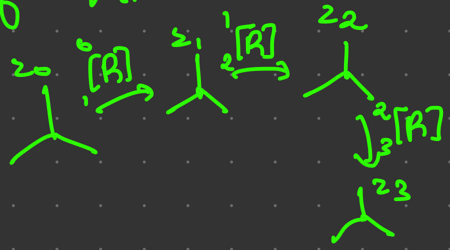
Composition of rotation

Given

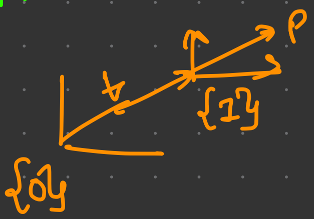


Given ${}^1p$ in frame $\{1\}$ transformed under ${}^1A$ to ${}^1q$. How does it look in same frame

$${}^1p = {}^1A\,{}^1q$$

$${}^0({}^1A) = {}^0_1[R]\,{}^1A\,({}^0_1[R])^T$$

Planar Translation:



$${}^1p = t + p$$

$$\begin{bmatrix} {}^1x \\ {}^1y \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = {}^1p = T(t)p$$

1 is Appended

$${}^1p = AP$$

Homogenous form is used

For inverse,

$$p = T^{-1}\,{}^1p$$

$$\hookrightarrow \begin{bmatrix} 1 & 0 & -t_x \\ 0 & 1 & -t_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$T(p)^{-1} = T(-p)$$

$$T(p)^{-1}T(p) = I$$

Planar Rotation:

$$p' = R(\ )p$$

$$\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} \quad \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Pose: Position & Orientation in plane

$\mathbb{R}^2$

$${}^A_BT = \begin{bmatrix} r_{11} & r_{12} & {}^A_Bx \\ r_{21} & r_{22} & {}^A_By_p \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^A_BR & {}^A_BO \\ 0 & 1 \end{bmatrix}$$

Rotation

Translation

special euclidean

For $\mathbb{R}^2$, ${}^A_BR \in SE(2)$

SAME APPLICABLE FOR $\mathbb{R}^3$

$$SE(2) = \{(p,R) \mid p \in \mathbb{R}^2,\ R \in SO(2)\} = \mathbb{R}^2 \times SO(2)$$

SE(3):

$SE(3) = \mathbb{R}^3 \times SO(3)$

→ Closure: $T_1, T_2 \in SE(3) \Rightarrow T_1 T_2 \in SE(3)$

→ Associative: $T_1(T_2 T_3) = (T_1 T_2) T_3$

→ Identity: $T, I \in SE(3) \Rightarrow TI = T$

→ Identity: For Every T, $T^{-1} \in SE(3)$ $TT^{-1} = I \in SE(3)$

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

Pure rotation:

$$\begin{bmatrix} R^T & 0 \\ 0 & 1 \end{bmatrix}$$

Pure translation

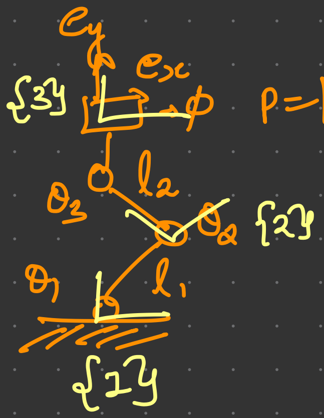
$$\begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$$

Composition:

$$^{A}_{C}T = \begin{bmatrix} ^{A}_{B}R \, ^{B}_{C}R & ^{A}_{B}R \, ^{B}_{C}O + {}^{A}O_B \\ 0 & 1 \end{bmatrix}$$

$$^{A}p = {}^{A}_{B}T \, ^{B}_{C}T \, ^{C}p$$

FORWARD KINEMATICS



$p = [x, y]^T$

Non-linear
↑
$p = f(\theta)$
↳ $\in \mathbb{R}^M$

↳ finding p given θ

→ read about this

Given $\{\theta_1, \theta_2\}$ $\{l_1, l_2\}$ find $[e_x, e_y, \phi]^T$

$e_{x_1} = l_1 c\theta_1 + l_2 c\theta_{12}$

$e_{y_1} = l_1 s\theta_1 + l_2 s\theta_{12}$

$\phi = \theta_1 + \theta_2$

$\theta_{12} = \theta_1 + \theta_2$

WORKSPACE:

→ Set of endeffector configurations

→ $g: Q \rightarrow SE(3)$ → Workspace
↓
Configspace

$W = \{g(\theta) \mid \theta \in Q\} \subseteq SE(3)$ Ignore θ part

⇓

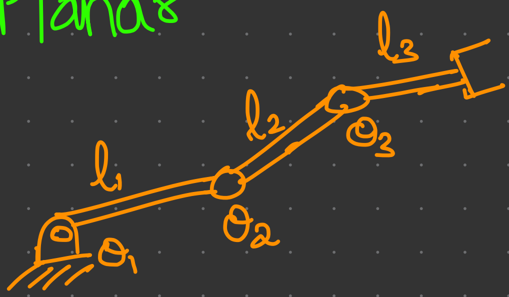
$W_R = \{p(\theta) \mid \theta \in Q\} \in \mathbb{R}^3$ [Reachable workspace]

Non-linear mapping b/w C-space & Workspace

Straight lines in C-space maps to curves in workspace

2R Workspace [Read this topic 0:50:00]

3D Planar



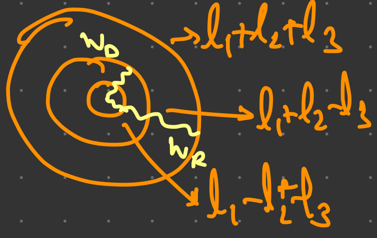
x, y, ϕ

$x = l_1 c\theta_1 + l_2 c\theta_2 - - - -$

$y = l_1 s\theta_1 + l_2 s\theta_2 - - - -$

If $l_1 > l_2 > l_3$

$l_1 > l_2 + l_3$



$W_R = \{P(\theta) | \theta \in Q$

W_R: Doesn't consider ability to arbitrarily orient the end-effector

W_D: Volume of space reached with arbitrary orientation [Dexterous Workspace]

$$W_D = \{p \in \mathbb{R}^3 \mid \forall R \in SO(3), \exists\, \theta \text{ with } f(\theta) = (p, R)\} \in \mathbb{R}^3$$

In above case, if $l_3 = 0$, $W_D = W_R$

nR Spatial Manipulator

We know $^{A}p = {}^{A}_{B}[T]\, {}^{B}p$

$$T = T(\hat{x}, L_1) \cdot R(\hat{z}, \theta_1) \quad (\text{or}) \quad T = R(\hat{z}, \theta_1)\, T(\hat{z}, L_1)$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & L_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 & \cos\theta_1 L_1 \\ \sin(\theta_1) & \cos\theta_1 & 0 & \sin\theta_1 L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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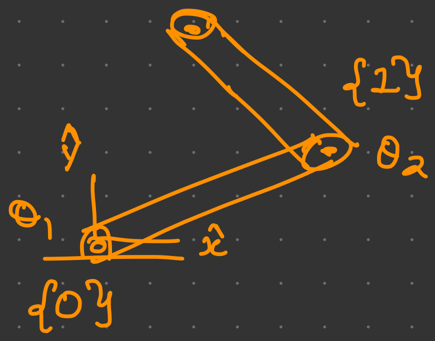


$${}^{1}_{0}T = [R(\hat{z},\theta_1).T(\hat{x},l_1)][R(\hat{z},\theta_2).T(\hat{x},l_2)$$

$\vdots$

2R forward dynamics: ${}^{0}_{2}[T] = {}^{0}_{1}[T]\,{}^{1}_{2}[T]$

$$T = \underbrace{R(\hat{z},\theta_1)T(\hat{x},L_1)}\ \underbrace{R(\hat{z},\theta_2)\,T(\hat{x},L_2)}$$

2R Forward Kinematics

$$\mathbf{T} = \mathbf{R}(\hat{\mathbf{z}},\theta_1)\mathbf{T}(\hat{\mathbf{x}},\mathrm{L}_1)\mathbf{R}(\hat{\mathbf{z}},\theta_2)\mathbf{T}(\hat{\mathbf{x}},\mathrm{L}_2)$$

$${}^{0}_{2}[\mathbf{T}] = {}^{0}_{1}[\mathbf{T}]{}^{1}_{2}[\mathbf{T}]$$

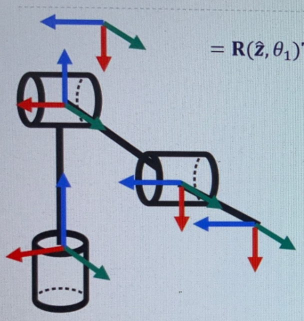
$$= \begin{pmatrix} \mathrm{Cos}[\theta_1] & -\mathrm{Sin}[\theta_1] & 0 & \mathrm{Cos}[\theta_1]L_1 \\ \mathrm{Sin}[\theta_1] & \mathrm{Cos}[\theta_1] & 0 & \mathrm{Sin}[\theta_1]L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Cos}[\theta_2] & -\mathrm{Sin}[\theta_2] & 0 & \mathrm{Cos}[\theta_2]L_2 \\ \mathrm{Sin}[\theta_2] & \mathrm{Cos}[\theta_2] & 0 & \mathrm{Sin}[\theta_2]L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Cos}[\theta_1]\mathrm{Cos}[\theta_2] - \mathrm{Sin}[\theta_1]\mathrm{Sin}[\theta_2] & -\mathrm{Cos}[\theta_2]\mathrm{Sin}[\theta_1] - \mathrm{Cos}[\theta_1]\mathrm{Sin}[\theta_2] & 0 & \mathrm{Cos}[\theta_1]L_1 + (\mathrm{Cos}[\theta_1]\mathrm{Cos}[\theta_2] - \mathrm{Sin}[\theta_1]\mathrm{Sin}[\theta_2])L_2 \\ \mathrm{Cos}[\theta_2]\mathrm{Sin}[\theta_1] + \mathrm{Cos}[\theta_1]\mathrm{Sin}[\theta_2] & \mathrm{Cos}[\theta_1]\mathrm{Cos}[\theta_2] - \mathrm{Sin}[\theta_1]\mathrm{Sin}[\theta_2] & 0 & \mathrm{Sin}[\theta_1]L_1 + (\mathrm{Cos}[\theta_2]\mathrm{Sin}[\theta_1] + \mathrm{Cos}[\theta_1]\mathrm{Sin}[\theta_2])L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{Cos}[\theta_1+\theta_2] & -\mathrm{Sin}[\theta_1+\theta_2] & 0 & \mathrm{Cos}[\theta_1]L_1 + \mathrm{Cos}[\theta_1+\theta_2]L_2 \\ \mathrm{Sin}[\theta_1+\theta_2] & \mathrm{Cos}[\theta_1+\theta_2] & 0 & \mathrm{Sin}[\theta_1]L_1 + \mathrm{Sin}[\theta_1+\theta_2]L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3R Spatial:

3R Spatial



$$= \mathbf{R}(\hat{\mathbf{z}},\theta_1)\mathbf{T}(\hat{\mathbf{z}},\mathrm{L}_1)\mathbf{R}(\hat{\mathbf{y}},\pi/2)\mathbf{T}(\hat{\mathbf{z}},0)\mathbf{R}(\hat{\mathbf{z}},\theta_2).\mathbf{T}(\hat{\mathbf{y}},\mathrm{L}_2)\mathbf{R}(\hat{\mathbf{z}},\theta_3).\mathbf{T}(\hat{\mathbf{y}},\mathrm{L}_3)$$

$$= \begin{pmatrix} -\mathrm{Sin}[\theta_1]\mathrm{Sin}[\theta_2+\theta_3] & -\mathrm{Cos}[\theta_2+\theta_3]\mathrm{Sin}[\theta_1] & \mathrm{Cos}[\theta_1] & -\mathrm{Sin}[\theta_1](\mathrm{Cos}[\theta_2]L_2 + \mathrm{Cos}[\theta_2+\theta_3]L_3) \\ \mathrm{Cos}[\theta_1]\mathrm{Sin}[\theta_2+\theta_3] & \mathrm{Cos}[\theta_1]\mathrm{Cos}[\theta_2+\theta_3] & \mathrm{Sin}[\theta_1] & \mathrm{Cos}[\theta_1](\mathrm{Cos}[\theta_2]L_2 + \mathrm{Cos}[\theta_2+\theta_3]L_3) \\ -\mathrm{Cos}[\theta_2+\theta_3] & \mathrm{Sin}[\theta_2+\theta_3] & 0 & L_1 + \mathrm{Sin}[\theta_2]L_2 + \mathrm{Sin}[\theta_2+\theta_3]L_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

DH Parameters: → Reduces no. of parameters needed for kinematic representation

Approach for forward kinematics: Attach ref frames to each link of open chain and then to derive the forward kinematics from the knowledge of relative displacements b/w adjacent link frames

Set of rules for assigning the frame

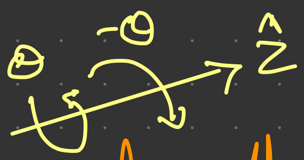
Two link parameters: Link length, link twist

Two joint parameters: Joint angle, link offset

DH Convention:

1. $\hat{z}_i$ coincides with joint axis $\{i\}$ and $\hat{z}_{i-1}$ with $\{i-1\}$

   Right hand rule determines direction of rotation



   Find unique line that mutually intersects both axis a_{i-1}

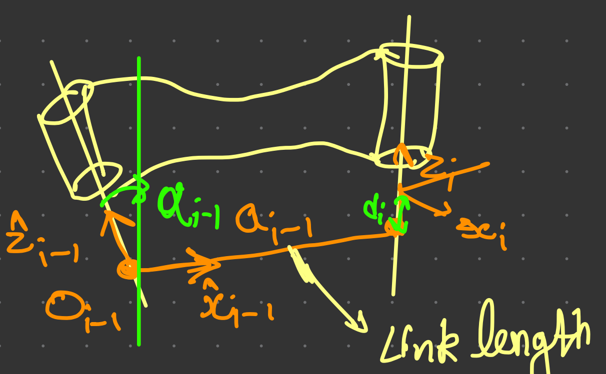
2. To fix origin of frame:

   Find line segment a_{i-1} that mutually intersects $\hat{z}_i, \hat{z}_{i-1}$

   Mutually $\perp^{lar}$ always exists and is unique except when both parallel

3. $\hat{x}_{i-1}$ is chosen to be in direction of mutually $\perp^{lar}$ line pointing from $\hat{z}_{i-1}$ & $\hat{z}_i$

   $\hat{y}_{i-1}$ determined by RH rule



Link length: Mutually $\perp^{lar}$ line b/w $\hat{z}_{i-1}$ & $\hat{z}_i$ is called link length a_{i-1} of link $i-1$

Link twist: Angle α_{i-1} is angle from $\hat{z}_{i-1}$ to $\hat{z}_i$ measured about $\hat{x}_{i-1}$

Link Offset: d_i is distance from intersection of $\hat{x}_{i-1}$ to $\hat{z}_i$ to origin of link-{i} frame

Example:

$$ {}^{i-1}_{i}[T] = \underbrace{R(\hat{x}_{i-1}, \alpha_{i-1})}_{\text{Link Twist}}\, \underbrace{T(\hat{x}_{i-1}, a_{i-1})}_{\text{Link Length}}\, \underbrace{T(\hat{z}_i, d_i)}_{\text{Link Offset}}\, \underbrace{R(\hat{z}_i, \theta_i)}_{\text{Joint Angle}} $$

$$ {}^{i-1}_{i}[T] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Cos}[\alpha_{i-1}] & -\mathrm{Sin}[\alpha_{i-1}] & 0 \\ 0 & \mathrm{Sin}[\alpha_{i-1}] & \mathrm{Cos}[\alpha_{i-1}] & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Cos}[\theta_i] & -\mathrm{Sin}[\theta_i] & 0 & 0 \\ \mathrm{Sin}[\theta_i] & \mathrm{Cos}[\theta_i] & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{i-1}_{i}[T] = \begin{pmatrix} \mathrm{Cos}[\theta_i] & -\mathrm{Sin}[\theta_i] & 0 & a_{i-1} \\ \mathrm{Cos}[\alpha_{i-1}]\mathrm{Sin}[\theta_i] & \mathrm{Cos}[\alpha_{i-1}]\mathrm{Cos}[\theta_i] & -\mathrm{Sin}[\alpha_{i-1}] & -\mathrm{Sin}[\alpha_{i-1}]d_i \\ \mathrm{Sin}[\alpha_{i-1}]\mathrm{Sin}[\theta_i] & \mathrm{Cos}[\theta_i]\mathrm{Sin}[\alpha_{i-1}] & \mathrm{Cos}[\alpha_{i-1}] & \mathrm{Cos}[\alpha_{i-1}]d_i \\ 0 & 0 & 0 & 1 \end{pmatrix} $$

Special Cases:

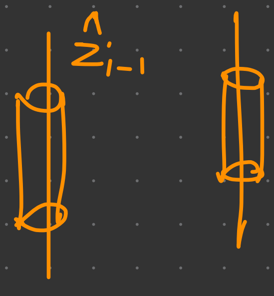
Case I: When $\hat{z}_i \perp \hat{z}_{i-1}$, no line segement exists

→ Link length = 0

→ $\hat{x}_{i-1} \perp$ (plane spanned by $\hat{z}_i$ & $\hat{z}_{i-1}$)

→ $\hat{x}_{i-1}$ could be on either direction by size of $\hat{\alpha}_{i-1}$ has to be changed accordingly

Case 2: When $\hat{z}_i \parallel \hat{z}_{i-1}$, there doesn't exist unique orthogonal line segement



Example of RRRP in 0:57:
R R Manipulator