Intro to R.V & Density Functions

Population: Set of all possible samples in problem

Unit selected from each **trial** of experiment is reffered to **sample**.

Sample space of x

Processing of selecting a sample is called sampling.

Sampling process can be repeated with replacement or without replacemen

Independent: Outcome of particular trial has no bearing on outcome of following trials i.e. samples are independent of each other.

Identical distribution: Probablity that any particular sample is drawn is unchanged across trials. Prob. is identical for all trials

I.I.D: Independent & Identical

Discrete R.V

→ A r.v x is called discrete R.V if set of all possible values that x can take is countable

→ $X = \{v_1, v_2 \dots v_n\}$ $n=$ finite

→ **Probablity Mass function (PMF):**

$$\forall_i \; P(v_i) \geq 0$$

$$\sum_{i=1}^{n} P(v_i) = \sum_{i=1}^{n} p_i = 1$$

→ $E[x] = \mu = \sum x P(x) = \sum_{i=1}^{n} v_i P(v_i)$ [Mean Weighted]

→ $Var[X] = \sigma^2 = E[(x-\mu)^2] = \sum_{i=1}^{n} (v_i-\mu)^2 P(v_i)$ ↳ Variance; Whole population

Continous R.V

→ Continous range of values infinite number of values $(-\infty, \infty)$

→ **Probablity Density function (PDF):**

$$\forall_x \; p(x) > 0,$$

$$Pr[x \in (a,b)] = \int_a^b p(x)\,dx$$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1$$

$p(x_t)$ → Limiting value for density of prob in small window around point x_t

$\rightarrow \sigma^2 = E[x^2] - (E[x])^2$

σ = Standard deviation

$\rightarrow E(x) = \mu = \int_{-\infty}^{\infty} x p(x) dx$

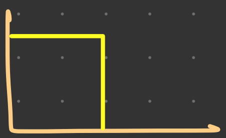
$\rightarrow Var(x) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x) dx$

Uniform Density:

$$U(a,b) = \begin{cases} \frac{1}{b-a} & , a \leq x \leq b \\ 0 & , \text{otherwise} \end{cases}$$

$$\mu = \frac{a+b}{2} \quad \sigma^2 = \frac{(b-a)^2}{12}$$

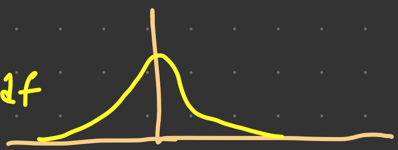
pdf



Normal Density:

$$N(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

pdf



Cumulative Distribution Function (CDF)

$$C(t) = \int_{-\infty}^{t} p(x) \rightarrow \text{Integral of density up to a point}$$

$$C(t) = \begin{cases} 0 & , t < a \\ \frac{t-a}{b-a} & , a \leq t \leq b \\ 1 & , t > b \end{cases}$$

CDF of normal:

$$cdf(x) = \frac{1}{2}\left(1 + erf\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right) \rightarrow \text{No closed form}$$

$$\hookrightarrow erf(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2} dt$$

Generating Numbers:

Use CDF to generate random numbers

$p(x)$, $y = C(x)$ $\quad [t-\delta t, t+\delta t]$, $r = C(t)$

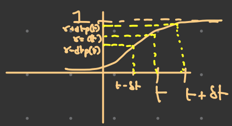
For $x = t + \delta t$,

$y = r + \delta t\, p(t)$

If δt = small,

$p(t+\delta t) \approx p(t)$

$C(t-\delta t) = r - \delta t . p(t)$



All samples of x within window of size $2\delta t$ around t will map to window of size $2\delta t . p(t)$ around $C(t)$ for y.

Density of $y = \frac{1}{p(t)}$ Density of x is unity i.e. y is of uniform $[0,1]$

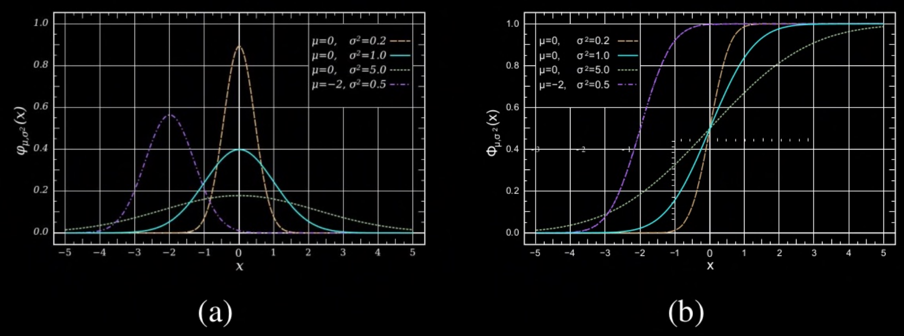


Figure 3. (a) normal densities with different parameters and (b) the their CDFs [?].

Given R.V x of any density, corresponding R.V $y = C(t)$ will be $U[0,1]$

When inversed,

R.V y that follows pdf $U[0,1]$, r.v $x = C^{-1}(y)$ will follow PDF with corresponding CDF as $C(\,)$

In other words,

given set of random numbers y_i, with uniform density $U(0,1)$, we can map it to set of R.V x_i with any desired PDF using inverse CDF function