Two-Channel Digital and Hybrid Analog/Digital Multirate Filter Banks With Very Low-Complexity Analysis or Synthesis Filters

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***Abstract*—Multirate filter banks make use of analysis and synthesis filter banks. This paper introduces two-channel digital and hybrid analog/digital multirate filter banks where either the analysis or synthesis filters have a very low complexity. Such filter banks find application, for example, in high-speed analog-to digital converters where it is essential to minimize the complexity of the discrete-time or analog filters. The proposed digital filter banks are approximately perfect reconstruction (PR) filter banks, whereas the hybrid analog/digital filter banks can be chosen to be either approximately PR or approximately perfect magnitude reconstruction filter banks. The design is performed by first optimizing the digital or analog analysis filters and then, with the analysis filters fixed, optimizing the digital synthesis filters. This design procedure makes it possible to obtain analysis filters of very low order and complexity. The overall complexity is also low. Further, the proposed filter banks are, in all cases, very easy to design by making use of well-known and reliable optimization techniques; in particular, as small distortion and aliasing as desired are readily obtained because they are controlled in a linear programming problem. Several design examples are included, illustrating the properties of the proposed filter banks.**

***Index Terms*—Hybrid analog/digital filter banks, linear programming, low complexity, two-channel filter banks.**

I. Introduction

Hybrid discrete-time/digital and analog/digital multirate filter banks incorporate an analysis filter bank and a synthesis filter bank where the analysis (synthesis) filter bank uses discrete-time and analog filters, respectively, whereas the synthesis (analysis) filter bank, in both cases, employs digital filters. The overall filter banks, with discrete-time and analog analysis filters, are shown in Figs. 1 and 2, respectively. These filter banks find applications in, e.g., high-speed analog-to-digital converters (ADCs). ADCs that utilize filter banks can be viewed as a generalization of time-interleaved ADCs where the use of filter banks can reduce the distortion caused by gain and time skew mismatch errors [2]–[5]. The filter banks in Figs. 1 and 2 are described by a distortion transfer function and one or several aliasing transfer functions. The filter bank is said to be a perfect reconstruction (PR) filter bank if all aliasing terms are zero and the distortion function is a pure delay. It is further said to be a perfect magnitude reconstruction (PMR) filter bank if the distortion function instead is a general all-pass function. In the latter case, we have phase distortion, but no magnitude distortion, i.e., the magnitude response of the distortion function is constant for all frequencies. The analysis and synthesis filters must be designed such that the filter bank achieves PR or PMR, or at least approximately PR or PMR.

Using a discrete-time analysis filter bank, the design problem is the same as for digital filter banks. In this case, it is, therefore, possible to obtain filter banks achieving PR or PMR, as well as approximately PR and PMR by employing any of the digital filter banks with these properties that have been developed during the past decades [6]–[8]. (If the application at hand allows it, it is often advantageous in terms of arithmetic complexity to consider filter banks that approximate PR or PMR [9].) However,

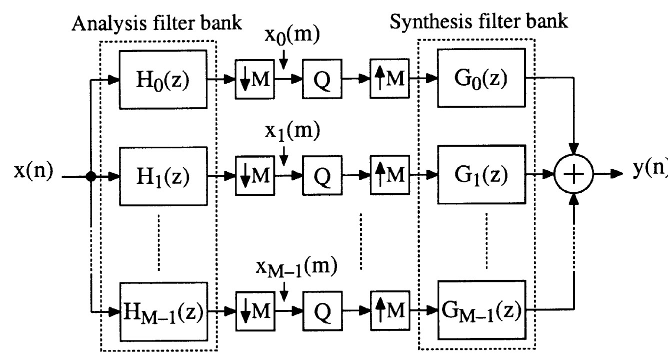

Fig. 1. M-channel hybrid discrete-time/digital maximally decimated filter bank ADC.

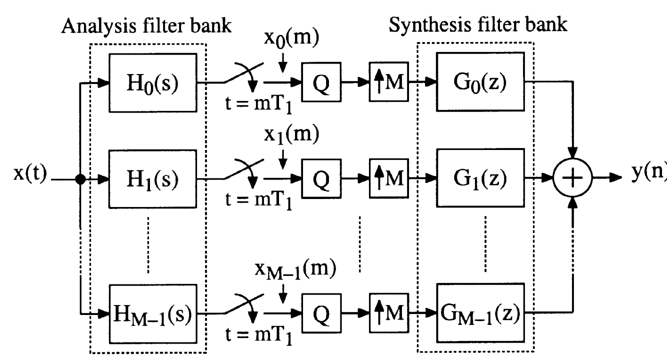

Fig. 2. M-channel hybrid analog/digital maximally decimated filter bank ADC with $T_1 = MT$.

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in these digital filter banks, the analysis and synthesis filters usually have about the same arithmetic complexity, which, of course, is of no concern if the objective is to minimize the overall complexity. However, in applications where it is important to minimize the complexity of either the analysis or synthesis filters, the traditional filter banks are not suitable. For example, in hybrid discrete-time/digital filter banks, it is essential to minimize the order and complexity of the discrete-time filters (e.g., switched capacitor filters) since it is much more difficult to implement such filters than digital filters for high frequencies with a high accuracy. It seems that no or little attention has been paid to this design problem in the literature.

Using an analog analysis filter bank, where the analog filters are to be implemented using lumped elements, it is to our knowledge not known how to achieve PR or PMR. Therefore, approximately PR and PMR filter banks are considered. (In practice, there is no reason to use exact PR or PMR filter banks anyway since the desired frequency responses of the analog filters cannot be implemented exactly.) Further, we can no longer rely upon design techniques for digital filter banks. As opposed to the area of digital filter banks, there exist only a few publications dealing with hybrid analog/digital filter banks [4], [5], [10]. Some of the design techniques such as those in [4] and [5] results, however, in analog filters of higher order and complexity than necessary because these filters are designed after the digital filters have been optimized and fixed. Again, this will cause problems since it is difficult to implement wide-band analog filters with a high accuracy, especially for high-order filters. Design approaches where the synthesis filters are designed after the analysis filters have been designed and fixed have been addressed in [4], [5], and [10], but in those approaches, only approximately PR filter banks are considered, not approximately PMR filter banks. This results in digital synthesis filters of higher order and complexity than necessary if the application at hand allows phase distortion. In addition, the digital filters are designed by using nonlinear optimization methods straightforwardly, which may cause problems. In particular, it may be difficult to find good enough solutions in terms of small distortion and aliasing since locally optimal solutions can be obtained that are far from the global optimal solution.

From the above, it is clear that there is a need for new filter banks and design techniques, both for digital and hybrid analog/digital filter banks. In this paper, we introduce two-channel digital and hybrid analog/digital filter banks with very low-complexity analysis filters. The main reason for considering both the digital and hybrid analog/digital banks in the same paper is that they are closely related to each other. Although we only consider the case in which the analysis filters have a low complexity, it is naturally possible to instead let the synthesis filters be the low-complexity filters by simply interchanging the analysis and synthesis filters. The proposed digital filter banks are approximately PR filter banks. They have magnitude distortion, but no phase distortion. Further, two different cases are treated in which the aliasing terms are, respectively, exactly zero and arbitrarily small. Given the analysis filters in these banks, it is also easy to obtain a PMR filter bank where the complexities of the analysis and synthesis filters are equal. However, this filter bank is well known [7], [11] and, therefore, are only briefly commented upon in Section II. For the proposed hybrid analog/digital filter banks, one can choose whether they should approximate either PR or PMR.

The filter bank design is performed by first optimizing the digital or analog analysis filters and then, with the analysis filters fixed, optimizing the digital synthesis filters. By designing the analysis and synthesis filters separately, it is possible to obtain analysis filters of very low order and complexity. The overall complexity is also low. Further, the proposed filter banks are, in all cases, easy to design by making use of well-known and reliable optimization techniques. In particular, we can readily achieve as good designs as desired since both the distortion and aliasing functions are controlled in a linear programming problem. In the digital filter bank, the analysis filters are equiripple (Cauer) half-band infinite impulse response (IIR) filters, which can be designed using closed-form solutions. In the hybrid analog/digital filter bank, the analysis filters are again equiripple filters, but with a fixed zero pair, which necessitates optimization since closed-form solutions do not exist for such filters. To this end, the well-known pole placer design program can be used [12]. Thus, in both filter banks, the analysis filters have a very low complexity. The digital synthesis filters are finite impulse response (FIR) filters in both the digital and hybrid analog/digital filter banks. In the digital and approximately PMR hybrid analog/digital filter banks, the synthesis filters can be designed using linear programming, which ensures optimal synthesis filters. In the approximately PR hybrid analog/digital filter banks, the design of the digital synthesis filters is split into two parts. First, nonlinear programming is used to equalize the phase distortion. Second, linear programming is used to equalize the magnitude distortion. The overall synthesis filters are suboptimal in the sense that optimal subfilters can be ensured in both design steps. In the first step, it can be guaranteed because it is known that the equiripple solution is the optimal solution to phase equalization if the phase error is considered [13]. In the second step, it can again be ensured because linear programming is used.

We consider the digital and hybrid analog/digital filter banks in Sections II and III, respectively. Several design examples are included, illustrating the properties and advantages of the proposed filter banks. Finally, some concluding remarks are given in Section IV.

II. Digital Filter Banks

In this section, we first recapitulate the basics of two-channel digital filter banks in Section II-A. Then, the proposed filter banks are considered in Sections II-B–D.

A. Review of Two-Channel Digital Filter Banks

Ignoring the quantizations, the z-transforms of the output $y(n)$ and input $x(n)$ of the filter bank in Fig. 1 with $M = 2$ are related according to

$$Y(z) = V_0(z)X(z) + V_1(z)X(-z) \tag{1}$$

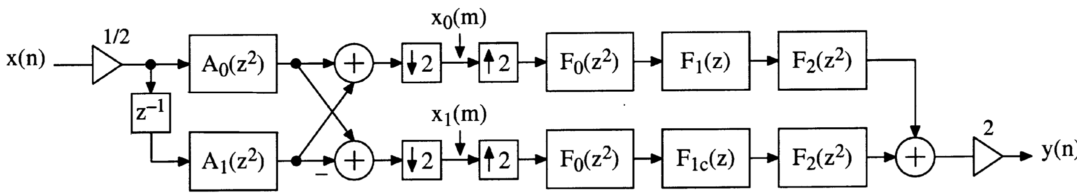


Fig. 3. Proposed filter bank.

where

$$V_0(z) = \frac{H_0(z)G_0(z) + H_1(z)G_1(z)}{2}$$
$$V_1(z) = \frac{H_0(-z)G_0(z) + H_1(-z)G_1(z)}{2}. \tag{2}$$

The functions $V_0(z)$ and $V_1(z)$ are the distortion and aliasing transfer functions, respectively. A PR filter bank is obtained if

$$V_0(z) = cz^{-d} \quad V_1(z) = 0 \tag{3}$$

for some integer d and nonzero constant c. In this case, the output signal is just a scaled and delayed version of the input signal, i.e., $y(n) = cx(n-d)$. A PMR filter bank is obtained if the magnitude response of the distortion function is constant and the aliasing function is zero, i.e.,

$$|V_0(e^{j\omega T})| = c \quad V_1(z) = 0. \tag{4}$$

In this case, we have no magnitude distortion, but phase distortion.

B. Proposed Filter Banks

This section introduces the proposed class of digital filter banks, and gives some expressions for the frequency responses that are useful in the filter design. The proposed overall filter bank is shown in Fig. 3.

Analysis Filter Bank: The analysis filters in the proposed filter bank are half-band IIR low-pass and high-pass filters, respectively. Their transfer functions can be written in polyphase forms as

$$H_0(z) = \frac{A_0(z^2) + z^{-1}A_1(z^2)}{2}$$
$$H_1(z) = \frac{A_0(z^2) - z^{-1}A_1(z^2)}{2} \tag{5}$$

where $A_0(z)$ and $A_1(z)$ are real causal stable all-pass filters and the order of $H_0(z)$ and $H_1(z)$, denoted here by K, is odd [7]. It is well known that, e.g., a low-pass and high-pass Cauer (elliptic) filter pair fall into this class of analysis filters [7], [14]–[17]. Thus, it is possible to obtain a very low-complexity analysis filter bank. Note that the downsamplers after the analysis bank can be moved to the input of $A_0(z)$ and $A_1(z)$, as illustrated in Fig. 4.

For convenience, in the following sections, we will introduce some useful expressions. Let

$$A_i(z) = \frac{z^{-K_i}D_i(z^{-1})}{D_i(z)}, \qquad i = 0, 1 \tag{6}$$

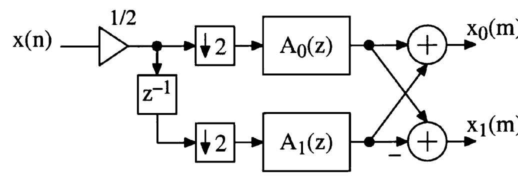


Fig. 4. Realization of the analysis filter bank.

where

$$D_i(z) = \sum_{k=0}^{K_i} d_{ik}z^{-k}, \qquad i = 0, 1 \tag{7}$$

with K_i denoting the order of $A_i(z)$ and $D_i(z)$. The analysis filters $H_0(z)$ and $H_1(z)$ can now be rewritten as

$$H_0(z) = \frac{N(z)}{D(z^2)} \quad H_1(z) = \frac{N_c(z)}{D(z^2)} \tag{8}$$

where

$$\begin{aligned} N(z) &= 0.5[z^{-2K_0}D_0(z^{-2})D_1(z^2) \\ &\quad + z^{-1}z^{-2K_1}D_0(z^2)D_1(z^{-2})] \\ N_c(z) &= 0.5[z^{-2K_0}D_0(z^{-2})D_1(z^2) \\ &\quad - z^{-1}z^{-2K_1}D_0(z^2)D_1(z^{-2})] \\ D(z^2) &= D_0(z^2)D_1(z^2). \end{aligned} \tag{9}$$

From (9), we see that

$$N_c(z) = N(-z). \tag{10}$$

It is known that $N(z)$ and $N_c(z)$ are linear-phase FIR filters with symmetric and antisymmetric impulse responses, respectively [16]. That is, if $h_N(n)$ and $h_{N_c}(n)$ denote the impulse responses of $N(z)$ and $N_c(z)$, respectively, then $h_N(n) = h_N(K-n)$ and $h_{N_c}(n) = -h_{N_c}(K-n)$ for $n = 0, 1, \ldots, K$, where $K = 2(K_0 + K_1) + 1$ is the order of $N(z)$ and $N_c(z)$, as well as $H_0(z)$ and $H_1(z)$. From (9), we also get

$$N^2(z) - N_c^2(z) = z^{-K}D(z^2)D(z^{-2}). \tag{11}$$

Synthesis Filter Bank: It is well known that, given the analysis filters above, PMR is obtained by choosing the synthesis filters according to $G_0(z) = 2H_0(z)$ and $G_1(z) = -2H_1(z)$ since the distortion and aliasing transfer functions then become $V_0(z) = z^{-1}A_0(z^2)A_1(z^2)$ and $V_1(z) = 0$ [7], [11]. This solution is very attractive when phase distortion can be tolerated since the overall complexity is very low. However, in cases where a linear phase is required, this synthesis filter bank cannot be used. One way to improve the phase response is to use an all-pass IIR filter (phase equalizer) in cascade with the filter

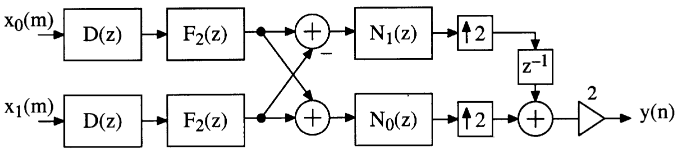


Fig. 5. Realization of the synthesis filter bank in *Case 1*.

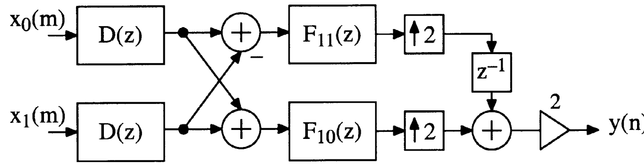


Fig. 6. Realization of the synthesis filter bank in *Case 2*.

bank. However, this solution can only achieve an approximately linear phase. Further, to obtain a small deviation from linear phase, the orders of the all-pass IIR filters become high, as will be demonstrated in Example 1.

In order to obtain a distortion function with an exact linear phase, we propose the following synthesis filter bank. Let the transfer functions of the synthesis filters be

$$\begin{aligned} G_0(z) &= 2F_0(z^2)F_1(z)F_2(z^2) \\ G_1(z) &= 2F_0(z^2)F_{1c}(z)F_2(z^2) \end{aligned} \tag{12}$$

where $F_0(z)$ is a nonlinear-phase FIR filter, whereas $F_1(z), F_{1c}(z)$, and $F_2(z)$ are linear-phase FIR filters. The role of $F_0(z)$ is to eliminate the phase distortion. Since $N(z)$ and $N_c(z)$ in (8) are linear-phase FIR filters, the phase distortion emanates from $D(z^2)$. The filter $F_0(z)$ is, therefore, chosen as

$$F_0(z) = D(z) \tag{13}$$

in order to obtain a distortion function with an exact linear phase. The role of $F_1(z), F_{1c}(z)$, and $F_2(z)$ is to eliminate or suppress aliasing and to equalize the magnitude distortion.

Distortion and Aliasing: Two different cases are considered, in which the aliasing function is exactly zero and arbitrarily small, respectively.

Case 1: In this case, the aliasing function is exactly zero. This is achieved by selecting $F_1(z)$ and $F_{1c}(z)$ as

$$F_1(z) = N(z) \quad F_{1c}(z) = -N_c(z). \tag{14}$$

From (2) and (8)–(14), the distortion and aliasing transfer functions are obtained as

$$V_0(z) = z^{-K}D(z^2)D(z^{-2})F_2(z^2) \quad V_1(z) = 0. \tag{15}$$

Hence, by selecting $F_1(z)$ and $F_{1c}(z)$ according to (14), we obtain exactly zero aliasing and a distortion function with a linear phase, provided that $F_2(z^2)$ is a linear-phase filter, since a function in the form of $H(z)H(z^{-1})$ is real for $z = e^{j\omega T}$. Further, $F_2(z)$ must be an even-order linear-phase filter with a symmetric impulse response, i.e., if $f_2(n)$ denotes the impulse response of $F_2(z)$, then $f_2(n) = f_2(K_{F2} - n)$ for $n = 0, 1, \ldots, K_{F2}$, where K_{F2} is even and denotes the order of $F_2(z)$. The reason is that $F_2(z)$ is used for equalizing the magnitude distortion of $D(z)D(z^{-1})$ (see Section II-C). Therefore, it cannot have any zeros on the unit circle. This means that the effective order of $F_2(z^2)$ is $4m$, where m is some positive integer. The synthesis bank is, in this case, efficiently realized as shown in Fig. 5, where $N_0(z)$ and $N_1(z)$ are the polyphase components of $N(z)$, i.e., $N(z) = N_0(z^2) + z^{-1}N_1(z^2)$.

Case 2: In this case, the aliasing function is not exactly zero, but it can be made arbitrarily small by properly designing the synthesis filters. The filter $F_2(z)$ is chosen as

$$F_2(z) = 1 \tag{16}$$

whereas $F_1(z)$ and $F_{1c}(z)$ are odd-order linear-phase FIR filters with symmetric and antisymmetric impulse responses, respectively. That is, if $f_1(n)$ and $f_{1c}(n)$ denote the impulse responses of $F_1(z)$ and $F_{1c}(z)$, respectively, then $f_1(n) = f_1(K_{F1} - n)$ and $f_{1c}(n) = -f_{1c}(K_{F1} - n)$ for $n = 0, 1, \ldots, K_{F1}$, where K_{F1} denotes the order of $F_1(z)$ and $F_{1c}(z)$. Further, since $H_0(z) = H_1(-z)$, it is possible and convenient to let $F_1(z)$ and $F_{1c}(z)$ be related as

$$F_{1c}(z) = (-1)^{(K_{F1}+K)/2}F_1(-z) \tag{17}$$

which implies that their impulse responses are related as

$$f_{1c}(n) = (-1)^{(K_{F1}+K)/2}(-1)^n f_1(n). \tag{18}$$

From (2) and (8)–(14), the distortion and aliasing transfer functions are obtained as

$$\begin{aligned} V_0(z) &= N(z)F_1(z) + N_c(z)F_{1c}(z) \\ V_1(z) &= N_c(z)F_1(z) + N(z)F_{1c}(z). \end{aligned} \tag{19}$$

The distortion and aliasing transfer functions have a linear phase because $N(z)$ and $F_1(z)$ are linear-phase filters with symmetric impulse responses, whereas $N_c(z)$ and $F_{1c}(z)$ are linear-phase filters with antisymmetric impulse responses. The synthesis bank is in this case efficiently realized, as shown in Fig. 6, where $F_{10}(z)$ and $F_{11}(z)$ are the polyphase components of $F_1(z)$, i.e., $F_1(z) = F_{10}(z^2) + z^{-1}F_{11}(z^2)$.

Some Comments: As we shall see later in the example section, the *Case 2* synthesis filters can have a lower order as compared to the filters in the *Case 1* banks, even though the difference is modest. The main reason for considering the *Case 2* banks is, however, that the hybrid analog/digital banks, to be introduced in Section III, follows as a natural solution. For those banks, *Case 1* is not appropriate since the frequency responses of the analog synthesis filters do not exhibit the symmetry as do the frequency responses of the digital analysis filters.

C. Filter Bank Design

The filter bank is designed in two steps by first optimizing the IIR analysis filters to meet their respective requirements, and then, with the analysis filters fixed, optimizing the FIR synthesis filters to meet the requirements on the distortion function.

Analysis Filters: Let the specification of the analysis filters $H_0(z)$ and $H_1(z)$ be

$$\begin{aligned} 1-\delta_s^2 \leq |H_0(e^{j\omega T})|^2 \leq 1, \qquad & \omega T \in [0, \pi - \omega_s T] \\ |H_0(e^{j\omega T})|^2 \leq \delta_s^2, \qquad & \omega T \in [\omega_s T, \pi] \end{aligned} \tag{20}$$

and

$$\begin{aligned} 1-\delta_s^2 \leq |H_1(e^{j\omega T})|^2 \leq 1, \qquad & \omega T \in [\omega_s T, \pi] \\ |H_1(e^{j\omega T})|^2 \leq \delta_s^2, \qquad & \omega T \in [0, \pi - \omega_s T] \end{aligned} \tag{21}$$

respectively. In order to obtain as low complexity as possible, $H_0(z)$ and $H_1(z)$ are optimized using a Cauer approximation to meet the specifications of (20) and (21). In this case, $H_0(z)$ and $H_1(z)$ can easily be designed using, e.g., explicit formulas, as given in [15] for cascaded low-order wave digital all-pass filters. After some minor modifications, these formulas can also be used for other types of all-pass filters.

Synthesis Filters: The optimization of the synthesis filters $G_0(z)$ and $G_1(z)$ is performed in two different ways for *Cases 1* and *2*, respectively.

Case 1: In this case, $F_0(z), F_1(z)$, and $F_{1c}(z)$ are given by (13) and (14). The remaining linear-phase FIR filter $F_2(z^2)$ is optimized to make the magnitude response of the distortion transfer function $V_0(z)$ approximate unity. The term z^{-K} in $V_0(z)$, as given by (15), does not affect the magnitude response and can, thus, be ignored in the optimization. The remaining part of $V_0(z), D(z^2)D(z^{-2})F_2(z^2)$ is a function of z^2 and, thus, its frequency response is periodic with a period of π. It, therefore, suffices to consider $D(z^2)D(z^{-2})F_2(z^2)$ on $\omega T \in [0, \pi/2]$, which is equivalent to considering $\hat{V}_0(z)$, as given by

$$\hat{V}_0(z) = D(z)D(z^{-1})F_2(z) \tag{22}$$

on $\omega T \in [0, \pi]$. From the design point of view, it is convenient to utilize $\hat{V}_0(z)$ instead of $V_0(z)$ since we can then optimize $F_2(z)$ instead of $F_2(z^2)$. Both $\hat{V}_0(z)$ and $F_2(z)$ are linear-phase functions and, hence, their frequency responses can be expressed with the aid of zero-phase functions, which is convenient in the filter design. The frequency response of $F_2(z)$ can be written as

$$F_2(e^{j\omega T}) = e^{-jK_{F2}\omega T/2}F_{2R}(\omega T) \tag{23}$$

where $F_{2R}(\omega T)$ is the zero-phase frequency response, as given by [18]

$$F_{2R}(\omega T) = f_2\left(\frac{K_{F2}}{2}\right) + 2\sum_{n=1}^{K_{F2}/2} f_2\left(\frac{K_{F2}}{2} - n\right)\cos(n\omega T). \tag{24}$$

The frequency response of $\hat{V}_0(z)$ can be written as

$$\hat{V}_0(e^{j\omega T}) = e^{-jK_{F2}\omega T/2}\hat{V}_R(\omega T) \tag{25}$$

where $\hat{V}_R(\omega T)$ is the zero-phase frequency response, as given by

$$\hat{V}_R(\omega T) = D_R(\omega T)F_{2R}(\omega T) \tag{26}$$

with $D_R(\omega T)$ being

$$D_R(\omega T) = D(e^{-j\omega T})D(e^{j\omega T}). \tag{27}$$

Let the specification of $V(e^{j\omega T})$ be

$$1-\delta \leq |V(e^{j\omega T})| \leq 1+\delta, \qquad \omega T \in [0, \pi]. \tag{28}$$

The specification of (28) can be restated with the aid of $\hat{V}_R(\omega T)$ according to

$$1-\delta \leq \hat{V}_R(\omega T) \leq 1+\delta, \qquad \omega T \in [0, \pi]. \tag{29}$$

The filter $F_2(z)$ can now easily be designed such that $\hat{V}_R(\omega T)$ satisfies (29) by solving the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & c_{M+1} \\ \text{subject to} \quad & -c_{M+1} + \sum_{p=0}^{M} c_p \cos(p\omega_k T)D_R(\omega_k T) \leq 1 \\ & -c_{M+1} - \sum_{p=0}^{M} c_p \cos(p\omega_k T)D_R(\omega_k T) \leq -1 \end{aligned} \tag{30}$$

where $c_p, p = 0, 1, \ldots, M+1$ are the unknown parameters, $M = K_{F2}/2$, and $\omega_k T \in [0, \pi], k = 1, 2, \ldots, L$ is ωT discretized into L grid points. The impulse response $f_2(n)$ is related to c_p through

$$\begin{aligned} f_2(n) &= f_2(K_{F2} - n) \\ &= \begin{cases} c_0, & n = M \\ 0.5c_{M-n}, & 0 \leq n \leq M-1. \end{cases} \end{aligned} \tag{31}$$

The linear programming problem in (30) can be efficiently solved with the aid of the simplex algorithm [19], which is implemented in, e.g., the function *lp.m* in MATLAB's Optimization Toolbox [20]. The resulting filter $F_2(z)$ is optimal in the minimax sense. In general, the problem in (30) must be solved a number of times, with increasing (decreasing) values of M [order of $F_2(z)$] in order to find the minimum-order filter that satisfies (30). Equation (30) is satisfied when $c_{M+1} \leq \delta$. An alternative to linear programming is to use the well-known Parks–McClellan–Rabiner algorithm [21], which is used in, e.g., the function *remez.m* in MATLAB's Signal Processing Toolbox [22]. However, *remez.m* cannot be used straightforwardly for the problem in (30), the reason being that this function assumes approximation of a piecewise linear function with transition bands.

Case 2: In this case, $F_1(z)$ is optimized such that the magnitude responses of the distortion and aliasing transfer functions simultaneously approximate one and zero, respectively. All of $V_0(z), V_1(z), N(z), N_c(z), F_1(z)$, and $F_{1c}(z)$ are linear-phase functions. Their frequency responses can, therefore, be expressed with the aid of zero-phase functions, which again, is convenient in the filter design. The frequency responses of $N(z), N_c(z), F_1(z)$, and $F_{1c}(z)$ can be written as

$$\begin{aligned} N(e^{j\omega T}) &= e^{-jK\omega T/2} N_R(\omega T) \\ N_c(e^{j\omega T}) &= je^{-jK\omega T/2} N_{cR}(\omega T) \\ F_1(e^{j\omega T}) &= e^{-jK_{F1}\omega T/2} F_{1R}(\omega T) \\ F_{1c}(e^{j\omega T}) &= je^{-jK_{F1}\omega T/2} F_{1cR}(\omega T) \end{aligned} \quad (32)$$

where $N_R(\omega T)$ and $N_{cR}(\omega T), F_{1R}(\omega T)$, and $F_{1cR}(\omega T)$ are the respective zero-phase frequency responses, as given by [18]

$$\begin{aligned} N_R(\omega T) &= \sum_{n=0}^{(K-1)/2} 2h_N(n,K)\cos\left(\left[n+\frac{1}{2}\right]\omega T\right) \\ N_{cR}(\omega T) &= \sum_{n=0}^{(K-1)/2} 2h_{Nc}(n,K)\sin\left(\left[n+\frac{1}{2}\right]\omega T\right) \\ F_{1R}(\omega T) &= \sum_{n=0}^{(K_{F1}-1)/2} 2f_1(n,K_{F1})\cos\left(\left[n+\frac{1}{2}\right]\omega T\right) \\ F_{1cR}(\omega T) &= \sum_{n=0}^{(K_{F1}-1)/2} 2f_{1c}(n,K_{F1})\sin\left(\left[n+\frac{1}{2}\right]\omega T\right) \end{aligned} \quad (33)$$

with $h_N(n,K) = h_N[(K-1)/2-n], h_{Nc}(n,K) = h_{Nc}[(K-1)/2-n]$, $f_1(n,K_{F1}) = f_1[(K_{F1}-1)/2-n]$, and $f_{1c}(n,K_{F1}) = f_{1c}[(K_{F1}-1)/2-n]$. Further, due to (17), $F_{1cR}(\omega T)$ is related to $F_{1R}(\omega T)$ according to

$$F_{1cR}(\omega T) = (-1)^{(N_{F1}+K)/2} F_{1R}(\omega T - \pi) \quad (34)$$

which means that there are only $(K_{F1}+1)/2$ free filter coefficients in the design.

Using (2), (8), (9), (12), (13), and (32), the frequency responses of $V_0(z)$ and $V_1(z)$ are now obtained as

$$\begin{aligned} V_0(e^{j\omega T}) &= e^{-j\theta(\omega T)} V_{0R}(\omega T) \\ V_1(e^{j\omega T}) &= je^{-j\theta(\omega T)} V_{1R}(\omega T) \end{aligned} \quad (35)$$

where

$$\begin{aligned} V_{0R}(\omega T) &= N_R(\omega T)F_{1R}(\omega T) - N_{cR}(\omega T)F_{1cR}(\omega T) \\ V_{1R}(\omega T) &= N_{cR}(\omega T)F_{1R}(\omega T) + N_R(\omega T)F_{1cR}(\omega T) \end{aligned} \quad (36)$$

and

$$\theta(\omega T) = (K_{F1}+K)\omega T/2. \quad (37)$$

From (35), we have

$$|V_0(e^{j\omega T})| = |V_{0R}(\omega T)| \quad |V_1(e^{j\omega T})| = |V_{1R}(\omega T)|. \quad (38)$$

Let the specifications of $V_0(e^{j\omega T})$ and $V_1(e^{j\omega T})$ be

$$\begin{aligned} 1-\delta_1 \le |V_0(e^{j\omega T})| &\le 1+\delta_1, \qquad \omega T \in [0,\pi] \\ |V_1(e^{j\omega T})| &\le \delta_2, \qquad \omega T \in [0,\pi]. \end{aligned} \quad (39)$$

Due to (38), we can restate the specifications of (39) with the aid of $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ according to

$$\begin{aligned} 1-\delta_1 \le V_{0R}(\omega T) &\le 1+\delta_1, \qquad \omega T \in [0,\pi] \\ -\delta_2 \le V_{1R}(\omega T) &\le \delta_2, \qquad \omega T \in [0,\pi]. \end{aligned} \quad (40)$$

Clearly, $|V_0(e^{j\omega T})|$ and $|V_1(e^{j\omega T})|$ satisfy (39) if $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ satisfy (40). The filters $F_1(z)$ and $F_{1c}(z)$ can, for given filter orders, easily be designed such that $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ satisfy (40) by solving the following linear programming problem:

$$\begin{aligned} &\text{minimize } c_{M+1} \\ &\text{subject to} \\ &-c_{M+1}\frac{\delta 1}{\delta 2} + \sum_{p=0}^{M} c_p R_1(p,\omega_k T) R_2(p,\omega_k T) \le 1 \\ &-c_{M+1}\frac{\delta 1}{\delta 2} - \sum_{p=0}^{M} c_p R_1(p,\omega_k T) R_2(p,\omega_k T) \le -1 \\ &-c_{M+1} + \sum_{p=0}^{M} c_p R_1(p,\omega_k T) R_3(p,\omega_k T) \le 0 \\ &-c_{M+1} - \sum_{p=0}^{M} c_p R_1(p,\omega_k T) R_3(p,\omega_k T) \le 0 \end{aligned} \quad (41)$$

with

$$R_1(p,\omega_k T) = \begin{cases} 2\cos\left[\left(p+\frac{1}{2}\right)\omega_k T\right], & 0 \le p \le \frac{M-1}{2} \\ 2\sin\left[\left(p-\frac{M}{2}\right)\omega_k T\right], & \frac{M+1}{2} \le p \le M \end{cases} \quad (42)$$

$$R_2(p,\omega_k T) = \begin{cases} N_R(\omega_k T), & 0 \le p \le \frac{M-1}{2} \\ -N_{cR}(\omega_k T), & \frac{M+1}{2} \le p \le M \end{cases} \quad (43)$$

$$R_3(p,\omega_k T) = \begin{cases} N_R(\omega_k T-\pi), & 0 \le p \le \frac{M-1}{2} \\ -N_{cR}(\omega_k T-\pi), & \frac{M+1}{2} \le p \le M \end{cases} \quad (44)$$

where $M = K_{F1}$ and $\omega_k T \in [0,\pi], k = 1,2,\ldots,L$ is ωT discretized into L grid points. Due to (18), $c_{M-p}(n) = (-1)^{(K_{F1}+K)/2}(-1)^p c_{(M-1)/2-p}(n)$, $p = 0,1,\ldots,(M-1)/2$. Thus, the problem of (41) has in total $(M+3)/2$ unknown parameters. The impulse responses $f_1(n)$ and $f_{1c}(n)$ are related to c_p through

$$\begin{aligned} f_1(n) &= f_1(M-n) \\ &= c_{(M-1)/2-n}, \qquad n = 0,1,\ldots,\frac{M-1}{2} \\ f_{1c}(n) &= -f_{1c}(M-n) \\ &= c_{M-n}, \qquad n = 0,1,\ldots,\frac{M-1}{2}. \end{aligned} \quad (45)$$

The resulting filters $F_1(z)$ and $F_{1c}(z)$ are optimal in the minimax sense. The minimum-order filters that satisfy (40) are found by solving (41) with different values of M [order of $F_1(z)$ and $F_{1c}(z)$]. Equation (40) is satisfied when $c_{M+1} < \delta_2$.

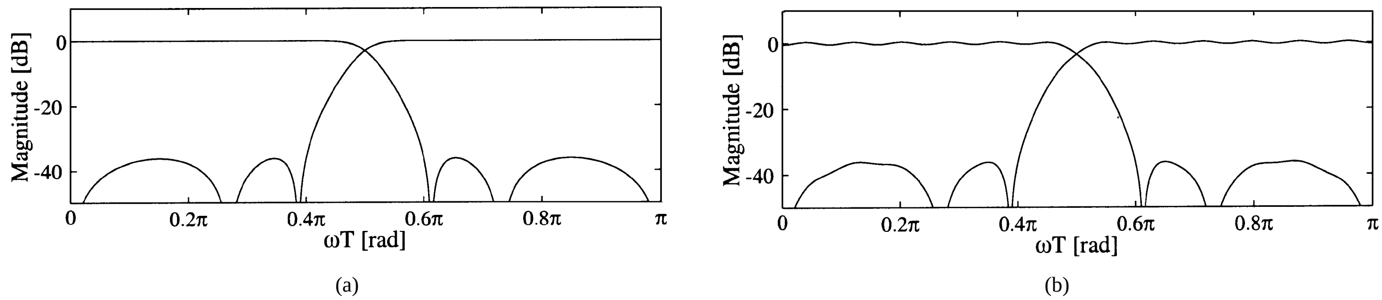


Fig. 7. Magnitude responses in *Example 1*. (a) Half-band IIR analysis filters $H_0(z)$ and $H_1(z)$. (b) FIR synthesis filters $G_0(z)$ and $G_1(z)$.

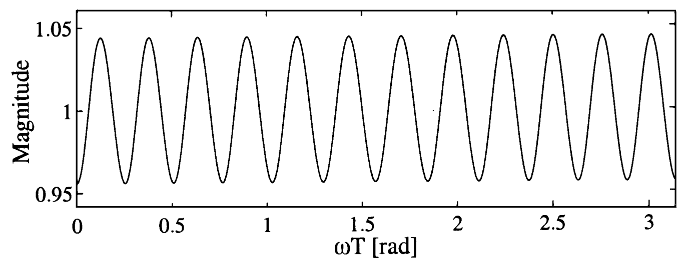


Fig. 8. Magnitude response of the distortion function $V_0(z)$ in *Example 1*.

The problem in (41) is stated in such a manner that the ripple for $V_{0R}(\omega T)$ in the final solution is δ_1/δ_2 times the ripple for $V_{1R}(\omega T)$.

D. Design Examples

In this section, we illustrate by means of two design examples the properties and advantages of the proposed filter banks.

Example 1: We start by designing the analysis filters $H_0(z)$ and $H_1(z)$ to meet (20) and (21) with $\omega_s T = 0.6\pi$ and $\delta_s = 0.02$ (-34 dB). The specifications are satisfied by using a fifth-order half-band Cauer filter pair. The design margin is allocated to the stopband, resulting in a ripple of 36.2 dB. The magnitude responses of $H_0(z)$ and $H_1(z)$ are shown in Fig. 7. Next, the synthesis filters $G_0(z)$ and $G_1(z)$ are for a *Case 1* design optimized so that the distortion function $V_0(z)$ satisfies (28) with $\delta = 0.05$. (Recall that the aliasing function $V_1(z)$ is exactly zero in *Case 1*.) First, $F_0(z)$ is chosen according to (13), which implies that its order is two. Second, $F_1(z)$ and $F_{1c}(z)$ are selected according to (14), which implies that they are fifth-order filters. Finally, $F_2(z)$ is optimized by solving the problem of (30). The specification of $V_0(z)$ is satisfied when the order of $F_2(z)[F_2(z^2)]$ is 22 (44). The magnitude responses of $G_0(z)$ and $G_1(z)$ are shown in Fig. 7 whereas the resulting magnitude response of $V_0(z)$ is shown in Fig. 8. Using the realization in Fig. 5, the analysis filters together require one multiplication per input/output sample, whereas the synthesis filters together require 18. The multiplication rate for the overall bank is thus 19.

As a comparison, we consider the filter bank class briefly discussed in Section II-B where both the analysis and synthesis filters are half-band IIR filters and an all-pass IIR filter, say, $H_{\mathrm{AP}}(z^2)$, is used to equalize the nonlinear phase response of the distortion function. (The special case of this filter bank class in which the analysis and synthesis filters have a pure delay branch, resulting in approximately linear-phase filters, has been considered in [23].) The analysis filters are thus exactly the same as in our filter banks. To obtain a phase error (i.e., the difference between the phase response and a linear-phase response) of 0.0314, 0.0082, 0.0021, and 0.000 56, the order of $H_{\mathrm{AP}}(z)$ becomes 12, 16, 20, and 24, respectively, for the equiripple solution. The overall synthesis bank thus requires 13, 17, 21, and 25 multiplications per input/output sample. Hence, to obtain a small phase error, the complexity of the synthesis filters exceeds that of our filter banks. Further, FIR filters have a simpler structure than all-pass filters. Therefore, FIR filters (and, thus, our filter banks) are often to prefer in very large scale integration (VLSI) implementations even when their complexities (in terms of the number of arithmetic operations) is somewhat higher than that of the corresponding IIR filters. One should, however, keep in mind that it is difficult to make comparisons between different filter bank classes since they have different properties. Our filter banks have an exact linear-phase response, but magnitude distortion. The all-pass equalized filter banks above have no magnitude distortion, but phase distortion. Nevertheless, the comparison does show that the proposed filter banks are attractive when it is desired to have a low-complexity analysis bank and nonlinear-phase analysis filters can be allowed.

Example 2: In this example, we compare the synthesis filter *Case 1* and *Case 2* designs. Recall that for a *Case 1* design, the aliasing function is exactly zero, whereas in *Case 2*, it is approximately zero. We consider the same specification as in Example 1, but here we allow -50-dB aliasing. We know from Example 1 that 53th-order synthesis filters are needed for the *Case 1* design. Using instead the *Case 2* design, it turns out that the order can be reduced to 51. Thus, the order and thereby also the delay can be decreased by employing *Case 2*, even though the decrease is modest. The complexity, in terms of arithmetic operations, is higher though for the *Case 2* design (27 compared to 18 multipliers in the synthesis bank). In that sense, the *Case 1* design is preferred. On the other hand, in a practical implementation, there are other factors to consider than the number of operations, such as finite wordlength effects, etc. It may, therefore, be worthwhile to investigate both alternatives. Further, as mentioned in Section II-A, the main reason for introducing the *Case 2* design is that the hybrid analog/digital banks in Section III follows as a natural solution. As this example shows, the *Case 2* design has a low complexity, even if it is higher than that of the *Case 1* design. Therefore, the complexity of the synthesis filters in the hybrid analog/digital banks will also be low since those banks can be regarded as "modified analog/digital versions" of the *Case 2* banks in this section.

III. Hybrid Analog/Digital Filter Banks

In this section, we first recapitulate the basics of two-channel hybrid analog/digital filter banks in Section III-A. The proposed filter banks are then considered in Sections III-B–D. As we shall see, these banks are closely related to the *Case 2* digital filter banks introduced in Section II.

A. Review of Two-Channel Hybrid Analog/Digital Filter Banks

It is assumed that the input signal is strictly bandlimited to π/T. In this case, the Nyquist criterion for sampling with an effective sampling frequency of $1/T$ without aliasing is fulfilled. Hence, it is possible to, in principle, eliminate aliasing in the filter bank ADC of Fig. 2 completely by properly designing the analysis and synthesis filters. If the filter bank has PR, then the actual aliasing in the filter bank ADC is determined by the antialiasing filter that must precede the filter bank. The aliasing will then be identical to that of a conventional ADC that uses the same antialiasing filter.

As opposed to digital filter banks, the use of transfer functions gives little insights into hybrid analog/digital filter banks since two complex variables are needed (s and z). It is, therefore, convenient to directly use the frequency responses since we then can use only one (frequency) variable (ω). Ignoring the quantizations, the frequency responses of the input and output signals of the filter bank in Fig. 2 are related according to

$$Y(e^{j\omega T}) = V_0(j\omega)\frac{1}{T}X(j\omega) + V_1(j\omega)\frac{1}{T}X\left(j\omega - j\frac{\pi}{T}\right), \quad \omega T \in [0,\pi] \tag{46}$$

where

$$\begin{aligned} V_0(j\omega) &= \frac{1}{2}H_0(j\omega)G_0(e^{j\omega T}) + \frac{1}{2}H_1(j\omega)G_1(e^{j\omega T}) \\ V_1(j\omega) &= \frac{1}{2}H_0\left(j\omega - j\frac{\pi}{T}\right)G_0(e^{j\omega T}) \\ &\quad + \frac{1}{2}H_1\left(j\omega - j\frac{\pi}{T}\right)G_1(e^{j\omega T}). \end{aligned} \tag{47}$$

The functions $V_0(j\omega)$ and $V_1(j\omega)$ are the distortion and aliasing frequency responses, respectively. A PR filter bank is obtained if

$$V_0(j\omega) = ce^{-jd\omega T} \quad V_1(j\omega) = 0 \tag{48}$$

for some constant d and nonzero constant c. In this case, the output signal is a sampled version of the scaled and shifted input signal, i.e., $y(n) = cx(nT - dT)$. A PMR filter bank is obtained if

$$|V_0(j\omega)| = c \quad V_1(j\omega) = 0. \tag{49}$$

In this case, we have no magnitude distortion, but phase distortion. For digital filter banks it is well known how to achieve PR and PMR [7]. For hybrid analog/digital filter banks, where the analog filters are to be implemented with lumped elements, we have a different situation. The frequency responses of the analog filters are, in this case, rational functions of $j\omega$, whereas those of the digital filters are rational functions of $e^{j\omega T}$. To our knowledge, it is not known how to achieve PR and PMR for such realizable filter banks. Therefore, we consider approximately PR and PMR hybrid filter banks. In practice, there is no reason to study exact PR or PMR filter banks anyway since the desired frequency responses of the analog filters cannot be implemented exactly.

B. Proposed Filter Banks

This section introduces the proposed class of hybrid analog/digital filter banks, and gives some expressions for the frequency responses that are useful in the filter design.

Analysis Filters: Let the transfer functions of the analog low-pass and high-pass analysis filters be given by

$$H_0(s) = \frac{A_0(s) + A_1(s)}{2} \quad H_1(s) = \frac{A_0(s) - A_1(s)}{2} \tag{50}$$

where $A_0(s)$ and $A_1(s)$ are stable all-pass filters with orders denoted by K_0 and K_1, respectively. Their transfer functions can be written as

$$A_i(s) = \frac{\prod_{m=0}^{K_i}(-s - s_m)}{\prod_{m=0}^{K_i}(s - s_m)}, \qquad i = 0, 1. \tag{51}$$

The order of $H_0(s)$ and $H_1(s)$, denoted here by K, is $K = K_0 + K_1$. For real low-pass and high-pass filters, one of K_0 and K_1 is always odd, whereas the other is always even, which means that K is always odd. Letting the analog filters be expressible as a sum and difference of two all-pass subfilters is not a severe restriction. For example, it is well known that the standard approximations Butterworth, Chebyshev, and Cauer in the odd-order case can be expressed in this way [15]. In fact, all analog filters with an odd characteristic function can be written in this manner [17]. One advantage of expressing the analog filters as a sum of two all-pass filters is that the number of free parameters in these filters is small, which is advantageous from the design point of view. However, the filters should not be implemented as two all-pass filters in parallel due to the severe stopband sensitivity. Instead, low-sensitivity structures such as ladder structures should be used.

The frequency responses of $H_0(s)$ and $H_1(s)$ can be written as

$$H_0(j\omega) = e^{j\Phi(\omega)}H_{0R}(\omega) \quad H_1(j\omega) = je^{j\Phi(\omega)}H_{1R}(\omega) \tag{52}$$

where $H_{0R}(\omega)$ and $H_{1R}(\omega)$ are real valued functions as given by

$$\begin{aligned} H_{0R}(\omega) &= \cos\left(\frac{\Phi_0(\omega) - \Phi_1(\omega)}{2}\right) \\ H_{1R}(\omega) &= \sin\left(\frac{\Phi_0(\omega) - \Phi_1(\omega)}{2}\right) \end{aligned} \tag{53}$$

and $\Phi(\omega)$ is

$$\Phi(\omega) = \frac{\Phi_0(\omega) + \Phi_1(\omega)}{2} \tag{54}$$

where $\Phi_0(\omega)$ and $\Phi_1(\omega)$ denote the phase responses of $A_0(s)$ and $A_1(s)$, respectively.

Synthesis Filters: Let the low-pass and high-pass digital synthesis filters be given by

$$G_0(z) = 2F_0(z)F_1(z) \quad G_1(z) = 2F_0(z)F_{1c}(z) \tag{55}$$

where $F_0(z)$ is a K_{F0}th-order nonlinear-phase FIR filter, and $F_1(z)$ and $F_{1c}(z)$ are K_{F1}th-order linear-phase FIR filters. The filter order K_{F1} of $F_1(z)$ and $F_{1c}(z)$ is restricted to be odd, and further, their impulse responses $f_1(n)$ and $f_{1c}(n)$ are symmetric and antisymmetric, respectively, i.e., $f_1(n) = f_1(K_{F1} - n)$ and $f_{1c}(n) = -f_{1c}(K_{F1} - n), n = 0, 1, \ldots, K_{F1}$. The role of $F_0(z)$ is to equalize the phase distortion, whereas $F_1(z)$ and $F_{1c}(z)$ are used to shape the magnitude response.

The frequency response of $F_0(z)$ can be written as

$$F_0(e^{j\omega T}) = |F_0(e^{j\omega T})|e^{j\Phi_{F0}(\omega T)} \tag{56}$$

where $|F_0(e^{j\omega T})|$ and $\Phi_{F0}(\omega T)$ are the magnitude and phase responses of $F_0(z)$, respectively. Since $F_1(z)$ and $F_{1c}(z)$ are odd-order filters with symmetric and antisymmetric impulse responses, respectively, their frequency responses can again be expressed as in (32).

Distortion and Aliasing: With the analysis and synthesis filters defined in the previous subsection, the distortion and aliasing frequency responses can be written as

$$\begin{aligned} V_0(j\omega) &= e^{j\Theta_0(\omega)}V_{0R}(\omega) \\ V_1(j\omega) &= e^{j\Theta_1(\omega)}V_{1R}(\omega) \end{aligned} \tag{57}$$

where $V_{0R}(\omega)$ and $V_{1R}(\omega)$ are real functions as given by

$$\begin{aligned} V_{0R}(\omega T) = &H_{0R}(\omega)F_{1R}(\omega T)|F_0(e^{j\omega T})| \\ &-H_{1R}(\omega)F_{1cR}(\omega T)|F_0(e^{j\omega T})| \\ V_{1R}(\omega T) = &H_{0R}\left(\omega - \frac{\pi}{T}\right)F_{1R}(\omega T)|F_0(e^{j\omega T})| \\ &-H_{1R}\left(\omega - \frac{\pi}{T}\right)F_{1cR}(\omega T)|F_0(e^{j\omega T})| \end{aligned} \tag{58}$$

and

$$\begin{aligned} \Theta_0(\omega) &= \Phi(\omega) + \Phi_{F0}(\omega T) - K_{F1}\omega T/2 \\ \Theta_1(\omega) &= \Phi\left(\omega - \frac{\pi}{T}\right) + \Phi_{F0}(\omega T) - K_{F1}\omega T/2. \end{aligned} \tag{59}$$

From (57), we get

$$|V_0(j\omega)| = |V_{0R}(\omega)| \quad |V_1(j\omega)| = |V_{1R}(\omega)|. \tag{60}$$

C. Filter Bank Design

The filter bank is designed by first optimizing the analysis filters. With the analysis filters fixed, the synthesis filters are then optimized.

Analysis Filters: Let the specifications of $H_0(s)$ and $H_1(s)$ be

$$\begin{aligned} 1 - \delta_s^2 \le |H_0(j\omega)|^2 \le 1, &\qquad \omega \in [0, \pi/T - \omega_s] \\ |H_0(j\omega)|^2 \le \delta_s^2, &\qquad \omega \in [\omega_s, \pi/T] \\ |H_0(j\omega)| = 0, &\qquad \omega = \pi/T \end{aligned} \tag{61}$$

and

$$\begin{aligned} 1 - \delta_s^2 \le |H_1(j\omega)|^2 \le 1, &\quad \omega \in [\omega_s, \pi/T] \\ |H_1(j\omega)|^2 \le \delta_s^2, &\quad \omega \in [0, \pi/T - \omega_s] \\ |H_1(j\omega)| = 0, &\qquad \omega = 0 \end{aligned} \tag{62}$$

respectively. The requirements on $H_0(s)$ and $H_1(s)$ in terms of frequency selectivity and passband and stopband attenuations are set by the application [4], [5]. The additional requirement that $|H_0(j\omega)|$ be zero for $\omega = \pi/T$ is due to our chosen class of synthesis filters. To see why, we first recognize that $V_{1R}(0) = H_{0R}(-j\pi/T)F_{0R}(0)$ because $F_{1R}(0) = 0$, which is due to the fact that $f_1(n)$ is antisymmetric. Thus, to make the aliasing $V_{1R}(\omega T)$ small when ωT is zero, or close to zero, $H_0(s)$ should have a zero pair at, or at least close to, $\pm j\pi/T$ since $F_0(z)$ is a low-pass filter, i.e., $F_{0R}(0) \approx 1$. Therefore, $H_0(s)$ must have a transmission zero at, or at least close to, this frequency. In this paper, we consider the case where one zero pair of $H_0(s)$ is restricted to be exactly at $\pm j\pi/T$. Similarly, $H_1(s)$ should have a zero at the origin, but this is automatically ensured for the filter class under consideration.

Since both $H_0(s)$ and $H_1(s)$ are determined by the two all-pass filters $A_0(s)$ and $A_1(s)$, it suffices to design either $H_0(s)$ or $H_1(s)$. This filter must, however, be designed in such a way that both filters satisfy their respective requirements. This is easily done by utilizing that the filters are power complementary, i.e., their magnitude responses are related as

$$|H_0(j\omega)|^2 + |H_1(j\omega)|^2 = 1 \tag{63}$$

which follows immediately from (52) and (53). Further, since low filter order is of high priority, the equiripple approximation should be used. The approximation problem of finding $H_0(s)$ and $H_1(s)$ can be solved in numerous ways. One approach is to use tabulated filters with suitable passband and stopband attenuation and then perform a frequency transformations in order to achieve a transmission zero pair at $\pm j\pi/T$. An alternative is to use nonlinear optimization with, e.g., the poles of $A_0(s)$ and $A_1(s)$ as optimization parameters. This will lead to an optimization problem with a small number of free parameters. The optimization can be performed using the transformed variable [12], [24] in order to improve the numerical accuracy.

Synthesis Filters: Let the specifications of $V_0(j\omega)$ and $V_1(j\omega)$ be

$$1 - \delta_1 \le |V_0(j\omega)| \le 1 + \delta_1, \qquad \omega \in [0, \pi/T] \tag{64}$$

$$|\Phi_e(\omega)| \le \delta_{e\,\max}, \qquad \omega \in [0, \pi/T] \tag{65}$$

and

$$|V_1(j\omega)| \le \delta_2, \qquad \omega \in [0, \pi/T] \tag{66}$$

respectively, where $\Phi_e(\omega)$ is the phase error, which is defined as the difference between the phase response of $V_0(j\omega)$ and the linear-phase function $-d\omega T$, where d is some constant [13]. That is, the phase error is given by

$$\Phi_e(\omega) = \Theta_0(\omega) + d\omega T. \tag{67}$$

The synthesis filters are designed in two steps. First, $F_0(z)$ is optimized in such a way that $\Phi_e(\omega)$ meets the requirements of

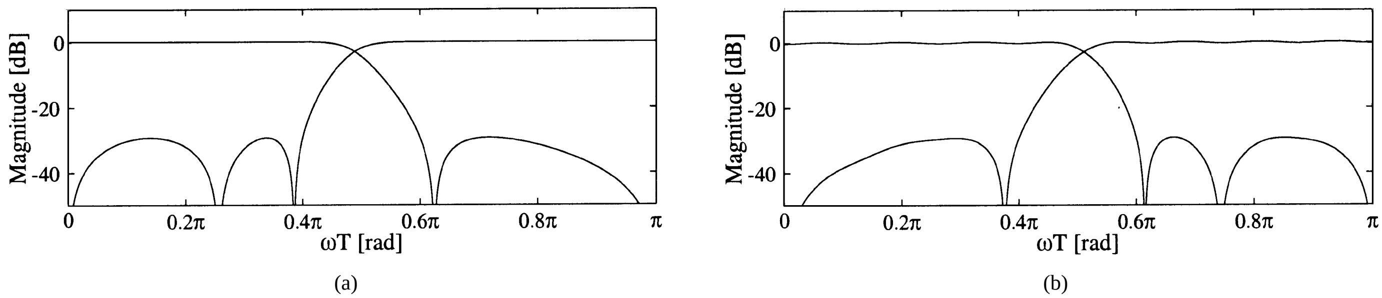


Fig. 9. Magnitude responses in *Example 3*. (a) Analog analysis filters $H_0(s)$ and $H_1(s)$. (b) FIR synthesis filters $G_0(z)$ and $G_1(z)$.

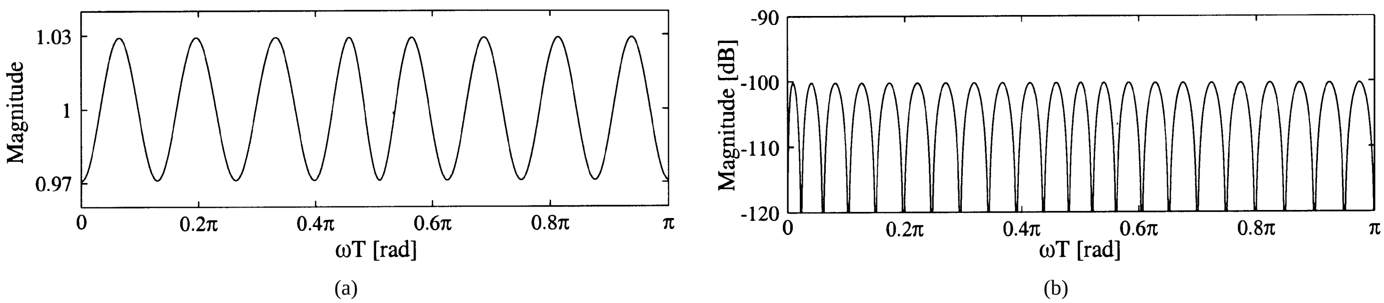


Fig. 10. (a) Magnitude of the distortion function $V_0(j\omega)$. (b) Magnitude of the aliasing function $V_1(j\omega)$.

(65). Then, $F_1(z)$ and $F_{1c}(z)$ are optimized in such a way that $|V_0(j\omega)|$ and $|V_1(j\omega)|$ meet the requirements of (64) and (66), respectively.

The filter $F_0(z)$ can, for a given filter order, easily be designed such that $\Phi_e(\omega)$ satisfies (40) by solving the following nonlinear programming problem:

$$\begin{aligned} &\text{minimize} \quad \delta_e \\ &\text{subject to} \quad |\Phi_e(\omega_k)| \leq \delta_e \end{aligned} \tag{68}$$

where $\omega_k \in [0, \pi], k = 1, 2, \ldots, L$ is ω discretized into L grid points. The unknown parameters are the impulse response values $f_0(n)$ (filter coefficients) of $F_0(z), d$, and δ_e. The optimal solution to the problem of (68) is the equiripple solution, i.e., when the phase error $\Phi_e(\omega)$ takes on the maximum error $\delta_{e\,\max}$ at $K_{F0} + 2$ (number of unknowns) different frequencies [13] with alternating signs. This is under the assumption that the delay d in (67) is a free parameter. However, the phase response values of the analog filters are generally not a multiple of π at $\omega = \pi/T$. This means that there are certain restrictions on d, which implies that the optimum solution in some cases may exhibit only $K_{F0} + 1$ ripples. One way to avoid restrictions on d is to reoptimize the analog filters so that their phase response values are a multiple of π at $\omega = \pi/T$. We have observed though that the equiripple solution in many cases indeed can be found without doing this reoptimization.

Since the problem of (68) is a nonlinear programming problem, one generally has to perform several optimizations with different initial solutions in order to obtain the optimum one. We stop once the equiripple solution has been found. Our experience is that it is relatively easy to find the optimum by simply use random initial solutions. In most cases, it then suffices to perform just a few optimizations. Naturally, one has to start with reasonable random initial solutions. We have randomly placed zeros within the unit circle around the imaginary axis. This can be motivated by studying the corresponding digital case in Section II. There, the zeros are on the imaginary axis, within the unit circle, since they cancel the poles of the half-band IIR analysis filters. Here, we have an "analog version" of that filter bank class, which makes our initial solutions reasonable.

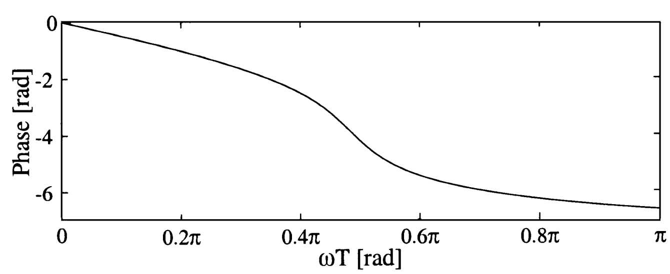
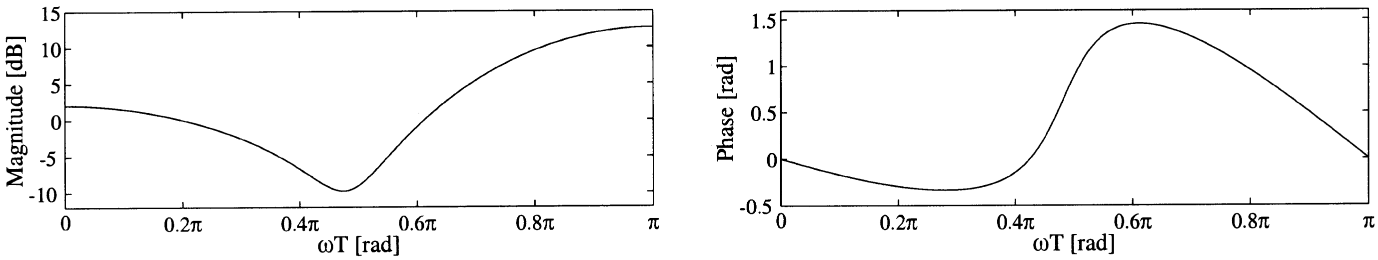


Fig. 11. Phase response of the analog analysis filters $H_0(s)$ and $H_1(s)$ in *Example 4*.

To optimize $F_1(z)$ and $F_{1c}(z)$, we first restate the specifications of (64) and (66) with the aid of $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ according to

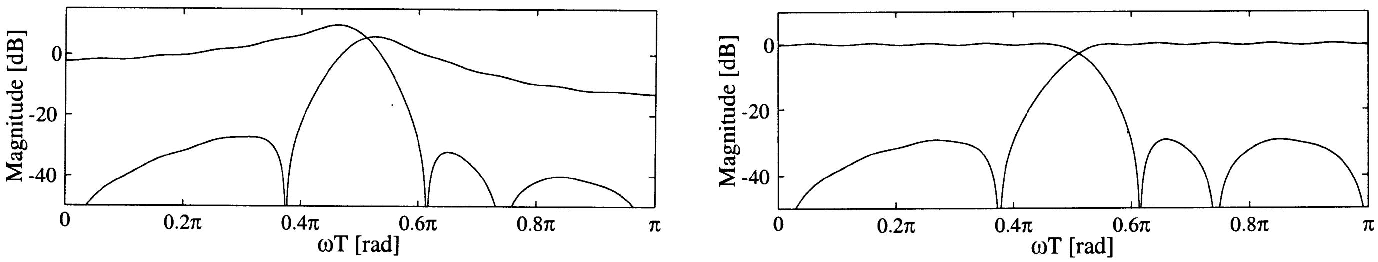
$$\begin{aligned} 1 - \delta_1 \leq V_{0R}(\omega T) \leq 1 + \delta_1, &\qquad \omega T \in [0, \pi] \\ -\delta_2 \leq V_{1R}(\omega T) \leq \delta_2, &\qquad \omega T \in [0, \pi]. \end{aligned} \tag{69}$$

Clearly, $|V_0(e^{j\omega T})|$ and $|V_1(e^{j\omega T})|$ satisfy (64) and (66) if $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ satisfy (69). The filters $F_1(z)$ and $F_{1c}(z)$ can, for given filter orders, easily be designed such that $V_{0R}(\omega T)$ and $V_{1R}(\omega T)$ satisfy (69) by solving the linear programming problem stated in (41) with $c_p, p = 0, 1, \ldots, M + 1$



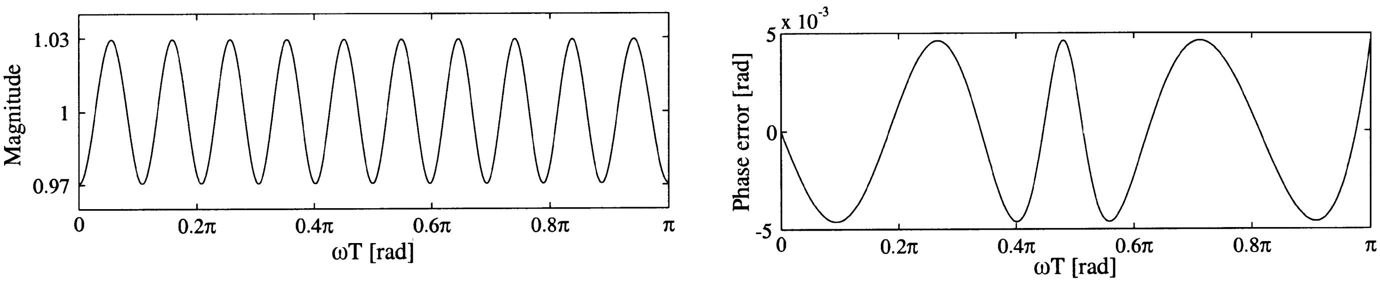
(a) (b)

Fig. 12. (a) Magnitude response and (b) phase response of the phase equalizer $F_0(z)$ in *Example 4*.



(a) (b)

Fig. 13. Magnitude responses in *Example 4*. (a) Magnitude equalizers $F_1(z)$ and $F_{1c}(z)$. (b) Overall synthesis filters $G_0(z)$ and $G_1(z)$.



(a) (b)

Fig. 14. (a) Magnitude response and (b) phase error of the distortion function $V_0(j\omega)$ in *Example 4*.

being free unknown parameters, and with $R_2(p, \omega_k T)$ and $R_3(p, \omega_k T)$ being

$$R_2(p, \omega_k T) = \begin{cases} H_{0R}(\omega_k)|F_0(e^{j\omega_k T})|, & 0 \le p \le \dfrac{M-1}{2} \\ -H_{1R}(\omega_k)|F_0(e^{j\omega_k T})|, & \dfrac{M+1}{2} \le p \le M \end{cases} \tag{70}$$

$$R_3(p, \omega_k T) = \begin{cases} H_{0R}\left(\omega_k - \dfrac{\pi}{T}\right)|F_0(e^{j\omega_k T})|, & 0 \le p \le \dfrac{M-1}{2} \\ -H_{1R}\left(\omega_k - \dfrac{\pi}{T}\right)|F_0(e^{j\omega_k T})|, & \dfrac{M+1}{2} \le p \le M. \end{cases} \tag{71}$$

Approximately PR and PMR Filter Banks: The special case in the above design procedure occurs when there are no requirements on the phase response of $V_0(j\omega)$, i.e., when the specification of (65) is not present. In this case, we set $F_0(z) = 1$ and the overall system is referred to as an approximately PMR filter bank. If, on the other hand, phase requirements are included, i.e., the specification of (65) must be satisfied, then the overall filter bank is referred to as an approximately PR bank.

D. Design Examples

This section illustrates by means of two design examples the properties of the proposed filter banks.

Example 3: This example considers the design of an approximately PMR filter bank. We start by designing the analysis filters $H_0(s)$ and $H_1(s)$ to meet (61) and (62) with $\omega_s = 0.6\pi/T$ and $\delta_s = 0.05$ (−26 dB). Further, the filters are constrained to be minimum phase and the low-pass filter to have a zero at $\omega = \pi/T$. The specifications are satisfied by using a fifth-order filter pair. The design margin is allocated to the stopband, resulting in a ripple of 0.0343 (−29.3 dB). The magnitude responses of $H_0(z)$ and $H_1(z)$ are shown in Fig. 9. Next, the synthesis filters $G_0(z)$ and $G_1(z)$ are optimized by solving (41) so that the distortion and aliasing functions $V_0(z)$ and $V_1(z)$ satisfy (64) and (66) with $\delta_1 = 0.025$ and $\delta_2 = 10^5$. The specifications are satisfied using a filter pair of order 37. The magnitude responses of $G_0(z)$ and $G_1(z)$ are shown in Fig. 9, whereas the resulting magnitude responses of the distortion and aliasing functions are shown in Fig. 10.

Example 4: This example considers the design of an approximately PR filter bank. We consider the same specification as in *Example 3*, but with the additional requirement that the phase error $\Phi_e(\omega T)$ of the distortion function should satisfy

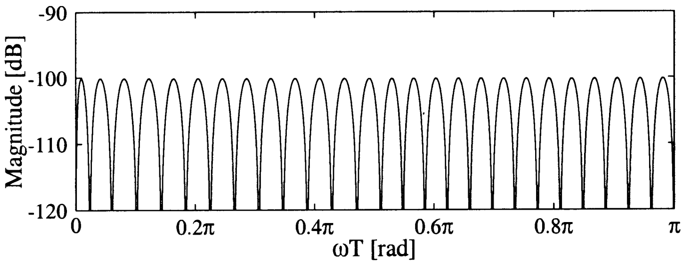


Fig. 15. Magnitude response of the aliasing function $V_1(j\omega)$ in *Example 4*.

(65) with $\delta_{e\max} = 0.005$ rad. The analog filters are the same as in *Example 3*. Their magnitude responses are shown in Fig. 9, whereas their phase response (except for the phase shift of $\pm\pi/2$ in the high-pass filter) is shown in Fig. 11. The difference from *Example 3* regarding the filter design is that the synthesis filters in this case are designed in two steps. First, $F_0(z)$ is optimized by solving (68) so that $\Phi_e(\omega T)$ meets its specification. A sixth-order filter is required to this end. The magnitude and phase response of $F_0(z)$ are shown in Fig. 12. Second, $F_1(z)$ and $F_{1c}(z)$ are optimized by solving (41) so that $V_0(z)$ and $V_1(z)$ satisfy their requirements. Their specifications are satisfied using a filter pair of order 45. The magnitude responses of $F_1(z)$ and $F_{1c}(z)$, as well as the overall synthesis filters $G_0(z)$ and $G_1(z)$ are shown in Fig. 13. The magnitude response and phase error of the distortion function are shown in Fig. 14. The magnitude response of the aliasing function is shown in Fig. 15.

IV. Conclusion

This paper has introduced two-channel digital and hybrid analog/digital multirate filter banks where either the analysis or synthesis filters have a very low complexity. This paper has only considered the case in which the analysis filters have a low complexity, but it is possible to instead let the synthesis filters be the low-complexity filters by simply interchanging the analysis and synthesis filters. The proposed digital filter banks are approximately PR filter banks, whereas the hybrid analog/digital filter banks can be selected to be either approximately PR or PMR filter banks. The analysis filters make use of digital IIR filters and analog filters in the digital and hybrid analog/digital filter banks, respectively, whereas the synthesis filters in both cases employ digital FIR filters.

The filter bank design is performed by first optimizing the analysis filters and then, with the analysis filters fixed, optimizing the digital synthesis filters. In all cases, the filters are easy to design by making use of well-known and reliable optimization techniques. For the digital filter banks, the IIR analysis filters and FIR synthesis filters are designed using closed-form solutions and linear programming, respectively. In the hybrid analog/digital filter bank, the analog analysis filters and parts of the FIR synthesis filters (in the approximately PR case) are optimized using nonlinear programming, whereas the remaining parts of the FIR synthesis filters again make use of linear programming. The nonlinear programming problems use, however, only a few unknown parameters, which makes these problems easy to solve as well.

By designing the analysis and synthesis filters separately, it is possible to obtain analysis filters of very low order and complexity. In addition, it was for the digital filter banks demonstrated by means of design examples that the overall complexity is low as well. For the hybrid analog/digital filter banks, it is difficult to make such comparisons. One reason is that there exist very few papers dealing with such filter banks. For example, we have not found any paper that treats approximately PMR analog/digital filter banks. Further, in the existing papers dealing with the approximately PR case, we have not found any designs that minimize the same cost function as we do. In addition, the existing techniques use nonlinear programming straightforwardly. We have observed that, using such an approach, it is, in many cases, difficult to find initial parameters so that the final solution is good enough; e.g., it is difficult to achieve small aliasing terms. A main advantage of using our approach is that we can then easily achieve as good designs as desired since both the distortion and aliasing functions are controlled in a linear programming problem. Further, the hybrid analog/digital filter bank can be seen as a "modified analog/digital version" of the *Case 2* digital filter banks considered in Section II. The analog/digital filter bank will, therefore, have similar properties in terms of complexity as has the digital filter bank. That is, the former will have a reasonably low complexity since the latter so has.

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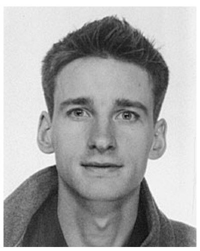

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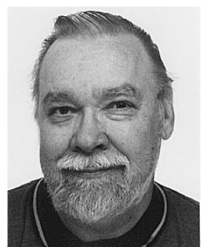

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