Machine learning HW 4

Understanding Machine Learning: From Theory to Algorithms[Exercises for chapters 10,11 and 18]

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chapter 10

Exercise 10.1:

$\epsilon, \delta \in (0,1)$ Pick k "chunks" of size $m_{\mathcal{H}}(\epsilon/2)$. Apply A on each of these chunks, to obtain $\hat{h}_1, \ldots, \hat{h}_k$ [1] Now, apply an ERM over $\hat{\mathcal{H}}$. Note that $\hat{\mathcal{H}} := \left\{\hat{h}_1, \ldots, \hat{h}_k\right\}$ with the training data being the last chunk of size $\left\lceil \frac{2}{\epsilon^2} . \log\left(\frac{4k}{\delta}\right) \right\rceil$ Denote the output hypothesis by $\hat{h}$.We also should claim that with probability at least $1-\delta/2$, $L_{\mathcal{D}}(\hat{h}) \leq \min_{i \in [k]} L_{\mathcal{D}}(h_i) + \frac{\epsilon}{2}$. Now we have: $L_{\mathcal{D}}(\hat{h}) \leq \min_{i \in [k]} L_{\mathcal{D}}(h) + \epsilon$

chapter 11

Exercise 11.1:

Consider a case in that the label is chosen at random according to $\mathbb{P}[y=1] = [y=0] = \frac{1}{2}$ Consider a learning algorithm that outputs the constant predictor $h(x) = 1$ if the parity of the labels on the training set is 1 and otherwise the algorithm outputs the constant predictor $h(x) = 0$. Prove that the difference between the leave-one-out estimate and the true error in such a case is always $\frac{1}{2}$.

first consider S set as a i.i.d sample

we know h as the out put of learning algorithm.

because h is a constant function we have $L_D(h) = \frac{1}{2}$ we want to calculate the $L_V(h)$. assume the parity of S is 1

then fix some fold $\{(x,y)\} \subseteq S$

we have two cases bellow:

- as the $S \setminus \{X\}$ is 1 so $Y = 0$ and since trained using $S \setminus \{X\}$ the algorithm oytputs the predictor $h(x) = 1$

  therefor the leave-one-out estimate using this fold is 1.

- as the $S \setminus \{X\}$ is 0 so $Y = 1$ and since trained using $S \setminus \{X\}$ the algorithm outputs the predictor $h(x) = 0$

  Therefore the leave-one-out estimate using this fold is 1.

  after averaging the two folds, we calculate, we find out that the estimated error of h is 1. and the difference between estimation error and the true error is $\frac{1}{2}$.

  for the other case (the parity be 0) analyze in the same way.

Exercise 11.2:

consider $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots \subseteq \mathcal{H}_k$ and also, $\forall i \in k, \quad |\mathcal{H}_i| = 2^i$.Learning $\mathcal{H}_k$ in the Agnostic-Pac model provides the following bound for an ERM hypothesis h:

$$L_{\mathcal{D}}(h) \leq \min_{h \in \mathcal{H}_k} L_{\mathcal{D}}(h) + \left(\frac{2}{m}.(k+1+\log(\frac{1}{\delta})\right)^{\frac{1}{2}}$$

[1]Note that the probability that $\min_{i \in [k]} L_{\mathcal{D}}(h) \leq \min L_{\mathcal{D}}(h) + \frac{\epsilon}{2}$ is at least $1 - \delta_0^k \geq 1 - \frac{\delta}{2}$

Alternatively, we can use model selection as we describe next. let us to assume that j is the minimal index which contains a hypothesis $h^* \in \text{argmin}_{h\in\mathcal{H}} L_{\mathcal{D}}(h)$. In tgis stage we try to fix some $r \in [k]$. Now according to Hoeffding's inequality, with probability at least $\frac{1-\delta}{2k}$, we have:

$$|L_{\mathcal{D}}(\hat{h}_r) - L_V(\hat{h}_r)| \leq \left(\frac{\log(4/\delta)}{2\alpha m}\right)^{\frac{1}{2}}$$

by applying the union bound we can claim that, In particular, with probability at least $1-\delta$ we have $L_{\mathcal{D}}(\hat{h}) \leq L_{\mathcal{D}}(\hat{h}_j) + \sqrt{\frac{2.\log(4k/\delta)}{\alpha m}}$ Using similar arguments, we obtain that with probability at least $1-\delta/2$,

$$L_{\mathcal{D}}(\hat{h}_j) \leq L_{\mathcal{D}}(h^*) + \sqrt{\frac{2\log(4|\mathcal{H}_j|/\delta)}{m - m\alpha}}$$

Combining the two last inequalities with the union bound, we obtain that with probability at least $1-\delta$:

$$L_{\mathcal{D}}(\hat{h}) \leq L_{\mathcal{D}}(h^*) + \sqrt{\frac{2\log(4k/\delta)}{\alpha m}} + \sqrt{\frac{2(j+\log(4/\delta))}{(1-\alpha)m}}$$

Comparing the two bounds, we see that when the "optimal index" j is significantly smaller than k, the bound achieved using model selection is much better. Being even more concrete, if j is logarithmic in k, we achieve a logarithmic improvement.

chapter 18

Exercise 18.2:

in first iteration we compute the information gain:

$$H(Y) = -\frac{1}{2}\log(\frac{1}{2}) - \frac{1}{2}\log(\frac{1}{2}) = 1$$

$$\begin{gathered}IG(X_1) = H(Y) - H(Y|X_1)\\ = 1 - \left[\left(\frac{3}{4}\right)\left(\left(-\frac{2}{3}\right)\log\left(\frac{2}{3}\right)\right) - \frac{1}{3}\log\frac{1}{3} + \frac{1}{4}(-0\log 0 - 1\log 1)\right]\\ = 1 - \left(\frac{3}{4}\right)\left[-\frac{2}{3}\log\left(\frac{2}{3}\right) - \frac{1}{3}\log\left(\frac{1}{3}\right)\right] > 0\end{gathered}$$

$$\begin{gathered}IG(X_2) = H(Y) - H(Y|X_2)\\ = 1 - \left[\left(\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)\log\left(\frac{1}{2}\right)\right) - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right)\right]\\ = 1 - \left[\frac{1}{2}(-1) + \frac{1}{2}(-1)\right] = 0\end{gathered}$$

$$\begin{gathered}IG(X_3) = H(Y) - H(Y|X_3)\\ = 1 - \left[\left(\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)\log\left(\frac{1}{2}\right)\right) - \frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2}\right)\right]\\ = 1 - \left[\frac{1}{2}(-1) + \frac{1}{2}(-1)\right] = 0\end{gathered}$$

so we choose $X_1 = 0$ for begin the tree. 1. for choosing the left node:

$$ID3\left(\left\{\left(\left(1,1,1\right),1\right),\left(\left(1,0,0\right),1\right),\left(\left(1,1,0\right),0\right)\right\},\left\{x_2,x_3\right\}\right)$$

we have to compute the info. gain again

$$H(Y)=-\frac{2}{3}\log(\frac{2}{3})-\frac{1}{3}\log(\frac{1}{3})$$

$$\begin{gathered}IG(X_2)=H(Y)-H(Y|X_2)\\=H(Y)-\left[\frac{2}{3}\left(-\frac{1}{2}\log\frac{1}{2}\right)-\frac{1}{2}\log\frac{1}{2}\right)\\=H(Y)-\frac{2}{3}\end{gathered}$$

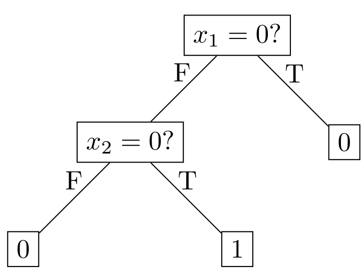
$$\begin{gathered}IG(X_3)=H(Y)-H(Y|X_3)\\=H(Y)-\left[\frac{2}{3}\left(-\frac{1}{2}\log\frac{1}{2}\right)-\frac{1}{2}\log\frac{1}{2}\right)\\=H(Y)-\frac{2}{3}\end{gathered}$$

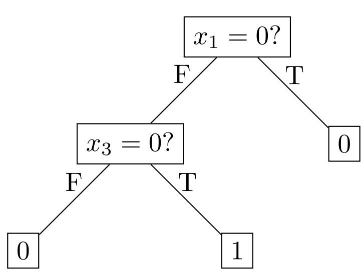
its possible to choose either X_2 or X_3 to have 2 different trees.

2. right node:

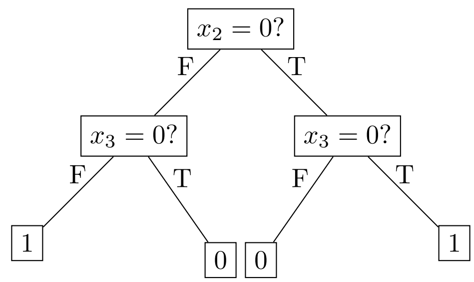
$$ID3\left(\left\{\left(\left(0,0,1\right),0\right)\right\},\left\{x_2,x_3\right\}\right)$$

the only possible label is 0. training error for FIRST tree is $\frac{1}{4}$ because the only mislabeled point is $\left(\left(1,1,1\right),1\right)$





and the training error for the second tree is also $\frac{1}{4}$ because the only mislabeled point is $((1,0,0),1)$
so the training error for any tree with the 2 depth with ID3 is at least $\frac{1}{4}$
We want to show the decision tree with the 0 training error.



$x_2 = 0?$
F
T
$x_3 = 0?$
$x_3 = 0?$
F
T
F
T
1
0
0
1