Bi-Arc Digraphs and Approximation of H-coloring

Abstract

We study the class of digraphs analogous to complement of circular arc bigraphs. They are defined via a geometric representation by two families of arcs from a circle satisfying the condition of complement of circular arc graphs with clique cover two (circular arc bigraphs); hence we call them *Bi-arc digraph*.

This digraphs introduced recently by Hell and Rafiey [17]. They admit a number of equivalent definitions, including an ordering characterization by so-called min-orderings and the existence of semi-lattice polymorphisms. Min-orderings arose in the study of list homomorphism problems. If H admits a min-ordering (or a certain extension of min-orderings), then the list homomorphism problem to H is known to admit a polynomial time algorithm. The bi-arc digraphs contain the class of monotone proper-interval bigraphs the digraphs that admit a min-max ordering. If a digraph H admits a min-max (and extended min-max) ordering then minimum cost homomorphism to H, MinHOM(H), is polynomial time solvable and NP-complete otherwise.

We show another importance of this class of digraphs by designing a constant approximation algorithm for minimum cost homomorphism. When H is a target digraphs and admits a min-ordering then MinHOM(H), can be approximated within a constant factor.

We define an extended version of min-ordering and give a constant approximation algorithm for the extended version. Moreover, we show that, given an almost satisfiable $((1-\epsilon)-satisfiable)$ instance of list homomorphism problem, there is a polynomial time algorithm that robustly approximate list homomorphism problem by giving $(1-O(\epsilon))$-satisfying homomorphism.

Keywords. Approximation algorithms, robust approximability, minimum cost homomorphism problems, min orderings, randomized rounding

1 Introduction

A graph H is an *interval graph* if it admits an interval representation, where an *interval representation* of H is a family of closed intervals $I_v, v \in V(H)$, such that u and v are adjacent in H if and only if I_u intersects I_v. (We take this definition to apply in the case $u = v$ as well, and hence an interval graph H is automatically *reflexive*, i.e., each vertex is adjacent to itself, via a *loop*.) If I is a closed interval, we denote by $\ell(I)$ (respectively $r(I)$) the left (respectively right) endpoint of I. Note that intervals I and I' intersect if and only if $r(I) \geq \ell(I')$ and $r(I') \geq \ell(I)$.

The study of interval graphs is one of the most beautiful and popular parts of graph theory, with interesting applications [10], elegant characterization theorems [9, 18], and ingenious and efficient recognition algorithms [2, 3, 12]. When it comes to digraphs, there is a consensus that instead of one family of intervals, one needs two families, one for the out-neighbourhoods, and one for the in-neighbourhoods [4, 21]. A digraph H is an *interval digraph* if it admits a bi-interval representation, where a *bi-interval representation* consists of two families of closed intervals $I_v, v \in V(H)$, and $J_v, v \in V(H)$, such that uv is an arc of H if and only if I_u intersects J_v. This concept has been investigated [4], but the results do not seem to have the appeal of interval graphs – for instance there is no known forbidden structure characterization analogous to [18]. The first recognition algorithm for this classes of digraphs has complexity $O(nm^6(n + m)\log n)$ [20] and then improved to $O(mn)$ just recently [22]. In the special case of reflexive digraphs, the authors of [7, 8] have proposed another class of interest: an interval digraph is called an *adjusted interval digraph* if it admits a bi-interval representation $I_v, J_v, v \in V(H)$, in which each pair of corresponding intervals I_v and J_v has the same left endpoint, $\ell(I_v) = \ell(J_v)$ for all $v \in V(H)$. Such a bi-interval representation is called *adjusted*. For adjusted interval digraphs, one can prove a forbidden structure characterization, which implies a faster recognition algorithm [7, 8]. The clue that this class may be a better analogue of interval graphs came from the fact that interval graphs and adjusted interval digraphs have the same ordering characterization, by the so-called min ordering [7, 8].

A *min-ordering* of a graph H is an ordering of its vertices $a_1, a_2, \ldots, a_n$, so that the existence of the edges $a_i a_j, a_{i'} a_{j'}$ with $i < i', j' < j$ implies the existence of the edge $a_i a_{j'}$. A *min-max ordering* of a graph H is an ordering of its vertices $a_1, a_2, \ldots, a_n$, so that the existence of the edges $a_i a_j, a_{i'} a_{j'}$ with $i < i', j' < j$ implies the existence of the edges $a_i a_{j'}, a_{i'} a_j$. For bigraphs, it is more convenient to speak of two orderings, and we define a *min ordering* of a bigraph H to be an ordering $a_1, a_2, \ldots, a_p$ of the white vertices and an ordering $b_1, b_2, \ldots, b_q$ of the black vertices, so that the existence of the edges $a_i b_j, a_{i'} b_{j'}$ with $i < i', j' < j$ implies the existence of the edge $a_i b_{j'}$; and a *min-max ordering* of a bigraph H to be an ordering of $a_1, a_2, \ldots, a_p$ of the white vertices and an ordering $b_1, b_2, \ldots, b_q$ of the black vertices, so that the existence of the edges $a_i b_j, a_{i'} b_{j'}$ with $i < i', j' < j$ implies the existence of the edges $a_i b_{j'}, a_{i'} b_j$. The bigraph H that admits a min ordering has an circular arc representation. A bigraph H is called co-circular arc bigraph if its complement is a circular arc graph(circular arc graph with clique cover two). Graph G is called circular arc if the vertices can be viewed of the arcs from a circle and two vertices are adjacent in G if their arcs intersect. The following Theorem proved in [13].

Theorem 1.1 *A bipartite graph H is a co-circular arc bigraph if and only if H admits a min ordering.*

A linear ordering $<$ of $V(D)$ is a *min ordering* for digraph D if it satisfies the following property: if $u < w$ and $z < v$ and both uv, wz are arcs of D, then uz is also an arc of D. In addition if wv is also an arc of H then $<$ is called *min-max* ordering. It is shown in [7, 8] that a reflexive digraph D has a min ordering if and only if it is an adjusted interval digraph. If we interpret a graph as a digraph by replacing each edge uv by the two opposite arcs uv, vu, then a reflexive graph has a min ordering if and only if it is an interval graph [5]. Thus, the concept of a min ordering suggests that, for reflexive digraphs, one natural generalization of the class of interval graphs is the class of adjusted interval digraphs. We often can view a digraph as a binary relation. We note that for structures with two binary relations (digraphs with two kinds of arcs), the recognition problem of having a min ordering is NP-complete, via a reduction (from a preliminary version of [1]) similar to that in the proof of Theorem 4.9 in [16]. This problem asks whether for two given digraphs H_1, H_2 with the same vertex set V one can find a linear ordering of V which is min ordering for both H_1 and H_2. The question that whether a binary relation R (or input digraph H) has an X-underbar property (min ordering) was asked in [1] and it was posed an as open problem in [16].

A *homomorphism* of a digraph D to a digraph H is a mapping $f : V(D) \rightarrow V(H)$ such that for any arc xy of D the pair $f(x)f(y)$ is an arc of H.

Let H be a fixed digraph. The *list homomorphism problem* to H, denoted by LHOM(H), seeks, for a given input digraph D and lists $L(x) \subseteq V(H), x \in V(D)$, a homomorphism f of D to H such that $f(x) \in L(x)$ for all $x \in V(D)$. It was proved in [6] that for irreflexive graphs, the problem LHOM(H) is polynomial time solvable if H is a co-circular arc bigraph, and is NP-complete otherwise. It was shown in [5] that for reflexive graphs H, the problem LHOM(H) is polynomial time solvable if H is an interval graph, and is NP-complete otherwise.

The *minimum cost homomorphism problem* to H, denoted by MinHOM(H), seeks, for a given input graph G and vertex-mapping costs $c(x, u), x \in V(G), u \in V(H)$, a homomorphism f of G to H that minimizes the total cost $\sum_{x \in V(G)} c(x, f(x))$. It was proved in [11] that for irreflexive graphs, the problem MinHOM(H) is polynomial time solvable if H is a proper interval bigraph, and it is NP-complete otherwise. It was also shown there that for reflexive graphs H, the problem MinHOM(H) is polynomial time solvable if H is a proper interval graph, and it is NP-complete otherwise.

In [13], the authors have found a dichotomy for approximating the MinHOM(H) when H is a bipartite graph. When H admits a min ordering (complement of H is a circular arc graph) then MinHOM(H) admits a constant approximation algorithm and otherwise there is no approximation for MinHOM(H).

Obtaining a dichotomy for approximation of MinHOM(H) when H is a digraph is a much more complex task. One can show that if LHOM(H) is not polynomial then there is no constant approximation algorithm for MinHOM(H). The authors in [14] showed that if H contains a DAT (digraph asteroidal triple) then LHOM(H) is NP-complete and polynomial otherwise. If H admits a min ordering then LHOM(H) is polynomial time solvable and if H admits a min-max ordering then MinHOM(H) is polynomial time solvable and NP-complete otherwise [15]. One of the first steps toward finding a dichotomy for approximation of MinHOM(H) is to decides whether there exists a constant approximation algorithm for MinHOM(H) when H admits a min ordering or not. Here we show that this is the case and there is a constant approximation algorithm for MinHOM(H) when H is a Bi-arc digraph (H has min ordering).

2 Min-Ordering and Bi-Arc Digraphs

In this section we define the Bi-arc digraphs and show that they admit min ordering (they have X-underbar property).

Definition 2.1 *Digraph $H = (V, A)$ is called* bi-arc digraph *if each vertex of v, is represented by a pair of arcs I_v, J_v on a circle with north pole N and south pole S with the following conditions :*

- *I_v contains N and not S and J_v contains S and not N.*
- *uv is an arc of H if and only if I_u and J_v do not intersect.*
- *the clockwise end of I_v precedes the clockwise end of I_w (I_v is closer to N than I_w in the clockwise direction) iff the the clockwise end of J_v precedes the clockwise end of J_w.*

Theorem 2.2 *Digraph H admits a min-ordering if and only if H is a bi-arc digraph.*

Proof. Suppose H is a bi-arc graph. The vertices of H correspond to the arcs containing the north pole N, are ordered according to the clockwise order of their clockwise extremes, i.e., u comes before u' if the clockwise end of I_u precedes the clockwise end of $I_{u'}$. It is now easy to see from the definitions that if $uv, u'v'$ are arcs of H with $u < u'$ and $v' < v$, then I_u and $J_{v'}$ must be disjoint, and so uv' must be an arc of H.

Now suppose $v_1, v_2, \ldots, v_n$ is a min ordering for H. We lay out southern arcs $J_1, J_2, \ldots, J_n$ and northern arcs $I_1, I_2, \ldots, I_n$ in clockwise order. The clockwise end of I_r (corresponding to v_r) is closer to N than the clockwise end of I_s (corresponding to v_s) for $r < s$. The clockwise end of J_r (corresponding to v_r) is closer to S than the clockwise end of J_s (corresponding to v_s) for $r < s$.

In the i-th stage, assign the counterclockwise endpoint of the i-th southern arc J_i so that it stops just after the last in-neighbour v_j (after the clockwise end of I_j) with $j \geq i$.

Note that if a vertex v_j has no in-neighbor then counterclockwise end of J_j extends up to clockwise end of I_1. We proceed analogously for the i-th northern arc I_i. All the non in-neighbor of v_i that came after the last in-neighbour v_j are now intersect I_i. But there may be some non in-neighbours v_k of v_i ($k \geq i$) which precede some in-neighbours of v_i and hence still have to be made to intersect I_i. These have no out-neighbours amongst $v_i, v_{i+1}, \ldots, v_n$, and we shall correspondingly extend the counterclockwise endpoint J_k up just after I_i. ■

3 Approximation Algorithm for MinHOM(H)

In this section, first we present an LP for MinHOM(H) when the target digraph H admits min-max ordering. Second, we show a one-to-one correspondence between integral solutions of the LP and homomorphisims from digraph D to H. Before presenting the LP, we give a procedure to modify lists associated to the vertices of D.

To each vertex $x \in D$, we associate a list $L(x)$ that initially contains $V(H)$. We think of $L(x)$ the set of possible images for x in a homomorphism from D to H. We apply the arc consistency as follows. Take an arbitrary arc $xy \in A(D)$ ($yx \in A(D)$) and let $a \in L(x)$. If there is no out-neighbor (in-neighbor) of a in $L(y)$ then we remove a from $L(x)$ and repeat.

For every pair $(x, y) \in V(D) \times V(D)$ we consider a list of possible pairs (a, b), $a \in L(x)$ and $b \in L(y)$. We perform pair consistency as follows. Consider three vertices $x, y, z \in V(D)$. For $(a, b) \in L(x, y)$ if there is no $c \in L(z)$ such that $(a, c) \in L(x, z)$ and $(c, b) \in L(z, y)$ then remove (a, b) from $L(x, y)$ and repeat.

Let $a_1, a_2, a_3, \ldots, a_p$ be a min-max ordering $<$ of target digraph H. Define $\ell^+(i)$ to be the smallest subscript j such that a_j is an out-neighbour of a_i (and $\ell^-(i)$ to be the smallest subscript j such that a_j is a in-neighbour of a_i).

We define the LP program as follows. For every vertex x of D and every vertex a_i of H define variable x_i with the following constraints.

$$
\begin{array}{lll}
\text{Minimize} & \sum_{x,i} c(x, a_i)(x_i - x_{i+1}) & \\
\text{Subject to:} & x_i \geq 0 & \\
 & x_1 = 1 & \\
 & x_{p+1} = 0 & \\
 & x_{i+1} \leq x_i & \\
 & x_{i+1} = x_i & \text{if } a_i \notin L(x) \\
 & u_i \leq v_{l^+(i)} & \forall\ uv \in A(D) \\
 & v_i \leq u_{l^-(i)} & \forall\ uv \in A(D)
\end{array}
$$

We denote the set of constraints of the above LP by $\mathcal{S}$. In what follows we prove that there is a one-to-one correspondence between integer solutions of $\mathcal{S}$ and homomorphisms from D to H when H admits a min-max ordering.

Theorem 3.1 *There is a one-to-one correspondence between homomorphisms of D to H and integer solutions of $\mathcal{S}$.*

Proof. For homomorphism $f : D \to H$, if $f(x) = a_t$ we set $x_i = 1$ for some $i \leq t$, otherwise we set $x_i = 0$. We set $x_1 = 1$ and $x_{p+1} = 0$ for all $x \in V(D)$. Now all the variables are nonnegative and we have $x_{i+1} \leq x_i$. Note that if $a_i \notin L(x)$ then $f(x) \neq a_i$ and we have $x_i = x_{i+1} = 0$. It remains to show that $u_i \leq v_{l^+(i)}$ for every uv arc in D. Suppose for contradiction that $u_i = 1$ and $v_{l^+(i)} = 0$ and let $f(u) = a_r$ and $f(v) = a_s$. This implies that $u_r = 1$, whence $i \leq r$; and $v_s = 1$, whence $s < l^+(i)$. Since $a_i a_{l^+(i)}$ and $a_r a_s$ both are arcs of H with $i \leq r$ and $s < l^+(i)$, the fact that H has a min ordering implies that $a_i a_s$ must also be an arc of H, contradicting the definition of $l^+(i)$. The proof for $v_j \leq u_{l^-(i)}$ is analogous.

Conversely, if there is an integer solution for $\mathcal{S}$, we define a homomorphism f as follows: we let $f(x) = a_i$ when i is the largest subscript with $x_i = 1$. We prove that this is indeed a homomorphism by showing that every arc of D is mapped to an arc of H. Let uv be an arc of D and assume $f(u) = a_r$, $f(v) = a_s$. We show that $a_r a_s$ is an arc in H. Observe that $1 = u_r \leq v_{l^+(r)} \leq 1$ and $1 = v_s \leq u_{l^-(s)} \leq 1$, therefore we must have $v_{l^+(r)} = u_{l^-(s)} = 1$. Since r and s are the largest subscripts such that $u_r = v_s = 1$ then $l^+(r) \leq s$ and $l^-(s) \leq r$. Since $a_r a_{l^+(r)}$ and $a_{l^-(s)} a_s$ are arcs of H, we must have the arc $a_r a_s$, as H admits a min-max ordering.

Furthermore, $f(x) = a_i$ if and only if $x_i = 1$ and $x_{i+1} = 0$, so, $c(x, a_i)$ contributes to the sum if and only if $f(x) = a_i$. ■

We have translated the minimum cost homomorphism problem to an integer linear program. In fact, this linear program corresponds to a minimum cut problem in an auxiliary network, and can be solved by network flow algorithms [11, 19]. In [13], a similar result to Theorem 3.1 was proved for the MinHOM(H) problem on undirected graphs when target graph H is bipartite and admits a min-max ordering. We shall enhance the above system $\mathcal{S}$ to obtain an approximation algorithm for the case H is only assumed to have a min ordering as well as for the case where H admits a k-min ordering.

4 The Approximation Bound

In what follows, we describe our approximation algorithm for MinHOM(H) in the case that the fixed digraph H has a min ordering (i.e., is a bi-arc digraph (Theorem 2.2)). Indeed, we give $2|V(H)|$-approximation algorithm for the problem. Note that, since H is fixed the ratio is constant.

Theorem 4.1 *If digraph H admits a min ordering, then MinHOM(H) has a $2|V(H)|$-approximation algorithm.*

Proof. Suppose H has a min ordering $a_1, a_2, \cdots, a_p$ of its vertices. Let E' denote the set of all pairs (a_i, a_j) such that $a_i a_j$ is not an arc of H, but there is an arc $a_i a_{j'}$ of H with

$j' < j$ and an arc $a_{i'}a_j$ of H with $i' < i$. Let $E = E(H)$ and define H' to be the graph with vertex set $V(H)$ and arc set $E \cup E'$. (Note that E and E' are disjoint sets.)

Observation 4.2 *The ordering $a_1, a_2, \cdots, a_p$ is a min-max ordering of H'.*

Proof. We show that for every pair of arcs $e = a_i a_{j'}$ and $e' = a_{i'}a_j$ in $E \cup E'$, with $i' < i$ and $j' < j$, both $g = a_i a_j$ and $g' = a_{i'}a_{j'}$ are in $E \cup E'$. If both e and e' are in E, $g \in E \cup E'$ and $g' \in E$.

If only one of the arcs e, e', say e, is in E', there is a vertex $a_{j''}$ with $a_i a_{j''} \in E$ and $j'' < j'$, and a vertex $a_{i''}$ with $a_{i''}a_{j'} \in E$ and $i'' < i$. Now, $a_{i'}a_j$ and $a_i a_{j''}$ are both in E, so $g \in E \cup E'$. We may assume that $i'' \neq i'$, otherwise $g' = a_{i''}a_{j'} \in E$. If $i'' < i'$, then $g' \in E \cup E'$ because $a_{i'}a_{j''} \in E$; and if $i'' > i'$, then $g' \in E$ because $a_{i'}a_j \in E$.

If both edges e, e' are in E', then the earliest out-neighbor of a_i and earliest in-neighbor of a_j in E imply that $g \in E \cup E'$, and the earliest out-neighbors of $a_{i'}$ and earliest in-neighbor of $a_{j'}$ in E imply that $g' \in E \cup E'$. ■

Observation 4.3 *Let $e = a_i a_j \in E'$. Then a_i does not have any out-neighbor after a_j, or a_j does not have any in-neighbor after a_i.*

Proof of Observation 4.3 easily follows from the fact that we have a min ordering.

Since H' has a min-max ordering, we can form system $\mathcal{S}$ of linear inequalities for H' as described above. Homomorphisms of D to H' are in a one-to-one correspondence with integer solutions of $\mathcal{S}$. However, we are interested in homomorphisms of D to H, not H'. Therefore we shall add further inequalities to $\mathcal{S}$ to ensure that we only admit homomorphisms of G to H, i.e., avoid mapping arcs of D to the arcs in E'.

For every arc $e = a_i a_j \in E'$ and every arc $uv \in A(D)$, two of the following set of inequalities will be added to $\mathcal{S}$.

i. if a_s is the first in-neighbour of a_j after a_i, we add the inequality

$$v_j \leq u_s + \sum_{a_t a_j \in E,\ t<i} (u_t - u_{t+1})$$

ii. else if a_j has no in-neighbour after a_i, we add the inequality

$$v_j \leq v_{j+1} + \sum_{a_t a_j \in E,\ t<i} (u_t - u_{t+1})$$

iii. if a_s is the first out-neighbour of a_i after a_j, we add the inequality

$$u_i \leq v_s + \sum_{a_i a_t \in E,\ t<j} (v_t - v_{t+1})$$

iv. else if a_i has no out-neighbour after a_j, we add the inequality

$$u_i \leq u_{i+1} + \sum_{a_i a_t \in E} (v_t - v_{t+1}).$$

For every two vertices u, v of D and two vertices $a_i, a_j \in H$ add the following :

v. $u_i - u_{i+1} \leq \sum_{(a_i,a_j)\in L(u,v)} (v_j - v_{j+1})$

Claim 4.4 *There is a one-to-one correspondence between homomorphisms of D to H and integer solutions of the extended system $\mathcal{S}$.*

Proof. Suppose f is a homomorphism of D to H', obtained from an integer solution for $\mathcal{S}$, and, for some arc uv of D, let $f(u) = a_i, f(v) = a_j$. We show that $a_ia_j \in E$. If this is not the case then we have $u_i = 1$, $u_{i+1} = 0$, $v_j = 1$, $v_{j+1} = 0$, and for all $a_ta_j \in E$ with $t < i$ we have $u_t - u_{t+1} = 0$. If a_s is the first in-neighbor of a_j after a_i, then we will also have $u_s = 0$, and so the first inequality (i.) fails. Else if a_j has no in-neighbor after a_i, and the second inequality (ii.) fails. The remaining two other cases are similar.

Conversely, suppose f is a homomorphism of D to H (i.e., f maps the edges of D to the edges in E). We show that the inequalities hold. For a contradiction, assume that the first inequalities fails (the other inequalities are similar). This means that for some arc $uv \in A(D)$ and some edge $a_ia_j \in E'$, we have $v_j = 1$, $u_s = 0$, and the sum of $(u_t - u_{t+1}) = 0$, summed over all $t < i$ such that a_t is an in-neighbor of a_j. The latter two facts easily imply that $f(u) = a_i$. Since a_j has an in-neighbor after a_i, Observation 4.3 tells us that a_i has no out-neighbors after a_j, whence $f(v) = a_j$ and thus $a_ia_j \in E$, contradicting the fact that $a_ia_j \in E'$. Note that if there is a homomorphism from D to H then the last two inequality are necessary conditions for having such a homomorphism.

■

At this point, our algorithm will minimize the cost function over extended $\mathcal{S}$ in polynomial time using a linear programming algorithm. This will generally result in a fractional solution. (Even though the original system $\mathcal{S}$ is known to be totally unimodular [19] and hence have integral optima, we have added inequalities, and hence lost this advantage.) We will obtain an integer solution by a randomized procedure called *rounding*. We choose a random variable $X \in [0, 1]$, and define the rounded values $u'_i = 1$ when $u_i \geq X$, and $u'_i = 0$ otherwise; and similarly for v'_j. It is easy to check that the rounded values satisfy the original inequalities, i.e., correspond to a homomorphism f of D to H'.

Now the algorithm will once more modify the solution f to become a homomorphism of D to H, i.e., to avoid mapping edges of D to the edges in E'. This will be accomplished by another randomized procedure, which we call *shifting*. We choose another random variable $Y \in [0, 1]$, which will guide the shifting. Let F denote the set of all arcs in E' to which some arcs of D is mapped by f. If F is empty, we need no shifting. Otherwise, let a_ia_j be an arc of F with maximum sum $i + j$ (among all arcs of F). By the maximality of $i + j$, we know that a_ia_j is the last arc of F from both a_i and a_j. Since $F \subseteq E'$, Observation 4.3 implies that either a_j has no in-neighbor after a_i or a_i has no out-neighbor after a_j. Suppose the first case happens (the shifting process is similar in the other case).

We now consider, one by one, vertices v in D such that $f(v) = a_j$ (i.e. $v'_j = 1$) and v has an in-neighbor u in D with $f(u) = a_i$ (i.e. $u'_i = 1$). (Such vertices v exist by the definition of F.) For such a vertex v, consider the set of all vertices a_t with $t < j$ such that $a_ia_t \in E$.

This set is not empty, since $a_i a_j$ is in E' because of two arcs of E. Suppose the set consists of a_t with subscripts t ordered as $t_1 < t_2 < \cdots < t_k$. The algorithm now selects one vertex from this set as follows. Let $P_{v,t} = \frac{v_t - v_{t+1}}{P_v}$, where

$$P_v = \sum_{a_i a_t \in E,\ t<j} (v_t - v_{t+1}).$$

Then a_{t_q} is selected if $\sum_{p=1}^{q} P_{v,t_p} < Y \leq \sum_{p=1}^{q+1} P_{v,t_p}$. Thus a concrete a_t is selected with probability $P_{v,t}$, which is proportional to the difference of the fractional values $v_t - v_{t+1}$.

When the selected vertex is a_t, we shift the image of the vertex v from a_j to a_t. Note that a_t is before a_j in the min-ordering. Now we might need to shift image of the neighbor of v. First consider in-neighbors of v. By our assumption, a_j does not have an in-neighbor after a_i. Thus, no in-neighbor of v is mapped to a vertex after a_i. Let a_l be an in-neighbor of a_j before a_i, and z be an in-neighbor of v such that z is mapped to a_l. Because of min-ordering $a_l a_t$ is an edge of H and no shift is required for image of z. Second, consider out-neighbors of v. Let z be an out-neighbor of v whose image is a_l. Observe that a_t must have an out-neighbor $a_{t'}$ and we can map z to $a_{t'}$. This is because, by the choice of a_t, we have $0 < v_t - v_{t+1} \leq \sum_{(a_t, a_{t'}) \in L(v,z)} (z_{t'} - z_{t'+1})$, where the second inequality is because of inequality v. Therefore, there must be at least one out-neighbor for a_t. We map z to a_l, an out-neighbor of a_t with $(a_t, a_l) \in L(v, z)$, with probability

$$\frac{z_l - z_{l+1}}{\sum_{(a_t, a_{t'}) \in L(v,z)} (z_{t'} - z_{t'+1})}.$$

This may cascade a chain of shifting, in other words we may need to shift an image of (in-neighbor) out-neighbor of z, to some (in-neighbor) out-neighbor of a_t and so on. We claim that the shifting modifies homomorphism f.

Claim 4.5 *The shifting procedure modifies homomorphism f.*

Proof. It it easy to see that, if there exists a homomorphism from G to H, then there is a homomorphism from G to H that maps every vertex of G to the smallest vertex in its list **??**. We show that, a sequence of shiftings, either stops at some point, or it keeps shifting to a smaller vertex in each list. In the later case, after finite (polynomially many) steps, we end up mapping every vertex of G to the smallest vertex in its list.

Consider an edge $vz \in E(G)$. Suppose $f(v) = a_t$ and $f(z) = a_l$. Assume that we have shifted the image of v from a_t to $a_{t'} \in L(v)$ where $a_{t'}$ is before a_t in the min-ordering. If $a_{t'} a_l$ is in $E(H)$ then we do not have to shift the image of z. Note that, since $a_{t'}$ is in $L(v)$ then it has to have an out-neighbor in $L(z)$. Let say $a_{l'} \in L(z)$ is an out-neighbor of $a_{t'}$.

If $a_{l'}$ is after a_l in the min-ordering then by min-ordering $a_{t'}a_{l'} \in E(H)$. Else, $a_{l'}$ is before a_l in the min-ordering and we shift the image of z to a smaller vertex in its list.

■

Claim 4.5 shows that this shifting modifies the homomorphism f, and hence the corresponding values of the variables. Namely, $v'_{t+1}, \ldots, v'_j$ are reset to 0, keeping all other values the same. Note that these modified values still satisfy the original constraints, i.e., the modified mapping is still a homomorphism.

We repeat the same process for the next v with these properties, until $a_i a_j$ is no longer in F (because no edge of D maps to it). This ends the iteration on $a_i a_j$, and we proceed to the next edge $a_{i'} a_{j'}$ with the maximum $i' + j'$ for the next iteration. Each iteration involves at most $|V(D)|$ shifts. After at most $|E'|$ iterations, the set F is empty and we no longer need to shift.

We now claim that because of the randomization, the cost of this homomorphism is at most $|V(H)|$ times the minimum cost of a homomorphism. We denote by w the value of the objective function with the fractional optimum u_i, v_j, and by w' the value of the objective function with the final values u'_i, v'_j, after the rounding and all the shifting. We also denote by w^* the minimum cost of a homomorphism of D to H. Obviously we have $w \leq w^* \leq w'$.

We now show that the expected value of w' is at most a constant times w. We focus on the contribution of one summand, say $v'_t - v'_{t+1}$, to the calculation of the cost. (The other case, $u'_s - u'_{s+1}$, is similar.)

In any integer solution, $v'_t - v'_{t+1}$ is either 0 or 1. The probability that $v'_t - v'_{t+1}$ contributes to w' is the probability of the event that $v'_t = 1$ and $v'_{t+1} = 0$. This can happen in the following situations.

1. v is mapped to a_t by rounding, and is not shifted away. In other words, we have $v'_t = 1$ and $v'_{t+1} = 0$ after rounding, and these values don't change by shifting.

2. v is first mapped to some $a_j, j > t$, by rounding, and then re-mapped to a_t by shifting. This happens if there exist i and u such that uv is an arc of D mapped to $a_i a_j \in F$, and then the image of v is shifted to a_t, where $a_i a_t \in E$. In other words, we have $u'_i = v'_j = 1$ and $u'_{i+1} = v'_{j+1} = 0$ after rounding; and then v is shifted from a_j to a_t.

3. $uv \in D$ is mapped to arc $a_i a_j \in E(H)$, then the image of v is shifted to a_t because the image of u was shifted to $a_{t'}$.

For the situation in 1, we compute the expectation as follows. The values $v'_t = 1, v'_{t+1} = 0$ are obtained by rounding if $v_{t+1} < X \leq v_t$, i.e., with probability $v_t - v_{t+1}$. The probability that they are not changed by shifting is at most 1, whence this situation occurs with probability at most $v_t - v_{t+1}$, and the expected contribution is at most $c(v, t)(v_t - v_{t+1})$.

For the situation in 2, consider a fixed a_j for which there exist a_i and u as described above. We give an upper bound on the probability of shifting v from a_j to a_t. The values

$u_i' = v_j' = 1$ and $u_{i+1}' = v_{j+1}' = 0$ are obtained by rounding if X satisfies $\max\{u_{i+1}, v_{j+1}\} < X \le \min\{u_i, v_j\}$, i.e., with probability $\min\{u_i, v_j\} - \max\{u_{i+1}, v_{j+1}\} \le u_i - v_{j+1} \le u_i - v_s \le P_v$. In the last two inequalities, we assume that a_s is the first out-neighbor of a_i after a_j, and use the third inequality added above the Claim 4.4. If a_i has no out-neighbors after a_j, the proof is analogous, using the fourth added inequality.

Having uv mapped to $a_i a_j$ in the rounding step, we shift v to a_t with probability $P_{v,t} = \frac{(v_t - v_{t+1})}{P_v}$. Therefore, for a fixed a_j and a_i, the probability that v is shifted from a_j to a_t is at most $\frac{v_t - v_{t+1}}{P_v} \cdot P_v = v_t - v_{t+1}$. Hence, the expected contribution for a fixed j (with its i and u) to the cost function is also at most $c(v,t)(v_t - v_{t+1})$.

For the situation in 3, this type of shifting can happen in the chain of shifting we discussed above. In other words, this shift for v can happen after shifting u. Shifting of u can happen after shifting one of its neighbors and so on. We prove the desire bound by an inductive argument. The base case is when $v'u$ is mapped to an edge from F and we shift the image of u and then we are forced to shift the image of v. As we discussed in the second situation, the shifting for u happens with probability at most $u_{t'} - u_{t'-1}$. Now, the shifting for v happens with probability

$$(u_{t'} - u_{t'-1}) \cdot \frac{v_t - v_{t+1}}{\sum\limits_{(a_i,a_j)\in L(u,v)} (v_j - v_{j+1})}.$$

Recall that, by inequality v., $u_{t'} - u_{t'-1} \le \sum\limits_{(a_i,a_j)\in L(u,v)} (v_j - v_{j+1})$. It implies, the image of v is shifted to a_t with probability at most $v_t - v_{t+1}$. Suppose, at the kth step of a chain of shifting, u was shifted to $a_{t'}$ with probability at most $u_{t'} - u_{t'-1}$. By the same argument, one can see that v is shifted to a_t with probability at most $v_t - v_{t+1}$.

Let r denote the number of vertices of H, that are incident with some edges of E'. Clearly the situation in 2 can occur for at most r different values of j, and the situation in 3 can can occur for at most $|V(H)|$ different values of j. Therefore, a fixed v in D contributes at most $(1+r+|V(H)|)c(v,t)(v_t - v_{t+1})$ to the expected value of w'. Thus the expected value of w' is at most

$$(1 + r + |V(H)|) \sum_{v,j} c(v,j)(v_j - v_{j+1}) \le (1 + r + |V(H)|)w.$$

Since we have $w \le w^*$, this means that the expected value of w' is at most $(1+r+|V(H)|)w^*$. Note that $1 + r + |V(H)| \le 1 + 2|E'|$, and also $1 + r + |V(H)| < 2|V(H)|$ because a_1 are not incident with any edges of E' by definition.

At this point we have proved that our two-phase randomized procedure produces a homomorphism whose expected cost is at most $(1 + r + |V(H)|)$ times the minimum cost. It can be transformed to a deterministic algorithm as follows. There are only polynomially many values v_t (at most $|V(D)| \cdot |V(H)|$). When X lies anywhere between two such consecutive values, all computations will remain the same. Thus we can derandomize the first phase by trying all these values of X and choosing the best solution. Similarly, there are only polynomially many values of the partial sums $\sum_{i=1}^{p} P_{u,t_i}$ (again at most $|V(D)|\dot{|}V(H)|$), and when Y lies between two such consecutive values, all computations remain the same. Thus we can also derandomize the second phase by trying all possible values and choosing the best. Since the expected value is at most $(1 + r + |V((H)|)$ times the minimum cost, this bound also applies to this best solution. ■

Lemma 4.6 *The probability that any of situations 1,2 or 3 happens is at most* $v_t - v_{t+1}$.

5 Approximation for Extended Min-Ordering

Let $H = (V, H)$ be a digraph such that $V(H)$ can be partitioned into k disjoints subsets $V_0, V_1, \ldots, V_{k-1}$ such that every arc of H belongs to $V_i \times V_{i+1}$, for some $0 \leq i \leq k-1$ (subscript addition modulo k). A *k-min-max ordering* of H is a linear ordering $<$ of each set V_i, so that the min-max ordering ($u < w, z < v$ and $uv, wz \in A(H)$ imply that $uz, wv \in A(H)$) is satisfied for u, w and v, z in any two circularly consecutive sets V_i and V_{i+1} ($u, w \in V_i$, $v, z \in V_{i+1}$, subscript addition modulo k). It is known that MinHOM(H) is polynomially solvable if H admits k-min-max ordering and it is NP-complete otherwise [15].

In a similar way, a *k-min ordering* of H is a linear ordering $<$ of each set V_i, so that the min ordering ($u < w, z < v$ and $uv, wz \in A(H)$ imply that $uz \in A(H)$) is satisfied for u, w and v, z in any two circularly consecutive sets V_i and V_{i+1} ($u, w \in V_i$, $v, z \in V_{i+1}$, subscript addition modulo k).

Definition 5.1 *We say digraph $H = (V, A)$ is k-arc digraph iff each vertex v of V is represented by a pair of arcs I_v, J_v from a circle with $2k$ poles $N_0, N_1, \ldots, N_{k-1}, S_0, S_1, \ldots, S_{k-1}$ (in this order and in the clockwise direction) with the following conditions :*

- *Each arc I_v contains $S_{i+1}, S_{i+2}, \ldots, S_{k-1}, N_0, N_1, \ldots, N_i$ and no other poles and each arc J_v contains $N_{i+1}, N_{i+2}, \ldots, N_{k-1}, S_0, S_1, \ldots, S_i$ for some $0 \leq i \leq k-1$ and no other points.*

- *If the clockwise end of I_v is before the clockwise end of I_w then the clockwise end of J_v is before the clockwise end of J_w.*

- *If $uv \in A$ then I_u and J_v do not intersect.*

Theorem 5.2 *Digraph $H = (V, A)$ is a k-arc digraphs if and only if it admits a k-min ordering.*

Proof. Suppose H has a k-min ordering. Then there are disjoint sets $V_0, V_1, \ldots, V_{k-1}$ which they partition V and there is an ordering of the vertices in V_i, $0 \leq i \leq i+1$ where the min ordering property is satisfied for any two circularly consecutive sets V_i and V_{i+1} respectively. We consider a circle with $2k$ distinct poles $N_0, N_1, \ldots, N_{k-1}, S_0, S_1, \ldots, S_{k-1}$ in this ordering in the clockwise direction. Each vertex $u \in V_i$ is represented by an pair of arcs I_u, J_u where I_u contains the poles $S_{i+1}, S_{i+2}, \ldots, S_{k-1}, N_0, N_1, \ldots, N_i$ only and J_u contains the poles $N_{i+1}, N_{i+2}, \ldots, N_{k-1}, S_0, S_1, \ldots, S_i$ only. Now by using the same procedure described in Theorem 2.2 we can extend the counterclockwise end of I_u, J_u such that does not intersect I_w, J_w where $uw \in A$. Conversely, if there exists an arc representation of the vertices of H then we define V_i be the set of vertices whose arcs contain N_i and not containing N_{i+1}. We put u before u' in V_i if clockwise end of I_u is before the clockwise end of u' (in the clockwise direction). It is not difficult to verify the correlation between the adjacency of vertices and disjointness of the arcs. ■

For the case where H admits k-min-max ordering, the same LP as before works out. The only difference is in proving the correctness of inequalities $u_i \leq v_{l^+(i)}$ and $v_j \leq u_{l^-(i)}$ for every $uv \in A(D)$. For homomorphism $f : D \to H$, suppose $f(u) = a_r$, $f(v) = a_s$ with $a_r a_s \in V_j \times V_{j+1}$. If $a_i a_{l^+(i)} \in V_j \times V_{j+1}$, then an identical argument as above yields to $u_i \leq v_{l^+(i)}$. Otherwise, if $a_i a_{l^+(i)} \in V_{j'} \times V_{j'+1}$ with $j \neq j'$, having $u_i \leq v_{l^+(i)}$ does effect not the way we transfer homomorphism f to an equivalent integer solution of $\mathcal{S}$. The rest of the proof for the following theorem is the same as proof of Theorem 3.1.

Theorem 5.3 *Suppose H is a digraph that admits k-min-max ordering, then there is a one-to-one correspondence between homomorphisms of D to H and integer solutions of $\mathcal{S}$.*

6 Conclusion and Future Work

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