Presentation Title

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Outline

Outline

Gain Scheduling[1]

Definition

Gain scheduling (GS) is a control technique in which the controller structure remains constant in operation, while the gains are conveniently updated according to the current value of scheduling signals that may be either exogenous or endogenous with respect to the plant.

- Initially it had no formal design framework or theoretical proofs. Early works were mainly application based.
- Allows the use of linear design tools to control nonlinear systems

[1] Wilson J. Rugh and Jeff S. Shamma. "Research on gain scheduling". In: *Automatica* 36.10 (Oct. 2000), pp. 1401–1425. DOI: 10.1016/S0005-1098(00)00058-3.

Linear Parameter Varying Systems

The Linear Parameter Varying (LPV) Paradigm concerns itself with a special case of Time Varying Systems (TVS).

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{1}$$

- Original paper by Jeff Shamma[2,3]
- Hot topic in systems and control
- Not widely spread in industrial applications
- Has become a standard in GS design

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[2] Jeff S. Shamma. "Analysis and design of gain scheduled control systems". PhD thesis. 1988, p. 201.

[3] Jeff S. Shamma and M. Athans. "Analysis of gain scheduled control for nonlinear plants". In: *IEEE Transactions on Automatic Control* 35.8 (1990), pp. 898–907. DOI: 10.1109/9.58498.

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$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A(\rho(t)) & B(\rho(t)) \\ C(\rho(t)) & D(\rho(t)) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \tag{2}$$

- Scheduling parameters vary with time and are unknown a priori, but measured during operation. They can either be exogenous or endogenous. In the latter case, some authors refer to the system as quasi-LPV[4]

[4] Olivier Sename, Péter Gáspár, and Jószef Bokor. *Robust Control and Linear Parameter Varying Approaches.* Ed. by Olivier Sename, Peter Gaspar, and József Bokor. Vol. 437. Lecture Notes in Control and Information Sciences. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013. DOI: 10.1007/978-3-642-36110-4.

System Behavior

Loss of stability:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 - 0.5\rho(t) & -0.2 \end{bmatrix} x \tag{3}$$

- if $-1 \leq \rho(t) \leq 1$, the eigenvalues remain on the left half plane.
- for some particular parameter trajectory, say $\rho(t) = \cos(2t)$, the following behavior occours

Non minimum phase behavior

$$\dot{x} = \begin{bmatrix} 0 & (2-\rho(t))^2 & 1+0.5\rho(t)+(2-\rho(t))^2 \\ 1 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \tag{4}$$

$$y(t) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x. \tag{5}$$

- For frozen values of ρ, the system has a pair of complex zeroes $-0.1 \pm \frac{j}{2}\sqrt{2\rho + \frac{99}{25}}$ and three poles $0, \pm(2-\rho)$.
- For a specific parameter trajectory, the plant gives a bounded output for an unbounded input (blocking effect of a zero[5])

[5] Jesse B. Hoagg and Dennis S. Bernstein. "Nonminimum-phase zeros - much to do about nothing - classical control - revisited part II". In: *IEEE Control Systems* 27.3 (June 2007), pp. 45–57. DOI: 10.1109/MCS.2007.365003.

Non minimum phase behavior[6]

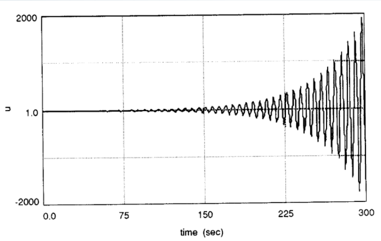


Fig.

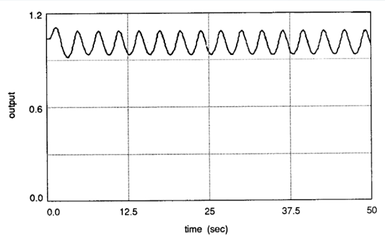


Fig.

[6] Jeff S. Shamma and Michael Athans. "Gain scheduling: potential hazards and possible remedies". In: *IEEE Control Systems* 12.3 (June 1992), pp. 101–107. DOI: 10.1109/37.165527.

Gain Scheduling Control Design

- Nonlinear systems that can be covered by the LPV framework: hybrid dynamical systems, jump linear systems and switched linear systems.[7]
- Algorithm:
  - model
  - design
  - implementation
  - performance assessment

[7] Christian Hoffmann and Herbert Werner. "A Survey of Linear Parameter-Varying Control Applications Validated by Experiments or High-Fidelity Simulations". In: *IEEE Transactions on Control Systems Technology* 23.2 (Mar. 2015), pp. 416–433. DOI: 10.1109/TCST.2014.2327584.

Field Overview - Modeling

- Affine

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \left(\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \sum_{i=1}^{N} \rho_i(t) \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} \tag{6}$$

  Polytopic LPV: if $A_0, B_0, C_0, D_0 = 0$, $\sum_{i=1}^{N} \rho_i(t) = 1, \rho_i(t) \geq 0$.

- Linear Fractional Transformation

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(P) & B_w(P) & B_u(P) \\ C_z(P) & D_{zw}(P) & D_{zu}(P) \\ C_y(P) & D_{yw}(P) & D_{yu}(P) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{7}$$

  with

$$\begin{bmatrix} A(P) & B_w(P) & B_u(P) \\ C_z(P) & D_{zw}(P) & D_{zu}(P) \\ C_y(P) & D_{yw}(P) & D_{yu}(P) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{bmatrix} + \begin{bmatrix} B_p \\ D_{zp} \\ D_{yp} \end{bmatrix} P(I - D_{lp}P)^{-1} \begin{bmatrix} C_l & D_{lw} & D_{lu} \end{bmatrix} \tag{8}$$

Field Overview

- Importance of representation choice: system characteristics, feasibility of design problem
- Types of representation can be transformed between each one
- Modeling techniques: Jacobian linearization, State transformation, nonlinear embedding, etc.

Field Overview

- Analysis: non minimum phase behavior, loss of stability, controllability and other characteristics
- Various blank spaces in the literature

Field Overview

- Synthesis: Lyapunov based methods, Small gain theorem, Invariant set methods
- State feedback, output feedback and dynamic controllers
- Type of Lyapunov function: quadratic, polynomial, fuzzy, etc.
- The field mainly focuses on decreasing conservativeness and obtaining alternative forms of design

Field Overview

- Computational issues: obtaining a tractable problem, with a finite number of constraints, convexity of the parameter space, etc.
- NP-hard problems, BMI problems, why are LMIs important
- Implementation issues: parameter measurement or estimation, controller causality, etc.

What's lacking in literature

- Updated survey and field overview
- Real applications and higher order systems
- Relationship between LPV and T-S Fuzzy
- Controllability of LPV systems

Outline

LPV vs Takagi-Sugeno Fuzzy

- There exists a close connection between the LPV and Takagi-Sugeno fuzzy frameworks[8] that has not been fully explored by the literature

[8] Damiano Rotondo. *Advances in Gain-Scheduling and Fault Tolerant Control Techniques*. Springer Theses. Cham: Springer International Publishing, 2018. DOI: 10.1007/978-3-319-62902-5.

A few citations

- Control design for a mobile robot: a fuzzy LPV approach[9]
- Flight Control Design For A STT Missile: A Fuzzy LPV Approach[10]
- State-Feedback H∞ Control for LPV System Using T-S Fuzzy Linearization Approach[11]

[9] A. Tsourdos et al. "Control design for a mobile robot: a fuzzy LPV approach". In: *Proceedings of 2003 IEEE Conference on Control Applications, 2003. CCA 2003.* IEEE, 2003, pp. 552–557. DOI: 10.1109/CCA.2003.1223496.

[10] **Blumel2017**.

[11] Jizhen Liu, Yang Hu, and Zhongwei Lin. "State-Feedback H∞ Control for LPV System Using T-S Fuzzy Linearization Approach". In: *Mathematical Problems in Engineering* 2013.May (2013), pp. 1–18. DOI: 10.1155/2013/169454.

A few citations

- Continuous quasi-LPV Systems: how to leave the quadratic Framework?[12]
- Automated generation and comparison of Takagi–Sugeno and polytopic quasi-LPV models[13]

[12] A. JAADARI. “Continuous quasi-LPV Systems: how to leave the quadratic Framework?” PhD thesis. UNIVERSITAT POLITECNICA DE VALENCIA, 2013.

[13] Damiano Rotondo et al. “Automated generation and comparison of Takagi-Sugeno and polytopic quasi-LPV models”. In: *Fuzzy Sets and Systems* 277 (2015), pp. 44–64. DOI: 10.1016/j.fss.2015.02.002.

Takagi-Sugeno Fuzzy systems

Definition

A Linear Takagi-Sugeno fuzzy model[a] is a representation of a nonlinear system, described by fuzzy IF-THEN rules of the form

$$\begin{aligned} &\text{IF } \big(z_1(t) \text{ is } M_{i1}\big) \text{ and } \big(z_2(t) \text{ is } M_{i2}\big) \text{ and } \dots \text{ and } \big(z_p(t) \text{ is } M_{ip}\big), \\ &\text{THEN } \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t), \\ y(t) = C_i x(t), \end{cases} \qquad i = 1, 2, \dots, r. \end{aligned} \tag{9}$$

where M_{ij} is the Fuzzy set, r the number of model rules and $z_i(t)$ is the i^{th} premise variable, which can either be a function of the states or external disturbances.

[a] Kazuo Tanaka and Hua O. Wang. *Fuzzy Control Systems Design and Analysis*. 2001. DOI: 10.1002/0471224596.

Takagi-Sugeno Fuzzy systems

The overall fuzzy model of the system is achieved by fuzzy blending of the local models, that is, given a pair $(x(t), u(t))$, the final output is inferred as

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\} && (10)\\ y &= \sum_{i=1}^{r} h_i(z(t)) C_i x(t) && (11)\end{aligned}$$

where $h_i(z(t))$ are called the membership functions and satisfy the following convexity properties

$$\sum_{i=1}^{r} h_i(z(t)) = 1, h_i(z(t)) \geq 0. \tag{12}$$

Automated generation and comparison of Takagi–Sugeno and polytopic quasi-LPV models[14]

Polytopic LPV	T-S fuzzy
$\sigma.x(\tau) = \sum_{i=1}^{N} \pi_i(\theta(\tau))(A_i x(\tau) + B_i u(\tau))$	$\sigma.x(\tau) = \sum_{i=1}^{N} \omega_i(\nu(\tau))(A_i x(\tau) + B_i u(\tau))$
$y = \sum_{i=1}^{N} \pi_i(\theta(\tau)) C_i x(\tau)$	$y = \sum_{i=1}^{N} \omega_i(\nu(\tau)) C_i x(t)$
$\sum_{i=1}^{N} \pi_i(\theta(\tau)) = 1$	$\sum_{i=1}^{N} \omega_i(\nu(\tau)) = 1$
$\pi_i(\theta(\tau)) \geq 0$	$\omega_i(\nu(\tau)) \geq 0$

[14] Rotondo et al., "Automated generation and comparison of Takagi-Sugeno and polytopic quasi-LPV models".

Automated generation and comparison of Takagi–Sugeno and polytopic quasi-LPV models

Problems:

- LPV notation is not standard, hindering a more profound analysis
- Sector nonlinearity application

Our proposal

The equivalence between T-S fuzzy and LPV goes beyond what is stated in Rotondo et al.

Polytopic LPV	T-S fuzzy
$\dot{x} = \sum_{i=1}^{N} \rho_i(t)\{A_i x(t) + B_i u(t)\}$	$\dot{x} = \sum_{i=1}^{r} h_i(z(t))\{A_i x(t) + B_i u(t)\}$
$y = \sum_{i=1}^{N} \rho_i(t) C_i x(t)$	$y = \sum_{i=1}^{r} h_i(z(t)) C_i x(t)$
$\sum_{i=1}^{N} \rho_i(t) = 1$	$\sum_{i=1}^{r} h_i(z(t)) = 1$
$\rho_i(t) \geq 0$	$h_i(z(t)) \geq 0$

Our proposal

- T-S fuzzy is a special case of LPV systems
- Polytopic LPV and T-S fuzzy are indistinguishable for control design.

Theorem

A T-S fuzzy system is a polytopic LPV system.

Proof.

To prove this affirmation, we will write the T-S fuzzy system (10) in the form of (6). First, we write it in vector form

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{r} h_i(z(t))A_i & \sum_{i=1}^{r} h_i(z(t))B_i \\ \sum_{i=1}^{r} h_i(z(t))C_i & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \tag{13}$$

Then, choose

$$\rho_i(t) := h_i(z(t)) \; \forall i \in \{1, ..., r\} \tag{14}$$

Proof.

So we can write

$$\sum_{i=1}^{r} \rho_i(t) A_i = \sum_{i=1}^{r} h_i(z(t)) A_i \tag{15}$$

$$\sum_{i=1}^{r} \rho_i(t) B_i = \sum_{i=1}^{r} h_i(z(t)) B_i \tag{16}$$

$$\sum_{i=1}^{r} \rho_i(t) C_i = \sum_{i=1}^{r} h_i(z(t)) C_i \tag{17}$$

Substituting these into (13) yields (6), thus completing the proof. □

Next Steps

- Even though this proof is simple, it has not been done before.
- We are working on proving that any LPV system can be represented by a T-S fuzzy system
- Further analyze the implications of the relationship between them

Outline

LaSalle Extension

- Extension of the LaSalle Invariance Principle[15] for T-S Fuzzy Systems
- Ultimate boundedness[16]
- S-procedure[17]

[15] Luís Fernando Costa Alberto. "O princípio de invariância de Lasalle estendido aplicado ao estudo de coerência de geradores e à análise de estabilidade transitória multi-swing". PhD thesis. 2000.

[16] Michele C. Valentino et al. "Ultimate boundedness sufficient conditions for nonlinear systems using TS fuzzy modelling". In: *Fuzzy Sets and Systems* 361 (2019), pp. 88–100. DOI: 10.1016/j.fss.2018.03.010.

[17] Michele C. Valentino et al. "Sufficient conditions in terms of linear matrix inequalities for guaranteed ultimately boundedness of solutions of switched Takagi-Sugeno fuzzy systems using the S-procedure". In: *Information Sciences* 572 (Sept. 2021), pp. 501–521. DOI: 10.1016/j.ins.2021.04.103.

LaSalle Invariance Principle

Theorem

Let $V : \mathbb{R}^n \to \mathbb{R}$ *and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be* C^1 *functions,* $L \in \mathbb{R}$ *be a constant such that* $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$ *is bounded. Also, let* $\dot{V}(x) \leq 0 \ \forall x \in \Omega_L$ *and* $E := \{x \in \Omega_L : \dot{V}(x) = 0\}$ *and define* B *as the greatest invariant set in* E*. Then, every solution of*

$$\dot{x} = f(x)$$

starting in Ω_L *converges to* B *as* $t \to \infty$

LaSalle Invariance Principle - extended

Theorem

Let $V : \mathbb{R}^n \to \mathbb{R}$ *and* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be* C^1 *functions,* $L \in \mathbb{R}$ *be a constant such that* $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$ *is bounded. Also, let* $C := \{x \in \Omega_L : \dot{V}(x) > 0\}$ *and* $\sup_{x \in C} V(x) = l < L$*, and define* $E := \{x \in \Omega_L : \dot{V}(x) = 0\} \cup \bar{\Omega}_l$*, where* $\bar{\Omega}_l = \{x \in \Omega_L : V(x) \leq l\}$ *and define* B *as the greatest invariant set in* E*. Then, every solution of*

$$\dot{x} = f(x)$$

starting in Ω_L *converges to* B *as* $t \to \infty$

LaSalle Extension

- Alternative way of analyzing the asymptotic behavior of solutions of nonlinear systems
- Applied to Switched T-S Fuzzy systems[18]

[18] Valentino et al., "Ultimate boundedness sufficient conditions for nonlinear systems using TS fuzzy modelling".

LaSalle Extension for T-S Fuzzy systems

Let us consider the following nonlinear system

$$\dot{x} = f(x) \tag{18}$$

where f is a $\mathcal{C}^1$ vector field in $\mathbb{R}^n$, $n \in \mathbb{N}^*$. We assume this system can be *exactly* represented by the T-S Fuzzy model[19]

$$\dot{x} = \sum_{i \in \mathcal{R}} h_i A_i x, \ \mathcal{R} = \{1, \ldots, r\} \tag{19}$$

in the following set of the state space

$$Z = \{x \in \mathbb{R}^n : |x_\nu| \leq \bar{x}_\nu, \ \forall \nu \in \{1, \ldots, n\}\} \tag{20}$$

where $A_i \in \mathbb{R}^{n \times n}$. The membership functions $h_i : Z \longrightarrow \mathbb{R}, \ \forall i \in \mathcal{R}$ have the following convex properties

$$\sum_{i \in \mathcal{R}} h_i(x) = 1, \ h_i(x) \geq 0 \ \forall i \in \mathcal{R}, \ \forall x \in Z. \tag{21}$$

[19] Tadanari Taniguchi et al. "Model construction, rule reduction, and robust compensation for generalized form of Takagi-Sugeno fuzzy systems". In: *IEEE Transactions on Fuzzy Systems* 9.4 (2001), pp. 525–538. DOI: 10.1109/91.940966.

LaSalle Extension for T-S Fuzzy systems

Obtain LMI conditions that guarantee

- Level set Ω_L is bounded
- $\dot{V}(x) > 0$ only in a bounded set $C \subset \Omega_L$

LaSalle Extension for T-S Fuzzy systems

- $\Omega_L = \{x \in \mathbb{R}^n : V(x) < L\}$
- $$V(x) = x' \sum_{k \in G} h_k P_k x,$$

  where $P_k = P_k' \in \mathbb{R}^{n \times n}\ \forall k \in G \subset \mathcal{R}$.
- $$\dot{V}(x) = x' \Big[\sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) + \sum_{k \in G} \dot{h}_k P_k \Big] x$$
- $$Z_\nu = \{x \in Z : x_\nu = \bar{x}_\nu\} \bigcup \{x \in Z : x_\nu = -\bar{x}_\nu\}\ \forall \nu \in \{1, ..., n\}$$

  $$\therefore \bigcup_{\nu \in \mathcal{N}} Z_\nu = \partial Z$$

LaSalle Extension for T-S Fuzzy systems

We will show that it is always possible to choose L such that $\Omega_L \subset Z$.

For every $P_k, k \in G$ and $x \in \mathbb{R}^n$,

$$x^T P_k x \geq \lambda_{min}(P_k)||x||^2. \tag{22}$$

Multiplying (22) by h_k and summing over k yields

$$\sum_{k \in G} h_k x^T P_k x = V(x) \geq \sum_{k \in G} h_k \lambda_{min}(P_k)||x||^2. \tag{23}$$

LaSalle Extension for T-S Fuzzy systems

Evaluating $V(x)$ in ∂Z

$$V(\partial Z) \geq \sum_{k \in G} h_k \lambda_{min}(P_k) ||x||^2 \tag{24}$$

$$\Longrightarrow V(\partial Z) \geq \min_{x \in \partial Z} \sum_{k \in G} h_k \lambda_{min}(P_k) ||x||^2 \geq \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k ||x||^2 \tag{25}$$

LaSalle Extension for T-S Fuzzy systems

If we restrict the norm of x to its smallest value at the border of Z, we can also write

$$V(\partial Z) \geq \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k \min_{\nu \in \{1,\dots,n\}} \bar{x}_\nu^2. \tag{26}$$

Which means that choosing

$$L < \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k \min_{\nu \in \{1,\dots,n\}} \bar{x}_\nu^2. \tag{27}$$

we guarantee that $\Omega_L \subset Z$

LaSalle Extension for T-S Fuzzy systems

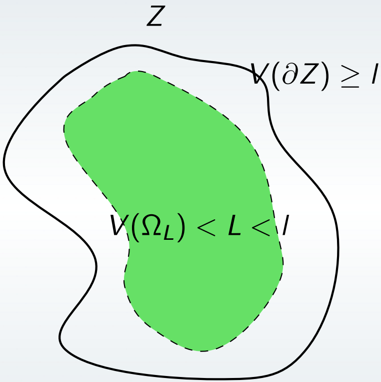


Fig.: Diagram showing that $\Omega_L \subset Z$, where $l = \sum_{k \in G} \min_{x \in \partial Z} \lambda_{min}(P_k) h_k \min_{\nu \in \{1,\dots,n\}} \bar{x}_\nu^2$

Next Steps

Defining or estimating the set

$$C = \{x \in \Omega_L : \dot{V}(x) > 0\} \tag{28}$$

$$\forall x \in C, x' \Big[\sum_{j \in \mathcal{R}} \sum_{k \in G} h_k h_j (A_j' P_k + P_k A_j) + \sum_{k \in G} \dot{h}_k P_k \Big] x > 0$$

Treating the two addends separately, we choose to

$$x' \sum_{k \in G} \sum_{j \in \mathcal{R}} h_k h_j \big(A_j' P_k + P_k A_j \big) x < 0$$

Next Steps

To guarantee this, it is sufficient that

$$\Upsilon_{kk} < 0, \forall k \in G \tag{29}$$

$$\Upsilon_{kj} + \Upsilon_{jk} < 0, \forall k, j \in \mathcal{R}, j < k, \tag{30}$$

where

$$\Upsilon_{kj} = \begin{bmatrix} L_k A_j + A_j' L_k' & * \\ P_k - L_k' + R_k A_j & -R_k - R_k' \end{bmatrix} \tag{31}$$

Next Steps

- Working on $x' \sum_{k \in G} \dot{h}_k P_k x > 0$
- Guarantee the $\sup_{x \in C} V(x)$ condition in the LMIs using the S-Procedure

Outline

UAV Application

Fig.: Bebop 2

- Unmanned Aerial Vehicles (UAVs) have been increasingly gaining popularity in robotics
- hover and vertical take off and landing
- do not require a runway: more flexible
- highly complex, nonlinear, underactuated system
- Parrot Bebop 2: off the shelf, low price, internal hover controller, open source driver

Trajectory tracking implementation

Fig.: Bebop 2

Fig.: Robot Operating System

Fig.: Vicon Motion Capture Systems

Closed loop

Outdoor x Indoor experiments

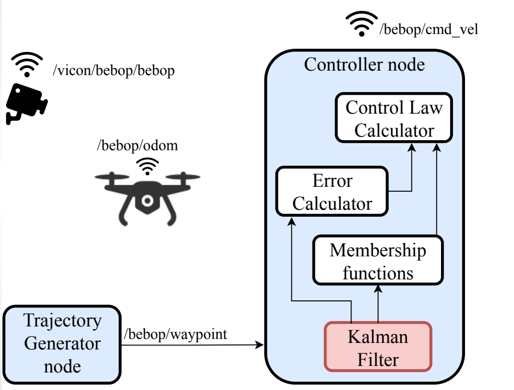


Fig.: Block diagram showing the controller loop

Quadcopter Model

Considering the the quadrotor's response to the input command vector $u = \begin{bmatrix} u_{\nu_x} & u_{\nu_y} & u_{\dot{z}} & u_{\dot{\psi}} \end{bmatrix}^T$, the dynamics can be modeled as the following linear system[a]

$$\ddot{q}_b = \Gamma_A \dot{q}_b + \Gamma_B u \tag{32}$$

where $q_b = \begin{bmatrix} x_b & y_b & z_b & \psi_b \end{bmatrix}^T$

$$\Gamma_A = -\begin{bmatrix} \gamma_2 & & & \\ & \gamma_4 & & \\ & & \gamma_6 & \\ & & & \gamma_8 \end{bmatrix}, \quad \Gamma_B = \begin{bmatrix} \gamma_1 & & & \\ & \gamma_3 & & \\ & & \gamma_5 & \\ & & & \gamma_7 \end{bmatrix} \tag{33}$$

[a] L. Vago Santana et al. "A Trajectory Tracking and 3D Positioning Controller for the AR . Drone Quadrotor". In: *2014 International Conference on Unmanned Aircraft Systems (ICUAS)* (2014), pp. 118–123. DOI: 10.1109/ICUAS.2014.6842321.

For control purposes, it's more convenient to work with position and orientation with respect to a global stationary frame.

Considering rotation only in the z axis, we have

$$\dot{q} = R\dot{q}_b \tag{34}$$

where

$$R = \begin{bmatrix} \cos\psi & -\sin\psi & 0 & 0 \\ \sin\psi & \cos\psi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{35}$$

Santana et. al. (2014) - Global Reference Frame

$$\ddot{q} + NR^T\dot{q} = Mu, \tag{36}$$

$$M = \begin{bmatrix} \gamma_1 \cos\psi & -\gamma_3 \sin\psi & 0 & 0 \\ \gamma_1 \sin\psi & \gamma_3 \cos\psi & 0 & 0 \\ 0 & 0 & \gamma_5 & 0 \\ 0 & 0 & 0 & \gamma_7 \end{bmatrix} \tag{37}$$

$$N = \begin{bmatrix} \gamma_2 \cos\psi & -\gamma_4 \sin\psi & 0 & 0 \\ \gamma_2 \sin\psi & \gamma_4 \cos\psi & 0 & 0 \\ 0 & 0 & \gamma_6 & 0 \\ 0 & 0 & 0 & \gamma_8 \end{bmatrix}. \tag{38}$$

Error Model

Given a twice differentiable desired state trajectory $\zeta_d(t) = \begin{bmatrix} \dot{q}_d^T & q_d^T \end{bmatrix}^T$, let us define the tracking error $e = \zeta - \zeta_d$,

$$\dot{e} = \begin{bmatrix} -NR^T & 0 \\ I & 0 \end{bmatrix} e + \begin{bmatrix} I \\ 0 \end{bmatrix} \nu \tag{39}$$

where $\nu = -\ddot{q}_d - NR^T \dot{q}_d + Mu$ is a virtual control input to the error system. The control input to be sent to the quadrotor system is given by

$$u = M^{-1}(\nu + \ddot{q}_d + NR^T \dot{q}_d) \tag{40}$$

System Analysis

- The origin is an unstable equilibrium point for system (39) and has an infinite number of equilibria located in $\{e \in \mathbb{R}^8 : e_1, e_2, e_3, e_4 = 0\}$
- Hartman-Grobman Theorem does not apply (linearized system with non hyperbolic equilibria)
- To make the closed loop system have an unique equilibrium point at the origin, we modify (40)

$$u = M^{-1}\left(\nu + \ddot{q}_d + NR^T\dot{q}_d - k(q - q_d)\right), \tag{41}$$

  where k is a positive scalar

- $$\Rightarrow \dot{e} = \begin{bmatrix} -NR^T & -kI \\ I & 0 \end{bmatrix} e + \begin{bmatrix} I \\ 0 \end{bmatrix} \nu \tag{42}$$

- We have not thoroughly studied the impact of this modification, but it seems to have increased the feasibility of the various LMI conditions we tested

LPV system

The error system is given by

$$\dot{e} = A(\psi)e + B\nu$$

where

$$A(\psi) = -\left[\begin{array}{cccc|c} \gamma_2\cos^2(\psi)+\gamma_4\sin^2(\psi) & \frac{\gamma_4-\gamma_2}{2}\sin(2\psi) & & & \\ \frac{\gamma_2-\gamma_4}{2}\sin(2\psi) & \gamma_4\cos^2(\psi)+\gamma_2\sin^2(\psi) & & & 0 \\ & & \gamma_6 & & \\ & & & \gamma_8 & \\ \hline & & -I & & 0 \end{array}\right]$$

and

$$B = \begin{bmatrix} I \\ 0 \end{bmatrix}\nu$$

Modeling alternatives

Different choices of parameter vectors yield different representations. For example, choosing

$$\rho(t) = \begin{bmatrix} \gamma_2 \cos^2(\psi) + \gamma_4 \sin^2(\psi) \\ \frac{\gamma_2 - \gamma_4}{2} \sin(2\psi) \end{bmatrix} \tag{43}$$

as the parameter vector, one can write

$$\dot{e} = (A_0 + \sum_{i=1}^{2} A_i \rho_i)e + B\nu \tag{44}$$

for some appropriate A_i, $i = 0, \ldots, 2$. The parameters bounds are given by

$$\begin{aligned} \rho_1 &\in [\min(\gamma_2, \gamma_4),\ \max(\gamma_2, \gamma_4)] \\ \rho_2 &\in [-\frac{\gamma_2 - \gamma_4}{2},\ \frac{\gamma_2 - \gamma_4}{2}]. \end{aligned} \tag{45}$$

Modeling alternatives

As an alternative we have

$$\rho(t) = \begin{bmatrix} \cos^2(\psi) \\ \sin^2(\psi) \\ \sin(2\psi) \end{bmatrix} \tag{46}$$

with bounds

$$\rho_1, \rho_2 \in [0, 1], \;\; \rho_3 \in [-1, 1] \tag{47}$$

which gives a representation of the form

$$\dot{e} = (A_0 + \sum_{i=1}^{3} A_i \rho_i) e + B\nu \tag{48}$$

Modeling alternatives

Using the sector nonlinearity approach, the regular choice of premises would be

$$z_1(t) = \gamma_2 \cos^2(\psi) + \gamma_4 \sin^2(\psi) \tag{49a}$$

$$z_2(t) = \frac{\gamma_2 - \gamma_4}{2} \sin(2\psi) \tag{49b}$$

However, choosing simpler premises

$$z_1(t) = \cos \psi(t) \tag{50a}$$

$$z_2(t) = \sin \psi(t) \tag{50b}$$

still gives an exact representation of the error system.

TS-fuzzy model

The tracking error system can be thus represented as

$$\dot{e}(t) = \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j A_{ij} e + B\nu = \sum_{i=1}^{4}\sum_{j=1}^{4} h_i h_j \begin{bmatrix} -N_i R_j^T & 0 \\ I & 0 \end{bmatrix} e + \begin{bmatrix} I \\ 0 \end{bmatrix} \nu \tag{51}$$

where N_i, R_j are the matrices of the local models of the system and

$$\forall\, i \in \{1, \dots, 4\},\ \ h_i(\psi) \geq 0 \text{ and } \sum_{i=1}^{r} h_i(\psi) = 1. \tag{52}$$

$$\begin{aligned} h_1(\psi) &= w_1^1(\psi) w_1^2(\psi), & h_2(\psi) &= w_1^1(\psi) w_0^2(\psi), \\ h_3(\psi) &= w_0^1(\psi) w_1^2(\psi), & h_4(\psi) &= w_0^1(\psi) w_0^2(\psi). \\ w_0^1 &= \frac{\cos(\psi(t)) + 1}{2}, & w_1^1 &= 1 - w_0^1 \\ w_0^2 &= \frac{\sin(\psi(t)) + 1}{2}, & w_1^2 &= 1 - w_0^2. \end{aligned} \tag{53}$$

Models summary

#	Description
1	4 subsystems
2	16 subsystems
3	10 subsystems
4	8 subsystems

Table: Summary of different modeling alternatives we developed

All the models are equivalent to each other and exactly represent the nonlinear system. For proof, check Rayza Araujo et al. "On the selection of membership functions of TS fuzzy models for a commercial quadrotor". In: *Anais da XIV Conferência Brasileira de Dinâmica, Controle e Aplicações*. Escola de Engenharia de São Carlos - USP, 2019

Controller Design via LMI

For all design procedures used, the controller is of the form

$$\nu = -\sum_{i=1}^{r} h_i K_i e \tag{54}$$

where r is the number of subsystems for each model

Remark

In implementation, the control signal sent to the system is

$$u = M^{-1}(-\sum_{i=1}^{r} h_i K_i e + \ddot{q}_d + NR^T \dot{q}_d) \tag{55}$$

Stabilizing State Feedback

$$V(\mathbf{e}(t)) : S \to \mathbb{R}^8,\ V(\mathbf{e}(t)) = \sum_{i=1}^{4} h_i \mathbf{e}(t)' \mathbf{P}_i \mathbf{e}(t),\ S \subset \mathbb{R}^8,\ \mathcal{R} = \{1, ..., 4\}$$

Theorem

Let ϕ_ρ be known positive real numbers satisfying $|\dot{h}_\rho| \leq \phi_\rho, \forall \rho \in \mathcal{R}$. If for a positive constant μ, there exist matrices $\mathbf{Z} \in \mathbb{R}^{8\times 8}$, $\mathbf{Y}_i \in \mathbb{R}^{4\times 8}$, $\mathbf{X} = \mathbf{X}' \in \mathbb{R}^{8\times 8}$ and $\mathbf{Q}_i = \mathbf{Q}_i' \in \mathbb{R}^{8\times 8}$ satisfying (56)-(59), *then system* (**??**), *with feedback $\nu = -\sum_{i=1}^{4} h_i K_i e$ and local gains $\mathbf{K}_i = \mathbf{Y}_i \mathbf{Z}^{-1}$, is asymptotically stable for any $\mathbf{e} \in \mathcal{D} := \{\mathbf{e} \in S : |\dot{h}_\rho| \leq \phi_\rho,\ \forall \rho \in \mathcal{R}\}$.*

$$\mathbf{Q}_i \succ \mathbf{0}, \quad \forall i \in \mathcal{R}, \tag{56}$$

$$\mathbf{Q}_\rho + \mathbf{X} \succeq \mathbf{0}, \quad \forall \rho \in \mathcal{R}, \tag{57}$$

$$\Psi_{ii} \prec \mathbf{0}, \quad \forall i \in \mathcal{R}, \tag{58}$$

$$\frac{1}{r-1}\Psi_{ii} + \Psi_{ij} + \Psi_{ji} \prec \mathbf{0}, \quad \forall i, j \in \mathcal{R}, i < j \tag{59}$$

Theorem (cont.)

where

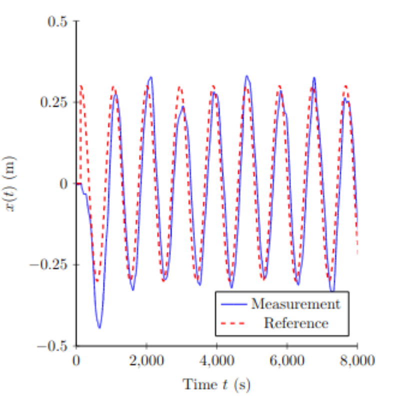
$$\Psi_{ij} = \begin{bmatrix} \mathbf{\mathring{Q}} - \mathbf{A}_{ij}\mathbf{Z} - \mathbf{Z}'\mathbf{A}_{ij}' + \mathbf{B}\mathbf{Y}_i + \mathbf{Y}_i'\mathbf{B}' & \star \\ \mathbf{Q}_i - \mu\left(\mathbf{A}_{ij}\mathbf{Z} - \mathbf{B}\mathbf{Y}_i\right) + \mathbf{Z}' & \mu\left(\mathbf{Z} + \mathbf{Z}'\right) \end{bmatrix},$$

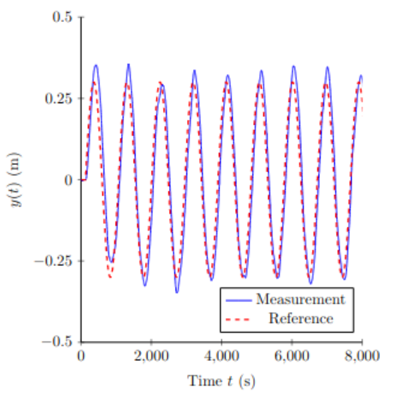
$$\mathbf{\mathring{Q}} = \sum_{\rho=1}^{4} \phi_\rho \left(\mathbf{Q}_\rho + \mathbf{X}\right).$$

Proof.

Check Rayza Araujo et al. “Trajectory Tracking for the Bebop Parrot quadrotor using Takagi-Sugeno fuzzy models”. In: *Anais do 14º Simpósio Brasileiro de Automação Inteligente.* Galoa, 2019, pp. 2449–2454. DOI: 10.17648/sbai-2019-111505 □

Experimental Results





- large initial error but the controller is able to correct it
- large error in the max and min values of the sinusoid caused by poor velocity estimation (will be discussed in the implementation section)
- otherwise, good tracking

Robust Stability - model uncertainty

Remark

Results were accepted for conference, but the publication was cancelled due to the pandemic

Polytopic Uncertainty:

$$\dot{\mathbf{e}} = \sum_{k=1}^{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \tilde{\alpha}_k h_i h_j (\mathbf{A}_{kij} - \mathbf{B}\mathbf{K}_i)\mathbf{e} + \mathbf{B}_w \mathbf{w} \tag{60}$$

$$\mathbf{y} = \mathbf{C}\mathbf{e} \tag{61}$$

$\tilde{\alpha}_k$ model the uncertainty in parameters $\gamma_i,\ i = 2, 4, 6, 8$. They satisfy the convex conditions $\tilde{\alpha}_1 \geq 0$, $\tilde{\alpha}_2 \geq 0$ and $\tilde{\alpha}_1 + \tilde{\alpha}_2 = 1$. Input w models the disturbance.

Control design goal: find K_i

Robust Stability

Theorem

If there exist a positive constant μ, *matrices* $\mathbf{W} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{Y}_i \in \mathbb{R}^{n_u \times n_x}$ *and* $\mathbf{Q}_i \succ \mathbf{0} \in \mathbb{R}^{n_x \times n_x}$, $\forall\ i \in \mathcal{R}$, *satisfying*

$$\Upsilon_{kii}^{\ell} \prec \mathbf{0} \tag{62}$$

$$\frac{2}{r-1}\Upsilon_{kii}^{\ell} + \Upsilon_{kij}^{\ell} + \Upsilon_{kji}^{\ell} \prec \mathbf{0}, \tag{63}$$

then, system (60), *with* $\mathbf{w(t)} \neq \mathbf{0}$, *local gains* $\mathbf{K}_i = \mathbf{Y}_i\mathbf{W}^{-1}$, *and* $\phi = \max_{i\in\mathcal{R}}\{\phi_i\}$, *is asymptotically stable for any solution contained in* $\mathcal{D} = \left\{\mathbf{e}(t) \in \mathbb{R}^8 : |\dot{h}_i| \leq \phi_i,\ \forall i \in \mathcal{R}\right\}$,

Robust Stability

Theorem (cont.)

where $k \in \{1,2\}$, $i,j \in \mathcal{R}$, $i \neq j$, $\ell \in \{1,\dots,\eta\}$

$$\Upsilon^{\ell}_{kij} = \begin{bmatrix} \tilde{\mathbf{Q}}_{\ell} - \mathbf{A}_{kij}\mathbf{W} - \mathbf{W}'\mathbf{A}'_{kij} + \mathbf{B}\mathbf{Y}_i + \mathbf{Y}'_i\mathbf{B}' & \star \\ \mathbf{Q}_i + \mathbf{W}' - \mu(\mathbf{A}_{kij}\mathbf{W} - \mathbf{B}\mathbf{Y}_i) & \mu(\mathbf{W} + \mathbf{W}') \end{bmatrix} \tag{64, 65}$$

$$\tilde{\mathbf{Q}}_{\ell} = \tilde{\phi} \sum_{i=1}^{4} \mathbf{G}_{(i,\ell)} \mathbf{Q}_i \tag{66}$$

and $\mathbf{G}_{(i,\ell)}$ is the element in the i^{th} row and ℓ^{th} column of matrix $\tilde{G}$

Matrix $\tilde{G}$

Define the set

$$\Omega := \{v \in \mathbb{R}^4 : -\phi_i \le v_i \le \phi_i,\ c'v = 0\} \tag{67}$$

where $c' = [1\ 1\ 1\ 1]$ and v_i is the i^{th} coordinate of v. It represents the region of valid combinations of $\dot{h}_i$. The vertices of (67) are organized in the following matrix:[20]

$$\tilde{\mathbf{G}} := \left[\ \tilde{\mathbf{g}}^1 \quad \tilde{\mathbf{g}}^2 \quad \dots \quad \tilde{\mathbf{g}}^\eta\ \right] \in \mathbb{R}^{4\times\eta}. \tag{68}$$

where $\eta = \dfrac{4!}{2!2!} = 6$.

- Less conservative than assuming $|\dot{h}_i| = \phi_i$

[20] Leonardo A. Mozelli and Ricardo L. S. Adriano. "On computational issues for stability analysis of LPV systems using parameter-dependent Lyapunov functions and LMIs". In: *International Journal of Robust and Nonlinear Control* 29.10 (July 2019), pp. 3267–3277. DOI: 10.1002/rnc.4528.

Results

There are both simulation and experimental results, but we decided to include only the simulation in the publication (timing reasons)

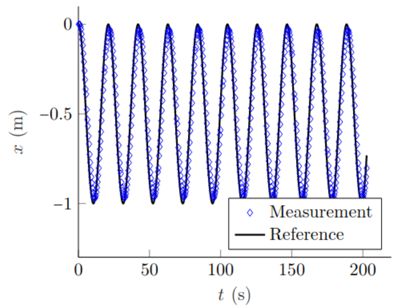


Fig. 2. Sphinx result for variable x. Control design solution using Theorem 1 and $\mu = 1$.

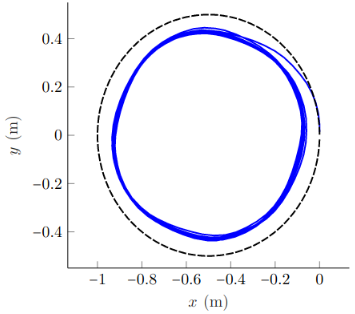


Fig. 1. Sphinx result. Desired trajectory (dashed line) and the control solution using Theorem 1 for $\mu = 1$ (continuous line).

- Performance loss due to parameter uncertainty (steady state error)
- This uncertainty model is too conservative

$\mathcal{H}\infty$ performance

Theorem

If there exist positive constants μ *and* γ*, matrices* $\mathbf{W} \in \mathbb{R}^{n_x \times n_x}$, $\mathbf{Y}_i \in \mathbb{R}^{n_u \times n_x}$ *and* $\mathbf{Q}_i \succ \mathbf{0} \in \mathbb{R}^{n_x \times n_x}$, $\forall\ i \in \mathcal{R}$*, satisfying*

$$\Psi^{\ell}_{kii} \prec \mathbf{0} \tag{69}$$

$$\frac{2}{r-1}\Psi^{\ell}_{kii} + \Psi^{\ell}_{kij} + \Psi^{\ell}_{kji} \prec \mathbf{0} \tag{70}$$

Then, system (60) *is stabilizable with local gains* $\mathbf{K}_i = \mathbf{Y}_i\mathbf{W}^{-1}$*, and* $\mathcal{H}\infty$ *guaranteed cost* γ *for any solution* $\mathbf{e}(t)$ *contained in* $\mathcal{D} = \left\{\mathbf{e}(t) \in \mathbb{R}^8 : |\dot{h}_i| \leq \phi_i,\ \forall i \in \mathcal{R}\right\}$,

$\mathcal{H}\infty$ performance

Theorem (cont.)

where $k \in \{1,2\}$, $i,j \in \mathcal{R}$, $i \neq j$, $\ell \in \{1,\ldots,\eta\}$, $\tilde{\phi} = \max_{i\in\mathcal{R}}\{\phi_i\}$

$$\Psi_{kij}^{\ell} = \begin{bmatrix} \tilde{\mathbf{Q}}_\ell - \mathbf{A}_{kij}\mathbf{W} - \mathbf{W}'\mathbf{A}_{kij}' + \mathbf{B}\mathbf{Y}_i + \mathbf{Y}_i'\mathbf{B}' & \star & \star & \star \\ \mathbf{Q}_i + \mathbf{W}' - \mu(\mathbf{A}_{kij}\mathbf{W} - \mathbf{B}\mathbf{Y}_i) & \mu(\mathbf{W}+\mathbf{W}') & \star & \star \\ -\mathbf{B}_w' & -\mu\mathbf{B}_w' & -\gamma^2 & \star \\ \mathbf{C}\mathbf{W} & \mathbf{0} & \mathbf{0} & -\mathbf{I} \end{bmatrix}$$

$$\tilde{\mathbf{Q}}_\ell = \tilde{\phi}\sum_{i=1}^{r}\mathbf{G}_{(i,\ell)}\mathbf{Q}_i$$

Results (simulation)

There are also experimental results (outdoor, wind measured)

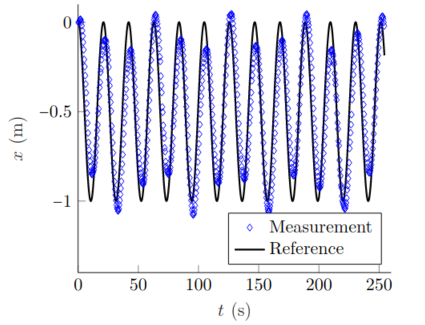


Fig. 7. Sphinx result for variable x. Control design solution using Theorem 2, $\gamma = 0.5, \mu = 0.5$ and wind disturbance of 3 m/s in both x and y directions.

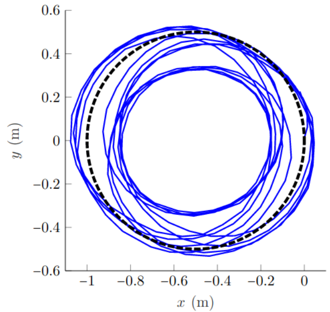


Fig. 6. Sphinx result. Desired trajectory (dashed line) and the control design solution using Theorem 2 (continuous line), $\gamma = 0.5, \mu = 0.5$ and wind disturbance of 3 m/s in both x and y directions.

- Uncertainty model is too conservative and hinders the disturbance rejection

Bounds on control input

Because of the application, we had to include bounds on the control input in all previous LMIs using the following lemma[21]

Lemma

Given positive constants ϑ_1 and ϑ_2, then the local gains $\mathbf{K}_i = \mathbf{Y}_i\mathbf{W}^{-1}$, $\forall\, i \in \mathcal{R}$, satisfies the condition $\mathbf{K}_i\mathbf{K}_i' < \vartheta_1 I/\vartheta_2^2$ if the following LMIs hold:

$$\begin{bmatrix} \vartheta_1 I & \mathbf{Y}_i \\ \mathbf{Y}_i' & I \end{bmatrix} > 0 \qquad (71)$$
$$\mathbf{Q}_i > \vartheta_2 I$$

- Control input is indirectly bounded by reducing the magnitude of entries of $\mathbf{K}_i$
- ϑ_1 and ϑ_2 are chosen appropriately

[21] E. Assunção et al. "Robust state-derivative feedback LMI-based designs for multivariable linear systems". In: *International Journal of Control* 80.8 (Aug. 2007), pp. 1260–1270. DOI: 10.1080/00207170701283899.

Static Output feedback

- Motivation: Not measuring all states + unsolved problem in the literature
- No feasibility
  - various LMI conditions (based on the previous results and others from literature) $\longrightarrow https://github.com/araujorayza/bebop\_control\_design$
  - different model types
- Controllability issues?

Controllability Study

- Gap in literature
- Linearization $\longrightarrow$ no significant results
- Lyapunov based LMI analysis $\longrightarrow$ infeasibility not proven

Outline

Publications (Journals)

- ELIAS, LEANDRO JOSE ; FARIA, FLAVIO ANDRADE ; ARAUJO, R. ; MAGOSSI, RAFAEL ; OLIVEIRA, VILMA ALVES . **Robust static output feedback H control for uncertain Takagi-Sugeno fuzzy systems**. IEEE Transactions on Fuzzy Systems, v. 1, p. 1-1, 2022.
- Leandro J. Elias, Flávio A. Faria, Rayza Araujo, Vilma A. Oliveira. **Stability analysis of Takagi–Sugeno systems using a switched fuzzy Lyapunov function**. Information Sciences, Volume 543, 2021, Pages 43-57, ISSN 0020-0255.
- ELIAS, LEANDRO J. ; FARIA, FLÁVIO A. ; ARAUJO, RAYZA ; OLIVEIRA, VILMA A. . **Stability conditions of TS fuzzy systems with switched polynomial Lyapunov functions.** IFAC-PAPERSONLINE, v. 53, p. 6352-6357, 2020.

Publications

- F. Q. MAGOSSI, RAFAEL; ARAÚJO, RAYZA ; J. ELIAS, LEANDRO ; A. FARIA, FLÁVIO ; A. OLIVEIRA, VILMA . **Projeto de controlador de ordem fixa via LMI com limitação da norma H-infinito e garantias de margens de estabilidade.** In: Congresso Brasileiro de Automática 2020, 2020, Porto Alegre. Anais do Congresso Brasileiro de Automática 2020, 2020.
- ARAUJO, R. ; ELIAS, L. J. ; FARIA, F. A. ; OLIVEIRA, V. A. . **On the selection of membership functions of TS fuzzy models for a commercial quadrotor.** In: XIV Conferência Brasileira de Dinâmica, Controle e Aplicações, 2019, São Carlos. Anais da XIV Conferência Brasileira de Dinâmica, Controle e Aplicações, 2019. p. 1-7.

Publications

- ARAUJO, R. ; FARIA, F. A. ; ELIAS, L. J. ; OLIVEIRA, V. A. . **Trajectory Tracking for the Bebop Parrot quadrotor using Takagi-Sugeno fuzzy models.** In: XIV Simpósio Brasileiro de Automação Inteligente, 2019, Ouro Preto. Anais do XIV Simpósio Brasileiro de Automação Inteligente. Campinas: SBA, 2019. p. 1-6.
- FARIA, FLAVIO A. ; ELIAS, LEANDRO J. ; ARAUJO, RAYZA ; OLIVEIRA, VILMA A. . **Less conservative state feedback design conditions for switched Takagi-Sugeno fuzzy systems.** In: 2019 18th European Control Conference (ECC), 2019, Naples. 2019 18th European Control Conference (ECC), 2019. p. 3698.
- MAGOSSI, R. F. Q.; BEZERRA, R. A. ; LEME, P. V. ; OLIVEIRA, V. A. . **PID Controller Design Based On H∞ Performance.** In: XIII Simpósio Brasileiro de Automação Inteligente, 2017, Porto Alegre. XIII Simpósio Brasileiro de Automação Inteligente, 2017. p. 1814-1820.

Unpublished results

- Survey on Gain Scheduling
- Control law modification
- "Robust Trajectory Tracking for the Bebop Parrot quadrotor using fuzzy PDC controllers"
  - accepted, but unpublished because of the pandemic
  - there is still experiment data to be added, so it can easily be expanded
- LPV and T-S Fuzzy relationship analysis*
- LaSalle extension for T-S Fuzzy systems*
- Static Output feedback design and controllability for quadrotor*

*still working on the results

Thank You!