Control problem

$$\inf_u J(u), \quad J(u) := \int_0^T C(s, x_s, u_s)\mathrm{d}s + g(x_T),\ \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$$

Value function

$$V(t,x) := \inf_u \int_t^T C(s, x_s, u_s)\mathrm{d}s + g(x_T),\ \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$$

HJ satisfied by value function

$$\begin{cases} 0 = V_t(t,x) + H\big(t, x, V_x(t,x)\big) \\ V(T,x) = g(x) \end{cases} \qquad H(t,x,p) := \inf_u \big\{C(t,x,u) + pf(t,x,u)\big\}$$

PDE solver

Find $V(t,x)$ and $V_x(t,x)$ numerically or analytically

Optimal control using $V_x(t,x)$

$$u^*(t,x) \in \underset{u}{\operatorname{argmin}} \Big\{C(t,x,u) + V_x(t,x) \cdot f\big(t,x,u\big)\Big\}$$

Optimal trajectory of state variable

$$\mathrm{d}x_s^* = f\big(x_s^*, u^*(t, x_s^*)\big)\mathrm{d}s$$