Combinatorics

Lecture 1

Basic Counting Principles

Sum, Product Rules

Website : https://www.goclasses.in/

Combinatorics

Problem: How to count without counting.

- How do you figure out how many things there are with a certain property without actually enumerating all of them.

Sometimes this requires a lot of cleverness and deep mathematical insights.

But there are some standard techniques.

- That's what we'll be studying.

Eg:

abc   6 Arrangements

abc , bac , cab
acb , bca , cba

abcd — 24 Arrangements

abcd - - -
abdc - - -
adbc - - -

24 Enumerations

2 Inclusion-Exclusion-Principle

3 Generating Functions

Recurrence Relations

Combinatorics

Next Topic

Sum Rule

Union of disjoint sets

(1) Set $A = \{a, b\}$ $|A| = 2$; $A \cup B = \{a, b, 1, 2\}$

Set $B = \{1, 2\}$ $|B| = 2$

$|A \cup B| = |A| + |B| = 2 + 2 = 4$

A, B are Disjoint ✓

$\boxed{A \cap B = \phi}$

Note: Set A, B ;

If A, B are Disjoint (nothing Common in A, B)

then

$$\boxed{|A \cup B| = |A| + |B|}$$

number

$$\#(A \cup B) = \#A + \#B$$

$$n(A \cup B) = n(A) + n(B)$$

same

Note:** If set A, set B are NOT Disjoint:

$$n(A \cup B) \neq n(A) + n(B)$$

Eg: $A = \{a, b\}$ $\quad A \cup B = \{a, b, c\}$

$B = \{a, c\}$ $\quad n(A \cup B) = 3 \neq \boxed{2+2}$

overcounting

(2) IISc

Students: a, b, c

faculties: x, y

Task T: choose one person: $3+2$

Task T_1: Student → 3 ways → a, b, c

OR → +

Task T_2: faculty → 2 ways → x, y

Sum Rule:

Task T $\rightarrow$ $\boxed{n_1 + n_2 + n_3 \text{ ways}}$

Task T_1 ① OR Task T_2 ① OR Task T_3

n_1 ways

n_2 ways

n_3 ways

① ②

② Mutually Disjoint

The Sum Rule: If there are $n(A)$ ways to do A and, distinct from them, $n(B)$ ways to do B, then the number of ways to do A *or* B is $n(A) + n(B)$.

- This rule generalizes: there are $n(A) + n(B) + n(C)$ ways to do A *or* B *or* C

Sum Rule :

If a first task can be performed in m ways, while a second task can be performed in n ways, and the two tasks cannot be performed simultaneously, then performing either task can be done in m+n ways.

Or

If you have m objects in one group, and n objects in a second group, which is completely separate from the first, (meaning that they share no common objects), then the total number of objects in both groups is m+n.

Or

Here is a set definition: The cardinality of the union of two disjoint sets is the sum of each set's cardinality.

Example 1: Suppose that you are in a restaurant, and are going to have either soup or salad but not both. There are two soups and four salads on the menu. How many choices do you have? By the Sum Rule, you have $2 + 4 = 6$ choices.

$\boxed{\text{Order 1 item}} \Rightarrow 2+4$

1

Order Soup OR Order Salad

2 ways 4 ways

2

Disjoint

Q: Haryana: {Chandigarh, Rohtak}

Punjab: { " , ludhiyana}

Task: Choose 1 city: $\Rightarrow 2+2 = 4$ ways?

H OR P

2 ways ⟷ Not Disjoint ⟵ 2 ways

wrong

Example. Let S be the set of students attending the combinatorics lecture. It can be partitioned into parts S_1 and S_2 where

$$S_1 = \text{set of students that } \textit{like} \text{ easy examples.}$$
$$S_2 = \text{set of students that } \textit{don't like} \text{ easy examples.}$$

If $|S_1| = 22$ and $|S_2| = 8$ then we can conclude $|S| = 30$.

Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?

Fac. OR Stu.

37 ways

83 ways

Disjoint

$37+83$

A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

→ Disjoints Lists.

IISc CSA: Theory System Int. Systems

Pool A: Algo, DM, Graph Theory — 23 ways

OR

Pool B: OS, CD — 15

OR

Pool C: ML, DS, Game theory — 19

Selective one project: 23 + 15 + 19 Choices

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

$k := 0$
for $i_1 := 1$ **to** n_1
    $k := k + 1$
for $i_2 := 1$ **to** n_2
    $k := k + 1$
        .
        .
        .
for $i_m := 1$ **to** n_m
    $k := k + 1$

What is the value of k after the following code, where $n_1, n_2, \ldots, n_m$ are positive integers, has been executed?

Every iteration of loop is increasing the value of k by 1.

```
k := 0
for i_1 := 1 to n_1
    k := k + 1
for i_2 := 1 to n_2
    k := k + 1
      .
      .
      .
for i_m := 1 to n_m
    k := k + 1
```

$k = n_1$

$k = n_1 + n_2$

$k = n_1 + n_2 + n_3$

$k = \boxed{n_1 + n_2 + n_3 + \cdots + n_m}$

$k = 0$

for ($\underline{i = 1 \text{ to } n_1}$)

$\boxed{k = k + 1}$

increasing the value of k by 1 in C Prog.

$\underbrace{i = 1}$ $\downarrow$ $k = 1$

$i = 2$ $\downarrow$ $k = 2$

$i = 3$ $\downarrow$ $k = 3$

$\ldots$

$i = n_1$ $\downarrow$ $k = n_1$

If we roll two dice, one green and one purple, how many ways are there to get a sum of 7 or 11?

Different Dice

If we roll two dice, one green and one purple, how many ways are there to get a sum of 7 or 11?

Task: 7 or 11 ⟹ 6 + 2 = 8 ways

7 → 6 ways

OR

11 → 2 ways

Disjoint

G	P
1	6
2	5
3	4
4	3
5	2
6	1

G	P
6	5
5	6

If we roll two dice, one green and one purple, how many ways are there to get a sum of 7 or 11?

First let's look at a sum of 7:

$$(1, 6)(6, 1)(2, 5)(5, 2)(3, 4)(4, 3)$$

And now a sum of 11:

$$(5, 6)(6, 5)$$

So since there are 6 ways to get a sum of 7, and 2 ways get a sum of 11, we have $6 + 2 = 8$ ways to get a sum of 7 or 11.

THE RULE OF SUM

If task A can be performed in m ways, and task B can be performed in n ways, and the two tasks cannot be performed simultaneously (disjoint), then tasks A **OR** B can be performed in $m + n$ ways.

Combinatorics

Next Topic

Product Rule

Cross Product of two sets

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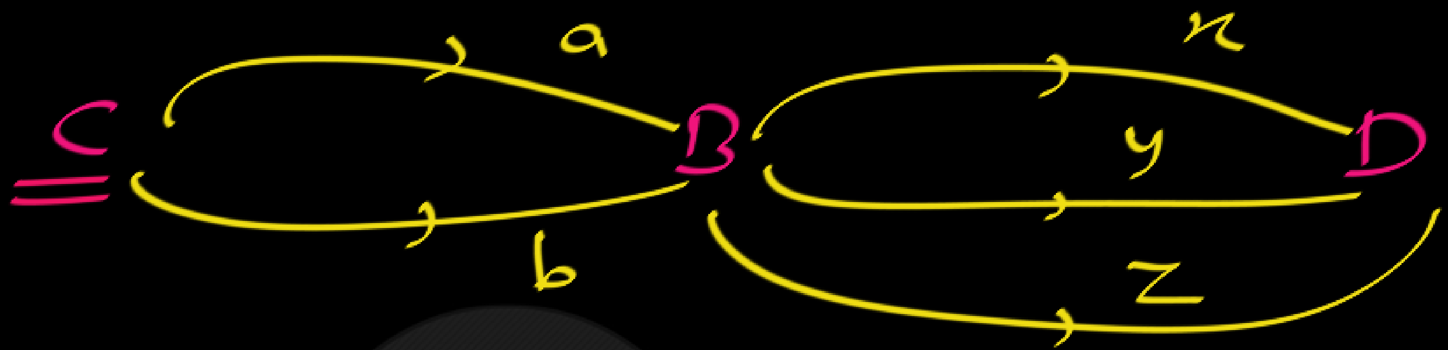
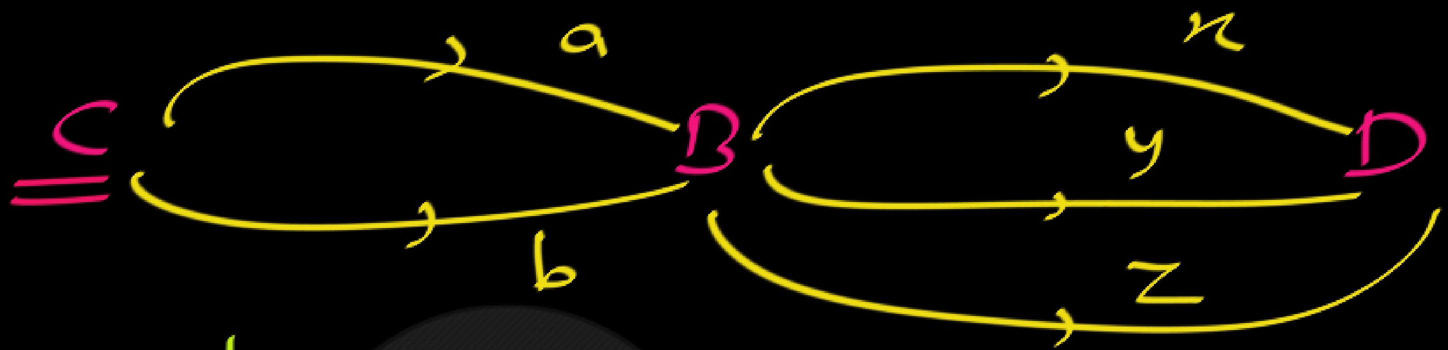


ways $\boxed{C \longrightarrow D \text{ via } B}$ Task T $\Rightarrow$ 2×3 ways.

$C \longrightarrow B$ AND $B \longrightarrow D$

2 ways

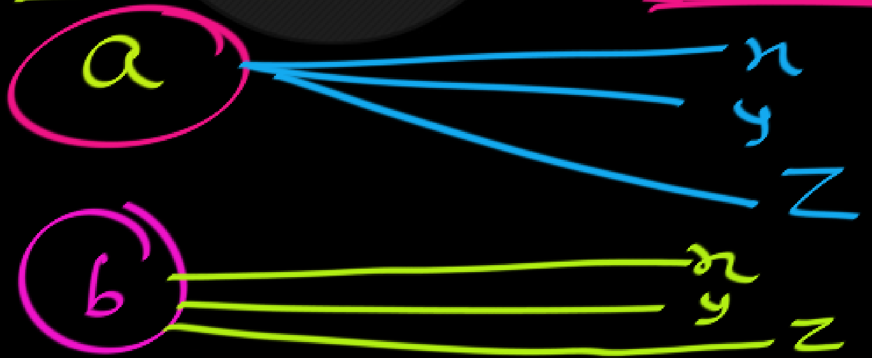
3 ways



ways $C \longrightarrow D$ via B :

$C \longrightarrow B$

$B \longrightarrow D$



Total ways $= 6 = 2 \times 3$

Note 1: Task T_1 Task 2

m ways

way w_1 — n_1 ways

way w_2 → n_1 ways

⋮

way w_m → n_1 ways

Task T_1

AND

Task T_2

#ways = $m \times n_1$ ways

Note 2: Task T_1 — Task 2

4 ways:

- way w_1 —— n_1 ways
- way w_2 $\rightarrow$ n_1 ways
- way w_3 $\rightarrow$ n_1 ways
- way w_4 $\rightarrow$ n_2 ways

Task T_1

AND

Task T_2

||

#ways =

$4n_1$

wrong.

Correct: #ways $= (3n_1) + n_2$

Note:

Product Rule

Task T

Task T_1 AND Task T_2

n ways — m ways

#ways for T = $n \times m$

Imp:

After T_1 is Done in Any way, there are m ways for T_2.

The Product Rule

Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 \cdot n_2$ ways to do the procedure.

THE PRODUCT RULE Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

Q: 4 letter English word; How many?

Task T

T_1 and T_2 and T_3 and T_4

a	t	a	b

26 ways × 26 ways × 26 ways × 26 ways $= (26)^4$

room
moon
babu

Q: 4 letter English word; How many without Repetition?

T_1 and T_2 and T_3 and T_4

t	a	z	b

25 ways × 26 ways × 24 ways × 23 ways

T_2
T_1
T_3
T_4

Q: 4 letter English word; How many without Repetition;

T_1	T_2	T_3	T_4
s	u	a	p
26 ways	25 ways	23 ways	24 ways

26 ways × 25 ways × 23 ways × 24 ways

T_1
T_2
T_4
T_3

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

S AND P

12 ways

D_3 office

11 ways

$\boxed{12 \times 11}$ ways

find mistake:

D_1 D_2 --- D_n

S — 2

or

P — 2

or

None — 3

wrong

Product Rule :

If a procedure can be broken down into first and second stages, and if there are m possible outcomes for the first stage, and after first stage is done/over, n possible outcomes for the second stage, then the total procedure can be carried out in mn ways.

Here's a definition using sets: The cardinality of the cartesian product of two sets is the product of each set's cardinality.

Note:

by Default Repetition

word	Yes Allowed (moon, root, cat)
4 Digit number	Allowed (4044, 4325)
Set	NOT Allowed

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1a_2...a_k$ of symbols from A, with repetition allowed.

≡ Distinct ≡ Different

(a) How many distinguishable strings of length 3 over the alphabet $\{A, B, \ldots, Z\}$ exist?

String = word

b	b	a

$26 \times 26 \times 26 = (26)^3$

Note:

whatever

26 ways

26 ways

26 ways

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1a_2...a_k$ of symbols from A, with repetition allowed.

(a) How many distinguishable strings of length 3 over the alphabet $\{A, B, \ldots, Z\}$ exist?

Solution: Since there are 26 choices for the first symbol, 26 for the second, and 26 for the third, then by the Product Rule there are $26^3 = 17576$ such strings of length 3.

Distinguishable strings with repetitions allowed:

Permutations of a subset (no repetitions):

If you are choosing 3 different symbols from 26 and arranging them, then you are considering permutations where the order matters and no symbol repeats. The number of such permutations is given by 26×25×24. This formula counts the arrangements of 3 symbols out of 26 where each symbol is used exactly once.

Distinguishable strings with repetitions allowed:
For distinguishable strings where symbols can be repeated, each of the 3 positions in the string can be occupied by any of the 26 symbols independently.
Thus, you have 26 choices for each of the 3 positions, leading to a total of 26^3 possible strings.

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1 a_2 \ldots a_k$ of symbols from A, with repetition allowed.

word

(b) How many strings of length 4 over the alphabet $\{0, 1, \ldots, 9\}$ do not begin with 0?

1×0

0

0 0 0

Whatever

9 ways

10 ways

10

10

Ans:

$9 \times 10 \times 10 \times 10$

Q: NOT Start with 0; No Repetition:

No 0 — | 8 | 0 | 0 | 0 |

9 ways × 9 ways × 8 ways × 7 ways

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1a_2...a_k$ of symbols from A, with repetition allowed.

(b) How many strings of length 4 over the alphabet $\{0, 1, \ldots, 9\}$ do not begin with 0?

Solution: There are nine choices for the first symbol, and ten for each of the second, third, and fourth. By the Product Rule, there are $9 \cdot 10^3 = 9000$ strings of length 4 that do not begin with 0. (Note: this is also the number of 4-digit positive integers with no leading zeros.)

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1a_2...a_k$ of symbols from A, with repetition allowed.

(c) The standard California license-plate number begins with a nonzero decimal digit that is followed by three uppercase alpha characters, which are in turn followed by three decimal digits. How many of these license numbers exist?

English Alphabet

Non Zero Digit

8	B	T	M			

$9 \times 26 \times 26 \times 26 \times 10 \times 10 \times 10$

ways

8	B	T	m	0	2	3

Non Zero Digit

$9 \times 26 \times 25 \times 24 \times 9 \times 8 \times 7$

9 ways

without Any Repetion

Example 2: The first two of these problems apply the generalized Product Rule. We begin with a couple of definitions: An *alphabet* is a finite set of symbols. A *string* of length k over an alphabet A is a finite sequence $a_1a_2...a_k$ of symbols from A, with repetition allowed.

(c) The standard California license-plate number begins with a nonzero decimal digit that is followed by three uppercase alpha characters, which are in turn followed by three decimal digits. How many of these license numbers exist?

Solution: By the preceding problems and the basic Product Rule, there are

$$9 \cdot 26^3 \cdot 10^3 = 158184000$$

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Chair:

| E | 98 |

26 ways

100 ways

(1 to 100)

26×100

A1	A2	A3	-	-	A100
B1	B2	B3	-	-	B100
⋮					
Z1	Z2	-	-	-	Z100

#Chairs: 26×100

There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?

32 Computers

c_1 — 24 Ports

c_2 — 24 "

⋮

c_{32} — 24 "

Total Ports = 32×24 Ports

Note: bit string — string of 0, 1

word

bit : 0, 1

Digit — 0, - - , 9

000 length = 3

00 ≠ 000

length 2

length 3

How many different bit strings of length seven are there?

0	1	0	1	0	1	1

2 ways

2 choices

2 2 2 2 2

2^7

How many different bit strings of length seven are there? without Repetition

0	1		1	1	1	1

?

0.

How many different bit strings of length seven are there?

Solution: Each of the seven bits can be chosen in two ways, because each bit is either 0 or 1. Therefore, the product rule shows there are a total of $2^7 = 128$ different bit strings of length seven. ◀

How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits (and no sequences of letters are prohibited, even if they are obscene)?

$26 \times 26 \times 26 \times 10 \times 10 \times 10$

Q :

A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this? (in How many orders)

1 2 - - - 50

c_{13} 1

50 choices

49 choices

1

$$\boxed{50 \times 49 \times 48 \times \cdots \times 3 \times 2 \times 1} = 50!$$

will study later

A traveling salesman wants to do a tour of all 50 state capitals. How many ways can he do this?

Ans :

50 choices for the first place to visit, 49 for the second, . . . , 1 for the last.

50*49*48*...*3*2*1

- Two basic very useful decomposition rules:

  1. Product rule: useful when task decomposes into a sequence of independent tasks

  2. Sum rule: decomposes task into a set of alternatives

Combinatorics

Next Topic

Subtraction Rule

Inclusion Exclusion of two sets

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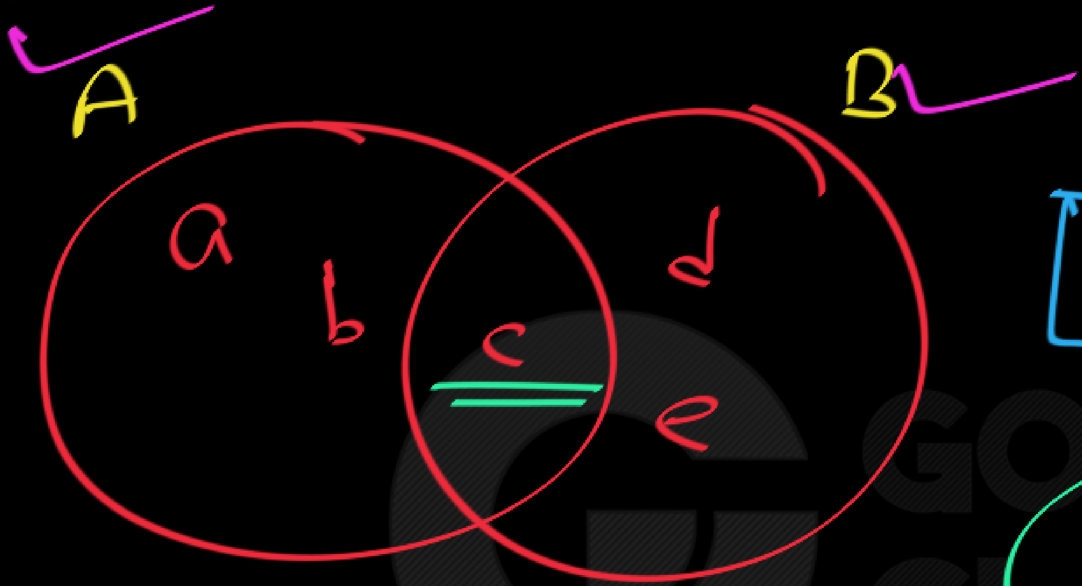
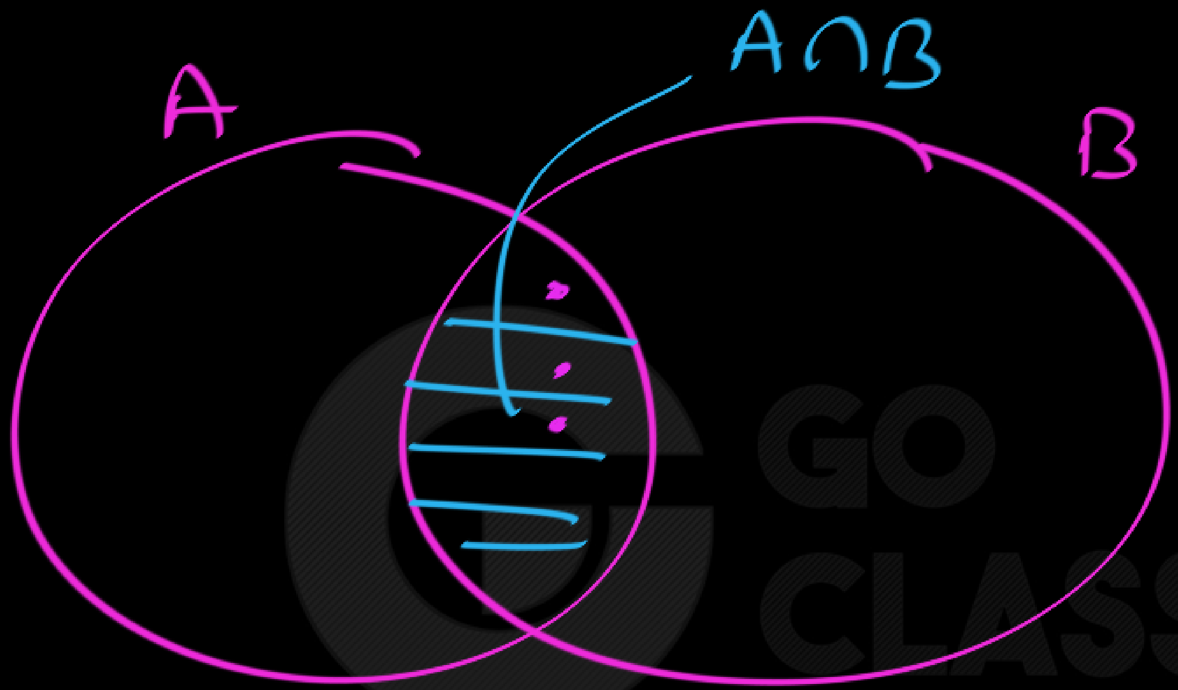


over counting

"c" Countes two times

$$n(A \cup B) = \boxed{n(A) + n(B)} - n(A \cap B)$$

$$3 + 3 - 1 = 5$$



Exclusion

inclusion

$$n(A \cup B) = \boxed{n(A) + n(B)} - n(A \cap B)$$

THE SUBTRACTION RULE If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

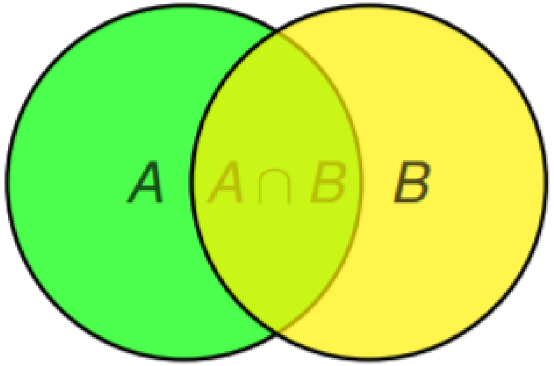
Subtraction Rule (Inclusion-Exclusion for two sets)

Subtraction Rule

For any finite sets A and B (not necessarily disjoint),

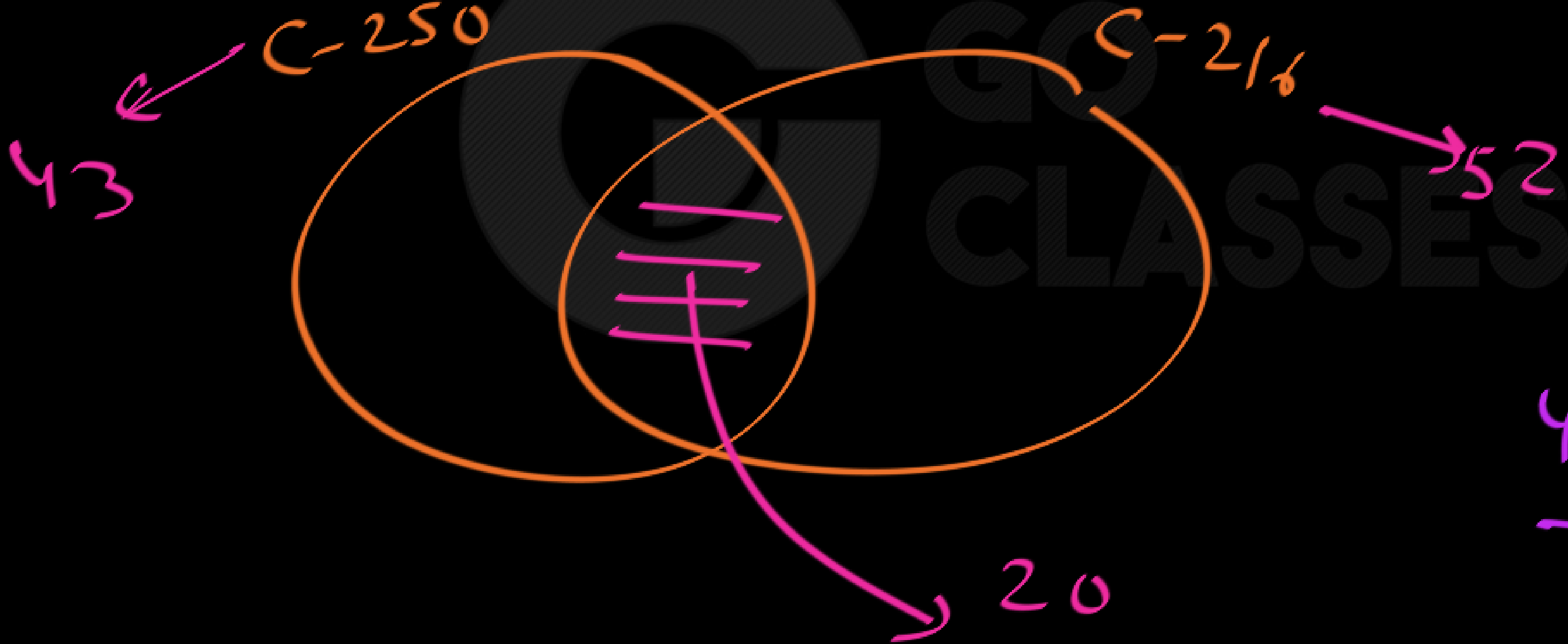
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof: Venn Diagram:



|A| + |B| overcounts (twice) exactly those elements in $A \cap B$. □

In a class of undergraduate Comp Sci students, 43 have taken C-250, 52 have taken C-216, while 20 have taken both. No student has taken any other courses. How many students are there?



$43 + 52 - 20$

A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Neither CS NoR B = Total Application − $(C \cup B)$

$= 350 - (220 + 147 - 51)$

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

8 length $\Rightarrow$ $2^7 + 2^6$

1 _ _ _ _

① OR

_ _ _ _ 0 0

2^7

② Disjoint ?

2^6

wrong

Not Disjoint

How many bit strings of length eight either start with a 1 bit or end with the two bits 00?

8 length $\Rightarrow$ $2^7 + 2^6 - 2^5$

$\boxed{1\_\_\_\_\_\_\_} + \boxed{\_\_\_\_\_\_00} - \boxed{1\_\_\_\_\_00}$

$2^7 + 2^6 - 2^5$

How many bitstrings of length eight either begin with 00 or end with 101?

How many bitstrings of length eight either begin with 00 or end with 101?

8 length $\Rightarrow 2^6 + 2^5 - 2^3$

$\boxed{00\_\ \_\ \_} + \boxed{\_\ \_\ \_\ .\ 101} - \boxed{00\ .\ .\ .\ .\ 101}$

$2^6 \quad + \quad 2^5 \quad - \quad 2^3$

Q: 8 length bit string ; Starting with 00 OR Starting with 101?

8 length $\Rightarrow$ $2^6 + 2^5$ ✓

00_ _ _ OR 101_ _ _

2^6 2^5

Disjoint

Q: 8 length bit string; starting with 00 OR starting with 000?

Ans:

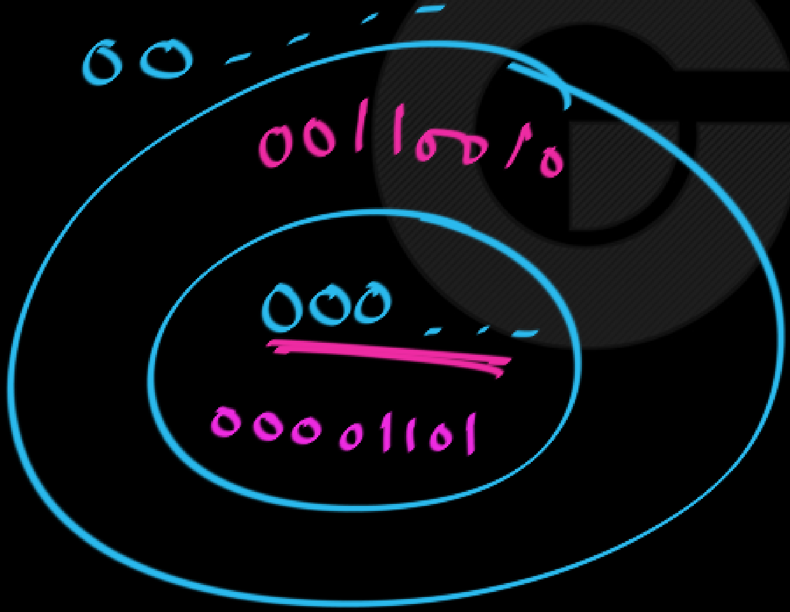
8 length $\Rightarrow 2^6 + 2^5 - 2^5 = 2^6$

$\boxed{00\_\_\_}$ OR $\boxed{000\ldots}$ $-$ $\boxed{000\ldots\ldots}$

2^6 $\quad$ 2^5 $\quad$ 2^5

Not Disjoint

Q: 8 length bit string ; starting with 00 OR starting with 000 ?



2^6

Q: 8 length bit string; starting with 00 OR starting with 10?

00 _ _ _ OR 10 _ _ _

Disjoint

$\underline{2^6 + 2^6} = 2^7$

Q: 8 length bit string ; starting with 00 and starting with 10 ?

Ans: 0

Q: 8 length bit string ; starting with 00 and starting with 000 ?

Ans: starting with 000

2^5

How many bitstrings of length eight either begin with 00 or end with 101?

Solution: There are 2^6 that begin with 00, 2^5 that end with 101, and 2^3 that start with 00 and end with 101. So the number of bitstrings with at least one of the two properties is $2^6 + 2^5 - 2^3 = 88$.

Palindrome: $\Rightarrow$ $\boxed{w = w^R}$

eye, madam, 101, 1001, 1111, 111

Note:

"Palindrome" is not a palindrome.

Example 1.6 Palindromes A *palindrome* is a list that reads the same from right to left as it does from left to right. For example, ignoring capitalization, punctuation and spaces, "Madam I'm Adam." becomes the palindrome madamimadam.

How many k-long palindromes can be formed from an n-set? The first $\lceil k/2 \rceil$ list elements are arbitrary and the remaining elements are determined.* Thus the answer is $n^{\lceil k/2 \rceil}$.

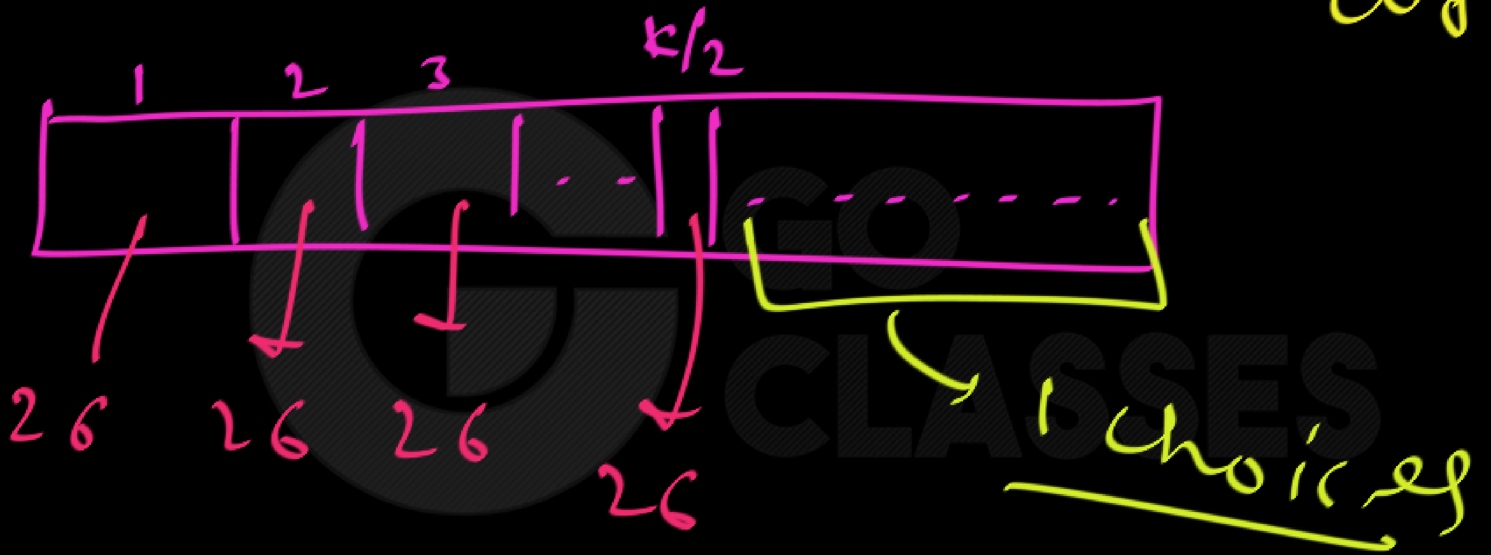
4-length English palindrome words:

k	z	z	k

$= (26)^2$

26 choices $\times$ 26 choices $\times$ 1 choices $\times$ 1 choices

K-length (K-even) Exp-Palindrome words



$= (26)^{K/2}$

k - length (k - odd)

Eng - Palindrome words

26 26 26

1 Choices

$(26)^{\frac{k+1}{2}}$

k-length English palindromes;

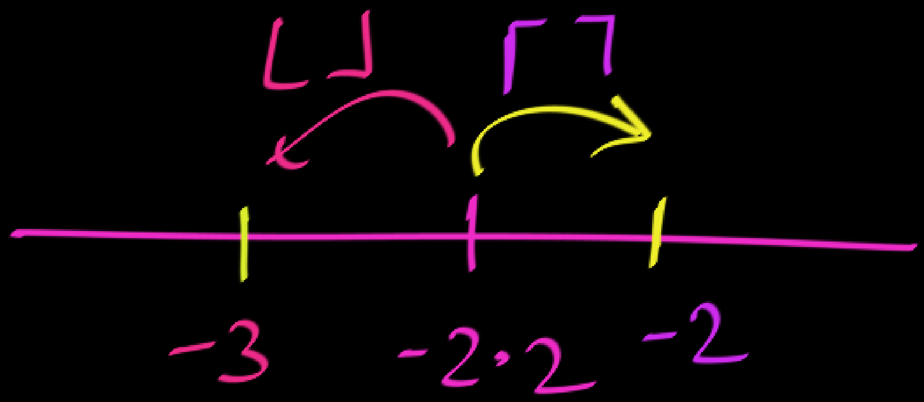
$$26^{\left\lceil \frac{k}{2} \right\rceil}$$

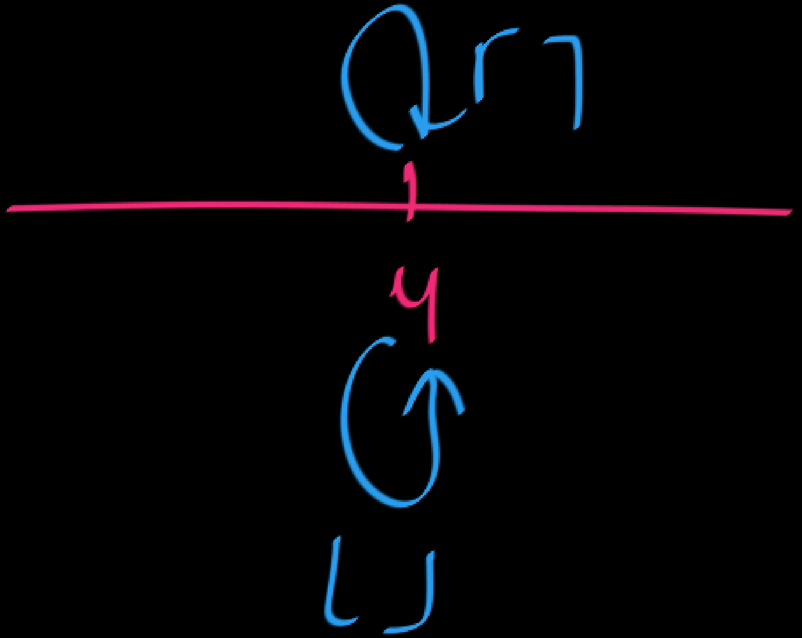
$\lceil 3.2 \rceil = 4$ ceil

$\lfloor 3.2 \rfloor = 3$ floor

$\lfloor\ \rfloor$ $\lceil\ \rceil$

3 3.2 4





$$\lceil -2.2 \rceil = -2$$

$$\lfloor -2.2 \rfloor = -3$$

$$\lceil 4 \rceil = 4$$

$$\lfloor 4 \rfloor = 4$$

Q: 4 Digit ; NOT starting with 0 ;
No Repetition

No
0

8 | 1 | 0 | 6

9 × 9 × 8 × 7

Choices   Choices   Choices   Choices

Know these things:

1. Cards — 4 suits — Diamond, club, hearts, Spade

13 cards of Every suit

(2, 3, -, -, 10, J, Q, K, A)

2. Coin → 2 sides

Head, Tail

Counting Functions How many functions are there from a set with m elements to a set with n elements?

Solution: A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $n \cdot n \cdot \cdots \cdot n = n^m$ functions from a set with m elements to one with n elements. For example, there are $5^3 = 125$ different functions from a set with three elements to a set with five elements. ◀

A student ID is made up of 3 letters followed by two digits.

a. How many possible ID's exist?

b. How many ID's are possible if duplicate letters or numbers aren't allowed?

c. How many student ID's are possible with an even number of "A"'s?