Let $\varphi(x) = f(x) - \frac{\alpha}{2}\|x\|_2^2$, f is α-strong convex $\Rightarrow \varphi$ is convex

$\varphi(y) - \varphi(x) - \nabla\varphi(x)^T(y-x)$ (HW1 (d))

$= f(y) - \frac{\alpha}{2}\|y\|^2 - f(x) + \frac{\alpha}{2}\|x\|^2 - (\nabla f(x)^T - \alpha x^T)(y-x)$

$= f(y) - f(x) - \nabla f(x)^T(y-x) - \frac{\alpha}{2}\|y\|_2^2 + \frac{\alpha}{2}\|x\|_2^2 + \alpha\langle x, y\rangle - \alpha\|x\|_2^2$

$\leq \frac{\beta}{2}\|x-y\|_2^2 - \frac{\alpha}{2}\|x-y\|^2$ (f β smooth)

$\Rightarrow \varphi$ is $(\beta-\alpha)$-smooth

$\Rightarrow (\nabla f(x) - \nabla f(y))^T(x-y) - \alpha\|x-y\|_2^2 \geq \frac{1}{\beta-\alpha}\|\nabla\varphi(x) - \nabla\varphi(y)\|_2^2$

$\frac{1}{\beta-\alpha}\|\nabla\varphi(x) - \nabla\varphi(y)\|_2^2 = ((\nabla f(x) - \nabla f(y))^T - \alpha(x-y)^T)$
$((\nabla f(x) - \nabla f(y)) - \alpha(x-y))\frac{1}{\beta-\alpha}$

$= (\|\nabla f(x) - f(y)\|_2^2 - 2\alpha(\nabla f(x) - \nabla f(y))^T(x-y) + \alpha^2\|x-y\|^2)\cdot\frac{1}{\beta-\alpha}$

$(1 + \frac{2\alpha}{\beta-\alpha})(\nabla f(x) - f(y))^T(x-y) \geq \underbrace{(\|\nabla f(x) - f(y)\|_2 + 2\alpha^2\|x-y\|_2^2)}_{\beta-\alpha}$

$(\frac{\beta+\alpha}{\beta-\alpha})\ \frown\ \geq \frac{\alpha^2}{\beta-\alpha}\|x-y\|_2^2 + \frac{\beta\alpha-\alpha^2}{\beta-\alpha}\cdot\|x-y\|_2^2$
$+ \frac{1}{\beta-\alpha}\|\nabla f(x) - \nabla f(y)\|_2^2$

Let $\varphi(x) = f(x) - \frac{\alpha}{2}\|x\|_2^2$ φ is convex as proved previously

Since f is α-strong convex

$f(x) - f(y) \geq \nabla f(y)^T(x-y)$

$\varphi(x) - \varphi(y) - \nabla\varphi(y)^T(x-y)$

$= f(x) - \frac{\alpha}{2}\|x\|_2^2 - f(y) + \frac{\alpha}{2}\|y\|^2 + (\nabla f(y)^T - \frac{\alpha}{2}y^T)(x-y)$

$f(x) - f(y) + \nabla f(y)^T(x-y) \leq \frac{\beta}{2}\|x-y\|^2$