ELEC-E8101 Digital and Optimal Control

Homework 3 - Solution

1. The characteristic equation of a system is given by
$$1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0$$
Determine the value of $K,\ K > 0$, for which the system is stable. [0.5p]

Solution. The characteristic equation is
$$z^2 - 1.2z + 0.2 + K(0.2z + 0.5) = 0 \Rightarrow z^2 + (0.2K - 1.2)z + (0.5K + 0.2) = 0$$

1st way: From the triangle rule:
$$\begin{cases} -1 < 0.5K + 0.2 < 1 \Rightarrow -1.2 < 0.5K < 0.8 \Rightarrow \underline{-2.4 < K < 1.6} \\ 0.2K - 1.2 - 1 < 0.5K + 0.2 \Rightarrow 0.3K > -2.4 \Rightarrow \underline{K > -8} \\ -(0.2K - 1.2) - 1 < 0.5K + 0.2 \Rightarrow -0.2K + 0.2 < 0.5K + 0.2 \Rightarrow \underline{K > 0} \end{cases}$$

From the inequalities above, $\boxed{0 < K < 1.6}$.

2nd way: Let's use the *Jury's stability test*:

$$\begin{array}{ccc|c} \boxed{1} & 0.2K - 1.2 & 0.5K + 0.2 & \\ 0.5K + 0.2 & 0.2K - 1.2 & 1 & b_2 = \dfrac{0.5K + 0.2}{1} \\ \hline \boxed{1 - (0.5K + 0.2)^2} & (0.2K - 1.2)(0.8 - 0.5K) & & \\ (0.2K - 1.2)(0.8 - 0.5K) & 1 - (0.5K + 0.2)^2 & & b_1 = \dfrac{0.2K - 1.2}{0.5K + 1.2} \\ \hline \boxed{\dfrac{0.7K(0.8 - 0.5K)(0.3K + 2.4)}{0.5K + 1.2}} & & & \end{array}$$

Stability conditions require that the boxed expressions are all greater than 0. First, $1 > 0$ holds. For the second to hold we need:
$$\begin{aligned} 1 - (0.5K + 0.2)^2 > 0 \Rightarrow & [1 - (0.5K + 0.2)][1 + (0.5K + 0.2)] > 0 \\ & (0.8 - 0.5K)(1.2 + 0.5K) > 0 \Rightarrow \underline{-2.4 < K < 1.6} \end{aligned}$$

For the third case, since the denominator is positive already (given that $0 < K$), we want to make sure that $(0.8 - 0.5K)(0.3K + 2.4) > 0$, which corresponds to: $-8 < K < 1.6$. Combining the two cases, we have that $\boxed{0 < K < 1.6}$, as before.

Remark. The closed-loop poles are the roots of equation $z^2 + (0.2K - 1.2)z + (0.5K + 0.2) = 0$, which are given by
$$p_{1,2} = \frac{-(0.2K - 1.2) \pm \sqrt{(0.2K - 1.2)^2 - 4(1)(0.5K + 0.2)}}{2(1)}.$$

For closed-loop stability we need the poles to be inside the unit disk. Solving the problem this way requires a lot of algebraic manipulations that need a lot of time!

2. The characteristic equation of a system is given by

$$\chi(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0$$

Determine whether the system is stable or not. [1p]

Solution. First, we check the *necessary* conditions:

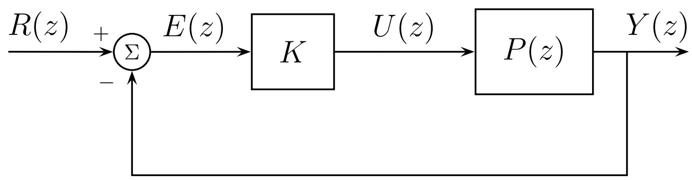
$$\chi(1) = 1 - 2 + 1.4 - 0.1 = 0.3 > 0$$
$$(-1)^3\chi(-1) = (-1)(-1 - 2 - 1.4 - 0.1) = 4.5 > 0$$

The necessary conditions are satisfied. Now we are going to do the Jury's test:

$$\begin{array}{cccc|c}
\boxed{1} & -2 & 1.4 & -0.1 & \\
-0.1 & 1.4 & -2 & 1 & b_3 = \dfrac{-0.1}{1} = -0.1 \\
\hline
(1 - (-0.1)(-0.1) =)\boxed{0.99} & (-2 - (-0.1)(1.4) =) - 1.86 & 1.2 & & \\
1.2 & -1.86 & 0.99 & & b_2 = \dfrac{1.2}{0.99} \\
\hline
\left(0.99 - \frac{1.2}{0.99}(1.2)\right)\boxed{< 0} & \bullet & & & \\
\bullet & \bullet & & & b_1 = \bullet \\
\hline
\bullet & & & &
\end{array}$$

Since the coefficient is negative, the system is unstable (despite satisfying the necessary conditions).

3. Consider the feedback system



where

$$P(z) = \frac{-1}{z^2 + z + 2}$$

and K is a constant.

a) Draw the pole/zero diagram (z-plane) for the *open-loop* system $P(z)$. Is the system stable? [0.5p]

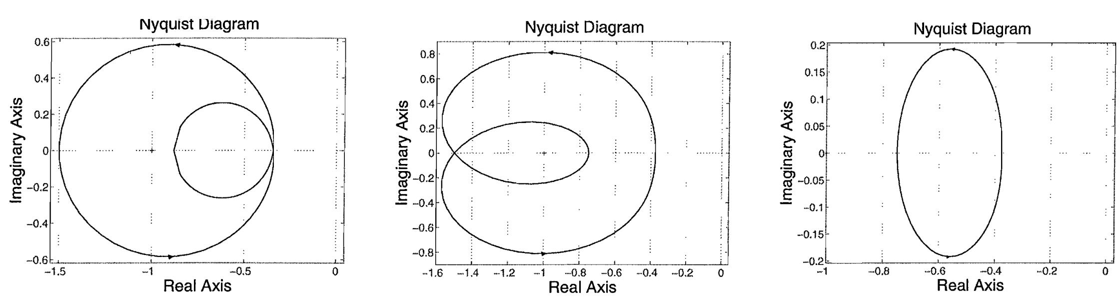
b) Show that the closed-loop transfer function from $R(z)$ to $Y(z)$ is given by [0.5p]

$$G(z) = \frac{-K}{z^2 + z + 2 - K}$$

c) For which values of K is the closed-loop stable? [0.5p]

d) Consider the closed-loop system and let the input $r[k]$ be a unit step. Find, as a function of gain K, the steady-state value of $y[k]$ (i.e., the $\lim_{k\to\infty} y[k]$) when this is finite, stating for which values of K the answer is valid. [0.5p]

e) Let $K = 1.5$. The figure below shows three Nyquist plots (A, B and C), but only one corresponds to $KP(z)$.



A **B** **C**

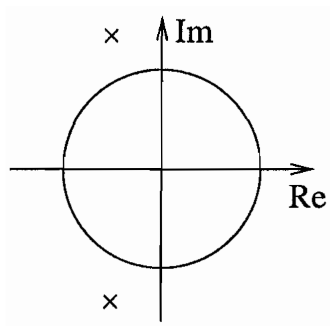
Choose the correct one, justifying your answer with respect to the Nyquist stability criterion. [1p]

Solution.

a) The open-loop poles of the system are the roots of the equation $z^2 + z + 2 = 0$, i.e.,

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \frac{-1 \pm j\sqrt{7}}{2}$$

The poles are outside the unit circle (see figure below), since $|p_{1,2}| > 1$, and therefore the system is unstable.



b) The closed-loop transfer function from $R(z)$ to $Y(z)$ is given by

$$G(z) = \frac{Y(z)}{R(z)} = \frac{KP(z)}{1 + KP(z)} = \frac{-K}{z^2 + z + 2 - K}.$$

c) **1st way:** The closed-loop poles are the roots of the equation $z^2 + z + 2 - K = 0$, which are given by

$$p_{1,2} = \frac{-1 \pm \sqrt{1^2 - 4(1)(2-K)}}{2(1)} = \frac{-1 \pm \sqrt{4K-7}}{2}.$$

For closed-loop stability we need the poles to be inside the unit disk.

<u>For $4K - 7 < 0$:</u>

$$\left(\frac{-1}{2}\right)^2 + \left(\frac{\sqrt{4K-7}}{2}\right)^2 < 1 \Rightarrow |4K - 7| < 3$$

1) Since we assume already that $4K - 7 < 0$, it holds that $4K - 7 < 3$. Hence, $K < 7/4$.
2) $-3 < 4K - 7 \Rightarrow K > 1$.

Therefore, for $4K - 7 < 0$, $1 < K < 7/4$.

<u>For $4K - 7 > 0$:</u>

$$-1 < \frac{-1 \pm \sqrt{4K-7}}{2} < 1$$

which gives $7/4 < K < 2$.

So, **combining both cases**, $1 < K < 2$.

2nd way: Let's use the *Jury's stability test*:

$$
\begin{array}{ccc}
\boxed{1} & 1 & 2-K \\
2-K & 1 & 1 \\
\hline
\boxed{1-(2-K)^2} & K-1 & \\
K-1 & 1-(2-K)^2 & \\
\hline
1-(2-K)^2 - \dfrac{(K-1)^2}{1-(2-K)^2} & &
\end{array}
\qquad
\begin{array}{l}
b_2 = \dfrac{2-K}{1} = 2-K \\[3em]
b_1 = \dfrac{K-1}{1-(2-K)^2}
\end{array}
$$

The last term can be written as:

$$
\begin{aligned}
1-(2-K)^2 - \frac{(K-1)^2}{1-(2-K)^2} &= (K-1)(3-K) - \frac{(K-1)^2}{(K-1)(3-K)} \quad \text{(difference of two squares)} \\
&= \frac{(K-1)^2(3-K)^2 - (K-1)^2}{(K-1)(3-K)} \\
&= \boxed{\frac{(K-1)^2\left[(3-K)^2-1\right]}{1-(2-K)^2}}
\end{aligned}
$$

Stability conditions require that the boxed expressions are all greater than 0. First, $1 > 0$ holds. For the second to hold we need:

$$
\begin{aligned}
1-(2-K)^2 > 0 \Rightarrow & [1-(2-K)][1+(2-K)] > 0 \\
& (K-1)(3-K) > 0 \Rightarrow 1 < K < 3
\end{aligned}
$$

For the third case, since the denominator is positive already (given that $1 < K < 3$ we want to make sure that $(3-K)^2 - 1 > 0$, which corresponds to: $K < 2$ or $K > 4$. Combining the two cases, we have that $\boxed{1 < K < 2}$.

3rd way: Using the triangle rule:

$$
\begin{cases}
-1 < 2-K < 1 \Rightarrow 1 < K < 3 \\
0 < 2-K \Rightarrow K < 2 \\
-2 < 2-K \Rightarrow K < 4
\end{cases}
$$

The solution is the intersection of the 3 sets given using the triangle rule, i.e., $\boxed{1 < K < 2}$.

d) When $K \notin (1,2)$, the system is unstable and therefore $y[k]$ will grow unbounded. When $k \in (1,2)$, the closed-loop system is stable and to find the steady-state value of $y[k]$, denoted here by y_{ss}, we use the Final Value Theorem to the closed-loop transfer function $G(z)$ we found in part b):

$$
\begin{aligned}
y_{ss} &= \lim_{k\to\infty} y[k] = \lim_{z\to 1}(z-1)Y(z) = \lim_{z\to 1}(z-1)G(z)U(z) \\
&= \lim_{z\to 1}(z-1)\frac{-K}{z^2+z+2-K}\frac{z}{z-1} = \frac{-K}{4-K}
\end{aligned}
$$

e) **1st way:** For $K = 1.5$, the closed-loop system is stable. Since the open-loop system has 2 unstable poles, the Nyquist diagram must have 2 counterclockwise encirclements of the point $-1 + j0$. Thus, plot B is the correct.

2nd way: Nyquist plot A shows that, for $z = 1$ or $z = -1$, $KP(z) = -1.5$. However, $KP(1) = -3/8$ and $KP(-1) = -3/4$, thus plot A cannot be the one. Nyquist plot C shows that the magnitude of $KP(z)$ is approximately always less than 0.75. However, $|KP(e^{j1.93})| = 1.6$. Also, there exists only one encirclement, and the system could never be stable. Therefore, plot C cannot be the one either. Plot B satisfies all of the above and it is the correct one.