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DIP Lecture 10

Frequency domain processing

RECAP

Inverse FT

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} F(\omega) e^{i\omega t} d\omega$$

$$F(\omega) = F\{f(t)\}$$

FT:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = F^{-1}\{F(\omega)\}$$

Intuitions:

→ Measures strength of presence of particular freq. within a signal

→ How dominant each freq component is in original signal

Unit impulse function:

→ Has integral property, shifting property

↳ picks up value at origin

↓

Also applicable to shifted impulse

$$\delta(t-\lambda), \delta(t+\lambda)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt = e^{-i\omega T}$$

→ F.T of shifted impulse is complex exponential

Sampling

→ Scaled version of time shifted impulses

$$s_{\Delta}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t)$$

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f_n \, \delta(t - n\Delta T)$$

↕ F.T

$$\tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F\left(\mu - \frac{n}{\Delta T}\right)$$

Periodicity

To digitise, we take only one sample ←

Discrete FT:

$$F[m] = \sum_{n=0}^{n=M-1} f_n e^{\frac{-j2\pi mn}{M}}, \quad m = 0, 1, 2 \ldots (M-1)$$

$$\text{Continous} \leftarrow u = \frac{\overset{\to \text{Distrete}}{m}}{M\Delta T}$$

DFT & IDFT

→ $$f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_M e^{\frac{j2\pi nm}{M}}, \quad n = 0, 1, 2 \ldots M-1)$$

→ $F[m]$ is complex value

→ Represents amplitude, phase of function

→ $F[\cdot]$'s content at angular freq, $\frac{2\pi m}{M}$

Records energy portion at various freq. bands present in input similar to continous F.T

Magnitude ← $$\|.\| = \sqrt{Re\{F[m]\}^2 + Im\{F[m]\}^2}$$

Angle / Phase ← $$\angle = \tan^{-1}\left[\frac{Im[F[m]\}}{Re\{F[m]\}}\right]$$

Implementation of DFT in practice:

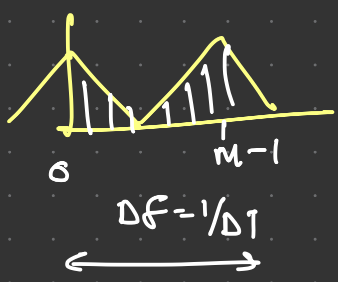
→ $$F[m] = \sum_{n=0}^{n=M-1} f_n e^{\frac{-j2\pi mn}{M}}, \quad m = 0, 1, 2 \ldots (M-1)$$

→ Can be expressed in vectorised form

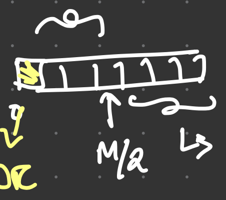
$$\underset{M\times 1}{F} = \underset{M\times M}{P} \cdot \underset{M\times 1}{f}$$

$$[P] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & p & p^2 & p^3 & p^4 \\ 1 & \vdots & \vdots & p^6 & p^8 \\ 1 & \vdots & \vdots & p^{11} & p^{12} \\ 1 & \vdots & \vdots & p^{13} & p^{16} \end{bmatrix} \leftarrow \text{for } M = 5, \quad p = e^{-j\frac{2\pi}{5}} = \text{cis}\left(\frac{-2\pi}{5}\right), \quad e^{-j\frac{4\pi}{5}} = p^2 \ldots$$

Centered DFTs:



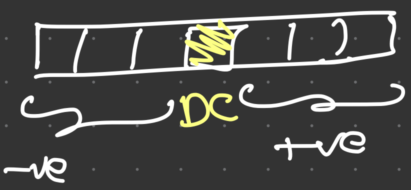
+ve freq



↳ -ve freq as the right side is exact replica for the region to the left of 0

↓

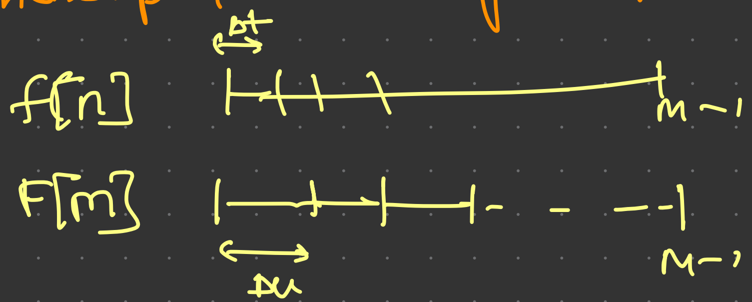
Now try to shift it where -ve lies on left



$$F[M] = \sum_{n=0}^{M-1} f[n]\, e^{\frac{-j2\pi\left(m-\frac{M-1}{2}\right)\left(n-\frac{M-1}{2}\right)}{M}}$$

↙ Center shift DFT formula

Relationship b/w sampling & freq intervals



Δu (Freq. resolution) $= \dfrac{1}{M \cdot \Delta T}$ } → Inverse relationship

↳ Time resolution

→ Entire freq. range spanned by M components, $B = M\Delta u = \dfrac{1}{\Delta T}$

2D DFT & IDFT

$$F[m,n] = \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} f[x,y]\, e^{-2\pi j \left(\frac{mx}{M} + \frac{nx}{N}\right)}$$

$$f[x,y] = \sum_{y=0}^{N-1} \sum_{x=0}^{M-1} \frac{1}{MN} F[m,n]\, e^{+2\pi j \left(\frac{mx}{M} + \frac{nx}{N}\right)}$$

Shifting origin,

$$f[x,y]\, e^{j2\pi\left[\frac{u_0 x}{M} + \frac{v_0 y}{M}\right]} \longleftrightarrow F(U-U_0, V-V_0)$$

With increasing spatial resolution, freq resolution decreases

Better visualisations

→ Amplitude rescaling:

$$G(k,l) = \log\left(1 + F(k,l)\right)$$

Magnitude vs Phase:

→ $|F(u,v)|$ decreases with higher spatial freq

→ Phase appears less informative

In general, natural images could have same mag. transform so we look into phase

Time complexity:

Direct Computation: $N = 2^n$

DFT: N^2

FFT: $N \log N$ → Fastest using Cooley Tukey Algorithm

Image enhancement & Filtering in Freq. Domain

→ Low pass filters:

→ Inverse of product of F & m

$$F^{-1}\{F.m\}$$

↳ Matrix m where threshold

$$m(x,y) = \begin{cases} 1, & \text{if } (x,y) \text{ is closer to center than threshold} \\ 0, & \text{" " " further than threshold} \end{cases}$$

Sharp filter cutoff can cause ringing effect

→ Gaussian filtering:

→ Can be applied in freq domain

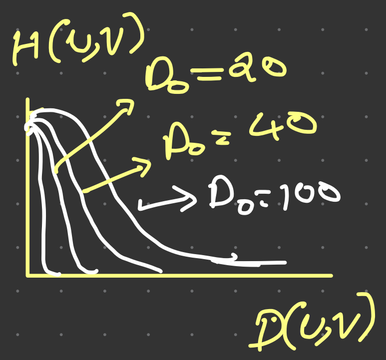
→ Similar steps as done in spatial domain

↳ Create filter

↳ Multiply DFT of img with filter

↳ Invert result

→ F.T of Gaussian = Gaussian



→ Gaussian low pass filters:

(GLPF) $H(u,v) = e^{-D^2(u,v)/(2D_0^2)}$

→ Ideal high pass filtering

→ Opposite of low pass filtering [removes center (low freq. values)]

→ Causes SHARPENING

→ If using circle, size effects

↳ Higher cutoff ⇒ More info. removed