The Fundamental Course

Practice Questions on Proof Techniques

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GATE CSE AIR 53; AIR 67; AIR 107; AIR 206

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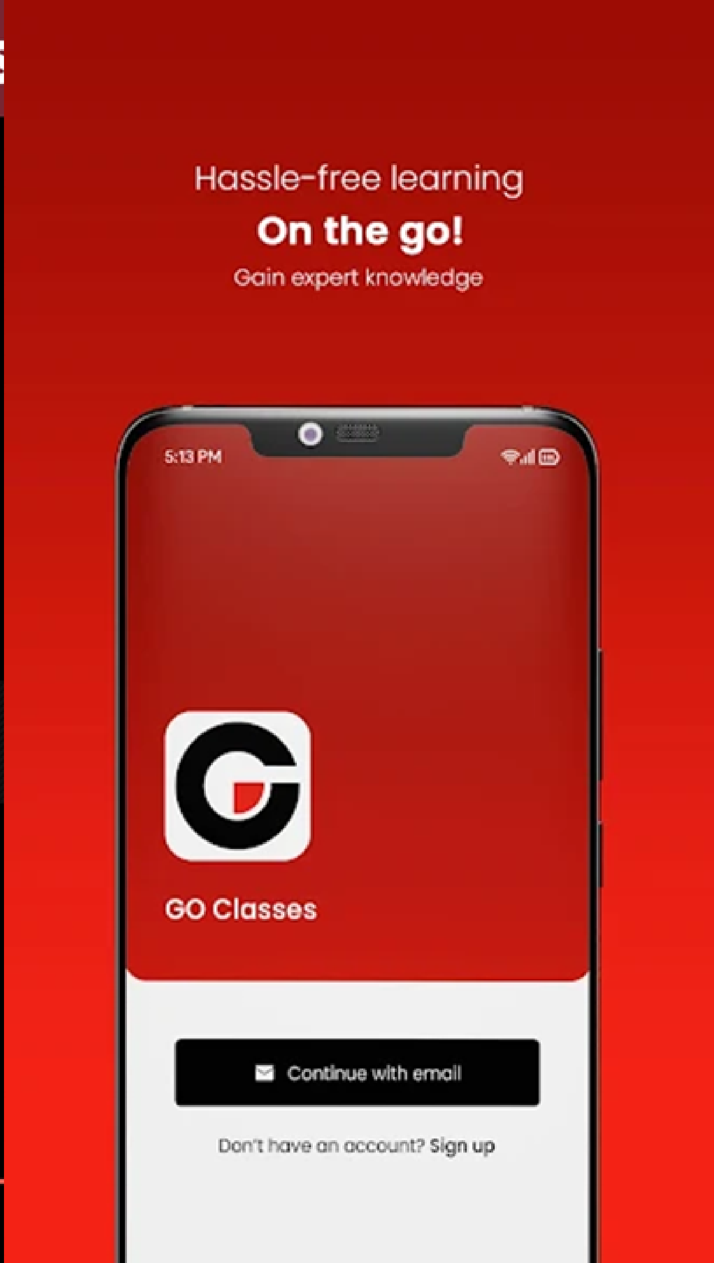

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The Fundamental Course

Practice Questions on Proof Techniques

Q :

Math 2420 (Spring 2024) **Quiz #06** (02/27) **Problems**

Problem 6.1 (5 points). Prove that, for all $m, n \in \mathbb{Z}$, if m is even and n is odd then $m - n$ is odd. *[That is, prove that the difference of an even number and an odd number is odd.]*

Q :

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}$

Math 2420 (Spring 2024) **Quiz #06** (02/27) **Problems**

Problem 6.1 (5 points). Prove that, for all $m, n \in \mathbb{Z}$, if m is even and n is odd then $m - n$ is odd. *[That is, prove that the difference of an even number and an odd number is odd.]*

$m, n \in \mathbb{Z}$

$m = 2$, $n = 2$ ✓

$m = 2$, $n = -4$ ✓

Prove that:

even − odd = Odd

To prove : $\boxed{\text{even} - \text{odd} = \text{odd}}$

Proof :

$2k$

$2k+1$

WRONG Proof

Only proving for Consecutive numbers

WRONG Proof

Direct proof:

Given: $m = \text{even}$; $n = \text{odd}$

Apply facts that you know are True

To prove: $m - n = \text{odd}$

Direct proof:

Given: $m = \text{even}$; $n = \text{odd}$

$m = 2k$ → some integer

$n = 2a + 1$ → some integer

$$m - n = 2k - (2a+1)$$

$$= 2k - 2a - 1 = \underbrace{2(\underbrace{k-a}_{\text{int}})}_{\text{even int}} - 1 = \text{Odd}$$

int int

To prove: $m - n = \text{odd}$

even

odd

5 : odd

−5 : odd

$\boxed{\text{even} - \text{odd} = \text{odd}}$

$2 - 5 = -3$ odd

$8 - 5 = 3$ odd

$\underbrace{(-4)}_{\text{even}} - \underbrace{(-7)}_{\text{odd}} = 3$ odd

Q :

Odd – Odd ?

Odd – Even ?

Even – Even ?

Even – Odd ?

Q :

Odd – Odd ? = even

$$\left(\underbrace{(2a+1)}_{} - (2b+1) \right) = 2(a-b) = \text{even}$$

(a: Some int, b: Some int, $a-b$: int)

Odd – Even ? = Odd (prove it: H/w)

Even – Even ? = even

$$\left(2a - 2b = 2(a-b) = \text{even} \right)$$

Even – Odd ? = Odd (Already done)

Q :

If j and k are integers and $j - k$ is even, which of the following must be even?

A. $jk - 2j$
B. jk
C. $2j + k$
D. $jk + j$

Q : (MSQ)

If j and k are integers and $j-k$ is even, which of the following must be even?

A. $jk-2j$

B. jk

C. $2j+k$

D. $jk+j$

in all cases, must be even

Ans : D

Given: $\underbrace{J - k = \text{even}}$ $\quad \underbrace{J, k \in \mathbb{Z}}$

J	k	$J-k$
even	even	even ✓
even	odd	odd
odd	even	odd
odd	odd	even ✓

J, k → both odd or both even

Option B: JK Two Cases

$\underbrace{J, K : \text{both even}}$

$\underbrace{JK = \text{even}}$

$\underbrace{J, K : \text{both odd}}$

$JK = \text{odd} \cdot \text{odd} = \text{odd}$

Option B: $\rightarrow$ Eliminated

Option D: $Jk + J$ Two Cases

J, k : both even

$\underbrace{Jk}_{\text{even}} + \underbrace{J}_{\text{even}} = \text{even}$

$\underbrace{\text{even} + \text{even}}_{\text{even}}$

J, k : both odd

$\underbrace{Jk}_{\text{odd}} + \underbrace{J}_{\text{odd}} = \text{even}$

$\underbrace{\text{odd} + \text{odd}}_{\text{even}}$

Option A: $JK - 2J$ Two Cases

J, K : both even

$\underbrace{JK}_{\text{even}} - \underbrace{2J}_{\text{even}} = \text{even}$

even − even → even

J, K : both odd

$\underbrace{JK}_{\text{odd}} - \underbrace{2J}_{\text{even}} = \text{odd}$

odd − even = odd

Option A: Eliminated

option c: $2J+k$ Two Cases

J, k : both even

$\downarrow$

$2J+k = \text{even}$

J, k : both odd

$\downarrow$

$2J+k = \text{odd}$

option c: Eliminated

Q :

8. (12 pts) Prove or disprove the following statements. (proof types: Contradiction, Direct, Contrapositive, and counterexample)

   a. If x is an irrational number, then so is $1/x$.

   b. The sum of an irrational number and a rational number is irrational.

Q :

8. (12 pts) Prove or disprove the following statements. (proof types: Contradiction, Direct, Contrapositive, and counterexample)

a. If x is an irrational number, then so is $1/x$.

True (prove it)

True (prove it)

b. The sum of an irrational number and a rational number is irrational.

Q :

True

8. (12 pts) Prove or disprove the following statements. (proof types: Contradiction, Direct, Contrapositive, and counterexample)

a. If x is an irrational number, then so is $1/x$.

$x \neq 0$

Prove by Contradiction: Given: x : irrational

To prove: $1/x$ is irrational

Contradiction

$\boxed{\text{Assume}}$ $1/x$ = rational

$\frac{1}{x} = \frac{a}{b}$ $a \rightarrow$ int, $b \rightarrow$ int $\neq 0$ $\Rightarrow$ $\boxed{x = \frac{b}{a}}$ $\rightarrow$ x : Rational

Contradiction

$a \rightarrow \neq 0$ (because $x \neq 0$)

b. The sum of an irrational number and a rational number is irrational.

0 : Rational

$$\frac{0 \to \text{int}}{1 \to \text{int} \neq 0}$$

(8a): **Given:** $\boxed{x \text{ is Irrational.}}$ $\rightarrow x \neq 0$

To prove: $1/x$ is irrational.

Prove by "Contradiction":

Assume: $\underbrace{1/x \text{ is Rational.}}$

$$\frac{1}{x} = \frac{a \rightarrow \text{int}}{b \rightarrow \text{int} \neq 0}$$

$\frac{1}{x} \rightarrow \neq 0$

$\Rightarrow x = \frac{b \rightarrow \text{int}}{a \rightarrow \text{int} \neq 0}$

x: Rational

Contradiction

(b) Irrational x + Rational r = Irrational

$r = \frac{a}{b}$ — $a \to$ int, $b \to$ int $\neq 0$

Proof by Contradiction:

Assume $x + r =$ Rational

$x + \frac{a}{b} = \frac{c}{d}$ — $c \to$ int, $d \to$ int $\neq 0$

$x = \frac{c}{d} - \frac{a}{b} = \frac{\boxed{cb - ad}}{bd}$ — $cb - ad \to$ int, $bd \to$ int $\neq 0$

$x =$ rational

Contradiction

So; $\underbrace{\text{our Assumption was false.}}$

So; Irrational + Rational = Irrational

Positive integers

Q :

2 If a, b, and c are [redacted] which of the following are necessarily true?

I. If $a < b$ and $ab \neq 0$, then $\frac{1}{a} > \frac{1}{b}$.

II. If $a < b$, then $ac < bc$ for all c

III. If $a < b$, then $a + c < b + c$ for all c.

IV. If $a < b$, then $-a > -b$

(A) I only (B) I and III only (C) III and IV only (D) II, III, and IV only (E) I, II, III, and IV

Positive integers : $Z^+ = \{1, 2, 3, --\} = N$

Q :

2 If a, b, and c are real numbers, which of the following are necessarily true?

I. If $a < b$ and $ab \neq 0$, then $\frac{1}{a} > \frac{1}{b}$.

II. If $a < b$, then $ac < bc$ for all c

III. If $a < b$, then $a + c < b + c$ for all c.

IV. If $a < b$, then $-a > -b$

(A) I only (B) I and III only (C) III and IV only (D) II, III, and IV only (E) I, II, III, and IV

$$\underbrace{a|b \ \& \ b|a \Rightarrow a=b}$$

If $\underbrace{a, b \in \mathbb{N}}$

Ans: True

If $a, b \in \mathbb{R}$ or $\underbrace{a, b \in \mathbb{Z}}$

Ans: false

$a = 1, b = -1$

Q :

Pay Attention to Small Small Details

2 If a, b, and c are real numbers, which of the following are necessarily true?

I. If $a < b$ and $ab \neq 0$, then $\frac{1}{a} > \frac{1}{b}$. Counter Ex: $a = -2$, $b = 2$

II. If $a < b$, then $ac < bc$ for all c

III. If $a < b$, then $a + c < b + c$ for all c.

IV. If $a < b$, then $-a > -b$

(A) I only (B) I and III only (C) III and IV only (D) II, III, and IV only (E) I, II, III, and IV

D: for all $a, b \in \mathbb{R}$

If $\underbrace{a < b}$ then $\underbrace{-a > -b}$.

inequality

(c): $a, b, c \in \mathbb{R}$

If $a < b$ then $a + c < b + c$.

$a+c < b+c$ (if $c =$ neg.)

(b) $\boxed{\text{If } a<b \text{ then } ac<bc \text{ for all } c.}$ → false

Prove by Counter Example:

$$\left\{\begin{array}{l} a=2 \\ b=3 \\ c=-1 \end{array}\right\} \longrightarrow \begin{array}{l} a<b \checkmark \\ \text{but } ac \not< bc \end{array}$$

Take c = negative value

Q :

23. $S(n)$ is a statement about positive integers n such that whenever $S(k)$ is true, $S(k+1)$ must also be true. Furthermore, there exists some positive integer n_0 such that $S(n_0)$ is not true. Of the following, which is the strongest conclusion that can be drawn?

(A) $S(n_0 + 1)$ is not true.
(B) $S(n_0 - 1)$ is not true.
(C) $S(n)$ is not true for any $n \leqq n_0$.
(D) $S(n)$ is not true for any $n \geqq n_0$.
(E) $S(n)$ is not true for any n.

Q :

Given

23. $S(n)$ is a statement about positive integers n such that whenever $S(k)$ is true, $S(k+1)$ must also be true. Furthermore, there exists some positive integer n_0 such that $S(n_0)$ is not true. Of the following, which is the strongest conclusion that can be drawn?

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Q : if

23. $S(n)$ is a statement about positive integers n such that whenever $S(k)$ is true, $S(k+1)$ must also be true. Furthermore, there exists some positive integer n_0 such that $S(n_0)$ is not true. Of the following, which is the strongest conclusion that can be drawn?

(A) $S(n_0 + 1)$ is not true.
(B) $S(n_0 - 1)$ is not true. → True
(C) $S(n)$ is not true for any $n \leqq n_0$.
(D) $S(n)$ is not true for any $n \geqq n_0$.
(E) $S(n)$ is not true for any n.

$\underbrace{\text{for all } n \leq n_0}$

$S(n) \neq \text{True}$

$S(n)$: $\boxed{5n < n^2}$

$n \in \mathbb{N}$

$n = 2$: $S(2)$: false

$n = 5$: $S(5)$: false

$n = 6$; $S(6)$: True

$n = 7$; $S(7)$: True

$n = 8$; $S(8)$: True

— — - - - -

We can prove that $S(n)$ is True, for $n \geq 6$, by induction.

$S(n): \quad 5n < n^2$

$n \in N$

$n_0 = 3$

$S(n_0) = S(3) = \text{false}$

option E: $S(7)$: True

Eliminated

$S(n): \quad 5n < n^2$

$n \in N$

$n_0 = 5$

$S(n_0) = S(5) = $ false

option A

$S(n_0 + 1)$: True

Eliminated

Q: $S(n)$: $\boxed{\text{- - - - -}}$

Given:

If $S(k)$ = True then $S(k+1)$ = True

Given: $S(n_0)$ = false, for some n_0

$\mathbb{N}$: 1, 2, 3, - - - -, $n_0 - 1$, n_0, n_0+1, - - - - - -

$S(k)$:

$S(n_0)$ = false

Given: If $S(k)$ = True then $S(k+1)$ = True

for all k.

$\mathbb{N}$: 1, 2, 3, ----, $n_0 - 1$, n_0, $n_0 + 1$, ------

$S(n_0) = \text{false}$

$S(k)$:

Q: Can $S(n_0 - 1)$: True ? $\Rightarrow$ No.

because if $S(n_0 - 1) = \text{True}$ then $S(n_0) = \text{True}$

$\mathbb{N}$: 1, 2, 3, a, - - - - n_0, - - - - -

If $a < n_0$ then

Can $S(a) = \text{True}$? → NO.

because if $S(a) = \text{True}$ then $S(a+1) = \text{True}$

$S(a+2) = \text{True}$, $S(a+3) = \text{True}$, - - . $S(n_0) = \text{True}$

$$- - - - - \; n_0 \;, \underbrace{- - - - -}$$

$S = \text{False}$

$S(n_0) = \text{false}$

can not say anything for these

Q :

Problem. Prove that, for all $d, m, n \in \mathbb{Z}$, if $d \mid m$ and $d \mid n$ then $d \mid (5m + 8n)$.

Problem. Prove that, for all $d, m \in \mathbb{Z}$, if $d \mid m$ then $d^2 \mid m^2$.