Graph convexity partizan games: complexity and winning strategies

Samuel N. Araújo[1,2], J Marcos Brito[2], Raquel Folz[3], Rosiane de Freitas[3], and Rudini Sampaio[2]

[1] Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Crato, Brazil
`samuel.araujo@ifce.edu.br`

[2] Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil
`jmbmaluno@alu.ufc.br, rudini@dc.ufc.br`

[3] Instituto de Computação, Universidade Federal do Amazonas, Manaus, Brazil
{`raquel.folz,rosiane`}`@icomp.ufam.edu.br`

Abstract. The first paper of convexity on general graphs, in English, is the paper "Convexity in graphs", published in 1981. One of its authors, Frank Harary, introduced in 1984 the first graph convexity games, focused on the geodesic convexity, which are impartial games and were investigated in a sequence of five papers until 2003. Only in 2023 the first PSPACE-hardness result on impartial convexity games were proved. In this paper, we introduce the partizan variants of these impartial games on the geodesic convexity and extend them to other graph convexities, obtaining winning strategies and complexity results. Among them, we obtain winning strategies for general convex geometries and winning strategies for trees from Conway's combinatorial game theory on partizan games. We also prove that the normal play and the misère play of the partizan hull game on the geodesic convexitiy is PSPACE-complete even in graphs with diameter two.

Keywords: Graph convexity, geodesic games, PSPACE-hard, Combinatorial game theory.

1 Introduction

In 1984, Harary introduced the first graph convexity games in his abstract "Convexity in graphs: achievement and avoidance games" [16]. This games are impartial and were investigated in a sequence of five papers [4, 5, 16, 17, 20] until 2003, all of them focused on the geodesic convexity. Only in 2023 the first PSPACE-hardness results on impartial convexity games were proved [1]. In this paper, we introduce the partizan variants of these impartial games on the geodesic convexity and extend them to other graph convexities, obtaining complexity and algorithmic results. In order to explain them, we need some terminology.

All graphs here are simple and finite. Let $N_G(v)$ be the set of neighbors of v in a given graph G and $N_G[v] = N_G(v) \cup \{v\}$. We may omit the subscript when G is clear from the context. Given a graph G and a set $S \subseteq V(G)$, let the *geodesic interval* $\mathrm{I_g}(S)$ be the set S and every vertex in a shortest path between two vertices of S. We say that S is *convex in the geodesic convexity* [13, 14] if $\mathrm{I_g}(S) = S$. The *geodesic convex hull* of S is the minimum convex set $\mathrm{hull_g}(S)$ containing S. It is known that $\mathrm{hull_g}(S)$ can be obtained by applying $\mathrm{I_g}(\cdot)$ from S until obtaining a convex set.

We define below the main games introduced by Harary [4, 5, 16] for the geodesic convexity. Under *normal play* (or *achievement variant*), the first player unable to move loses the game. Under *misère play* (or *avoidance variant*), the first player unable to move wins the game. From the classical Zermelo-von Neumann theorem [22], one of the two players has a winning strategy in each one of these games, since they are finite perfect-information games without draw. So, the decision problem of these games is whether Alice has a winning strategy.

Definition 1. *Let G be a graph. In the games defined below, the set L of labeled vertices is initially empty and $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex v which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.*

– *In the geodesic hull game* $\mathrm{HG_g}$*: $f_1(L) = L$ and $f_2(L) = \mathrm{hull_g}(L)$.*
– *In the geodesic interval game* $\mathrm{IG_g}$*: $f_1(L) = L$ and $f_2(L) = \mathrm{I_g}(L)$.*
– *In the geodesic closed hull game* $\mathrm{CHG_g}$*: $f_1(L) = f_2(L) = \mathrm{hull_g}(L)$.*
– *In the geodesic closed interval game* $\mathrm{CIG_g}$*: $f_1(L) = f_2(L) = \mathrm{I_g}(L)$.*

A two-person combinatorial game under the normal or misère play convention is *impartial* if the set of moves available from any position (or configuration) is the same for both players. As an example, the Nim game and the Hackenbush game are very important in the Sprague-Grundy theory on impartial games. It is easy to see that the games of Definition 1 are impartial games.

In 1985, the normal and misère variants of the geodesic interval game were solved for cycles, wheels and complete bipartite graphs [5]. In 1988, the result of [5] regarding wheel graphs was improved in [20] and, in 2003, Haynes et al. [17] obtained results for trees, complete multipartite graphs and block graphs in the normal and misère variants. In 2023, Araújo et al. [1] also extended these games to any graph convexity and obtained complexity and algorithmic results.

In this paper, we introduce in Section 2 the natural partizan variants of the impartial convexity games of Definition 1. We obtain winning strategies in the class of Ptolemaic graphs to the geodesic games PARTIZAN $\mathrm{IG_g}$ and PARTIZAN $\mathrm{HG_g}$. Regarding the closed games PARTIZAN $\mathrm{CIG_g}$ and PARTIZAN $\mathrm{CHG_g}$ on trees, we obtain a polynomial time algorithm based on Conway's combinatorial game theory. Finally, we prove that the normal and the misère variants of PARTIZAN $\mathrm{HG_g}$ and PARTIZAN $\mathrm{CHG_g}$ are PSPACE-complete. We also extend these games to any graph convexity and obtain a general result on convex geometries in partizan convexity games.

2 Partizan convexity games

We say that a two-person combinatorial game under the normal or the misère play convention is *partizan* if it is not impartial. A rich combinatorial game theory on partizan games was developed by John H. Conway in two classical books, "*On numbers and games*" [7] and "*Winning ways for your mathematical plays*" [3], in which impartial games such as Nim and Hackenbush play a central role. Many partizan games are defined by partitioning the set of possible moves of an impartial game in two sets, each of which corresponds to the possible moves of each player. As an example, in the Blue-Red Hackenbush game defined below, the segments of the impartial Hackenbush game are colored either blue or red and one player called Blue can only cut blue segments and the other player called Red can only cut red segments. From this beautiful combinatorial game theory on partizan games, we introduce the following partizan convexity games, which are the natural partizan variants of the impartial convexity games of Definition 1.

Definition 2. *Let Ψ be a game of Definition 1. Let* PARTIZAN Ψ *be the game obtained from the following modifications on Ψ: the vertices of the graph G are colored either A or B, and Alice (resp. Bob) can only label vertices colored A (resp. B). Also let* SIMPLIFIED PARTIZAN Ψ *be defined by the following: in addition to the colored graph G, the instance contains a vertex v of G which is already labeled at the beginning of the game.*

We show an interesting relation between the classical Blue-Red Hackenbush and the games SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$. In the Blue-Red Hackenbush game, there are points connected by line segments that are colored either blue or red. See Figure 1 for some examples. Some of the points are on a "ground line" represented in green. Two players, Blue and Red, alternately play by cutting segments. Blue (resp. Red) can only cut one blue (resp. red) segment per turn. The cut segment is deleted together with any other segments that are no longer connected to the ground. Following the normal play convention, the first player unable to move loses. A Blue-Red Hackenbush position can be winning for either Blue, or Red, or the second player. No position is winning for the first player, independently if the first player is Blue or Red, which would be called a "*fuzzy*" position. That is, Blue-Red Hackenbush is a non-fuzzy partizan game.

Conway associates a dyadic number (rational number whose denominator is a power of two) to each Blue-Red Hackenbush position in terms of advantage (spare moves) for Blue. A position in which the first player always loses has value zero. Positive values are winning for Blue and negative values are winning for Red. Unlike impartial games, which uses the bitwise xor operation in the Sprague-Grundy theory, the value of a Blue-Red Hackenbush position consisting of n disjoint positions (a move in one of them does not affect the others) is the sum of their values.

For example, the position (a) of Figure 1 has value 0, since the first player loses, independently if the first player is Blue or Red. The position (b) has value -1, since Blue loses immediately and Red has 1 spare move. On the other hand, the positions (c) and (d) have value 1/2 each, since the sum of the positions (b)+(c)+(d) gives a position with value 0: the first to play loses. Finally, the positions (f) and (g) have value 1/4 each, since position (e) has value -1/2 and the sum of the positions (e)+(f)+(g) gives a position with value 0.

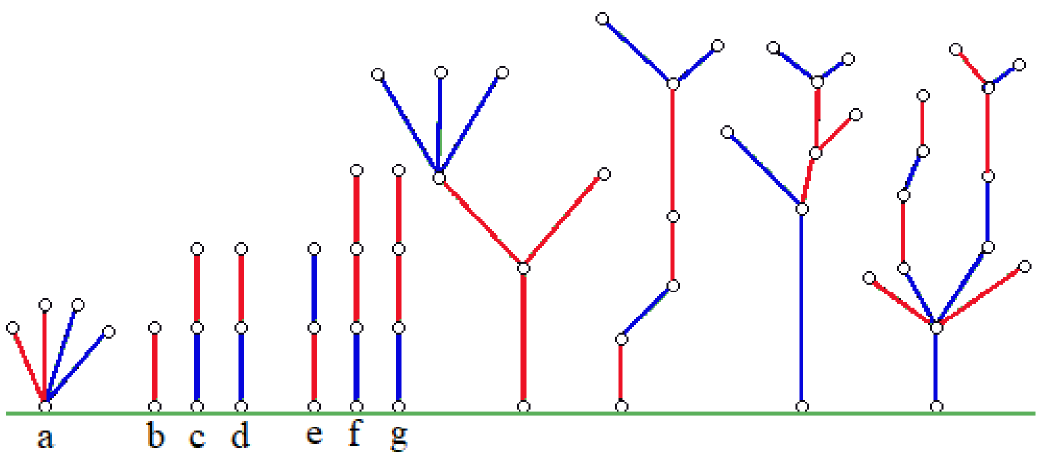


Fig. 1. Blue-Red Hackenbush positions.

The following lemma shows an interesting relation between the Blue-Red Hackenbush game and SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$. A subdivided star is the tree obtained by subdividing each edge of a star $K_{1,p}$ as many times as we want. See Figure 2 for an example.

Lemma 1. *Let G a subdivided star with center c whose vertices (except c) are colored A or B. Then the value of* SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ *and* $\mathrm{CIG_g}$ *on (G, c) is the value of Blue-Red Hackenbush on the disjoint union of strings, where each string represents a maximal path on G starting at c such that the k-th first segment is blue if and only if the k-th last vertex of the path is colored A.*

Proof (sketch). In a Hackenbush string, if the k-th first segment is cut, all the segments from the k-th to the last are deleted. In SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$ on (P_{n+1}, v_{n+1}), where the path P_n has vertices $v_1, \dots, v_{n+1}$, if v_k is labeled, then the vertices in the path $v_k, \dots, v_{n+1}$ can no longer be labeled in the next rounds. This shows the equivalence between strings in Hackenbush and paths in SIMPLIFIED PARTIZAN CIG_g. " Then the game on G is a disjoint union of the games on the branches, and we are done. □

As an example, consider SIMPLIFIED PARTIZAN CIG_g in the subdivided star G of Fig 2, where the black vertex is already labeled at the beginning, and its equivalent Hackenbush position with value $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}-1=0$. Note that the first player loses, independently if it is Alice or Bob.

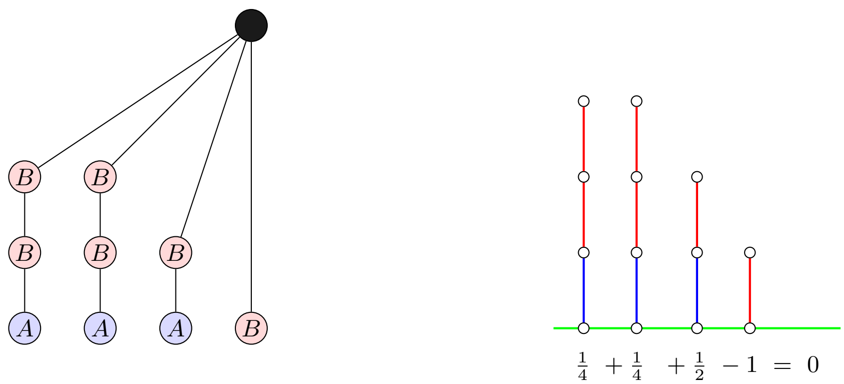


Fig. 2. SIMPLIFIED PARTIZAN CIG_g with the black vertex already labeled at the beginning. Instance with value 0: the first to play loses. The equivalent Hackenbush position is shown (Alice is associated with blue and Bob with red).

This relation between SIMPLIFIED PARTIZAN CIG_g and Blue-Red Hackenbush works for subdivided stars, but is not valid for any tree. Moreover, unlike Blue-Red Hackenbush positions, a position of the game PARTIZAN CIG_g can be fuzzy, that is, winning for the first player whoever it is. For example, in the cycle C_4 with vertices v_1, v_2, v_3, v_4 where v_1 and v_3 are colored A and v_2 and v_4 are colored B, the first to play wins in all the games PARTIZAN IG_g, PARTIZAN HG_g, PARTIZAN CIG_g and PARTIZAN CHG_g.

Despite this, the theorem of the next section shows that PARTIZAN CIG_g and PARTIZAN CHG_g positions cannot be fuzzy in trees and solves these games in those graphs, which are not as easy as Blue-Red Hackenbush in trees.

3 Partizan convexity games in trees

In order to deal with the games PARTIZAN CIG_g and PARTIZAN CHG_g in trees, we need to talk about *surreal numbers*, which were created in 1969 by John H. Conway and were introduced to the public for the first time in the 1974 mathematical novelette "*Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*" of Donald Knuth [19].

Instead of defining them in all their generality, let us focus on surreal numbers obtained from a Blue-Red Hackenbush position. The main notation has the form $\{B|R\}$ with $B < R$, where B is the greatest possible value of a position obtained after a move of Blue and R is the smallest possible value of a position obtained after a move of Red. Recall that every position of Blue-Red Hackenbush is associated to a dyadic number. Other possible notations are $\{B|\cdot\}$ and $\{\cdot|R\}$ with

$B \geq 0$ and $R \leq 0$, where $\cdot$ means *empty* (the player has no move). For example, the simplest notations include $\{\cdot|\cdot\} = 0$ (no player can move and then the first loses), $\{n|\cdot\} = n+1$ ($n+1$ blue segments and no red segment), $\left\{0 \;\middle|\; \frac{1}{2^{k-1}}\right\} = \frac{1}{2^k}$ (a line with 1 blue segment joined to the ground followed by k red segments, as in Figure 1(f)) and $\{n|n+1\} = n + \{0|1\} = n + \frac{1}{2}$ (n blue segments plus a position (c) of Figure 1).

It is shown in [3,7] how to determine the exact value of $\{B|R\}$, which satisfies $B < \{B|R\} < R$, using the so called "*simplicity rule*", and how to calculate recursively the value of a Hackenbush position using the values of subpositions. For Hackenbush trees (as the positions of Figure 1), this procedure can be done in polynomial time. However, calculating the value of a general Hackenbush position is NP-hard [3], and it is still an open problem if Blue-Red Hackenbush is PSPACE-hard.

Theorem 1. *In trees,* PARTIZAN $\mathrm{CIG_g}$ *and* $\mathrm{CHG_g}$ *are non-fuzzy and are polynomial time solvable.*

Proof (sketch). Let T be a tree. As in the proof of Thm 1, we may consider that a vertex in a path between two labeled vertices of T is also labeled. First consider SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v) where v is a vertex of T. If $V(T) = \{v\}$, the value of this position is zero: the first to play loses. So let u be a neighbor of v. We write (T, v, u) for the instance $(T_{v,u}, v)$ of SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$, where $T_{v,u}$ is the subtree of T containing v obtained after the removal of all edges of v except vu. Note that v is a leaf of $T_{v,u}$. Let us first solve the game for the instance (T, v, u).

If u has no neighbor, the value is $\{0|\cdot\} = 1$ (resp. $\{\cdot|0\} = -1$) if u is colored A (resp. B). Let $u_1, \ldots, u_k$ be the neighbors of u distinct from v. Assume, by induction, we know the values $\{A_i|B_i\}$ of (T, u, u_i) in SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$, where positive (resp. negative) values are winning for Alice (resp. Bob). Recall that $A_i < \{A_i|B_i\} < B_i$ for any i with non-empty A_i and B_i.

Let us determine the greatest value A_{vu} after an Alice's move on (T, v, u). If no vertex of (T, v, u) is colored A, then Alice has no move and then A_{vu} is empty. If u has color A, then the best option for Alice is to label vertex u, since $\{A_i|B_i\} > A_i$ for every i, and consequently the value $A_{vu} = \sum_{i=1}^{k} \{A_i|B_i\}$, since the games in the subtrees are independent after this move. Otherwise, assuming that u has color B, Alice must play in a vertex $w_j \neq u$ in a subtree (T, u, u_j), also labeling the vertex u, and consequently

$$A_{vu} \;=\; \max_{j=1}^{k} \left\{ A_j - \{A_j|B_j\} + \sum_{i=1}^{k} \{A_i|B_i\} \;:\; A_j \text{ is not empty} \right\}$$

Now let us determine the smallest value B_{vu} after a Bob's move on (T, v, u). If no vertex of (T, v, u) is colored B, then Bob has no move and then B_{vu} is empty. If u has color B, then Bob labels u, since $\{A_i|B_i\} < B_i$ for every i, and the value $B_{vu} = \sum_{i=1}^{k} \{A_i|B_i\}$. Otherwise, assuming that u is colored A, Bob must play in a subtree (T, u, u_j), also labeling the vertex u, and consequently

$$B_{vu} \;=\; \min_{j=1}^{k} \left\{ B_j - \{A_j|B_j\} + \sum_{i=1}^{k} \{A_i|B_i\} \;:\; B_j \text{ is not empty} \right\}$$

From this, we solved SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v, u) by calculating $\{A_{vu}|B_{vu}\}$. Regarding SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v), note that the game on the subtrees of v are independent and then it is possible to calculate the value $\{A_v|B_v\} = \sum_{u \in N(v)} \{A_{vu}|B_{vu}\}$.

On PARTIZAN $\mathrm{CIG_g}$ on T, let us calculate the best value A_T for Alice and B_T for Bob. If no vertex of T is colored A (resp. B), then A_T (resp. B_T) is empty. Otherwise, $A_T = \max_{v \in V(G)} \{A_v|B_v\}$ and $B_T = \min_{v \in V(G)} \{A_v|B_v\}$. This leads to a polytime recursive algorithm. □

4 PSPACE-hardness of partizan hull games

In this section, we prove that the normal and misère variants of the partizan hull games PARTIZAN HG_g and PARTIZAN CHG_g and their simplified versions are PSPACE-complete (see Definitions 2). As mentioned before, we consider the games as decision problems: given a graph, does Alice have a winning strategy? Since the number of turns is at most n and, in each turn, the number of possible vertices to label is at most n, all these games are polynomially bounded two player games, which implies that they are in PSPACE [18].

One of the main difficulties in proving PSPACE-hardness is to find a suitable PSPACE-hard problem for the reduction, since many more problems are known to be NP-hard than PSPACE-hard. Fortunately, PARTIZAN NODE KAYLES and PARTIZAN CLIQUE FORMING, which are PSPACE-complete [21], turned out to be very useful for both reductions. In these games, Alice and Bob (starting with Alice) alternately select vertices of a graph G, whose vertices are colored either A or B. Alice (resp. Bob) can only select vertices with color A (resp. B). In PARTIZAN NODE KAYLES, the selected vertices during the game must induce an independent set. In the PARTIZAN CLIQUE FORMING game, the selected vertices during the game must induce a clique.

Recall that, in the misère variant, the first player unable to move wins the game.

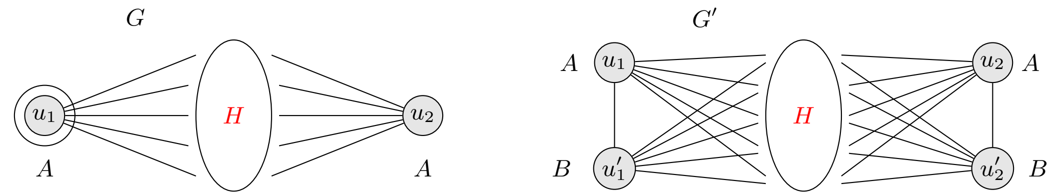
Theorem 2. *The misère variants of* PARTIZAN HG_g *and* SIMPLIFIED PARTIZAN HG_g *are PSPACE-complete even in graphs with diameter two.*

Proof (sketch). Let H be an instance of the PARTIZAN CLIQUE FORMING game: a graph whose vertices are colored either A or B. We may assume that H is not complete. We first obtain a reduction for SIMPLIFIED PARTIZAN HG_g.

Let G be the graph obtained from H by adding two non-adjacent new vertices u_1 and u_2 with color A which are adjacent to all vertices of H. See Figure 3(a). Notice that G has diameter 2. Also let u_1 be the vertex which is already labeled in SIMPLIFIED PARTIZAN HG_g. We prove that Alice has a winning strategy in the PARTIZAN CLIQUE FORMING game on H if and only if she has a winning strategy in the misère variant of SIMPLIFIED PARTIZAN HG_g on (G, u_1). If Alice labels u_2, she loses immediately, since $\text{hull}_g(\{u_1, u_2\}) = V(G)$. Moreover, if a player labels a vertex v_j of H in the hull game HG_g and there is a non-adjacent labeled vertex v_i in H, then the player loses immediately, since $\text{hull}_g(\{v_i, v_j\}) = V(G)$. So, we may assume that the set L of labeled vertices form a clique in all turns, except the last one. This is directly related to the PARTIZAN CLIQUE FORMING game on H. If Alice has a winning strategy in the PARTIZAN CLIQUE FORMING game on H, then Bob is the first to label a vertex of G with a labeled non-neighbor, implying that he loses the misère variant of SIMPLIFIED PARTIZAN HG_g. Analogously, if Bob has a winning strategy in the PARTIZAN CLIQUE FORMING game on H.

Now we obtain a reduction for PARTIZAN HG_g. Let the graph G' be obtained from G by adding two new vertices u'_1 and u'_2 with color B, adjacent to all vertices of H, and the edges u'_1u_1 and u'_2u_2. See Fig 3(b). Note that G' has diameter 2. If Alice has a winning strategy in PARTIZAN CLIQUE FORMING on H, then she plays PARTIZAN HG_g in G' using her strategy in H, unless Bob selects u'_1 (resp. u'_2) causing Alice to select u_1 (resp. u_2). Analogously if Bob has a winning strategy in the PARTIZAN CLIQUE FORMING game on H. If there are two non-adjacent selected vertices during the game, then PARTIZAN HG_g is over and the last to play loses. Then, as before, the set L of labeled vertices form a clique in all turns, except the last one, and we are done. □

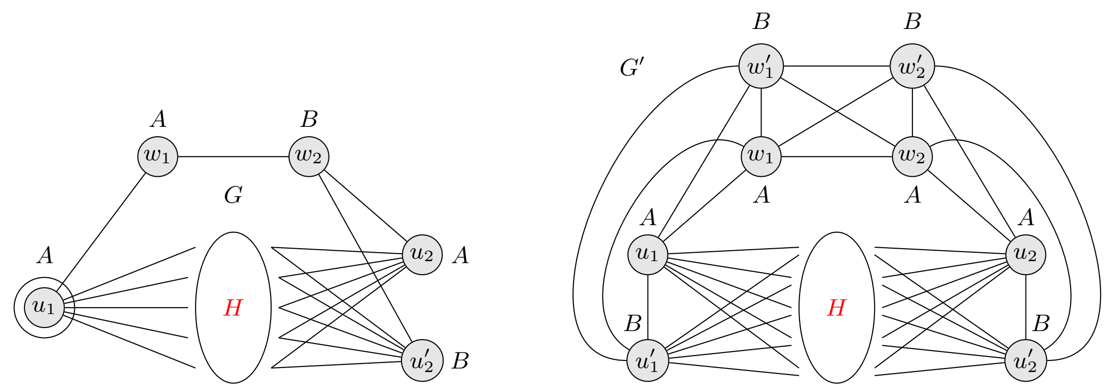
Theorem 3. *The normal variants of* PARTIZAN HG_g *and* SIMPLIFIED PARTIZAN HG_g *are PSPACE-complete even in graphs with diameter two.*



(a) Graph G of the reduction to SIMPLIFIED PARTIZAN $\mathrm{HG_g}$

(b) Graph G' of the reduction to PARTIZAN $\mathrm{HG_g}$

Fig. 3. Graphs of the reductions in Theorem 2 on the misère variants.



(a) Graph G of reduction to SIMPLIFIED $\mathrm{HG_g}$

(b) Graph G' of reduction to $\mathrm{HG_g}$

Fig. 4. Graphs of the reductions in Theorem 3 on the normal variants.

Proof (sketch). Similar to the proof of Theorem 2, but using the gadgets of Figure 4. ◻

It is not difficult to check that the arguments in the proofs of Theorems 2 and 3 are also valid for PARTIZAN $\mathrm{CHG_g}$.

Corollary 1. *The normal and misère variants of the hull games* PARTIZAN $\mathrm{HG_g}$, PARTIZAN $\mathrm{CHG_g}$ *and their simplified versions are PSPACE-complete even in graphs with diameter two.*

5 Partizan convexity games in Ptolemaic graphs

Given a graph G and a convex set S of G in the geodesic convexity, a vertex $v \in S$ is *extreme* in S if $S \setminus \{x\}$ is also convex. The *extreme vertices of the graph* G are the extreme vertices of $V(G)$. In 1986, Farber and Jamison [15] proved that a vertex is extreme in the geodesic convexity of a graph if and only if it is simplicial (its closed neighborhood forms a clique). Let $\mathrm{Ext_g}(G)$ be the set of extreme vertices of G in the geodesic convexity, which are the simplicial vertices.

Farber and Jamison [15] also defined a graph convexity as *geometric* (also called *convex geometry*) if it satisfies the *Minkowski-Krein-Milman* property: every convex set is the convex hull of its extreme vertices. The research on graph convex geometries is usually concentrated in determining the graph class in which a given graph convexity is geometric. They proved that the geodesic convexity is geometric in a graph if and only if the graph is Ptolemaic [15], that is, it is chordal (every induced cycle is a triangle) and distance hereditary (every induced path is a shortest path).

Ptolemaic graphs are also defined as the graphs satisfying the Ptolemy's inequality: for any four vertices v_1, v_2, v_3, v_4, we have $d(v_1, v_2)d(v_3, v_4) + d(v_1, v_4)d(v_2, v_3) \geq d(v_1, v_3)d(v_2, v_4)$, where $d(x, y)$ is the distance between two vertices x and y in the graph. They are also the gem-free chordal graphs, where the gem is the graph obtained by adding a universal vertex in the P_4 path.

In this section, we use these results to obtain winning strategies for PARTIZAN $\mathrm{HG_g}$ and PARTIZAN $\mathrm{IG_g}$ on Ptolemaic graphs. We first prove a general result on the geodesic convexity. Let $\mathrm{Ext}_\mathrm{g}^A(G)$ and $\mathrm{Ext}_\mathrm{g}^B(G)$ be the subsets of vertices of $\mathrm{Ext_g}(G)$ colored A and B, respectively, when the vertices of G are colored either A or B.

Theorem 4. *Let G be a graph whose vertices are colored either A or B such that $\mathrm{Ext_g}(G)$ is a geodesic hull (resp. interval) set. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN $\mathrm{HG_g}$ *(resp.* PARTIZAN $\mathrm{IG_g}$*) if and only if $\mathrm{Ext}_\mathrm{g}^B(G) = \emptyset$, or $\mathrm{Ext}_\mathrm{g}^A(G) \neq \emptyset$ and $|V_A| > |V_B|$. Moreover Alice wins the misère variant of* PARTIZAN $\mathrm{HG_g}$ *(resp.* PARTIZAN $\mathrm{IG_g}$*) if and only if $|\mathrm{Ext}_\mathrm{g}^A(G)| \leq |\mathrm{Ext}_\mathrm{g}^B(G)|$.*

Proof (sketch). Recall that $V(G) \setminus \{s\}$ is convex for every $s \in \mathrm{Ext_g}(G)$. That is, there are two options for the end of the games PARTIZAN $\mathrm{HG_g}$ and PARTIZAN $\mathrm{IG_g}$: either a player has no move or every vertex of $\mathrm{Ext_g}(G)$ was labeled.

Let us first consider the normal variant. If every vertex of $\mathrm{Ext_g}(G)$ has color A (resp. B), then Alice (resp. Bob) wins since she (resp. he) is the last to select a vertex of $\mathrm{Ext_g}(G)$. So assume that there is a vertex colored A and a vertex colored B in $\mathrm{Ext_g}(G)$. If $\mathrm{Ext_g}(G)$ is a geodesic hull set (resp. geodesic interval set), then PARTIZAN $\mathrm{HG_g}$ (resp. PARTIZAN $\mathrm{IG_g}$) ends when the last vertex of $\mathrm{Ext_g}(G)$ is labeled or one of the players has no move. Then Alice (resp. Bob) plays avoiding her (resp. his) last unlabelled vertex of $\mathrm{Ext_g}(G)$, and then Alice wins if and only if $|V_A| > |V_B|$.

Now consider the misère variant. If $|\mathrm{Ext}_\mathrm{g}^A(G)| \leq |\mathrm{Ext}_\mathrm{g}^B(G)|$, then Alice can label all vertices of $\mathrm{Ext}_\mathrm{g}^A(G)$ before Bob can label all the vertices of $\mathrm{Ext}_\mathrm{g}^B(G)$. Then the misère games end in two possible situations: (a) Alice is unable to play because all vertices of V_A were labeled, or (b) Alice is unable to play because Bob had to label the last unlabeled vertex of $\mathrm{Ext}_\mathrm{g}^B(G)$. In both, Alice wins. Analogously, if $|\mathrm{Ext}_\mathrm{g}^A(G)| > |\mathrm{Ext}_\mathrm{g}^B(G)|$, Bob wins the misère games. □

As an easy example of Thm 4, consider the graph G in the proof of Theorem 2 by adding two non-adjacent vertices u_1 and u_2 adjacent to all vertices of a given graph H (whose vertices are colored either A or B), shown in Fig 3(a). Now let G' be obtained from G by adding two new vertices u_1' and u_2' and the edges $u_1'u_1$ and $u_2'u_2$. Notice that $\mathrm{Ext_g}(G') = \{u_1', u_2'\}$ and $\mathrm{hull_g}(\{u_1', u_2'\}) = \mathrm{I_g}(\{u_1', u_2'\}) = V(G')$, that is, $\{u_1', u_2'\}$ is a geodesic hull set and a geodesic interval set of G'. Then, independently of the structure and the colors in the graph G in the construction of this example, if u_1' and u_2' are colored A, Alice (resp. Bob) wins PARTIZAN $\mathrm{HG_g}$ and PARTIZAN $\mathrm{IG_g}$ on the graph G' in the normal (resp. misère) variant. Moreover, if u_1' is colored A and u_2' is colored B, then Alice always wins the misère variant and she wins the normal variant if and only if the number of vertices colored A in G is greater than the number of vertices colored B.

Now we find winning strategies in polynomial time for PARTIZAN $\mathrm{IG_g}$ and PARTIZAN $\mathrm{HG_g}$ in Ptolemaic graphs. The theorem below is the partizan variant of a result of Haynes, Henning and Tiller [17] in 2003 on block graphs, since every block graph is a Ptolemaic graph.

Theorem 5. *Let G be a Ptolemaic graph whose vertices are colored either A or B. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN $\mathrm{IG_g}$ *and* PARTIZAN $\mathrm{HG_g}$ *if and only if $\mathrm{Ext}_\mathrm{g}^B(G) = \emptyset$, or $\mathrm{Ext}_\mathrm{g}^A(G) \neq \emptyset$ and $|V_A| >$*

$|V_B|$. *Moreover Alice wins the misère variant of* PARTIZAN $\mathrm{IG_g}$ *and* PARTIZAN $\mathrm{HG_g}$ *if and only if* $|\operatorname{Ext}_{\mathrm{g}}^{A}(G)| \leq |\operatorname{Ext}_{\mathrm{g}}^{B}(G)|$.

Proof (sketch). Regarding the game PARTIZAN $\mathrm{HG_g}$, the result follows from Theorem 4 and the fact that the geodesic convexity is geometric in Ptolemaic graphs [15]. Regarding the game PARTIZAN $\mathrm{IG_g}$, the result holds since Ptolemaic graphs are chordal and distance-hereditary, every vertex of a chordal graph is in an induced path between two simplicial vertices and every induced path is a shortest path in a distance hereditary graph. □

6 Other convexities and other variations

Other contribution of this paper is the generalization of these games to any graph convexity, obtaining very natural convexity games and enriching the prolific research area of graph convexity. We also prove general results. For this, let us define general convexity in graphs.

A *convexity* $\mathcal{C}$ on a finite set $V \neq \emptyset$ is a family of subsets of V such that $\emptyset, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. That is, $S_1, S_2 \in \mathcal{C}$ implies $S_1 \cap S_2 \in \mathcal{C}$. A member of $\mathcal{C}$ is said to be a *$\mathcal{C}$-convex set*. Given $S \subseteq V$, the *$\mathcal{C}$-convex hull* of S is the smallest $\mathcal{C}$-convex set $\mathrm{hull}_{\mathcal{C}}(S)$ containing S. We say that S is a *$\mathcal{C}$-hull set* if $\mathrm{hull}_{\mathcal{C}}(S) = V$. It is easy to see that $\mathrm{hull}_{\mathcal{C}}(\cdot)$ is a *closure operator*, that is, for every $S, S' \subseteq V$: (a) $S \subseteq \mathrm{hull}_{\mathcal{C}}(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{hull}_{\mathcal{C}}(S) \subseteq \mathrm{hull}_{\mathcal{C}}(S')$ [monotonicity], (c) $\mathrm{hull}_{\mathcal{C}}(\emptyset) = \emptyset$ [normalization] and (d) $\mathrm{hull}_{\mathcal{C}}(\mathrm{hull}_{\mathcal{C}}(S)) = \mathrm{hull}_{\mathcal{C}}(S)$ [idempotence].

We say that $\mathrm{I} : 2^V \to 2^V$ is an *interval function* on V if, for every $S, S' \subseteq V$, (a) $S \subseteq \mathrm{I}(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{I}(S) \subseteq \mathrm{I}(S')$ [monotonicity] and (c) $\mathrm{I}(\emptyset) = \emptyset$ [normalization]. It is known that any interval function *induces* a unique convexity, containing each set $S \subseteq V$ such that $\mathrm{I}(S) = S$. Moreover, every convexity is induced by an interval function. We then assume that every convexity $\mathcal{C}$ on V is defined by an explicitly given interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ on V. It is also known that the convex hull of a set S in a convexity $\mathcal{C}$ can be obtained by applying the corresponding interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ until obtaining a $\mathcal{C}$-convex set. A set S is a *$\mathcal{C}$-interval set* if $\mathrm{I}_{\mathcal{C}}(S) = V$.

Given a graph G, a *graph convexity* on G is simply a convexity $\mathcal{C}$ on $V(G)$ with a given interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ on $V(G)$. A standard way to define a *graph convexity* $\mathcal{C}$ on a graph G is by fixing a family $\mathcal{P}$ of paths of G and taking the interval function $\mathrm{I}_{\mathcal{C}}(S)$ as the set S with all vertices lying on some path of $\mathcal{P}$ whose endpoints are in S. The most studied graph convexities are path convexities, such as the *geodesic convexity* [13,14], the *P_3 convexity* [2], the *monophonic convexity* [9,12,15], the *m^3 convexity* [11] and the *triangle path convexity* [6,10], where $\mathcal{P}$ is the family of all geodesics (shortest paths) of the graph, of all paths of order three, of all induced paths, of all induced paths of size at least 3 and of all paths $v_1, \ldots, v_k$ with no edge $v_i v_j$ with $|j - i| > 2$, respectively.

This definition also applies to directed graphs. For example, there are the geodesic convexity, the monophonic convexity and the P_3 convexity for directed graphs, related to shortest paths, induced paths and paths with three vertices in the directed graph, respectively, where the paths must respect the orientation of the edges ("*arcs*").

Given a graph convexity $\mathcal{C}$ on a graph G, let the *$\mathcal{C}$-hull number* $\mathrm{hn}_{\mathcal{C}}(G)$ be the size of a minimum $\mathcal{C}$-hull set of G and let the *$\mathcal{C}$-interval number* $\mathrm{in}_{\mathcal{C}}(G)$ be the size of a minimum $\mathcal{C}$-interval set of G.

In the following, we generalize the convexity games of Definition 1 to any graph convexity.

Definition 3. *Given a graph convexity $\mathcal{C}$ on a graph G, we introduce four convexity games. In the games defined below, the set L of labeled vertices is initially empty and the definitions of $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex v which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.*

- *In the $\mathcal{C}$-hull game* $\mathrm{HG}_{\mathcal{C}}$*: $f_1(L) = L$ and $f_2(L) = \mathrm{hull}_{\mathcal{C}}(L)$.*

- *In the $\mathcal{C}$-interval game* $\mathrm{IG}_{\mathcal{C}}$*:* $f_1(L) = L$ *and* $f_2(L) = \mathrm{I}_{\mathcal{C}}(L)$*.*
- *In the closed $\mathcal{C}$-hull game* $\mathrm{CHG}_{\mathcal{C}}$*:* $f_1(L) = f_2(L) = \mathrm{hull}_{\mathcal{C}}(L)$*.*
- *In the closed $\mathcal{C}$-interval game* $\mathrm{CIG}_{\mathcal{C}}$*:* $f_1(L) = f_2(L) = \mathrm{I}_{\mathcal{C}}(L)$*.*

As before, the last to play wins in the normal variant of these games, the last to play loses in the misère variants and the decision problem associated to these games is whether Alice has a winning strategy.

As in Definition 1, we may also have the SIMPLIFIED and the PARTIZAN versions of these games. As an example, we have the following lemma on directed graphs, where the subscript m is for the monophonic convexity.

Lemma 2. *The Blue-Red Hackenbush game in trees is equivalent to the game* SIMPLIFIED PARTIZAN $\mathrm{CIG}_{\mathrm{m}}$ *in DAGs (directed acyclic graphs) with one source, one sink and all other vertices with in-degree 1, where the sink is labeled at the beginnning of the game.*

Proof (sketch). Given a Blue-Red Hackenbush tree T, we obtain the directed graph $\overrightarrow{G}$ from the following. For every edge x of T, create a vertex v_x on $\overrightarrow{G}$. If x and y are adjacent edges of T and x is under y, then create the arc $x_x v_y$ on $\overrightarrow{G}$. Moreover, create a vertex z in $\overrightarrow{G}$ and all arcs $v_x z$, where x has no adjacent edge above it in T, and consider that z is already labeled at the beginning. Finally, color v_x with color A in $\overrightarrow{G}$ if and only if the edge x is colored blue in T. Figure 5 shows an example. Notice that $\overrightarrow{G}$ is a DAG and has exactly one sink (the vertex z) and one source (the vertex associated to the edge of T connected to the ground). Moreover every vertex of $\overrightarrow{G}$ has in-degree 1, except the source and the sink. Also note that labeling a vertex v_x of $\overrightarrow{G}$ in the game SIMPLIFIED PARTIZAN $\mathrm{CIG}_{\mathrm{m}}$ makes impossible to label the vertices associated to the edges of T above x in the next rounds, which is equivalent to Hackenbush when cutting the edge x.

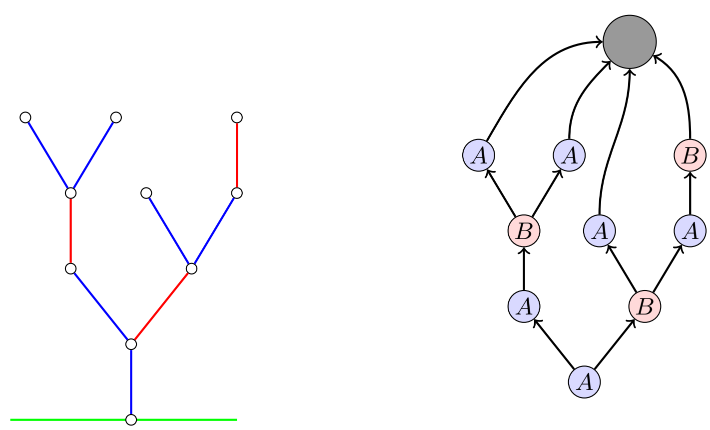


Fig. 5. Blue-Red Hackenbush in trees can be modeled as the game SIMPLIFIED PARTIZAN $\mathrm{CIG}_{\mathrm{m}}$ in DAGs with one source, one sink and all other vertices with in-degree 1, where the sink (black vertex) is already labeled at the beginning and m corresponds to the monophonic convexity.

Finally, let $\overrightarrow{G}$ be a DAG (directed acyclic graph) with one source s, one sink z and all other vertices with in-degree 1, where the sink is labeled at the beginning of the game SIMPLIFIED PARTIZAN $\mathrm{CIG}_{\mathrm{m}}$. We obtain a Blue-Red Hackenbush tree T from the following. For every vertex

$v \neq z$ of $\overrightarrow{G}$, create a vertex v in T. For every arc uv of $\overrightarrow{G}$ with $v \neq z$, create the edge uv in T. Finally, create a vertex g in T which is on the ground in the Blue-Red Hackenbush game and create the edge gs, where s is the vertex of T corresponding to the source s of $\overrightarrow{G}$. Finally, color the edge uv of T with blue if and only if the vertex v is colored A in $\overrightarrow{G}$. Notice that T is a Hackenbush tree. Also note that cutting the edge uv of T makes impossible to select the edges above uv in T, which is equivalent to SIMPLIFIED PARTIZAN CIG_m in $\overrightarrow{G}$, since the sink z is already labeled. □

7 General convexity games in convex geometries

Recall that a convexity $\mathcal{C}$ on a graph G is *geometric* (also called *antimatroid* or *convex geometry*) if it satisfies the *Minkowski–Krein–Milman* property: every convex set is the convex hull of its extreme vertices in the convexity $\mathcal{C}$, where a vertex v is an *extreme vertex* of S in the convexity $\mathcal{C}$ if $S \setminus \{v\}$ is also a $\mathcal{C}$-convex set. Let $\text{Ext}_\mathcal{C}(S)$ be the set of extreme vertices of S.

The research on graph convex geometries (or geometric convexities) is usually concentrated in determining the graph class in which a given graph convexity is geometric. In 1986, Farber and Jamison [15] proved that the monophonic (resp. geodesic) convexity is geometric if and only if the graph is chordal (resp. Ptolemaic). In 1999, Dragan et al. [11] proved that the m^3 convexity is geometric if and only if the graph is weak bipolarizable. Recently, it was proved that the triangle path (resp. P_3) convexity is geometric if and only if the graph is a forest (resp. forest of stars) [8].

Regarding the partizan variants, we extend the results of Section 5 to any convexity.

Theorem 6. *Let $\mathcal{C}$ be a graph convexity on a graph G, whose vertices are colored either A or B, such that $\text{Ext}_\mathcal{C}(G)$ is a $\mathcal{C}$-hull (resp. $\mathcal{C}$-interval) set. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN $\text{HG}_\mathcal{C}$ *(resp.* PARTIZAN $\text{IG}_\mathcal{C}$*) if and only if $\text{Ext}^B_\mathcal{C}(G) = \emptyset$, or $\text{Ext}^A_\mathcal{C}(G) \neq \emptyset$ and $|V_A| > |V_B|$. Moreover Alice wins the misère variant of* PARTIZAN $\text{HG}_\mathcal{C}$ *(resp.* PARTIZAN $\text{IG}_\mathcal{C}$*) if and only if $|\text{Ext}^A_\mathcal{C}(G)| \leq |\text{Ext}^B_\mathcal{C}(G)|$.*

Proof (sketch). The proof follows the same structure of the proof of Theorem 4, replacing the geodesic convexity "g" with the convexity $\mathcal{C}$. □

We also find winning strategies in polytime for PARTIZAN $\text{HG}_\mathcal{C}$ in general convex geometries.

Theorem 7. *Let $\mathcal{C}$ be a convex geometry on a graph G, whose vertices are colored either A or B. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN $\text{HG}_\mathcal{C}$ *if and only if $\text{Ext}^B_\mathcal{C}(G) = \emptyset$, or $\text{Ext}^A_\mathcal{C}(G) \neq \emptyset$ and $|V_A| > |V_B|$. Moreover Alice wins the misère variant of* PARTIZAN $\text{HG}_\mathcal{C}$ *if and only if $|\text{Ext}^A_\mathcal{C}(G)| \leq |\text{Ext}^B_\mathcal{C}(G)|$.*

Proof (sketch). The result follows from Theorem 6 and the fact that $\text{Ext}_\mathcal{C}(G)$ is a $\mathcal{C}$-hull set when the convexity $\mathcal{C}$ is a convex geometry on G, from the Minkowski-Krein-Milman property. □

8 Conclusions and Final Remarks

In this paper, we introduced the partizan variants of the geodesic convexity games IG_g, HG_g, CIG_g and CHG_g, introduced by Harary [4,5,16] in 1984 (see Definition 2). Below we list our results:

– We proved in Theorem 1 that PARTIZAN CIG_g and PARTIZAN CHG_g are polynomial time solvable in trees, using Conway's combinatorial game theory and surreal numbers;
– We proved that PARTIZAN IG_g and PARTIZAN HG_g are polytime solvable in Ptolemaic graphs in Theorem 5, using results on convex geometries and the Minkowski-Krein-Milman property;

– We prove PSPACE-hardness for the normal and misère variants of PARTIZAN HG_g, PARTIZAN CHG_g and their simplified versions, even in graphs with diameter two, in Corollary 1, using reductions from the PARTIZAN CLIQUE FORMING game;

Finally, in Section 6, we generalize these games to any graph convexity, such as the P_3 and monophonic convexities. We finish with Theorem 7 with winning strategies on PARTIZAN $HG_{\mathcal{C}}$ for any convex geometry $\mathcal{C}$, which are generalizations of Theorem 5. Below we list some open problems:

– PROBLEM 1: The geodesic interval games PARTIZAN IG_g and CIG_g are PSPACE-complete?
– PROBLEM 2: Let GREEN CIG_g be the extension of PARTIZAN CIG_g allowing green vertices, which can be labeled by any player. If all vertices are green, we have CIG_g. If no vertex is green, we have PARTIZAN CIG_g. Is the game GREEN CIG_g also polytime solvable in trees?

References

1. Samuel N. Araújo, J. Marcos Brito, Raquel Folz, Rosiane de Freitas, and Rudini Sampaio. Graph convexity impartial games: complexity and winning strategies. *Theor. Comput. Sci.*, 998:114534, 2024.
2. R M Barbosa, E M M Coelho, M C Dourado, D Rautenbach, and J L Szwarcfiter. On the Carathéodory number for the convexity of paths of order three. *SIAM J Disc Math*, 26(3):929–939, 2012.
3. E. R. Berlekamp, J. H. Conway, and R. K. Guy. *Winning Ways for Your Mathematical Plays*, volume 1. Academic Press, 1982.
4. Fred Buckley and Frank Harary. Closed geodetic games for graphs. *Congressus Numerantium*, 47:131–138, 1985. Proc 16th Southeastern Conference on Combinatorics, Graph Theory and Computing.
5. Fred Buckley and Frank Harary. Geodetic games for graphs. *Quaest Math*, 8:321–334, 1985.
6. Manoj Changat and Joseph Mathew. On triangle path convexity in graphs. *Discrete Mathematics*, 206(1):91–95, 1999.
7. John H. Conway. *On numbers and games.* L.M.S. monographs ; no. 6. Academic Press, London, 1976.
8. M. C. Dourado, M. Gutierrez, F. Protti, R. Sampaio, and S. Tondato. Characterizations of graph classes via convex geometries: A survey. arXiv, 2023.
9. Mitre C Dourado, Fábio Protti, and Jayme L Szwarcfiter. Complexity results related to monophonic convexity. *Discrete Applied Mathematics*, 158(12):1268–1274, 2010.
10. Mitre C. Dourado and Rudini M. Sampaio. Complexity aspects of the triangle path convexity. *Discrete Applied Mathematics*, 206:39–47, 2016.
11. Feodor F. Dragan, Falk Nicolai, and Andreas Brandstädt. Convexity and HHD-free graphs. *SIAM Journal on Discrete Mathematics*, 12(1):119–135, 1999.
12. Pierre Duchet. Convex sets in graphs, II. Minimal path convexity. *Journal of Combinatorial Theory, Series B*, 44(3):307–316, 1988.
13. Martin G Everett and Stephen B Seidman. The hull number of a graph. *Discrete Mathematics*, 57(3):217–223, 1985.
14. M Farber and R E Jamison. On local convexity in graphs. *Discrete Mathematics*, 66(3):231–247, 1987.
15. Martin Farber and Robert E Jamison. Convexity in graphs and hypergraphs. *SIAM Journal on Algebraic Discrete Methods*, 7(3):433–444, 1986.
16. Frank Harary. Convexity in graphs: Achievement and avoidance games. In *Annals of Discrete Mathematics (20)*, volume 87, page 323. North-Holland, 1984.
17. T.W. Haynes, M.A. Henning, and C. Tiller. Geodetic achievement and avoidance games for graphs. *Quaestiones Mathematicae*, 26:389–397, 2003.
18. R. Hearn and E. Demaine. *Games, Puzzles and Computation.* A. K. Peters Ltd, 2009.
19. D. E. Knuth. *Surreal Numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness: a Mathematical Novelette*. Addison-Wesley Publishing Company, 1974.
20. M. Nečásková. A note on the achievement geodetic games. *Quaest Math*, 12:115–119, 1988.
21. T. Schaefer. On the complexity of some two-person perfect-information games. *Journal of Computer and System Sciences*, 16(2):185–225, 1978.
22. E. Zermelo. Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. In *Proc. 5th Int. Congress of Mathematicians*, pages 501–504, 1913.