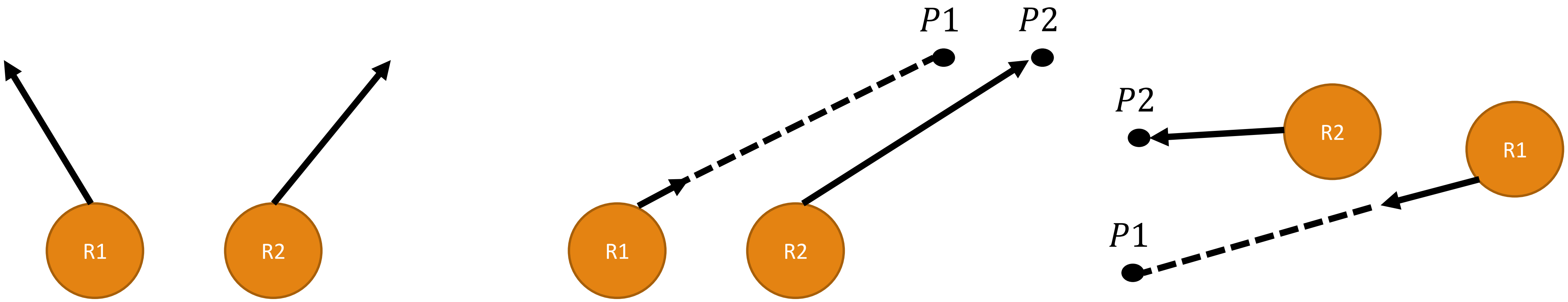
Theorem 1(proof)

- In C_1 , R_1 **active** and instructed to stay and $R_2$is **passive** i.e. doesn't move
- Since R_2 in C_2 is in the same state as R_1 in C_1, it stays
- Adversary can switch from C_1 to C_2, forcing R_1 and R_2 to stay in place indefinitely.



Definition 2

- An algorithm is $N-diverging$ if the distance of two non-faulty robots increases after a cycle

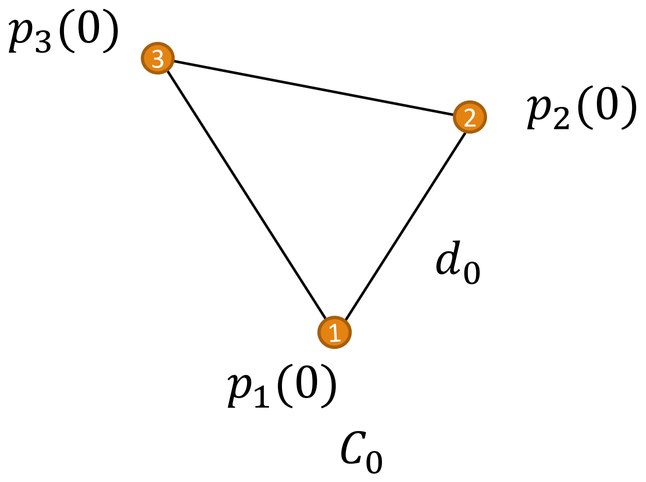


Lemma 1

- In the **SSYNC** (or even **FSYNC**) model a $3 - diverging$ algorithm will fail to achieve gathering or convergence.

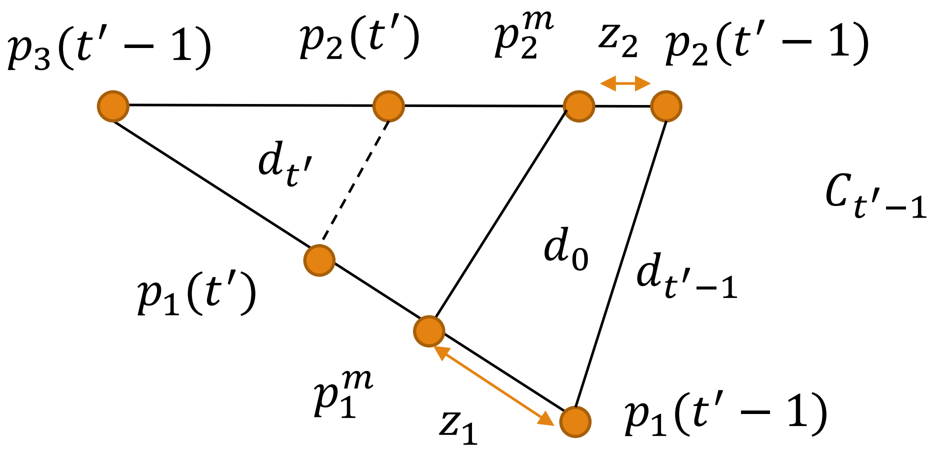
Lemma 1(proof)

- Suppose ∃ an algorithm A
- Consider a $(3,1) - Byzantine$ system T with robots R_1 , R_2 and R_3
- $\sigma = \{ C_0, C_1, \dots, C_k\}$ is a sequence of configurations
- Adversary only intervenes in $C_0\ to\ C_1$ and increases $dist(R_1, R_2)$, i.e. $d_1 > d_0$
- At C_k all robots are gathered in one point



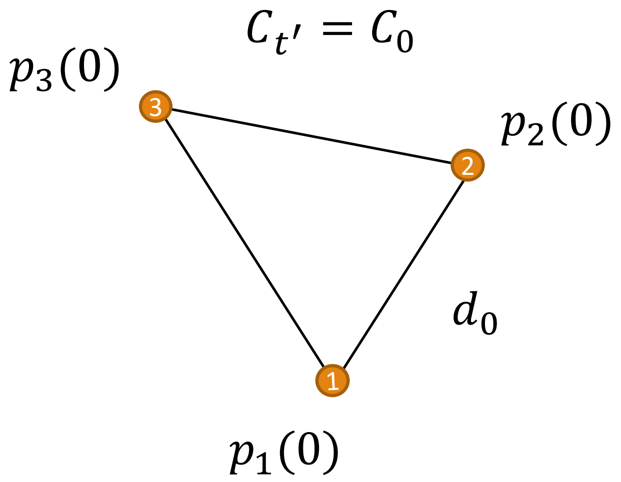
Lemma 1(proof)

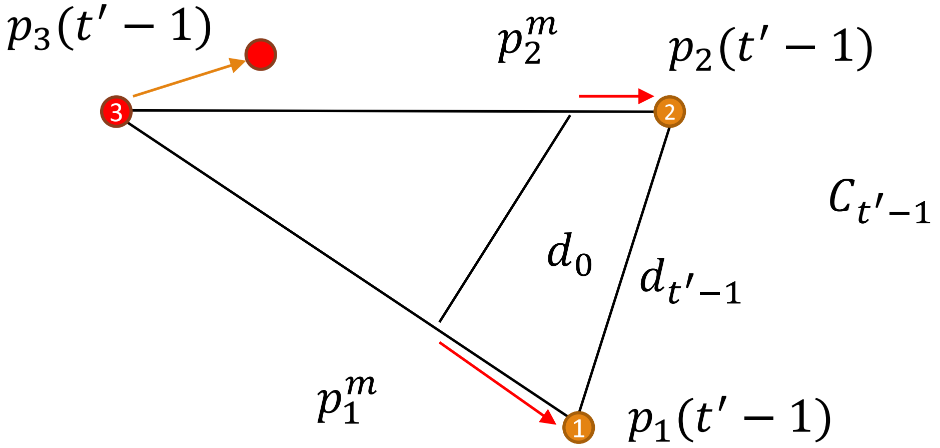
- Assume a system T'



Lemma 1(proof)

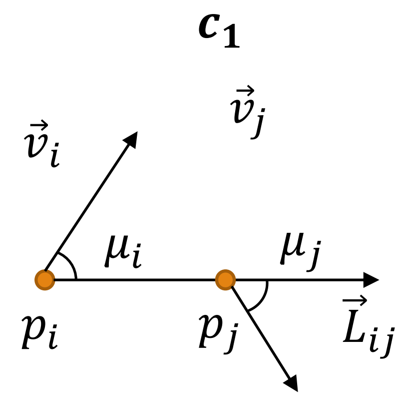
- R_3 is faulty and R_1 and R_2 are stopped at p_1 and p_2.

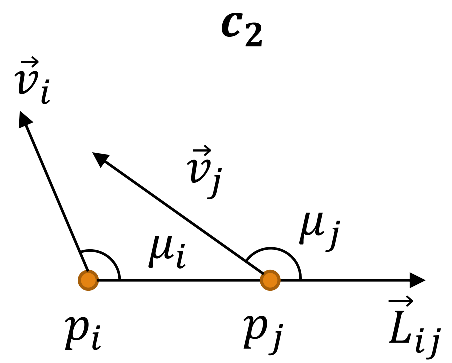




Observation

- Let A be an algorithm operating in $(3,1) - Byzantine$ system
- In any of the following scenarios, A will be 3-diverging:
  - **C1-** $\boldsymbol{0 \leq \mu_i \leq \pi \leq \mu_j \leq 2\pi \ OR\ 0 \leq \mu_j \leq \pi \leq \mu_i \leq 2\pi}$
  - **C2-** $\boldsymbol{0 \leq \mu_i < \mu_j \leq \pi \ AND\ \mu_i \geq \frac{\pi}{2}\ OR\ \mu_j \leq \frac{\pi}{2}}$
  - C3- $0 \leq \mu_j \leq \mu_i \leq \pi$
  - C4- $\pi \leq \mu_i \leq \mu_j \leq 2\pi$
  - C5- $\pi \leq \mu_j < \mu_i \leq 2\pi\ and\ either\ \mu_i \leq \frac{3\pi}{2}\ or\ \mu_j \leq \frac{3\pi}{2}$



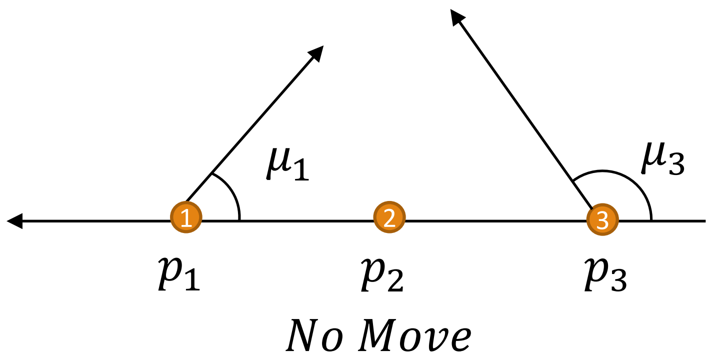


Theorem 2

- In a $(3,1) - Byzantine$ system under the **SSYNC** model it is impossible to perform successful gathering or convergence.

Theorem 2(proof)

- Algorithm A in which R_1, R_2 and R_3 are collinear, and R_2 is in middle.
- If R_2 = **stationary** => non-hyperactive => by **Theorem 1** gathering is not possible.



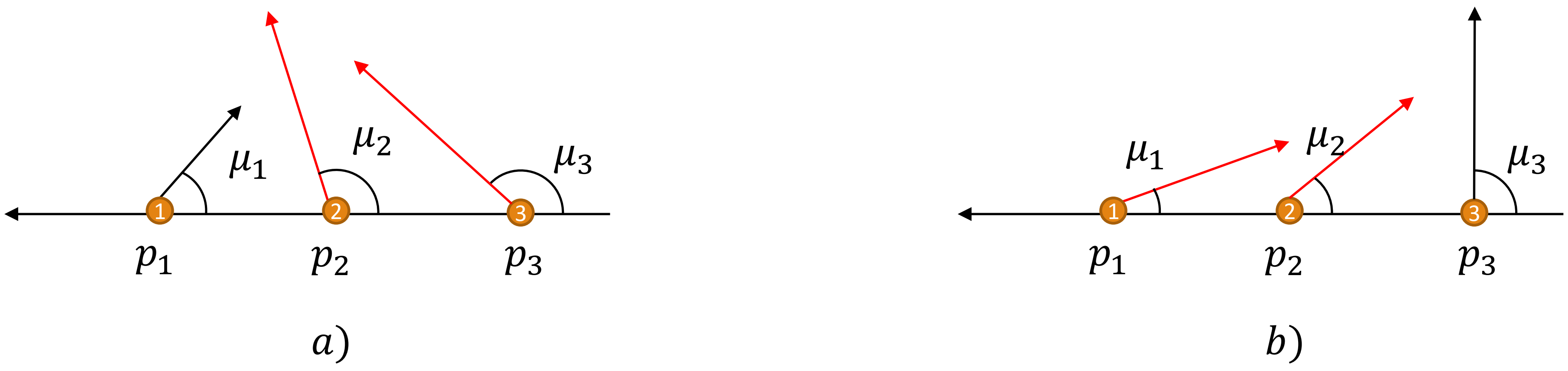
Theorem 2(proof)

- From Observation ⇒ if $0 \le \mu_1, \mu_2, \mu_3 \le \pi$, to avoid 3-diverging, necessarily $\mu_3 > \mu_2 > \mu_1$.

a) If $\mu_2 \ge \frac{\pi}{2}$ ⇒ by C2, p_2 and p_3 are diverging.

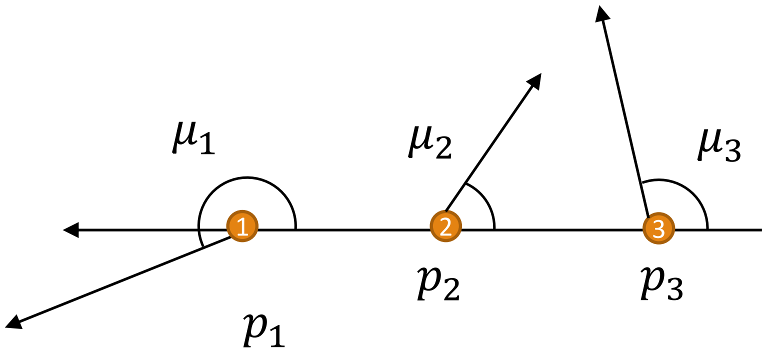
b) If $\mu_2 \le \frac{\pi}{2}$ ⇒ by C2, p_1 and p_2 are diverging.

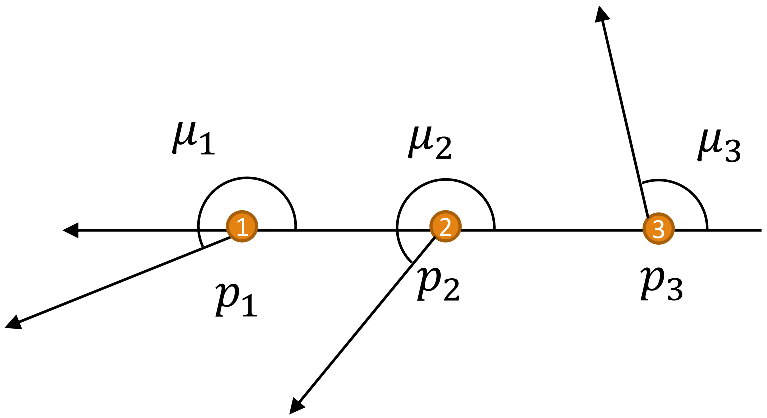
- Similar argument for $\pi \le \mu_1, \mu_2, \mu_3 \le 2\pi$.



Theorem 2(proof)

- By $C_1 \Rightarrow$ If $\mu_1 > \pi$ and $\mu_2, \mu_3 < \pi$ **OR** If $\mu_1, \mu_2 > \pi$ and $\mu_3 < \pi$ A is diverging





- By **Lemma 1**, A fails to achieve gathering or convergence.

Fault tolerant gathering in the FSYNC model

- **Geometric span** of the set of point P:

$$Span(P) = \max\{dist(p,q)|p,q \in P\}$$

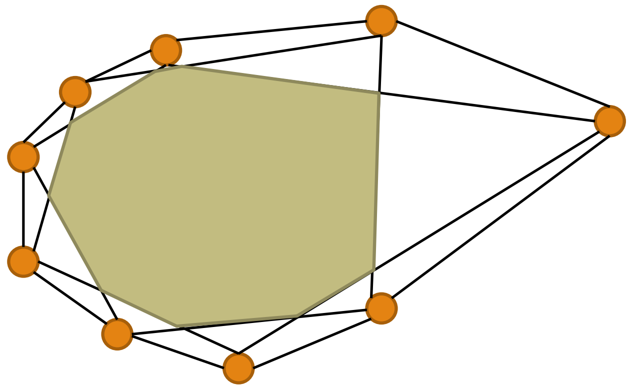
- **The center of gravity** of a multiset P of $n \geq 3$ points $p_i = (x_i, y_i)$:

$$C_{grav}(P) = \left(\frac{\sum_{i=1}^{N} x_i}{N}, \frac{\sum_{i=1}^{N} y_i}{N}\right)$$

Definitions

- A distributed robot algorithm is **Concentrating** if,
  1. It is non-diverging
  2. Exist a constant $c > 0$ at least one pair of non-faulty robots get closer by c.

- The **Hull Intersection** $H_{int}^{k}(P)$ is the convex set created as the intersection of all $\binom{N}{k}$ sets $H(P \backslash \{p_{i1}, \ldots, p_{ik}\})$, for $1 \leq k \leq N$, $p_{ij} \in P$.



A gathering algorithm for $N \geq 3f+1$ in the FSYNC model

- **The Algorithm**

Procedure $\boldsymbol{Gather_{Byz}(P)}$

1. Compute $Q \leftarrow V_H(H_{int}^{f}(P))$.

2. Set $p_G \leftarrow C_{grav}(Q)$.

- V_H denotes the set of vertices of $H_{int}^{f}(P)$.

Analysis

- The Objective is to show if,
  - K Robots at points $P = \{p_1, \dots, p_K\}$ move towards a point p_G in their convex hull $H(P)$
  - Their **geometric span** decreases by at least cS for some constant $\boldsymbol{c} \geq \mathbf{1/4}$
  - The robots meet within finite states

Some Lemmas

- **Lemma 2:** Two robots R_1 and R_2, and let $\alpha = \angle p_1 p_G p_2$.
- If $\alpha \leq \frac{\pi}{2}$ then the distance between them decreases by at least $\boldsymbol{S'(1 - \cos\alpha)}$.

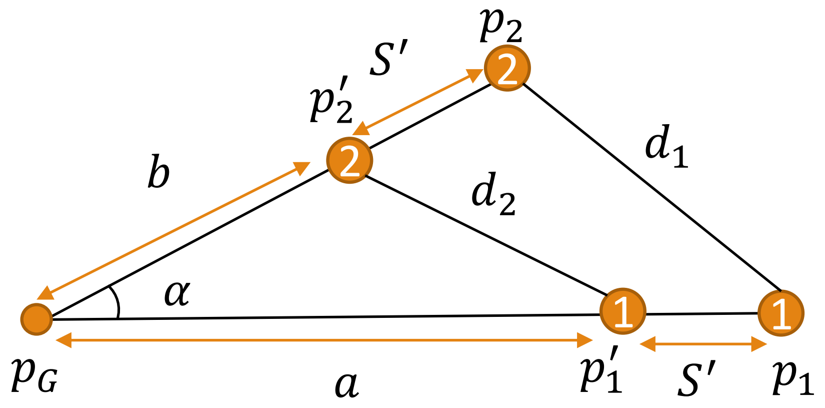
$d_1^2 = (a + S')^2 + (b + S')^2 - 2(a + S')(b + S')cos\alpha$

$d_2^2 = a^2 + b^2 - 2abcos\alpha$

$d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2) \Rightarrow d_1 - d_2 = \frac{2a+2b+2S'}{d_1+d_2} . S'(1 - cos\alpha)$

$\Delta p_1 p_G p_2 \Rightarrow a + b + 2S' > d_1,\ \Delta p_1' p_G p_2' \Rightarrow a + b > d_2$

$\frac{2a+2b+2S'}{d_1+d_2} > 1 \Rightarrow d_1 - d_2 > S'(1 - cos\alpha)$

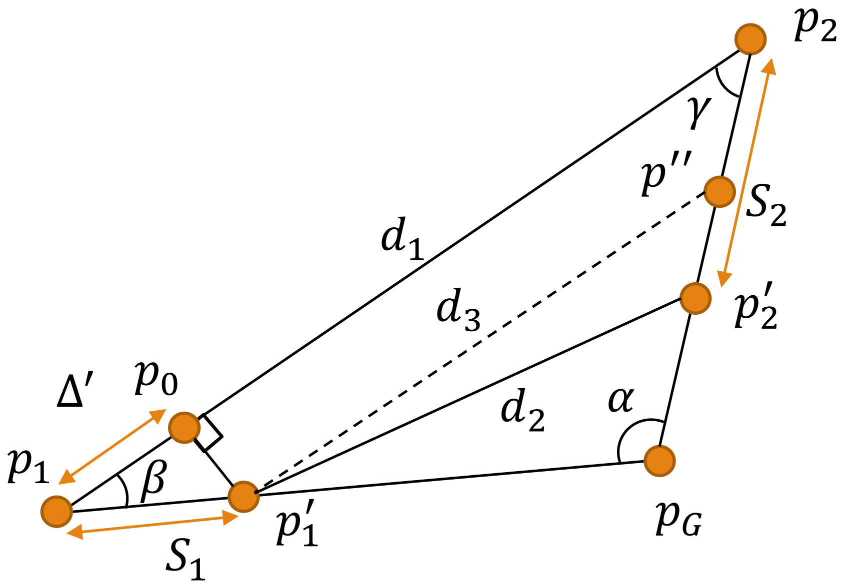


Some Lemmas

- **Lemma 3 :** $\alpha \geq \frac{\pi}{2} \Rightarrow dist(p_1, p_2)$ decreases by at least $0.7S$
- **Proof:** given $d_2 \leq d_3$ => suffice to show $\Delta = d_1 - d_3 > 0.7S$
- $\Delta' = dist(p_0, p_1) \leq \Delta$ show $\Delta' \geq 0.7S$
- $\alpha \geq \frac{\pi}{2} \Rightarrow \beta + \gamma \leq \frac{\pi}{2}$. w.l.o.g. assume $\beta \leq \frac{\pi}{4}$
- Given $S_i \geq S$, by **sine theorem** on triangle $\Delta p_1 p_1' p_0$,

$$S \leq S_1 = \frac{S_1}{\sin\left(\frac{\pi}{2}\right)} = \frac{\Delta'}{\sin\left(\frac{\pi}{2} - \beta\right)}$$

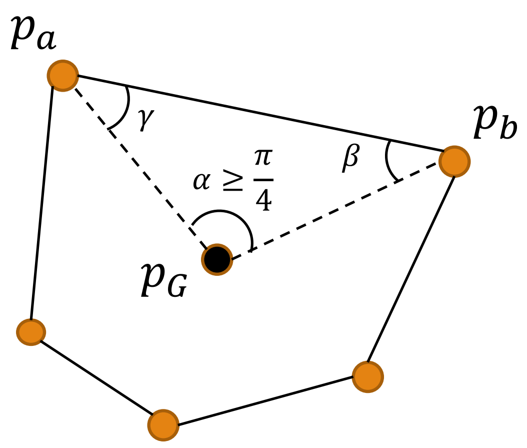
- Hence, $\Delta' \geq S.\cos(\beta) \geq S.\cos\left(\frac{\pi}{4}\right) \geq 0.7S$



Some Lemmas

- **Lemma 4:** $Span(P) = dist(p_a, p_b) \Rightarrow$ for every point P_G in $H(P)$, $\angle p_a p_G p_b \geq \pi/4$.

- **Proof:**



- By contradiction assume $\alpha < \frac{\pi}{4}$ and $w.l.o.g\ \beta \geq \gamma$
- $\alpha < \frac{\pi}{4} < \frac{3\pi}{8} < \frac{\pi-\alpha}{2} = \frac{\beta+\gamma}{2} < \beta < \beta + \gamma = \pi - \alpha \Rightarrow sin\beta > \sin\alpha$
- On $\Delta p_a p_b p_G$, $\frac{dist(p_a, p_b)}{dist\ (p_a, p_G)} = \frac{sin\alpha}{sin\beta}$
- $dist\ (p_a, p_G)$ > $dist(p_a, p_b)$ = $Span\ (P)$ => contradiction

Lemma 5

- K robots $R_1, \ldots, R_K$ at $P = \{p_1, \ldots, p_K\}$
- Traverse same distance S towards a point p_G in the convex hull $H(P)$,
- New positions are $P' = \{p'_1, \ldots, p'_K\} \Rightarrow$
- $Span(P') \leq Span(P) - cS,\, c \geq 1/4$

Lemma 5(proof)

- $p_a, p_b \in \mathrm{V_H}(H(P))$ and $Span(P) = dist(p_a, p_b)$
- $p'_a, p'_b \in \mathrm{V_H}(H(P'))$ and $Span(P') = dist(p'_a, p'_b)$
- By Lemma 4 $\alpha = \angle p_a p_G p_b \geq \pi/4$.
- If $\frac{\pi}{4} \leq \alpha < \frac{\pi}{2}$, according to Lemma 2,

$$dist(p'_a, p'_b) \leq dist(p_a, p_b) - (1 - \cos\alpha)\,S \leq dist(p_a, p_b) - 0.25S.$$

- If $\alpha \geq \pi/2$, by Lemma 3

$$dist(p'_a, p'_b) \leq dist(p_a, p_b) - 0.7S$$

- So in any case we have,

$$dist(p'_a, p'_b) \leq dist(p_a, p_b) - 0.25S \Rightarrow Span(P') \leq Span(P) - 0.25S$$

Corollary 1

- From Lemma 5 we conclude,
- If a set of K robots traverse at least by S,
- $\boldsymbol{Span\ (P') \le Span(P) - cS}$.

Lemma 6

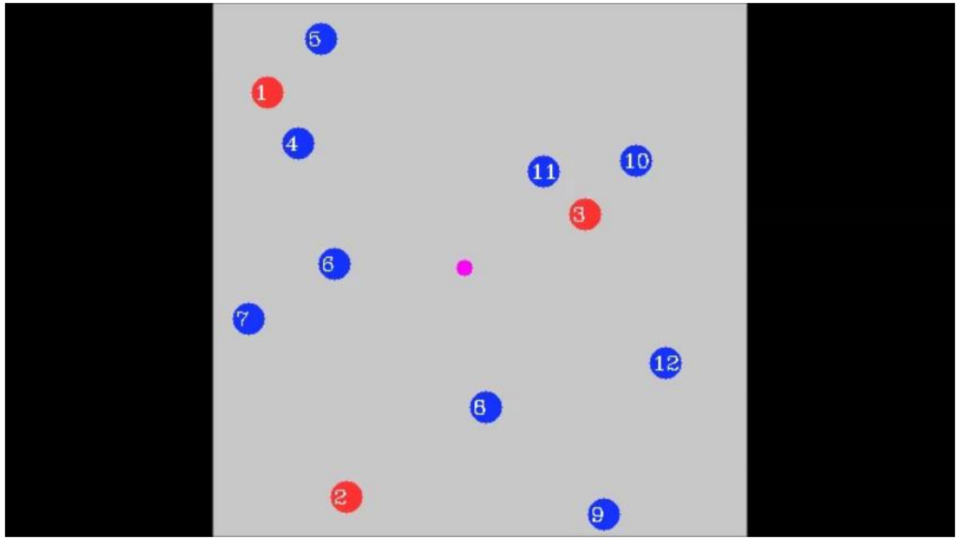
- Using the algorithm, Robots meet finite number of cycles
- **Proof:** for $t \geq 1$, H_t = convex hull at the beginning of cycle t.
- Robots move at least S in each cycle towards p_G in the convex hull
- By Corollary 1 $\Rightarrow Span\ (H_{t+1}) \leq Span\ (H_t) - 0.25S$ for every t
- At most $4.Span(H_1)/S$ cycles $\Rightarrow Span(P) = 0$.
- Thus all robots meet.

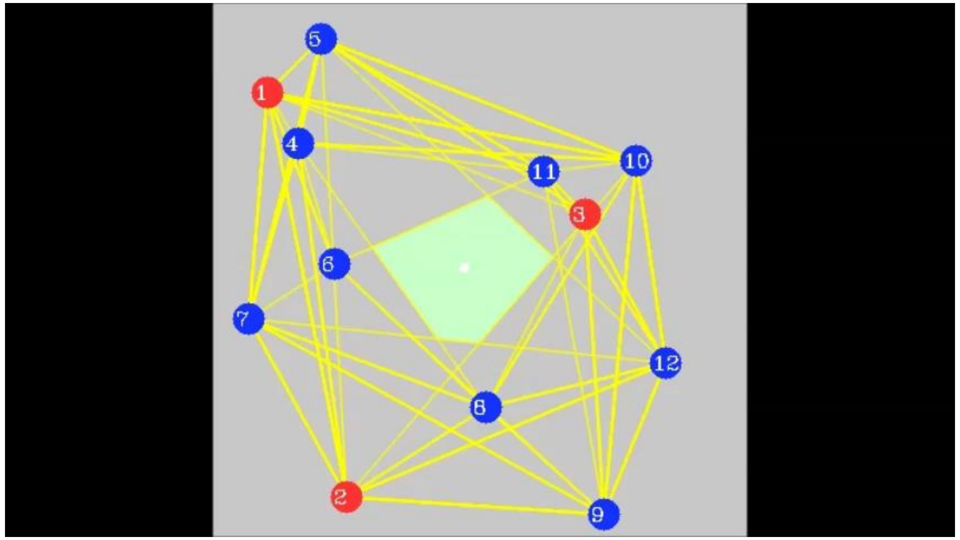
Theorem 3

- Algorithm $\boldsymbol{Gather_{byz}}$ solves $(N, f) - Byzantine$ gathering for any $N \geq 3f + 1$, in ***FSYNC***
- **Proof:** by Lemma 6 it is sufficient to show that $p_G \in H(R_{NF})$
- Set $H_{int}^{f}(P) \subseteq H(P)$ as well as every $N - f$ subsets of $P \Rightarrow H_{int}^{f}(P) \subseteq H(R_{NF})$
- $C_{grav}(\mathrm{P}) \in H(P) \Rightarrow p_G \in H(R_{NF})$
- Therefore, C_{grav} of the set $V_H\left(H_{int}^{f}(P)\right)$ is well defined.

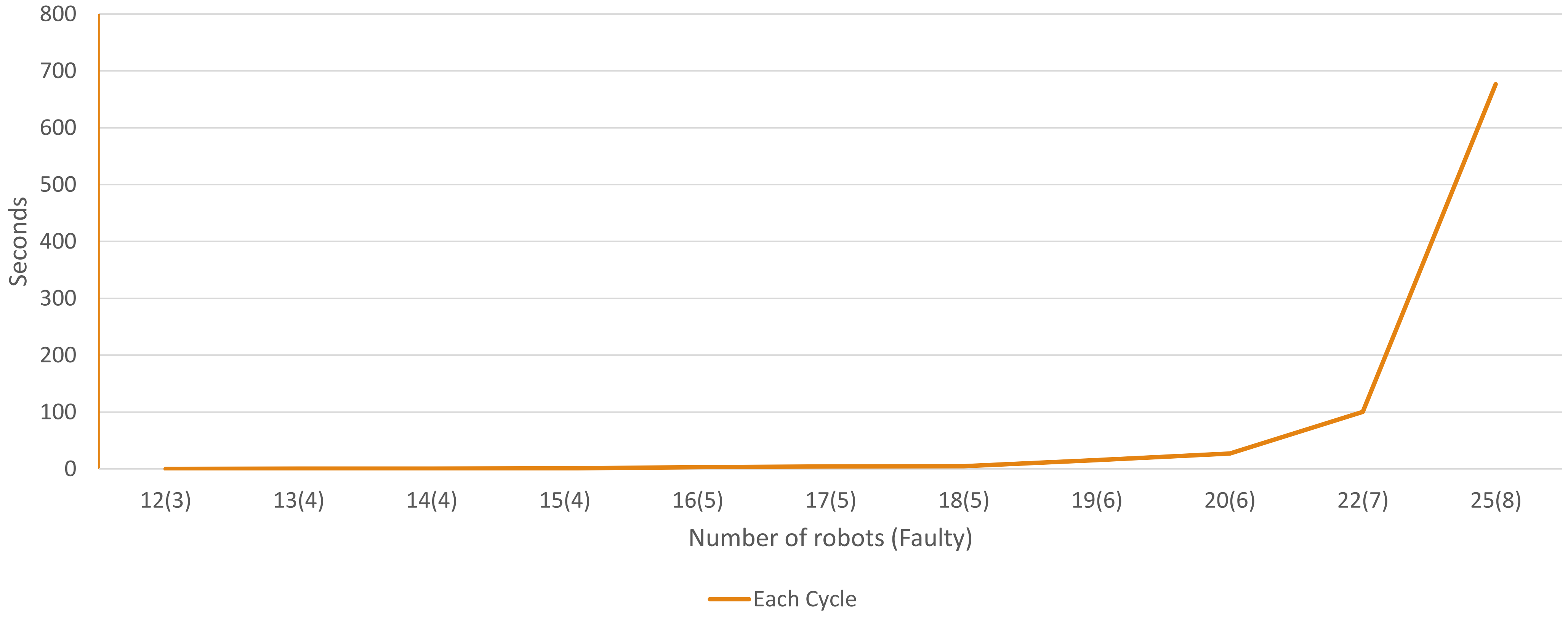
This concludes the proof of the algorithm

The algorithm in practice





Timing of the algorithm



Remarks

- A Formal analysis of Gathering problem
- Fault tolerance for crash and Byzantine
- Offers solutions for $N \geq 3f$ **crash** and $N \geq 3f + 1$ **Byzantine** failure models
- In practice, only viable for few number of failures
- No speculation on upper bound for **move**

Helly's theorem

Helly's Theorem for $d = 2$ (cf. [27, Theorem.4.1.1]): Let $\mathcal{A}$ be a finite family of at least three convex sets in $\mathbb{R}^2$. If every three members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.

Lemma 6.1. *For a multiset $P = \{p_1, \dots, p_N\}$, $N \geq 3k+1$, $H^k_{int}(P)$ is convex and nonempty.*

Proof: $H^k_{\text{int}}(P)$ is convex as it is the intersection of $\binom{N}{k}$ convex sets. We prove that it is nonempty by Helly's Theorem. Consider three arbitrary sets $P^l = \{p^l_1, \dots, p^l_k\} \subseteq P, 1 \leq l \leq 3$, and let $Q^l = H(P \setminus P^l)$, $1 \leq l \leq 3$. Then $Q^1 \cap Q^2 \cap Q^3$ contains at least $P' = P \setminus (P^1 \cup P^2 \cup P^3)$. As $|P| \geq 3k+1$, $|P'| \geq 1$. It follows that the intersection of every three such sets is nonempty, and by Helly's Theorem $V_{\mathrm{H}}(H^k_{\text{int}}(P))$ is nonempty as well. ∎