Methods Of Optimal Control

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Structure of the course

- Short lecture (10-15 minutes)
- Team work (50 minutes)
    - Problem solving
    - Programming (Python)
    - Managerial skills
    - Reporting skills
- Delivery (5 minutes)

Structure of the course

Python

Each student is expected to bring a computer to the classroom with a *Python 3.12*, *IPython*, and *Jupyter Notebook* installed.

- https://www.python.org/downloads
- https://www.anaconda.com/docs/main
- Virtual environment:

  https://docs.python.org/3/library/venv.html
  https://www.anaconda.com/docs/tools/working-with-conda/environments

GitHub

Each student is required to have a *GitHub* account.

Rules of teamwork

- Be respectful.
- Allow all team members to engage and express their opinion. No interruption.
- Arrange chair to make sure everyone is involved.
- No one knows everything. Make sure that everyone learns what you know.
- Break down the tasks between group members based on ones abilities.
- Ask for feedback from me frequently.

Composition of teams

Each team requires one or two members to take the following roles:

- Problem solving
- Programming (Python)
- Managerial skills
- Reporting skills

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

Optimization versus control

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$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

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where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

$$x_t^* = \frac{\alpha_t}{2} = \operatorname*{argmin}_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T \left(x_t^2 - \alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x : [0, T] \to\in \mathbb{R}$ such that for some function $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Optimization versus control

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Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)

Optimization versus control

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Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)
Check it for $\alpha_t = \mathbb{1}_{\{0 \leq t \leq \frac{T}{2}\}}$.

Optimization versus control

Dynamic x_t

$$\inf\left\{\int_0^T\left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}$$

Infimum is over all functions $x:[0,T]\to\in\mathbb{R}$ such that for some function $u:[0,T]\to\mathbb{R}$

$$\mathrm{d}x_t=(-\beta x_t+u_t)\mathrm{d}t,\quad x_0=x$$

Can we find u_t such that $x_t=\frac{\alpha_t}{2}$? (For simplicity, take $\beta=0$.)
Check it for $\alpha_t=\mathbb{1}_{\{0\le t\le\frac{T}{2}\}}$.
For $\alpha_t=\mathbb{1}_{\{0\le t\le\frac{T}{2}\}}$, what is the value of the infimum? Is it

$$\inf\left\{\int_0^T\left(x_t^2-\alpha_t x_t\right)\mathrm{d}t\right\}=-\frac{1}{4}\int_0^T\alpha_t^2\mathrm{d}t=-\frac{T}{8}?$$

Optimization versus control

A control problem without a myopic solution

$$\inf \int_0^T \left(x_t^2 - \alpha_t x_t + u_t^2\right) \mathrm{d}t \qquad (1)$$

Infimum is over all functions $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Trade-off:

- Trying to send $x_t \to \frac{\alpha_t}{2}$ may cause $\int_0^T u_t^2 dt$ to grow.
- Trying to keep cost $\int_0^T u_t^2 dt$ near zero, does not bring x_t close to $\frac{\alpha_t}{2}$.

What is the sweet spot for u_t?

A generic control problem

Definition

$$\inf_{u\in\mathcal{U}}\int_0^T C(t,x_t,u_t)\mathrm{d}t+g(x_T),\quad \mathrm{d}x_t=f(x_t,u_t)\mathrm{d}t \tag{2}$$

- $C:\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^n\to\mathbb{R}$: *running cost*
- $g:\mathbb{R}^d\to\mathbb{R}$: *terminal cost*
- $\mathcal{U}$: *an admissible* set of functions $u:[0,T]\to\mathbb{R}^n$, *control variable.*

A generic control problem

Admissible controls

$\mathcal{U}$ is chosen to fit the proper application and/or to make the control problem wellposed.

Admissibility for wellposedness

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2)\mathrm{d}t = -\infty, \qquad \mathrm{d}x_t = (x_t - u_t)\mathrm{d}t \tag{3}$$

$\mathcal{U}$ to be the set of all functions $u : [0, T] \to \mathbb{R}$
If we restrict $\mathcal{U}$ to the set of functions $u : [0, T] \to [-1, \infty)$ (some lower bound on the value), then

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2)\mathrm{d}t > -\infty, \qquad \mathrm{d}x_t = (x_t - u_t)\mathrm{d}t. \tag{4}$$

Infinite horizon

Infinite horizon

An infinite horizon control problem is accommodated by setting $T = \infty$. For example,

$$\inf_{u \in \mathcal{U}} \int_0^\infty e^{-t}(x_t^2 + u_t^2)\mathrm{d}t, \;\; C(t,x,u) = e^{-t}(x^2+u^2) \tag{5}$$

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t,x,u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \; : \; x_t \notin D\} \tag{6}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Infinite horizon

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \ : \ x_t \notin D\} \tag{7}$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Solution

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} \mathrm{d}t, \ \mathrm{d}x_t = u_t \mathrm{d}t \ \text{ with } \ |u_t| \leq 1$$

An optimal control is described by existing D as fast as possible, $|u| = 1$, and stop as soon as we exit, $|u| = 0$.

Dynamic programming principle (DPP)

Value function

Fix $x_t = x$.

$$V(t,x) := \inf_{u \in \mathcal{U}_t} \int_t^T C(s, x_s, u_s) \mathrm{d}s + g(x_T), \qquad \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$$

$\mathcal{U}_t$: the set of admissible controls restricted to $[t, T]$.

Dynamic programming principle (DPP)

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \qquad \mathrm{d}x_r = f(x_r, u_r)\mathrm{d}r$$

$\mathcal{U}_{t,s}$: the set of admissible controls restricted to $[t, s]$.

DPP

Balance of cost in DPP

$$V(t,x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s), \qquad x_s = x + \int_t^s f(x_r, u_r)\mathrm{d}r$$

Proof of DPP

$$
\begin{aligned}
V(t,x) &= \inf_{u\in\mathcal{U}_t} \int_t^T C(r,x_r,u_r)\mathrm{d}r + g(x_T) \\
&= \inf_{u_1\in\mathcal{U}_{t,s}} \inf_{u_2\in\mathcal{U}_s} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T) \\
&= \inf_{u_1\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + \inf_{u_2\in\mathcal{U}_s} \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T)
\end{aligned}
$$

Note that $\inf_{u_2\in\mathcal{U}_s} \int_s^T C(r,x_r,u_2(r))\mathrm{d}r + g(x_T) = V(s,x_s)$. Therefore,

$$
V(t,x) = \inf_{u_1\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_1(r))\mathrm{d}r + V(s,x_s)
$$

Numerical DPP

Discretization

$\Delta t = \frac{T}{N}$

$$
\begin{cases}
V(t_N, x) = g(x) \\
V(t_i, x) = \inf_{u \in \mathcal{U}_{t_i, t_{i+1}}} \int_{t_i}^{t_{i+1}} C(r, x_r, u_r) \mathrm{d}r + V(t_{i+1}, x_{t_{i+1}}), \\
\qquad\qquad\qquad\qquad\qquad x_{t_{i+1}} = x + \int_{t_i}^{t_{i+1}} f(s, x_s, u_s) \mathrm{d}s
\end{cases}
$$

Approximation

$$
\begin{cases}
\hat{V}(t_N, x) = g(x) \\
\hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u)\Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \;\; x_{t_{i+1}} = x + f(t_i, x, u)\Delta t
\end{cases}
$$

Numerical DPP

Approximation

$$\begin{cases} \hat{V}(t_N, x) = g(x) \\ \hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u)\Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \; x_{t_{i+1}} = x + f(t_i, x, u)\Delta t \end{cases}$$

Simplification of one-step approximate DPP

The approximation is not over the control $u : [t_i, t_{i+1}] \to \mathbb{R}^m$, but over values $u \in \mathbb{R}^m$. The optimal value $\hat{u}^*$ is a constant approximately optimal control over $[t_i, t_{i+1}]$.

Algorithm

Algorithm 1: Numerical DPP

Parameter : T, N, $f(t,x,u)$, $C(t,x,u)$, and $g(x)$;

$\Delta t = \frac{T}{N}$

Data: $\hat{V}(t_N, x) = g(x)$;

x_i^j for $j = 1, ..., J$ and $i = 0, ..., N-1$;

(x_i^j means the jth discrete point at time t_i.)

for $i \leftarrow N-1$ *to* 0 **do**

- $\hat{x}_{i+1}^j = x_i^j + f(t_i, x_i^j, u)\Delta t$;
- $\tilde{V}(t_i, x_i^j) \leftarrow \inf_u C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{t_{i+1}}^j)$;
- $\hat{V}(t_i, x)$ obtained from interpolation on $\tilde{V}(t_i, x_i^j)$ for $j = 1, ..., J$;
- $\hat{u}^*(t_i, x_i^j) \in \underset{u}{\operatorname{argmin}}\, C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{i+1}^j)$;

return $\hat{V}(t_i, \cdot)$ *and* $\hat{u}^*(t_i, \cdot)$ *for* $i = 0, ..., N-1$.

DPP algorithm

Exercise

Why interpolation is required in Algorithm 1?
Can we perform the algorithm by only knowing $\hat{V}(t_{i+1}, x^j_{i+1})$ for all $j = 1, ..., J$?
Note the difference between $\hat{V}(t_{i+1}, x^j_{i+1})$ and $\hat{V}(t_{i+1}, \hat{x}^j_{i+1})$ and the difference between x^j_{i+1} and $\hat{x}^j_{i+1} = x^j_i + f(t_i, x^j_i, u)\Delta t$.

$$\inf_u C(t_i, x^j_i, u) + \hat{V}(t_i, x^j_i + f(t_i, x^j_i, u)\Delta t)$$

Quadratic example

Example

Value function:

$$V(t,x) := \inf_{u \in \mathcal{U}_t} \int_t^T \left(x_s^2 + u_s^2\right) \mathrm{d}s + \frac{1}{2}x_T^2 - x_T, \qquad \mathrm{d}x_s = (x_s - u_s)\mathrm{d}s. \quad (8)$$

We cannot find value functions using a myopic argument.

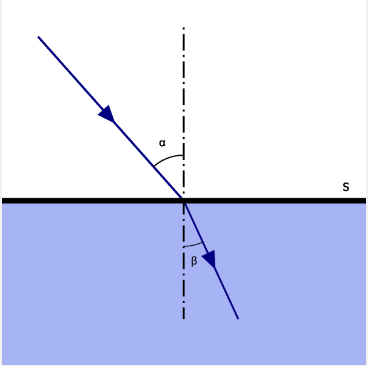
Exercise

1) In example above, write the approximate DPP from time t_i to t_{i+1}.
2) Assume that $\hat{V}(t_{i+1}, x) = a_{i+1}x^2 + b_{i+1}x + c_{i+1}$ for some known values a_{i+1}, b_{i+1}, and c_{i+1}. Use optimization of a quadratic function to find $\hat{V}(t_i, x)$. Note that you need to use $\hat{x}_{t_{i+1}} = x + (x - u)\Delta t$.
3) Does $\hat{V}(t_i, x)$ is of the form $a_i x^2 + b_i x + c_i$? What is the relation between (a_i, b_i, c_i) and $(a_{i+1}, b_{i+1}, c_{i+1})$?

Hamilton-Jacobi equation

Hamiltonian and Lagrangian

Hamilton: principle of minimum action



Hamilton-Jacobi equation

Recall the DPP

$$V(t,x) = \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s), \qquad x_s = x + \int_t^s f(x_r, u_r)\mathrm{d}r$$

Taylor expansion

$$V(s, x_s) = V(t,x) + V_t(t,x)(s-t) + V_x(t,x)(x_s - x) + R_2$$

$$\begin{aligned} V(t,x) &= \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\ &= \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\ &\qquad + V(t,x) + V_t(t,x)(s-t) + V_x(t,x)\int_t^s f(x_r, u_r)\mathrm{d}r + R_2 \end{aligned}$$

Hamilton-Jacobi equation

Taylor expansion

$$V(s, x_s) = V(t,x) + V_t(t,x)(s-t) + V_x(t,x)(x_s - x) + R_2$$

$$\begin{aligned}
V(t,x) &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\
&= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V(t,x) + V_t(t,x)(s-t) + V_x(t,x)\int_t^s f(x_r, u_r)\mathrm{d}r + R_2 \\
&= V(t,x) + V_t(t,x)(s-t) + \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V_x(t,x)\int_t^s f(x_r, u_r)\mathrm{d}r + R_2
\end{aligned}$$

Hamilton-Jacobi equation

Taylor expansion

$$
\begin{aligned}
\cancel{V(t,x)} &= \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_r)\mathrm{d}r + V(s,x_s)\\
&= \cancel{V(t,x)} + V_t(t,x)(s-t) + \inf_{u\in\mathcal{U}_{t,s}} \int_t^s C(r,x_r,u_r)\mathrm{d}r\\
&\quad + V_x(t,x)\int_t^s f(r,x_r,u_r)\mathrm{d}r + R_2
\end{aligned}
$$

Dividing both sides by $s-t$ and sending $s \to t$.

Hamilton-Jacobi equation

Taylor expansion

$$
\begin{aligned}
0 &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\
&= V_t(t,x) + \inf_{u \in \mathcal{U}_{t,s}} \lim_{s \to t} \frac{\int_t^s C(r, x_r, u_r)\mathrm{d}r}{s-t} \\
&\quad + V_x(t,x) \lim_{s \to t} \frac{\int_t^s f(r, x_r, u_r)\mathrm{d}r}{s-t} + \lim_{s \to t} \frac{R_2}{s-t}
\end{aligned}
$$

$R_2 = o(s-t)$: $\lim_{s \to t} \frac{R_2}{s-t} = 0$.

Hamilton-Jacobi equation

HJ equation

$$0 = V_t(t,x) + \inf_u \Big\{ C(t,x,u) + V_x(t,x) f(t,x,u) \Big\}$$

$$\begin{cases} 0 = V_t(t,x) + H(t,x,V_x(t,x)) \\ V(T,x) = g(x) \end{cases}$$

Hamiltonian: $H(t,x,p) = \inf_u \Big\{ C(t,x,u) + p \cdot f(t,x,u) \Big\}$

LQC

A linear-quadratic control problem

Consider the control problem:

$$\inf_u \left\{ \int_0^T \left(x_t^2 + u_t^2 \right) \mathrm{d}t \right\}, \quad \mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t \tag{9}$$

$C(t, x, u) = x^2 + u^2$ and $f(t, x, u) = -\beta x + u$.
Write the HJ equation.
After writing the HJ, plug in $V(t, x) = a(t)x^2 + b(t)x + c(t)$the HJ and find ODEs for $a(t)$, $b(t)$, and $c(t)$. What are $a(T)$, $b(T)$, and $c(T)$?

Eikonal equation

Fastest exit

Recall the fastest exit problem.

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} dt, \; dx_t = u_t dt \; \text{ with } \; |u_t| \leq 1$$

Write the definition of value function for initial state $x_0 = x \in D$. Write the HJ equation. Is there any boundary condition?

Solution to Eikonal equation

Write the HJ equation and boundary condition for the special case where $D = [-1, 1] \subset \mathbb{R}$. Which one of the following functions satisfy the HJ equation? Which one matches the value function?

$$v_1(x) = 1 - |x|, v_2(x) = \begin{cases} \frac{1}{2} - |x - \frac{1}{2}| & 0 \leq x \leq 1 \\ \frac{1}{2} - |x + \frac{1}{2}| & -1 \leq x < 0 \end{cases}$$

SIR model in epidemiology

ODEs

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta I_t S_t - \gamma I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma I_t \mathrm{d}t \end{cases}$$

Controlled state variables

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

SIR model in epidemiology

Controlled state variables

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

$$\inf_{\beta_t,\gamma_t} \int_0^T ((b_1 - \beta_t)^2 + \gamma_t^2)\mathrm{d}t + I_T^2$$

Write the HJ equation in variables $x = (S, I, R)$.

SIR model in epidemiology

Controlled state variables

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

$$\inf_{\beta_t, \gamma_t} \int_0^T ((b_1 - \beta_t)^2 + \gamma_t^2)\mathrm{d}t + I_T^2$$

Notice that $\mathrm{d}(S_t + I_t + R_t) = 0$. This should allow us to reduce the number of state variables $x_t = (S_t, I_t, R_t)$ to two, in place of three. Assume that the population size is given by N, $S_t + I_t + R_t = N$. Remove the variable R_t and write the HJ equation in terms of (S_t, I_t). Write the HJ equation in (S, I).

Consumption

Savings account

$$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

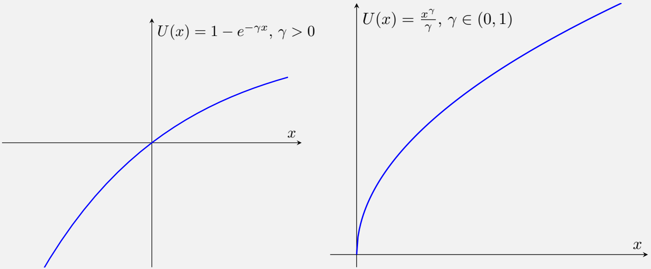
$$\sup_{c_t \geq 0} \int_0^T U(c_t)\mathrm{d}t + U(x_T)$$

U is a given function called utility function.

Consumption

Utility function

Utility function is a concave function that represent our enjoyment from consumption or wealth. The concavity signifies the fact that if our consumption or wealth level is low, increasing one more unit grants more joy compared to when our consumption or wealth level is higher and we obtain one more unit. Example of a utility function:



Consumption

Savings account

$$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T U(c_t)\mathrm{d}t + U(x_T)$$

HJ equation is given by:

$$\begin{cases} V_t + H(x, V_x) = 0 \\ V(T, x) = U(x) \end{cases}$$

with $H(x,p) = \sup_{c \geq 0}\{U(x) - cp\} + rxp$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

Show that the supremum in Hamiltonian is attained at $c^*(p) = p^{\frac{1}{\gamma-1}}$ and

$$H(x,p) = \sup_{c\geq 0}\left\{\frac{c^\gamma}{\gamma} - cp\right\} + rxp = \frac{1-\gamma}{\gamma}p^{\frac{1}{\gamma-1}} + rxp$$

Verify that $V(t,x) = f(t)\frac{c^\gamma}{\gamma}$ solves the HJ equation

$$\begin{cases} 0 = V_t + V_x^{\frac{\gamma}{\gamma-1}} + rxV_x \\ V(T,x) = \frac{c^\gamma}{\gamma} \end{cases}$$

and $f(t)$ satisfies

$$\begin{cases} 0 = f' + f^{\frac{\gamma}{\gamma-1}} + \frac{r}{\gamma}f \\ f(T) = 1 \end{cases}$$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

We solve the ODE $0 = f' + f^{\frac{\gamma}{\gamma-1}} + \frac{r}{\gamma} f$ by the change of variable $f = y^{1-\gamma}$.
y satisfies $y' + \frac{r}{\gamma(1-\gamma)} y + \frac{1}{1-\gamma} = 0$ and $y(t) = \left(1 + \frac{\gamma}{r}\right) e^{\frac{r}{\gamma(1-\gamma)}(T-t)} - \frac{\gamma}{r}$.

$$f(t) = \left(\left(1 + \frac{\gamma}{r}\right) e^{\frac{r}{\gamma(1-\gamma)}(T-t)} - \frac{\gamma}{r}\right)^{1-\gamma}$$

Since $1-\gamma < 0$, $f(t) > 1$ unless $t = T$. The optimal consumption $c^*(t,x) = \frac{1}{f^{\frac{1}{1-\gamma}}(t)} x$ or

$$\begin{cases} c_t^* = \frac{1}{f^{\frac{1}{1-\gamma}}(t)} x_t \\ \mathrm{d}x_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)}\right) x_t \mathrm{d}t \end{cases}$$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

$$\begin{cases} c_t^* = \frac{1}{f^{\frac{1}{1-\gamma}}(t)} x_t \\ \mathrm{d}x_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)}\right) x_t \mathrm{d}t \end{cases}$$

We learn from the above solution that $\frac{1}{f^{\frac{1}{1-\gamma}}(t)} < 1$, therefore the optimal consumption rate percentage of wealth and the account balance never goes negative.

Coding exercise

Plot the solution of $\mathrm{d}x_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)}\right) x_t \mathrm{d}t$ for different values for $r > 0$, $\gamma \in (0, 1)$, T, and different initial balance $x > 0$.

Consumption with no terminal wealth

Savings account

$$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T U(c_t)\mathrm{d}t \cancel{+ U(x_T)}$$

Can we consume infinitely by borrowing? . If we restrict consumption to case where borrowing is not allowed, $x_t \geq 0$ for all t, does this solve the issue? (Relation to admissible control.)
Write the HJ equation with boundary condition that reflects $x_t \geq 0$.

Consumption with decay

Savings account

$$dx_t = (rx_t - c_t)dt$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T e^{-kt} U(c_t)dt + e^{-kT} U(x_T)$$

Write the HJ equation. Hint: $C(t, x, u) = e^{-kt} U(u)$

Dynamic programming equation with decay

Control problem with decay

$$\inf_u \int_0^T e^{-kt}\bar{C}(x_t, u_t)\mathrm{d}t + e^{-kT}g(x_T), \qquad \mathrm{d}x_t = f(x_t, u_t)\mathrm{d}t$$

Value function

$$V(t,x) := \inf_u \int_t^T e^{-k(s-t)}\bar{C}(x_s, u_s)\mathrm{d}s + e^{-k(T-t)}g(x_T)$$

DPP

$$V(t,x) = \inf_u \int_t^s e^{-k(r-t)}\bar{C}(x_r, u_r)\mathrm{d}r + e^{-k(s-t)}V(s, x_s)$$

Dynamic programming equation with decay

Value function

$$V(t,x) := \inf_u \int_t^T e^{-k(s-t)} \bar{C}(x_s, u_s)\mathrm{d}s + e^{-k(T-t)} g(x_T)$$

DPP

$$V(t,x) = \inf_u \int_t^s e^{-k(r-t)} \bar{C}(x_r, u_r)\mathrm{d}r + e^{-k(s-t)} V(s, x_s)$$

HJB with decay

Write the first three terms of the Taylor polynomial for $e^{-k(s-t)} V(s, x_s)$ about point (t,x).

Dynamic programming equation with decay

HJB with decay

$$
\begin{aligned}
e^{-k(s-t)}V(s,x_s) =& V(t,x) + \Big(\big(V_t(t,x) - kV(t,x)\big)(s-t) \\
&+ V_x(t,x)\int_t^s f(x_r,u_r)\mathrm{d}r\Big) + O((s-t)^2)
\end{aligned}
$$

$$
\begin{aligned}
\cancel{V(t,x)} = \inf_u \int_t^s & e^{-k(r-t)}\bar{C}(x_r,u_r)\mathrm{d}r \\
&+ \cancel{V(t,x)} + \Big(\big(V_t(t,x) - kV(t,x)\big)(s-t) \\
&+ V_x(t,x)\int_t^s f(x_r,u_r)\mathrm{d}r\Big) + O((s-t)^2)
\end{aligned}
$$

Dynamic programming equation with decay

HJB with decay

Divide by $s - t$ and $s \to t$:

$$0 = \inf_u \bar{C}(x, u) + V_t(t, x) - kV(t, x) + V_x(t, x) \cdot f(x, u)$$

$$0 = V_t(t, x) - kV(t, x) + \inf_u \{ \bar{C}(x, u) + V_x(t, x) \cdot f(x, u) \}$$

Hamiltonian:

$$H(x, p) := \inf_u \{ \bar{C}(x, u) + p \cdot f(x, u) \}$$

Purpose of different HJ for decay

Infinite horizon

Time-homogeneity

$$V(x) = \inf_u \int_t^{\infty} e^{-k(s-t)} \bar{C}(x_s, u_s) \mathrm{d}s = \inf_u \int_0^{\infty} e^{-ks} \bar{C}(x_s, u_s) \mathrm{d}s$$

HJ equation

$$0 = -kV(x) + \inf_u \{ \bar{C}(x, u) + V_x(x) f(x, u) \}$$

Consumption with decay

Savings account

$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$. $c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^\infty e^{-kt} U(c_t)\mathrm{d}t$$

where consumption strategy must satisfy $x_t \geq 0$ for all $t \geq 0$. Write the HJ equation with proper boundary condition. Solve the HJ equation for $U(c) = \frac{c^\gamma}{\gamma}$ for $\gamma \in (0,1)$. Hint: find a constant f such that $V(x) = f\frac{x^\gamma}{\gamma}$ solves the HJ equation. Is the optimal consumption a constant multiple of x?

Coding exercise

Plot optimal wealth $\mathrm{d}x_t = \Big(r - c^*\Big)x_t\mathrm{d}t$ for different values for $r > 0$, $\gamma \in (0,1)$, and different initial balance $x > 0$.

Summary of HJ method

Control problem

$\inf_u J(u), \quad J(u) := \int_0^T C(s, x_s, u_s)\mathrm{d}s + g(x_T),\ \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$

↓

Value function

$V(t,x) := \inf_u \int_t^T C(s, x_s, u_s)\mathrm{d}s + g(x_T),\ \mathrm{d}x_s = f(x_s, u_s)\mathrm{d}s$

↓

HJ satisfied by value function

$\begin{cases} 0 = V_t(t,x) + H\big(t, x, V_x(t,x)\big) \\ V(T,x) = g(x) \end{cases} \quad H(t,x,p) := \inf_u \{C(t,x,u) + pf(t,x,u)\}$

↓

PDE solver

Find $V(t,x)$ and $V_x(t,x)$ numerically or analytically

↓

Optimal control using $V_x(t,x)$

$u^*(t,x) \in \underset{u}{\operatorname{argmin}} \Big\{ C(t,x,u) + V_x(t,x) \cdot f\big(t,x,u\big) \Big\}$

↓

Optimal trajectory of state variable

$\mathrm{d}x_s^* = f\big(x_s^*, u^*(t, x_s^*)\big)\mathrm{d}s$

Lagrange multiplier

Constrained optimization

$$\inf_x f(x) \quad \text{subject to} \quad g(x) = 0$$

Lagrangian

$$L(x, \lambda) := f(x) - \lambda \cdot g(x)$$

Saddle point problem

$$\sup_\lambda \inf_x L(x, \lambda) = \sup_\lambda H(\lambda)$$

with $H(\lambda) := \inf_x L(x, \lambda)$ called Hamiltonian.

Lagrange multiplier

Saddle point problem

$$\sup_{\lambda} \inf_{x} L(x, \lambda)$$

Strong duality

If $\sup_\lambda \inf_x L(x, \lambda) = \inf_x \sup_\lambda L(x, \lambda)$ holds, we call it strong duality.

$$\inf_{x} \sup_{\lambda} L(x, \lambda) = \inf_{x} \sup_{\lambda} f(x) - \lambda \cdot g(x) = \inf_{x} f(x) - \inf_{\lambda} \lambda \cdot g(x)$$

If at a point x, $g(x) \neq 0$, then $\inf_\lambda \lambda \cdot g(x) = -\infty$. Therefore, x is not a saddle point.

Geometric interpretation of Lagrange multiplier

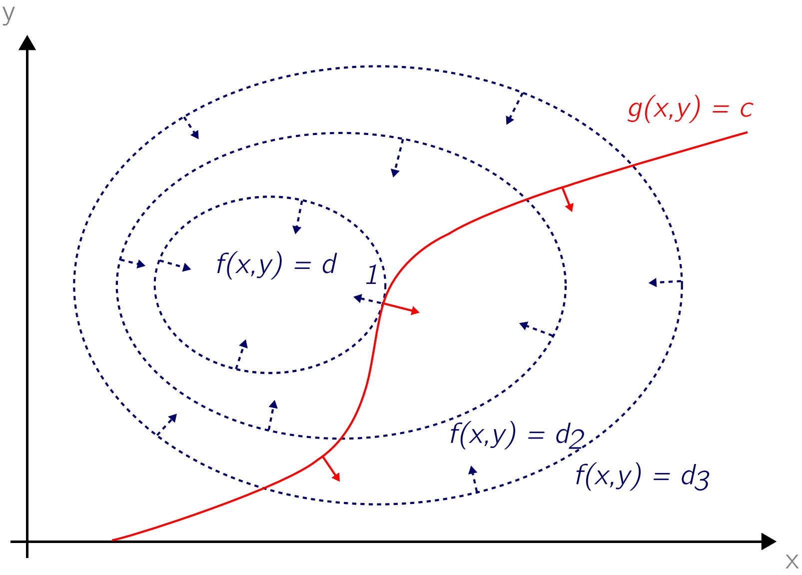


Figure: At the infimum point, ∇f and ∇g are parallel, $\nabla f = \lambda \nabla g$.

Karush-Kuhn-Tucker (KKT) condition

KKT

Assume the differentiability of f and g. If x^* solves $\inf_x f(x)$ subject to $g(x) = 0$ and λ^* solves $\sup_\lambda H(\lambda)$ such that

$$\begin{cases} \nabla f(x^*) - \lambda^* \cdot \nabla g(x^*) = 0 \\ g(x^*) = 0 \end{cases} \tag{10}$$

then, strong duality holds and (x^*, λ^*) is a saddle point for $\sup_\lambda \inf_x L(x, \lambda)$.
Conversely, if strong duality holds, then any saddle point (x^*, λ^*) satisfies (10). In particular, x^* solve

$$\inf_x f(x) \quad \text{subject to} \quad g(x) = 0$$

Constrained optimal control problem

Simple example

$$\inf_{u} \int_0^T \Big(x_t^2 - \alpha_t x_t\Big) \mathrm{d}t$$

where $\mathrm{d}x_t = u_t \mathrm{d}t$ subject to $\int_0^T x_t \mathrm{d}t = 0$. Lagrangian

$$L(u,\lambda) := \int_0^T \Big(x_t^2 - \alpha_t x_t\Big) \mathrm{d}t - \lambda \int_0^T x_t \mathrm{d}t = \int_0^T \Big(x_t^2 - (\alpha_t + \lambda) x_t\Big) \mathrm{d}t$$

Myopic solution with KKT:

$$x_t^* = (\alpha_t + \lambda^*)/2 \quad \text{and} \quad \int_0^T x_s^* \mathrm{d}s = 0$$

Therefore,

$$\lambda^* = -\frac{1}{T} \int_0^T \alpha_s \mathrm{d}s \quad \text{and} \quad x_t^* = \frac{\alpha_t - \frac{1}{T}\int_0^T \alpha_s \mathrm{d}s}{2}$$

Pontryagin principle: real application of Lagrange multiplier

What is the constraint?

$$\inf_{u} \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t + x_T^2 \right\} \tag{11}$$

Constraint: $\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t$

Lagrangian

$$L(u, \lambda) = \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t + x_T^2 - \int_0^T \lambda_t \cdot (\mathrm{d}x_t - (-\beta x_t + u_t)\mathrm{d}t)$$

Integration by parts

Apply integration by parts on $\int_0^T \lambda_t \cdot \mathrm{d}x_t$ and substitute it into Lagrangian.

Pontryagin principle: real application of Lagrange multiplier

Simplify Lagrangian

Integration by parts

$$\int_0^T \lambda_t \cdot \mathrm{d}x_t = \lambda_T x_T - \lambda_0 x_0 - \int_0^T x_t \cdot \mathrm{d}\lambda_t$$

$$\begin{aligned}
L(u,\lambda) = &\int_0^T \Big(x_t^2 - \alpha_t x_t\Big)\mathrm{d}t + x_T^2 \\
&- \Big(\lambda_T x_T - \lambda_0 x_0 - \int_0^T x_t \cdot (\mathrm{d}\lambda_t - (-\beta x_t + u_t)\mathrm{d}t)\Big) \\
= &\int_0^T \Big(\big(x_t^2 - \alpha_t x_t - x_t \cdot (-\beta x_t + u_t)\big)\mathrm{d}t + x_t \cdot \mathrm{d}\lambda_t\Big) \\
&+ x_T^2 - \lambda_T x_T + \lambda_0 x_0
\end{aligned}$$

Pontryagin principle: real application of Lagrange multiplier

Hamiltonian (new)

$$L(u,\lambda) = \int_0^T \Big(\big(x_t^2 - \alpha_t x_t - x_t \cdot (-\beta x_t + u_t) \big) \mathrm{d}t + x_t \cdot \mathrm{d}\lambda_t \Big) \\ + x_T^2 - \lambda_T x_T + \lambda_0 x_0$$

$H(t,x,\lambda,u) = x^2 - \alpha_t x - x \cdot \big(-\beta x + u \big)$

Myopically perform KKT

Minimizing $x_T^2 - \lambda_T x_T$ wrt x_T: $\lambda_T^* = 2x_T^*$
Minimizing Hamiltonian wrt u: $H(t, x_t^*, \lambda_t^*, u_t^*) = \inf_u H(t, x_t^*, \lambda_t^*, u)$
Minimizing integrand wrt x_t:

$$\mathrm{d}\lambda_t^* + H_x(t, x_t^*, \lambda_t^*, u_t^*) = 0$$

Putting all together

Pontriagyn maximum principle

If we can find functions x_t^*, λ_t^*, and u_t^* such that

$$\begin{cases} d\lambda_t^* + \partial_x H(t, x_t^*, \lambda_t^*, u_t^*)dt = d\lambda_t^* + (-\lambda_t^*\beta + 2x_t^* - \alpha_t)dt = 0 \\ \hfill \text{(minimize integrand wrt } x) \\ \lambda_T^* = 2x_T^* \text{ (minimize terminal wrt } x_T) \\ H(t, x_t^*, \lambda_t^*, u_t^*) \le H(t, x_t^*, \lambda_t^*, u) \text{ for all } u \text{ (minimize integrand wrt } u) \\ dx_t^* = (-\beta x_t^* + u_t^*)dt \text{ (constraint)} \end{cases}$$

Then, u_t^* is an optimal control and x_t^* is an optimal trajectory.

General case

Pontriagyn maximum principle

Define

$$H(t, x, \lambda, u) := C(t, x, u) - \lambda \cdot f(t, x, u)$$

If we can find functions x_t^*, λ_t^*, and u_t^* such that

$$\begin{cases} d\lambda_t^* + \partial_x H(t, x_t^*, \lambda_t^*, u_t^*)dt = 0, \text{ (minimize integrand wrt } x) \\ \lambda_T^* = g_x(x_T^*) \text{ (minimize terminal wrt } x_T) \\ H(t, x_t^*, \lambda_t^*, u_t^*) \leq H(t, x_t^*, \lambda_t^*, u) \text{ for all } u \text{ (minimize integrand wrt } u) \\ dx_t^* = f(t, x_t^*, u_t^*)dt \text{ (constraint)} \end{cases}$$

Then, u_t^* is an optimal control and x_t^* is an optimal trajectory.

Pontriagyn maximum principle

Relation between Pontriagyn principle and value function

If the value function is differentiable and the condition of Pontriagyn principle holds, then

$$\lambda_t^* = V_x(t, x_t^*)$$

Example

Write Pontriagyn maximum principle for the control problem with

$$\inf_u \int_0^T (x_t^2 + u_t^2)\mathrm{d}t + x_T^2 + x_T, \quad \text{subject to} \quad \mathrm{d}x_t = (x_t + u_t)\mathrm{d}t$$

Can you solve the equations and find x^*, λ^*, and u^*?

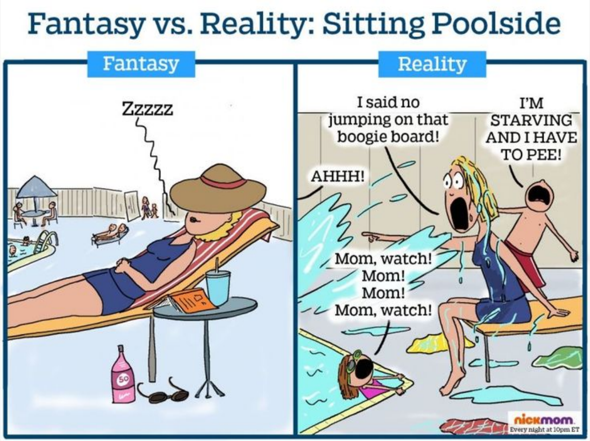
Individual project

Due end October

In your area of study, find an optimal control problem. Then, write the cost functions and the control variable and determine a set of admissible controls.

Noise

Observations are often noisy!



Noise

Observations are often noisy!

Fantasy: $\mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t$
Real world: $\mathrm{d}X_t = b(t, X_t, u_t)\mathrm{d}t + \text{noise}$

white noise

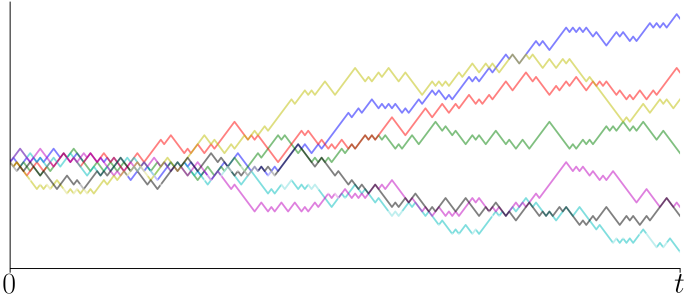
Denoted by $\mathrm{d}W_t$, at two different times t_1 and t_2, $\mathrm{d}W_{t_1}$ and $\mathrm{d}W_{t_2}$ are independent.
To many's disappointment, white noise does not exist in a conventional sense. **Brownian motion, a.k.a. Wiener process describes the noise**

Stochastic

Stochastic process

A stochastic process $\{X_t\}_t$ is a set of random variables indexed by time $t \in [0, \infty)$. It represents evolution of an uncertain quantity over time. At each time t, X_t is a random variable with a certain distribution that depends on t. Furthermore, the joint distribution of $X_{t_1}, \dots, X_{t_n}$ for different times $t_1, \dots, t_n$ is important in shaping a stochastic process. An observation, denoted by ω, is one *realization* of a stochastic process. The function $X(\omega) : [0, \infty) \to \mathbb{R}^d$ is called a sample path of X.

Stochastic



Wiener process or Brownian motion

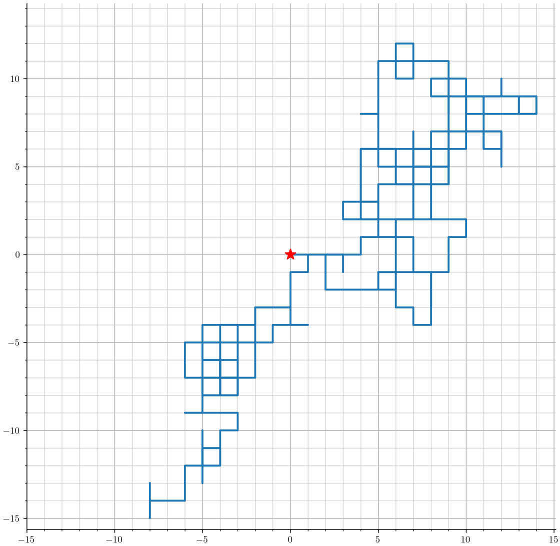
A Wiener process or Brownian motion is a stochastic process with the following property.

1) W_0 is distributed according to a known distribution, μ. μ can be a Dirac delta at a point, $W_0 = 0$.

2) For any $t_1 < \cdots < t_n$, $W_{t_n} - W_{t_{n-1}}, \ldots, W_{t_2} - W_{t_1}$ are independent and have Gaussian distribution with mean zero and variance $t_n - t_{n-1}, \ldots, t_2 - t_1$.

3) Sample paths of W are continuous functions of t.

Random walk

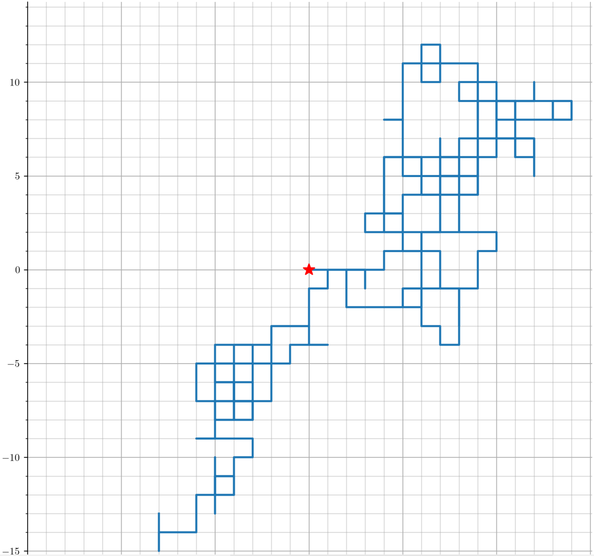


Random walk to Wiener process

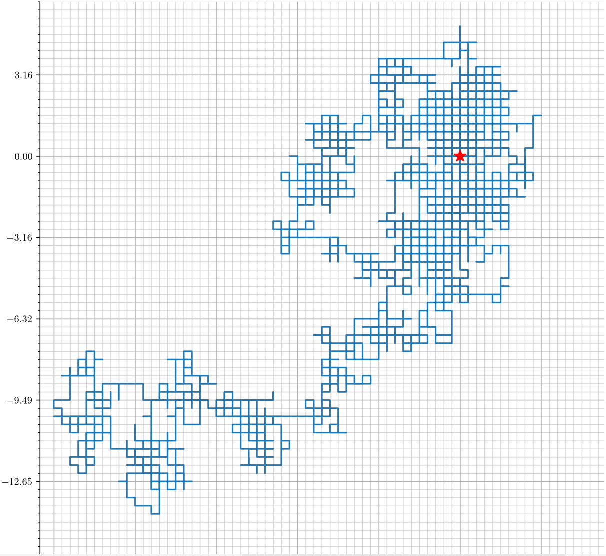
Step size

For $N \in \mathbb{N}$, we modify the step size to $\sqrt{1/N}$ and modify the time between two steps by $1/N$

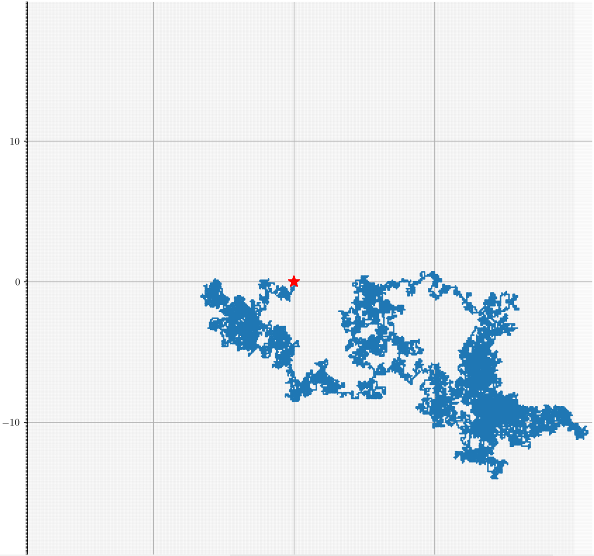
Random walk 1



Random walk 0.1



Random walk 0.01



Random walk 0.001