Region Covariance Matrix and PCA Analysis;

Example: Horizontal Edge

1	1	1	1	1	1
1	**1**	**1**	**1**	**1**	1
1	**1**	**1**	**1**	**1**	1
0	**0**	**0**	**0**	**0**	0
0	**0**	**0**	**0**	**0**	0
0	0	0	0	0	0

We are interested in calculating the region covariance matrix of the highlighted 4-by-4 area above. It corresponds to a horizontal edge in and image.

For each pixel we will use the vector

z_k = [$I_{k,}$ |horizontal derivative|, |vertical derivative|]T

The intensity matrix is:

$$I = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain the vertical edges by convolving the image with the kernel $[1 \quad -1]$. We have:

$$|V| = |I * [1. \quad -1]| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain the horizontal edges features by convolving the image with the kernel $[1 \quad -1]^T$. We have:

$$|H| = |I * [1 \quad -1]^T| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now construct our features for each pixel. We do row scanning over the 4x4 region and construct a 3-by-1 feature vector per pixel. We have 16 pixels in total, so our feature matrix is of size 3 by 16 as follows:

$$X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now calculate the feature mean vector μ, the vector has size 3 by 1:

$$\mu = \frac{1}{N}\begin{bmatrix} \sum I_i \\ \sum |V_i| \\ \sum |H_i| \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/4 \end{bmatrix}$$

Afterwards, we subtract the mean vector from the features matrix, so we obtain:

$$X - \mu = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

We now calculate the region covariance matrix

$$C \coloneqq \frac{1}{N}(X-\mu)(X-\mu)^T = \frac{1}{16}\begin{bmatrix} \frac{16}{4} & 0 & -\frac{1}{8} * 8 - \frac{3}{8} * 4 + \frac{1}{8} * 4 \\ 0 & 0 & 0 \\ -\frac{1}{8} * 8 - \frac{3}{8} * 4 + \frac{1}{8} * 4 & 0 & \frac{12}{16} + \frac{36}{16} \end{bmatrix}$$

$$C = \frac{1}{16}\begin{bmatrix} 4 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 3 \end{bmatrix}$$

In order to find the eigen-decomposition of C, we find the eigenvalues by solving the characteristic polynomial

$$\det(C - \lambda I) = (4-\lambda)(-\lambda)(3-\lambda) - 2(-2\lambda) = 0$$

Our eigenvalues are $\lambda_0 = 0, \lambda_1 = \frac{7+\sqrt{17}}{2}, \lambda_2 = \frac{7-\sqrt{17}}{2}$

The eigenvectors are $v_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_1 = \begin{bmatrix} -1-\sqrt{17} \\ 0 \\ 4 \end{bmatrix}, v_2 = \begin{bmatrix} -1+\sqrt{17} \\ 0 \\ 4 \end{bmatrix}$

Example 2:

Example: Vertical Edge

0	0	0	1	1	1
0	**0**	**0**	**1**	**1**	1
0	**0**	**0**	**1**	**1**	1
0	**0**	**0**	**1**	**1**	1
0	**0**	**0**	**1**	**1**	1
0	0	0	1	1	1

We are interested in calculating the region covariance of the highlighted 4-by-4 area above.

We follow the same procedure used in the previous example. We first find the intensity matrix:

$$I = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We obtain the vertical edges by convolving the image with the kernel $[1 - 1]$. We have:

$$|V| = |I * [1 - 1]| = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

We obtain the horizontal edges features by convolving the image with the kernel $[1 - 1]^T$. We have:

$$|H| = |I * [1 - 1]^T| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now construct the feature matrix of size 3 by 16.

$$X = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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We now calculate the feature mean vector μ, the vector has size 3 by 1:

$$\mu = \frac{1}{N}\begin{bmatrix} \sum I_i \\ \sum |V_i| \\ \sum |H_i| \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/4 \\ 0 \end{bmatrix}$$

Afterwards, we subtract the mean vector from the features matrix, so we obtain:

$$X - \mu = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now calculate the region covariance matrix

$$C := \frac{1}{N}(X-\mu)(X-\mu)^T = \frac{1}{16}\begin{bmatrix} \frac{16}{4} & 4*\frac{1}{8}-4*\frac{3}{8}-\frac{1}{8}*8 & 0 \\ 4*\frac{1}{8}-4*\frac{3}{8}-\frac{1}{8}*8 & \frac{12}{16}+\frac{36}{16} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \frac{1}{16}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of C are:

$$\lambda_0 = 0, \lambda_1 = \frac{7+\sqrt{17}}{2}, \lambda_2 = \frac{7-\sqrt{17}}{2}$$

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The eigenvectors are $v_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} -1-\sqrt{17} \\ 4 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} -1+\sqrt{17} \\ 4 \\ 0 \end{bmatrix}$