Parameterisations

R matrix → Implicit

other meth → Explicit

Inverse Problem

Euler Angles on lecture 7

$$R = R(\hat{z}_A, \alpha)\, R(\hat{x}_B, \beta) * R(\hat{z}_C, \gamma)$$

$$R = {}^{A}_{B}R\, {}^{B}_{C}R\, {}^{C}_{D}R$$

→ Can lead to gumbal lock

→ Loss of 1 DOF

Shouldn't use matrices:

→ Numerical issue

→ Storage issues

→ User interaction issues

→ Interpolation issues

Lecture 8

Composition of rotation

$${}^{0}({}^{1}A) = {}^{0}_{1}R\, {}^{1}A\, ({}^{0}_{1}R)^T$$

If

$${}^{a}({}^{b}_{c}R) = {}^{a}_{b}R\, {}^{b}_{c}R\, ({}^{a}_{b}R)^T$$

Role of eigen values, vectors

Rotation vectors and Axis/Angle

Arbitrary orientation as rotation about some unit axis by some angle

Axis-Angle form

Rotation Vector - Scale axis by angle and compact it down to single 3D vector

Axis-Angle Theorem:

$$z' = e^{i\theta} z$$

$$\left\{ \begin{array}{l} n = \text{Normal} \\ \theta = \text{Theta} \end{array} \right\} \text{where} \quad v \cdot n = 0$$

$$\boxed{v' = \cos\theta\, v + \sin\theta\,(n \times v)}$$

INVERSE

Given,

$$v = v_{\parallel} + v_{\perp}$$

$$v' = v_{\parallel} + v_{\perp}'$$

Sol:

$$v' = v_{\parallel} + \cos\theta\, v_{\perp} + \sin\theta\,(n \times v_{\perp})$$

$$v' = v_{\parallel} + \cos\theta\,(v - v_{\parallel}) + \sin\theta\,(n \times v)$$

$$v' = (1 - \cos\theta)\, v_{\parallel} + \cos\theta\, v + \sin\theta\,(n \times v)$$

Rodrigue's rotation formula

$$v' = (1 - \cos\theta)(v \cdot n)\, n + \cos\theta\, v + \sin\theta\,(n \times v)$$

If $\theta = 0$, $v' = v$

$R(n,\theta) = I + \sin\theta\,\hat{n} + (1-\cos\theta)\hat{n}^2$

Rot Vector Convention

$k = n\theta$

Minimal way to represent = 3

In above case 4 parameters

Inverse:

As $\text{trace}(R) = 1 + 2\cos\theta$

$\theta = \cos^{-1}\left(\frac{\text{tr}(R)-1}{2}\right)$

$n = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{bmatrix}$

Singularity

If $R = I$, $\theta = 0$

$R \neq I$, two possible $\hat{n}, \theta$

$n = -n$, $\theta' = 2\pi - \theta$

QUARTERIONS

→ Non-commutative

→ $q = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$

→ $q = (q_0, q)$

→ $i^2 = j^2 = k^2 = ijk = -1$

→ Multiplication done using $ij = k$, $jk = i$, $ki = j$, $ijk = -1$

$p * q = p_0 q_0 - p\cdot q + p_0 q + q_0 p + p \times q$

→ Quat Properties

→ $q^* = q_0 - q$

→ $|q| = |q|^2 = qq^* = q_0^2 + q\cdot q$

→ Iden: $qq^{-1} = q^{-1}q = 1$

→ Forms group under multiplication

→ Inverse: $q^* q q^{-1} = q^{-1} q q^* = q^*$

$q^{-1} = \frac{q^*}{|q|}$

If normalized $q^{-1} = q^*$

Prod of 2 quat → another quat

$\left.\begin{matrix} q = c\alpha + us\alpha \\ p = c\beta + us\beta \end{matrix}\right\} \Rightarrow r = pq = c(\alpha+\beta) + us(\alpha+\beta) \Rightarrow \mathbb{R}^4$

↳ Same unit vector

→ Triple product produces pure quat

Quat ⟷ Rot

Pure quat: $q_0 = 0$

To associate angles: $q = q_0 + q$

$q_0^2 + q^2 = 1$

$c\theta^2 + s\theta^2 = 1$

$q = \cos\left(\frac{\theta}{2}\right) + u\sin\left(\frac{\theta}{2}\right)$

where $u = \frac{q}{|q|}$

$p' = qpq^*$

↳ length is preserved also holds for $p' = q^* p q$ in $\mathbb{R}^3$

↳ Angle of rotation is twice angle Above associated with quaternion

$p = 0 + p$

$p' = 0 + p'$

$$p' = qpq^* = (q_0 + q)(0 + p)(q_0 - q)$$

$$= (q_0^2 - |q|^2)p + 2(q \cdot p)q + 2q_0(q \times p)$$

	Params	Inverse	Singularity
R	9	R^T	None
Euler	$3(\alpha, \beta, \gamma)$	$(-\gamma, -\beta, -\alpha)$	$\beta = \pm \pi/2$
Axis-Angle	$4(\hat{n}, \theta)$	$(-\hat{n}, \theta)$	$\theta = 0, \pi$
Rotational plane	$K = \hat{n}\theta$	$-\hat{K}$	$\theta = \pi$
Quat	4 param		None