Methods Of Optimal Control

Arash Fahim

FSU Fall 2025

September 1, 2025

Structure of the course

- Short lecture (10-15 minutes)
- Team work (50 minutes)
    - Problem solving
    - Programming (Python)
    - Managerial skills
    - Reporting skills
- Delivery (5 minutes)

Structure of the course

Python

Each student is expected to bring a computer to the classroom with a *Python 3.12*, *IPython*, and *Jupyter Notebook* installed.

- `https://www.python.org/downloads`
- `https://www.anaconda.com/docs/main`
- Virtual environment:

  `https://docs.python.org/3/library/venv.html`
  `https://www.anaconda.com/docs/tools/working-with-conda/environments`

GitHub

Each student is required to have a *GitHub* account.

Rules of teamwork

- Be respectful.
- Allow all team members to engage and express their opinion. No interruption.
- Arrange chair to make sure everyone is involved.
- No one knows everything. Make sure that everyone learns what you know.
- Break down the tasks between group members based on ones abilities.
- Ask for feedback from me frequently.

Composition of teams

Each team requires one or two members to take the following roles:

- Problem solving
- Programming (Python)
- Managerial skills
- Reporting skills

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) dt \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

$$x_t^* = \frac{\alpha_t}{2} = \operatorname*{argmin}_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2 - \alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x : [0, T] \to\in \mathbb{R}$ such that for some function $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Can we find u_t such that $x_t=\frac{\alpha_t}{2}$? (For simplicity, take $\beta=0$.)

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2 - \alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x : [0, T] \to\in \mathbb{R}$ such that for some function $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)
Check it for $\alpha_t = \mathbb{1}_{\{0 \le t \le \frac{T}{2}\}}$.

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T\Big(x_t^2-\alpha_t x_t\Big)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Can we find u_t such that $x_t=\frac{\alpha_t}{2}$? (For simplicity, take $\beta=0$.)
Check it for $\alpha_t=\mathbb{1}_{\{0\le t\le\frac{T}{2}\}}$.
For $\alpha_t=\mathbb{1}_{\{0\le t\le\frac{T}{2}\}}$, what is the value of the infimum? Is it

$$\inf\left\{\int_0^T\Big(x_t^2-\alpha_t x_t\Big)\mathrm{d}t\right\}=-\frac{1}{4}\int_0^T\alpha_t^2\mathrm{d}t=-\frac{T}{8}?$$

Optimization versus control

A control problem without a myopic solution

$$\inf \int_0^T \left(x_t^2 - \alpha_t x_t + u_t^2\right) \mathrm{d}t \tag{1}$$

Infimum is over all functions $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Trade-off:

- Trying to send $x_t \to \frac{\alpha_t}{2}$ may cause $\int_0^T u_t^2 dt$ to grow.
- Trying to keep cost $\int_0^T u_t^2 dt$ near zero, does not bring x_t close to $\frac{\alpha_t}{2}$.

What is the sweet spot for u_t?

A generic control problem

Definition

$$\inf_{u \in \mathcal{U}} \int_0^T C(t, x_t, u_t)\mathrm{d}t + g(x_T), \quad \mathrm{d}x_t = f(x_t, u_t)\mathrm{d}t \tag{2}$$

- $C : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$: *running cost*
- $g : \mathbb{R}^d \to \mathbb{R}$: *terminal cost*
- $\mathcal{U}$: *an admissible* set of functions $u : [0, T] \to \mathbb{R}^n$, *control variable*.

A generic control problem

Admissible controls

$\mathcal{U}$ is chosen to fit the proper application and/or to make the control problem wellposed.

Admissibility for wellposedness

$$\inf_{u\in\mathcal{U}} \int_0^T (x_t - u_t^2)\mathrm{d}t = -\infty, \qquad \mathrm{d}x_t = (x_t - u_t)\mathrm{d}t \tag{3}$$

$\mathcal{U}$ to be the set of all functions $u : [0, T] \to \mathbb{R}$

If we restrict $\mathcal{U}$ to the set of functions $u : [0, T] \to [-1, \infty)$ (some lower bound on the value), then

$$\inf_{u\in\mathcal{U}} \int_0^T (x_t - u_t^2)\mathrm{d}t > -\infty, \qquad \mathrm{d}x_t = (x_t - u_t)\mathrm{d}t. \tag{4}$$

Infinite horizon

Infinite horizon

An infinite horizon control problem is accommodated by setting $T = \infty$. For example,

$$\inf_{u \in \mathcal{U}} \int_0^\infty e^{-t}(x_t^2 + u_t^2)\mathrm{d}t, \;\; C(t,x,u) = e^{-t}(x^2 + u^2) \tag{5}$$

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t,x,u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \;:\; x_t \notin D\} \tag{6}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Infinite horizon

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \ : \ x_t \notin D\} \tag{7}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Solution

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} \mathrm{d}t, \ \mathrm{d}x_t = u_t \mathrm{d}t \ \text{ with } \ |u_t| \leq 1$$

An optimal control is described by existing D as fast as possible, $|u| = 1$, and stop as soon as we exit, $|u| = 0$.

Dynamic programming principle (DPP)

Value function

Fix $x_t = x$.

$$V(t,x) := \inf_{u \in \mathcal{U}_t} \int_t^T C(s, x_s, u_s) \mathrm{d}s + g(x_T), \qquad \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$$

$\mathcal{U}_t$: the set of admissible controls restricted to $[t, T]$.

Dynamic programming principle (DPP)

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \qquad \mathrm{d}x_r = f(x_r, u_r)\mathrm{d}r$$

$\mathcal{U}_{t,s}$: the set of admissible controls restricted to $[t, s]$.

DPP

Balance of cost in DPP

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s), \qquad \mathrm{d}x_s = x + \int_t^s f(x_r, u_r)\mathrm{d}r$$

Individual project

Due end October

In your area of study, find an optimal control problem. Then, write the cost functions and the control variable and determine a set of admissible controls.