Methods Of Optimal Control

Arash Fahim

FSU Fall 2025

September 4, 2025

Structure of the course

- Short lecture (10-15 minutes)
- Team work (50 minutes)
    - Problem solving
    - Programming (Python)
    - Managerial skills
    - Reporting skills
- Delivery (5 minutes)

Structure of the course

Python

Each student is expected to bring a computer to the classroom with a *Python 3.12*, *IPython*, and *Jupyter Notebook* installed.

- `https://www.python.org/downloads`
- `https://www.anaconda.com/docs/main`
- Virtual environment:

  `https://docs.python.org/3/library/venv.html`
  `https://www.anaconda.com/docs/tools/working-with-conda/environments`

GitHub

Each student is required to have a *GitHub* account.

Rules of teamwork

- Be respectful.
- Allow all team members to engage and express their opinion. No interruption.
- Arrange chair to make sure everyone is involved.
- No one knows everything. Make sure that everyone learns what you know.
- Break down the tasks between group members based on ones abilities.
- Ask for feedback from me frequently.

Composition of teams

Each team requires one or two members to take the following roles:

- Problem solving
- Programming (Python)
- Managerial skills
- Reporting skills

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

$$x_t^* = \frac{\alpha_t}{2} = \operatorname*{argmin}_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Can we find u_t such that $x_t=\frac{\alpha_t}{2}$? (For simplicity, take $\beta=0$.)

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2 - \alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x : [0, T] \to\in \mathbb{R}$ such that for some function $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)
Check it for $\alpha_t = \mathbb{1}_{\{0\le t\le \frac{T}{2}\}}$.

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Can we find u_t such that $x_t=\frac{\alpha_t}{2}$? (For simplicity, take $\beta=0$.)
Check it for $\alpha_t=\mathbb{1}_{\{0\le t\le \frac{T}{2}\}}$.
For $\alpha_t=\mathbb{1}_{\{0\le t\le \frac{T}{2}\}}$, what is the value of the infimum? Is it

$$\inf\left\{\int_0^T \left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}=-\frac{1}{4}\int_0^T\alpha_t^2\mathrm{d}t=-\frac{T}{8}?$$

Optimization versus control

A control problem without a myopic solution

$$\inf \int_0^T \left(x_t^2 - \alpha_t x_t + u_t^2\right) \mathrm{d}t \tag{1}$$

Infimum is over all functions $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Trade-off:

- Trying to send $x_t \to \frac{\alpha_t}{2}$ may cause $\int_0^T u_t^2 dt$ to grow.
- Trying to keep cost $\int_0^T u_t^2 dt$ near zero, does not bring x_t close to $\frac{\alpha_t}{2}$.

What is the sweet spot for u_t?

A generic control problem

Definition

$$\inf_{u\in\mathcal{U}} \int_0^T C(t, x_t, u_t)\mathrm{d}t + g(x_T), \quad \mathrm{d}x_t = f(x_t, u_t)\mathrm{d}t \tag{2}$$

- $C : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$: *running cost*
- $g : \mathbb{R}^d \to \mathbb{R}$: *terminal cost*
- $\mathcal{U}$: *an admissible* set of functions $u : [0, T] \to \mathbb{R}^n$, *control variable.*

A generic control problem

Admissible controls

$\mathcal{U}$ is chosen to fit the proper application and/or to make the control problem wellposed.

Admissibility for wellposedness

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2) \mathrm{d}t = -\infty, \qquad \mathrm{d}x_t = (x_t - u_t) \mathrm{d}t \tag{3}$$

$\mathcal{U}$ to be the set of all functions $u : [0, T] \to \mathbb{R}$
If we restrict $\mathcal{U}$ to the set of functions $u : [0, T] \to [-1, \infty)$ (some lower bound on the value), then

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2) \mathrm{d}t > -\infty, \qquad \mathrm{d}x_t = (x_t - u_t) \mathrm{d}t. \tag{4}$$

Infinite horizon

Infinite horizon

An infinite horizon control problem is accommodated by setting $T = \infty$. For example,

$$\inf_{u \in \mathcal{U}} \int_0^\infty e^{-t}(x_t^2 + u_t^2)\mathrm{d}t, \;\; C(t, x, u) = e^{-t}(x^2 + u^2) \tag{5}$$

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \;:\; x_t \notin D\} \tag{6}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Infinite horizon

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \ : \ x_t \notin D\} \tag{7}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Solution

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} \mathrm{d}t, \ \mathrm{d}x_t = u_t \mathrm{d}t \ \text{ with } \ |u_t| \leq 1$$

An optimal control is described by existing D as fast as possible, $|u| = 1$, and stop as soon as we exit, $|u| = 0$.

Dynamic programming principle (DPP)

Value function

Fix $x_t = x$.

$$V(t,x) := \inf_{u \in \mathcal{U}_t} \int_t^T C(s, x_s, u_s) \mathrm{d}s + g(x_T), \qquad \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$$

$\mathcal{U}_t$: the set of admissible controls restricted to $[t, T]$.

Dynamic programming principle (DPP)

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \qquad \mathrm{d}x_r = f(x_r, u_r)\mathrm{d}r$$

$\mathcal{U}_{t,s}$: the set of admissible controls restricted to $[t, s]$.

DPP

Balance of cost in DPP

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s), \qquad x_s = x + \int_t^s f(x_r, u_r)\mathrm{d}r$$

Proof of DPP

$$
\begin{aligned}
V(t,x) &= \inf_{u\in\mathcal{U}_t} \int_t^T C(r,x_r,u_r)\mathrm{d}r + g(x_T) \\
&= \inf_{u_1\in\mathcal{U}_{t,s}} \inf_{u_2\in\mathcal{U}_s} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T) \\
&= \inf_{u_1\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + \inf_{u_2\in\mathcal{U}_s} \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T)
\end{aligned}
$$

Note that $\inf_{u_2\in\mathcal{U}_s} \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T) = V(s,x_s)$. Therefore,

$$
V(t,x) = \inf_{u_1\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + V(s,x_s)
$$

Numerical DPP

Discretization

$\Delta t = \frac{T}{N}$

$$\begin{cases} V(t_N, x) = g(x) \\ V(t_i, x) = \inf_{u \in \mathcal{U}_{t_i, t_{i+1}}} \int_{t_i}^{t_{i+1}} C(r, x_r, u_r) \mathrm{d}r + V(t_{i+1}, x_{t_{i+1}}), \\ \qquad\qquad\qquad\qquad x_{t_{i+1}} = x + \int_{t_i}^{t_{i+1}} f(s, x_s, u_s) \mathrm{d}s \end{cases}$$

Approximation

$$\begin{cases} \hat{V}(t_N, x) = g(x) \\ \hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u)\Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \;\; x_{t_{i+1}} = x + f(t_i, x, u)\Delta t \end{cases}$$

Numerical DPP

Approximation

$$\begin{cases} \hat{V}(t_N, x) = g(x) \\ \hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u)\Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \;\; x_{t_{i+1}} = x + f(t_i, x, u)\Delta t \end{cases}$$

Simplification of one-step approximate DPP

The approximation is not over the control $u : [t_i, t_{i+1}] \to \mathbb{R}^m$, but over values $u \in \mathbb{R}^m$. The optimal value $\hat{u}^*$ is a constant approximately optimal control over $[t_i, t_{i+1}]$.

Algorithm

Algorithm 1: Numerical DPP

Parameter : T, N, $f(t,x,u)$, $C(t,x,u)$, and $g(x)$;

$\Delta t = \frac{T}{N}$

Data: $\hat{V}(t_N, x) = g(x)$;

x_i^j for $j = 1, ..., J$ and $i = 0, ..., N-1$;

(x_i^j means the jth discrete point at time t_i.)

for $i \leftarrow N-1$ *to* 0 **do**

- $\hat{x}_{i+1}^j = x_i^j + f(t_i, x_i^j, u)\Delta t$;
- $\tilde{V}(t_i, x_i^j) \leftarrow \inf_u C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{t_{i+1}}^j)$;
- $\hat{V}(t_i, x)$ obtained from interpolation on $\tilde{V}(t_i, x_i^j)$ for $j = 1, ..., J$;
- $\hat{u}^*(t_i, x_i^j) \in \underset{u}{\operatorname{argmin}}\, C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{i+1}^j)$;

return $\hat{V}(t_i, \cdot)$ *and* $\hat{u}^*(t_i, \cdot)$ *for* $i = 0, ..., N-1$.

DPP algorithm

Exercise

Why interpolation is required in Algorithm 1?
Can we perform the algorithm by only knowing $\hat{V}(t_{i+1}, x^j_{i+1})$ for all $j = 1, ..., J$?
Note the difference between $\hat{V}(t_{i+1}, x^j_{i+1})$ and $\hat{V}(t_{i+1}, \hat{x}^j_{i+1})$ and the difference between x^j_{i+1} and $\hat{x}^j_{i+1} = x^j_i + f(t_i, x^j_i, u)\Delta t$.

$$\inf_u C(t_i, x^j_i, u) + \hat{V}(t_i, x^j_i + f(t_i, x^j_i, u)\Delta t)$$

Quadratic example

Example

Value function:

$$V(t,x) := \inf_{u\in\mathcal{U}_t} \int_t^T \left(x_s^2 + u_s^2\right) \mathrm{d}s + \frac{1}{2}x_T^2 - x_T, \qquad \mathrm{d}x_s = (x_s - u_s)\mathrm{d}s. \quad (8)$$

We cannot find value functions using a myopic argument.

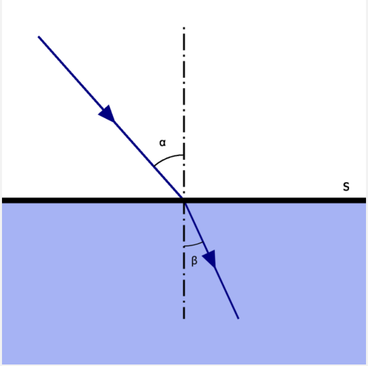
Exercise

1) In example above, write the approximate DPP from time t_i to t_{i+1}.
2) Assume that $\hat{V}(t_{i+1}, x) = a_{i+1}x^2 + b_{i+1}x + c_{i+1}$ for some known values a_{i+1}, b_{i+1}, and c_{i+1}. Use optimization of a quadratic function to find $\hat{V}(t_i, x)$. Note that you need to use $\hat{x}_{t_{i+1}} = x + (x-u)\Delta t$.
3) Does $\hat{V}(t_i, x)$ is of the form $a_i x^2 + b_i x + c_i$? What is the relation between (a_i, b_i, c_i) and $(a_{i+1}, b_{i+1}, c_{i+1})$?

Hamilton-Jacobi equation

Hamiltonian and Lagrangian

Hamilton: principle of minimum action



Lagrangian: running cost $C(t, x, u)$.

Hamilton-Jacobi equation

Recall the DPP

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s), \qquad x_s = x + \int_t^s f(x_r, u_r)\mathrm{d}r$$

Taylor expansion

$$V(s, x_s) = V(t,x) + V_t(t,x)(s-t) + V_x(t,x)(x_s - x) + R_2$$

$$\begin{aligned} V(t,x) &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\ &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\ &\qquad + V(t,x) + V_t(t,x)(s-t) + V_x(t,x) \int_t^s f(x_r, u_r)\mathrm{d}r + R_2 \end{aligned}$$

Hamilton-Jacobi equation

Taylor expansion

$$V(s, x_s) = V(t,x) + V_t(t,x)(s-t) + V_x(t,x)(x_s - x) + R_2$$

$$\begin{aligned}
V(t,x) &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\
&= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V(t,x) + V_t(t,x)(s-t) + V_x(t,x) \int_t^s f(x_r, u_r)\mathrm{d}r + R_2 \\
&= V(t,x) + V_t(t,x)(s-t) + \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V_x(t,x) \int_t^s f(x_r, u_r)\mathrm{d}r + R_2
\end{aligned}$$

Hamilton-Jacobi equation

Taylor expansion

$$
\begin{aligned}
\cancel{V(t,x)} &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\
&= \cancel{V(t,x)} + V_t(t,x)(s-t) + \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V_x(t,x) \int_t^s f(r, x_r, u_r)\mathrm{d}r + R_2
\end{aligned}
$$

Dividing both sides by $s - t$ and sending $s \to t$.

Hamilton-Jacobi equation

Taylor expansion

$$
\begin{aligned}
0 &= \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_r)\mathrm{d}r + V(s,x_s) \\
&= V_t(t,x) + \inf_{u\in\mathcal{U}_{t,s}} \lim_{s\to t} \frac{\int_t^s C(r,x_r,u_r)\mathrm{d}r}{s-t} \\
&\quad + V_x(t,x) \lim_{s\to t} \frac{\int_t^s f(r,x_r,u_r)\mathrm{d}r}{s-t} + \lim_{s\to t} \frac{R_2}{s-t}
\end{aligned}
$$

$R_2 = o(s-t)$: $\lim_{s\to t} \frac{R_2}{s-t} = 0$.

Hamilton-Jacobi equation

HJ equation

$$0 = V_t(t,x) + \inf_u \Big\{ C(t,x,u) + V_x(t,x) f(t,x,u) \Big\}$$

$$\begin{cases} 0 = V_t(t,x) + H(t,x,V_x(t,x)) \\ V(T,x) = g(x) \end{cases}$$

Hamiltonian: $H(t,x,p) = \inf_u \Big\{ C(t,x,u) + p \cdot f(t,x,u) \Big\}$

LQC

A linear-quadratic control problem

Consider the control problem in Section **??**:

$$\inf_u \left\{ \int_0^T \left(x_t^2 + u_t^2\right) \mathrm{d}t \right\}, \quad \mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t \tag{9}$$

$C(t, x, u) = x^2 + u^2$ and $f(t, x, u) = -\beta x + u$.
Write the HJ equation.
After writing the HJ, plug in $V(t, x) = a(t)x^2 + b(t)x + c(t)$the HJ and find ODEs for $a(t)$, $b(t)$, and $c(t)$. What are $a(T)$, $b(T)$, and $c(T)$?

Eikonal equation

Fastest exit

Recall the fastest exit problem.

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} \mathrm{d}t, \;\; \mathrm{d}x_t = u_t \mathrm{d}t \;\; \text{with} \;\; |u_t| \leq 1$$

Write the HJ equation. Is there any boundary condition?

Eikonal equation

Solution to Eikonal equation

Write the HJ equation and boundary condition for $D = [-1, 1] \subset \mathbb{R}$. Which one of the following functions satisfy the HJ equation? Which one is the value function?

$$v_1(x) = 1 - |x|, v_2(x) = \begin{cases} \frac{1}{2} - |x - \frac{1}{2}| & 0 \le x \le 1 \\ \frac{1}{2} - |x + \frac{1}{2}| & -1 \le x < 0 \end{cases}$$

Individual project

Due end October

In your area of study, find an optimal control problem. Then, write the cost functions and the control variable and determine a set of admissible controls.