Modeling data with probability distributions

ECE 407

Distributional modeling

A useful way to summarize a data set:

- Fit a probability distribution to it.
- Simple and compact, and captures the big picture while smoothing out the wrinkles in the data.
- In subsequent application, use distribution as a proxy for the data.

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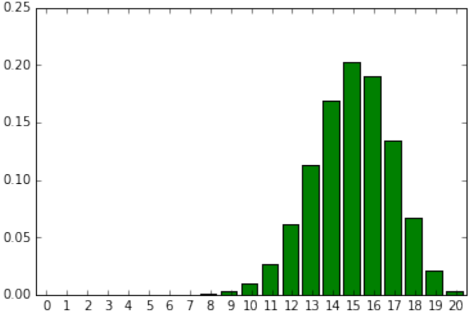
Well, this is true in one dimension. For higher-dimensional data, we'll use combinations of 1-d models: **products** and **mixtures**.

The binomial distribution

Binomial(n, p): the number of heads when n coins of bias (heads probability p) are tossed, independently.



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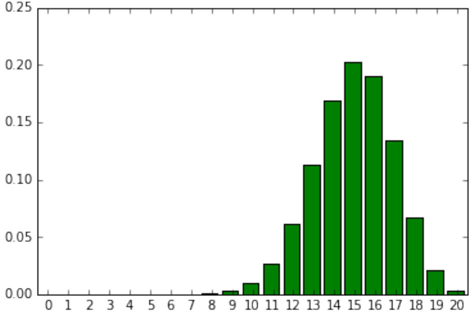

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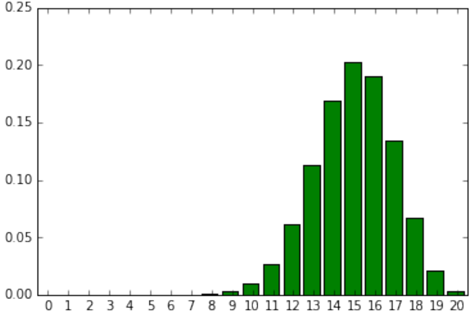

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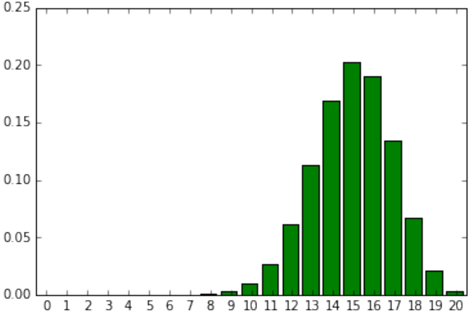

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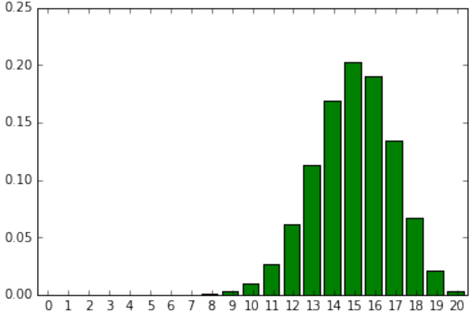

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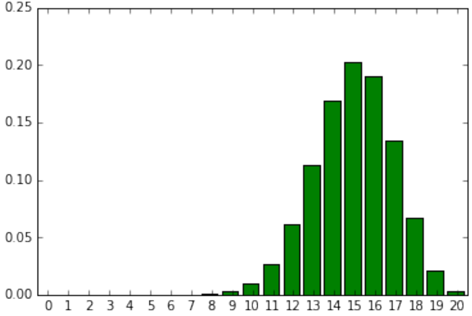

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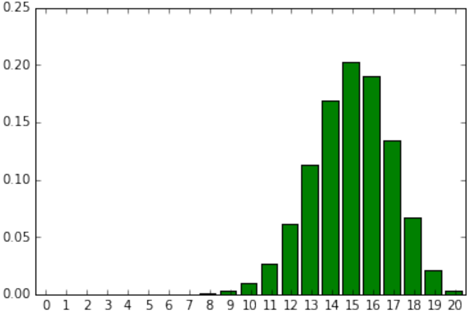

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$$\mathrm{var}(X) = np(1-p)$$
$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Fitting a binomial distribution to data

Example: Upcoming election in a two-party country.

- You choose 1000 people at random and poll them.
- 600 say Tory (Conservative Party in the UK)

What is a good estimate for the fraction of votes the Conservative Party will get in the election?

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More generally, you observe n tosses of a coin of unknown bias. k of them are heads. How to estimate the ``bias`` of the coin?

$$\hat{p} = \frac{k}{n}$$

p is the estimate of the probability of "success" in Bernoulli distribution.

Maximum likelihood estimation

Let $\mathcal{P}$ be a class of probability distributions (Gaussians, Poissons, etc).

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E.g. Suppose $\mathcal{P}$ is the class of binomials. We observe n coin tosses, and k of them are heads.

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$$\mathrm{LL}(p) = k \ln p + (n-k) \ln(1-p).$$

LL: Log-Likelihood

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- Set the derivative to zero.

$$\text{LL}'(p) = \frac{k}{p} - \frac{n-k}{1-p} = 0 \quad \Rightarrow \quad p = \frac{k}{n}.$$

Maximum likelihood: a small caveat

You have two coins of unknown bias.

- You toss the first coin 10 times, and it comes out heads every time.

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The likelihood principle would choose the second coin. Is this right?

Laplace smoothing

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Laplace's law of succession: What is the probability that the sun won't rise tomorrow?

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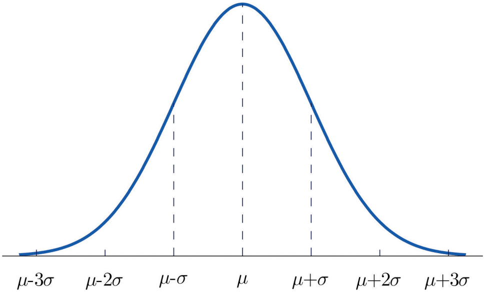
Laplace's law of succession: What is the probability that the sun won't rise tomorrow?

- Let p be the probability that the sun won't rise on a randomly chosen day. We want to estimate p.
- For the past 5000 years ($= 1825000$ days), the sun has risen every day. Using Laplace smoothing, estimate

$$p = \frac{1}{1825002}.$$

The normal distribution



The normal (or *Gaussian*) $N(\mu, \sigma^2)$ has mean μ, variance σ^2, and density function

$$f(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

- 66% of the distribution lies within one standard deviation of the mean, i.e. in the range $\mu \pm \sigma$
- 95% lies within $\mu \pm 2\sigma$
- 99% lies within $\mu \pm 3\sigma$

Maximum likelihood estimation of the normal

Suppose you see n data points $x_1, \ldots, x_n \in \mathbb{R}$, and you want to fit a Gaussian $N(\mu, \sigma^2)$ to them. How to choose μ, σ?

- Maximum likelihood: pick μ, σ to maximize

$$\mathrm{L}(\text{data}|\mu, \sigma^2) = \prod_{i=1}^{n} \left(\frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x_i - \mu)^2}{2\sigma^2} \right) \right)$$

- Work with the log-likelihood, since it makes things easier:

$$\mathrm{LL}(\mu, \sigma^2) = \frac{n}{2} \ln \frac{1}{2\pi\sigma^2} - \sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2\sigma^2}.$$

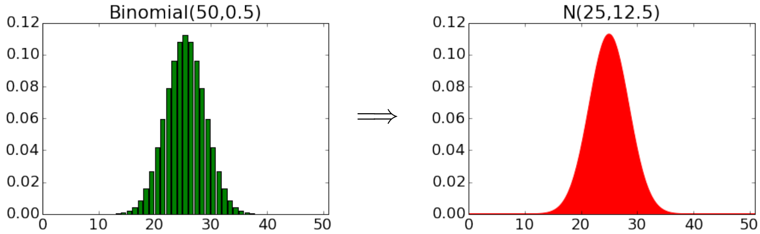
- Setting the derivatives to zero, we get

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

These are simply the empirical mean and variance.

Normal approximation to the binomial



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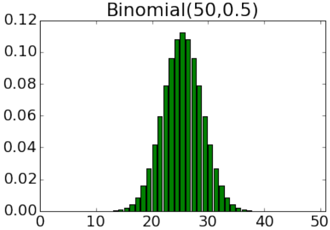
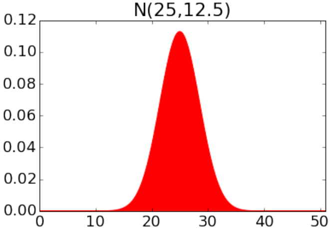


$$\Longrightarrow$$



When a coin of bias p is tossed n times, let X be the number of heads.

- We know X has mean np and variance $np(1-p)$.
- As n grows, the distribution of X looks increasingly like a Gaussian with this mean np and variance $\mathrm{p}(1-\mathrm{p})$.

Application to sampling

We want to find out what fraction p of San Diegans know how to surf. So we poll n random people, and find that k of them surf. Our estimate:

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$$\begin{aligned} \text{mean} &= p \\ \text{variance} &= \frac{p(1-p)}{n} \ \leq \ \frac{1}{4n} \end{aligned}$$

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$$\text{mean} = p$$

$$\text{variance} = \frac{p(1-p)}{n} \leq \frac{1}{4n}$$

Confidence intervals:

- With 95% confidence, our estimate is accurate within $\pm 1/\sqrt{n}$.
- With 99% confidence, our estimate is accurate within $\pm 3/2\sqrt{n}$.

The multinomial distribution

A k-sided die:

- A fair coin has two possible outcomes, each equally likely.
- A fair die has six possible outcomes, each equally likely.
- Imagine a k-faced die, with probabilities $p_1, \ldots, p_k$.

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- Parameters: $p_1, \ldots, p_k \geq 0$, with $p_1 + \cdots + p_k = 1$.
- $\mathbb{E}X = (np_1, np_2, \ldots, np_k)$.
- $\Pr(n_1, \ldots, n_k) = \binom{n}{n_1, n_2, \ldots, n_k} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where

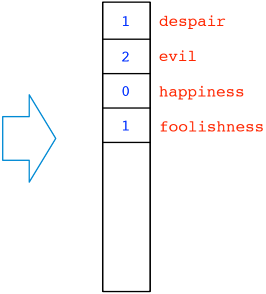
$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$$

  is the number of ways to place balls numbered $\{1, \ldots, n\}$ into bins numbered $\{1, \ldots, k\}$.

Example: text documents

Bag-of-words: vectorial representation of text documents.

It was the best of times, it was the worst of times, it was the age of wisdom, it was the age of foolishness, it was the epoch of belief, it was the epoch of incredulity, it was the season of Light, it was the season of Darkness, it was the spring of hope, it was the winter of despair, we had everything before us, we had nothing before us, we were all going direct to Heaven, we were all going direct the other way – in short, the period was so far like the present period, that some of its noisiest authorities insisted on its being received, for good or for evil, in the superlative degree of comparison only.



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Define the random vector $X = (x1, x2, \ldots, x_{|V|})$,
where xi = # of times the ith word appears in the document.

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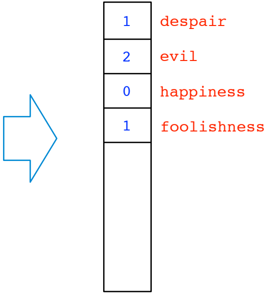


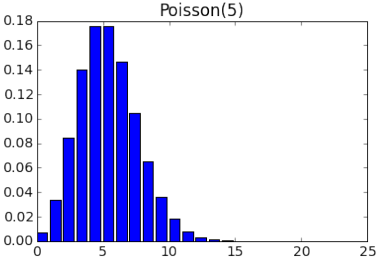
- Fix V = some vocabulary.
- Treat the words in a document as independent draws from a multinomial distribution over V:

$$p = (p_1, \ldots, p_{|V|}), \quad \text{such that} \quad p_i \geq 0 \text{ and } \sum_i p_i = 1$$

Treat each document as a random vector of length $|V|$

The Poisson distribution

A distribution over the non-negative integers $\{0, 1, 2, \ldots\}$



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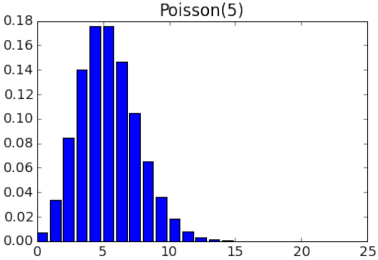

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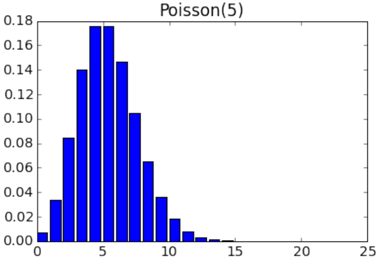

The Poisson has parameter $\lambda > 0$, with $\Pr(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}$

- Mean: $\mathbb{E}X = \lambda$
- Variance: $\mathbb{E}(X - \lambda)^2 = \lambda$
- Maximum likelihood fit: set λ to the empirical mean

How the Poisson arises

Count the number of events (collisions, phone calls, etc) that occur in a certain interval of time. Call this number X, and say it has expected value λ.

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Now suppose we divide the interval into small pieces of equal length.

If the probability of an event occurring in a small interval is:

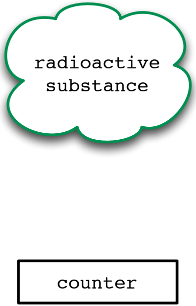
- independent of what happens in other small intervals, and
- the same across small intervals,

then $X \sim \text{Poisson}(\lambda)$.

Example: The number of hits at a website in any time interval is a Poisson random variable. A particular site has on average λ=2 hits/sec
What is the probability that there are no hits in an interval of 1 second?

Poisson: examples

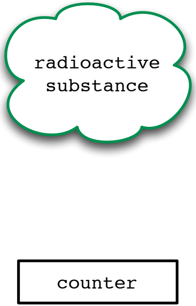
Rutherford's experiments with radioactive disintegration (1920)



- $N = 2608$ intervals of 7.5 seconds
- $N_k = \#$ intervals with k particles
- Mean: 3.87 particles per interval

Poisson: examples

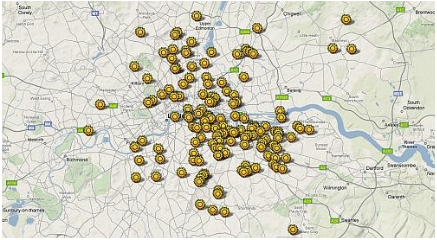
Rutherford's experiments with radioactive disintegration (1920)



- $N = 2608$ intervals of 7.5 seconds
- $N_k = \#$ intervals with k particles
- Mean: 3.87 particles per interval

k	0	1	2	3	4	5	6	7	8	≥ 9
N_k	57	203	383	525	532	408	273	139	45	43
P(3.87)	54.4	211	407	526	508	394	254	140	67.9	46.3

Flying bomb hits on London in WWII



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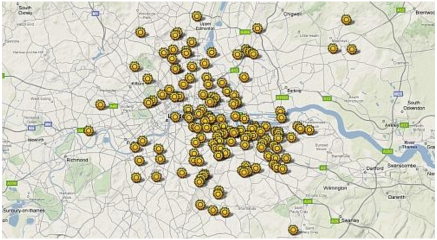

- Area divided into 576 regions, each 0.25 km^2
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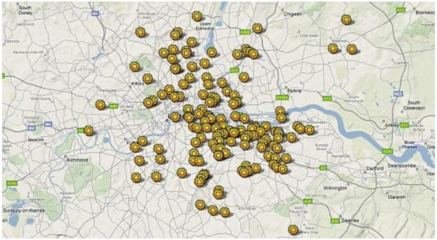

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- Mean: 0.93 hits per region

k	0	1	2	3	4	≥ 5
N_k	229	211	93	35	7	1
P(0.93)	226.8	211.4	98.54	30.62	7.14	1.57

Multivariate distributions

Almost all distributions we've considered are for one-dimensional data.

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1. Model each coordinate separately and treat them as independent.
   For $x = (x_1, \ldots, x_p)$, fit separate models Pr_i to each x_i, and assume

   $$\mathrm{Pr}(x_1, \ldots, x_p) = \mathrm{Pr}_1(x_1)\mathrm{Pr}_2(x_2) \cdots \mathrm{Pr}_p(x_p).$$

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3. More general graphical models.
   Arbitrary dependencies between coordinates.