2.7 Eigendecomposition

Notes are created by Busra Tugce Gurbuz based on *Deep Learning* book (Goodfellow et al.)

- Many mathematical objects can be understood better by breaking them into constituent parts, or finding some properties of them that are universal, not caused by the way we choose to represent them.
  - For example, integers can be decomposed into prime factors. The way we represent the number 12 will change depending on whether we write it in base ten or in binary, but it will always be true that 12 = 2x2x3. From this representation, we can conclude useful properties, such as that 12 is not divisible by 5, or that any integer multiple of 12 will be divisible by 3.
- Likewise, we can decompose matrices in ways that show us information about their functional properties that is not obvious from the representation of the matrix as an array of elements. One of the most widely used kinds of matrix decomposition is called **Eigendecomposition**, in which we decompose a matrix into a set of **eigenvectors and eigenvalues.**

Eigenvector of a square matrix A is a non-zero vector v such that multiplication by A alters only the scale of v. The scalar which scales the eigenvector is called **eigenvalue**.

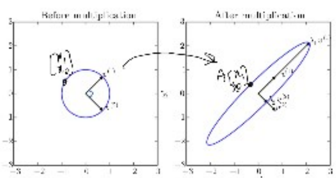
transformation matrix, eigenvalue

$$A\vec{v} = \lambda\vec{v}$$

Eigenvector

This means that matrix-vector multiplication of $A\vec{v}$ gives the same result with scaling eigenvector $\vec{v}$ with some value

$v^T A = \lambda v^T \rightsquigarrow$ left eigenvectors



- How we would calculate the eigenvalues and eigenvectors by hand given the definition?

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \qquad \text{Definition: } Ax = \lambda x, \; x \in \mathbb{R}, \; x \neq 0$$

$$Ax = \lambda I x, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Ax - \lambda I x = 0$$

$$\underbrace{(A - \lambda I)}_{M} x = 0 \to x \neq 0$$ from the definition we know that

$M \rightsquigarrow$ M is not invertable because there is some non-zero vector x in the null space of matrix M. If M is not invertable, its determinant is 0.

$$\det(A - I\lambda) = 0$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix} = 3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda_1 = -2 \quad \lambda_2 = -1$$ → two eigenvalues

The expression $\det(A - \lambda I)$ is called the **characteristic polynomial** of A and eigenvalues are defined to be the roots of this polynomial. In general, the characteristic polynomial of an nxn matrix is an n^{th} degree polynomial which means that there will be at most n roots of the polynomial. The set of eigenvalues is what we called **spectrum** of A.

How to find eigen vectors?

Let's take $\lambda_2 = -1$

$$Ax = \lambda x$$

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$x_2 = -x_1$$

$$-2x_1 - 3x_2 = -x_2$$

$\to \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix}$ family of eigenvectors

Unit eigenvectors

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Because the criteria is $x_1 = -x_2$, you normalize it with their norm

We can do the same for λ_1.

$$\begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix} \longrightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{pmatrix}$$

$$x_2 = -2x_1 \to \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- If v is an eigenvector of A, then so is any rescaled vector sv ($s \in \mathbb{R}$, $s \neq 0$). Moreover, sv still has the same eigenvalue. For this reason, we usually only look for unit eigenvectors.

$$A(sv) = s(Av) = s(\lambda v) = \lambda(sv)$$

- Eigenvalues and eigenvectors of a matrix help us to find subspaces which are invariant under A (if we see matrix A as a linear transformation). As a result, only square matrices have eigenvalues and eigenvectors.
  - If A is non square, it would be $A: \mathbb{R}^m \to \mathbb{R}^n$. Then, $Ax = \lambda x$ would not make sense since $Ax \notin \mathbb{R}^m$
  - For non square matrices, we can define **singular values**
- By definition, $v=0$ cannot be an eigenvector of a matrix, lambda = 0 can be an eigenvalue.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Lambda is an eigenvalue of A if and only if lambda is an eigenvalue of A transpose.
  - By using this theorem, we can show that left eigenvector of A is the right eigenvector of A transpose.

Left eigenvector: $v^T A = \lambda v^T$

$$(v^T A)^T = (\lambda v^T)^T$$

$$A^T v = \lambda v$$

- Suppose that a matrix A has n linearly independent eigenvectors such as. $\{v^{(1)}, \ldots, v^{(n)}\}$ with corresponding eigenvalues. $\{\lambda_1, \ldots \lambda_n\}$. We may concatenate all of the eigenvectors to form a matrix V with one eigenvector per column. Likewise, we can concatenate the eigenvalues to form a vector. In this case, **eigendecomposition** is defined:

$$A = V \underbrace{\operatorname{diag}(\lambda)}_{} V^{-1}$$

eigenvectors matrix

$$\begin{bmatrix} | & & | \\ v^{(1)} & \cdots & v^{(n)} \\ | & & | \end{bmatrix}$$

eigenvalues vector

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ 0 & 0 & \cdots & & \lambda_n \end{bmatrix}$$

Proof:

$$Av = \lambda v$$

$AQ = Q\Lambda$ (Q: eigenvector matrix, Λ: diagonal eigenvalue)

$$AQQ^T = Q\Lambda Q^T$$

$$A = Q\Lambda Q^T$$

  - Not every matrix can be decomposed into eigenvalues and eigenvectors. In some cases, the decomposition exists, but may involve complex rather than real numbers. Every symmetric matrix can be decomposed into an expression using only real-valued eigenvectors and eigenvalues.

$$A = Q\Lambda Q^T$$

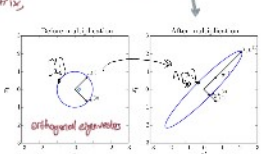
diagonal eigenvalue matrix

symmetric matrix $A = A^T$

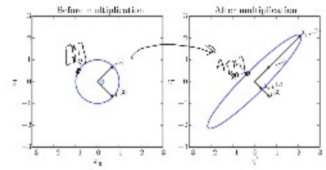
orthogonal matrix composed of eigenvectors of A.

Because of the property of the symmetric matrix, eigenvectors are orthogonal

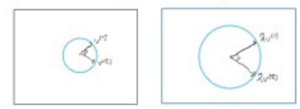
The eigenvalue $\Lambda_{i,i}$ is associated with the eigenvector in column i of Q, denoted as $Q_{:,i}$. Because Q is orthogonal matrix, we can think of A as scaling space by λ_i in direction $v^{(i)}$.



- While any real symmetric matrix A is guaranteed to have an eigendecomposition, the eigendecomposition might not be **unique.**
  - If any two or more eigenvectors share the same eigenvalue, then any set of orthogonal vectors lying in their span are also eigenvectors with that eigenvalue, and we could equivalently choose a Q using those eigenvectors instead.
  - By convention, they sort the entries of Λ in descending order. **Under this convention, the eigendecomposition is unique only if all the eigenvalues are unique.**



Based on the definition of $Ax = \lambda v$, here eigenvectors of A looks unique.
What wouldn't be unique?



- The eigendecomposition of a matrix tells us many useful facts about the matrix.
  - The matrix is **singular** if and only if any of the eigenvalues are zero.

Proof: We know that a system has infinitely many solutions when there is more than one solution.

We also know that a matrix is singular when it is not invertable. This is the case when there is no solution or infinitely many solutions.

$$Av = \lambda v \to \lambda = 0 \Rightarrow Av = 0, \; v \neq 0$$

So, A has infinitely many solutions. As a result, A is singular

  - The eigendecomposition of a real symmetric matrix can also be used to optimize quadratic expressions of the form $f(x) = x^T A x$ subject to $\|x\|_2 = 1$
    - Whenever x is equal to an eigenvector of A, f takes on the value of the corresponding eigenvalue.
    - The maximum value of f within the constraint region is the maximum eigenvalue and its minimum value within the constraint region is the minimum eigenvalue.
- A matrix whose eigenvalues are all positive is called **positive definite.** A matrix whose eigenvalues are all positive or zero-valued is called **positive semi definite.**
  - Likewise if all eigenvalues are negative, the matrix is **negative definite,** and if all eigenvalues are negative or zero-valued, it is **negative semi definite.**
  - Positive semidefinite matrices guarantee that: $\forall x, \; x^T A x \geq 0$
  - In addition to above property, the positive definite matrices guarantee $x^T A x = 0 \Rightarrow x = 0$