Homework 1 – Solutions

The answers to the homework problems should be returned to the corresponding folder on the Robotics-course MyCourses-platform. The answer files should be named as "solutions_homework problems 1_Firstname_Lastname.pdf". Deadline for returning the solutions is Thursday 5.10, 12:00 am, (*before the lecture when the solutions are presented*).

Note: some equations, that might appear useful for solving the problems, are given at the end of this document (the same set of equations accompanies the final exam).

1. The rotation matrix $\boldsymbol{R}$ describes the orientation of a new coordinate frame with respect to the world frame: $\boldsymbol{R} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

a) The x-axis of the new coordinate frame is parallel to the: a) world frame negative x-axis, b) negative y-axis, c) z-axis or d) negative z-axis ? (1 point)

b) The y-axis of the new coordinate frame is parallel to the: a) world frame negative x-axis, b) y-axis, c) negative y-axis or d) negative z-axis ? (1 point)

c) The z-axis of the new coordinate frame is parallel to the: a) world frame x-axis, b) negative x-axis, c) y-axis or d) z-axis ? (1 point)

d) Does R describe a right-handed coordinate frame? Answer YES/NO (1 point)

Solution:

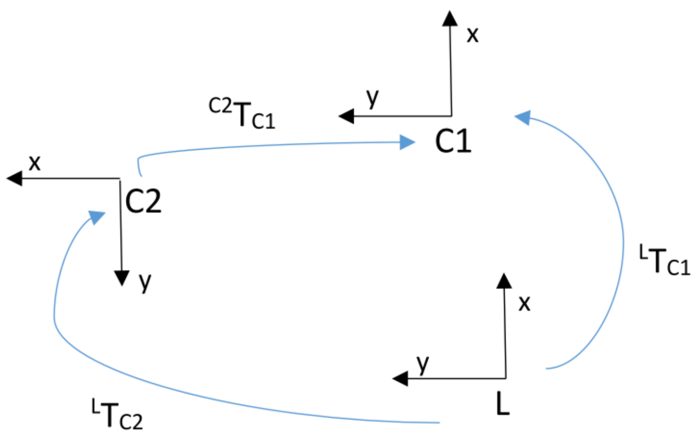
a) b)

b) d)

c) a)

d) YES

2. The task is to solve the unknown relative location of frame C1 with respect to frame C2 by utilizing the known relative locations ${}^{L}\mathbf{T}_{C2}$ and ${}^{L}\mathbf{T}_{C1}$. The setup is illustrated in the figure below.



Determine first, on the matrix symbol level, the equation for the unknown transformation matrix ${}^{C2}\mathbf{T}_{C1}$. Thereafter calculate the numerical form of the 4x4 homogenous transformation matrix ${}^{C2}\mathbf{T}_{C1}$. (8 points)

The known transformation matrices are:

$$
{}^{L}\mathbf{T}_{C2} = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{L}\mathbf{T}_{C1} = \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Solution:

First equate the two paths of relative transformations describing the location of frame C1 with respect to frame L:

${}^{L}\mathbf{T}_{C2}\,{}^{C2}\mathbf{T}_{C1} = {}^{L}\mathbf{T}_{C1}$, which is then multiplied from the left with the inverse of ${}^{L}\mathbf{T}_{C2}$ (because ${}^{L}\mathbf{T}_{C2}^{-1}\,{}^{L}\mathbf{T}_{C2} = \mathbf{I}$) to "separate" the unknown transformation to the left side of the equation

<u>${}^{C2}\mathbf{T}_{C1} = {}^{L}\mathbf{T}_{C2}^{-1}\,{}^{L}\mathbf{T}_{C1}$, which the answer to the first part of the question</u>

To calculate the numerical form of the transformation matrix we utilize the equation for the inverse of a homogenous transformation matrix

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 3} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0}_{1\times 3} & 1 \end{pmatrix} \tag{2.21}$$

<u>So, we get for the numerical form of the matrix equation</u>

$$^{C2}\mathbf{T}_{C1} = \begin{bmatrix} 0 & 1 & 0 & -7 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -7 \\ -1 & 0 & 0 & 5-6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -7 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. The 3D-frame {B} is located initially coincident with the frame {A}. We first translate the origin of {B} frame 5 units in negative y-direction with respect to frame {A}. Next, we rotate the translated frame {B} about its yB-axis by -30 degrees and then translate 3 units in z-direction with respect to the rotated frame {B}.

a) Give the 4x4 homogenous transformation matrix which describes the position and orientation of frame {B} with respect to frame {A}. (4 points)

b) Give the 4x4 homogenous transformation matrix which describes the position and orientation of frame {B} with respect to frame {A} in the case when the order of transformations is 1st transl(x=0,y=−5,z=0), 2nd transl(x=0,y=0,z=3) and 3rd rot_y(−30°) (4 points)

c) The coordinates of a point **P** with respect to frame {B} be are [x=0,y=6,z=3]. What are the coordinates of point **P** given with respect to frame {A} in both cases **a)** and **b)** of the relative location of frame {B} w.r.t frame {A}? (4 points)

Solution:

a) For each of the individual transformations we first form a 4x4 homogenous transformation matrix of their own:

transl(x=0,y=−5,z=0)= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, rot_y(−30°)= $\begin{bmatrix} \cos(-30) & 0 & \sin(-30) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-30) & 0 & \cos\ (-30) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

transl(x=0,y=0,z=3)= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Then we form the matrix equation by placing the matrices of individual transformations, in the correct order, in the equation (starting from the left with the first transformation):

$$^{A}\boldsymbol{T}_{B}=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.866 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

after the multiplication we get

$$^{A}\boldsymbol{T}_{B}=\begin{bmatrix} 0.866 & 0 & -0.5 & -1.5 \\ 0 & 1 & 0 & -5 \\ 0.5 & 0 & 0.866 & 2.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) He we will see that the order in which the transformations are linked really matters. The individual transformation matrices are the same as in part a), the order in which they form the matrix equation just differs:

$$^{A}\boldsymbol{T}_{B}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & -5\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0.866 & 0 & -0.5 & 0\\ 0 & 1 & 0 & 0\\ 0.5 & 0 & 0.866 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

after the multiplication we get

$$^{A}\boldsymbol{T}_{B}=\begin{bmatrix}0.866 & 0 & -0.5 & 0\\ 0 & 1 & 0 & -5\\ 0.5 & 0 & 0.866 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}$$

c) And now we can calculate the coordinates of point **P** w.r.t. frame {A}, first for the case of part **a)**

$$^{A}\mathbf{P}={}^{A}\boldsymbol{T}_{B}*{}^{B}\mathbf{P}=\begin{bmatrix}0.866 & 0 & -0.5 & -1.5\\ 0 & 1 & 0 & -5\\ 0.5 & 0 & 0.866 & 2.6\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 6\\ 3\\ 1\end{bmatrix}=\begin{bmatrix}-1.5-1.5\\ 6-5\\ 2.6+2.6\\ 1\end{bmatrix}=\begin{bmatrix}-3\\ 1\\ 5.2\\ 1\end{bmatrix}$$

and then for the case of part **b)**

$$^{A}\mathbf{P}={}^{A}\boldsymbol{T}_{B}*{}^{B}\mathbf{P}=\begin{bmatrix}0.866 & 0 & -0.5 & 0\\ 0 & 1 & 0 & -5\\ 0.5 & 0 & 0.866 & 3\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 6\\ 3\\ 1\end{bmatrix}=\begin{bmatrix}-1.5\\ 6-5\\ 2.6+3\\ 1\end{bmatrix}=\begin{bmatrix}-1.5\\ 1\\ 5.6\\ 1\end{bmatrix}$$

4. Equations for computing the "Z-Y-Z Euler angle" three angle orientation representation (*i.e. z-rot, y-rot, z-rot rotations executed in this order from left to right each w.r.t to the corresponding axis of the current frame*) from a generic rotation matrix can be derived from the matrix equation: (**For more details see Craig's text book pages 45-46**)

$$ {}^A_BR_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \tag{2.73} $$

where (note in 2.72 cα=cos(α), sγ=sin(γ) and so on)

$$ {}^A_BR_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}. \tag{2.72} $$

By equating matrix elements of the two 3x3 rotation matrices we get equations for the three angles, α, β, γ, (*i.e. z-rot, y-rot, z-rot*) (*valid in the case when* $\beta \neq 0.0°$ *or* $180.0°$)

$$ \begin{aligned} \beta &= \mathrm{Atan2}(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}), \\ \alpha &= \mathrm{Atan2}(r_{23}/s\beta, r_{13}/s\beta), \\ \gamma &= \mathrm{Atan2}(r_{32}/s\beta, -r_{31}/s\beta). \end{aligned} \tag{2.74} $$

where

$$ \operatorname{atan2}(y,x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0, \\ \arctan(\frac{y}{x}) + \pi & \text{if } x < 0 \text{ and } y \geq 0, \\ \arctan(\frac{y}{x}) - \pi & \text{if } x < 0 \text{ and } y < 0, \\ +\frac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0, \\ -\frac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0, \\ \text{undefined} & \text{if } x = 0 \text{ and } y = 0. \end{cases} $$

If β=0.0 we can calculate the solution by choosing α=0.0 in which case γ-angle can be computed as

$$ \begin{aligned} \beta &= 0.0, \\ \alpha &= 0.0, \\ \gamma &= \mathrm{Atan2}(-r_{12}, r_{11}). \end{aligned} \tag{2.75} $$

Correspondingly, if we have β=180.0 the angles can be computed as follows:

$$ \begin{aligned} \beta &= 180.0°, \\ \alpha &= 0.0, \\ \gamma &= \mathrm{Atan2}(r_{12}, -r_{11}). \end{aligned} \tag{2.76} $$

By using these equations calculate the corresponding Z-Y-Z Euler angle triplets for the following three rotation matrices:

a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (4 points)

b) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ (4 points)

c) $\begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0 & 0 & -1 \\ -0.7071 & 0.7071 & 0 \end{bmatrix}$ (4 points)

d) Demonstrate that the computed angles in **a)**, **b)** and **c)** actually give us the original rotation matrices when assigned back to equation 2.72. (4 points)

Solution:

a) β=Atan2(1,0)= 90˚ α= Atan2(0,1)=0˚ γ= Atan2(1,0)=90˚

b) β= Atan2(1,0)=90˚ α= Atan2(1,0)= 90˚ γ= Atan2(0,1)= 0˚

c) β= Atan2(1,0)= 90˚ α= Atan2(-1,0)= -90˚ γ= Atan2(0.7071, 0.7071)=45˚

d) For **a)** we have $R_{ZYZ}(\alpha = 0, \beta = 90, \gamma = 90) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ which equals the original rotation matrix

For **b)** we have $R_{ZYZ}(\alpha = 90, \beta = 90, \gamma = 0) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ which equals the original rotation matrix

For **c)** we have $R_{ZYZ}(\alpha = -90, \beta = 90, \gamma = 45) = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \\ -0.7071 & 0.7071 & 0 \end{bmatrix}$ which equals the original rotation matrix

ELEC-C1320 Robotiikka - Equations

Link parameters and the corresponding elementary transformations as well as the symbolic form of the link matrix according to the **standard** Denavit and Hartenberg parameter convention:

$$^{j-1}\boldsymbol{A}_j(\theta_j, d_j, a_j, \alpha_j) = \boldsymbol{T}_{Rz}(\theta_j)\boldsymbol{T}_z(d_j)\boldsymbol{T}_x(a_j)\boldsymbol{T}_{Rx}(\alpha_j)$$

$$^{j-1}\boldsymbol{A}_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j\cos\alpha_j & \sin\theta_j\sin\alpha_j & a_j\cos\theta_j \\ \sin\theta_j & \cos\theta_j\cos\alpha_j & -\cos\theta_j\sin\alpha_j & a_j\sin\theta_j \\ 0 & \sin\alpha_j & \cos\alpha_j & d_j \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Link parameters and the corresponding elementary transformations as well as the symbolic form of the link matrix according to the **modified** Denavit and Hartenberg parameter convention:

$$^{j-1}\boldsymbol{A}_j = \boldsymbol{R}_x(\alpha_{j-1})\boldsymbol{T}_x(a_{j-1})\boldsymbol{R}_z(\theta_j)\boldsymbol{T}_z(d_j)$$

$$^{j\text{-}1}\boldsymbol{A}_j = \begin{bmatrix} c\theta_j & -s\theta_j & 0 & a_{j-1} \\ s\theta_j c\alpha_{j-1} & c\theta_j c\alpha_{j-1} & -s\alpha_{j-1} & -s\alpha_{j-1}d_j \\ s\theta_j s\alpha_{j-1} & c\theta_j s\alpha_{j-1} & c\alpha_{j-1} & c\alpha_{j-1}d_j \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Elementary rotation transformations (i.e. rotations about principal axis by θ):

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse of a 4x4 transformation matrix:

$$\boldsymbol{T}^{-1} = \begin{pmatrix} \boldsymbol{R} & \boldsymbol{t} \\ \boldsymbol{0}_{1\times 3} & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{t} \\ \boldsymbol{0}_{1\times 3} & 1 \end{pmatrix} \tag{2.21}$$

Derivation of trigonometric functions:

Dsinx = cosx

Dcosx = -sinx

Definition of (manipulator) Jacobian matrix:

If $\boldsymbol{y} = F(\boldsymbol{x})$ and $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^m$ then the Jacobian is the $m \times n$ matrix

$$J = \frac{\partial F}{\partial \boldsymbol{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

Jacobian transpose transforms a wrench applied at the end-effector, ${}^0\boldsymbol{g}$ to torques and forces experienced at the joints $\boldsymbol{Q}$:

$$\boldsymbol{Q} = {}^0\boldsymbol{J}(\boldsymbol{q})^T \, {}^0\boldsymbol{g}$$