Discrete Random Variables

Given an experiment:

- $\mathcal{S}$: sample space
- $\mathcal{F}$: collection of events
- $\mathcal{P}$ is a probability measure on $\mathcal{F}$.
  - probability is a number between 0 and 1,
  - assigned to events, consistent with axioms

Discrete Random Variables

Some informal observations:

- Sample space outcomes may not be real numbers, which are easier to deal with. So we create a mapping from outcomes to numbers and refer to it as a *random variable. (RV)*
- RV: "Numerical view" of experiment outcomes
- Denoted by capital letters (i.e. X, Y)
- The set of possible values of X is the *range* of X.
- Range is denoted by S_X.

Example-1

- Coin tossing $\Rightarrow \{H, T\}$
- $X = \#$ of heads: $1, 0$
- Range of values $S_X = \{0, 1\}$

Example-2

- Rolling a die: 1-1, 1-2,.. 1-6..2-1..2-6,.. 6-6
- More than one RV possible for the same experiment:
- $X =$ sum of face values
- $S_X = \{2, 3, 4, ..., 12\}$ *(finite)*
- $Y =$ face value of first roll
- $S_Y = \{1, 2, 3, 4, 5, 6\}$ *(finite)*

Example-3

Transmit a packet, if lost or corrupted transmit again. Repeat until packet is correctly received.

- RV: $X = \#$ of transmissions of packet, until packet is correctly received.
- Range: $S_X = \{1, 2, 3, 4...\}$ *countably infinite*

Example-4

Record the time of start and end of a phone call.
Assume no call can exceed 1000 minutes.

- RV: $X =$ duration of phone call in minutes.
- Range: $S_X = (0, 1000]$ *uncountably infinite*

Definitions

Random Variable: Assigns real numbers to each outcome, satisfying "some" conditions.
Examples 1, 2 and 3 are Discrete RV.

- *Discrete RV:* Range of RV is countable (finite or countably infinite)
- *Finite RV:* Range of RV is finite.

Example 1-RV

Bits 0 and 1 are equally likely. Three bits are transmitted. Define RV:

- $X = \#$ of 1's transmitted
- $Y = \#$ of 0's transmitted
- $R =$ (No of 1's)(No of 0's)$= XY$

Example 1-RV

Outcome	000	001	010	011	100	101	110	111
P[·]	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
X	0	1	1	2	1	2	2	3
Y	3	2	2	1	2	1	1	0
$R = XY$	0	2	2	2	2	2	2	0

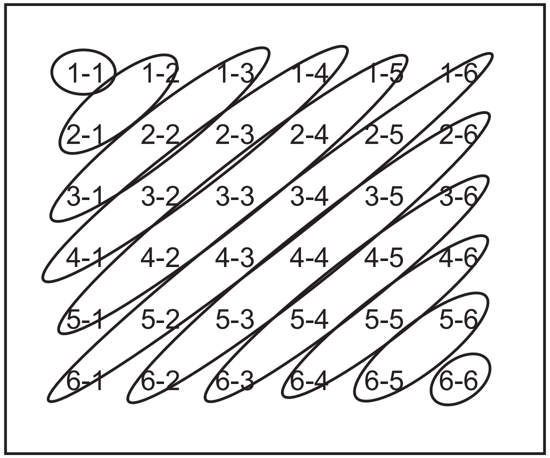
Example 1-RV

$$S_X = \{0,1,2,3\};\ S_Y = \{0,1,2,3\};\ S_R = \{0,2\}$$

- $P[X = 0] = 1/8; P[X = 1] = 3/8$
  $P[X = 2] = 3/8; P[X = 3] = 1/8;$
- $P[Y = 0] = 1/8; P[Y = 1] = 3/8$
  $P[Y = 2] = 3/8; P[Y = 3] = 1/8;$
- $P[R = 0] = 2/8; P[R = 2] = 6/8$

Example 2-RV: Die rolled twice

Experiment: Six-faced die is rolled twice and face values are recorded.



Define R.V. X as sum of rolled values. $S_X = \{2, 3, \ldots, 12\}$

$P[X = k] =?$ for $k = 6$. $P[X = k] =?$ for $k = 6.5$.

$P[X = k] =?$ for each $k \in S_X$

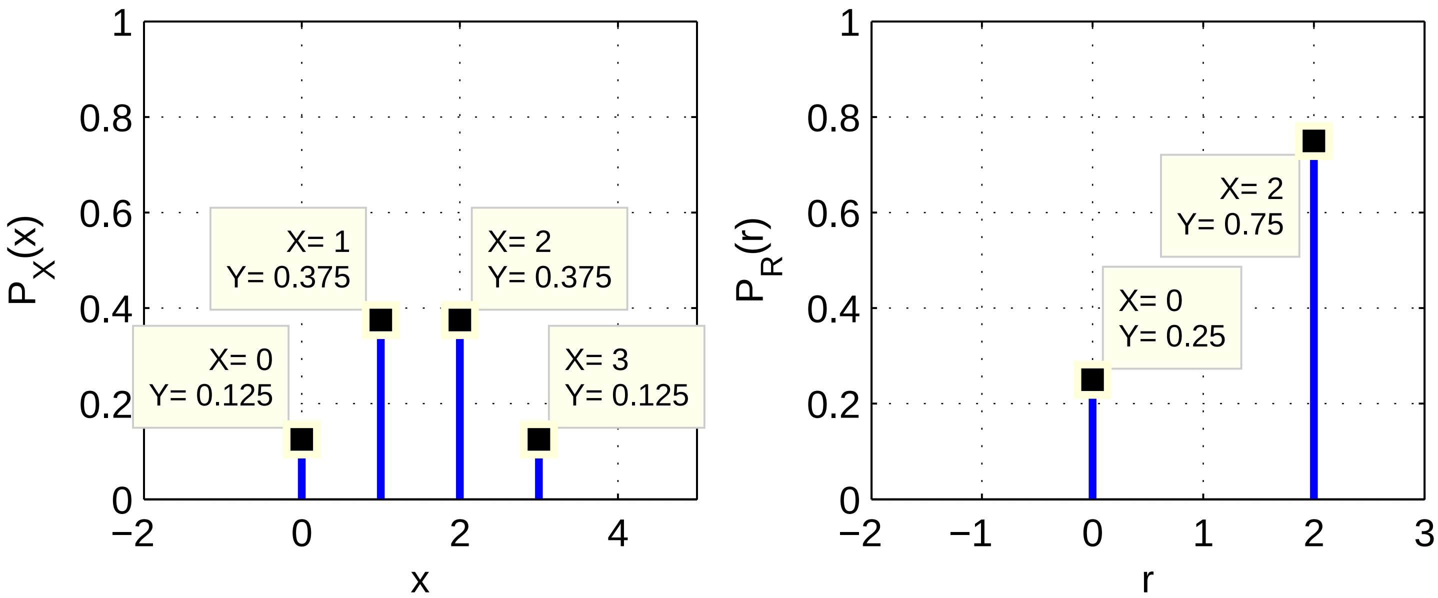
What is the probability of the event $[6 \leq X \leq 8] =$?

What is the probability of the event $[2^X \leq 10] =$?

$\longrightarrow$ Probability Mass Function

PMF: Probability Mass Function

PMF of RV X is defined as $P_X(x) = P[X = x]$ for all real X.
In the example 1-RV above:



Properties of PMF (Theorem 2.1)

- $P_X(x) \geq 0$
- $\sum_{x \in S_X} P_X(x) = 1 \Rightarrow P(S) = 1$
- $B \subset S_X$ corresponds to an event (subset of S)

$$P[X \in B] = \sum_{x \in B} P_X(x)$$

Each $x \in S_X$ is associated with one or more outcomes in sample space S and each outcome s in S is associated with some x in S_X

Example of PMF

Consider a random variable X with PMF

$$P_X(x) = \begin{cases} c(1+x^2) & x \in \{0,2,3\} \\ 0 & else \end{cases}$$

Event A = "X is even", Event B = "$\sin(\pi X/2) \neq 0$"

- $c =$?
- $P[A] =$?
- $P[B] =$?

Properties of PMF (Theorem 2.1)

In the example 1-RV above:
Consider event B = "3-bit outcome has at least one 1"
$\Rightarrow B = \{000\}^c$

- $P[B] = P[X \in \{1, 2, 3\}]$
- $= \sum_{x=1}^{3} P_X(x) = 3/8 + 3/8 + 1/8 = 7/8$
- Alternately $P[B] = 1 - P[B^c]$
- $= 1 - P[X = 0] = 1 - P_X(0) = 1 - 1/8 = 7/8$

Commonly Used Models-1

Bernoulli RV:
Number of successes in one trial

$$P_X(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \\ 0 & else \end{cases}$$

Note: $S_X = \{0, 1\}$

Commonly Used Models-2

Geometric RV:
X First success is on x-th trial, in sequence of Bernoulli trials.

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, .. \\ 0 & else \end{cases}$$

Note: $S_X = \{1, 2, 3...\}$

Commonly Used Models-3

Binomial RV:

$X =$ Number of successes in n Bernoulli trials.

$$P_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x} & x = 0, 1, 2, ..n \\ 0 & else \end{cases}$$

Note: $S_X = \{0, 1, 2, 3...n\}$

Useful Results

- Geometric series: $K, K\alpha, K\alpha^2, K\alpha^3 ...$
- Finite Sum: $S_M = K + K\alpha + K\alpha^2 + ... + K\alpha^{M-1}$
  $\alpha S_M = K\alpha + K\alpha^2 + ... + K\alpha^M$
  $S_M - \alpha S_M = K - K\alpha^M$
  $S_M(1-\alpha) = K(1-\alpha^M) \Rightarrow S_M = K\frac{1-\alpha^M}{1-\alpha}$, if $\alpha \neq 1$.
- If $\alpha = 1$ then $S_M = MK$

Useful Results

If $|\alpha| < 1$ Then

$$\lim_{M\to\infty} S_M \triangleq S_\infty = \frac{K}{1-\alpha} = \sum_{x=0}^{\infty} K\alpha^x$$

Therefore for $0 < p < 1$
$\sum_{x=1}^{\infty} p(1-p)^{x-1} = \sum_{k=0}^{\infty} p(1-p)^k = \frac{p}{1-(1-p)} = 1$

- Binomial Theorem:
  $(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x} \Rightarrow \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} =$
  $(p+1-p)^n = 1$

Poisson RV:

Count the number of occurrences/arrivals/episodes during an observation interval of duration T.

- $X =$ # of packets arriving at a node in 0.2 secs.
- $X =$ # of customers arriving in a 20-min period at a bank
- $X =$ # of database queries in 10 secs.

Poisson RV:

- Average arrival rate of packets = λ (packets/sec)
- Observation period=T (sec)
- Average # of arrivals in T secs $= \alpha = \lambda T$
- Actual number of arrivals in a specific T-second interval=X (RV)

Poisson model: $\boxed{P_X(x) = \frac{\alpha^x e^{-\alpha}}{x!}}$ $x = 0, 1, 2..$

$S_X = \{0, 1, 2...\}$

Poisson RV-Example

- $\lambda = 10$ packets/sec, $T = 0.2$ secs.
- $\alpha =$ mean arrival rate in 0.2 sec=10x0.2=2 packets
- $P_X(x) = \frac{2^x e^{-2}}{x!}$

P[at least two packets arrive in 0.2 secs]=?
= 1 - P[zero or one packet arrives in 0.2 secs]
=$1 - (P_X(0) + P_X(1)) = 1 - \frac{2^0 e^{-2}}{0!} - \frac{2^1 e^{-2}}{1!} = 1 - e^{-2} - 2e^{-2}$

Commonly Used Models-5

Uniform RV:

$$P_X(x) = \begin{cases} \frac{1}{l-k+1} & x = k, k+1, .., l \\ 0 & else \end{cases}$$

Die rolling: X = face value $S_X = \{1, 2, 3, 4, 5, 6\}$. Here:

$$P_X(x) = \begin{cases} \frac{1}{6} & x = 1, 2, \ldots, 6 \\ 0 & else \end{cases}$$

Cumulative Distribution Function (CDF)

For any real number x the CDF is the probability that the random variable X is no larger than x

$$\boxed{F_X(x) = P[X \leq x]}$$

Note that inequality sign is $(\leq)$, which means the function includes the probability of $X = x$.

Cumulative Distribution Function (CDF)

For any *discrete* random variable X with range $S_X = \{x_1, x_2 ...\}$ satisfying $x_1 \leq x_2 \leq \ldots < \infty$

- $F_X(-\infty) = 0$ and $F_X(\infty) = 1$
- For all $x \leq x'$, $F_X(x) \leq F_X(x')$
- For $x_i \in S_X$ and $0 < \epsilon < (x_i - x_{i-1})$, $F_X(x_i) - F_X(x_i - \epsilon) = P_X(x_i)$
- $F_X(x) = F_X(x_i)$ for all x such that $x_i \leq x < x_{i+1}$

CDF Theorem 2.3

For all $b > a$

$$F_X(b) - F_X(a) = P[a < X \leq b]$$

Example-CDF

Example: Consider the RV R with the following PDF:

$$P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & otherwise \end{cases}$$

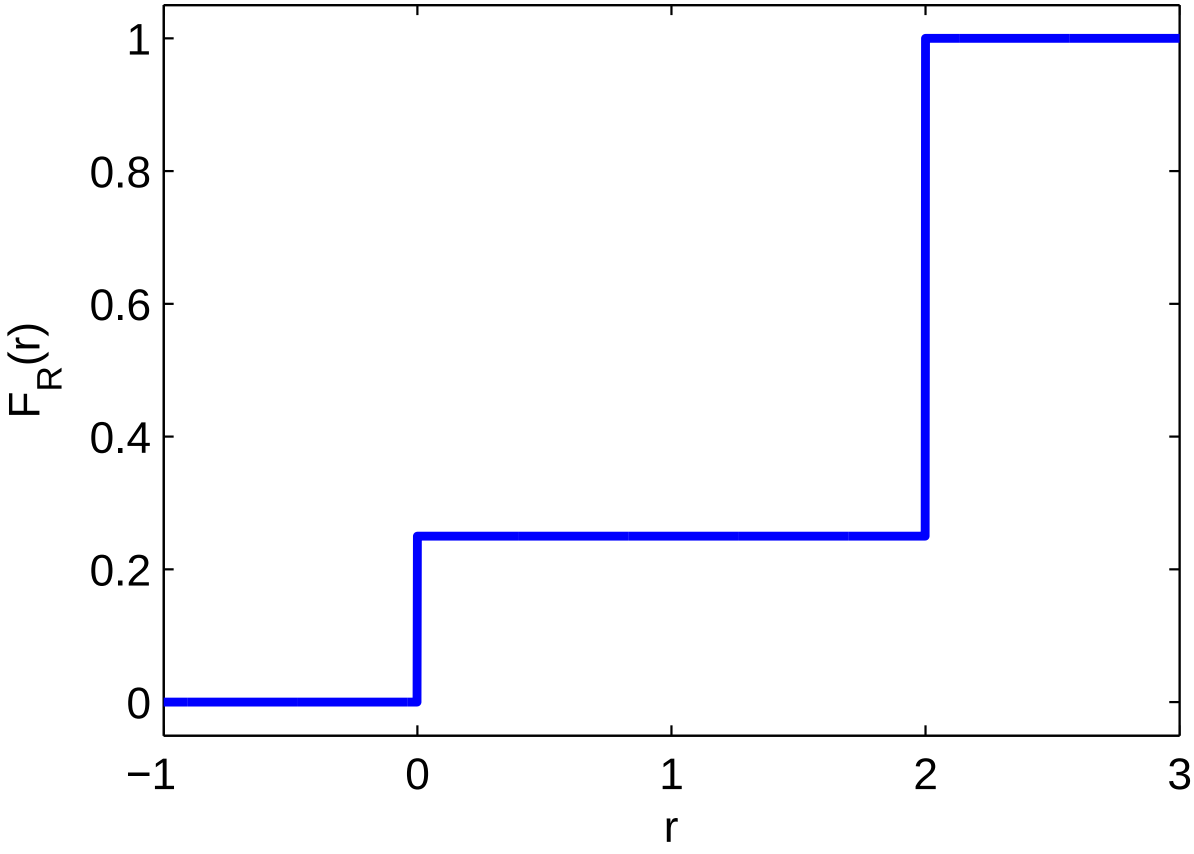
Find and sketch the CDF of the random variable R

Example-CDF

The discontinuities are at $r = 0$ and $r = 2$. At these points: $F_R(0) = 1/4$ and $F_R(2) = 1$. The complete expression is:

$$F_R(r) = P[R \leq r] = \begin{cases} 0 & r < 0 \\ 1/4 & 0 \leq r < 2 \\ 1 & r \geq 2 \end{cases}$$

Example-CDF



Averages (Expected Values)

Consider a die with six faces, one face with "1", two faces with "2" and three faces with "3".

- $P_X(1) = P[X = 1] = 1/6$
- $P_X(2) = P[X = 2] = 2/6$
- $P_X(3) = P[X = 3] = 3/6$

Roll this die 600 times. What is the average value of the face value?
Suppose "1" occurs 100 times, "2" occurs 200 times, "3" occurs 300 times. (these numbers can be different in different trials of 600 rolls)

Averages (Expected Values)

Average computed from sample

$$\begin{aligned} \text{Sample} - \text{ave} = & \; \frac{(100 \cdot 1) + (200 \cdot 2) + (300 \cdot 3)}{600} \\ = & \; (1/6 \cdot 1) + (2/6 \cdot 2) + (3/6 \cdot 3) = 14/6 = 2.333 \end{aligned}$$

Definition: Expected value:

$$\mu_X = E[X] = \sum_{x \in S_X} x P_X(x)$$

Sometimes expected value is called *mean value* and is denoted by μ_X.

Expected value

For the biased die example:

$$P_X(x) = \begin{cases} x/6 & x = 1, 2, 3 \\ 0 & else \end{cases}$$

$$\mu_X = E[X] = \sum_{x=1}^{3} x P_X(x) = 1 * 1/6 + 2 * 2/6 + 3 * 3/6$$

Expected value examples

Bernoulli RV:

$$E[X] = \sum_{x=0}^{1} x P_X(x) = 0 * (1-p) + 1 * p = p$$

Geometric RV:

$$P_X(x) = p(1-p)^{x-1} \quad x = 1, 2, 3, \ldots$$

$$E[X] = \sum_{x=1}^{\infty} x p(1-p)^{x-1} = \frac{1}{p}$$

Functions of a Random Variable

- Given: RV X, with PMF $P_X(x)$, range S_X
- Define: $Y = g(X)$, new random variable
- Question: What is the PMF of Y, $P_Y(y)$
- Approach:
  - Find range of $Y : S_Y$
  - For each y in S_Y, find $P[Y = y]$

$$P_Y(y) = \sum_{x:g(x)=y} P_X(x)$$

Example

Consider an experiment where a biased coin is tossed, with Prob[Heads]=$3/4$.
Let X be number of heads in a single toss. Then $S_X = \{0, 1\}$. X is Bernoulli.

$$P_X(x) = \begin{cases} 1/4 & x = 0 \\ 3/4 & x = 1 \\ 0 & else \end{cases}$$

Example (continued)

Someone promises to give you give you $100(X+1)^2$ dollars for each outcome with X heads, which is Bernoulli, as described. The money you will get in this experiment is a random variable Y where

$$Y = 100(X+1)^2$$

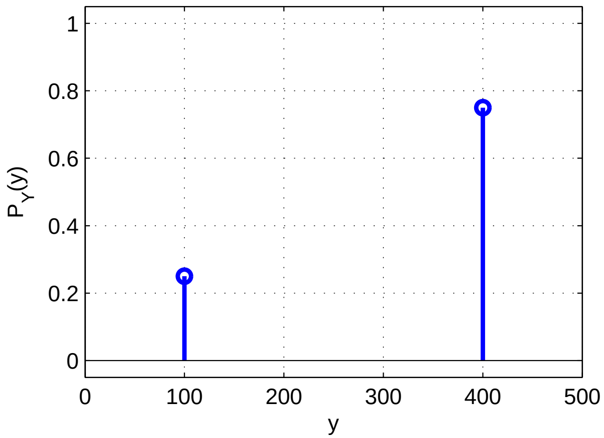
X	0	1
Y	100	400

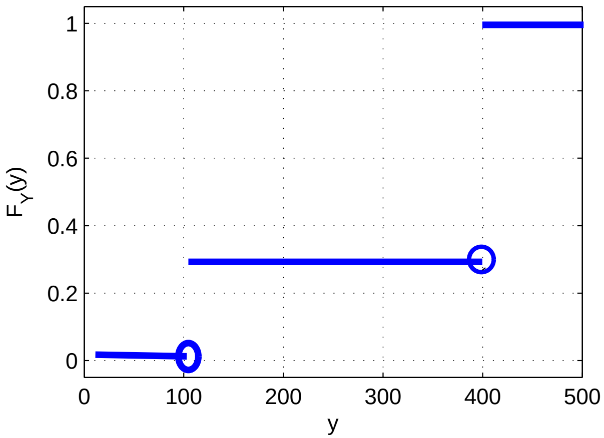
$$\Rightarrow S_Y = \{100, 400\}$$

Example (continued)

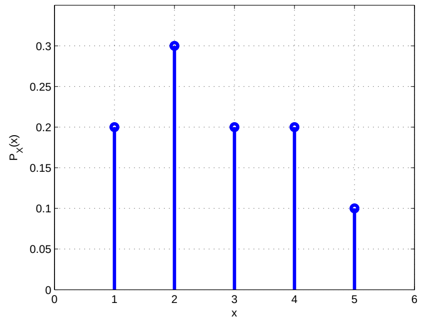
$$\begin{aligned} P[Y = 100] &= P[100(X+1)^2 = 100] \\ &= P[(X+1)^2 = 1] = P[X = 0] \\ &= P[X = 0] = 1/4 \end{aligned}$$

Similarly: $P[Y = 400] = P[X = 1] = 3/4$





Example

Given $P_X(x)$: 

$S_X = \{1, 2, 3, 4, 5\}$

$Y = g(X) = (X - 3)^2 \quad S_Y = \{0, 1, 4\}$

- $P_Y(y) = ?$

$P_Y(0) = P[Y = 0] = P[(X - 3)^2 = 0] = P[X = 3] = P_X(3) = 0.2$.

$P_Y(1) = P[Y = 1] = [X - 3 = \pm 1] = P[X \in \{2, 4\}] = \sum_{x=2,4} P_X(x) = 0.3 + 0.2 = 0.5$

Example (continued)

$$P_Y(4) = \sum_{(x-3)^2=4} P_X(x) = P_X(1) + P_X(5) = 0.2 + 0.1 = 0.3$$

$$P_Y(y) = \begin{cases} 0.2 & y = 0 \\ 0.5 & y = 1 \\ 0.3 & y = 4 \\ 0 & else \end{cases}$$

Expected value of Derived RV

$$\begin{aligned} E[Y] &= \sum_{y \in S_Y} y P_Y(y) = \sum_{y \in S_Y} y \sum_{x : g(x) = y} P_X(x) \\ &= \sum_{y \in S_Y} \sum_{x : g(x) = y} g(x) P_X(x) = \sum_{x \in S_X} g(x) P_X(x) \end{aligned}$$

Exp. value of Derived RV (Example)

Example:

$$E[Y] = \sum_{y \in S_Y} y P_Y(y) = 0 * 0.2 + 1 * 0.5 + 4 * 0.3 = 1.7$$

$$\begin{aligned} E[Y] &= \sum_{x \in S_X} (x-3)^2 P_X(x) \\ &= 4 * 0.2 + 1 * 0.3 + 0 * 0.2 + 1 * 0.2 + 4 * 0.1 \\ &= 1.7 \end{aligned}$$

Exp. value of Derived RV - Examples

Given $Y = g(X) = X(X-1)$. Find $E[g(X)]$ when X is

- Geometric
- Binomial
- Poisson

$$E[g(X)] = \sum_{x \in S_X} g(x) P_X(x) = \sum_{x \in S_X} x(x-1) P_X(x)$$

Some results

For $0 < q < 1$

$$T_1 = \sum_{x=0}^{\infty} q^x = \frac{1}{1-q}$$

$$\frac{d^2 T_1}{dq^2} = \sum_{x=2}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}$$

Geometric (Example Continued)

$q = 1 - p$

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)q^{x-1}p = pq\sum_{x=0}^{\infty} x(x-1)q^{x-2}$$

$$E[X(X-1)] = pq\frac{2}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}$$

Binomial (Example Continued)

$$
\begin{aligned}
E[X(X-1)] &= \sum_{x=0}^{n} x(x-1)\frac{n!}{(n-x)!x!}p^x q^{n-x} \\
&= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!(x-2)!}p^{x-2}q^{n-x} \\
&= n(n-1)p^2 \sum_{y=0}^{m} \binom{m}{y} p^y q^{m-y} \\
&= n(n-1)p^2
\end{aligned}
$$

Poisson (Example Continued)

$$E[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\alpha}\alpha^x}{x!} = \alpha^2$$

Example

$E[X^2] = ?$

- Geometric: $E[X^2 - X] = E[X^2] - E[X] = \frac{2q}{p^2}$
  $\Rightarrow E[X^2] = \frac{2q}{p^2} + \frac{1}{p} = \frac{2q+p}{p^2} = \frac{1+q}{p^2}$
- Binomial: $E[X^2] - E[X] = n(n-1)p^2$
  $\Rightarrow E[X^2] = n^2p^2 - np^2 + np = n^2p^2 + npq$
- Poisson: $E[X^2] = E[X] + E[X(X-1)]$
  $E[X^2] = \alpha + \alpha^2$

Moments of Random Variable

- $E[X] = \sum_{x \in S_X} x P_X(x) = \mu_X \Rightarrow$ *mean* or *first moment*
- $E[X^2] = \sum_{x \in S_X} x^2 P_X(x) \Rightarrow$ *second moment*
- $E[X^m] = \sum_{x \in S_X} x^m P_X(x) \Rightarrow$ *m-th moment*

Example

$$P_X(x) = \begin{cases} 0.25 & x = -1 \\ 0.5 & x = 0 \\ 0.25 & x = 1 \\ 0 & else \end{cases}$$

$$E[X] = 0.25 * (-1) + 0.5 * 0 + 0.25 * 1 = 0$$

$$E[X^2] = 0.25 * (1) + 0.5 * 0 + 0.25 * 1 = 0.5$$

Example (Continued)

Define $Y = X + 2$, then $S_Y = \{1, 2, 3\}$

$$P_Y(y) = \begin{cases} 0.25 & y = 1 \\ 0.5 & y = 2 \\ 0.25 & y = 3 \\ 0 & else \end{cases}$$

$$\begin{aligned} E[Y] &= 0.25 * 1 + 0.5 * 2 + 0.25 * 3 = 2 \\ E[Y^2] &= 0.25 * 1 + 0.5 * 4 + 0.25 * 9 = 4.5 \\ E[Y^4] &= 0.25 * 1 + 0.5 * 16 + 0.25 * 81 = 28.5 \end{aligned}$$

Central Moment

Spread about the mean $\Rightarrow$ Central moments
m-th central moment:

$$E[(X-\mu_X)^m] = \sum_{x\in S_X}(x-\mu_X)^m P_X(x)$$

Variance of X: $m=2$, second central moment

$$Var[X] = \sum_{x\in S_X}(x-\mu_X)^2 P_X(x) = \sigma_X^2$$

Standard deviation: σ_X

Variance

$$Var[X] = \sum_{x \in S_X} (x - \mu_X)^2 P_X(x) = \sum_{x \in S_X} (x^2 - 2\mu_X x + \mu_X^2) P_X(x)$$

$$= \sum_{x \in S_X} x^2 P_X(x) - 2\mu_X \sum_{x \in S_X} x P_X(x) + \mu_X^2 \sum_{x \in S_X} P_X(x)$$

$$= E[X^2] - 2\mu_X^2 + \mu_X^2 = E[X^2] - \mu_X^2$$

Ex: $Var[Y] = E[(Y-2)^2] = 0.5$ and $\sigma_Y = \sqrt{0.5}$

Conditional Probability Mass Function

Given the event B with $P[B] > 0$, the *conditional PMF* of X is

$$P_{X|B}(x) = P[X = x|B]$$

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & else \end{cases}$$

Conditional PMF Example

Discrete RV, X has the CDF:

$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1 & 7 \leq x < \infty \end{cases}$$

Example Continued

- Find $P_X(x), S_X$
- If $B = \{X > 0\}$, find $P[B]$
- Find $P_{X|B}(x) = P[X = x|B]$
- Find $E[X|B], E[X^2|B]$
- Find $E[X|B^C]$

Example Continued (a,b)

- (a)

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \end{cases}$$

$$S_X = \{-3, 5, 7\}$$

- (b)
$P[B] = P[X > 0] = \sum_{x>0} P_X(x) = P_X(5) + P_X(7) = 0.6$

Example Continued (c)

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & else \end{cases} = \begin{cases} \frac{P_X(5)}{0.6} & x = 5 \\ \frac{P_X(7)}{0.6} & x = 7 \\ 0 & else \end{cases}$$

Therefore:

$$P_{X|B}(x) = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & else \end{cases}$$

Example Continued (d)

$$E[X|B] = \sum_{x \in S_{X|B}} x P_{X|B}(x) = 5 * 2/3 + 7 * 1/3 = 17/3$$

$$E[X^2|B] = \sum_{x \in S_{X|B}} x^2 P_{X|B}(x) = 5^2 * 2/3 + 7^2 * 1/3 = 33$$

$$B^C = \{-3\}, P_{X|B^C}(x) = \begin{cases} 1 & x = -3 \\ 0 & else \end{cases}$$

$$E[X|B^C] = (-3) * 1 = -3$$