Polynomiography: The Art and Science of Visualizing Polynomials

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Abstract

A complex polynomial p(z) has several important properties, one of which is the maximum of its modulus over a unit disc D, denoted by $\|p\|_\infty$. By observing the images generated by some basic iterative methods, it would suggest that to compute $\|p\|_\infty$, we can work backwards by first computing the roots of p(z) and applying customized iterative methods. By using members of the Basic Family, an infinite family of iteration functions for polynomial root finding, I generated a variety of "polynomiographs", or images. It was found that generating images that were uncomplicated, attractive and showed the roots and regions of attraction clearly was the result of using the right combination of the iteration method and choice of polynomial. With more research, these will result in practical methods to approximate the modulus of a complex polynomial and generate meaningful images. The visual appeal of polynomiography means it can be used as the basis of a technology that connects younger students to mathematics and helps them learn difficult concepts.

Background

- Common iteration methods, like the Newton's method were not usually used for complex functions, until Cayley (1897) attempted to use the method for finding the roots of polynomial with iterations over the complex numbers.
- The visualization of a polynomial equation is called polynomiography, and the resulting image is called a polynomiograph.
- The basins of attraction is the set of points in the complex plane whose iterates converge to a root of the underlying polynomial.
- By showing that a point z* is a local maximum of |p(z)| if and only if
  z* = (p(z*)/p'(z*))/(|p(z*)/p'(z*)|), we can proceed to use innovative iterative methods in the Basic Family, and generate interesting polynomiographs by solving the equation
  G(z) = 0; where G(z) = p(z)|p'(z)| - zp'(z)|p(z)|.

References

[1] Kalantari, Bahman. (2016). A Necessary and Sufficient Condition for Local Maxima of Polynomial Modulus Over Unit Disc.

[2] Kalantari, Bahman. *Polynomial Root-Finding and Polynomiography*. World Scientific, 2009.

Methods

- Using the Basic Family, an infinite family of iteration functions for polynomial root-finding, and the appropriate complex polynomial, we can generate a host of polynomiographs.
- Picking a seed z, which is a complex number, we solve G(z) = p(z)|p'(z)| - zp'(z)|p(z)| = 0
- Compute $z_{k+1} = B_m(z_k)$, where $B_m(z_k)$ is any member of the Basic Family
- The images were generated using Wolfram Mathematica

$B_2(z)$ (Pseudo-Newton method)

$B_2(z) = z - p(z)/p'(z)$
Substituting, we get
$z_{k+1} = z_k - G_k(z_k)/G'_k(z_k)$

$B_3(z)$ (Halley's method)

$B_3(z) = z - (2p'(z)*p(z))/(2(p'(z))^2 - p''(z)*p(z))$
Substituting, we get
$z_{k+1} = z_k - (2G'_k(z_k)*G_k(z_k))/(2(G'_k(z_k))^2 - (G''_k(z_k)*G_k(z_k))$

$B_4(z)$

$B_4(z) = z - (6p'(z)^2*p(z) - 3p''(z)*p(z)^2)/(p'''(z)*p(z)^2 + 6p'(z)^3 - 6p''(z)*p'(z)*p(z))$
Substituting, we get
$z_{k+1} = z_k - (6G'_k(z_k)^2*G_k(z_k) - 3G''_k(z_k)*G_k(z_k)^2)/(G'''_k(z_k)*G_k(z_k)^2 + 6G'_k(z_k)^3 - 6G''_k(z_k)*G'_k(z_k)*G_k(z_k))$

Applications and Future Direction

- Polynomiography can be used as a tool of education, visually representing the concepts behind iteration functions, complex numbers and polynomials.
- The polynomiographs can be used to study the advantage in performance of one iteration function against another.
- As an artistic tool, it can stimulate a mathematically minded user's artistic side, and its visual appeal can encourage young students to take up more of an interest in mathematics.
- Further work needs to be conducted in order to investigate the rates of convergence and theoretical performance of these iterative methods, and the use of other members of the Basic Family to solve G(z) = 0.

Acknowledgements

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Results

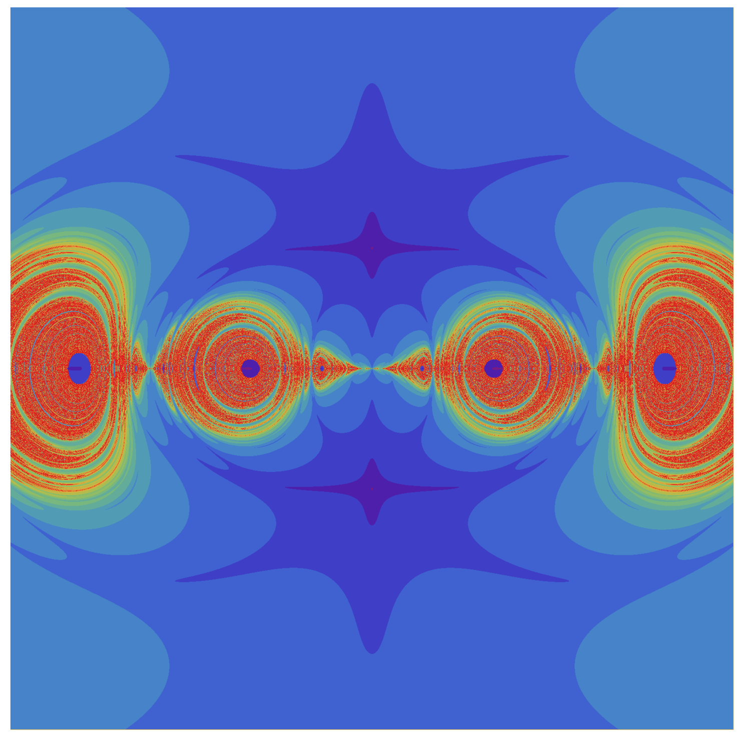

FIGURE 1: Generated using $B_2(z)$, p(z) = z^2 - 1

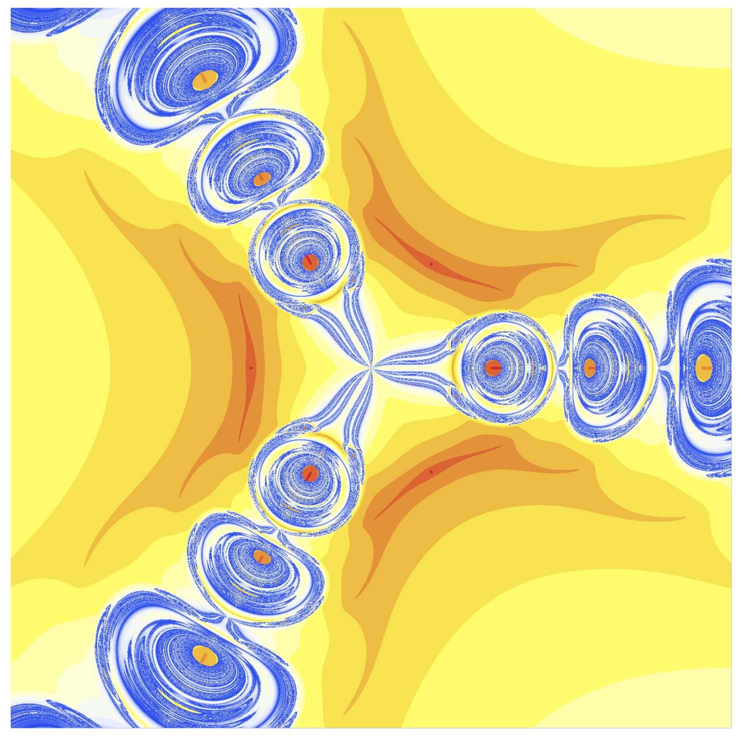

FIGURE 2: Generated using $B_2(z)$, p(z) = z^3 - 1

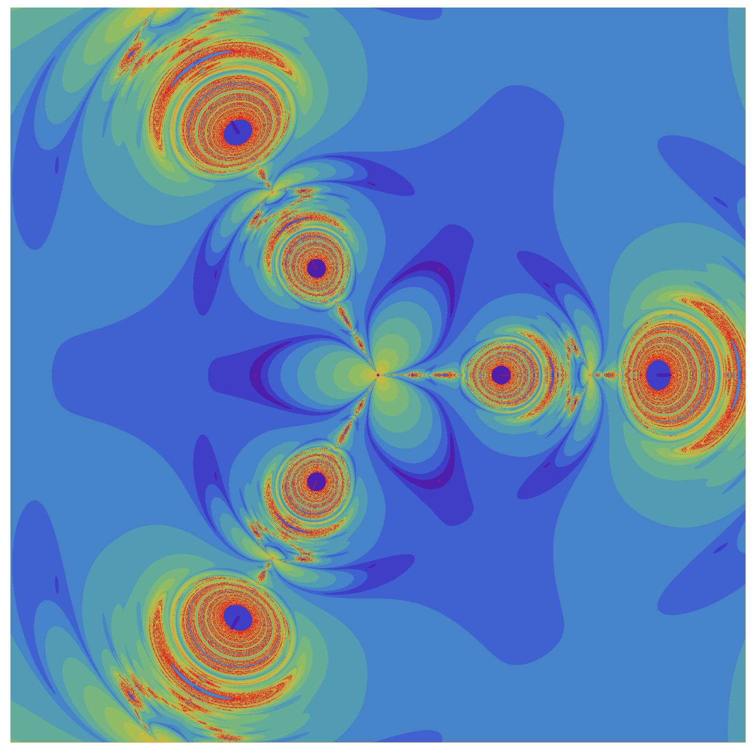

FIGURE 3: Generated using $B_3(z)$, p(z) = z^3 - 1

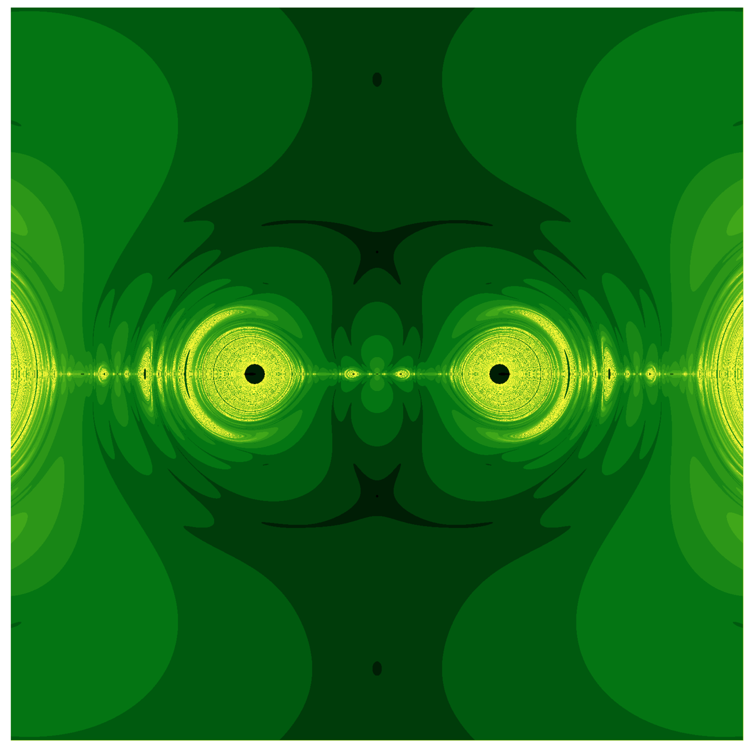

FIGURE 4: Generated using $B_3(z)$, p(z) = z^2 - 1

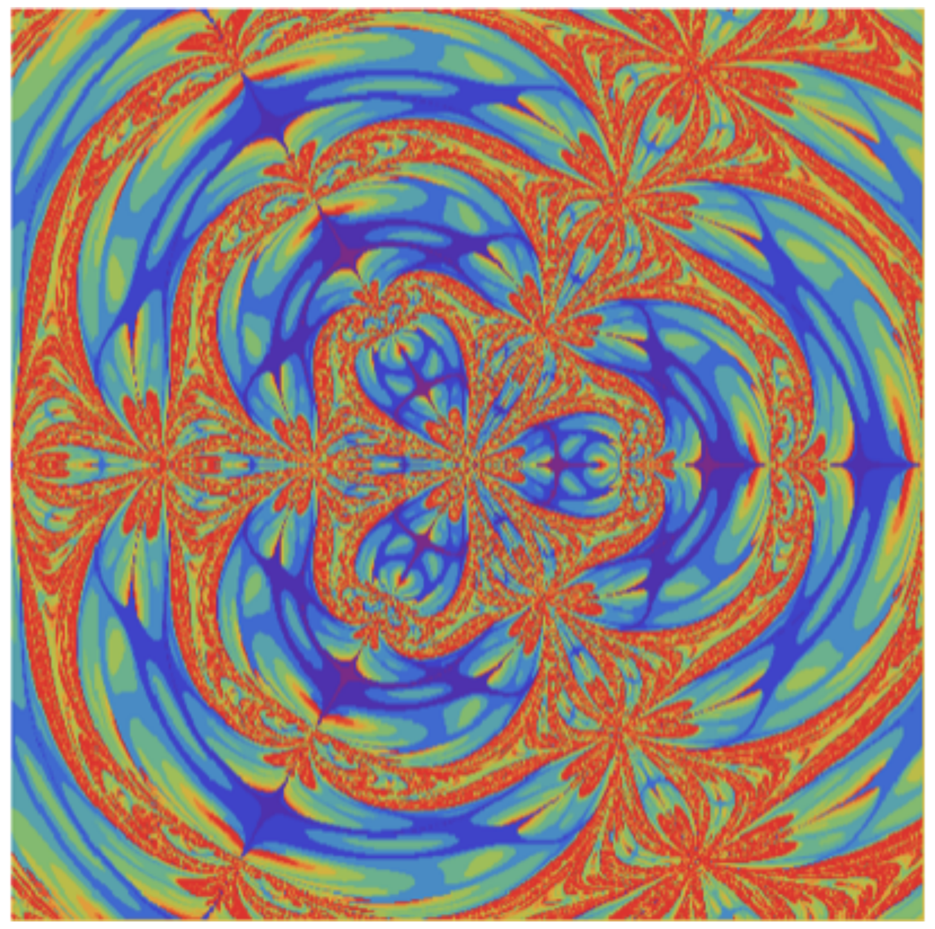

FIGURE 5 (left): Polynomiograph of fixed-point iteration of z^3-1

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