Chap-3
Analytic Geometry

Norm:

$\|\cdot\| : V \to \mathbb{R}$

→ Absolutely homo: $\|\lambda x\| = |\lambda| \|x\|$

→ Tri. Inequality: $\|x+y\| \leq \|x\| + \|y\|$

→ +ve def: $\|x\| \geq 0$ & $\|x\| = 0 \iff x = 0$

Manhattan Norm: [l_1 norm]

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

Euclidean Norm: (l_2 norm)

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{x^T x}$$

Inner Product:

→ Dot product:

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

→ Bilinear mapping:

$$\Omega(\lambda x + \psi y, z) = \lambda \Omega(x, z) + \psi \Omega(y, z)$$

$$\Omega(x, \lambda y + \psi z) = \lambda \Omega(x, y) + \psi \Omega(x, z)$$

Takes 2 vectors & maps to real numbers

Sym: $\Omega(x, y) = \Omega(y, x)$

+ve def:

$$\Omega(x, x) > 0, \; \Omega(0, 0) = 0$$

$$x = \sum_{i=1}^{n} \psi_i b_i$$

$$y = \sum_{j=1}^{n} \lambda_j b_j$$

$$\langle x, y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_i \langle b_i, b_j \rangle \lambda_j$$

$\Downarrow$

$$\langle x, y \rangle = \hat{x}^T A \hat{y}$$

Length & Distance:

$\|x\| := \sqrt{\langle x, x\rangle}$

Manhattan norm is ex. of norm without corr. inner product

Distance & Metrics:

$d(x,y) := \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$

Euc. dist

$d : V \times V \to \mathbb{R}$

$(x,y) \to d(x,y)$ Metric

+ve def: $d(x,y) \geq 0$ $d(x,y)=0 \Leftrightarrow x=y$

Symmetric: $d(x,y) = d(y,x)$

Δ inequ: $d(x,z) \leq d(x,y) + d(y,z)$

→ Transformations don't change length & angle too

Orthonormal Basis

For $\{b_1, b_2 \ldots b_n\}$ of V, if $\langle b_i, b_j\rangle = 0$

$\langle b_i, b_i\rangle = 1$

then orthonormal basis

Each basis vector has length/norm.

To construct orthonormal basis, called Gram-Schmidt Process

Orthogonal Complement

Cauchy-Schwartz Ineq:

$|\langle x,y\rangle| \leq \|x\|\|y\|$

$(V, \langle \cdot, \cdot \rangle)$

$\|x\| = \sqrt{x^T x}$

Angles & Orthogon.

ω - Angle b/w two vectors

$$-1 \leq \frac{\langle x,y\rangle}{\|x\| \ \|y\|} \leq 1$$

unique $\omega \in [0, \pi]$:

$$\cos(\omega) = \frac{\langle x,y\rangle}{\|x\| \ \|y\|}$$

$$\cos(\omega) = \frac{\langle x,y\rangle}{\langle x,x\rangle\langle y,y\rangle} = \frac{x^T y}{\sqrt{x^T x \, y^T y}}$$

x, y are orthogonal if

$\langle x, y\rangle = 0$

x, y are orthonormal if

$\|x\| = 1 = \|y\|$

Orthogonal matrix

$AA^T = I = A^T A \Rightarrow A^{-1} = A^T$

Inner Product of Functions

$$\langle u, v \rangle := \int_a^b u(x) v(x)\, dx$$

Orthogonal Projection.

Linear mapping: $\pi : V \to U$ is called projection if

$$\pi^2 = \pi \circ \pi = \pi$$

Matrix Decomposition

Determinant:

A is invertible only if $det(A) \neq 0$

Signed volume of an n-dimensional parallelopiped formed by columns of matrix A.

Area of $\|^{elo}$gram: $|det(b,g)|$

Area of $\|^{elo}$piped: $|det(r,g,b)|$

Laplace Expansion:

Row

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{jk} det(A_{jk})$$

Column

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j} a_{kj} det(A_{kj})$$

Submatrix:

$$A_{kj} \in \mathbb{R}^{(n-1)\times(n-1)}$$

$$det(A) = det(A^T)$$

$$det(A^{-1}) = \frac{1}{det(A)}$$

$$det(\lambda A) = \lambda^n det(A)$$

Trace:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

$tr(A+B) = tr(A) + tr(B)$

$tr(\alpha A) = \alpha\, tr(A)$

$tr(I_n) = n$

$tr(AB) = tr(BA)$

Characteristic Polynomial

$$P_A(\lambda) = det(A - \lambda I)$$

$$= c_0 + c_1\lambda + c_2\lambda^2 \ldots + c_{n-1}\lambda^{n-1} + (-1)^n \lambda^n$$

Characteristic polynomial

$c_0 = det(A)$

$c_{n-1} = (-1)^{n-1} tr(A)$

EigenValues & Eigenvectors

If A is square-matrix

λ = eigen-value of A

$x \in \mathbb{R}^n \setminus \{0\}$ = Eigenvectors of A

$$\underbrace{Ax = \lambda x}_{\text{Eigenvalue function}}$$

Largest eigen value → First eigenvalue

$(A - \lambda I_n)x = 0$ $\qquad$ $det(A - \lambda I_n) = 0$

$rk(A - \lambda I_n) < n$

Codirected: Same direction

Colinear: Same/Opposite direction

If x is E.Vec of with val λ then cx is E.Vec of value $c\lambda$ as

$$A(cx) = cAx = c\lambda x = \lambda(cx)$$

All vectors collinear to x are also eigenvectors of A.

→ Set of all eigenvalues of A is called eigenspectrum or just spectrum of A.

IMP Properties:

→ A, A^T has same λ's but not same x's

→ E_λ is null space of $A - \lambda I$ since

$$Ax = \lambda x \Rightarrow (A - \lambda)x = 0$$
$$(A - \lambda I)x = 0$$
$$x \in \ker(A - \lambda I)$$

→ Symmetric, +ve definite always have +ve, real eigenvalues

→ λ is root of characteristic poly. $p_A(\lambda)$ of A.

→ Algebraic multiplicity of λ_i is number of times root appears in char. polynomial

→ Eigenspace set of all eigenvectors of A associated with λ spans subspace of $\mathbb{R}^n$ denoted by E_λ.

→ Geometric multiplicity

GM of λ_i is no. of L.I eigenvectors associated with λ_i

Must be atleast 1.