Combinatorics

Next Topic:

Division Rule

Life is so much better if we ignore small differences.

Website : https://www.goclasses.in/

To count the number of cows in your field, first count the number of *legs* and then divide by four.

1 The Division Rule

The division rule is a common way to ignore “unimportant” differences when you are counting things. You can count distinct objects, and then use the division rule to “merge” the ones that are not significantly different.

Q.

A group of kids at a kids party, all leave their shoes at the door. How to count the kids?

Q.

A group of kids at a party all leave their shoes at the door. How to count the kids?

Ans = Count the shoes and divide by two.

the k-to-1 rule

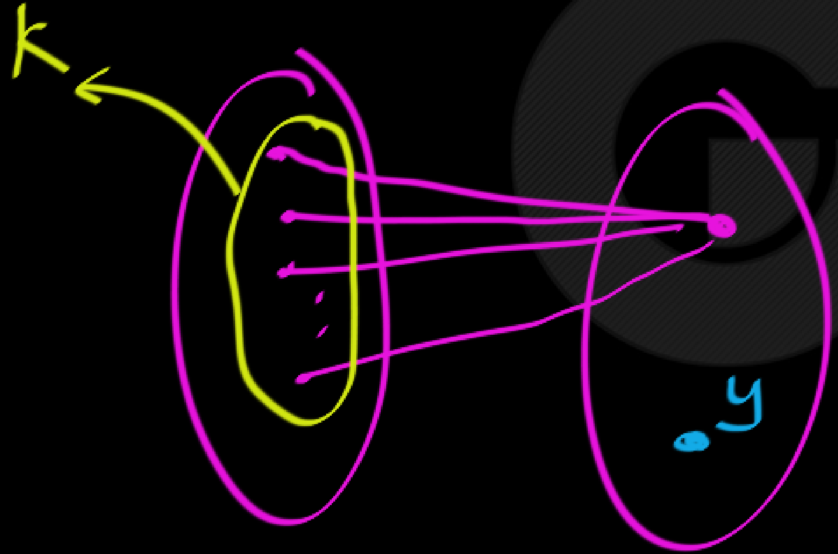
A group of kids at a slumber party all leave their shoes in a big pile at the door. How to count the kids? Count the shoes and divide by two.

This assumes a well defined function that maps each shoe to the kid who owns it. This is an example of a k-to-1 correspondence:

Let X and Y be finite sets. A function $f:X \to Y$ is a k-to-1 correspondence if for every $y \in Y$, there are exactly k different $x \in X$ such that $f(x) = y$.

k to 1 Correspondance: function

k to 1 AND onto



$\forall y \in$ Co-Domain

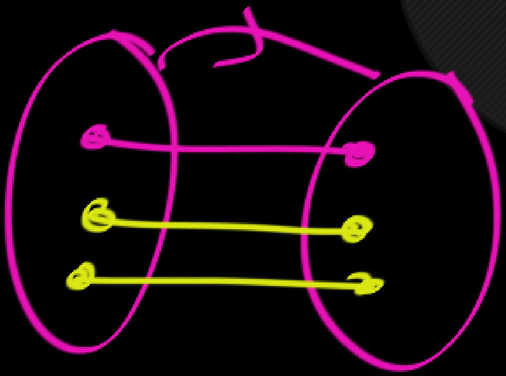
k-preimages of y

1 to 1 Correspondance :

function

1 to 1 AND onto

bijection



$\forall$ y $\in$ Co-Domain ;

has Exactly one pre-image

the k-to-1 rule

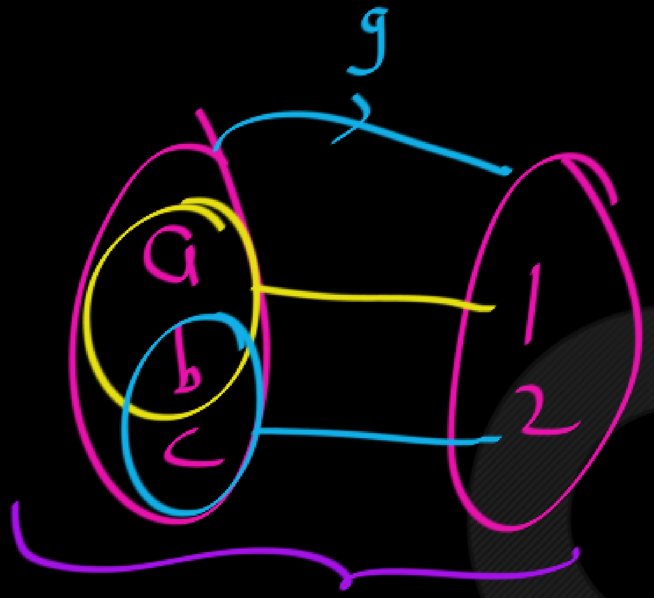
Let X and Y be finite sets. A function $f:X \to Y$ is a k-to-1 correspondence if for every $y \in Y$, there are exactly k different $x \in X$ such that $f(x) = y$.

The k-to-1 rule.

Suppose there is a k-to-1 correspondence from a finite set A to a finite set B. Then $|B| = |A|/k$

Division Rule: If there is a k-to-1 correspondence between of objects of type A with objects of type B, and there are $n(A)$ objects of type A, then there are $n(A)/k$ objects of type B.

A *k-to-1 correspondence* is an onto mapping in which every B object is the image of exactly k A objects.

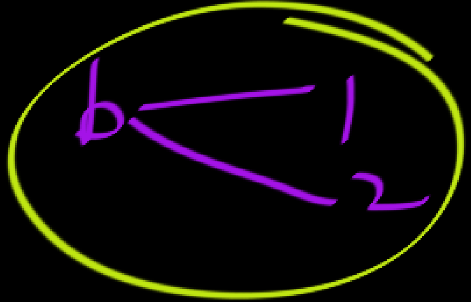


g : 2 to 1 Correspondance

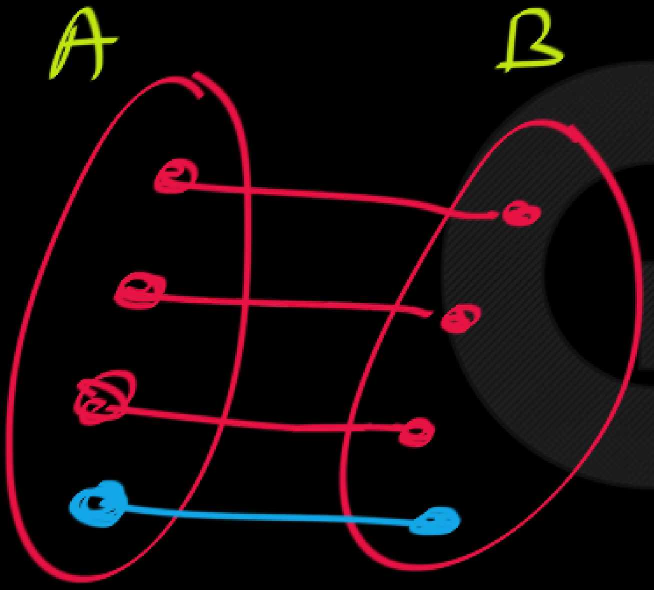
NOT Even a function

NOT 2-to-1 Correspondance

g is NOT a function

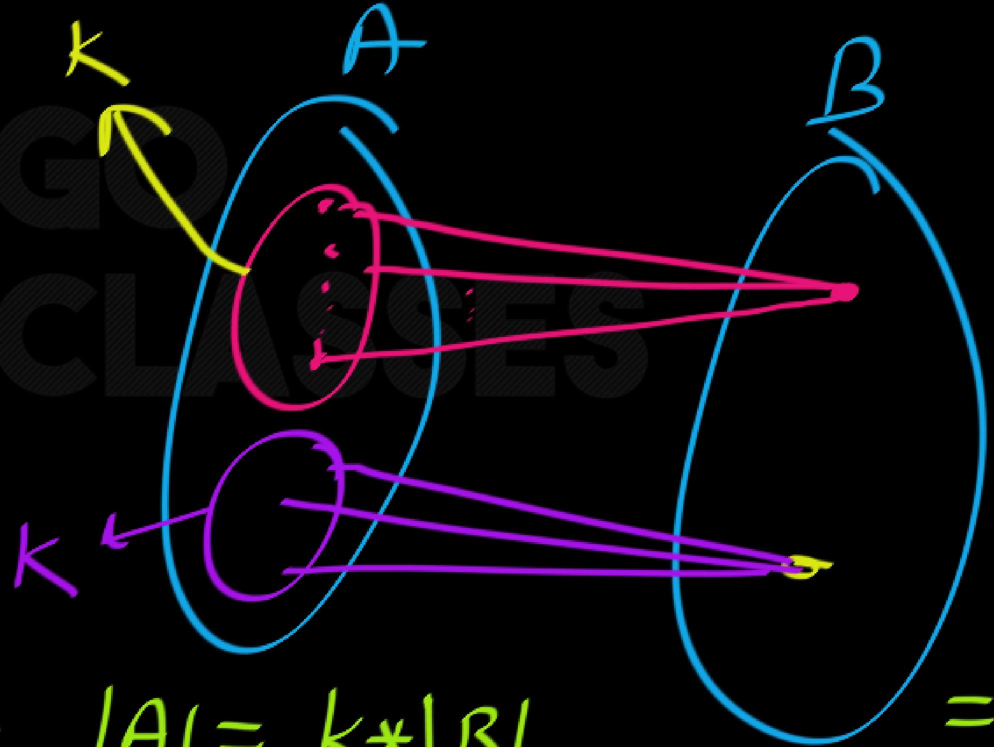


1-1 Correspondance:



$|A| = |B|$

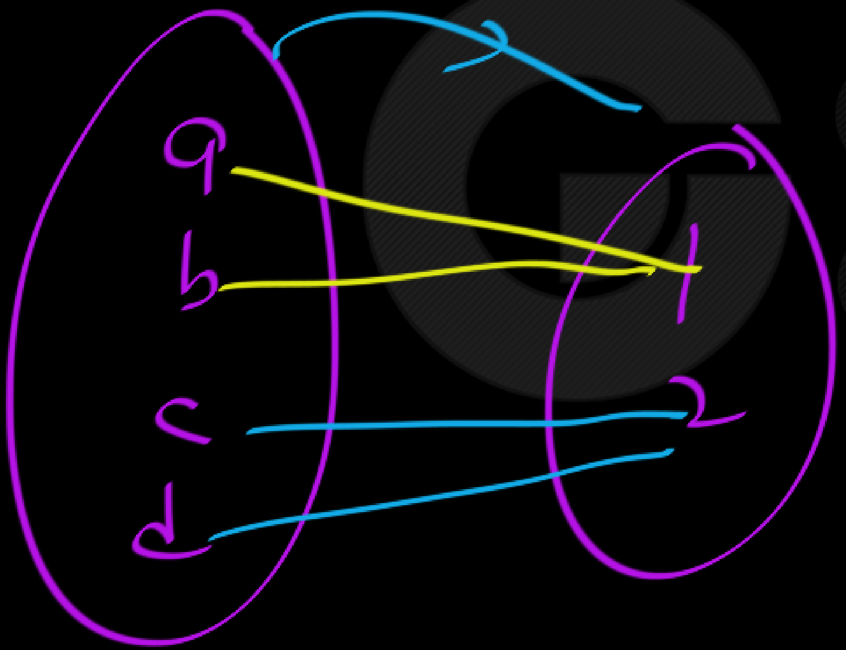
K-1 Correspondance

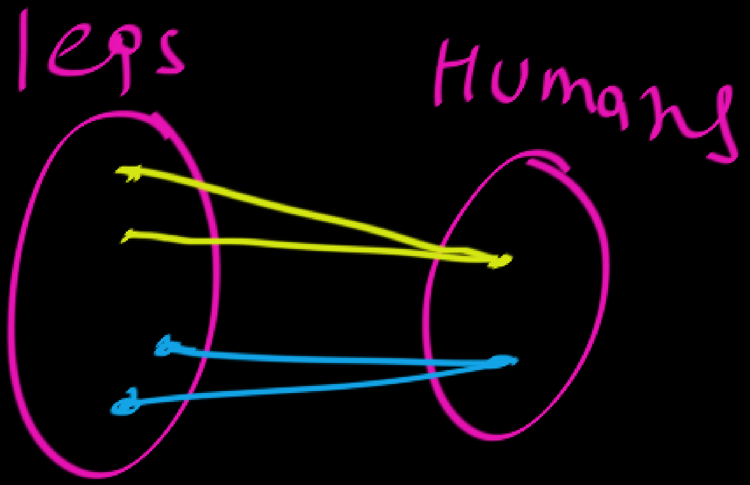


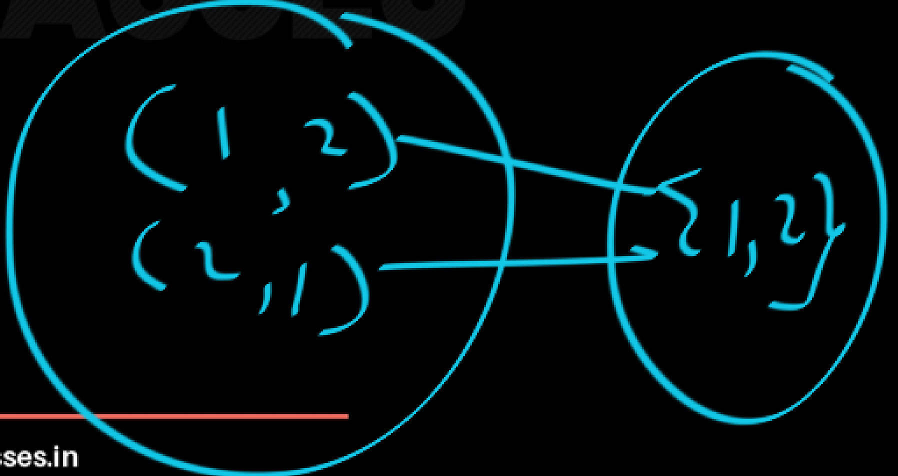
$|A| = k * |B|$

$|B| = |A|/k$

2 to 1 Correspondance:







For example, suppose A is a set of students, B is a set of tutorials, and f defines the assignment of students to tutorials. If 12 students are assigned to every tutorial, then the Division Rule says that there are 12 times as many students as tutorials.

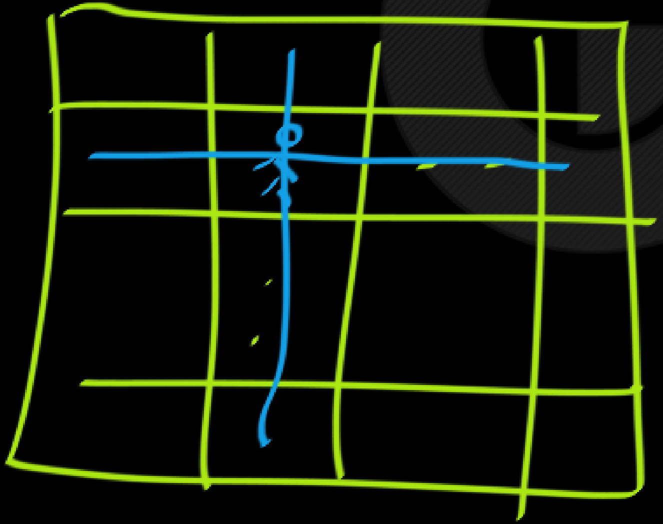
1 The division rule

[redacted]

elephant

In how many ways can we place two [redacted] rooks on an 8 by 8 chessboard so that they occupy different rows and different columns?

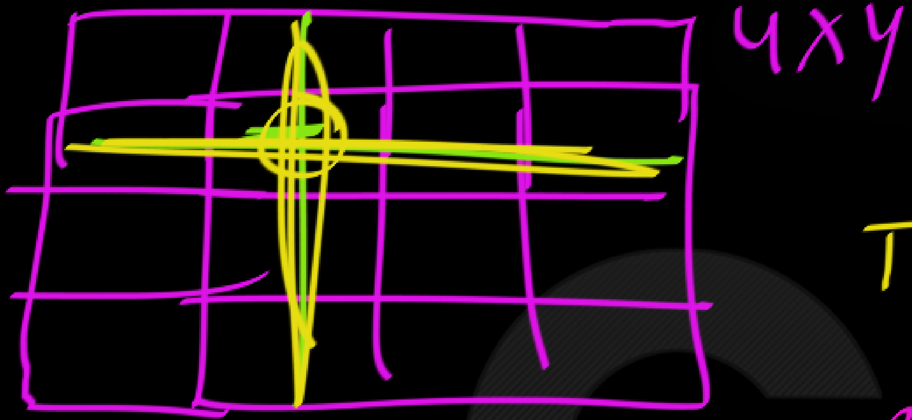
Different $\Rightarrow$ $\boxed{\text{Ans: } 64 \times 49}$



Black Rook AND White Rook

64 Choices

$(64-15) = 49$ Choices



Total Positions = $4 \times 4 = 16$

Remaining Positions = 9

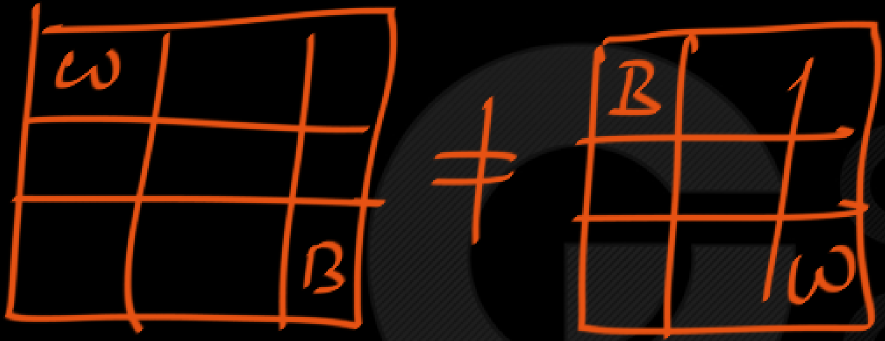
$= 16 - 7$

$= 9$

3x3

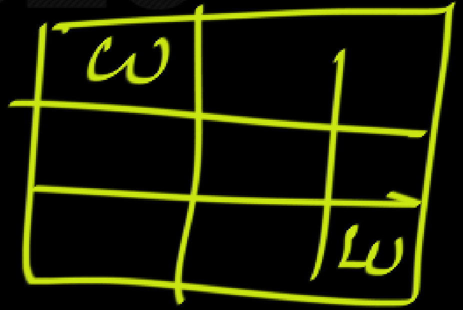
Chess board

Different Rooks



Identical Rooks

Same

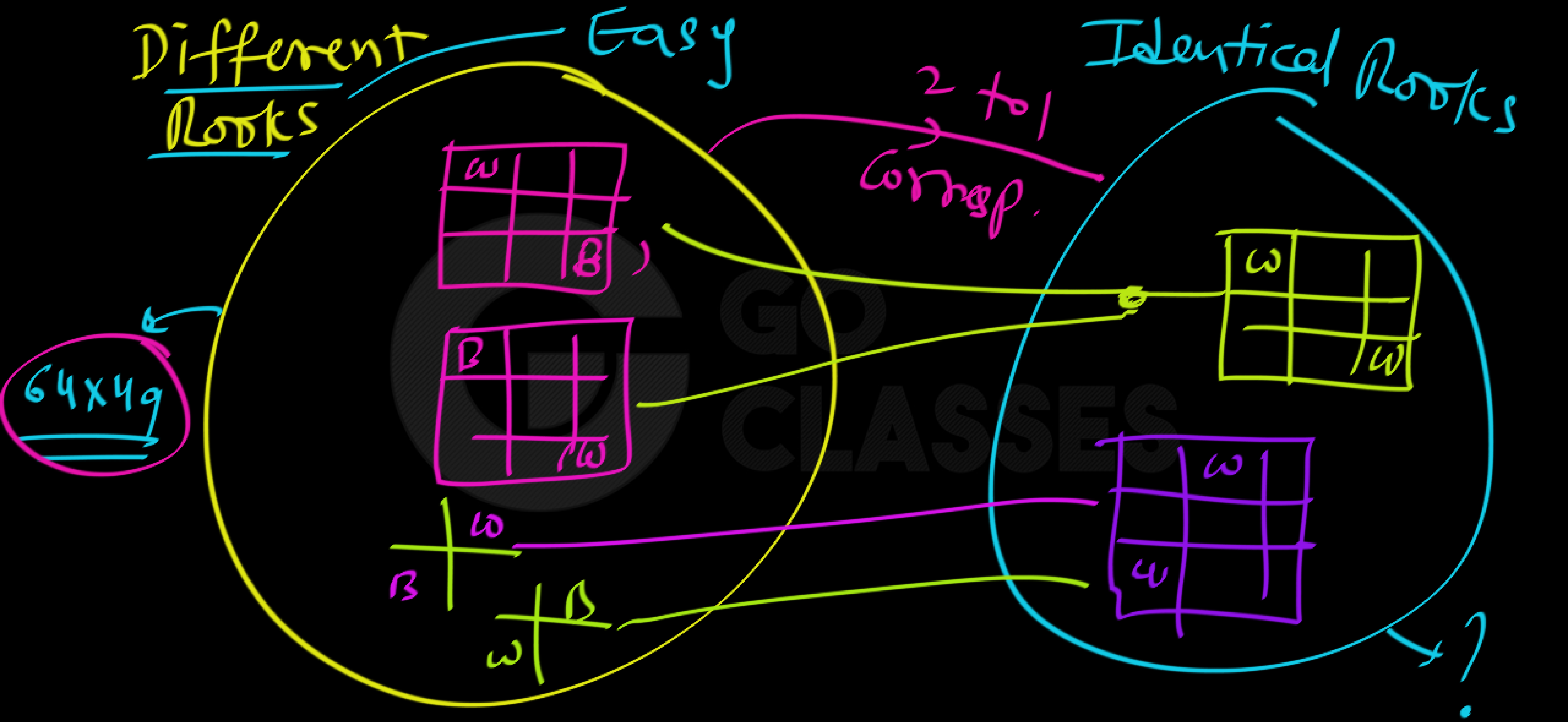


1 The division rule

same elephant

In how many ways can we place two *identical* rooks on an 8 by 8 chessboard so that they occupy different rows and different columns?

$$= \frac{64 \times 49}{2}$$


Different Rooks
Easy
Identical Rooks
2 to 1 corresp.
64x49
?

Q. How many   of exactly 2 elements are there for a set of n elements?

ordered Pairs

Different

(2,3), (3,2), (1,2) - - - - -

Ans:

(_ _)

n   n-1

Choices

= (n)(n-1)

Q. How many subsets of exactly 2 elements are there for a set of n elements?

2 to 1 Correspondance

Easy

$(n)(n-1)$

(1,2)
(2,1)
(3,4)
(4,3)
(2,3)
(3,2)

Actual

$\{1,2\}$
$\{3,4\}$
$\{2,3\}$
⋮

Ans

$\frac{n(n-1)}{2}$

Q. How many of exactly 3 elements are there for a set of n elements?

Ordered Triples

Different

$a \neq b \neq c \neq a$

$(a, b, c) = (n)(n-1)(n-2)$

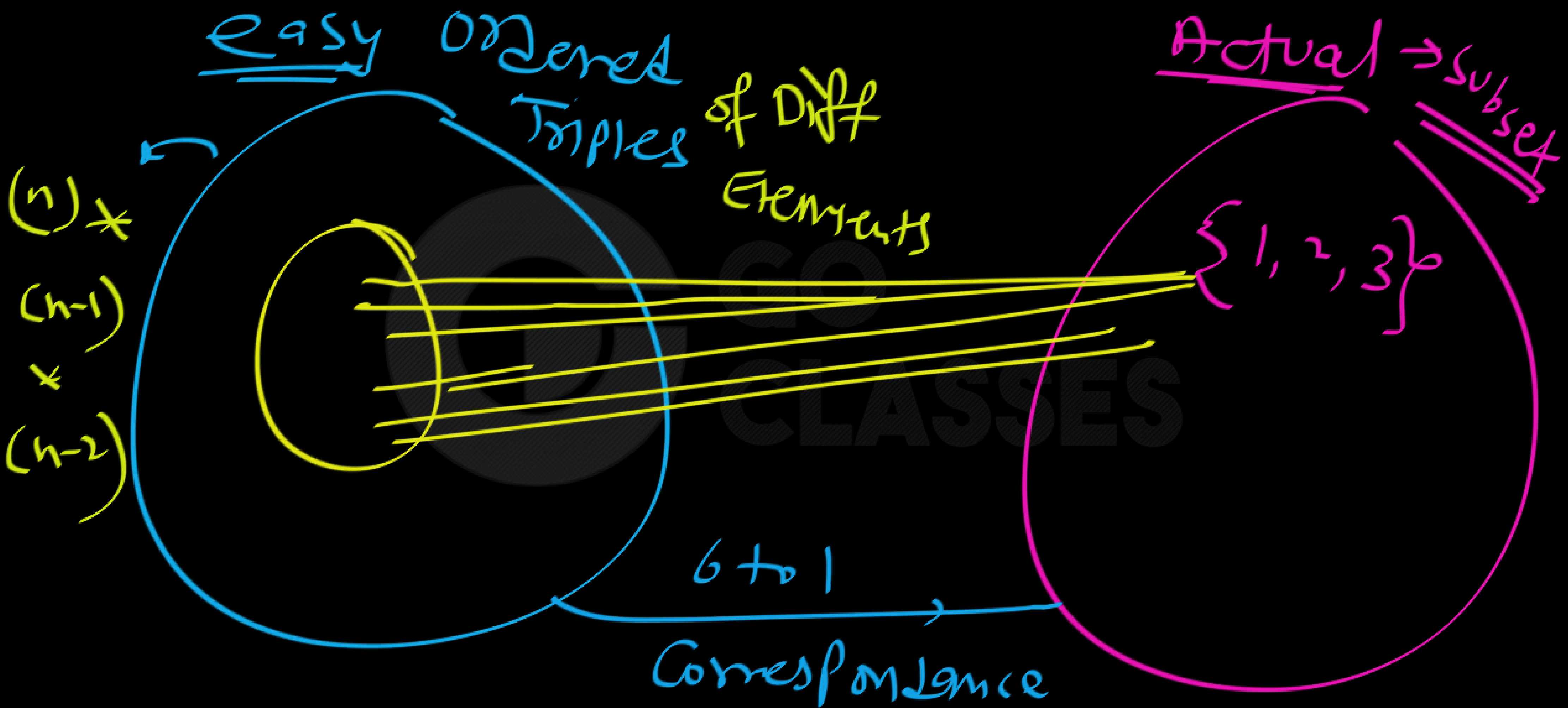
n $n-1$ $n-2$

choice

Q. How many subsets of exactly 3 elements are there for a set of n elements?

(1,2,3)
(1,3,2)
(2,1,3)
(2,3,1)
(3,1,2)
(3,2,1)

$\{1,2,3\}$


easy Ordered Triples of Diff Elements
(n) *
(n-1)
*
(n-2)
Actual → Subset
{1, 2, 3}
6 to 1
Correspondance

subsets of size 3 = $\dfrac{n(n-1)(n-2)}{6}$

Q. How many linear orders of 6 elements a,b,c,d,e,f ?

Q. How many linear orders of 6 elements a,b,c,d,e,f ?

d	a				
6	5	4	3	2	1

Choices

$$= \boxed{6 \times 5 \times 4 \times 3 \times 2 \times 1} = 6!$$

Q. How many linear orders of 6 elements a,b,c,d,e,f are there such that "a" comes before "b" (not necessarily immediately) ?

_ _ a _ _ _ _ b _ _ _ ✓

_ _ b _ _ _ a _ _ ✗

$\boxed{ab = \text{subsequence}}$

Easy → without a _ _ b Condition

With Condition

2-1 Correspondance

6!

a b c d e f
b a c d e f

a c d b e f
b c d a e f

a b c d e f

a _ _ _ b _ _

e f c a d b

$\frac{6!}{2}$

with Condition: $\boxed{\_\ \_\ a\ \_\ \_\ \_\ b\ \_\ \_\ \_}$

$\{a, b\}$

e f c _ d _

with Condition

e f c a d b

without Condition

e f c a d b

e f c b d a

Sub-sequence:

A subsequence of a string is a new string that is formed from the original string by deleting some(can be none) of the characters without disturbing the relative positions of the remaining characters. (i.e., "acd" is a subsequence of "abcde" while "aec" is not).

What is a subsequence?

ABCD

DC X

BDC X

CAB X

- A, B, C, D
- AB, AC, AD, BC, BD
- ABC, ABD, ACD
- ABCD

Subsequence

- "*cop*" is a subsequence of "computer"
- "*prom*" is a subsequence of "*programmer*"
- "*ping*" is a subsequence of "programming"
- *"sting" is a subsequence of "substring"*

Substring :— Consecutive Subsequence

$\omega = ABCD$

① AC — subsequence ✓ ; substring ✗

② BC — substring ✓

③ CB — substring ✗ ; subsequence ✗

GO CLAS

Classes

Sub-s

What is a sub strings ?

ABCD

- A, B, C, D
- AB, ~~AC~~, ~~AD~~, BC, ~~BD~~
- ABC, ~~ABD~~, ~~ACD~~, BCD
- ABCD , BC, CD

DC X

BDC X

CAB X

Q. How many linear orders of 6 elements a,b,c,d,e,f are there such that "a" comes before "b", and "b" comes before "c" (not necessarily immediately) ?

So, basically, we want a subsequence "abc" a, b, c, d, e, f

without Condition

e _ _ f d _ _

abc
subsequence

e a f d b c

→ 6 strings

a, b, c, d, e, f

#orders with $\underline{abc}$ subsequence

$$= \frac{6!}{3!}$$

<u>NOTE</u>: mostly, "Division Principle" is not used directly But in the form of "Combination" this is used.

Combinatorics

Next Topic:

Factorial, C, P

n! ; nPr ; nCr

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factorial (!)

n = Non-negative integer

n ∈ W

$$n! = \begin{cases} (n)(n-1)! \\ 0! = 1 \end{cases}$$

Definition

$$(n+1)! = (n+1)n!,$$

$$P(n,r) = \frac{n!}{(n-r)!} = n(n-1)\cdots(n-r+1)$$

$$C(n,r) = \frac{n!}{(n-r)!r!} = n(n-1)\cdots(n-r+1)/r!$$

$4! = 4(3!) = 4 * 3 * (2!) = 4 \times 3 \times 2 \times 1!$

$= \underline{4 \times 3 \times 2 \times 1} \times 0!$

$= \underline{\underline{24}}$

1

$5! = 5(4!) = 5 \times 24 = 120$

$3! = 3 \times 2 \times 1 = 6$

$1! = 1$

$2! = 2 \times 1 = 2$

$0! = 1$

$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

n = Non-Negative Integer

$n! = (n)(n-1)(n-2)(n-3) \cdots 3 \cdot 2 \cdot 1$

→ multiplication of integers from 1 to n.

$0! = 1$

$1! = 1$

$2! = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = (6)(5!) = 720$

$(n+1)! = (n+1)\,n!$

$$\boxed{\frac{n!}{(n-r)!} = {}^nP_r}$$

n, r = non-neg. integers

$r \le n$

Definition of ${}^nP_r$

$$\boxed{\frac{n!}{(n-r)!\,r!} = {}^nC_r}$$

Definition of ${}^nC_r$

$${}^{4}P_{2} = \frac{4!}{(4-2)!} = \frac{4!}{2!} = \frac{24}{2} = 12$$

$${}^{4}C_{2} = \frac{4!}{(4-2)!\,2!} = \frac{4!}{2!\,2!} = 6$$

$${}^{4}C_{0} = \frac{4!}{4!\,0!} = 1 \quad ; \quad {}^{4}P_{0} = \frac{4!}{4!} = 1$$

$$^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{^nP_r}{r!}$$

$$\boxed{^nP_r = r!\;{}^nC_r}$$

Notation:

$$^nC_r = \binom{n}{r} = C(n,r) = nCr = C^n_r$$

$$^nP_r = P(n,r) = P^n_r$$

Q:

$$^{10}C_2 = \frac{10!}{8!\,2!} = \frac{10 \times 9 \times \cancel{8!}}{\cancel{8!} \times 2!}$$

$$= 45$$

$$^{10}P_2 = 2!\,{}^{10}C_2 = 2(45) = 90$$

$$^6C_2 = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!}\; 2!} = 15 \; ; \; ^nC_1 = \frac{n!}{(n-1)!!!} = n$$

$$\boxed{^nC_1 = n} \; **$$

$$^nP_1 = 1! \; ^nC_1 = n$$

$$^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}\,2!}$$

$$\boxed{^nC_2 = \frac{n(n-1)}{2}}$$

① $^{n}C_2 = \underbrace{\dfrac{n(n-1)}{2}}$

② $^{n}C_1 = n$

$^{n}P_1 = n$

③ $^{n}C_n = 1$ ✓

$$\boxed{^{n}P_n = n! \; ^{n}C_n = n!}$$

$$\boxed{^{n}P_r = r! \; ^{n}C_r}$$

Combinatorics

Next Topic

Permutation & Combination

P(Permutation, Arrangement) , C(Combination, Selection)

Website : https://www.goclasses.in/

permute

/pəˈmjuːt/

verb **TECHNICAL**

submit to a process of alteration, rearrangement, or permutation.
"we wish to permute the order of the bytes"

permutation

/pəːmjʊˈteɪʃ(ə)n/

noun

each of several possible ways in which a set or number of things can be ordered or arranged.
"his thoughts raced ahead to fifty different permutations of what he must do"

Similar: arrangement | order | grouping | disposition | presentation | sorting

- **MATHEMATICS**
  the action of changing the arrangement, especially the linear order, of a set of items.

a, b, c

abc
acb

two Different Permutation

|||

order

Virat, Rohit, Pant

select 2 Players

V, P
P, V

— Same Selection

order NOT matter

V, R
R, P

a, b, c

abc

acb

two Different Permutation = Order

bca
bac
cab
cba

Virat, Rohit, Pant

select 2 Players for Cap, ViceCap

V, P
P, V — NOT Same
P, R — Different

Does "Order matter" ?

1. "My fruit salad is a combination of apples, grapes and bananas". — order Doesn't matter
2. "The combination to the safe is 472". — order matters
3. Password
4. Set — order Doesn't matter
5. Team, Committee, Group

"My fruit salad is a combination of apples, grapes and bananas“.

We don't care what order the fruits are in, they could also be "bananas, grapes and apples" or "grapes, apples and bananas", its the same fruit salad.

Sometimes in life Order of elements matter:

Eg: Arrangement

Eg:

1^{st}	2^{nd}	3^{rd}

Eg: English words

RAM $\neq$ ARM

Eg: Strings; sequence

Sometimes order of elements Doesn't matter:

1. selecting a team
2. Set unordered
3. meeting
4. Selection

Difference Between Permutation and Combination

Permutation = ordered Arrangement	**Combination** = Selection
Arranging people, digits, numbers, alphabets, letters, and colours	Selection of menu, food, clothes, subjects, team.
Picking a team captain, pitcher and shortstop from a group.	Picking three team members from a group.
Picking two favourite colours, in order, from a colour brochure.	Picking two colours from a colour brochure.
Picking first, second and third place winners.	Picking three winners.

When order of elements matters	When order of elements doesn't matter
Permutation / Arrangement	**Combination** / **Selection**
What matters?	What matters?
1. What elements	1. What elements (what set of elements)
2. What order	

a, b, c, d → I want 2

Permutation

a b
b a
Diff

b a
b c
Diff

Combination/selection

a b
a c
Diff

a c
c a
Same

Permutation: eg: string; sequence, ordered pairs, ordered tuples

Combination: Set; Committee, team, Group _ _

Combinatorics

Next Topic:

Permutation (A Fancy application of Product Rule)

P(Permutation, Arrangement, Order, Sequence, String)

Website : https://www.goclasses.in/

Permutations and Combinations

Permutations

Definition 1. *A permutation of a set of (distinct) objects is an ordering of the objects in row.*

Example 1. *In how many different orders can three runners finish a race if no ties are allowed?*

Solution. The different orders for elements a, b, and c are

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

There are six permutations. □

Q.

Number of permutations/orderings/arrangements of 4 distinct elements are there?

$\underbrace{a, b, c, d}$

c	a	d	b

$4 \times 3 \times 2 \times 1 = 4!$

Q.

Number of permutations/orderings/arrangements of 5 distinct elements are there?

$$5 \times 4 \times 3 \times 2 \times 1 = 5!$$

Q.

Number of permutations/orderings/arrangements of n distinct elements are there?

$n \times n-1 \times \; -- \times 1 = n!$

NOTE:

Number of permutations/orderings/arrangements of n distinct elements = n!

In general, given a set of n objects, how many permutations does the set have? We are free to select the first element, so we have n choices. Once we have chosen the first element, there are $n-1$ objects remaining, so we would have $n-1$ options for the second element. In general, the number of ways to perform each successive step is one less than the number of ways to perform the preceding step. This process continues until the last element, for which we have only one choice. By the product rule, we have the following.

Theorem 1. *For any integer n, with $n \geq 1$, the number of permutations of a set with n elements is $n!$.*

a, b, c, d, e, f $\Rightarrow$ How many orderings are possible such that f = 2nd position, e = 4th position

_ f _ e _ _

$4 \times 3 \times 2 \times 1$

$= 4! = 24$

4 $\Rightarrow$ orderings $= 4!$

$= 24$

Q.

Number of permutations/orderings/arrangements of 2 distinct elements from a set of 5 elements are there?

Q.

Number of permutations/orderings/arrangements of 2 distinct elements from a set of 5 elements are there?

$\underbrace{a, b, c, d, e}$

$5 \times 4 = 20$

$$\boxed{{}^5P_2 = 20}$$

Note: $${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

$$= \frac{(n)(n-1)(n-2)\cdots(n-r+1)\cancel{(n-r)!}}{\cancel{(n-r)!}}$$

$${}^{n}P_{r} = (n)(n-1)(n-2)\cdots(n-r+1)$$

$${}^{n}P_{r} = \underbrace{(n)(n-1)(n-2)\,---\,(n-r+1)}_{\#\text{terms} = r}$$

$${}^{7}P_{4} = (7)(6)(5)(4)$$

$${}^{7}P_{2} = (7)(6) = 42$$

$${}^{10}P_{3} = (10)(9)(8)$$

$$= 720$$

$$ {}^{n}P_{r} = \underbrace{(n)(n-1)(n-2)(n-3)\cdots(n-r+1)}_{r \text{ terms}} $$

$$ {}^{20}P_{4} = 20 \times 19 \times 18 \times 17 $$

Q.

Number of permutations/orderings/arrangements of r **distinct** elements from a set of **n elements** are there?

1 2 3 — — — r

$$(n-0)\times(n-1)\times(n-2)\times\cdots\times(n-r+1) = {}^{n}P_{r}$$

$$n-(r-1) = n-r+1$$

Note: n Distinct elements

Permute

$r \leq n$

r Distinct elements

ways $= {}^nP_r$


Product Rule
Permutation

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

Permutations

A *permutation* of n things taken r at a time, written $P(n, r)$, is an arrangement in a row of r things, taken from a set of n distinct things. Order matters.

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

abc
bac } Different Photos

$5 \longrightarrow 3$

Permute

$^5P_3 = \underline{5 \times 4 \times 3} = 60$

$5 \times 4 \times 3$

$5!$

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

100 Permute 3 $= {}^{100}P_3 = \underbrace{100 \times 99 \times 98}$

1st	2nd	3rd

$100 \times 99 \times 98$

$$^{n}P_{r} = \underbrace{(n)(n-1)(n-2)\cdots(n-r+1)}_{r \text{ terms}}$$

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

$$^{8}P_{3} = \underbrace{8 \times 7 \times 6}$$

Suppose that a saleswoman has to visit eight different cities. She must begin her trip in a specified city, but she can visit the other seven cities in any order she wishes. How many possible orders can the saleswoman use when visiting these cities?

7!

Combinatorics

Next Topic:

Combination (A Fancy application of Division Rule)

C(Combination, Selection, Unordered, Set, Committee, Team)

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Q.

Number of ways to select 3 people from a class of 5 people.

Q.

Number of ways to order 3 people from a class of 5 people.

life is Good.

$5 \times 4 \times 3$

$= {}^5P_3 = 5 \times 4 \times 3 = 60$

Q.

Number of ways to select 3 people from a class of 5 people.

a, b, c, d, e

order

Permutations → 60

abc
acb
bac
bca
cab
cba

a, b, c

Combination / selection

$\frac{60}{6} = 10$

Permutation

Combination

Product Rule

Division Rule

Q.

Number of ways to select r people from a set of n people.

Permutation

r! to 1 Correspondance

Combination

^nP_r

$r!$

$a_1, a_2, \ldots, a_r$

$r!$

$a_2, a_3, \ldots, a_{r+1}$

$\frac{^nP_r}{r!}$

$= {^nC_r}$


Division Rule
Combination


Division Rule
Combination
Permutation

Combinations

Now we want to count *unordered* selection of objects.

Suppose we want to choose r elements from a set with n elements, in no particular order, that is, we want to select a subset with r elements from a set with n elements. In how many ways can we do so?

Definition 3. *Let n and r be integers with $0 \leq r \leq n$. The symbol*

$$\binom{n}{r}$$

is read "n choose r" and represents the number of subsets of size r that can be chosen from a set with n elements.

Remark. *There are several notations for an r-combination from a set of n distinct elements: $C(n, r)$, nCr (n, choose r), and $\binom{n}{r}$, the binomial coefficient, which is the topic of the next section.*

Source : Berkeley

Note: n Distinct elements

Choose/select/Combination

r elements

#ways

$= {}^{n}C_{r}$

Theorem 3. *The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \le r \le n$, equals*

$$C(n,r) = nCr = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Proof. We relate r-combinations to r-permutations. The $P(n,r)$ r-permutations of the set can be obtained by forming the $C(n,r)$ r-combinations of the set, and then ordering the elements in each r-combination, which can be done in $P(r,r)$ ways. Consequently, by the product rule,

$$P(n,r) = C(n,r) \cdot P(r,r).$$

Therefore

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{(n-r)!} \div r! = \frac{n!}{(n-r)!} \cdot \frac{1}{r!} = \frac{n!}{r!(n-r)!}. \qquad \square$$

$$^nC_r = n \underline{\text{Choose}}\ r$$

$$^nP_r = n \underline{\text{Permute}}\ r$$

How many different committees of three students can be formed from a group of four students?

Solution: To answer this question, we need only find the number of subsets with three elements from the set containing the four students. We see that there are four such subsets, one for each of the four students, because choosing three students is the same as choosing one of the three students to leave out of the group. This means that there are four ways to choose the three students for the committee, where the order in which these students are chosen does not matter.

$$^4C_3 = \frac{^4P_3}{3!} = \frac{4 \times 3 \times 2}{6} = 4$$

Note:

$${}^{n}P_{r} = \underbrace{(n)(n-1)(n-2)\cdots(n-r+1)}_{r \text{ terms}}$$

$${}^{n}C_{r} = \frac{\underbrace{(n)(n-1)\cdots(n-r+1)}_{r \text{ terms}}}{r!}$$

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Card hand : Order Doesn't matter.

$^{52}C_5 = {}^{52}C_{47}$

52 Cards

Select 5 Cards to keep = selecting 47 Cards to reject

$^{52}C_5$

$^{52}C_{47}$

$^{n}\text{Choose}_{r} = {}^{n}\text{Choose}_{n-r}$

$$\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$$

$$^{n}C_{r} = \frac{(n)(n-1)\cdots(n-r+1)}{r!}$$

$$^{100}C_{98} = \frac{(100)(99)(98)\cdots(3)}{98!}$$

$$|||$$

$$^{100}C_{2} = \frac{(100)(99)}{2!}$$

Combinatorics

Next Sub-Topic:

Why 0! = 1 ? Why nC(n+1) = 0?

Website : https://www.goclasses.in/

Q:

Why $0! = 1$?

Q:
Why
$0! = 1$?

CORRECT Answer: By Convention. By Definition.

So, the right question that should be asked is "Why this convention"?

Q.

Number of subsets of size 0 of a set of size n?

1

ϕ

Q.

select r

Number of subsets of size r of a set of size n?

$= {}^nC_r$

So, ${}^nC_0 = 1 \Rightarrow \frac{\cancel{n!}}{\cancel{n!}\,0!} = 1 \Rightarrow \boxed{0! = 1}$

Q.

Number of subsets of size n+1 of a set of size n?

$= 0$

So, ${}^{n}C_{n+1} = 0$

Q.

Number of subsets of size r (>n) of a set of size n?

$= 0$

so, ${}^{n}C_{r} = 0$

$r > 0$