Diffusion

Last time we considered images as functions

$$I : \Omega \to \mathbb{R}^n$$

Now we consider image evolutions over time

$$I : \Omega \times [0, t] \to \mathbb{R}^n$$

There are two physical models we are considering
First model: Fick's law

Differences of concentration in a field cause a flux in the direction opposing the concentration gradient

$$\vec{j} = -D\nabla I \tag{1}$$

Image case: 2D-field in Ω:

$$j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$$

D is called *Diffusion Tensor*, a positive, symmetric matrix in the 2D case.

Second model: Continuity equation

$$\frac{\partial I}{\partial t} + \text{div } j = c$$

Where the divergence is defined as

$$\text{div} j = \frac{\partial}{\partial x} j_1 + \frac{\partial}{\partial y} j_2 \tag{2}$$

and c is a constant for approaching/vanishing quantity.
Combining (1) and (2)

$$\frac{\partial I}{\partial t} = \text{div } (D \nabla I) \tag{3}$$

where ∇ is only with respect to x and y, i.e. $\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$.

Types of Diffusion

- Linear Diffusion D does not depend on Image I
  - isotropic:
$$D = \mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{div}\,(D\nabla I) = \Delta I \tag{4}$$
  - anisotropic: $\Rightarrow$ the resulting flux is not parallel to ∇I
- Nonlinear Diffusion: D depends on image I:
  - isotropic:
$$D = \mathbb{I} \cdot \varphi(I) = \begin{pmatrix} \varphi(I) & 0 \\ 0 & \varphi(I) \end{pmatrix} \Rightarrow \text{div}\,(D\nabla I) = \text{div}\,(\varphi(I) \cdot \nabla I) \tag{5}$$

    where typically $\varphi(I)$ is in the range of $0 \leq \varphi(I) \leq 1$ for well-posedness purposes, but there are φ not fulfilling this constraint, which we are going to use.
  - anisotropic: D is only a positive definite matrix (with some restrictions on the eigenvalues)

Implementation and Discretization of the nonlinear isotropic diffusion

$$\frac{\partial I}{\partial t} = \mathrm{div}(\varphi \nabla I) = \frac{\partial}{\partial x}(\varphi \cdot \frac{\partial I}{\partial x}) + \frac{\partial}{\partial y}(\varphi \cdot \frac{\partial I}{\partial y})$$

Discretization of the temporal derivative by forward differences:

$$\frac{\partial I}{\partial t} \approx \frac{I(x, y, t+\tau) - \overbrace{I(x, y, t)}^{I}}{\tau}$$

Discretization of the outer spatial derivatives by means of central differences on half-pixel values:

$$\begin{aligned}
\frac{\partial}{\partial x}(\varphi \cdot \frac{\partial I}{\partial x}) &\approx (\varphi \cdot \frac{\partial I}{\partial x})(x + 1/2, y, t) - (\varphi \cdot \frac{\partial I}{\partial x})(x - 1/2, y, t) \\
\frac{\partial}{\partial y}(\varphi \cdot \frac{\partial I}{\partial y}) &\approx (\varphi \cdot \frac{\partial I}{\partial y})(x, y + 1/2, t) - (\varphi \cdot \frac{\partial I}{\partial y})(x, y - 1/2, t)
\end{aligned}$$

On the half-pixel values the function φ is approximated by arithmetic means and the derivatives are approximated by central differences of the adjacent pixels:

$$
\begin{aligned}
(\varphi \cdot \frac{\partial I}{\partial x})(x + 1/2, y, t) &\approx \underbrace{\frac{\varphi(x+1, y, t) + \varphi(x, y, t)}{2}}_{\varphi_r} \cdot (I(x+1, y) - I(x, y)) \\
(\varphi \cdot \frac{\partial I}{\partial x})(x - 1/2, y, t) &\approx \underbrace{\frac{\varphi(x-1, y, t) + \varphi(x, y, t)}{2}}_{\varphi_l} \cdot (I(x, y) - I(x-1, y)) \\
(\varphi \cdot \frac{\partial I}{\partial y})(x, y + 1/2, t) &\approx \underbrace{\frac{\varphi(x, y+1, t) + \varphi(x, y, t)}{2}}_{\varphi_u} \cdot (I(x, y+1) - I(x, y)) \\
(\varphi \cdot \frac{\partial I}{\partial y})(x, y - 1/2, t) &\approx \underbrace{\frac{\varphi(x, y-1, t) + \varphi(x, y, t)}{2}}_{\varphi_d} \cdot (I(x, y) - I(x, y-1))
\end{aligned}
$$

The final discretized scheme reads

$$
\begin{aligned}
& \frac{I(x,y,t+\tau)-I(x,y,t)}{\tau} \\
= \quad & \varphi_r I(x+1,y,t)+\varphi_l I(x-1,y,t) \\
+ \quad & \varphi_u I(x,y+1,t)+\varphi_d I(x,y-1,t)-(\varphi_r+\varphi_l+\varphi_u+\varphi_d)I(x,y,t)
\end{aligned}
$$

For $\varphi \equiv 1$ one gets the discretized Laplacian equation:

$$
\begin{aligned}
& \frac{I(x,y,t+\tau)-I(x,y,t)}{\tau} \\
= \quad & \underbrace{I(x+1,y,t)+(x-1,y,t)+I(x,y+1,t)+I(x,y-1,t)-4I(x,y,t)}_{\approx \Delta I}
\end{aligned}
$$

Assuming we know the image at time t and want to compute it at time $t+1$:

$$
\begin{aligned}
& I(x,y,t+\tau) \\
= \quad & I(x,y,t)+\tau\cdot(\varphi_r I(x+1,y,t)+\varphi_l I(x-1,y,t) \\
+ \quad & \varphi_u I(x,y+1,t)+\varphi_d I(x,y-1,t)-(\varphi_r+\varphi_l+\varphi_u+\varphi_d)I(x,y,t))
\end{aligned}
$$

A natural assumption is to have the gradient vanish at the image boundaries, meaning that $\frac{\partial}{\partial x} I = 0$ at the left and right boundary and $\frac{\partial}{\partial y} I = 0$ at the top and bottom boundary. This ensures, that the average grey value of the image is preserved. you implement this by setting

$$
\begin{aligned}
I(-1, y, t) &:= I(0, y, t) \\
I(w, y, t) &:= I(w-1, y, t) \\
I(x, -1, t) &:= I(x, 0, t) \\
I(x, h, t) &:= I(x, h-1, t)
\end{aligned}
$$

If we see the image as a stacked-up vector:

$$I(t+\tau) = (\mathbb{I} + \tau A(\varphi(I(t))) \cdot I(t) \tag{6}$$

We want the matrix $(\mathbb{I} + A)$ to be non-negative, therefore, if $\varphi < 1$, $\varphi > 0$, we have $\varphi_r, \varphi_l, \varphi_u, \varphi_d \geq 0$, and the restriction of the time step size $\tau \leq 1/4$.

There are three ways of discretization of the continuous diffusion equation (3):

- Explicit: $I(t+\tau)$ is computed with spatial relations and diffusivity of time t:

$$\begin{aligned} \frac{I(t+\tau)-I(t)}{\tau} &= A(\varphi(I(t)))\cdot I(t) \\ I(t+\tau) &= (\mathbb{I}+\tau A(\varphi(I(t)))\cdot I(t) \end{aligned}$$

- Semi-Implicit: $I(t+\tau)$ is computed with spatial relations of time $t+\tau$ and diffusivity of time t:

$$\begin{aligned} \frac{I(t+\tau)-I(t)}{\tau} &= A(\varphi(I(t)))\cdot I(t{\color{red}+\tau}) \\ I(t) &= (\mathbb{I}-\tau A(\varphi(I(t)))\cdot I(t+\tau) \end{aligned}$$

- Fully Implicit: $\Rightarrow$ Variational Method

Implement the linear diffusion with

$$\varphi(I) = 1$$

and the nonlinear diffusion with

$$\varphi(I) = \frac{1}{\sqrt{|\nabla I|^2 + \epsilon}}$$