Theoretical Computer Science 998 (2024) 114534

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

# Graph convexity impartial games: Complexity and winning strategies ☆

Samuel N. Araújo [a,b], João Marcos Brito [b], Raquel Folz [c], Rosiane de Freitas [c], Rudini M. Sampaio [b,*]

[a] *Instituto Federal de Educação, Ciência e Tecnologia do Ceará, Crato, Brazil*
[b] *Departamento de Computação, Universidade Federal do Ceará, Fortaleza, Brazil*
[c] *Instituto de Computação, Universidade Federal do Amazonas, Manaus, Brazil*

ARTICLE INFO

Communicated by M. Mavronicolas

*Keywords:*
Graph convexity
Geodesic games
PSPACE-hard
Sprague-Grundy theory
Impartial games

ABSTRACT

Accordingly to Duchet (1987), the first paper of convexity on general graphs, in english, is the 1981 paper "Convexity in graphs". One of its authors, Frank Harary, introduced in 1984 the first graph convexity games, focused on the geodesic convexity, which were investigated in a sequence of five papers that ended in 2003. In this paper, we continue this research line, extend these games to other graph convexities, and obtain winning strategies and complexity results. Among them, we obtain winning strategies for general convex geometries and winning strategies for trees from the Sprague-Grundy theory on impartial games. We also obtain the first PSPACE-hardness results on convexity games, by proving that the normal play and the misère play of the impartial hull game on the geodesic convexity is PSPACE-complete even in graphs with diameter two.

## 1. Impartial convexity games

Convexity is a classical topic, studied in many different branches of mathematics. The study of convexities applied to graphs has started recently, about 50 years ago. The 1972 paper of Erdős et al. [15] is one of the first in this topic, focused in tournaments. Accordingly to Duchet [13], the first paper on convexity in graphs written in English, published in 1981, is the paper "Convexity in graphs" due to Frank Harary and Juhani Nieminem [21]. In 1984, Harary introduced the first graph convexity games in his abstract "Convexity in graphs: achievement and avoidance games" [20]. This research line on graph convexity games ended in 2003 after a sequence of five papers [4,5,20,22,26], all of them focused on the geodesic convexity.[1] In this paper, we continue this research line, by obtaining complexity and algorithmic results on convexity games for the geodesic convexity and also for other graph convexities. In order to explain them, we need some terminology related to the geodesic convexity.

All graphs considered in this paper are simple and finite. Let $N_G(v)$ be the set of neighbors of $v$ in a given graph $G$ and $N_G[v] = N_G(v) \cup \{v\}$. We may omit the subscript when $G$ is clear from the context. Given a graph $G$ and a set $S \subseteq V(G)$, let the *geodesic interval* $I_g(S)$ be the set $S$ and every vertex in a shortest path between two vertices of $S$. We say that $S$ is *convex in the*

☆ This article belongs to Section A: Algorithms, automata, complexity and games, Edited by Paul Spirakis.
* Corresponding author.
*E-mail addresses:* samuel.araujo@ifce.edu.br (S.N. Araújo), jmbmaluno@alu.ufc.br (J.M. Brito), raquel.folz@icomp.ufam.edu.br (R. Folz), rosiane@icomp.ufam.edu.br (R. de Freitas), rudini@dc.ufc.br (R.M. Sampaio).

[1] Some authors such as Pelayo [27] use the term "*geodetic*" (instead of "*geodesic*"), while other authors such as van de Vel [30] do not use the term "*geodetic*".

https://doi.org/10.1016/j.tcs.2024.114534
Received 15 November 2023; Received in revised form 19 March 2024; Accepted 22 March 2024

Available online 27 March 2024

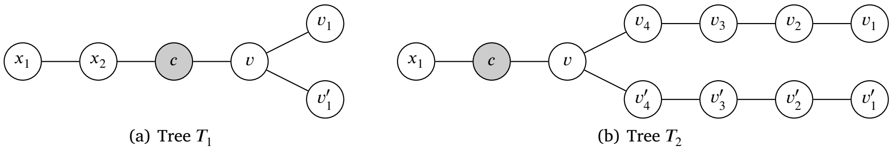



**Fig. 1.** Alice has a winning strategy in both variants of $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$ in the trees $T_1$ and $T_2$ by labeling the vertex $c$ in her first move.

*geodesic convexity* [16,18] if $\mathrm{I_g}(S) = S$. The *geodesic convex hull* of $S$ is the minimum convex set $\mathrm{hull_g}(S)$ containing $S$. It is known that $\mathrm{hull_g}(S)$ can be obtained by applying $\mathrm{I_g}(\cdot)$ from $S$ until obtaining a convex set.

Among the games introduced by Harary [4,5,20] in 1984 for the geodesic convexity, we have the *geodesic interval game*, the *geodesic hull game*, the *geodesic closed interval game* and the *geodesic closed hull game*, defined below. Fig. 1 shows some examples.

**Definition 1.1.** Let $G$ be a graph. In the games defined below, the set $L$ of labeled vertices is initially empty and the definitions of $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex $v$ which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.

- In the geodesic hull game $\mathrm{HG_g}$: $f_1(L) = L$ and $f_2(L) = \mathrm{hull_g}(L)$.
- In the geodesic interval game $\mathrm{IG_g}$: $f_1(L) = L$ and $f_2(L) = \mathrm{I_g}(L)$.
- In the geodesic closed hull game $\mathrm{CHG_g}$: $f_1(L) = f_2(L) = \mathrm{hull_g}(L)$.
- In the geodesic closed interval game $\mathrm{CIG_g}$: $f_1(L) = f_2(L) = \mathrm{I_g}(L)$.

Under the *normal play convention* (*normal variant or achievement variant*), the first player unable to move loses the game. Under the *misère play convention*, also called *avoidance variant*, the first player unable to move wins the game. From the classical Zermelo-von Neumann theorem [31], one of the two players has a winning strategy in each one of these games, since they are finite perfect-information games without draw. So, the decision problem of these games is whether Alice has a winning strategy.

As an example of the interval game $\mathrm{IG_g}$, consider the tree $T_1$ of Fig. 1(a). A possible sequence of labeled vertices (alternating Alice and Bob) in the game could be $v_1 - x_2 - c - v - x_1 - v'_1$, Bob winning the normal variant and losing the misère variant. Notice that, in the trees $T_1$ and $T_2$ of Fig. 1, the game ends when all leaves $x_1, v_1, v'_1$ are labeled. Then, in normal play, the players will avoid the last two unlabeled leaves, until the opponent is forced to label the penultimate leaf. In the misère variant, the players will avoid the last unlabeled leaf. Thus, since the number of vertices of $T_1$ and $T_2$ are even and odd, respectively, then Alice loses in $T_1$ (resp. $T_2$) and wins in $T_2$ (resp. $T_1$) in the normal (resp. misère) variant. In Section 4, this argument on the leaves of trees is generalized to simplicial vertices in Ptolemaic graphs for the interval game $\mathrm{IG_g}$ and hull game $\mathrm{HG_g}$ under the geodesic convexity and, more generally in Section 6, to "extreme vertices" for any convex geometry, which is a graph convexity satisfying the Minkowski-Krein-Milman property.

As an example of the closed interval game $\mathrm{CIG_g}$, consider the normal variant on the tree $T_1$ of Fig. 1(a). Alice has a winning strategy by labeling the vertex $c$ in her first move: if Bob labels $x_1$ (resp. $v$), Alice labels $v$ (resp. $x_1$); otherwise if Bob labels $x_2$ (resp. $v_1$ or $v'_1$), Alice labels $v_1$ (resp. $x_2$). Possible Alice winning sequences of labeled vertices (alternating Alice and Bob) in the normal variant on $T_1$ could be $c - x_1 - v - v_1 - v'_1$ and $c - x_2 - v_1 - v'_1 - x_1$. Alice also has a winning strategy in the normal variant on the tree $T_2$ of Fig. 1(b) by labeling the vertex $c$ in her first move, but the strategy is a bit more complicated and is related to the well known Nim game. In Section 2, we use the Sprague-Grundy theory on impartial games to obtain a polynomial time algorithm to decide the winner of the closed interval game $\mathrm{CIG_g}$ and the closed hull game $\mathrm{CHG_g}$ on trees in the normal and misère variants.

In 1985, the normal and misère variants of the geodesic interval game were solved for cycles, wheels and complete bipartite graphs [5]. In 1988, the result of [5] regarding wheel graphs was improved in [26] and, in 2003, Haynes, Henning and Tiller [22] obtained results for trees (exemplified before), complete multipartite graphs and block graphs in the normal and misère variants of the (non-closed) geodesic interval game.

In this paper, as mentioned before, we generalize the results of [22] by obtaining winning strategies in the class of Ptolemaic graphs (which contains the block graphs) to the geodesic games $\mathrm{IG_g}$ and $\mathrm{HG_g}$. Regarding the closed games $\mathrm{CIG_g}$ and $\mathrm{CHG_g}$ on trees, we obtain a polynomial time algorithm based on the Sprague-Grundy theory on impartial games. Finally, we also obtain the first PSPACE-hardness results on convexity games. We also extend these games to any graph convexity and obtain a general result on convex geometries.

## 2. Closed geodesic games in trees

We say that a two-person combinatorial game under the normal or the misère play convention is *impartial* if the set of moves available from any given position (or configuration) is the same for both players. It is easy to see that the games of Definition 1.1 are impartial games.

Nim is one of the most important impartial games in which two players take turns removing objects from disjoint heaps until there are no objects remaining. On each turn, a player chooses a non empty heap and must remove a positive number of objects from this heap. An instance of Nim is given by a sequence $\Phi = (h_1, h_2, \ldots, h_k)$, where $k$ is the number of heaps and $h_i \geq 1$ is the $i$-th heap's size. Nim plays a fundamental role in the Sprague-Grundy game theory and was mathematically solved [3] by Bouton in 1901 for

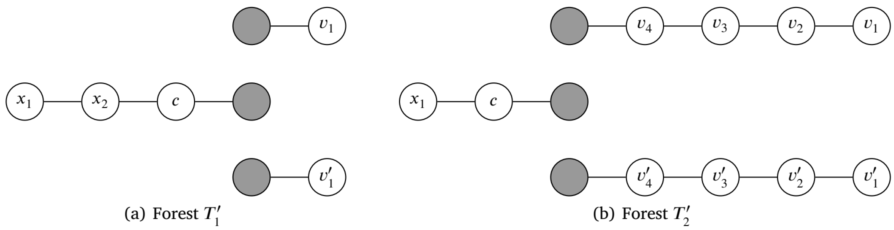



**Fig. 2.** Positions obtained after the first move of Alice in the vertex $v$, instead of $c$, on the trees $T_1$ and $T_2$ of Fig. 1 in the closed interval game $\mathrm{CIG_g}$. Bob has a winning strategy after this bad first move of Alice in the normal and misère variants.

any instance $\Phi$. In other words, the decision problem of deciding which player has a winning strategy is polynomial time solvable in both Nim variants. This is related to **nim-sum**$(\Phi) = h_1 \oplus \ldots \oplus h_k$, where $\oplus$ is the bitwise xor operation. Alice has a winning strategy in the normal play of Nim if and only if **nim-sum**$(\Phi) > 0$. Moreover, Alice has a winning strategy in the misère play of Nim if and only if **nim-sum**$(\Phi) > 0$ and there is at least one heap with more than one object, or **nim-sum**$(\Phi) = 0$ and all non-empty heaps have exactly one object. The winning strategies in both variants is to finish every move with a nim-sum of 0, except in the misère variant when all non-empty heaps have exactly one object.

The Sprague-Grundy theory, independently developed by R. P. Sprague [29] and P. M. Grundy [19] in the 1930s, states that it is possible to associate a number (called *nimber*) to every finite position of an impartial game, associating it to a one-heap game of Nim with that size. As expected from this association with heaps, after one move on a position with nimber $h > 0$ of an impartial game, it is possible to obtain a position with any nimber in $\{0, 1, \ldots, h-1\}$. One of the key elements of the Sprague-Grundy theory is that the nimber of a position can be calculated by the value $\mathrm{mex}\{h_1, \ldots, h_k\}$, where $\{h_1, \ldots, h_k\}$ contains all nimbers of the positions obtained after one move and the *minimum excludant* mex is the minimum non-negative integer not included in the set. Moreover, a position obtained by the disjoint union of $k$ positions with nimbers $h_1, \ldots, h_k$ has nimber $h_1 \oplus \ldots \oplus h_k$. Note that the operator $\oplus$ is commutative and associative, and that $h_i \oplus h_j = 0$ if and only if $h_i = h_j$.

From these tools, we first solve a simplified version of the closed interval game $\mathrm{CIG_g}$, defined below, which will be very useful to the original game $\mathrm{CIG_g}$.

**Definition 2.1.** Given a graph $G$ and a vertex $v$ of $G$, let SIMPLIFIED $\mathrm{CIG_g}$ on the instance $(G, v)$ be $\mathrm{CIG_g}$ on the graph $G$ with the exception that $v$ is already labeled at the beginning. That is, instead of being empty at the beginning, the set $L$ of labeled vertices is initially equal to $L = \{v\}$. Analogously, we define the simplified versions of the other games in Definition 1.1.

As an example, in SIMPLIFIED $\mathrm{CIG_g}$ with instance $(P_{n+1}, p_0)$, where the path $P_{n+1}$ has vertices $p_0 p_1 \ldots p_n$, the nimber is clearly $n$, since it is directly associated to a heap with $n$ objects $p_1, \ldots, p_n$. In Fig. 2, the forest $T'_1$ has nimber $1 \oplus 1 \oplus 3 = 3$ and the forest $T'_2$ has nimber $2 \oplus 4 \oplus 4 = 2$, which are winning for the first player in both normal and misère variants. In the example of Fig. 1, if the first vertex labeled by Alice in the trees $T_1$ and $T_2$ is $v$, instead of $c$, we obtain positions similar to the forests $T'_1$ and $T'_2$ of Fig. 2. That is, Alice loses if she labels $v$ first.

**Theorem 2.2.** *The decision problems related to the normal and misère variants of* SIMPLIFIED $\mathrm{CIG_g}$ *and* SIMPLIFIED $\mathrm{CHG_g}$ *are linear time solvable in trees.*

**Proof.** Let $T$ be a tree and let $v$ be a vertex of $T$. Since the simplified game with instance $(T, v)$ is in a tree, we may consider that, whenever a new vertex $w$ is labeled, the vertices in the path between $v$ and $w$ are also labeled during the game. This is because there is only one path between every pair of vertices in a tree and this path is minimum.

If $v$ has no neighbor, then the nimber of $(T, v)$ in SIMPLIFIED $\mathrm{CIG_g}$ is 0 (since there is no unlabeled vertex) and Alice loses the normal variant and wins the misère variant. So, let $u$ be a neighbor of $v$. We write $(T, v, u)$ for the instance $(T_{v,u}, v)$ of the simplified game, where $T_{v,u}$ is the subtree of $T$ containing $v$ obtained after the removal of all edges incident to $v$ except $vu$. Note that $v$ is a leaf of $T_{v,u}$.

Let us first solve SIMPLIFIED $\mathrm{CIG_g}$ for the instance $(T, v, u)$. Let $u_1, \ldots, u_k$ be the neighbors of $u$ distinct from $v$. Assume that we have the nimber $h_i$ of $(T, u, u_i)$ in SIMPLIFIED $\mathrm{CIG_g}$. If Alice labels $u$, then the games on $(T, u, u_1), \ldots, (T, u, u_k)$ become independent of each other, since a move in one of them does not affect the others. From the Sprague-Grundy theory, the resulting position has nimber $h_1 \oplus \ldots \oplus h_k$.

If Alice labels a vertex $w_i$ in $(T, u, u_i)$, which is distinct from $u$, then $u$ is also labeled and again the games on the subtrees become independent of each other. Let $h'_i$ be the nimber of the resulting position on $(T, u, u_i)$. Then, from the Sprague-Grundy theory, the resulting position on $(T, v, u)$ after the move on $w_i$ has nimber $h_1 \oplus \ldots \oplus h'_i \oplus \ldots \oplus h_k$. Moreover, $h'_i$ can be any value in $\{0, 1, \ldots, h_i - 1\}$.

Let $N$ be the set of all nimber values $h'_1 \oplus \ldots \oplus h'_k$ where $h'_i = h_i$ for every $i = 1, \ldots, k$, except at most one $h'_i \in \{0, \ldots, h_i - 1\}$. Therefore, from the Sprague-Grundy theory, the nimber of $(T, v, u)$ is exactly $\mathrm{mex}(N)$, since $N$ contains all possible nimbers after one move on $(T, v, u)$. In the specific case of $k = 1$, we have that $N = \{0, 1, \ldots, h_1\}$ and then $\mathrm{mex}(N) = h_1 + 1$. Clearly $N$ contains at most $h_1 + \ldots + h_k + 1$ values, which is at most $n_1 + \ldots + n_k + 1$, where $n_i$ is the number of vertices in $(T, u, u_i)$. This leads to the recursive polynomial time Algorithm 1 in a tree $T$ rooted at a leaf $v$ with neighbor $u$ which calculates the nimber of $(T, v, u)$ in linear time in the size of the subtree $T_{v,u}$.

**Algorithm 1** SIMPLIFIED-$\mathrm{CIG_g}$-leaf$(T, v, u)$.

**if** $u$ has degree 1 in $T$ **then return** 1
**let** $u_1, \dots, u_k$ be the neighbors of $u$ distinct from $v$ in $T$
**for** $i = 1, \dots, k$ **do**
  **let** $h_i \leftarrow$ SIMPLIFIED-$\mathrm{CIG_g}$-leaf$(T, u, u_i)$
**let** $H \leftarrow h_1 \oplus \dots \oplus h_k$
**if** $k > 1$ **then**
  $N \leftarrow \{H\}$
  **for** $i = 1, \dots, k$ **do**
    **for** $h'_i = 0, \dots, h_i - 1$ **do**
      $N \leftarrow N \cup \{H \oplus h_i \oplus h'_i\}$
  **return** mex$(N)$
**else**
  **return** $h_1 + 1$

Now consider SIMPLIFIED $\mathrm{CIG_g}$ on the entire tree $T$ and let $v_1, \dots, v_k$ be the neighbors of $v$ with $k \geq 2$. From the above, it is possible to determine the nimber $\ell_i$ of $(T, v, v_i)$ in SIMPLIFIED $\mathrm{CIG_g}$ from Algorithm 1. Then, as before, we have that the games in the subtrees are independent and the nimber of $(T, v)$ is exactly $h = \ell_1 \oplus \dots \oplus \ell_k$. This leads to Algorithm 2 in the tree $T$ rooted at $v$ to calculate the nimber $h$ of $(T, v)$ in SIMPLIFIED $\mathrm{CIG_g}$ in linear time on the size of $T$. Following the direct association with the Nim game in the Sprague-Grundy theory, we have that Alice has a winning strategy in the normal play if and only if $h > 0$. Moreover, Alice has a winning strategy in the misère play if and only if $h > 0$ and there is at least one subtree with more than one non-labeled vertex, or $h = 0$ and all subtrees with non-labeled vertices have exactly one non-labeled vertex. $\square$

**Algorithm 2** SIMPLIFIED-$\mathrm{CIG_g}$-tree$(T, v)$.

**let** $h \leftarrow 0$ and **let** $v_1, \dots, v_k$ be the neighbors of $v$ in $T$
**for** $i = 1, \dots, k$ **do**
  $h \leftarrow h \oplus$ SIMPLIFIED-$\mathrm{CIG_g}$-leaf$(T, v, v_i)$
**if** (normal play) **then**
  **if** $h > 0$ **then** Alice wins **else** Bob wins
**if** (misère play) **then**
  **if** all subtrees of $v$ have exactly one non-labeled vertex **then**
    **if** $h = 0$ **then** Alice wins **else** Bob wins
  **else**
    **if** $h > 0$ **then** Alice wins **else** Bob wins

In the example of the tree $T_2$ of Fig. 1, assuming that Alice labeled $c$ in her first move in the closed interval game $\mathrm{CIG_g}$, we have the position $(T_2, c)$ of SIMPLIFIED $\mathrm{CIG_g}$ where Bob is the first to play. Let $T'_2$ be obtained from the removal of $x_1$. Let us determine the nimber of $(T'_2, c)$. If Bob labels $v$, then we have two "heaps" with sizes $(4, 4)$, whose nimber is $4 \oplus 4 = 0$. If Bob labels other vertex of $T'_2$, we have a "heap" with size $k \leq 3$ and other "heap" with size 4. Then the nimber of $(T'_2, c)$ is $\mathrm{mex}\{4 \oplus 4, 4 \oplus 3, 4 \oplus 2, 4 \oplus 1, 4 \oplus 0\} = 1$. Finally, considering the vertex $x_1$ again, the nimber of $(T_2, c)$ is $1 \oplus 1 = 0$, which is winning in both normal and misère variants for the second player (which is Alice in this example).

**Corollary 2.3.** *The decision problems related to the normal and misère variants of the closed interval game* $\mathrm{CIG_g}$ *and the closed hull game* $\mathrm{CHG_g}$ *are solvable in time* $O(n^2)$ *on trees.*

**Proof.** Let $T$ be a tree. Alice has a winning strategy in $\mathrm{CIG_g}$ on $T$ if and only if there is a vertex $v$ such that the second player of SIMPLIFIED $\mathrm{CIG_g}$ on $(T, v)$ has a winning strategy, and we are done from Theorem 2.2. Moreover, in a tree, the closed hull game is equivalent to the closed interval game. From Theorem 2.2, SIMPLIFIED $\mathrm{CIG_g}$ on $(T, v)$ is linear time solvable for any $v$ and therefore, by checking for every vertex $v$, we can decide $\mathrm{CIG_g}$ and $\mathrm{CHG_g}$ on $T$ in time $O(n^2)$. $\square$

From the proof of Corollary 2.3, it seems more likely that Alice wins in trees. Then a natural question arises: for which trees does Bob have a winning strategy? The next lemma presents a large class of symmetric trees for which Bob wins $\mathrm{CIG_g}$ and $\mathrm{CHG_g}$ in both normal and misère variants.

**Theorem 2.4.** *Let $T'$ be a tree with vertices $v'_1, \dots, v'_n$ with $n \geq 2$ and let $T''$ be a copy of $T'$ with vertices $v''_1, \dots, v''_n$. For $i \in \{1, \dots, n\}$, let $T_i$ be the tree obtained from $T'$ and $T''$ by adding the edge $v'_i v''_i$. Then Bob wins the normal and misère variants of* $\mathrm{CIG_g}$ *and* $\mathrm{CHG_g}$ *in each tree $T_i$.*

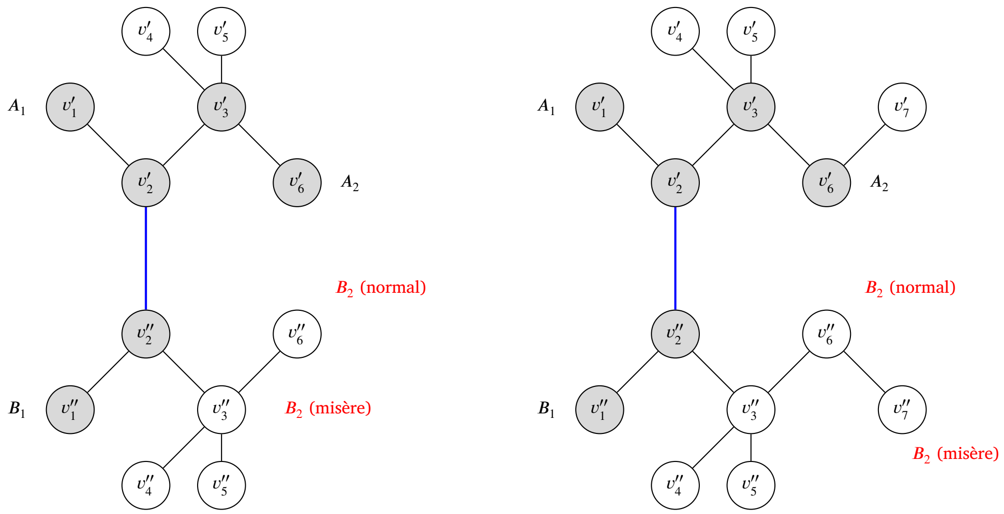



**Fig. 3.** Example of two symmetric trees of Theorem 2.4. Bob wins in both normal and misère variants. $A_1, A_2, B_1, B_2$ indicate the first and the second moves of Alice and Bob, respectively.

**Proof.** Let $i \in \{1, \dots, n\}$ and, to simplify notation, let $T = T_i$. We say that $v_j'$ and $v_j''$ are *brothers* for every $j = 1, \dots, n$. If Alice labels a vertex, Bob can always label its brother. This argument is sufficient for the normal variant.

Now consider the misère variant. As in the proof of Theorem 2.2, we may consider that vertices in the path between two labeled vertices are also labeled, since the games are also in a tree. An *unlabeled subtree* at some moment of the game is a maximal subtree of $T$ whose vertices are unlabeled. Suppose that Alice labeled the vertex $v_j'$. If all unlabeled subtrees have exactly one vertex, Bob labels $v_j''$, winning at the end. Then assume that there is an unlabeled subtree of at least two vertices. Bob considers the possibility of labeling $v_j''$. If all unlabeled subtrees would have exactly one vertex after this, then, instead of labeling $v_j''$, Bob labels the unlabeled neighbor of $v_j''$ for which, after this move, all unlabeled subtrees will have exactly one vertex. See Fig. 3 for an example. With this, all unlabeled subtrees will still have exactly one vertex, but the number of unlabeled vertices will be odd, and then Alice is the last to play and Bob wins the misère variant. □

In the examples of Fig. 3, if Alice labels $v_1'$ in the first turn, Bob labels $v_1''$ and then the vertices $v_2'$ and $v_2''$ cannot be labeled anymore. Suppose that Alice labels $v_6'$ in the second turn (and then $v_3'$ cannot be labeled anymore). In the normal variant, Bob continues to label the *brother* vertex ($v_6''$). In the misère variant, Bob realizes that labeling the brother vertex $v_6''$ will only generate unlabeled subtrees of size one and so, instead of labeling $v_6''$, he labels the neighbor of the brother $v_6''$ which guarantees that all unlabeled subtrees have exactly one vertex ($v_3''$ for the first tree and $v_7''$ for the second tree of Fig. 3).

## 3. PSPACE-hardness of impartial hull games

In this section, we prove that the normal and misère variants of hull games $\mathrm{HG_g}$ and $\mathrm{CHG_g}$ and their simplified versions are PSPACE-complete (see Definitions 1.1 and 2.1). As mentioned before, we consider the games as decision problems: given a graph, does Alice have a winning strategy? Since the number of turns is at most $n$ and, in each turn, the number of possible vertices to label is at most $n$, all these games are polynomially bounded two player games, which implies that they are in PSPACE [23].

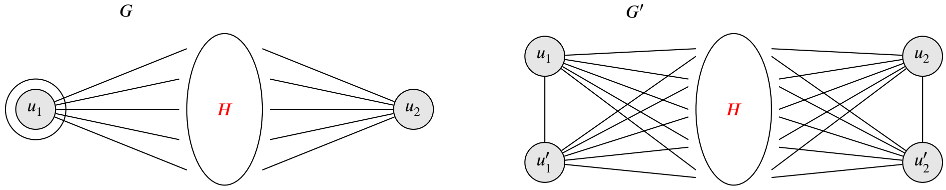
One of the main difficulties in proving PSPACE-hardness is to find the more suitable PSPACE-hard problem for the reduction, since many more problems are known to be NP-hard than PSPACE-hard. Fortunately, the CLIQUE FORMING game, which is PSPACE-complete [28], turned out to be very useful for both reductions. This game was recently used in reductions of other graph convexity games [6].

In this game, given a graph $G$, two players, Alice and Bob, starting by Alice, alternately select vertices and the subset of the chosen vertices must induce a clique. The last to play wins. This is related to the NODE KAYLES game, that is also PSPACE-complete [28], in which the objective is to get an independent set, instead of a clique. The CLIQUE FORMING game is the NODE KAYLES game played on the complement of the graph, and vice-versa.

**Theorem 3.1.** *The misère variants of* $\mathrm{HG_g}$ *and* SIMPLIFIED $\mathrm{HG_g}$ *are PSPACE-complete even in graphs with diameter two.*

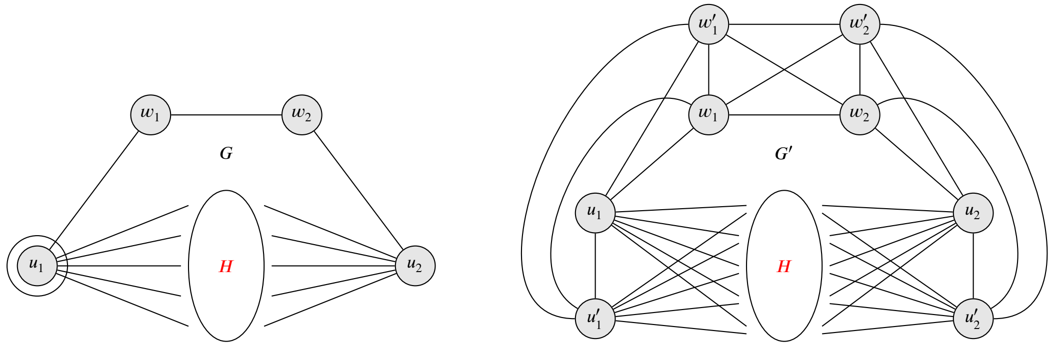
**Proof.** Let $H$ be an instance of the CLIQUE FORMING game. We may assume that $H$ is not complete. We first obtain a reduction for SIMPLIFIED $\mathrm{HG_g}$.

Let $G$ be the graph obtained from $H$ by adding two non-adjacent new vertices $u_1$ and $u_2$ adjacent to all vertices of $H$. See Fig. 4(a). Notice that $G$ has diameter 2. Also let $u_1$ be the vertex which is already labeled in SIMPLIFIED $\mathrm{HG_g}$. We prove that Alice has a winning strategy in the CLIQUE FORMING game on $H$ if and only if she has a winning strategy in the misère variant of SIMPLIFIED $\mathrm{HG_g}$ on




(a) Graph $G$ of the reduction to SIMPLIFIED $\mathrm{HG_g}$ (b) Graph $G'$ of the reduction to $\mathrm{HG_g}$

**Fig. 4.** Graphs of the reductions in Theorem 3.1 on the misère variants.




(a) Graph $G$ of the reduction to SIMPLIFIED $\mathrm{HG_g}$ (b) Graph $G'$ of the reduction to $\mathrm{HG_g}$

**Fig. 5.** Graphs of the reductions in Theorem 3.2 on the normal variants.

$(G, u_1)$. If a player labels $u_2$, he/she loses immediately, since $\mathrm{hull_g}(\{u_1, u_2\}) = V(G)$. Moreover, if a player labels a vertex $v_j$ of $H$ in the hull game $\mathrm{HG_g}$ and there is a non-adjacent labeled vertex $v_i$ in $H$, then the player loses immediately, since $\mathrm{hull_g}(\{v_i, v_j\}) = V(G)$. So, we may assume that the set $L$ of labeled vertices form a clique in all turns, except the last one. This is directly related to the CLIQUE FORMING game on $H$. If Alice has a winning strategy in the CLIQUE FORMING game on $H$, then Bob is the first to label a vertex of $G$ with a labeled non-neighbor, implying that he loses the misère hull game. Analogously, if Bob has a winning strategy in the CLIQUE FORMING game on $H$.

Now we obtain a reduction for $\mathrm{HG_g}$. Let the graph $G'$ be obtained from $G$ by adding two new vertices $u_1'$ and $u_2'$, adjacent to all vertices of $H$, and the edges $u_1'u_1$ and $u_2'u_2$. See Fig. 4(b). Notice that $G'$ has diameter 2. If Alice has a winning strategy in the CLIQUE FORMING game on $H$, then she plays $\mathrm{HG_g}$ in $G'$ according to her strategy in the vertices of $H$, unless Bob selects one of the vertices $u_1$ or $u_1'$ (resp. $u_2$ and $u_2'$) causing Alice to select the other. The same if Bob has a winning strategy in the CLIQUE FORMING game on $H$. If there are two non-adjacent selected vertices during the game, then $\mathrm{HG_g}$ is over and the last to play loses. Then, as before, the set $L$ of labeled vertices form a clique in all turns, except the last one, and we are done. □

**Theorem 3.2.** *The normal variants of* $\mathrm{HG_g}$ *and* SIMPLIFIED $\mathrm{HG_g}$ *are PSPACE-complete even in graphs with diameter two.*

**Proof.** Consider the same reduction of Theorem 3.1, but adding to $G$ two new vertices $w_1$ and $w_2$ and the edges $w_1w_2$, $w_1u_1$ and $w_2u_2$. See Fig. 5(a). Notice that $G$ has diameter 2.

We first obtain a reduction for SIMPLIFIED $\mathrm{HG_g}$. Let $u_1$ be the vertex which is already labeled in $G$. If a player labels $u_2$ (resp. $w_1$ or $w_2$), he/she loses immediately, since the opponent labels $w_1$ or $w_2$ (resp. $u_2$), winning the normal game because $\mathrm{hull_g}(\{u_1, u_2, w_1\}) = \mathrm{hull_g}(\{u_1, u_2, w_2\}) = V(G)$. Moreover, if a player labels a vertex $v_j$ of $H$ in the hull game and there is a non-adjacent labeled vertex $v_i$ in $H$, then the player loses immediately, since the opponent labels $w_1$ or $w_2$, winning the game because $\mathrm{hull_g}(\{v_i, v_j, w_1\}) = \mathrm{hull_g}(\{v_i, v_j, w_2\}) = V(G)$. So, we may assume that set $L$ of labeled vertices form a clique in all turns, except the last two. As before, this is directly related to the CLIQUE FORMING game on $H$: Alice has a winning strategy in the normal variant of SIMPLIFIED $\mathrm{HG_g}$ on $G$ if and only if she has a winning strategy in the CLIQUE FORMING game on $H$.

Now we obtain a reduction for the normal variant of $\mathrm{HG_g}$. Let the graph $G'$ be obtained from $G$ by adding four vertices $u_1'$, $u_2'$, $w_1'$ and $w_2'$, in such a way that $N[u_1'] = N[u_1]$, $N[u_2'] = N[u_2]$, $N[w_1'] = N[w_1]$ and $N[w_2'] = N[w_2]$. We say that vertices with the same closed neighborhood are twin vertices. See Fig. 5(b). Notice that $G'$ has diameter 2. In the game, if the set $L$ of labeled vertices is a clique, we say that we are in Phase 1; otherwise we are in Phase 2.

First suppose that Alice has a winning strategy in the CLIQUE FORMING game on $H$. Consider that the game is in Phase 1. Then Alice plays $\mathrm{HG_g}$ in $G'$ according to her winning strategy in the vertices of $H$. Then the first labeled vertex $v_0$ is in $H$. If Bob labels one of the vertices $u_1$ or $u_1'$ (resp. $u_2$ or $u_2'$), Alice labels $w_2$ (resp. $w_2'$), winning the game since $\mathrm{hull_g}(\{v_0, u_1, w_2\}) = V(G)$. Now consider that the game is in Phase 2. If Bob labels $w_1$ or $w_1'$ (resp. $w_2$ or $w_2'$), Alice labels $w_2$ (resp. $w_1$), winning the game, since $\mathrm{hull_g}(\{v_0, w_1, w_2\}) = V(G')$. Moreover, if Bob labels a vertex $v_j$ of $H$ and there is a non-adjacent labeled vertex $v_i$ in $H$, then Alice labels $w_1$ or $w_2$, winning the game because $\mathrm{hull_g}(\{v_i, v_j, w_1\}) = \mathrm{hull_g}(\{v_i, v_j, w_2\}) = V(G)$. With this, Alice is always the last to play.

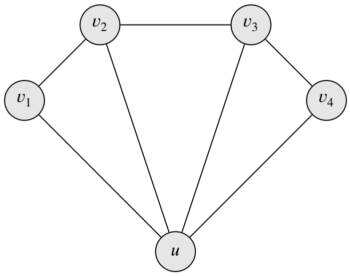



**Fig. 6.** Gem graph, obtained by including the universal vertex $u$ in the $P_4$ path $v_1 - v_2 - v_3 - v_4$. Notice that the gem is chordal, but it is not distance hereditary.

Now suppose that Bob has a winning strategy in the CLIQUE FORMING game on $H$. First consider that the first vertex $x$ labeled by Alice is not in $H$. Then Bob labels the twin $x'$ of $x$. In the second move of Bob, he selects either

- $w_1$ if $L \subseteq \{u_2, u'_2\} \cup V(H)$, or
- $w_2$ if $L \subseteq \{u_1, u'_1\} \cup V(H)$, or
- $u_1$ if $L \subseteq \{w_2, w'_2\} \cup V(H)$ or $L \subseteq \{u_2, u'_2, w_1, w'_1\}$ or $L \subseteq \{u_2, u'_2, w_2, w'_2\}$, or
- $u_2$ if $L \subseteq \{w_1, w'_1\} \cup V(H)$ or $L \subseteq \{u_1, u'_1, w_1, w'_1\}$ or $L \subseteq \{u_1, u'_1, w_2, w'_2\}$,

winning the game immediately. Finally, consider that the first labeled vertex is in $H$. Then the same procedure of the last paragraph can be applied changing the roles of Alice and Bob. With this, Bob is always the last to play. □

It is not difficult to check that the arguments in the proofs of Theorems 3.1 and 3.2 are also valid for the closed hull game $\mathrm{CHG_g}$.

**Corollary 3.3.** *The normal and misère variants of the hull game* $\mathrm{HG_g}$*, the closed hull game* $\mathrm{CHG_g}$ *and their simplified versions are PSPACE-complete even in graphs with diameter two.*

## 4. Impartial convexity games in Ptolemaic graphs

Given a graph $G$ and a convex set $S$ of $G$ in the geodesic convexity, we say that a vertex $v \in S$ is *extreme* in $S$ if $S \setminus \{v\}$ is also convex. The *extreme vertices of the graph* $G$ are the extreme vertices of $V(G)$. In 1986, Farber and Jamison [17] proved that a vertex is extreme in the geodesic convexity of a graph if and only if it is simplicial (its closed neighborhood forms a clique). Let $\mathrm{Ext_g}(G)$ be the set of extreme vertices of $G$ in the geodesic convexity, which are the simplicial vertices.

Farber and Jamison [17] also defined a graph convexity as *geometric* (also called *convex geometry*) if it satisfies the *Minkowski-Krein-Milman* property: every convex set is the convex hull of its extreme vertices. The research on graph convex geometries is usually concentrated in determining the graph class in which a given graph convexity is geometric. They proved that the geodesic convexity is geometric in a graph if and only if the graph is Ptolemaic [17], that is, it is chordal (every induced cycle is a triangle) and distance hereditary (every induced path is a shortest path).

Ptolemaic graphs are also characterized as the graphs satisfying the Ptolemy's inequality: for any four vertices $v_1, v_2, v_3, v_4$, we have $d(v_1, v_2)d(v_3, v_4) + d(v_1, v_4)d(v_2, v_3) \geq d(v_1, v_3)d(v_2, v_4)$, where $d(x, y)$ is the distance between two vertices $x$ and $y$ in the graph. They are also the gem-free chordal graphs, where the gem is the graph of Fig. 6, obtained by including a universal vertex in the $P_4$ path.

In this section, we use these results to obtain winning strategies for $\mathrm{HG_g}$ and $\mathrm{IG_g}$ on Ptolemaic graphs. We first prove a general result on the geodesic convexity. We say that $S \subseteq V(G)$ is a *geodesic hull set* (resp. *geodesic interval set*) if $\mathrm{hull_g}(S) = V(G)$ (resp. $\mathrm{I_g}(S) = V(G)$).

**Theorem 4.1.** *Let $G$ be a graph such that* $\mathrm{Ext_g}(G)$ *is a geodesic hull set (resp. geodesic interval set). Then Alice wins* $\mathrm{HG_g}$ *(resp.* $\mathrm{IG_g}$*) if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant.*

**Proof.** Let $S = \mathrm{Ext_g}(G)$. Recall that $V(G) \setminus \{s\}$ is convex for every $s \in S$. That is, every vertex of $S$ must be labeled in $\mathrm{HG_g}$ and $\mathrm{IG_g}$. If $\mathrm{Ext_g}(G)$ is a geodesic hull set (resp. geodesic interval set), then $\mathrm{HG_g}$ (resp. $\mathrm{IG_g}$) ends when the last vertex of $S$ is labeled. The following arguments are valid for both $\mathrm{HG_g}$ and $\mathrm{IG_g}$. First consider the normal variant. Alice wins if she labels the last unlabeled vertex of $S$. If $n$ is odd, Alice plays avoiding the last two vertices of $S$ forcing Bob to label the penultimate vertex of $S$, Alice winning the game. If $n$ is even, Bob wins (the argument is the same, replacing Alice by Bob). Now consider the misère variant. Alice wins if Bob labels the last unlabeled vertex of $S$. If $n$ is even, Alice plays avoiding the last vertex of $S$ forcing Bob to label it, Alice winning the game. If $n$ is odd, Bob wins (the argument is the same, replacing Alice by Bob). □

As an easy example of Theorem 4.1, shown in Fig. 7, consider again the graph $G$ obtained in the proof of Theorem 3.1 by adding two non-adjacent vertices $u$ and $u_1$ adjacent to all vertices of a given graph $H$, shown in Fig. 4. Now let $G'$ be obtained from $G$

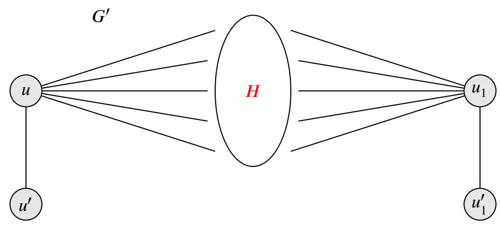



**Fig. 7.** Graph $G'$ of the example.

by adding two new vertices $u'$ and $u'_1$ and the edges $u'u$ and $u'_1u_1$. Notice that $\mathrm{Ext}_g(G') = \{u', u'_1\}$ and $\mathrm{hull}_g(\{u', u'_1\}) = \mathrm{I}_g(\{u', u'_1\}) = V(G')$, that is, $\{u', u'_1\}$ is a geodesic hull set and a geodesic interval set of $G'$. Then, independently of the structure of the graph $H$ in the construction of this example, we have that Alice has a winning strategy in $\mathrm{HG}_g$ and $\mathrm{IG}_g$ on the graph $G'$ in the normal (resp. misère) variant if and only if $H$ has an odd (resp. even) number of vertices. This is because the player forced to label the first (resp. the last) vertex of $\{u', u'_1\}$ loses the normal (resp. misère) variant of $\mathrm{HG}_g$ and $\mathrm{IG}_g$ on $G'$.

With this, we find an interesting connection between convex geometries and winning strategies for $\mathrm{HG}_g$. The theorem below extends a result of Haynes, Henning and Tiller [22] in 2003 on block graphs, since every block graph is a Ptolemaic graph.

**Theorem 4.2.** *Let $G$ be a Ptolemaic graph. Then Alice wins $\mathrm{IG}_g$ and $\mathrm{HG}_g$ if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant.*

**Proof.** Regarding the game $\mathrm{HG}_g$, the result follows from Theorem 4.1 and the fact that the geodesic convexity is geometric in Ptolemaic graphs [17]. Regarding the game $\mathrm{IG}_g$, the result holds since Ptolemaic graphs are chordal and distance-hereditary, every vertex of a chordal graph is in an induced path between two simplicial vertices and every induced path is a shortest path in a distance hereditary graph. □

## 5. Other convexities and other variations

Other contribution of this paper is the generalization of these games to any graph convexity, obtaining very natural convexity games and enriching the prolific research area of graph convexity. We also prove general results. For this, let us define general convexity in graphs.

A *convexity* $\mathcal{C}$ [30] on a finite set $V \neq \emptyset$ is a family of subsets of $V$ such that $\emptyset, V \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. That is, $S_1, S_2 \in \mathcal{C}$ implies $S_1 \cap S_2 \in \mathcal{C}$. A member of $\mathcal{C}$ is said to be a *$\mathcal{C}$-convex set*. Given $S \subseteq V$, the *$\mathcal{C}$-convex hull* of $S$ is the smallest $\mathcal{C}$-convex set $\mathrm{hull}_{\mathcal{C}}(S)$ containing $S$. We say that $S$ is a *$\mathcal{C}$-hull set* if $\mathrm{hull}_{\mathcal{C}}(S) = V$. It is easy to see that $\mathrm{hull}_{\mathcal{C}}(\cdot)$ is a *closure operator*, that is, for every $S, S' \subseteq V$: (a) $S \subseteq \mathrm{hull}_{\mathcal{C}}(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{hull}_{\mathcal{C}}(S) \subseteq \mathrm{hull}_{\mathcal{C}}(S')$ [monotonicity], (c) $\mathrm{hull}_{\mathcal{C}}(\emptyset) = \emptyset$ [normalization[2]] and (d) $\mathrm{hull}_{\mathcal{C}}(\mathrm{hull}_{\mathcal{C}}(S)) = \mathrm{hull}_{\mathcal{C}}(S)$ [idempotence].

We say that $\mathrm{I} : 2^V \to 2^V$ is an *interval function* on $V$ if, for every $S, S' \subseteq V$, (a) $S \subseteq \mathrm{I}(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{I}(S) \subseteq \mathrm{I}(S')$ [monotonicity] and (c) $\mathrm{I}(\emptyset) = \emptyset$ [normalization]. It is known that every interval function *induces* a unique convexity, containing each set $S \subseteq V$ such that $\mathrm{I}(S) = S$. Moreover, every convexity is induced by an interval function. We then assume that every convexity $\mathcal{C}$ on $V$ is defined by an explicitly given interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ on $V$. It is also known that the convex hull of a set $S$ in a convexity $\mathcal{C}$ can be obtained by exhaustively applying the corresponding interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ until obtaining a $\mathcal{C}$-convex set. We say that $S$ is a *$\mathcal{C}$-interval set* if $\mathrm{I}_{\mathcal{C}}(S) = V$.

Given a graph $G$, a *graph convexity* on $G$ is simply a convexity $\mathcal{C}$ on $V(G)$ with a given interval function $\mathrm{I}_{\mathcal{C}}(\cdot)$ on $V(G)$. A standard way to define a *graph convexity* $\mathcal{C}$ on a graph $G$ is by fixing a family $\mathcal{P}$ of paths of $G$ and taking the interval function $\mathrm{I}_{\mathcal{C}}(S)$ as the set $S$ with all vertices lying on some path of $\mathcal{P}$ whose endpoints are in $S$. The most studied graph convexities are path convexities, such as the *geodesic convexity* [16,18], the *$P_3$ convexity* [1], the *monophonic convexity* [10,14,17], the *$m^3$ convexity* [12] and the *triangle path convexity* [7,11], where $\mathcal{P}$ is the family of all geodesics (shortest paths) of the graph, of all paths of order three, of all induced paths, of all induced paths of size at least 3 and of all paths $v_1, \ldots, v_k$ with no edge $v_iv_j$ with $|j - i| > 2$, respectively.

This definition also applies to directed graphs. For example, there are the geodesic convexity, the monophonic convexity and the $P_3$ convexity for directed graphs, related to shortest paths, induced paths and paths with three vertices in the directed graph, respectively, where the paths must respect the orientation of the edges ("*arcs*").

Given a graph convexity $\mathcal{C}$ on a graph $G$, let the *$\mathcal{C}$-hull number* $\mathrm{hn}_{\mathcal{C}}(G)$ be the size of a minimum $\mathcal{C}$-hull set of $G$ and let the *$\mathcal{C}$-interval number* $\mathrm{in}_{\mathcal{C}}(G)$ be the size of a minimum $\mathcal{C}$-interval set of $G$.

In the following, we generalize the convexity games of Definition 1.1 to any graph convexity.

[2] Here we follow the definition of closure operator from van de Vel [30], which includes the normalization property.

**Definition 5.1.** Given a graph convexity $C$ on a graph $G$, we introduce four convexity games. In the games defined below, the set $L$ of labeled vertices is initially empty and the definitions of $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex $v$ which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.

- In the $C$-hull game $\mathrm{HG}_C$: $f_1(L) = L$ and $f_2(L) = \mathrm{hull}_C(L)$.
- In the $C$-interval game $\mathrm{IG}_C$: $f_1(L) = L$ and $f_2(L) = \mathrm{I}_C(L)$.
- In the closed $C$-hull game $\mathrm{CHG}_C$: $f_1(L) = f_2(L) = \mathrm{hull}_C(L)$.
- In the closed $C$-interval game $\mathrm{CIG}_C$: $f_1(L) = f_2(L) = \mathrm{I}_C(L)$.

As before, the last to play wins in the normal variant of these games, the last to play loses in the misère variants and the decision problem associated to these games is whether Alice has a winning strategy.

Finally, there is one more variant that is usually studied in the literature of games, the *optimization variant*. In addition to the graph $G$, the instance also has a positive integer $k$, and Alice wins if the set $L$ of labeled vertices at the end is at most $k$. That is, Alice is the cooperative player which wants to optimize $L$ (by minimizing it), while Bob is the non-cooperative player which wants to disturb Alice (by maximizing $L$), no matter who ends the game. As an example of optimization variant of a well known game, we mention the *graph coloring game* and the *game chromatic number* [2,8,24,25]. The optimization variant leads to four natural graph parameters on convexity games, related to the optimum value such that Alice has a winning strategy.

**Definition 5.2.** Given a convexity $C$ on a graph $G$, let the four convexity game parameters

- *game $C$-hull number* $\mathrm{ghn}_C(G)$,
- *game $C$-interval number* $\mathrm{gin}_C(G)$,
- *closed game $C$-hull number* $\mathrm{cghn}_C(G)$,
- *closed game $C$-interval number* $\mathrm{cgin}_C(G)$

be the minimum $k$ (size of the set $L$ of labeled vertices) such that Alice has a winning strategy in the optimization variant of the corresponding convexity games of Definition 5.1 on the graph $G$.

The next theorem shows general inequalities with the game parameters for any graph convexity.

**Theorem 5.3.** *Let $C$ be a convexity on a graph $G$. Then*

- $\mathrm{hn}_C(G) \le \mathrm{cghn}_C(G) \le \mathrm{ghn}_C(G) \le \min\{\, 2\cdot \mathrm{hn}_C(G) - 1,\ n \,\}$,
- $\mathrm{in}_C(G) \le \mathrm{gin}_C(G) \le \min\{\, 2\cdot \mathrm{in}_C(G) - 1,\ n \,\}$,
- $\mathrm{in}_C(G) \le \mathrm{cgin}_C(G) \le n$.

**Proof.** First consider the hull games. Since the set $L$ of labeled vertices in the optimization variant of the games $\mathrm{CHG}_C$ and $\mathrm{HG}_C$ must be a hull set in the convexity $C$ on the graph $G$, according to Definition 5.1, we have that $|L| \ge \mathrm{hn}_C(G)$. Also, note that the closed game constraint only affects Bob, as Alice is not interested in playing on a vertex already belonging to the convex hull of $L$, since she wants to minimize $|L|$. From this, we have $\mathrm{hn}_C(G) \le \mathrm{cghn}_C(G) \le \mathrm{ghn}_C(G)$. Finally, if Alice always labels a vertex of a minimum hull set $S$, the size of $L$ at the end will be at most $2\cdot|S| - 1$, since Alice labels at most $|S|$ vertices of $S$, Bob labels at most $|S| - 1$ vertices outside $S$ and the game ends after the last vertex of $S$ is labeled. Then $\mathrm{ghn}_C(G) \le 2\cdot \mathrm{hn}_C(G) - 1$. The result for the interval games is analogous. The difference is that the constraint of the closed interval game also affects Alice (see Fig. 8 for an example). □

As an example on the geodesic convexity, consider the graph $G_k$ of Fig. 8 with $4k$ black vertices. In the game $\mathrm{IG_g}$, Alice wants to label the vertices $\{a_1, \ldots, a_8\}$, which is the unique minimum geodesic interval set, with size $\mathrm{in}(G_k) = 8$. Bob can always label a vertex outside this set, and then $\mathrm{gin_g}(G_k) = 2\cdot 8 - 1 = 15$. In the game $\mathrm{CIG_g}$, Bob can always forbid the vertices $\{v_1, \ldots, v_4, a_1, \ldots, a_8\}$ in his first move. For example, if Alice labels $a_1$, Bob labels a black vertex between $a_5$ and $a_6$, forbidding all those vertices, except $a_2$. Alice then labels $a_2$ and, from this point, all moves must be on black vertices, summing at most $\mathrm{cgin_g}(G_k) = 3k + 2$, which is greater than 15 for $k \ge 5$. In the geodesic hull games $\mathrm{HG_g}$ and $\mathrm{CHG_g}$, Alice wins in her second move and then $\mathrm{hn_g}(G_k) = 2 < 3 = \mathrm{cghn_g}(G_k) = \mathrm{ghn_g}(G_k)$.

For examples in the $P_3$ convexity, see Table 1. Notice that $K_n (n \ge 3)$ and $C_5$ achieve the lower bound of Theorem 5.3, $C_4$ and $C_6$ achieve the upper bound, and $C_7$ is in the middle. We use the subscripts "m", "g" and "p3" to indicate the monophonic, geodesic and $P_3$ convexities, respectively.

The next lemmas present results on simple graph classes.

**Lemma 5.4.** *Let $n \ge 2$. Alice loses the normal variant and wins the misère variant of the four convexity games in the $P_3$ convexity on the complete graph $K_n$. Moreover, all the game numbers of Definition 5.2 are $n$ in the monophonic and geodesic convexities and are 2 in the $P_3$ convexity.*

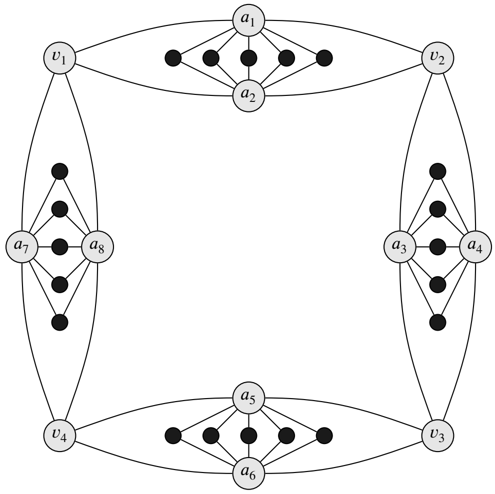



**Fig. 8.** The graph $G_k$ with $4k$ black vertices: $\mathrm{cgin_g}(G_k) > \mathrm{gin_g}(G_k)$ for $k \geq 5$.

**Table 1**
Game parameters $\mathrm{ghn_{p3}}(\cdot)$ and $\mathrm{cghn_{p3}}(\cdot)$ for the cycles $C_3, \ldots, C_7$ and $K_n$ for $n \geq 3$. Lower and upper bounds of Theorem 5.3 in the first and last row, respectively.

| Game parameter | $K_n$ | $C_4$ | $C_5$ | $C_6$ | $C_7$ |
|---|---|---|---|---|---|
| *$P_3$-hull number* $\mathrm{hn_{p3}}(\cdot)$ | 2 | 2 | 3 | 3 | 4 |
| *closed game $P_3$-hull number* $\mathrm{cghn_{p3}}(\cdot)$ | 2 | 3 | 3 | 4 | 5 |
| *game $P_3$-hull number* $\mathrm{ghn_{p3}}(\cdot)$ | 2 | 3 | 3 | 5 | 5 |
| upper bound of Theorem 5.3 | 3 | 3 | 5 | 5 | 7 |

**Proof.** In the monophonic and geodesic convexities, every vertex must be labeled in all games. In the $P_3$ convexity, all games finish in the second move. □

**Lemma 5.5.** *Let $n \geq 4$. Bob always wins the normal and the misère variants of the four monophonic convexity games on the cycle $C_n$. Moreover,* $\mathrm{ghn_m}(C_n) = \mathrm{gin_m}(C_n) = \mathrm{cghn_m}(C_n) = \mathrm{cgin_m}(C_n) = 3$.

**Proof.** In the normal variant, Bob just labels a vertex non-adjacent to the first vertex labeled by Alice. In the misère variant, Bob just labels a vertex adjacent to the first vertex labeled by Alice. □

Regarding the geodesic convexity on cycles, Buckley and Harary [5] proved the following.

**Lemma 5.6** *([5]). Let $n \geq 3$. Alice wins the normal* $\mathrm{IG_g}$ *on $C_n$ if and only if $n$ is odd. Moreover, Alice wins the misère* $\mathrm{IG_g}$ *on $C_n$ if and only if $n \mod 4$ is 1 or 2.*

We then prove the following lemma regarding the closed variants $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$.

**Lemma 5.7.** *Let $n \geq 4$. Alice wins the normal variant of* $\mathrm{CHG_g}$ *and* $\mathrm{CIG_g}$ *on $C_n$ if and only if $n$ is odd. Moreover, Bob always wins the misère variant of* $\mathrm{CHG_g}$ *and $CIG_g$ on $C_n$. Finally,* $\mathrm{cghn_g}(C_n) = \mathrm{cgin_g}(C_n) = \mathrm{ghn_g}(C_n) = \mathrm{gin_g}(C_n) = 3$.

**Proof.** Consider $C_n$ as the cycle $v_1, v_2, \ldots, v_n$ and assume that Alice labels $v_1$ in her first turn. In the normal variants, Bob labels $v_{\lfloor n/2 \rfloor + 1}$. If $n$ is even, he wins; otherwise, he loses. In the misère variants, Bob labels $v_{\lceil n/2 \rceil}$ and Alice loses in the next move. □

## 6. General convexity games in convex geometries

Recall that a convexity $C$ on a graph $G$ is *geometric* (also called *antimatroid* or *convex geometry*) if it satisfies the *Minkowski–Krein–Milman* property: every convex set is the convex hull of its extreme vertices in the convexity $C$, where a vertex $v$ is an *extreme vertex* of $S$ in the convexity $C$ if $S \setminus \{v\}$ is also a $C$-convex set. Let $\mathrm{Ext}_C(S)$ be the set of extreme vertices of $S$.

The research on graph convex geometries (or geometric convexities) is usually concentrated in determining the graph class in which a given graph convexity is geometric. In 1986, Farber and Jamison [17] proved that the monophonic (resp. geodesic) convexity is geometric if and only if the graph is chordal (resp. Ptolemaic). In 1999, Dragan et al. [12] proved that the $m^3$ convexity is geometric if and only if the graph is weak bipolarizable. Recently, it was proved that the triangle path (resp. $P_3$) convexity is geometric if and only if the graph is a forest (resp. forest of stars) [9].

**Theorem 6.1.** *Let $C$ be a graph convexity on a graph $G$ such that $\mathrm{Ext}_C(G)$ is a $C$-hull set (resp. $C$-interval set). Then Alice wins $\mathrm{HG}_C$ (resp. $\mathrm{IG}_C$) if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant. Moreover, in the optimization variant, $\mathrm{ghn}_C(G) = \ell$ (resp. $\mathrm{gin}_C(G) = \ell$), where $\ell = \min\left\{\, 2 \cdot |\mathrm{Ext}_C(G)| - 1,\ n \,\right\}$.*

**Proof.** Regarding the normal and misère variants, the proof follows the same structure of the proof of Theorem 4.1, replacing the geodesic convexity g with the convexity $C$. Regarding the optimization variant, notice that Alice wants to label the extreme vertices as soon as possible, and Bob wants to avoid the extreme vertices. Then the number of labeled vertices is equal to $\ell' = 2 \cdot |\mathrm{Ext}_C(G)| - 1$ if $\ell' \le n$, since we have $|\mathrm{Ext}_C(G)|$ vertices labeled by Alice, who is the first and the last to play, and $|\mathrm{Ext}_C(G)| - 1$ vertices labeled by Bob. □

With this, we find an interesting connection between general convex geometries and winning strategies of hull games. The next theorem is a generalization of Theorem 4.1, which was focused on the geodesic convexity.

**Theorem 6.2.** *Let $C$ be a convex geometry on a graph $G$. Then Alice wins $\mathrm{HG}_C$ if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant. Moreover, $\mathrm{ghn}_C(G) = \min\left\{\, 2 \cdot |\mathrm{Ext}_C(G)| - 1,\ n \,\right\}$.*

**Proof.** Since $C$ is a convex geometry on $G$ and, from the definition of convexity, $V(G)$ is a $C$-convex set, then $\mathrm{Ext}_C(G)$ is a $C$-hull set, and we are done from Theorem 6.1. □

As a consequence, we have the following.

**Corollary 6.3.** *In the monophonic convexity of chordal graphs, Alice wins $\mathrm{HG}_{\mathrm{m}}$ and $\mathrm{IG}_{\mathrm{m}}$ if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant. Moreover, in the $P_3$ convexity of any rooted tree $T$ in which every non-leaf node has at least two children, $\mathrm{ghn}_{\mathrm{p3}}(T) = n$, and Alice wins $\mathrm{HG}_{\mathrm{p3}}$ on $T$ if and only if $n$ is odd in the normal variant or $n$ is even in the misère variant.*

**Proof.** The results on the hull games follow from Theorem 6.1. Regarding the interval games, Farber and Jamison [17] proved that the extreme vertices of a graph in the geodesic and monophonic convexities are the simplicial vertices, which are the vertices whose neighborhood is a clique. Moreover, it is known that the simplicial vertices of a chordal graph form a monophonic interval set (every vertex is in an induced path between two simplicial vertices). Then the result follows from Theorem 6.2.

Finally, notice that $\mathrm{Ext}_{\mathrm{p3}}(T)$ is the set of leaves of $T$ and is a $P_3$-hull set of $T$, since every non-leaf node has two children, and we are done from Theorem 6.1. Furthermore, $n \le 2 \cdot |\mathrm{Ext}_{\mathrm{p3}}(T)| - 1$. □

## 7. Conclusions and final remarks

In this paper, we investigated the impartial geodesic convexity games $\mathrm{IG}_{\mathrm{g}}$, $\mathrm{HG}_{\mathrm{g}}$, $\mathrm{CIG}_{\mathrm{g}}$ and $\mathrm{CHG}_{\mathrm{g}}$ of Definition 1.1, introduced by Harary [4,5,20] in 1984. Below we list our results:

- We proved that $\mathrm{CIG}_{\mathrm{g}}$ and $\mathrm{CHG}_{\mathrm{g}}$ are polynomial time solvable in trees, in Theorem 2.3, using the Sprague-Grundy theory;
- We proved that $\mathrm{IG}_{\mathrm{g}}$ and $\mathrm{HG}_{\mathrm{g}}$ are polynomial time solvable in Ptolemaic graphs, extending the result of [22] on block graphs, in Theorem 4.2, using results on convex geometries, which are graph convexities satisfying the Minkowski-Krein-Milman property;
- We obtained the first PSPACE-hardness results on convexity games by proving that the normal and misère variants of $\mathrm{HG}_{\mathrm{g}}$ and $\mathrm{CHG}_{\mathrm{g}}$ and their simplified versions are PSPACE-complete even in graphs with diameter two, in Corollary 3.3, using reductions from the CLIQUE FORMING game;

Finally, in Section 5, we also generalize these games to any graph convexity, such as the $P_3$ convexity and the monophonic convexity, and define the optimization variants of them, introducing four natural game convexity parameters, such as the game hull number and the game interval number. We find general lower and upper bounds for these parameters and we determine their values for well known graph convexities in simple graph classes. We finish with Theorems 6.2 with winning strategies on $\mathrm{HG}_C$ for any convex geometry $C$, which is a generalization of Theorem 4.2.

Below we list some open problems:

- PROBLEM 1: For which trees Bob wins the game $\mathrm{CIG}_{\mathrm{g}}$?
- PROBLEM 2: The game $\mathrm{CIG}_{\mathrm{g}}$ is polynomial time solvable in superclasses of trees, such as unicyclic graphs or cactus graphs?
- PROBLEM 3: The geodesic interval games $\mathrm{IG}_{\mathrm{g}}$ and $\mathrm{CIG}_{\mathrm{g}}$ are PSPACE-complete?
- PROBLEM 4: The game convexity parameters $\mathrm{ghn}_{\mathrm{g}}(G)$, $\mathrm{gin}_{\mathrm{g}}(G)$, $\mathrm{cghn}_{\mathrm{g}}(G)$ and $\mathrm{cgin}_{\mathrm{g}}(G)$ are PSPACE-hard?
- PROBLEM 5: The game convexity parameters $\mathrm{ghn}_C(G)$, $\mathrm{gin}_C(G)$, $\mathrm{cghn}_C(G)$ and $\mathrm{cgin}_C(G)$ are PSPACE-hard when $C$ is the $P_3$ convexity or the monophonic convexity?

## CRediT authorship contribution statement

**Samuel N. Araújo:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing – original draft, Writing – review & editing. **João Marcos Brito:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft. **Raquel Folz:** Conceptualization, Investigation, Methodology, Validation. **Rosiane de Freitas:** Conceptualization, Formal analysis, Funding acquisition, Investigation, Validation. **Rudini M. Sampaio:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

## References

[1] Rommel M. Barbosa, Erika M.M. Coelho, Mitre C. Dourado, Dieter Rautenbach, Jayme L. Szwarcfiter, On the Carathéodory number for the convexity of paths of order three, SIAM J. Discrete Math. 26 (3) (2012) 929–939.
[2] Hans L. Bodlaender, On the Complexity of Some Coloring Games, Int. J. Found. Comput. Sci. 2 (2) (1991) 133–147. WG-1990, LNCS, vol. 484, 1990, pp. 30–40.
[3] Charles L. Bouton, Nim, a game with a complete mathematical theory, Ann. Math. 3 (1/4) (1901) 35–39.
[4] Fred Buckley, Frank Harary, Closed geodetic games for graphs, in: Proceedings of the 16th Southeastern Conference on Combinatorics, Graph Theory and Computing, Congr. Numer. 47 (1985) 131–138.
[5] Fred Buckley, Frank Harary, Geodetic games for graphs, Quaest. Math. 8 (1985) 321–334.
[6] Ullas Chandran S.V., Sandi Klavžar, Neethu P.K., Rudini Sampaio, The general position avoidance game and hardness of general position games, Theor. Comput. Sci. 988 (2024) 114370.
[7] Manoj Changat, Joseph Mathew, On triangle path convexity in graphs, Discrete Math. 206 (1) (1999) 91–95.
[8] Eurinardo Costa, Victor Lage Pessoa, Rudini Sampaio, Ronan Soares, PSPACE-completeness of two graph coloring games, Theor. Comput. Sci. 824–825 (2020) 36–45.
[9] Mitre C. Dourado, Marisa Gutierrez, Fábio Protti, Rudini Sampaio, Silvia Tondato, Characterizations of graph classes via convex geometries: a survey, 2023, arXiv.
[10] Mitre C. Dourado, Fábio Protti, Jayme L. Szwarcfiter, Complexity results related to monophonic convexity, Discrete Appl. Math. 158 (12) (2010) 1268–1274.
[11] Mitre C. Dourado, Rudini M. Sampaio, Complexity aspects of the triangle path convexity, Discrete Appl. Math. 206 (2016) 39–47.
[12] Feodor F. Dragan, Falk Nicolai, Andreas Brandstädt, Convexity and HHD-free graphs, SIAM J. Discrete Math. 12 (1) (1999) 119–135.
[13] Pierre Duchet, Convexity in combinatorial structures, in: Proc. 14th Winter School on Abstract Analysis, Circolo Matematico di Palermo, 1987, pp. 261–293.
[14] Pierre Duchet, Convex sets in graphs, II. Minimal path convexity, J. Comb. Theory, Ser. B 44 (3) (1988) 307–316.
[15] P. Erdős, E. Fried, A. Hajnal, E.C. Milner, Some remarks on simple tournaments, Algebra Univers. 2 (1972) 238–245.
[16] Martin G. Everett, Stephen B. Seidman, The hull number of a graph, Discrete Math. 57 (3) (1985) 217–223.
[17] Martin Farber, Robert E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebraic Discrete Methods 7 (3) (1986) 433–444.
[18] Martin Farber, Robert E. Jamison, On local convexity in graphs, Discrete Math. 66 (3) (1987) 231–247.
[19] P.M. Grundy, Mathematics and games, Eureka 2 (1939) 6–8.
[20] Frank Harary, Convexity in graphs: achievement and avoidance games, in: Annals of Discrete Mathematics (20), vol. 87, North-Holland, 1984, p. 323.
[21] Frank Harary, Juhani Nieminem, Convexity in graphs, J. Differ. Geom. 16 (1) (1981) 185–190.
[22] T.W. Haynes, M.A. Henning, C. Tiller, Geodetic achievement and avoidance games for graphs, Quaest. Math. 26 (2003) 389–397.
[23] R. Hearn, E. Demaine, Games, Puzzles and Computation, A. K. Peters Ltd, 2009.
[24] Carlos V.G.C. Lima, Thiago Marcilon, Nicolas Martins, Rudini Sampaio, PSPACE-hardness of variants of the graph coloring game, Theor. Comput. Sci. 909 (2022) 87–96.
[25] Carlos V.G.C. Lima, Thiago Marcilon, Nicolas Martins, Rudini Sampaio, The connected greedy coloring game, Theor. Comput. Sci. 940 (2023) 1–13.
[26] M. Nečásková, A note on the achievement geodetic games, Quaest. Math. 12 (1988) 115–119.
[27] I.M. Pelayo, Geodesic Convexity in Graphs, Springer, 2013.
[28] T. Schaefer, On the complexity of some two-person perfect-information games, J. Comput. Syst. Sci. 16 (2) (1978) 185–225.
[29] R. Sprague, Über mathematische Kampfspiele, Tohoku Math. J. 41 (1936) 438–444.
[30] Marcel L.J. van de Vel, Theory of Convex Structures, vol. 50, Elsevier, 1993.
[31] E. Zermelo, Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels, in: Proc. 5th Int. Congress of Mathematicians, 1913, pp. 501–504.