Theoretical Computer Science 1044 (2025) 115267

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Algorithms and complexity of graph convexity partizan games

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ARTICLE INFO

Section Editor: Paul G. Spirakis

Handling Editor: Yong Chen

Keywords:
Graph convexity
Geodesic games
PSPACE-hard
Combinatorial game theory
Partizan games
Surreal

ABSTRACT

In 2024, the first PSPACE-hardness result on impartial convexity games was proved, after a gap of 40 years from the introduction of these games by Frank Harary, in 1984. In this paper, we introduce the partizan variants of these impartial games on the geodesic convexity and extend them to other graph convexities, obtaining winning strategies and complexity results. We first prove that the geodesic partizan hull game is PSPACE-complete, for both normal and misère variants. We also obtain a polynomial time algorithm to decide the winner of the geodesic partizan hull game in trees, using surreal numbers and Conway's combinatorial game theory on partizan games. Finally, we obtain winning strategies for partizan convexity games when the graph convexity forms a convex geometry.

1. Introduction

The first graph convexity games were introduced in 1984 by Frank Harary [20], which are impartial games focused on the geodesic convexity. In this paper, we introduce the partizan variants of these impartial games on the geodesic convexity and extend them to other graph convexities, obtaining complexity and algorithmic results. In order to explain them, we need some terminology.

Given a graph G and a set $S \subseteq V(G)$, let the *geodesic interval* $\mathrm{I}_g(S)$ be the set S and every vertex in a shortest path between two vertices of S. We say that S is *convex in the geodesic convexity* [17,19] if $\mathrm{I}_g(S) = S$. The *geodesic convex hull* of S is the minimum convex set $\mathrm{hull}_g(S)$ containing S. It is known that $\mathrm{hull}_g(S)$ can be obtained by applying $\mathrm{I}_g(\cdot)$ from S until obtaining a convex set.

Among the games introduced by Harary [6,7,20], we have the *geodesic interval game*, the *geodesic hull game*, the *geodesic closed interval game* and the *geodesic closed hull game*, defined below.

Definition 1.1. Let G be a graph. In the games defined below, the set L of labeled vertices is initially empty and the definitions of $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex v which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.

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[1] A preliminary extended abstract of this article appeared in the proceedings of COCOON'24.

https://doi.org/10.1016/j.tcs.2025.115267
Received 14 November 2024; Received in revised form 21 February 2025; Accepted 16 April 2025

Available online 22 April 2025

- In the geodesic hull game HG_g: $f_1(L) = L$ and $f_2(L) = \mathrm{hull}_g(L)$.
- In the geodesic interval game IG_g: $f_1(L) = L$ and $f_2(L) = \mathrm{I}_g(L)$.
- In the geodesic closed hull game CHG_g: $f_1(L) = f_2(L) = \mathrm{hull}_g(L)$.
- In the geodesic closed interval game CIG_g: $f_1(L) = f_2(L) = \mathrm{I}_g(L)$.

Under the *normal play convention* (*normal variant or achievement variant*), the first player unable to move loses the game. Under the *misère play convention*, also called *avoidance variant*, the first player unable to move wins the game. When the convention is not explicitly mentioned, we consider that it is the normal play convention. From the classical Zermelo-von Neumann theorem [27], one of the two players has a winning strategy in each one of these games, since they are finite perfect-information games without draw. So, the decision problem of these games is whether Alice has a winning strategy. Other convexity games were studied recently in [3,5,8].

We say that a two-person combinatorial game is *impartial* if the set of moves available from any given position (or configuration) is the same for both players. The games of Definition 1.1 are examples of impartial games.

Regarding the geodesic interval game IG_g, the normal and misère variants were solved for cycles and complete bipartite graphs [7] in 1985, for wheels [24] in 1988 and for trees, complete multipartite graphs and block graphs [21] in 2003. Regarding the geodesic closed interval game CIG_g, the normal and misère variants were solved in 2024 for trees [1] and for cacti and block graphs [11]. Regarding the hull games HG_g and CHG_g, they were proved PSPACE-hard even in graphs with diameter two [1].

Our contributions. We first introduce in Section 2 the natural partizan variants of these four impartial convexity games (Definition 1.1) and present some examples. We then prove in Section 4 that the normal and the misère variants of PARTIZAN HG_g and PARTIZAN CHG_g are PSPACE-complete. In Section 5, we determine the winner of the geodesic games PARTIZAN IG_g and PARTIZAN HG_g in the class of Ptolemaic graphs. In Section 3, we determine the winner of the normal variant of the closed games PARTIZAN CIG_g and PARTIZAN CHG_g on trees, by obtaining a polynomial time algorithm based on surreal numbers and Conway's combinatorial game theory. Finally, we extend these games to any graph convexity in Section 6 and obtain a general result on convex geometries in partizan convexity games in Section 7.

2. Partizan convexity games

We say that a two-person combinatorial game is *partizan* if it is not impartial. A rich combinatorial game theory on partizan games was developed by John H. Conway in two classical books, "*On numbers and games*" [10] and "*Winning ways for your mathematical plays*" [4], in which impartial games such as Nim and Hackenbush play a central role. Many partizan games are defined by partitioning the set of possible moves of an impartial game in two sets, each of which corresponds to the possible moves of each player. As an example, in the Blue-Red Hackenbush game defined below, the segments of the impartial Hackenbush game are colored either blue or red and one player called Blue can only cut blue segments and the other player called Red can only cut red segments. From this beautiful combinatorial game theory on partizan games, we introduce the following partizan convexity games, which are the natural partizan variants of the impartial convexity games of Definition 1.1.

Definition 2.1. Let Ψ be a game of Definition 1.1. Let PARTIZAN Ψ be the game obtained by the following modifications on Ψ: the graph G of the instance has all vertices colored either A or B, and Alice (resp. Bob) can only label vertices colored A (resp. B). Also let SIMPLIFIED PARTIZAN Ψ be defined by the following: in addition to the colored graph G, the instance contains a vertex v of G which is already labeled at the beginning of the game.

As an example, consider the graph of Fig. 1, whose vertices are already colored A or B. Alice (resp. Bob) can only label vertices colored A (resp. B). Since the graph is a tree, PARTIZAN HG_g is equivalent to PARTIZAN IG_g, and PARTIZAN CHG_g is equivalent to PARTIZAN CIG_g.

First, notice that Bob wins the normal and the misère variants of PARTIZAN IG_g in the graph of Fig. 1 (see Theorem 5.2). In the normal game, he only plays in his only leaf after Alice has played in all her vertices, winning the game, since he has more vertices than Alice. In the misère game, he plays first in his only leaf and then Alice will finish the game after some turns, losing the misère game, since she has more leaves than Bob.

Regarding the closed game PARTIZAN CIG_g, Alice wins the normal variant in the graph of Fig. 1 by labeling her highest vertex first (vertex c), but the argument is more complicated (see below the relation with the Hackenbush game).

Blue-Red Hackenbush is the main game used in the book "*Winning ways for your mathematical plays*" [4] to explain the Combinatorial Game Theory on partizan games. We show an interesting relation between this classical game and the games SIMPLIFIED PARTIZAN CHG_g and CIG_g. For this, let us define Blue-Red Hackenbush. In this game, there are points connected by line segments that are colored either blue or red. See Fig. 2 for some examples. Some of the points are on a "ground line" represented in green. Two players, Blue and Red, alternately play by cutting segments. Blue (resp. Red) can only cut one blue (resp. red) segment per turn. The cut segment is deleted together with any other segments that are no longer connected to the ground. Following the normal play convention, the first player unable to move loses. A Blue-Red Hackenbush position is defined by a configuration of the segments and by the information of which player will be the next to play (Blue or Red). A position can be winning for either Blue, or Red, or the second player to move in this position. No position is winning for the first player, independently if the first player is Blue or Red, which would be called a "*fuzzy*" position. That is, Blue-Red Hackenbush is a non-fuzzy partizan game.

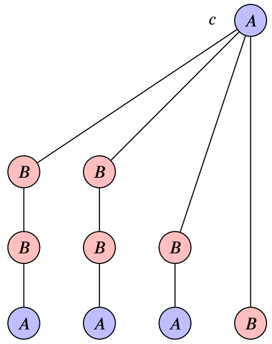


Fig. 1. Instance of partizan games: a graph G with all vertices colored either A or B, which are also associated to the colors blue and red, respectively. Alice (resp. Bob) can only label vertices colored A (resp. B).

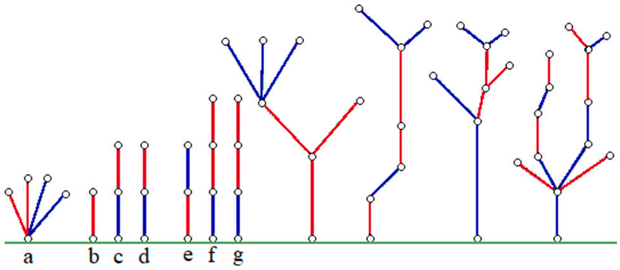


Fig. 2. Blue-Red Hackenbush positions. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Conway associates a dyadic number (rational number whose denominator is a power of two) to each Blue-Red Hackenbush position in terms of advantage (spare moves) for Blue. A position in which the first player always loses has value zero. Positive values are winning for Blue and negative values are winning for Red. Unlike impartial games, which uses the bitwise xor operation in the Sprague-Grundy theory, the value of a Blue-Red Hackenbush position consisting of n disjoint positions (a move in one of them does not affect the others) is the sum of their values.

For example, the position (a) of Fig. 2 has value 0, since the first player loses, independently if the first player is Blue or Red. The position (b) has value -1, since Blue loses immediately and Red has 1 spare move. On the other hand, the positions (c) and (d) have value 1/2 each, since the sum of the positions (b)+(c)+(d) gives a position with value 0: the first to play loses. Finally, the positions (f) and (g) have value 1/4 each, since position (e) has value -1/2 and the sum of the positions (e)+(f)+(g) gives a position with value 0.

The following lemma shows an interesting relation between the Blue-Red Hackenbush game and SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$. A subdivided star is the tree obtained by subdividing each edge of a star $K_{1,p}$ as many times as we want. See Fig. 3 for an example.

Lemma 2.2. *Let G be a subdivided star with center c whose vertices (except c) are colored either A or B. Then the value of SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$ on (G,c) in normal play is equal to the value of the Blue-Red Hackenbush game on the disjoint union of strings, where each string represents a maximal path on G starting at c such that the k-th first segment is blue if and only if the k-th last vertex of the path is colored A.*

Proof. In a Hackenbush string, if the k-th first segment is cut, all the segments from the k-th to the last are deleted. In SIMPLIFIED PARTIZAN $\mathrm{CHG_g}$ and $\mathrm{CIG_g}$ on (P_{n+1}, v_{n+1}), where the path P_n has vertices $v_1,\dots,v_{n+1}$, if v_k is labeled, then the vertices in the path $v_k,\dots,v_{n+1}$ can no longer be labeled in the next rounds. This shows the equivalence between strings in Hackenbush and paths in SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$. Regarding the subdivided star G, notice that a move in one branch of G does not affect the other branches. Then the game on G is a disjoint union of the games on the branches, and we are done. □

As an example, consider SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ in the subdivided star G of Fig. 3, where the black vertex c is already labeled at the beginning, and its equivalent Hackenbush position with value $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}-1=0$. Notice that the first player loses, independently if the first player is Alice or Bob. Therefore, Alice wins the game PARTIZAN $\mathrm{CIG_g}$ in the graph of Fig. 1, by labeling her highest vertex (vertex c), since, after her first move, the resulting graph is the graph of Fig. 3, which is losing for the next player, which is Bob.

This relation between SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ and Blue-Red Hackenbush works for subdivided stars, but is not valid for any tree. Moreover, unlike Blue-Red Hackenbush positions, a position of the game PARTIZAN $\mathrm{CIG_g}$ can be fuzzy, that is, winning for the first player whoever it is. For example, in the cycle C_4 with vertices v_1,v_2,v_3,v_4 where v_1 and v_3 are colored A and v_2 and v_4 are colored B, the first to play wins in all the games PARTIZAN $\mathrm{IG_g}$, PARTIZAN $\mathrm{HG_g}$, PARTIZAN $\mathrm{CIG_g}$ and PARTIZAN $\mathrm{CHG_g}$.

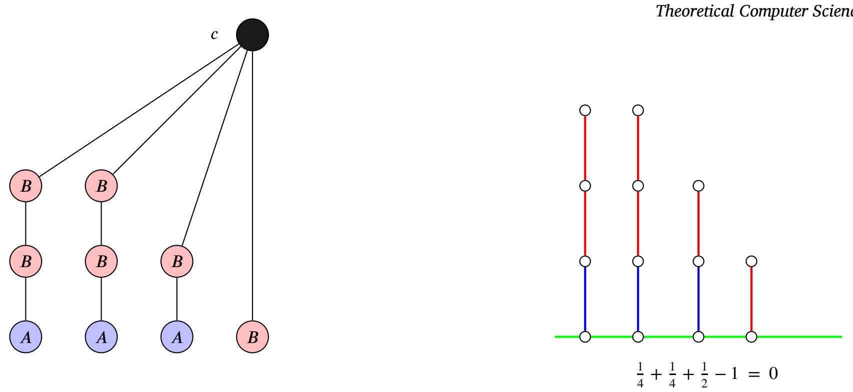


Fig. 3. SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ with the black vertex already labeled at the beginning. Instance with value 0: the first to play loses. The equivalent Hackenbush position is shown (Alice is associated with blue and Bob with red).

Despite this, we show in the next section that PARTIZAN $\mathrm{CIG_g}$ and PARTIZAN $\mathrm{CHG_g}$ positions cannot be fuzzy in trees and we obtain a polynomial time algorithm to decide the winner of these games in those graphs, which are not as easy as Blue-Red Hackenbush in trees.

3. Partizan convexity games in trees

In order to deal with the normal play of the games PARTIZAN $\mathrm{CIG_g}$ and PARTIZAN $\mathrm{CHG_g}$ in trees, we need to talk about *surreal numbers*, which were created in 1969 by John H. Conway and were introduced to the public for the first time in the 1974 mathematical novelette "*Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*" of Donald Knuth [23].

Instead of defining them in all their generality, let us focus on surreal numbers obtained from a Blue-Red Hackenbush position. The main notation has the form $\{B|R\}$ with $B < R$, where B is the greatest possible value of a position obtained after a move of Blue and R is the smallest possible value of a position obtained after a move of Red. Recall that every position of Blue-Red Hackenbush is associated to a dyadic number. Other possible notations are $\{B|\cdot\}$ and $\{\cdot|R\}$ with $B \geq 0$ and $R \leq 0$, where $\cdot$ means *empty* (the player has no move). For example, the simplest notations include $\{\cdot|\cdot\} = 0$ (no player can move and then the first loses), $\{n|\cdot\} = n+1$ ($n+1$ blue segments and no red segment), $\left\{0 \;\middle|\; \frac{1}{2^{k-1}}\right\} = \frac{1}{2^k}$ (a line with 1 blue segment joined to the ground followed by k red segments, as in Fig. 2(f)) and $\{n|n+1\} = n + \{0|1\} = n + \frac{1}{2}$ (n blue segments plus a position (c) of Fig. 2).

It is shown in [4,10] how to determine the exact value of $\{B|R\}$, which satisfies $B < \{B|R\} < R$, using the so-called "*simplicity rule*", and how to calculate recursively the value of a Hackenbush position using the values of subpositions. For Hackenbush trees (as the positions of Fig. 2), this procedure can be done in polynomial time. However, calculating the value of a general Hackenbush position is NP-hard [4], and it is still an open problem if Blue-Red Hackenbush is a PSPACE-hard game.

Theorem 3.1. *In the normal play convention on trees,* PARTIZAN $\mathrm{CIG_g}$ *and* $\mathrm{CHG_g}$ *are non-fuzzy games and are polynomial time solvable.*

Proof. Let T be a tree and let v be a vertex of T. First consider SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v). Since the simplified game with instance (T, v) is in a tree, we may consider that, whenever a new vertex w is labeled, the vertices in the path between v and w are also labeled during the game. This is because there is only one path between every pair of vertices in a tree and this path is minimum.

If v is the only vertex of T, then the value of this position is zero (the first to play loses), since there is no vertex colored A and no vertex colored B. So let u be a neighbor of v. We write (T, v, u) for the instance $(T_{v,u}, v)$ of SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$, where $T_{v,u}$ is the subtree of T containing v obtained after the removal of all edges incident to v except vu. Note that v is a leaf of $T_{v,u}$. Let us first solve the game for the instance (T, v, u).

If u has no neighbor and is colored A (resp. B), then the value is $\{0|\cdot\} = 1$ (resp. $\{\cdot|0\} = -1$). So let $u_1, \ldots, u_k$ be the neighbors of u distinct from v. Assume, by induction, that we know the values $\{A_i|B_i\}$ of (T, u, u_i) in SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$, where positive (resp. negative) values are winning for Alice (resp. Bob). Recall that $A_i < \{A_i|B_i\} < B_i$ for every i when A_i and B_i are not empty.

Let us determine the greatest value A_{vu} after an Alice's move on (T, v, u). If no vertex of (T, v, u) is colored A, then Alice has no move and then A_{vu} is empty. If u is colored A, then the best option for Alice is to label vertex u, since $\{A_i|B_i\} > A_i$ for every i, and consequently the value $A_{vu} = \sum_{i=1}^{k} \{A_i|B_i\}$, since the games in the subtrees are independent after this move of Alice on u. Otherwise, assuming that u is colored B, Alice must label a vertex $w_j \neq u$ in a subtree (T, u, u_j), which will also label the vertex u, and consequently

$$A_{vu} \;=\; \max_{j=1}^{k} \left\{ A_j - \{A_j|B_j\} + \sum_{i=1}^{k} \{A_i|B_i\} \;:\; A_j \text{ is not empty} \right\},$$

since, after choosing the best choice A_j, the games in the subtrees become independent (because u will also be labeled).

Now let us determine the smallest value B_{vu} after a Bob's move on (T, v, u). If no vertex of (T, v, u) is colored B, then Bob has no move and B_{vu} is empty. If u is colored B, then Bob labels u, since $\{A_i|B_i\} < B_i$ for every i, and the value $B_{vu} = \sum_{i=1}^{k}\{A_i|B_i\}$. Otherwise, assuming that u is colored A, Bob must play in a subtree (T, u, u_j), also labeling the vertex u, and consequently

$$B_{vu} \;=\; \min_{j=1}^{k}\left\{\, B_j - \{A_j|B_j\} + \sum_{i=1}^{k}\{A_i|B_i\} \;:\; B_j \text{ is not empty} \right\}$$

From this, we solved SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v, u) by calculating $\{A_{vu}|B_{vu}\}$. Regarding SIMPLIFIED PARTIZAN $\mathrm{CIG_g}$ on (T, v), note that the game on the subtrees of v is independent and then it is possible to calculate the value $\{A_v|B_v\} = \sum_{u\in N(v)}\{A_{vu}|B_{vu}\}$, where $N(v)$ contains the neighbors of v.

Finally, consider PARTIZAN $\mathrm{CIG_g}$ on T and let us calculate the best value A_T for Alice and B_T for Bob. If no vertex of T is colored A (resp. B), then A_T (resp. B_T) is empty. Otherwise, $A_T = \max_{v\in V_A}\{A_v|B_v\}$ and $B_T = \min_{v\in V_B}\{A_v|B_v\}$, where V_A (resp. V_B) is the set of vertices colored A (resp. B). If $\{A_T|B_T\} > 0$, Alice wins. If $\{A_T|B_T\} < 0$, Bob wins. If $\{A_T|B_T\} = 0$, the first player loses (independently if it is Alice or Bob). This leads to a polynomial time recursive algorithm. $\square$

Algorithm 1 $\mathrm{CIG_g}$-leaf(T, v, u).

let n_{vu} be the number of vertices in T_{vu}, excluding v
if all vertices of T are colored A **then return** $\{n_{vu} - 1 \mid \cdot\}$
if all vertices of T are colored B **then return** $\{\cdot \mid -(n_{vu} - 1)\}$
let $u_1, \dots, u_k$ be the neighbors of u distinct from v
for $i = 1, \dots, k$ **do**
  $\{A_i|B_i\} \leftarrow \mathrm{CIG_g}$-leaf$(T, u, u_i)$
$A_{vu} \leftarrow B_{vu} \leftarrow \sum_{i=1}^{k}\{A_i|B_i\}$
if u is colored A **then**
  $B_{vu} \leftarrow B_{vu} + \min_{j=1}^{k}\{B_j - \{A_j|B_j\} \;:\; B_j \text{ is not empty}\}$
else
  $A_{vu} \leftarrow A_{vu} + \max_{j=1}^{k}\{A_j - \{A_j|B_j\} \;:\; A_j \text{ is not empty}\}$
return $\{A_{vu}|B_{vu}\}$

Algorithm 2 $\mathrm{CIG_g}$-node(T, v).

let $v_1, \dots, v_k$ be the neighbors of v
$r \leftarrow 0$
for $i = 1, \dots, k$ **do**
  $r \leftarrow r + \mathrm{CIG_g}$-leaf$(T, v, v_i)$
return r

Algorithm 3 PARTIZAN-$\mathrm{CIG_g}$-tree(T).

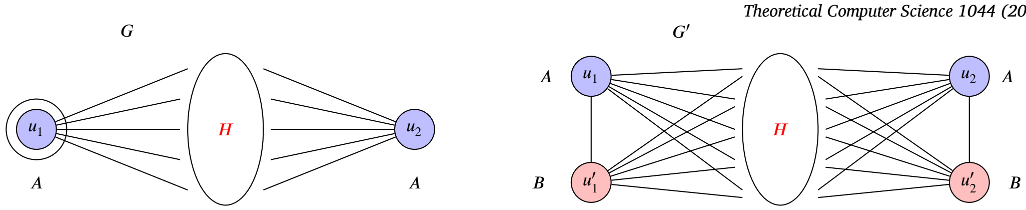
let n be the number of vertices of T
let V_A (resp. V_B) be the set of vertices colored A (resp. B)
if $V_B = \emptyset$ **then return** $\{n - 1 \mid \cdot\}$
if $V_A = \emptyset$ **then return** $\{\cdot \mid -(n - 1)\}$
for every vertex v of T **do**
  $\{A_v|B_v\} \leftarrow \mathrm{CIG_g}$-node$(T, v)$
$A_T = \max_{v\in V_A}\{A_v|B_v\}$; $B_T = \min_{v\in V_B}\{A_v|B_v\}$
return $\{A_T \mid B_T\}$

4. PSPACE-hardness of partizan hull games

In this section, we prove that the normal and misère variants of the partizan hull games PARTIZAN $\mathrm{HG_g}$ and PARTIZAN $\mathrm{CHG_g}$ are PSPACE-complete (see Definition 2.1). As mentioned before, we consider the games as decision problems: given a graph, does Alice have a winning strategy? Since the number of turns is at most n and, in each turn, the number of possible vertices to label is at most n, all these games are polynomially bounded two player games, which implies that they are in PSPACE [22].

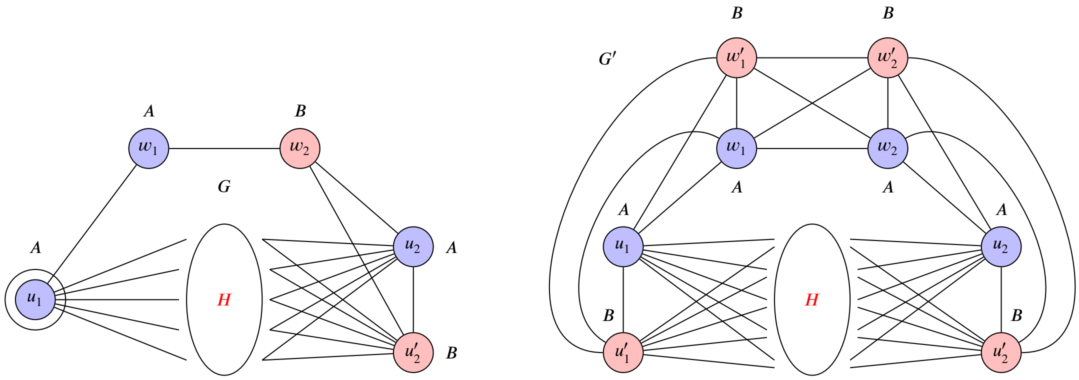
In the theoretical computer science literature, many more NP-hard problems are known than PSPACE-hard. Moreover, in order to find a PSPACE-hard problem for the reduction, few are known to be related to partizan games. Fortunately, we found the suitable problems PARTIZAN NODE KAYLES and PARTIZAN CLIQUE FORMING, which are PSPACE-complete [25], which turned out to be very useful for both reductions. In these games, Alice and Bob (starting with Alice) alternately select vertices of a graph G, whose vertices are colored either A or B, in such a way that Alice (resp. Bob) can only select vertices with color A (resp. B). In PARTIZAN NODE KAYLES, the selected vertices during the game must induce an independent set. In the PARTIZAN CLIQUE FORMING game, the selected vertices during the game must induce a clique.

Recall that, in the misère variant, the first player unable to move wins the game.



(a) Graph G of the reduction to SIMPLIFIED PARTIZAN $\mathrm{HG_g}$ (b) Graph G' of the reduction to PARTIZAN $\mathrm{HG_g}$

Fig. 4. Graphs of the reductions in Theorem 4.1 on the misère variants, where H is a graph whose vertices are colored A or B.



(a) Graph G of the reduction to SIMPLIFIED $\mathrm{HG_g}$ (b) Graph G' of the reduction to $\mathrm{HG_g}$

Fig. 5. Graphs of the reductions in Theorem 4.2 on the normal variants, where H is a graph whose vertices are colored A or B.

Theorem 4.1. *The misère variants of* PARTIZAN $\mathrm{HG_g}$ *and* SIMPLIFIED PARTIZAN $\mathrm{HG_g}$ *are PSPACE-complete even in graphs with diameter two.*

Proof. Let H be an instance of PARTIZAN CLIQUE FORMING game: a graph whose vertices are colored either A or B. We may assume that H is not complete. We first obtain a reduction for SIMPLIFIED PARTIZAN $\mathrm{HG_g}$.

Let G be the graph obtained from H by adding two non-adjacent new vertices u_1 and u_2 with color A which are adjacent to all vertices of H. See Fig. 4(a). Notice that G has diameter 2. Also let u_1 be the vertex which is already labeled in SIMPLIFIED PARTIZAN $\mathrm{HG_g}$. We prove that Alice has a winning strategy in PARTIZAN CLIQUE FORMING game on H if and only if she has a winning strategy in the misère variant of SIMPLIFIED PARTIZAN $\mathrm{HG_g}$ on (G, u_1). If Alice labels u_2, she loses immediately, since $\mathrm{hull_g}(\{u_1, u_2\}) = V(G)$. Moreover, if a player labels a vertex v_j of H in the hull game $\mathrm{HG_g}$ and there is a non-adjacent labeled vertex v_i in H, then the player loses immediately, since $\mathrm{hull_g}(\{v_i, v_j\}) = V(G)$. So, we may assume that the set L of labeled vertices forms a clique in all turns, except the last one. This is directly related to the PARTIZAN CLIQUE FORMING game on H. If Alice has a winning strategy in the PARTIZAN CLIQUE FORMING game on H, then Bob is the first to label a vertex of G with a labeled non-neighbor, implying that he loses the misère variant of SIMPLIFIED PARTIZAN $\mathrm{HG_g}$. Analogously, if Bob has a winning strategy in the PARTIZAN CLIQUE FORMING game on H.

Now we obtain a reduction for PARTIZAN $\mathrm{HG_g}$. Let the graph G' be obtained from G by adding two new vertices u_1' and u_2' with color B, adjacent to all vertices of H, and the edges $u_1'u_1$ and $u_2'u_2$. See Fig. 4(b). Notice that G' has diameter 2. If Alice has a winning strategy in the PARTIZAN CLIQUE FORMING game on H, then she plays PARTIZAN $\mathrm{HG_g}$ in G' according to her strategy in the vertices of H, unless Bob selects u_1' (resp. u_2') causing Alice to select u_1 (resp. u_2). Analogously if Bob has a winning strategy in the PARTIZAN CLIQUE FORMING game on H. If there are two non-adjacent selected vertices during the game, then PARTIZAN $\mathrm{HG_g}$ is over and the last to play loses. Then, as before, the set L of labeled vertices forms a clique in all turns, except the last one, and we are done. □

Theorem 4.2. *The normal variants of* PARTIZAN $\mathrm{HG_g}$ *and* SIMPLIFIED PARTIZAN $\mathrm{HG_g}$ *are PSPACE-complete even in graphs with diameter two.*

Proof. Let us begin with the normal variant of SIMPLIFIED PARTIZAN $\mathrm{HG_g}$. Given a graph H as an instance of PARTIZAN CLIQUE FORMING, with vertices colored either A or B, consider the graph G in Fig. 5(a)). Let u_1 be the vertex which is already labeled in G in SIMPLIFIED PARTIZAN $\mathrm{HG_g}$. Notice that G has diameter 2.

If Alice labels u_2 (resp. w_1), she loses immediately, since Bob labels w_2 (resp. u_2'), winning the normal game. If Bob labels u_2' (resp. w_2), he loses immediately, since Alice labels w_1 (resp. u_2), winning the normal game. This is because $\mathrm{hull_g}(\{u_1, u_2, w_1\}) = \mathrm{hull_g}(\{u_1, u_2, w_2\}) = \mathrm{hull_g}(\{u_1, u_2', w_1\}) = \mathrm{hull_g}(\{u_1, u_2', w_2\}) = V(G)$. Moreover, if a player labels a vertex v_j of H in SIMPLIFIED PARTIZAN $\mathrm{HG_g}$ and there is a non-adjacent labeled vertex v_i in H, then the player loses immediately, since the opponent labels w_1 or

w_2, winning the game because $\text{hull}_g(\{v_i, v_j, w_1\}) = \text{hull}_g(\{v_i, v_j, w_2\}) = V(G)$. So, we may assume that the set $L \subseteq V(H) \cup \{u_1\}$ of labeled vertices forms a clique in all turns, except the last two. This is directly related to the PARTIZAN CLIQUE FORMING game on H: Alice has a winning strategy in the normal variant of PARTIZAN SIMPLIFIED HG_g on G if and only if she has a winning strategy in the PARTIZAN CLIQUE FORMING game on H.

Now we obtain a reduction for the normal variant of PARTIZAN HG_g. Let the graph G' as in Fig. 5(b). Notice that G' has diameter 2. In the game, if the set L of labeled vertices is a clique, we say that we are in Phase 1; otherwise we are in Phase 2.

First suppose that Alice has a winning strategy in the PARTIZAN CLIQUE FORMING game on H. Consider that the game is in Phase 1. Then Alice plays PARTIZAN HG_g in G' according to her winning strategy in the vertices of H. Then the first labeled vertex v_0 is in H. If Bob labels u'_1 (resp. u'_2), Alice labels w_2 (resp. w_1), winning the game since $\text{hull}_g(\{v_0, u'_1, w_2\}) = V(G)$. Now consider that the game is in Phase 2. If Bob labels w'_1 (resp. w'_2), Alice labels w_2 (resp. w_1), winning the game, since $\text{hull}_g(\{v_0, w_1, w_2\}) = V(G')$. Moreover, if Bob labels a vertex v_j of H and there is a non-adjacent labeled vertex v_i in H, then Alice labels w_1 or w_2, winning the game because $\text{hull}_g(\{v_i, v_j, w_1\}) = \text{hull}_g(\{v_i, v_j, w_2\}) = V(G)$. With this, Alice is always the last to play.

Now suppose that Bob has a winning strategy in the CLIQUE FORMING game on H. First consider that the first vertex x labeled by Alice is not in H. Then Bob labels the twin x' of x. In the second move of Bob, he selects either u'_1 if $L \subseteq \{u_2, u'_2, w_2, w'_2\}$, or u'_2 if $L \subseteq \{u_1, u'_1, w_1, w'_1\}$, or w'_1 if $L \subseteq \{u_2, u'_2\} \cup V(H)$, or w'_2 if $L \subseteq \{u_1, u'_1\} \cup V(H)$ or $L \subseteq \{u_1, u'_1, u_2, u'_2\} \cup V(H)$, winning the game immediately. Then consider that the first labeled vertex is in H. The same procedure of the last paragraph can be applied changing the roles of Alice and Bob. With this, Bob is always the last to play. □

It is not difficult to check that the arguments in the proofs of Theorems 4.1 and 4.2 are also valid for PARTIZAN CHG_g.

Corollary 4.3. *The normal and misère variants of the hull games PARTIZAN* HG_g*, PARTIZAN* CHG_g *and their simplified versions are PSPACE-complete even in graphs with diameter two.*

5. Partizan convexity games in Ptolemaic graphs

Given a graph G and a convex set S of G in the geodesic convexity, we say that a vertex $v \in S$ is *extreme* in S if $S \setminus \{x\}$ is also convex. The *extreme vertices of the graph* G are the extreme vertices of $V(G)$. In 1986, Farber and Jamison [18] proved that a vertex is extreme in the geodesic convexity of a graph if and only if it is simplicial (its closed neighborhood forms a clique). Let $\text{Ext}_g(G)$ be the set of extreme vertices of G in the geodesic convexity, which are the simplicial vertices.

Farber and Jamison [18] also defined a graph convexity as *geometric* (also called *convex geometry*) if it satisfies the *Minkowski-Krein-Milman* property: every convex set is the convex hull of its extreme vertices. The research on graph convex geometries is usually concentrated in determining the graph class in which a given graph convexity is geometric. They proved that the geodesic convexity is geometric in a graph if and only if the graph is Ptolemaic [18], that is, it is chordal (every induced cycle is a triangle) and distance hereditary (every induced path is a shortest path).

In this section, we use these results to obtain winning strategies for PARTIZAN HG_g and PARTIZAN IG_g on Ptolemaic graphs. We first prove a general result on the geodesic convexity. Let $\text{Ext}^A_g(G)$ and $\text{Ext}^B_g(G)$ be the subsets of vertices of $\text{Ext}_g(G)$ colored A and B, respectively, when the vertices of G are colored either A or B.

Theorem 5.1. *Let G be a graph whose vertices are colored either A or B such that* $\text{Ext}_g(G)$ *is a geodesic hull (resp. interval) set. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of PARTIZAN* HG_g *(resp. PARTIZAN* IG_g*) if and only if* $\text{Ext}^B_g(G) = \emptyset$*, or* $\text{Ext}^A_g(G) \neq \emptyset$ *and* $|V_A| > |V_B|$*. Moreover Alice wins the misère variant of PARTIZAN* HG_g *(resp. PARTIZAN* IG_g*) if and only if* $|\text{Ext}^A_g(G)| \leq |\text{Ext}^B_g(G)|$.

Proof. Recall that $V(G) \setminus \{s\}$ is convex for every $s \in \text{Ext}_g(G)$. That is, there are two options for the end of the games PARTIZAN HG_g and PARTIZAN IG_g: either a player has no move or every vertex of $\text{Ext}_g(G)$ was labeled.

Let us first consider the normal variant. If every vertex of $\text{Ext}_g(G)$ has color A (resp. B), then Alice (resp. Bob) wins since she (resp. he) is the last to select a vertex of $\text{Ext}_g(G)$. So assume that there is a vertex colored A and a vertex colored B in $\text{Ext}_g(G)$. If $\text{Ext}_g(G)$ is a geodesic hull set (resp. geodesic interval set), then PARTIZAN HG_g (resp. PARTIZAN IG_g) ends when the last vertex of $\text{Ext}_g(G)$ is labeled or one of the players has no move. Then Alice (resp. Bob) plays avoiding her (resp. his) last unlabeled vertex of $\text{Ext}_g(G)$, and then Alice wins if and only if $|V_A| > |V_B|$.

Now consider the misère variant. If $|\text{Ext}^A_g(G)| \leq |\text{Ext}^B_g(G)|$, then Alice can label all vertices of $\text{Ext}^A_g(G)$ before Bob can label all the vertices of $\text{Ext}^B_g(G)$. Then the misère games end in two possible situations: (a) Alice is unable to play because all vertices of V_A were labeled, or (b) Alice is unable to play because Bob had to label the last unlabeled vertex of $\text{Ext}^B_g(G)$. In both, Alice wins. Analogously, if $|\text{Ext}^A_g(G)| > |\text{Ext}^B_g(G)|$, Bob wins the misère games. □

As an example of Theorem 5.1, recall the graph of Fig. 1 in the beginning of the paper. Bob wins both normal and misère variants of PARTIZAN IG_g and PARTIZAN HG_g. In the normal game, he only plays in his only leaf after Alice has played in all her vertices, winning the game, since he has more vertices than Alice. In the misère game, he plays first in his only leaf and then Alice will finish the game after some turns, losing the misère game, since she has more leaves than Bob.

Now we find winning strategies in polynomial time for PARTIZAN $\mathrm{IG_g}$ and PARTIZAN $\mathrm{HG_g}$ in Ptolemaic graphs. The theorem below is the partizan variant of a result of Haynes, Henning and Tiller [21] in 2003 on block graphs, since every block graph is a Ptolemaic graph.

Theorem 5.2. *Let G be a Ptolemaic graph whose vertices are colored either A or B. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN $\mathrm{IG_g}$ *and* PARTIZAN $\mathrm{HG_g}$ *if and only if* $\mathrm{Ext}_g^B(G) = \emptyset$, *or* $\mathrm{Ext}_g^A(G) \neq \emptyset$ *and* $|V_A| > |V_B|$. *Moreover Alice wins the misère variant of* PARTIZAN $\mathrm{IG_g}$ *and* PARTIZAN $\mathrm{HG_g}$ *if and only if* $|\mathrm{Ext}_g^A(G)| \leq |\mathrm{Ext}_g^B(G)|$.

Proof. Regarding the game PARTIZAN $\mathrm{HG_g}$, the result follows from Theorem 5.1 and the fact that the geodesic convexity is geometric in Ptolemaic graphs [18]. Regarding the game PARTIZAN $\mathrm{IG_g}$, the result holds since Ptolemaic graphs are chordal and distance-hereditary, every vertex of a chordal graph is in an induced path between two simplicial vertices and every induced path is a shortest path in a distance hereditary graph. □

6. Other convexities and other variations

Other contribution of this paper is the generalization of the games of Definition 2.1 to any graph convexity, obtaining very natural convexity games and enriching the prolific research area of graph convexity. We also prove general results. For this, let us define general convexity in graphs.

A *convexity* C [26] on a finite set $V \neq \emptyset$ is a family of subsets of V such that $\emptyset, V \in C$ and C is closed under intersections. That is, $S_1, S_2 \in C$ implies $S_1 \cap S_2 \in C$. A member of C is said to be a *C-convex set*. Given $S \subseteq V$, the *C-convex hull* of S is the smallest C-convex set $\mathrm{hull}_C(S)$ containing S. We say that S is a *C-hull set* if $\mathrm{hull}_C(S) = V$. It is easy to see that $\mathrm{hull}_C(\cdot)$ is a *closure operator*, that is, for every $S, S' \subseteq V$: (a) $S \subseteq \mathrm{hull}_C(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{hull}_C(S) \subseteq \mathrm{hull}_C(S')$ [monotonicity], (c) $\mathrm{hull}_C(\emptyset) = \emptyset$ [normalization[2]] and (d) $\mathrm{hull}_C(\mathrm{hull}_C(S)) = \mathrm{hull}_C(S)$ [idempotence].

We say that $\mathrm{I} : 2^V \to 2^V$ is an *interval function* on V if, for every $S, S' \subseteq V$, (a) $S \subseteq \mathrm{I}(S)$ [extensivity], (b) $S \subseteq S' \Rightarrow \mathrm{I}(S) \subseteq \mathrm{I}(S')$ [monotonicity] and (c) $\mathrm{I}(\emptyset) = \emptyset$ [normalization]. It is known that every interval function *induces* a unique convexity, containing each set $S \subseteq V$ such that $\mathrm{I}(S) = S$. Moreover, every convexity is induced by an interval function. We then assume that every convexity C on V is defined by an explicitly given interval function $\mathrm{I}_C(\cdot)$ on V. It is also known that the convex hull of a set S in a convexity C can be obtained by exhaustively applying the corresponding interval function $\mathrm{I}_C(\cdot)$ until obtaining a C-convex set. We say that S is a *C-interval set* if $\mathrm{I}_C(S) = V$.

Given a graph G, a *graph convexity* on G is simply a convexity C on $V(G)$ with a given interval function $\mathrm{I}_C(\cdot)$ on $V(G)$. A standard way to define a *graph convexity* C on a graph G is by fixing a family $\mathcal{P}$ of paths of G and taking the interval function $\mathrm{I}_C(S)$ as the set S with all vertices lying on some path of $\mathcal{P}$ whose endpoints are in S. The most studied graph convexities are path convexities, such as the *geodesic convexity* [17,19], the *P_3 convexity* [2], the *monophonic convexity* [13,16,18], the *m^3 convexity* [15] and the *triangle path convexity* [9,14], where $\mathcal{P}$ is the family of all geodesics (shortest paths) of the graph, of all paths of order three, of all induced paths, of all induced paths of size at least 3 and of all paths $v_1, \dots, v_k$ with no edge $v_i v_j$ with $|j - i| > 2$, respectively.

This definition also applies to oriented graphs. For example, there are the geodesic convexity, the monophonic convexity and the P_3 convexity for oriented graphs, related to shortest paths, induced paths and paths with three vertices in the oriented graph, respectively, where the paths must respect the orientation of the edges ("*arcs*").

Given a graph convexity C on a graph G, let the *C-hull number* $\mathrm{hn}_C(G)$ be the size of a minimum C-hull set of G and let the *C-interval number* $\mathrm{in}_C(G)$ be the size of a minimum C-interval set of G.

In the following, we generalize the convexity games of Definition 1.1 to any graph convexity.

Definition 6.1. Given a graph convexity C on a graph G, we introduce four convexity games. In the games defined below, the set L of labeled vertices is initially empty and the definitions of $f_1(L)$ and $f_2(L)$ depend on the game. Two players (Alice and Bob, starting by Alice) alternately label one unlabeled vertex v which is not in $f_1(L)$. The game ends when $f_2(L) = V(G)$.

- In the C-hull game HG_C: $f_1(L) = L$ and $f_2(L) = \mathrm{hull}_C(L)$.
- In the C-interval game IG_C: $f_1(L) = L$ and $f_2(L) = \mathrm{I}_C(L)$.
- In the closed C-hull game CHG_C: $f_1(L) = f_2(L) = \mathrm{hull}_C(L)$.
- In the closed C-interval game CIG_C: $f_1(L) = f_2(L) = \mathrm{I}_C(L)$.

As before, the last to play wins in the normal variant of these games, the last to play loses in the misère variants and the decision problem associated to these games is whether Alice has a winning strategy.

As in Definition 1.1, we may also have the SIMPLIFIED and the PARTIZAN versions of these games. As an example, we have the following lemma on oriented graphs, where the subscript m is for the monophonic convexity.

Lemma 6.2. *The Blue-Red Hackenbush game in trees is equivalent to the game* SIMPLIFIED PARTIZAN $\mathrm{CIG_m}$ *in DAGs (directed acyclic graphs) with one source, one sink and all other vertices with in-degree* 1*, where the sink is labeled at the beginning of the game.*

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[2] Here we follow the definition of closure operator from van de Vel [26], which includes the normalization property.

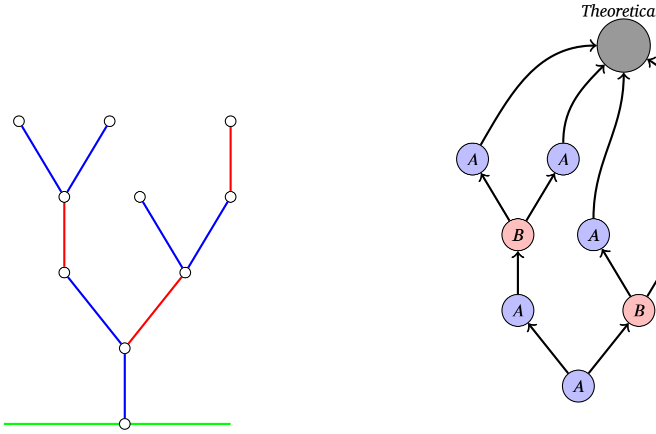

Fig. 6. Blue-Red Hackenbush in trees can be modeled as the game SIMPLIFIED PARTIZAN $\mathrm{CIG_m}$ in DAGs with one source, one sink and all other vertices with in-degree 1, where the sink (black vertex) is already labeled at the beginning and m corresponds to the monophonic convexity.

Proof. Given a Blue-Red Hackenbush tree T, we obtain the oriented graph $\vec{G}$ from the following. For every edge x of T, create a vertex v_x on $\vec{G}$. If x and y are adjacent edges of T and x is under y, then create the arc $x_x v_y$ on $\vec{G}$. Moreover, create a vertex z in $\vec{G}$ and all arcs $v_x z$, where x has no adjacent edge above it in T, and consider that z is already labeled at the beginning. Finally, color v_x with color A in $\vec{G}$ if and only if the edge x is colored blue in T. Fig. 6 shows an example. Notice that $\vec{G}$ is a DAG and has exactly one sink (the vertex z) and one source (the vertex associated to the edge of T connected to the ground). Moreover every vertex of $\vec{G}$ has in-degree 1, except the source and the sink. Also note that labeling a vertex v_x of $\vec{G}$ in the game SIMPLIFIED PARTIZAN $\mathrm{CIG_m}$ makes impossible to label the vertices associated to the edges of T above x in the next rounds, which is equivalent to Hackenbush when cutting the edge x.

Finally, let $\vec{G}$ be a DAG (directed acyclic graph) with one source s, one sink z and all other vertices with in-degree 1, where the sink is labeled at the beginning of the game SIMPLIFIED PARTIZAN $\mathrm{CIG_m}$. We obtain a Blue-Red Hackenbush tree T from the following. For every vertex $v \neq z$ of $\vec{G}$, create a vertex v in T. For every arc uv of $\vec{G}$ with $v \neq z$, create the edge uv in T. Finally, create a vertex g in T which is on the ground in the Blue-Red Hackenbush game and create the edge gs, where s is the vertex of T corresponding to the source s of $\vec{G}$. Finally, color the edge uv of T with blue if and only if the vertex v is colored A in $\vec{G}$. Notice that T is a Hackenbush tree. Also note that cutting the edge uv of T makes impossible to select the edges above uv in T, which is equivalent to SIMPLIFIED PARTIZAN $\mathrm{CIG_m}$ in $\vec{G}$, since the sink z is already labeled. $\square$

7. General convexity games in convex geometries

Recall that a convexity C on a graph G is *geometric* (also called *antimatroid* or *convex geometry*) if it satisfies the *Minkowski–Krein–Milman* property: every convex set is the convex hull of its extreme vertices in the convexity C, where a vertex v is an *extreme vertex* of S in the convexity C if $S \setminus \{v\}$ is also a C-convex set. Let $\mathrm{Ext}_C(S)$ be the set of extreme vertices of S.

As mentioned before, the research on graph convex geometries is usually concentrated in determining the graph class in which a given graph convexity is geometric. In 1986, Farber and Jamison [18] proved that the monophonic (resp. geodesic) convexity is geometric if and only if the graph is chordal (resp. Ptolemaic). In 1999, Dragan et al. [15] proved that the m^3 convexity is geometric if and only if the graph is the so-called weak bipolarizable. Recently, it was proved that the triangle path (resp. P_3) convexity is geometric if and only if the graph is a forest (resp. forest of stars) [12].

Regarding the partizan version of the convexity games, we extend the results of Section 5 to any convexity.

Theorem 7.1. *Let C be a graph convexity on a graph G, whose vertices are colored either A or B, such that $\mathrm{Ext}_C(G)$ is a C-hull (resp. C-interval) set. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN HG_C *(resp.* PARTIZAN IG_C*) if and only if $\mathrm{Ext}_C^B(G) = \emptyset$, or $\mathrm{Ext}_C^A(G) \neq \emptyset$ and $|V_A| > |V_B|$. Moreover Alice wins the misère variant of* PARTIZAN HG_C *(resp.* PARTIZAN IG_C*) if and only if $|\mathrm{Ext}_C^A(G)| \leq |\mathrm{Ext}_C^B(G)|$.*

Proof. The proof follows the same structure of the proof of Theorem 5.1, replacing the geodesic convexity g with the graph convexity C. $\square$

We also find winning strategies in polynomial time for PARTIZAN HG_C in general convex geometries.

Theorem 7.2. *Let C be a convex geometry on a graph G, whose vertices are colored either A or B. Let V_A and V_B be the set of vertices colored A and B, respectively. Then Alice wins the normal variant of* PARTIZAN HG_C *if and only if $\mathrm{Ext}_C^B(G) = \emptyset$, or $\mathrm{Ext}_C^A(G) \neq \emptyset$ and $|V_A| > |V_B|$. Moreover Alice wins the misère variant of* PARTIZAN HG_C *if and only if $|\mathrm{Ext}_C^A(G)| \leq |\mathrm{Ext}_C^B(G)|$.*

Proof. The result follows from Theorem 7.1 and the fact that the set $\mathrm{Ext}_C(G)$ of extreme vertices is a C-hull set when the convexity C is a convex geometry on G, from the Minkowski-Krein-Milman property. □

8. Conclusions and final remarks

In this paper, we introduced in Definition 2.1 the partizan versions of the impartial geodesic convexity games $\mathrm{IG_g}$, $\mathrm{HG_g}$, $\mathrm{CIG_g}$ and $\mathrm{CHG_g}$ of Definition 1.1, introduced by Harary [6,7,20] in 1984. Below we list our results:

- We proved that PARTIZAN $\mathrm{CIG_g}$ and PARTIZAN $\mathrm{CHG_g}$ are polynomial time solvable in trees, in Theorem 3.1, using surreal numbers and Conway's combinatorial game theory;
- We proved that PARTIZAN $\mathrm{IG_g}$ and PARTIZAN $\mathrm{HG_g}$ are polynomial time solvable in Ptolemaic graphs, using results on convex geometries, which are graph convexities satisfying the Minkowski-Krein-Milman property;
- We obtained the first PSPACE-hardness results on partizan convexity games by proving that the normal and misère variants of PARTIZAN $\mathrm{HG_g}$, PARTIZAN $\mathrm{CHG_g}$ and their simplified versions are PSPACE-complete even in graphs with diameter two, in Corollary 4.3, using reductions from the PARTIZAN CLIQUE FORMING game;

Finally, in Section 6, we also generalize these games to any graph convexity, such as the P_3 convexity and the monophonic convexity. We finish with Theorem 7.2 with winning strategies on PARTIZAN HG_C for any convex geometry C, which are generalizations of Theorem 5.2.

Below we list some open problems:

- PROBLEM 1: For which trees Bob wins the game PARTIZAN $\mathrm{CIG_g}$?
- PROBLEM 2: Are the geodesic interval games PARTIZAN $\mathrm{IG_g}$ and PARTIZAN $\mathrm{CIG_g}$ PSPACE-complete?
- PROBLEM 3: Let GREEN $\mathrm{CIG_g}$ be the extension of PARTIZAN $\mathrm{CIG_g}$ allowing green vertices, which can be labeled by any player. If all vertices are green, we have $\mathrm{CIG_g}$. If no vertex is green, we have PARTIZAN $\mathrm{CIG_g}$. Is the game GREEN $\mathrm{CIG_g}$ also polynomial time solvable in trees?

CRediT authorship contribution statement

Samuel N. Araújo: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **João Marcos Brito:** Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Raquel Folz:** Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Rosiane de Freitas:** Writing – original draft, Validation, Supervision, Resources, Methodology, Investigation, Funding acquisition, Conceptualization. **Rudini M. Sampaio:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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