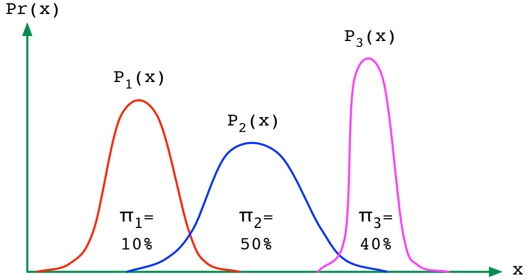
Classification with generative models II

ECE407

Based on Sanjoy Dasgupta's notes.

https://cseweb.ucsd.edu/~dasgupta/

Recall: generative model framework



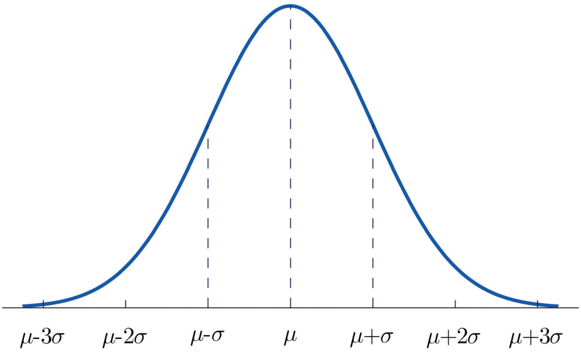
Labels $\mathcal{Y} = \{1, 2, \ldots, k\}$, density $\Pr(x) = \pi_1 P_1(x) + \cdots + \pi_k P_k(x)$.

where $P_1(x) = \Pr(x|Y=1), \ldots, P_k(x) = \Pr(x|Y=k)$

Approximate each P_j with a simple, parametric distribution:

- Product distributions.
  Assume coordinates are independent: naive Bayes.
- Multivariate Gaussians.
  Linear and quadratic discriminant analysis.
- More general graphical models.

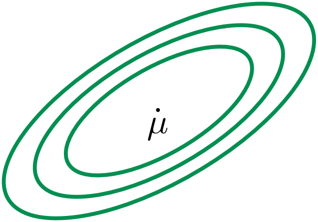
The univariate Gaussian



The Gaussian $N(\mu, \sigma^2)$ has mean μ, variance σ^2, and density function

$$\mathrm{f_x}(x) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

The multivariate Gaussian



$N(\mu, \Sigma)$: Gaussian in $\mathbb{R}^p$

- mean: $\mu \in \mathbb{R}^p$
- covariance: $p \times p$ matrix Σ

$$f_x(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{p/2}} \exp\left(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)\right)$$

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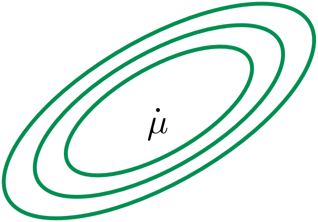

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Let $X = (X_1, X_2, \ldots, X_p)$ be a random draw from $N(\mu, \Sigma)$.

- $\mathbb{E}X = \mu$. That is, $\mathbb{E}X_i = \mu_i$ for all $1 \leq i \leq p$.
- $\mathbb{E}(X-\mu)(X-\mu)^T = \Sigma$. That is, for all $1 \leq i, j \leq p$,

$$\mathrm{cov}(X_i, X_j) = \mathbb{E}(X_i - \mu_i)(X_j - \mu_j) = \Sigma_{ij}$$

In particular, $\mathrm{var}(X_i) = \mathbb{E}(X_i - \mu_i)^2 = \Sigma_{ii}$.

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The X_i are independent and all have the same variance σ^2. Thus

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Note that: 1- The value of the pdf at x_o $f_x(x_o)$ is not a probability.
2- I_p is the p-dimensional identity matrix.

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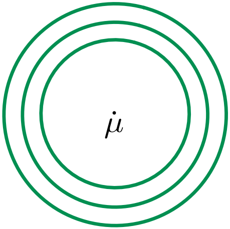

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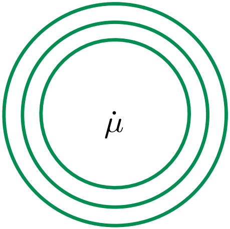

Each X_i is an independent univariate Gaussian $N(\mu_i, \sigma^2)$.

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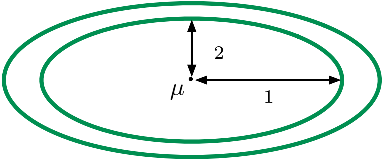
Contours of equal density are axis-aligned ellipsoids centered at μ:



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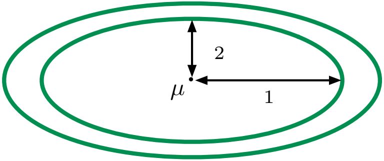

mean is a vector!

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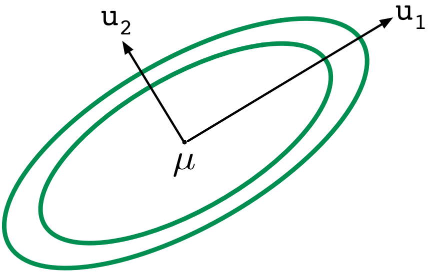
The general Gaussian

$N(\mu, \Sigma)$:



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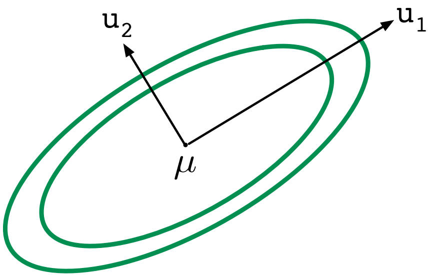

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- Eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$
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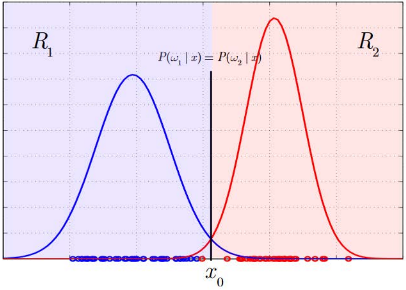
$N(\mu, \Sigma)$ is simply a rotated version of $N(\mu, \mathrm{diag}(\lambda_1, \ldots, \lambda_p))$.

Binary classification with a Gaussian generative model

Example: Two classes: ω_1 and ω_2 with $N(\mu_i,\sigma_i)$, respectively



- *Red and blue dots are the training data. Estimate the mean and variance of distributions of each class.*
- *Decision threshold is x_o (assuming that class prob are equal π_1*

Binary classification with Gaussian generative model

Estimate class probabilities π_1, π_2 and fit a Gaussian to each class:

$$P_1 = N(\mu_1, \Sigma_1), P_2 = N(\mu_2, \Sigma_2)$$

E.g. If data points $x^{(1)}, \ldots, x^{(m)} \in \mathbb{R}^p$ are class 1:

$$\mu_1 \; = \; \frac{1}{m}\left(x^{(1)} + \cdots + x^{(m)}\right) \quad \text{and} \quad \Sigma_1 \; = \; \frac{1}{m}\sum_{i=1}^{m}(x^{(i)} - \mu_1)(x^{(i)} - \mu_1)^T$$

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π_1, π_2, μ_2 and Σ_2 are estimated from the training data in a similar manner .

Given a new point x, predict class 1 iff:

$$\pi_1 P_1(x) > \pi_2 P_2(x) \quad \Leftrightarrow \quad x^T M x + 2w^T x \geq \theta,$$

where:

$$M = \frac{1}{2}(\Sigma_2^{-1} - \Sigma_1^{-1})$$

$$w = \Sigma_1^{-1}\mu_1 - \Sigma_2^{-1}\mu_2$$

and θ is a constant depending on the various parameters.

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or divide by (m-1) instead of m

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$\Sigma_1 = \Sigma_2$: linear decision boundary. Otherwise, quadratic boundary.

Common covariance: $\Sigma_1 = \Sigma_2 = \Sigma$

Linear decision boundary: choose class 1 iff

$$x \cdot \underbrace{\Sigma^{-1}(\mu_1 - \mu_2)}_{w} \geq \theta.$$

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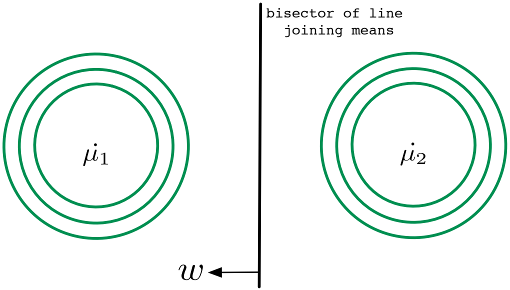
Example 1: Spherical Gaussians with $\Sigma = I_p$ and $\pi_1 = \pi_2$.



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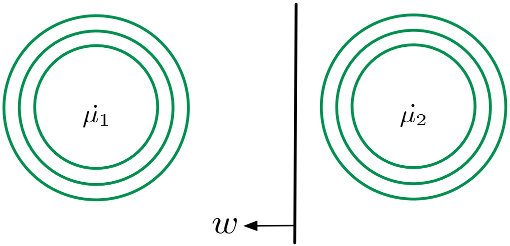

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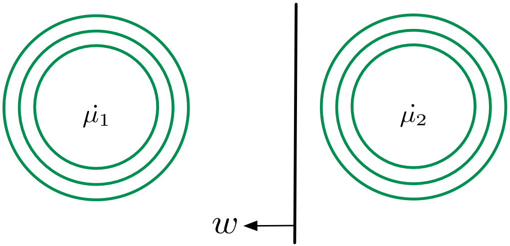
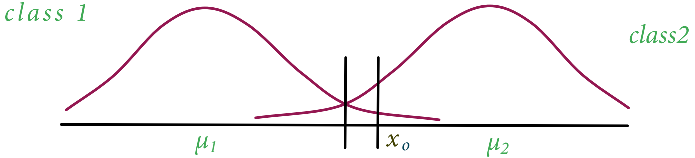
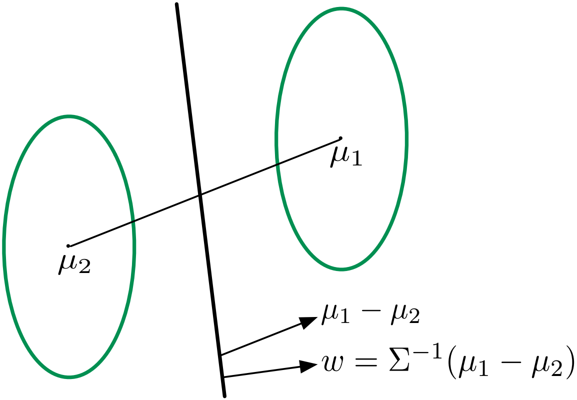


1-D example:



Decision boundary is x_o is closer to the mean of class 2 because $\pi_1 > \pi_2$

Example 3: Non-spherical.



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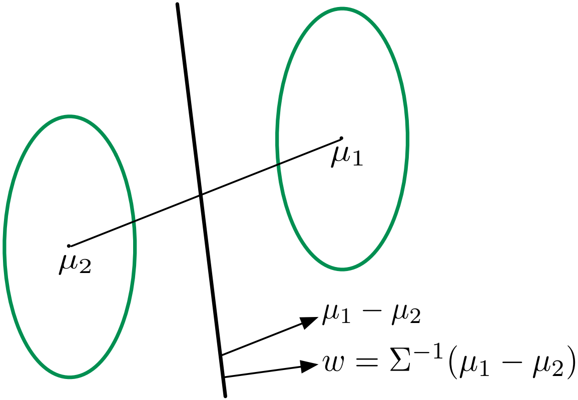

Rule: $w \cdot x \geq \theta$

- w, θ dictated by probability model, assuming it is a perfect fit
- Common practice: choose w as above, but fit θ to minimize training/validation error

Different covariances: $\Sigma_1 \neq \Sigma_2$

Quadratic boundary: choose class 1 iff $x^T M x + 2w^T x \geq \theta$, where:

$$M = \frac{1}{2}(\Sigma_2^{-1} - \Sigma_1^{-1})$$
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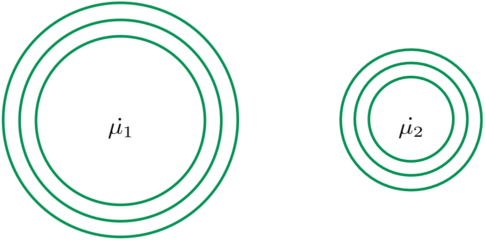

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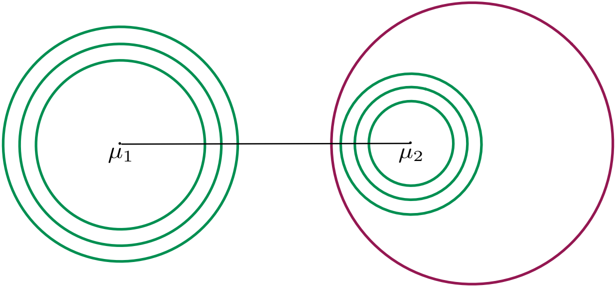
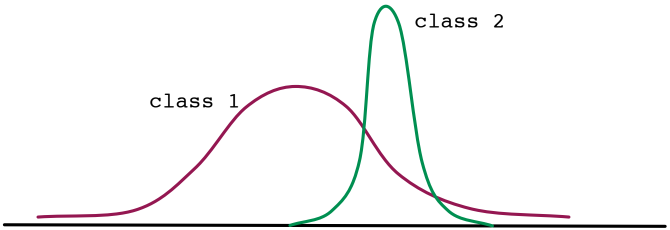
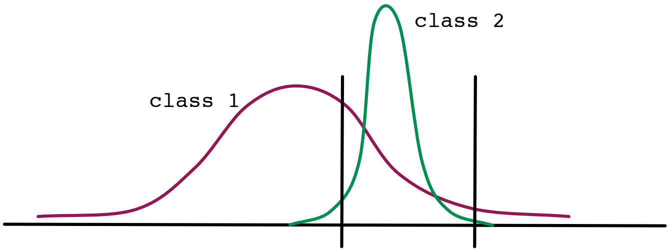


Example 2: Same thing in 1-d. $\mathcal{X} = \mathbb{R}$.

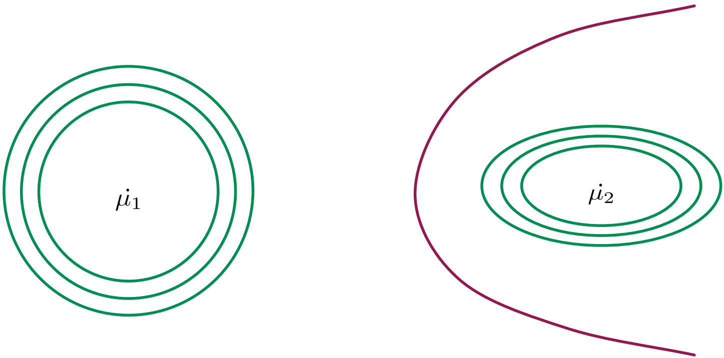


Example 2: 1-D example. $\mathcal{X} = \mathbb{R}$.



You may have two decision thresholds

Example 3: A parabolic boundary.



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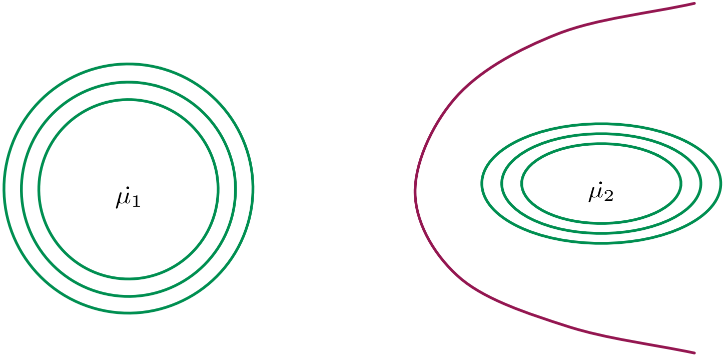

Many other possibilities!

Multiclass discriminant analysis

k classes: weights π_j, class-conditional distributions $P_j = N(\mu_j, \Sigma_j)$.

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To class a point x, pick $\arg\max_j f_j(x)$.

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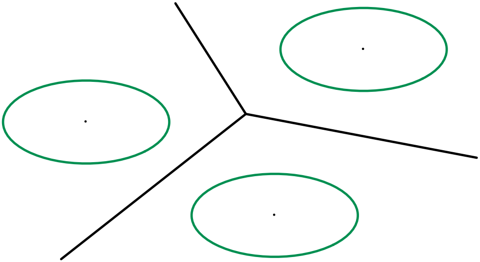
Example: If $\Sigma_1 = \cdots = \Sigma_k$, the boundaries are **linear**.



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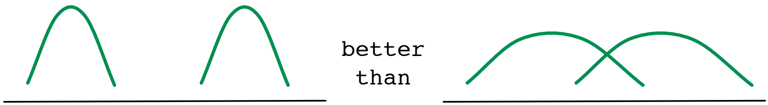
1. Projected means are well-separated, i.e. $(w \cdot \mu_1 - w \cdot \mu_2)^2$ is large.
2. Projected within-class variance is small.



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- Projected means: $w \cdot \mu_1$ and $w \cdot \mu_2$
- Projected variances: $w^T \Sigma_1 w$ and $w^T \Sigma_2 w$
- Average projected variance:

$$\frac{n_1(w^T \Sigma_1 w) + n_2(w^T \Sigma_2 w)}{n_1 + n_2} = w^T \Sigma w,$$

where $\Sigma = (n_1 \Sigma_1 + n_2 \Sigma_2)/(n_1 + n_2)$.

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Find w to maximize $J(w) = \dfrac{(w \cdot \mu_1 - w \cdot \mu_2)^2}{w^T \Sigma w}$

Solution: $w \propto \Sigma^{-1}(\mu_1 - \mu_2)$. Look familiar?