Methods Of Optimal Control

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FSU Fall 2025

September 15, 2025

Structure of the course

- Short lecture (10-15 minutes)
- Team work (50 minutes)
  * Problem solving
  * Programming (Python)
  * Managerial skills
  * Reporting skills
- Delivery (5 minutes)

Structure of the course

Python

Each student is expected to bring a computer to the classroom with a *Python 3.12*, *IPython*, and *Jupyter Notebook* installed.

- https://www.python.org/downloads
- https://www.anaconda.com/docs/main
- Virtual environment:

    https://docs.python.org/3/library/venv.html
    https://www.anaconda.com/docs/tools/working-with-conda/
    environments

GitHub

Each student is required to have a *GitHub* account.

Rules of teamwork

- Be respectful.
- Allow all team members to engage and express their opinion. No interruption.
- Arrange chair to make sure everyone is involved.
- No one knows everything. Make sure that everyone learns what you know.
- Break down the tasks between group members based on ones abilities.
- Ask for feedback from me frequently.

Composition of teams

Each team requires one or two members to take the following roles:

- Problem solving
- Programming (Python)
- Managerial skills
- Reporting skills

Optimization versus control

Example1

$\alpha : [0, T] \to \mathbb{R}$ is given.

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

Optimization versus control

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$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

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where the infimum is over all functions $x : [0, T] \to \mathbb{R}$.

$$\inf_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

$$x_t^* = \frac{\alpha_t}{2} = \operatorname*{argmin}_{x \in \mathbb{R}^d} \left\{ x^2 - \alpha_t x \right\}$$

Optimization versus control

Dynamic x_t

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) \mathrm{d}t \right\}$$

Infimum is over all functions $x : [0, T] \to \in \mathbb{R}$ such that for some function $u : [0, T] \to \mathbb{R}$

$$\mathrm{d}x_t = (-\beta x_t + u_t)\mathrm{d}t, \quad x_0 = x$$

Optimization versus control

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Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)

Optimization versus control

Dynamic x_t

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Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)
Check it for $\alpha_t = \mathbb{1}_{\{0 \leq t \leq \frac{T}{2}\}}$.

Optimization versus control

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Can we find u_t such that $x_t = \frac{\alpha_t}{2}$? (For simplicity, take $\beta = 0$.)

Check it for $\alpha_t = \mathbb{1}_{\{0 \le t \le \frac{T}{2}\}}$.

For $\alpha_t = \mathbb{1}_{\{0 \le t \le \frac{T}{2}\}}$, what is the value of the infimum? Is it

$$\inf \left\{ \int_0^T \left(x_t^2 - \alpha_t x_t \right) dt \right\} = -\frac{1}{4} \int_0^T \alpha_t^2 dt = -\frac{T}{8}?$$

Optimization versus control

A control problem without a myopic solution

$$\inf \int_0^T \left(x_t^2 - \alpha_t x_t + u_t^2 \right) dt \tag{1}$$

Infimum is over all functions $u : [0, T] \to \mathbb{R}$

$$dx_t = (-\beta x_t + u_t)dt, \quad x_0 = x$$

Trade-off:

- Trying to send $x_t \to \frac{\alpha_t}{2}$ may cause $\int_0^T u_t^2 dt$ to grow.
- Trying to keep cost $\int_0^T u_t^2 dt$ near zero, does not bring x_t close to $\frac{\alpha_t}{2}$.

What is the sweet spot for u_t?

A generic control problem

Definition

$$\inf_{u \in \mathcal{U}} \int_0^T C(t, x_t, u_t) dt + g(x_T), \quad dx_t = f(x_t, u_t) dt \qquad (2)$$

- $C : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$: *running cost*
- $g : \mathbb{R}^d \to \mathbb{R}$: *terminal cost*
- $\mathcal{U}$: *an admissible* set of functions $u : [0, T] \to \mathbb{R}^n$, *control variable.*

A generic control problem

Admissible controls

$\mathcal{U}$ is chosen to fit the proper application and/or to make the control problem wellposed.

Admissibility for wellposedness

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2)dt = -\infty, \quad dx_t = (x_t - u_t)dt \tag{3}$$

$\mathcal{U}$ to be the set of all functions $u : [0, T] \to \mathbb{R}$

If we restrict $\mathcal{U}$ to the set of functions $u : [0, T] \to [-1, \infty)$ (some lower bound on the value), then

$$\inf_{u \in \mathcal{U}} \int_0^T (x_t - u_t^2)dt > -\infty, \quad dx_t = (x_t - u_t)dt. \tag{4}$$

Infinite horizon

Infinite horizon

An infinite horizon control problem is accommodated by setting $T = \infty$. For example,

$$\inf_{u \in \mathcal{U}} \int_0^\infty e^{-t}(x_t^2 + u_t^2)dt, \ C(t, x, u) = e^{-t}(x^2 + u^2) \tag{5}$$

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \ : \ x_t \notin D\} \tag{6}$$

where $dx_t = u_t dt$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Infinite horizon

Exercise

Write the following problem as a generic control problems by associating the horizon T, the running cost $C(t, x, u)$ and terminal cost $g(x)$

(Shortest time to exit a bounded domain) Given a bounded domain $D \subset \mathbb{R}^d$, find

$$\inf_u \{t \geq 0 \ : \ x_t \notin D\} \tag{7}$$

where $dx_t = u_t dt$ with control $|u_t| \leq 1$ and $u_t \in \mathbb{R}^d$ and initial position $x_0 = x \in D$.

Solution

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} dt, \ dx_t = u_t dt \ \text{ with } \ |u_t| \leq 1$$

An optimal control is described by existing D as fast as possible, $|u| = 1$, and stop as soon as we exit, $|u| = 0$.

Dynamic programming principle (DPP)

Value function

Fix $x_t = x$.

$$V(t, x) := \inf_{u \in \mathcal{U}_t} \int_t^T C(s, x_s, u_s) \mathrm{d}s + g(x_T), \quad \mathrm{d}x_s = f(x_s, u_s) \mathrm{d}s$$

$\mathcal{U}_t$: the set of admissible controls restricted to $[t, T]$.

Dynamic programming principle (DPP)

$$V(t, x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \quad \mathrm{d}x_r = f(x_r, u_r) \mathrm{d}r$$

$\mathcal{U}_{t,s}$: the set of admissible controls restricted to $[t, s]$.

DPP

Balance of cost in DPP

$$V(t, x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \quad x_s = x + \int_t^s f(x_r, u_r) \mathrm{d}r$$
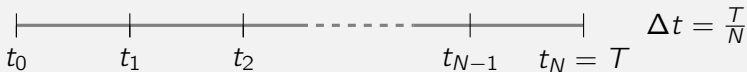
Proof of DPP

$$V(t,x) = \inf_{u \in \mathcal{U}_t} \int_t^T C(r, x_r, u_r) \mathrm{d}r + g(x_T)$$

$$= \inf_{u_1 \in \mathcal{U}_{t,s}} \inf_{u_2 \in \mathcal{U}_s} \int_t^s C(r, x_r, u_1(r)) \mathrm{d}r + \int_s^T C(r, x_r, u_2(r)) \mathrm{d}r + g(x_T)$$

$$= \inf_{u_1 \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_1(r)) \mathrm{d}r + \inf_{u_2 \in \mathcal{U}_s} \int_s^T C(r, x_r, u_2(r)) \mathrm{d}r + g(x_T)$$

Note that $\inf_{u_2 \in \mathcal{U}_s} \int_s^T C(r, x_r, u_2(r)) \mathrm{d}r + g(x_T) = V(s, x_s)$. Therefore,

$$V(t,x) = \inf_{u_1 \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_1(r)) \mathrm{d}r + V(s, x_s)$$

Numerical DPP

Discretization

$$\Delta t = \frac{T}{N}$$

$$\begin{cases} V(t_N, x) = g(x) \\ V(t_i, x) = \inf_{u \in \mathcal{U}_{t_i, t_{i+1}}} \int_{t_i}^{t_{i+1}} C(r, x_r, u_r) \mathrm{d}r + V(t_{i+1}, x_{t_{i+1}}), \\ \qquad\qquad\qquad x_{t_{i+1}} = x + \int_{t_i}^{t_{i+1}} f(s, x_s, u_s) \mathrm{d}s \end{cases}$$

Approximation

$$\begin{cases} \hat{V}(t_N, x) = g(x) \\ \hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u) \Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \ x_{t_{i+1}} = x + f(t_i, x, u) \Delta t \end{cases}$$

Numerical DPP

Approximation

$$\begin{cases} \hat{V}(t_N, x) = g(x) \\ \hat{V}(t_i, x) := \inf_u C(t_i, x_{t_i}, u)\Delta t + \hat{V}(t_{i+1}, x_{t_{i+1}}), \ x_{t_{i+1}} = x + f(t_i, x, u)\Delta t \end{cases}$$

Simplification of one-step approximate DPP

The approximation is not over the control $u : [t_i, t_{i+1}] \to \mathbb{R}^m$, but over values $u \in \mathbb{R}^m$. The optimal value $\hat{u}^*$ is a constant approximately optimal control over $[t_i, t_{i+1}]$.

Algorithm

Algorithm 1: Numerical DPP

Parameter T, N, $f(t, x, u)$, $C(t, x, u)$, and $g(x)$;

:
$\Delta t = \frac{T}{N}$

Data: $\hat{V}(t_N, x) = g(x)$;

x_i^j for $j = 1, ..., J$ and $i = 0, ..., N - 1$;

(x_i^j means the jth discrete point at time t_i.)

1 **for** $i \leftarrow N - 1$ *to* 0 **do**

2 $\quad \hat{x}_{i+1}^j = x_i^j + f(t_i, x_i^j, u)\Delta t$;

3 $\quad \tilde{V}(t_i, x_i^j) \leftarrow \inf_u C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{t_{i+1}}^j)$;

4 $\quad \hat{V}(t_i, x)$ obtained from interpolation on $\tilde{V}(t_i, x_i^j)$ for $j = 1, ..., J$;

5 $\quad \hat{u}^*(t_i, x_i^j) \in \underset{u}{\text{argmin}}\, C(t_i, x_i^j, u)\Delta t + \hat{V}(t_{i+1}, \hat{x}_{i+1}^j)$;

6 **return** $\hat{V}(t_i, \cdot)$ *and* $\hat{u}^*(t_i, \cdot)$ *for* $i = 0, ..., N - 1$.

DPP algorithm

Exercise

Why interpolation is required in Algorithm 1?

Can we perform the algorithm by only knowing $\hat{V}(t_{i+1}, x_{i+1}^j)$ for all $j = 1, ..., J$?

Note the difference between $\hat{V}(t_{i+1}, x_{i+1}^j)$ and $\hat{V}(t_{i+1}, \hat{x}_{i+1}^j)$ and the difference between x_{i+1}^j and $\hat{x}_{i+1}^j = x_i^j + f(t_i, x_i^j, u)\Delta t$.

$$\inf_u C(t_i, x_i^j, u) + \hat{V}(t_i, x_i^j + f(t_i, x_i^j, u)\Delta t)$$

Quadratic example

Example

Value function:

$$V(t, x) := \inf_{u \in \mathcal{U}_t} \int_t^T \left(x_s^2 + u_s^2 \right) ds + \frac{1}{2} x_T^2 - x_T, \quad dx_s = (x_s - u_s) ds. \quad (8)$$

We cannot find value functions using a myopic argument.

Exercise

1) In example above, write the approximate DPP from time t_i to t_{i+1}.

2) Assume that $\hat{V}(t_{i+1}, x) = a_{i+1} x^2 + b_{i+1} x + c_{i+1}$ for some known values a_{i+1}, b_{i+1}, and c_{i+1}. Use optimization of a quadratic function to find $\hat{V}(t_i, x)$. Note that you need to use $\hat{x}_{t_{i+1}} = x + (x - u)\Delta t$.

3) Does $\hat{V}(t_i, x)$ is of the form $a_i x^2 + b_i x + c_i$? What is the relation between (a_i, b_i, c_i) and $(a_{i+1}, b_{i+1}, c_{i+1})$?

Hamilton-Jacobi equation

Hamiltonian and Lagrangian

Hamilton: principle of minimum action
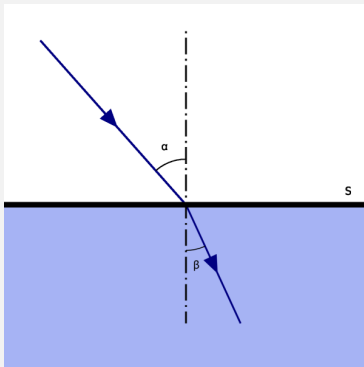
Hamilton-Jacobi equation

Recall the DPP

$$V(t, x) = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s), \quad x_s = x + \int_t^s f(x_r, u_r) \mathrm{d}r$$

Taylor expansion

$$V(s, x_s) = V(t, x) + V_t(t, x)(s - t) + V_x(t, x)(x_s - x) + R_2$$

$$
\begin{aligned}
V(t, x) &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s) \\
&= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r \\
&\quad + V(t, x) + V_t(t, x)(s - t) + V_x(t, x) \int_t^s f(x_r, u_r) \mathrm{d}r + R_2
\end{aligned}
$$

Hamilton-Jacobi equation

Taylor expansion

$$V(s, x_s) = V(t, x) + V_t(t, x)(s - t) + V_x(t, x)(x_s - x) + R_2$$

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\begin{aligned}
V(t, x) &= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r + V(s, x_s) \\
&= \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V(t, x) + V_t(t, x)(s - t) + V_x(t, x) \int_t^s f(x_r, u_r)\mathrm{d}r + R_2 \\
&= V(t, x) + V_t(t, x)(s - t) + \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r)\mathrm{d}r \\
&\quad + V_x(t, x) \int_t^s f(x_r, u_r)\mathrm{d}r + R_2
\end{aligned}
$$

Hamilton-Jacobi equation

Taylor expansion

$$\cancel{V(t,x)} = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r + V(s, x_s)$$

$$= \cancel{V(t,x)} + V_t(t,x)(s-t) + \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) \mathrm{d}r$$

$$+ V_x(t,x) \int_t^s f(r, x_r, u_r) \mathrm{d}r + R_2$$

Dividing both sides by $s - t$ and sending $s \to t$.

Hamilton-Jacobi equation

Taylor expansion

$$0 = \inf_{u \in \mathcal{U}_{t,s}} \int_t^s C(r, x_r, u_r) dr + V(s, x_s)$$

$$= V_t(t, x) + \inf_{u \in \mathcal{U}_{t,s}} \lim_{s \to t} \frac{\int_t^s C(r, x_r, u_r) dr}{s - t}$$

$$+ V_x(t, x) \lim_{s \to t} \frac{\int_t^s f(r, x_r, u_r) dr}{s - t} + \lim_{s \to t} \frac{R_2}{s - t}$$

$R_2 = o(s - t)$: $\lim_{s \to t} \frac{R_2}{s-t} = 0$.

Hamilton-Jacobi equation

HJ equation

$$0 = V_t(t,x) + \inf_u \left\{ C(t,x,u) + V_x(t,x)f(t,x,u) \right\}$$

$$\begin{cases} 0 = V_t(t,x) + H(t,x,V_x(t,x)) \\ V(T,x) = g(x) \end{cases}$$

Hamiltonian: $H(t,x,p) = \inf_u \left\{ C(t,x,u) + p \cdot f(t,x,u) \right\}$

A linear-quadratic control problem

Consider the control problem:

$$\inf_u \left\{ \int_0^T \left(x_t^2 + u_t^2 \right) dt \right\}, \quad dx_t = (-\beta x_t + u_t) dt \tag{9}$$

$C(t, x, u) = x^2 + u^2$ and $f(t, x, u) = -\beta x + u$.

Write the HJ equation.

After writing the HJ, plug in $V(t, x) = a(t)x^2 + b(t)x + c(t)$ the HJ and find ODEs for $a(t)$, $b(t)$, and $c(t)$. What are $a(T)$, $b(T)$, and $c(T)$?

Eikonal equation

Fastest exit

Recall the fastest exit problem.

$$\inf_u \int_0^\infty \mathbb{1}_{\{x_t \in D\}} dt, \ dx_t = u_t dt \ \text{ with } \ |u_t| \le 1$$

Write the definition of value function for initial state $x_0 = x \in D$. Write the HJ equation. Is there any boundary condition?

Solution to Eikonal equation

Write the HJ equation and boundary condition for the special case where $D = [-1, 1] \subset \mathbb{R}$. Which one of the following functions satisfy the HJ equation? Which one matches the value function?

$$v_1(x) = 1 - |x|, v_2(x) = \begin{cases} \frac{1}{2} - |x - \frac{1}{2}| & 0 \le x \le 1 \\ \frac{1}{2} - |x + \frac{1}{2}| & -1 \le x < 0 \end{cases}$$

SIR model in epidemiology

ODEs
Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta I_t S_t - \gamma I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma I_t \mathrm{d}t \end{cases}$$

Controlled state variables
Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

SIR model in epidemiology

Controlled state variables

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

$$\inf_{\beta_t, \gamma_t} \int_0^T ((b_1 - \beta_t)^2 + \gamma_t^2)\mathrm{d}t + I_T^2$$

Write the HJ equation in variables $x = (S, I, R)$.

SIR model in epidemiology

Controlled state variables

Susceptible, infected, and recovered:

$$\begin{cases} \mathrm{d}S_t = -\beta_t S_t I_t \mathrm{d}t \\ \mathrm{d}I_t = (\beta_t I_t S_t - \gamma_t I_t)\mathrm{d}t \\ \mathrm{d}R_t = \gamma_t I_t \mathrm{d}t \end{cases}$$

$\beta_t \in [b_0, b_1]$ and $\gamma_t \in [c_0, c_1]$ all positive.

$$\inf_{\beta_t, \gamma_t} \int_0^T ((b_1 - \beta_t)^2 + \gamma_t^2)\mathrm{d}t + I_T^2$$

Notice that $\mathrm{d}(S_t + I_t + R_t) = 0$. This should allow us to reduce the number of state variables $x_t = (S_t, I_t, R_t)$ to two, in place of three. Assume that the population size is given by N, $S_t + I_t + R_t = N$. Remove the variable R_t and write the HJ equation in terms of (S_t, I_t). Write the HJ equation in (S, I).

Consumption

Savings account

$$dx_t = (rx_t - c_t)dt$$
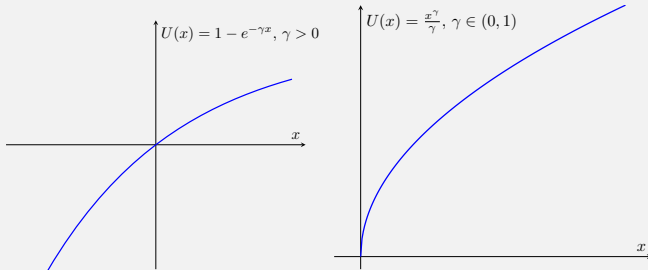
$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T U(c_t)dt + U(x_T)$$

U is a given function called utility function.

Consumption

Utility function

Utility function is a concave function that represent our enjoyment from consumption or wealth. The concavity signifies the fact that if our consumption or wealth level is low, increasing one more unit grants more joy compared to when our consumption or wealth level is higher and we obtain one more unit. Example of a utility function:

Consumption

Savings account

$$dx_t = (rx_t - c_t)dt$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T U(c_t)dt + U(x_T)$$

HJ equation is given by:

$$\begin{cases} V_t + H(x, V_x) = 0 \\ V(T, x) = U(x) \end{cases}$$

with $H(x, p) = \sup_{c \geq 0}\{U(x) - cp\} + rxp$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

Show that the supremum in Hamiltonian is attained at $c^*(p) = p^{\frac{1}{\gamma-1}}$ and

$$H(x,p) = \sup_{c \geq 0} \left\{ \frac{c^\gamma}{\gamma} - cp \right\} + rxp = \frac{1-\gamma}{\gamma} p^{\frac{1}{\gamma-1}} + rxp$$

Verify that $V(t,x) = f(t)\frac{c^\gamma}{\gamma}$ solves the HJ equation

$$\begin{cases} 0 = V_t + V_x^{\frac{\gamma}{\gamma-1}} + rxV_x \\ V(T,x) = \frac{c^\gamma}{\gamma} \end{cases}$$

and $f(t)$ satisfies

$$\begin{cases} 0 = f' + f^{\frac{\gamma}{\gamma-1}} + \frac{r}{\gamma}f \\ f(T) = 1 \end{cases}$$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

We solve the ODE $0 = f' + f^{\frac{\gamma}{\gamma-1}} + \frac{r}{\gamma}f$ by the change of variable $f = y^{1-\gamma}$.

y satisfies $y' + \frac{r}{\gamma(1-\gamma)}y + \frac{1}{1-\gamma} = 0$ and $y(t) = \left(1 + \frac{\gamma}{r}\right)e^{\frac{r}{\gamma(1-\gamma)}(T-t)} - \frac{\gamma}{r}$.

$$f(t) = \left(\left(1 + \frac{\gamma}{r}\right)e^{\frac{r}{\gamma(1-\gamma)}(T-t)} - \frac{\gamma}{r}\right)^{1-\gamma}$$

Since $1 - \gamma < 0$, $f(t) > 1$ unless $t = T$. The optimal consumption $c^*(t,x) = \frac{1}{f^{\frac{1}{1-\gamma}}(t)}x$ or

$$\begin{cases} c_t^* = \frac{1}{f^{\frac{1}{1-\gamma}}(t)}x_t \\ \mathrm{d}x_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)}\right)x_t\mathrm{d}t \end{cases}$$

Solving HJ equation for consumption

Utility $U(c) = \frac{c^\gamma}{\gamma}$ for $c \geq 0$

$$\begin{cases} c_t^* = \frac{1}{f^{\frac{1}{1-\gamma}}(t)} x_t \\ dx_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)} \right) x_t dt \end{cases}$$

We learn from the above solution that $\frac{1}{f^{\frac{1}{1-\gamma}}(t)} < 1$, therefore the optimal consumption rate percentage of wealth and the account balance never goes negative.

Coding exercise

Plot the solution of $dx_t = \left(r - \frac{1}{f^{\frac{1}{1-\gamma}}(t)} \right) x_t dt$ for different values for $r > 0$, $\gamma \in (0, 1)$, T, and different initial balance $x > 0$.

Consumption with no terminal wealth

Savings account

$$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T U(c_t)\mathrm{d}t \cancel{+ U(x_T)}$$

Can we consume infinitely by borrowing? . If we restrict consumption to case where borrowing is not allowed, $x_t \geq 0$ for all t, does this solve the issue? (Relation to admissible control.)
Write the HJ equation with boundary condition that reflects $x_t \geq 0$.

Consumption with decay

Savings account

$$\mathrm{d}x_t = (rx_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

$$\sup_{c_t \geq 0} \int_0^T e^{-kt} U(c_t)\mathrm{d}t + e^{-kT} U(x_T)$$

Write the HJ equation.

Dynamic programming equation with decay

Control problem with decay

$$\inf_u \int_0^T e^{-kt} C(x_t, u_t) \mathrm{d}t + e^{-kT} g(x_T), \quad \mathrm{d}x_t = f(x_t, u_t) \mathrm{d}t$$

Value function

$$V(t, x) := \inf_u \int_t^T e^{-k(s-t)} C(x_s, u_s) \mathrm{d}s + e^{-k(T-t)} g(x_T)$$

DPP

$$V(t, x) = \inf_u \int_t^s e^{-k(r-t)} C(x_r, u_r) \mathrm{d}r + e^{-k(s-t)} V(s, x_s)$$

Dynamic programming equation with decay

Value function

$$V(t, x) := \inf_u \int_t^T e^{-k(s-t)} C(x_s, u_s) \mathrm{d}s + e^{-k(T-t)} g(x_T)$$

DPP

$$V(t, x) = \inf_u \int_t^s e^{-k(r-t)} C(x_r, u_r) \mathrm{d}r + e^{-k(s-t)} V(s, x_s)$$

HJB with decay

Write the first three terms of the Taylor polynomial for $V(s, x_s)$ about point (t, x).

Dynamic programming equation with decay

HJB with decay

$$e^{-k(s-t)}V(s, x_s) = V(t, x) + e^{-k(s-t)}\Big(\big(V_t(t, x) - kV(t, x)\big)(s - t)$$

$$+ V_x(t, x) \int_t^s f(x_r, u_r)\mathrm{d}r\Big) + o((s - t)^2)$$

$$\cancel{V(t, x)} = \inf_u \int_t^s e^{-k(r-t)} C(x_r, u_r)\mathrm{d}r$$

$$+ \cancel{V(t, x)} + e^{-k(s-t)}\Big(\big(V_t(t, x) - kV(t, x)\big)(s - t)$$

$$+ V_x(t, x) \int_t^s f(x_r, u_r)\mathrm{d}r\Big) + o((s - t)^2)$$

Dynamic programming equation with decay

HJB with decay

Divide by $s - t$ and $s \to t$:

$$0 = \inf_u C(x, u) + V_t(t, x) - kV(t, x) + V_x(t, x)f(x, u)$$

$$0 = V_t(t, x) - kV(t, x) + \inf_u \{C(x, u) + V_x(t, x)f(x, u)\}$$

Hamiltonian:

$$H(x, p) := \inf_u \{C(x, u) + pf(x, u)\}$$

Purpose of different HJ for decay

Consumption with decay

Savings account

$$\mathrm{d}X_t = (rX_t - c_t)\mathrm{d}t$$

$c_t \geq 0$ is the rate of consumption.

$$\inf_{c_t \geq 0} \int_0^T e^{-kt} U(c_t)\mathrm{d}t$$

Write the HJ equation by including the decay in the DPP.

Lagrange multiplier

Constrained optimization

$$\inf_x f(x) \quad \text{subject to} \quad g(x) = 0$$

Lagrangian

$$L(x, \lambda) := f(x) - \lambda \cdot g(x)$$

Saddle point problem

$$\sup_\lambda \inf_x L(x, \lambda)$$

Individual project

Due end October

In your area of study, find an optimal control problem. Then, write the cost functions and the control variable and determine a set of admissible controls.