ARAVIND

DIP Lecture 9
Freq Domain Processing

→ Frequency Domain:

$\boxed{\text{INP}} \rightarrow \boxed{\text{Transform}} \rightarrow \boxed{\text{Processing}} \rightarrow \boxed{\text{Inverse Transform}} \rightarrow \boxed{\text{Output}}$

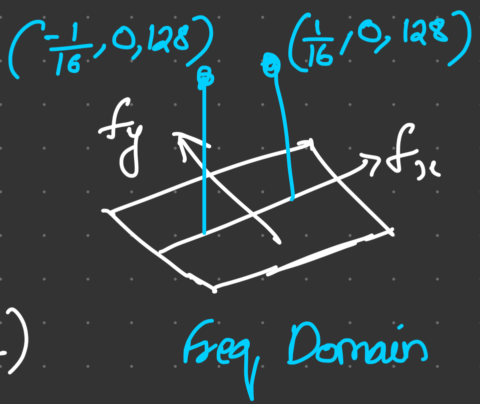
→ Spatial period:

Minimum no. of pixels b/w identical patterns in periodic image

→ Representation of spatial in freq domain:

→ Expressed in tuples

for $I(x,y) = 128\cos\left(\frac{2\pi x}{16}\right)$



Now for $I(x,y) = 128\sin\left(\frac{2\pi x}{4}\right)$

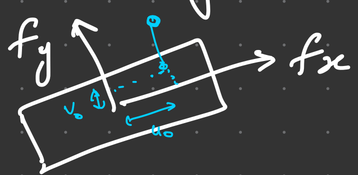
Any compression time/spatial domain is expansion in time domain and vice-versa

If freq along y-axis then in freq domain, impulse are observed in f_y axis

If we have $I(x,y) = \underbrace{\text{constant}}_{\text{DC component}} + \text{periodic}$

so impulse at origin

$s(x,y) = \sin\left[2\pi(u_0 x + v_0 y)\right]$



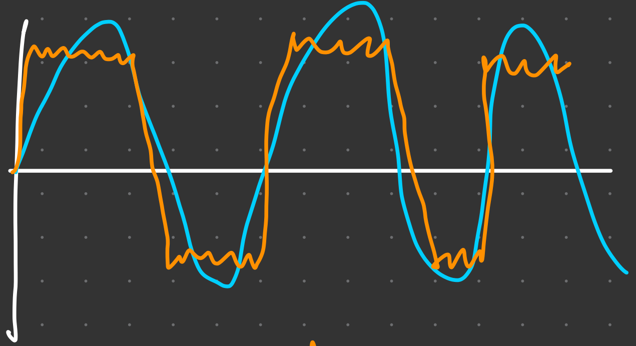
Lots of natural phenomenon follow periodic signals

↓

Use fourier series

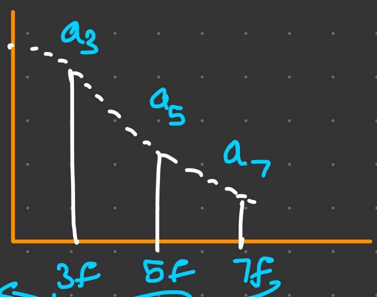
Approximate periodic signals with sines & cosines

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_n \cos\left(\frac{m\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$



Signal

⇒



Harmonics

Freq Spectrum

where f is the fundamental

As no. of terms increases, the approximation improves

To simplify this,

Using $e^{i\theta} = \cos\theta + i\sin\theta$

→ unit circle in counter-clockwise

So, $e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t)$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}, \quad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

So representing fourier series in complex coefficients

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\left(\frac{2\pi n t}{T}\right)}$$

$$c_n = \frac{1}{T}\int_0^T f(t) e^{-j\frac{2\pi n t}{T}} dt$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_m = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi m t}{T}\right) dt$$

$$b_m = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right) dt$$

Fourier Transform [Generalisable for non-periodic and periodic]

Approximate non-periodic signals with complex sinusoids

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

→ Centrosymmetric (symmetric about origin)

Phasor vectors spans all 4 quadrants due to which we get negative frequency

FT:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

Inv FT:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)]$$

Unit Impulse:

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

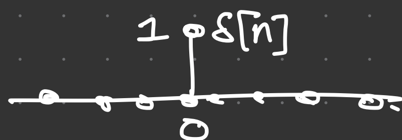
→ Area = 1

↳ Can be scaled too

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

Discrete Impulse Function:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$



Properties:

1) $\int_a^b \delta(t)\,dt = \begin{cases} 1, & a < 0 < b \\ 0, & \text{otherwise} \end{cases}$ → Integral property

2) $\int_a^b \delta(t) f(t)\,dt = \int_a^b \delta(t) \cdot f(0)\,dt = f(0) \int_a^b \delta(t)\,dt$

$= \begin{cases} f(0), & a < t < b \\ 0, & \text{otherwise} \end{cases}$

Sifting property

Shifted Unit Impulse

$\delta(t-\lambda) = \begin{cases} \infty, & t = \lambda \\ 0, & t \neq \lambda \end{cases}$

$\int_a^b \delta(t-\lambda)\,dt = \begin{cases} 1, & a < \lambda < b \\ 0, & \text{otherwise} \end{cases}$

$\int_a^b \delta(t-\lambda) f(t)\,dt$

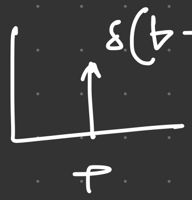
$= \begin{cases} f(\lambda), & a < t < b \\ 0, & \text{otherwise} \end{cases}$

FT of impulse functions

what freq components are present

→ DC signal

FT of shifted impulse:



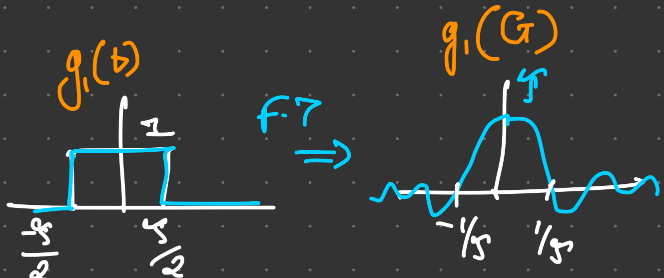
$F(\omega) = e^{-j\omega T}$

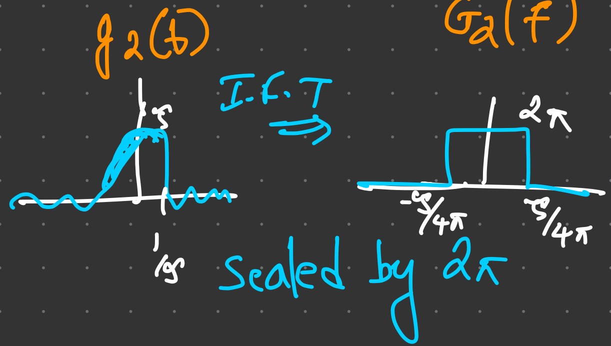
$\int_a^b \delta(t-T)\, x(t)\,dt = \begin{cases} x(T), & a < T < b \\ 0, & \text{otherwise} \end{cases}$

Duality property:

$$F[f(t)] = F(\omega)$$

$$F[F(t)] = 2\pi f(-\omega)$$

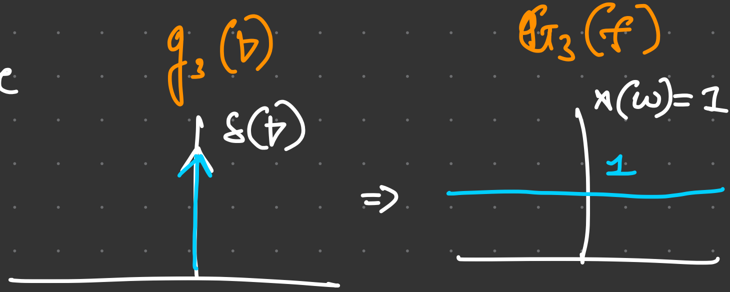


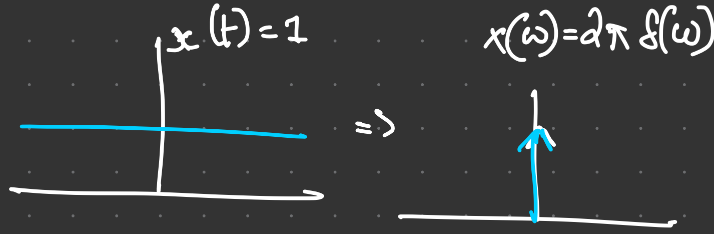


Advantages:

we only need to know one direction of transform.

That way we can get reverse transform





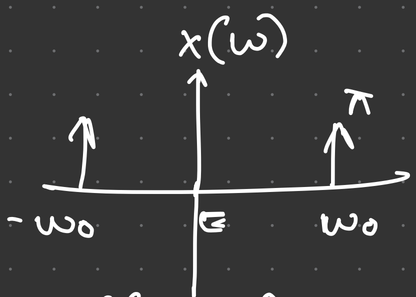
FT of complex exponential

$$e^{i\omega_0 t} \xleftrightarrow{F} 2\pi\delta(\omega-\omega_0)$$

$F(x_1(t)) + F(x_2(t))$
$= F(x_1(t) + x_2(t))$
↓
Not applicable for multiplication

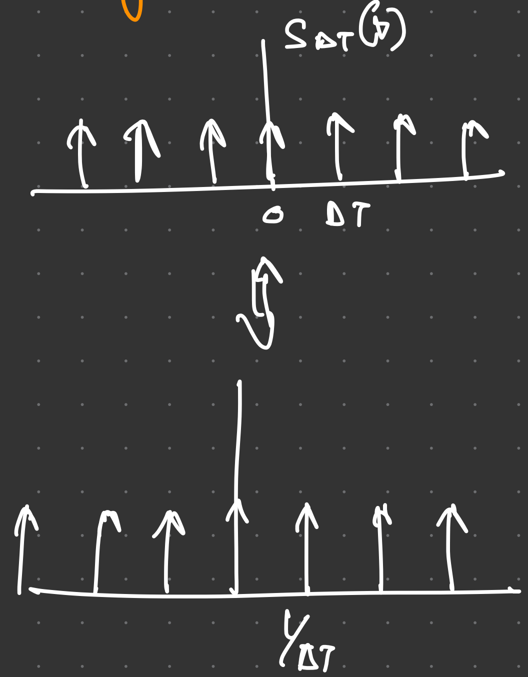
FT of real even function is also real

FT of cosine



$$F\{\cos(\omega_0 t)\} = \pi\delta(\omega+\omega_0) + \pi\delta(\omega-\omega_0)$$

FT of impulse train:



$$s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T)$$

$$\mathcal{F}(s_{\Delta T}(t)) = \sum_{n=-\infty}^{\infty} \delta(\mu - n/\Delta T)$$

$\downarrow$

$S(\mu)$

Impulse train is used for sampling for $f(t)$,

$$\bar{f}(t) = \sum_{n=-\infty}^{\infty} f_n \, \delta(t - n\Delta T)$$

$\uparrow$ Sampled values

FT of sampled functions

$$\bar{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

$\uparrow$ Continous copies

For m-sampling

$$\bar{f}(t) = \sum_{n=0}^{(m-1)\Delta T} f_n \, \delta(t - n\Delta T)$$

Digital processing of freq.

→ Need discrete freq. samples from continous

→ Need only one period value

→ $$\bar{F}(\mu) = \sum_{n=0}^{M-1} f_n e^{-j 2\pi \mu n \Delta T}$$

$$\mu = \frac{m}{M\Delta T}$$

$m = 0, \ldots, M-1$ $\uparrow$ Correction

Take $m = M-1$, $\mu = 1/\Delta T$

$$F[m] = \sum_{n=0}^{M-1} f_n e^{\frac{-j 2\pi m n}{M}}, \quad m = 0, 1, \ldots (M-1)$$