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Let P, Q be point clouds and P_i & Q_i be i^{th} points

$$\underset{R \in SO(3),\, t \in \mathbb{R}^3}{\operatorname{argmin}} \|(RP+t) - Q\|^2$$

above formula gives best aligning transforma between P and Q given known correspondences. This is because, with known correspondences it tries to find a transform minimizing distance between P_i & Q_i for all i. When correspondences are known, all P_i & Q_i aligning gives minimum value for above function since

$$\|RP+t - Q\|^2 \geq 0$$

and $\|RP+t - Q\|^2 = 0$ only if all P_i & Q_i coincide

$$F(t) = \sum_{i=1}^{n} \|(RP_i+t) - Q_i\|^2$$

$$\frac{\partial F(t)}{\partial t} = 2\sum_{i=1}^{n} RP_i + t - Q_i = 0 \quad \text{(for minima)}$$

$$\Rightarrow 2R\sum_{i=1}^{n} P_i + 2t\sum_{i=1}^{n} 1 - 2\sum_{i=1}^{n} Q_i = 0$$

$$t = \frac{1}{n}\sum_{i=1}^{n} Q_i - R\frac{1}{n}\sum_{i=1}^{n} P_i$$

$$\therefore \boxed{t = \bar{Q} - R\bar{P}} \quad - (1)$$

Assuming R is known, we see that above value of t is optimal.

$$\bar{Q} = \frac{1}{n}\sum_{i=1}^{n} Q_i \qquad \bar{P} = \frac{1}{n}\sum_{i=1}^{n} P_i$$

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Substituting it back in our original function

$$R = \underset{R \in SO(3)}{\operatorname{argmin}} \| R(P_i - \bar{P}) - (Q_i - \bar{Q}) \|^2$$

We can think of $\bar{P}$ & $\bar{Q}$ as centroids of P & Q. hence $P_i - \bar{P}$ & $Q_i - \bar{Q}$ is basically Centering our point clouds on origin

$\therefore$ take centred point clouds

$$X = P_i - \bar{P} \qquad Y = Q_i - \bar{Q}$$

and let rotated X be $X' = RX$

$$\therefore R = \underset{R \in SO(3)}{\operatorname{argmin}} \| X_i' - Y_i \|^2$$

$$Tr((X'-Y)^T (X'-Y)) = \| X_i' - Y_i \|^2 .$$

$$Tr((X'-Y)^T (X'-Y)) = Tr(X'^T X' + Y^T Y - 2Y^T X')$$

by property of trace $Tr(A+B) = Tr(A) + Tr(B)$ and $Tr(\lambda A) = \lambda Tr(A)$ we get

~~we get $\|X_i' - Y_i\|^2 = Tr(X'^T X') + Tr(Y^T Y) - 2Tr(\ldots)$~~

$$R = \underset{R \in SO(3)}{\operatorname{argmin}} Tr(X'^T X') + Tr(Y^T Y) - 2Tr(Y^T X')$$

we know $X' = RX$. Since R is orthonormal lengths are preserved $\Rightarrow Tr(X'^T X') = \sum_{i=1}^{n} |X_i'|^2 = \sum_{i=1}^{n} |X_i|^2$

$$\therefore R = \underset{R \in SO(3)}{\operatorname{argmin}} \sum_{i=1}^{n} |X_i|^2 + |Y_i|^2 - 2Tr(Y^T X')$$

only variable term above containing R is $Tr(Y^T X')$

$$\therefore R = \underset{R \in SO(3)}{\operatorname{argmax}} Tr(Y^T X') = \underset{R \in SO(3)}{\operatorname{argmax}} Tr(Y^T R X)$$

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by property of trace ~~Rell~~ $Tr(ABC) = Tr(CAB)$

$\Rightarrow R = \underset{R \in SO(3)}{\operatorname{argmax}} \; Tr(XY^T R)$

using SVD let $XY^T = UDV^T$

$\therefore Tr(XY^T R) = Tr(UDV^T R) = Tr(DV^T RU)$
(property : $Tr(ABCD) = Tr(DABC)$)

if $M = V^T R U$

$Tr(XY^T R) = \sum_{i=1}^{3} d_i M_{ii} \leq \sum_{i=1}^{3} d_i$

above inequality comes from the fact that V^T, R, U are all orthonormal. hence, M which is their product is also orthonormal.
Each column must have distance (ie norm) as 1 and each element should not exceed 1.
Hence max values for all $M_{ii} = 1$ and M must identity matrix.

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$\therefore R = \underset{R \in SO(3)}{\arg\max}\ Tr(XY^TR) \Rightarrow V^TRU = I$

$\therefore \quad R = VU^T$

But we haven't considered ~~the~~ the case where $|R|$ might be -1 using the above.

In the above case we ~~to~~ take second largest possible value for $\sum_{i=1}^{n} d_i M_{ii}$ which occur when $M_{33} = -1$. To take this into account we add a correction term C.

Hence we add a correction term to finally get

(2) $\boxed{R = VCU^T}$ ~~where C = [1 0 0; 0 1 0; 0 0 sgn(det(UV^T))·1]~~

Where

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \text{sgn}(\det(UV^T)) \cdot 1 \end{bmatrix}$$

$\therefore$ Using $\underset{R \in SO(3),\, t \in \mathbb{R}^3}{\arg\min} \| RP + t - Q \|^2$ — (A)

definitions of $\bar{Q}, \bar{P}, V, C, U$ discussed earlier

we found (1) $t = \bar{Q} - R\bar{P}$ and

(2) $R = VCU^T$ (Steps (1) & (2))

This is exactly procrustes alignment?

Since function (A) gives best alignment for known correspondences, so does procrustes

Hence Proved