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Complexity and winning strategies of graph convexity games

(Brief Announcement)

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Abstract

Accordingly to Duchet (1987), the first paper of convexity on general graphs, in english, is the 1981 paper "Convexity in graphs". One of its authors, Frank Harary, introduced in 1984 the first graph convexity games, focused on the geodesic convexity, which were investigated in a sequence of five papers that ended in 2003. In this paper, we continue this research line, extend these games to other graph convexities, and obtain winning strategies and complexity results. Among them, we obtain winning strategies for general convex geometries in graphs. We also obtain the first PSPACE-hardness results on convexity games, by proving that the normal play and the misère play of the hull game on the geodesic and the monophonic convexities are PSPACE-complete.

Keywords: Graph convexity; convex geometry; combinatorial games; PSPACE-hardness; geodesic game

1. Introduction

In 1984, Harary introduced the first graph convexity games in his abstract "Convexity in graphs: achievement and avoidance games" [1]. This research line on graph convexity games ended in 2003 after a sequence of five papers [1, 2, 3, 4, 5], all of them focused on the geodesic convexity. In this paper, we continue this research line, by defining new natural convexity games and obtaining results on the old and the new convexity games for many graph convexities. In order to explain them, we need some terminology. A *graph convexity* C on a finite graph G is a family of subsets of $V = V(G)$ such that $\emptyset, V \in C$, and C is closed under intersections. The *convex hull* of $S \subseteq V$ is the smallest member $\text{hull}_C(S) \supseteq S$ of C. A set S is *convex* if $S \in C$ and is a *hull set* if $\text{hull}_C(S) = V$. The interval function $\text{I}_C(\cdot)$ is such that S is convex if and only if $\text{I}_C(S) = S$. The convex hull of S can be obtained by iteratively applying $\text{I}_C(\cdot)$ until obtaining a convex set. The *geodesic*, *monophonic* and P_3 classical convexities are defined by considering $\text{I}_C(S)$ as S and any vertex on a path between 2 vertices of S which is minimum, induced or with 3 vertices, respectively.

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Definition 1.1. *Given a graph convexity $\mathcal{C}$ on G, we introduce 4 convexity games. In all of them, the set L of labeled vertices is initially empty and $f(L)$ and $g(L)$ depend on the game. Alice and Bob alternately label one unlabeled vertex v which is not in $f(L)$. The game ends when $g(L) = V(G)$. In the $\mathcal{C}$-hull-game: $f(L) = L$ and $g(L) = \mathrm{hull}_{\mathcal{C}}(L)$. In the $\mathcal{C}$-interval-game: $f(L) = L$ and $g(L) = \mathrm{I}_{\mathcal{C}}(L)$. In the closed $\mathcal{C}$-hull game: $f(L) = g(L) = \mathrm{hull}_{\mathcal{C}}(L)$. In the closed $\mathcal{C}$-interval-game: $f(L) = g(L) = \mathrm{I}_{\mathcal{C}}(L)$.*

Each game has 3 variants: *normal* (last to play wins), *misère* (last to play loses), and the *optimization* variant, in which it is given an additional parameter k and Alice wins if $|L| \leq k$ at the end (no matter who ends the game).

Definition 1.2. *The optimization variants define four new parameters:* game $\mathcal{C}$-hull number $\mathrm{ghn}_{\mathcal{C}}(G)$, game $\mathcal{C}$-interval number $\mathrm{gin}_{\mathcal{C}}(G)$, closed game $\mathcal{C}$-hull number $\mathrm{cghn}_{\mathcal{C}}(G)$ *and* closed game $\mathcal{C}$-interval number $\mathrm{cgin}_{\mathcal{C}}(G)$, *as the minimum k s.t. Alice has a winning strategy in the optimization variant of the corresponding convexity games on the graph G.*

From Zermelo's Theorem [6], one player has a winning strategy in each one of these games and their variants, since they are finite perfect information without draw. So, the decision problem is to decide if Alice has a winning strategy. The normal and misère variants are also called *achievement* and *avoidance* variants.

In 1985, Buckley et al. [2] introduced the geodesic interval game and, in 2003, Haynes et al. [5] obtained results for trees and complete multipartite graphs in the normal and misère variants of this game. In his 1984 abstract on graph convexity games, Harary [1] mentioned "*games involving the convex hull*", but no game of this type has been defined or investigated in the literature before.

2. Preliminary results

Lemma 2.1. *Let $\mathcal{C}$ be a convexity on a graph G. Then*

- $\mathrm{hn}_{\mathcal{C}}(G) \leq \mathrm{cghn}_{\mathcal{C}}(G) \leq \mathrm{ghn}_{\mathcal{C}}(G) \leq \min\left\{2 \cdot \mathrm{hn}_{\mathcal{C}}(G) - 1,\ n\right\}$ *and*
- $\mathrm{in}_{\mathcal{C}}(G) \leq \mathrm{cgin}_{\mathcal{C}}(G) \leq \mathrm{gin}_{\mathcal{C}}(G) \leq \min\left\{2 \cdot \mathrm{in}_{\mathcal{C}}(G) - 1,\ n\right\}$.

Lemma 2.2. *Consider the complete graph K_n with $n \geq 2$. In the monophonic and geodesic convexities, Alice wins all games of Definition 1.1 if and only if n is odd in the normal variant or n is even in the misère variant. In the P_3 convexity, Alice loses the normal and wins the misère variant of all games. All game numbers of Definition 1.2 are n in the monophonic and geodesic convexities and are 2 in the P_3 convexity.*

We use the subscripts "m" and "g" to indicate the geodesic and monophonic convexities, respectively.

Theorem 2.3. *Let $n \geq 4$. In the monophonic convexity, Bob always wins the normal and misère variants of the four games on the cycle C_n, and $\mathrm{ghn_m}(C_n) = \mathrm{gin_m}(C_n) = \mathrm{cghn_m}(C_n) = \mathrm{cgin_m}(C_n) = 3$. In the geodesic convexity, Alice wins the normal variant of the closed hull and interval games on C_n if and only if n is odd. In the misère variant, Bob always wins the closed geodesic hull and interval games on C_n. Moreover, $\mathrm{cghn_g}(C_n) = \mathrm{cgin_g}(C_n) = \mathrm{ghn_g}(C_n) = \mathrm{gin_g}(C_n) = 3$.*

3. PSPACE-hardness of the monophonic and geodesic hull games and closed hull games

We prove that simplified versions of the normal and misère variants of the monophonic hull game are PSPACE-complete. In the *simplified hull game*, the input consists of a graph G and a vertex v, which is already labelled before the beginning of the game. We obtain reductions from the clique-forming game, in which Alice and Bob take turns selecting vertices which must induce a clique (the last to play wins). This problem is PSPACE-complete [7].

Theorem 3.1. *The misère variant of the simplified monophonic hull game is PSPACE-complete.*

Proof. [sketch] Let H be an instance of the clique-forming game. Let G be the graph of diameter 2 obtained from H by adding two new vertices u_1 and u_2 adjacent to all vertices of H. Also let u_1 be the vertex which is already labelled. We prove that Alice has a winning strategy in the clique-forming game on H if and only if she has a winning strategy in the misère variant of the simplified monophonic hull game on (G, u_1). If a player labels u_2, he/she loses immediately, since $\mathrm{hull}_m(\{u_1, u_2\}) = V(G)$. Moreover, if a vertex v_j of H is labelled in the hull game and there is a non-adjacent labelled vertex v_i in H, the player loses immediately, since $\mathrm{hull}_m(\{v_i, v_j\}) = V(G)$. Thus we may assume that the set L of labelled vertices form a clique in all turns, except the last one, and we are done. □

Theorem 3.2. *The normal variant of the simplified monophonic hull game is PSPACE-complete.*

Proof. [sketch] Consider the same reduction above, adding to G an isolated new vertex w. If a player labels w (resp. u_2) when u_2 (resp. w) is not labelled, he/she loses immediately, since the opponent labels u_2 (resp. w) and wins the normal variant because $\mathrm{hull}_m(\{u_1, u_2, w\}) = V(G)$. Moreover, if a player labels a vertex v_j of H in the hull game and there is a non-adjacent labelled vertex v_i in H, then the player loses immediately, since the opponent labels w, winning the game because $\mathrm{hull}_m(\{v_i, v_j, w\}) = V(G)$. So, we may assume that the set L of labelled vertices form a clique in all turns, except the last two. We conclude that Alice wins the normal variant of the simplified hull game on (G, u_1) if and only if she has a winning strategy in the clique-forming game on H. □

Corollary 3.3. *The normal and misère variants of the hull game and the closed hull game on the monophonic and the geodesic convexities are PSPACE-complete.*

4. Winning strategies on general hull games and interval games for convex geometries

A graph convexity is *geometric* (or is a *convex geometry*) if it satisfies the *Minkowski–Krein–Milman* property: every convex set S is the convex hull of its extreme points, which are the vertices $v \in S$ such that $S \setminus \{v\}$ is also convex. Let $\mathrm{Ext}_C(G)$ be the set of extreme points of $V(G)$ in the convexity C. The research on graph convex geometries is usually concentrated in determining the graph class in which a given graph convexity is geometric. In 1986, Farber and Jamison [8] proved that the monophonic (resp. geodesic) convexity is geometric if and only if the graph is chordal (resp. Ptolemaic). We find an interesting connection between general convex geometries and winning strategies of hull games.

Theorem 4.1. *Let C be a convex geometry on G. Alice wins the C-hull game if and only if n is odd in the normal variant or n is even in the misère variant. Moreover,* $\mathrm{ghn}_C(G) = \min\left\{ 2 \cdot |\mathrm{Ext}_C(G)| - 1,\ n \right\}$.

Theorem 4.2. *Let T be a rooted tree s.t. any non-leaf node has at least two children. Alice wins the P_3 hull game on T if and only if n is odd in the normal or n is even in the misère variant. Moreover,* $\mathrm{ghn}_{P_3}(T) = n$.

The next theorem deals with hull games and interval games in the geodesic and monophonic convexities. It also extends a result of [5] on complete block graphs in the geodesic convexity, since they form a subclass of Ptolemaic graphs. Also notice that complete graphs are chordal and Ptolemaic (recall Lemma 2.2).

Theorem 4.3. *In chordal graphs, Alice wins the monophonic hull and interval games if and only if n is odd in the normal or n is even in the misère variant. In Ptolemaic graphs, Alice wins the geodesic hull and interval games if and only if n is odd in the normal or n is even in the misère variant.*

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