ELEC-E8101 Digital and Optimal Control

Homework 1 - Solution

1. Consider the difference equation

$$x[k+2] - 1.5x[k+1] + 0.54x[k] = u[k] \tag{1}$$

with initial conditions x[0]=x[1]=0 and

$$u[k] = \begin{cases} 1, & k = 1, \\ 0, & k = 0, 2, 3, 4, \dots \end{cases}$$

a) By using z-transforms calculate an analytical solution for $x[k]$. [0.5p]

b) Write a MATLAB code (script or function, as you wish), which uses difference equation (1) and solves the solution for x pointwise as $k = 0, 1, 2, 3, \dots, 10$. Verify that the solution is in accordance to that obtained in part a). [0.5p]

Solution.

a) First we find the z-transform of $u[k]$:

$$\begin{aligned} U(z) = \sum_{k=0}^{\infty} u[k]z^{-k} &= u[0] + u[1]z^{-1} + u[2]z^{-2} + \dots \\ &= 0 + z^{-1} + 0 + 0 + \dots = z^{-1} \end{aligned}$$

Taking the z-transform of difference equation (1):

$$\begin{aligned} &\left(z^2X(z) - z^2\underbrace{x[0]}_{=0} - z\underbrace{x[1]}_{=0}\right) - 1.5\left(zX(z) - z\underbrace{x[0]}_{=0}\right) + 0.54X(z) = U(z) \\ &X(z)\left(z^2 - 1.5z + 0.54\right) = z^{-1} \\ &X(z) = \frac{z^{-1}}{z^2 - 1.5z + 0.54} = \frac{1}{z(z-0.9)(z-0.6)} = \frac{A}{z} + \frac{B}{z-0.9} + \frac{C}{z-0.6} \end{aligned}$$

Using the Heavyside method for partial fractions:

$$\begin{aligned} A &= \lim_{z\to 0} \frac{1}{(z-0.9)(z-0.6)} = \frac{1}{(-0.9)\times(-0.6)} = \frac{1}{0.54} \\ B &= \lim_{z\to 0.9} \frac{1}{z(z-0.6)} = \frac{1}{0.9\times 0.3} = \frac{1}{0.27} \\ C &= \lim_{z\to 0.6} \frac{1}{z(z-0.9)} = \frac{1}{0.6\times(-0.3)} = -\frac{1}{0.18} \end{aligned}$$

Hence,

$$X(z) = \frac{\frac{1}{0.54}}{z} + \frac{\frac{1}{0.27}}{z-0.9} - \frac{\frac{1}{0.18}}{z-0.6} = z^{-1}\left(\frac{1}{0.54} + \frac{1}{0.27}\frac{z}{z-0.9} - \frac{1}{0.18}\frac{z}{z-0.6}\right)$$

Taking the inverse z-transform

$$x[k] = \begin{cases} 0, & k = 0, \\ \frac{1}{0.54}\delta[k-1] + \frac{1}{0.27}(0.9)^{k-1} - \frac{1}{0.18}(0.6)^{k-1}, & k = 1, 2, \dots \end{cases}$$

where $\delta[k]$ is the pulse function (i.e., $\delta[0] = 1$ and 0 otherwise).

b) MATLAB Code:

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% Solution code for Homework 1: Problem 1. 2019 %

% Part a)
k=[2,3,4,5,6,7,8,9,10];
x1=(1/0.27)*0.9.^(k-1)-(1/0.18)*0.6.^(k-1);
x0=[0 0];
x=[x0 x1];
disp(x)

% Part b)
x2(1)=0; x2(2)=0;
for k=3:1:11
        if k==4,
                e=1;
        else
                e=0;
        end
        x2(k)=1.5*x2(k-1)-0.54*x2(k-2)+e;
end
disp(x2)
```

Both parts a) and b) give:

0 0 0 1.0000 1.5000 1.7100 1.7550 1.7091 1.6160 1.5010 1.3789

2. Consider the following difference equation:

$$y[k+2] - 1.3y[k+1] + 0.4y[k] = u[k+1] - 0.4u[k].$$

a) Determine the pulse transfer function. [0.5p]

b) Is the system stable? Justify your answer. [0.5p]

c) Determine the step response. [1p]

Solution.

a) Taking the z-transform (assuming zero initial conditions):

$$z^2Y(z) - 1.3zY(z) + 0.4Y(z) = zU(z) - 0.4U(z)$$
$$G(z) = \frac{Y(z)}{U(z)} = \frac{z-0.4}{z^2-1.3z+0.4}$$

b) We want to find the poles of the transfer function, i.e.,

$$G(z) = \frac{Y(z)}{U(z)} = \frac{z-0.4}{z^2-1.3z+0.4} = \frac{z-0.4}{(z-0.8)(z-0.5)}.$$

The poles are: $p_1 = 0.8$ and $p_2 = 0.5$. The poles are within the unit circle and therefore, the system is stable.

c) From the difference equation:

<u>Up to $k=-3$:</u> $\ldots = y[-3] = y[-2] = y[-1] = 0.$

<u>For $k=-2$:</u>

$$y[0] - 1.3\underbrace{y[-1]}_{=0} + 0.4\underbrace{y[-2]}_{=0} = \underbrace{u[-1]}_{=0} - 0.4\underbrace{u[-2]}_{=0} \Rightarrow y[0] = 0$$

<u>For $k=-1$:</u>

$$y[1] - 1.3\underbrace{y[0]}_{=0} + 0.4\underbrace{y[-1]}_{=0} = \underbrace{u[0]}_{=1} - 0.4\underbrace{u[-1]}_{=0} \Rightarrow y[1] = 1$$

Taking the z-transform (*without* assuming zero initial conditions):

$$\big(z^2Y(z) - z^2\underbrace{y[0]}_{=0} - z\underbrace{y[1]}_{=1}\big) - 1.3\big(zY(z) - z\underbrace{y[0]}_{=0}\big) + 0.4Y(z) = \big(zU(z) - z\underbrace{u[0]}_{=1}\big) - 0.4U(z)$$
$$z^2Y(z) - \cancel{z} - 1.3zY(z) + 0.4Y(z) = zU(z) - \cancel{z} - 0.4U(z)$$

Therefore, $Y(z)$ is given by

$$Y(z) = \frac{z-0.4}{z^2-1.3z+0.4}U(z) = \frac{z-0.4}{(z-0.8)(z-0.5)}U(z)$$
$$= \frac{z-0.4}{(z-0.8)(z-0.5)}\frac{z}{z-1} = \frac{z(z-0.4)}{(z-0.8)(z-0.5)(z-1)}$$

We do partial fractions:

$$\frac{Y(z)}{z} = \frac{z-0.4}{(z-0.8)(z-0.5)(z-1)} \equiv \frac{A}{z-0.8} + \frac{B}{z-0.5} + \frac{C}{z-1}$$

$$\left.\begin{aligned} A &= \frac{0.4}{0.3(-0.2)} = -\frac{20}{3} = -6.67 \\ B &= \frac{0.1}{(-0.3)(-0.5)} = \frac{2}{3} = 0.67 \\ C &= \frac{0.6}{(0.2)(0.5)} = 6 \end{aligned}\right\} \Rightarrow Y(z) = -6.67\frac{z}{z-0.8} + 0.67\frac{z}{z-0.5} + 6\frac{z}{z-1}$$

Therefore,

$$y[k] = -6.67(0.8)^k - 0.67(0.5)^k + 6u[k].$$

Remark 1. As $k \to \infty$, the first 2 terms go to zero and the final value $y[\infty] = 6$. This can be deduced directly from $Y(z)$ using the Final Value Theorem:

$$y[\infty] = \lim_{z\to1}(z-1)Y(z) = \lim_{z\to1}\cancel{(z-1)}\frac{z(z-0.4)}{(z-0.8)(z-0.5)\cancel{(z-1)}} = \frac{0.6}{0.2\times0.5} = 6$$

Remark 2. The initial value $y[0]$ can be deduced directly from $Y(z)$ using the Initial Value Theorem:

$$y[0] = \lim_{z\to\infty} Y(z) = \lim_{z\to\infty}\frac{z(z-0.4)}{(z-0.8)(z-0.5)(z-1)} = 0$$

3. Your manager has given a task for you. Your assignment is to replace the old analog PID controller with a new digital PID controller. The transfer function of the continuous time process is:

$$P(s) = \frac{e^{-0.7s}}{s^2+0.8s+0.5}.$$

The old analog PID controller is presented in the form:

$$G_{\text{PID}}(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right),$$

where $K = 1$, $T_i = 1.5$ and $T_d = 1$. Develop a discrete PID controller approximation for the continuous-time PID controller $G_{\text{PID}}(s)$:

a) using backward difference approximation. [0.3p]

b) using Tustin's transformation. [0.3p]

Now, a practical continuous PID controller (use $N = 10$) is considered, given by

$$\hat{G}_{\text{PID}}(s) = K\left((Y_{\text{ref}}(s) - Y(s)) + \frac{1}{T_i s}(Y_{\text{ref}}(s) - Y(s)) - \frac{T_d s}{1 + T_d s/N}Y(s)\right).$$

Develop a discrete PID controller approximation for the continuous-time PID controller $\hat{G}_{\text{PID}}(s)$:

c) by replacing the integral by summation and using backward difference approximation in the derivative action. [0.4p]

d) What are the facts that affect the choosing of the sampling rate h? By simulations, compare how the different approximations work. In the simulations use the continuous time process and the discrete PID controllers which you have defined in (a)–(c). Compare the control signals and the responses of the controlled systems. Use different reference signals. What is the best approximation? Try using different sampling periods in the discrete controllers. What happens if the sampling period changes? [1p]

Solution. The problem is to discretize the transfer function of PID controller

$$G_{\text{PID}}(s) = K\left(1 + \frac{1}{T_i s} + T_d s\right),$$

by using different methods and to compare discrete controllers with the continuous time PID controller by simulating them.

a) When *backward difference approximation* is used in the discretization of a transfer function the Laplace variable s is replaced with $\frac{z-1}{zh}$. Hence, the discrete PID controller is:

$$\begin{aligned}
H_{\text{back}}(z) = G_{\text{PID}}(s)\Big|_{s\to\frac{z-1}{zh}} &= K\left(1 + \frac{1}{T_i\left(\frac{z-1}{zh}\right)} + T_d\left(\frac{z-1}{zh}\right)\right)\\
&= K\frac{\left(1 + \frac{T_d}{h} + \frac{h}{T_i}\right)z^2 + \left(-1 - 2\frac{T_d}{h}\right)z + \frac{T_d}{h}}{z^2 - z}
\end{aligned}$$

b) When the *Tustin's transformation* is used in the discretization of a continuous time transfer function the Laplace variable s is replaced with $\frac{2}{h}\frac{z-1}{z+1}$. Hence, the discrete PID controller is:

$$\begin{aligned}
H(z) = G_{\text{PID}}(s)\Big|_{s\to\frac{2}{h}\frac{z-1}{z+1}} &= K\left(1 + \frac{1}{T_i\left(\frac{2}{h}\frac{z-1}{z+1}\right)} + T_d\left(\frac{2}{h}\frac{z-1}{z+1}\right)\right)\\
&= K\frac{\left(1 + \frac{2T_d}{h} + \frac{h}{2T_i}\right)z^2 + \left(\frac{h}{T_i} - \frac{4T_d}{h}\right)z + \left(-1 + \frac{h}{2T_i} + \frac{2T_d}{h}\right)}{z^2 - 1}
\end{aligned}$$

c) The practical PID controller can be rewritten as

$$\begin{aligned}
\hat{G}_{\text{PID}}(s) &= K\left((Y_{\text{ref}}(s) - Y(s)) + \frac{1}{T_i s}(Y_{\text{ref}}(s) - Y(s)) - \frac{T_d s}{1 + T_d s/N}Y(s)\right)\\
&= K\big(G_P(s) + G_I(s) + G_D(s)\big),
\end{aligned}$$

where $G_P(s)$, $G_I(s)$, and $G_D(s)$ are the P-, I- and D-part of the controller. Thus, if the integral is replaced with sum, the discrete I-part $H_I(z)$ of the controller can be written as:

$$u_I[k] = u_I[k-1] + \frac{h}{T_i} e[k]$$
$$U_I(z) = z^{-1}U(z) + \frac{h}{T_i}E(z) \Rightarrow H_I(z) = \frac{U(z)}{E(z)} = \frac{hz}{T_i(z-1)}.$$

From the previous assignment we remember that if the backward difference approximation is used in the discretization the Laplace variable s is replaced with $\frac{z-1}{zh}$. Hence, the derivative part $H_D(z)$ of the discrete PID controller is:

$$H_D(z) = -\frac{T_d\left(\frac{z-1}{zh}\right)}{1 + \frac{T_d\left(\frac{z-1}{zh}\right)}{N}} = -\frac{T_d(z-1)}{\left(1+\frac{T_d}{N}\right)z - \frac{T_d}{N}}.$$

Hence, in the aggregate, the discrete PID controller is:

$$U_{\text{practical}}(z) = K\left[1 + \frac{hz}{T_i(z-1)}\right]E(z) - K\frac{T_d(z-1)}{\left(1+\frac{T_d}{N}\right)z - \frac{T_d}{N}}Y(z).$$

What about sampling time? If we want that the discrete PID controller behaves like the continuous time PID controller the sampling rate should be high. Usually in commercial unit controllers, the sampling time is short and constant (for example, 200 ms). The following rules of thumb can be used when you choose the sampling time for PID controller.

$$\frac{hN}{T_d} \approx 0.2 \cdots 0.6, \quad N \approx 10, \quad \frac{h}{L} \approx 0.01 \cdots 0.06,$$

where L is delay. Hence, the sampling time for the system is about $h \approx 0.007 \cdots 0.042$. In the simulation below we use sampling time $h = 0.014$.