



# Graph Theory

# Rooted Tree



# Graph Theory :

Recap :

Questions on Trees, Forests

Website : <https://www.goclasses.in/>



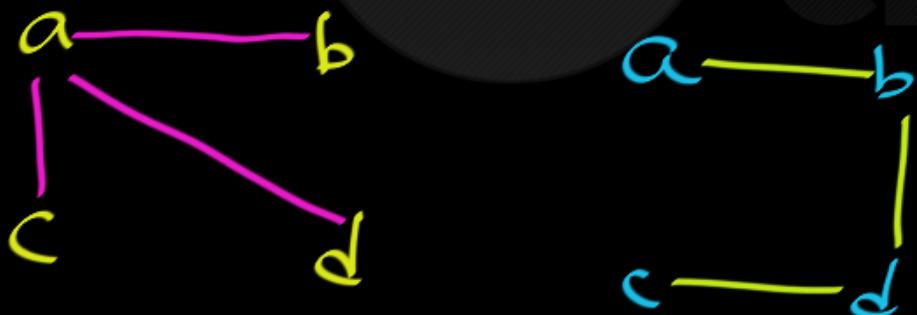
Q:

What is the maximum number of edges in a  
Forest on  $n$  vertices?



Q:

What is the maximum number of edges in a  
Forest on n vertices? =  $n-1$





NOTE:

Acyclic Graph

forest

with maximum number of Edges

$\equiv$  Tree

$\equiv$  maximally Acyclic

$\equiv$  Maximally forest



Q:

What is the maximum number of edges in a Forest which is NOT Tree, on  $n$  vertices?



Q:

What is the maximum number of edges in a  
Forest which is NOT Tree, on  $n$  vertices?

$$= n - 2$$

If  $(n-1)$  Edges then forest will be a Tree.



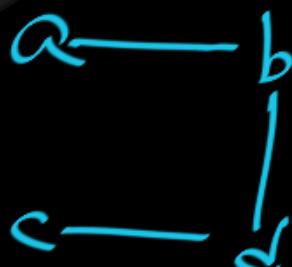
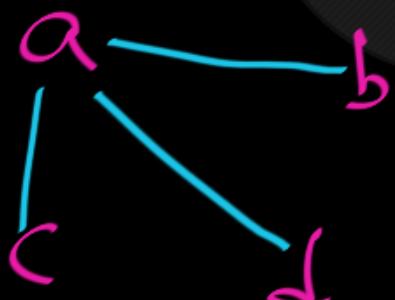
Q:

What is the minimum number of edges in a  
Connected Graph on  $n$  vertices?



Q:

What is the minimum number of edges in a  
Connected Graph on  $n$  vertices? =  $n - 1$



So Tree.



Tree  $\equiv$  Connected Graph with minimum  
number of edges  
 $\equiv$  minimally Connected



Q:

What is the minimum number of edges in a  
Connected Cyclic Graph on  $n$  vertices?



Q:

What is the minimum number of edges in a  
Connected Cyclic Graph on  $n$  vertices? =  $n$

Connected with  $n-1$  Edges  $\equiv$  Tree



Q:

What is the minimum number of edges in a  
Connected Acyclic Graph on  $n$  vertices?



Q:

What is the minimum number of edges in a  
Connected Acyclic Graph on  $n$  vertices?

Tree

$$= \underline{\underline{n-1}}$$



Q: Is it possible?

A tree with 9 vertices and the sum of the degrees of all the vertices is 18.



Q: Is it possible? — No

A tree with 9 vertices and the sum of the  
degrees of all the vertices is 18.  $\Rightarrow$  No

Tree with n vertices  $\Rightarrow |E| = n - 1$

Total Degree =  $2(n-1)$       for  $|V|=9$  ;  
Total Degree =  $2(8) = 16$



Q: Is it possible?

A graph with 9 vertices, 9 edges, and no cycles.





Q: Is it possible? — **No**

A graph with 9 vertices, 9 edges, and no cycles.  $\Rightarrow$  forest

$$\text{In a forest, } \max_{\text{---}} |E| = n - 1 = 9 - 1 = 8$$

In a Tree on  $n$  vertices ;

$$\min |E| = n-1$$

$$\max |E| = n-1 \text{ edges}$$



In a forest on  $n$  vertices ;

$$\min |E| = 0 \text{ (Empty)} \quad (E_n)$$

$$\max |E| = n-1 \text{ (Tree)} \quad (E_{\text{Tree}})$$



Q: Is it possible?

Graph has 8 vertices, 8 edges, and no cycles.





Q: Is it possible?  $\Rightarrow$  No

Graph has 8 vertices, 8 edges, and no cycles.

GO CLASSES forest

$$\underline{\max |E|} = n - 1$$



Q: Is it necessarily a Tree??

Graph G with n vertices, one vertex of degree  $n-1$  and rest of the vertices of degree 1.



Q: Is it necessarily a Tree??

$\alpha_1 \rightarrow$  Yes.

Graph G with n vertices, one vertex of degree n-1 and rest of the vertices of degree 1.

Graph , n vertices



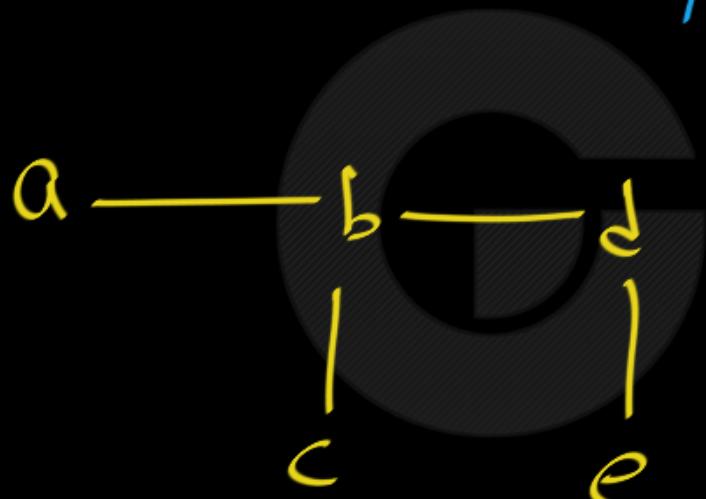
$q_1, q_2, q_3, \dots, q_n$

Def = n-1 means  $q_1$  is Adjacent to all others.





Star Graph → Tree



GO  
CLASSES

Tree ✓  
NOT Star Graph

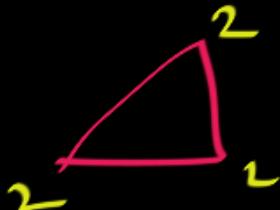


Q:

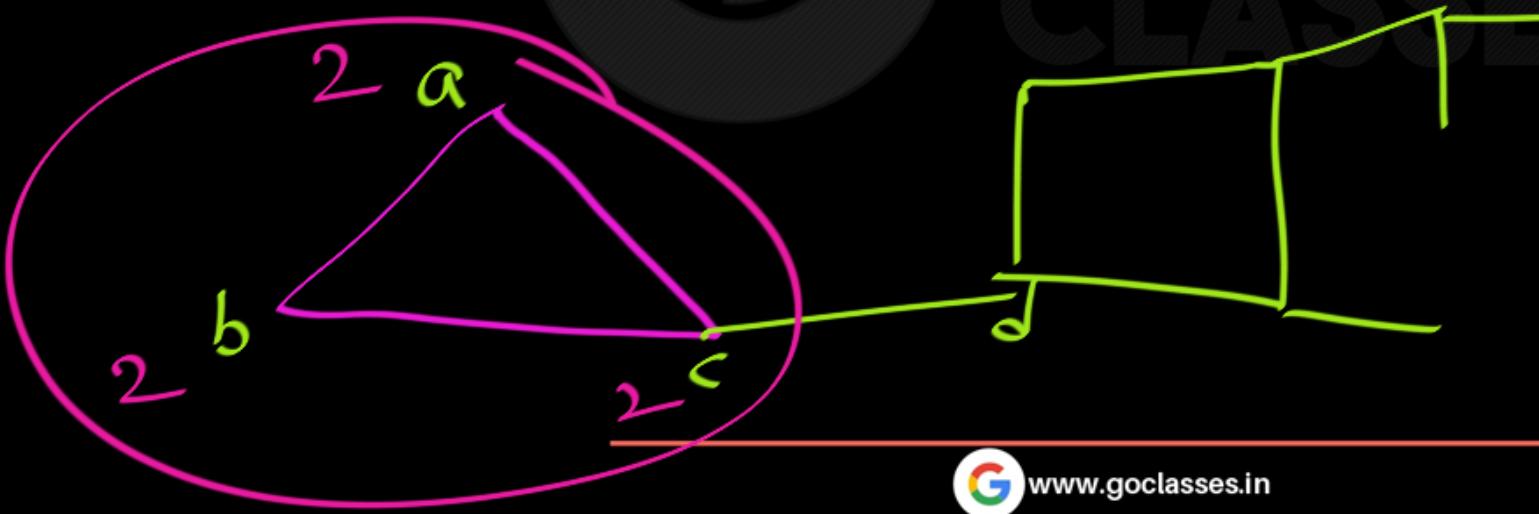
A cyclic graph must have at least 3 vertices of degree at least 2? True or False?



Q:

Smallest Cycle Possible

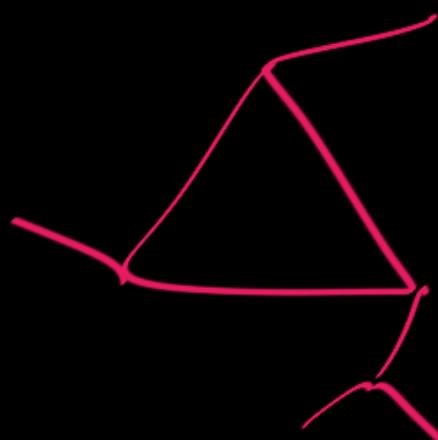
A cyclic graph must have at least 3 vertices of degree at least 2? True or False?





Ans:

In any cycle of a cyclic graph, every vertex in it has a degree greater or equal to 2 (since we have to both "enter" and "exit" the vertex).





Q: True or False?

Consider a graph  $G$  such that at least one vertex  $v$  is connected to all other vertices.  
Such  $G$  is not bipartite.

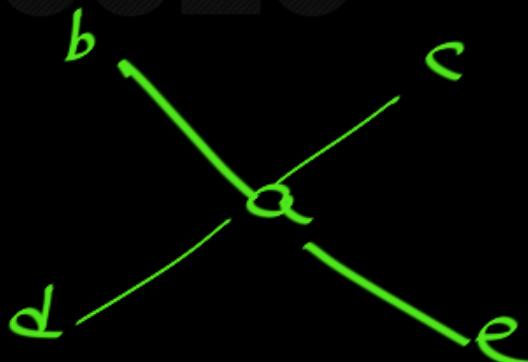


Q: True or False?

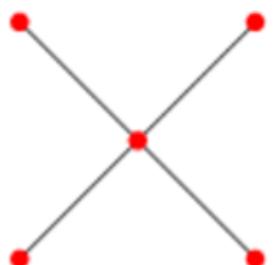
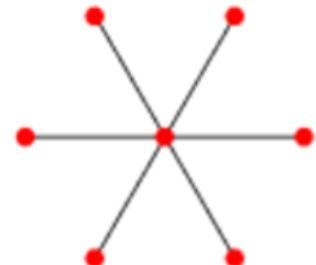


Consider a graph G such that at least one vertex v is connected to all other vertices.

Such G is not bipartite.



There are infinitely-many counterexamples to the claim as stated. Consider the graph on  $n + 1$  vertices where  $v$  is connected to each of the other  $n$  vertices but no other edges are included (this is sometimes called the [star graph](#)  $S_{n+1}$ ). The graph is bipartite: put  $v$  in one set and the remaining  $n$  vertices in another.

 $S_1$  $S_2$  $S_3$  $S_4$  $S_5$  $S_6$  $S_7$  $S_8$ 



Q: True or False?

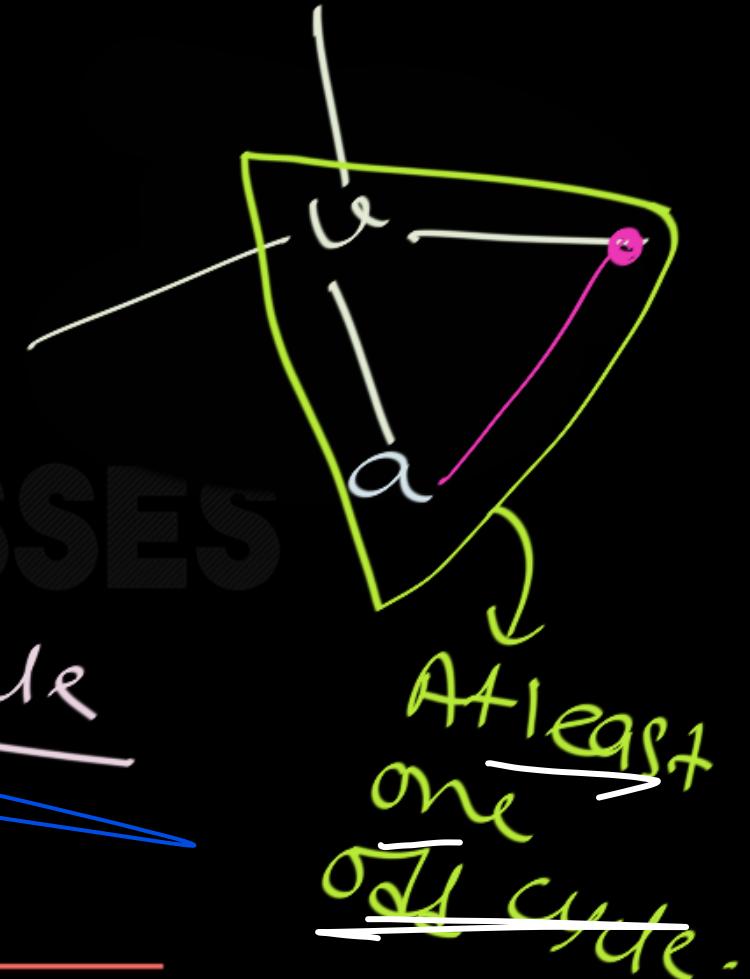
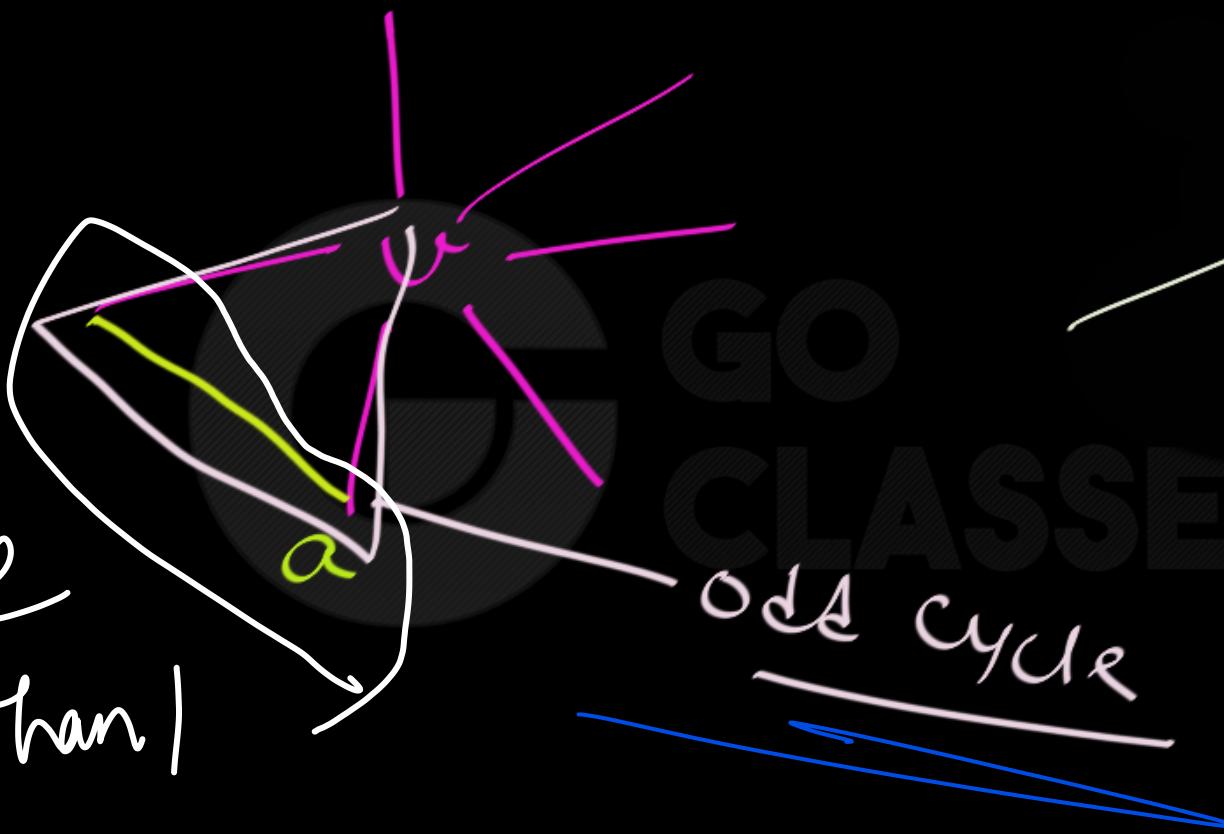
Consider a graph  $G$  such that at least one vertex  $v$  is connected to all other vertices, and there is some other vertex of degree more than 1. Such  $G$  is not bipartite.

Q: True or False?

Consider a graph  $G$  such that at least one vertex  $v$  is connected to all other vertices, and there is some "other" vertex of degree more than 1. Such  $G$  is not bipartite.



degree  
more than 1





Bipartite

iff

NO

odd cycle.

Bipartite iff

All cycles are even.



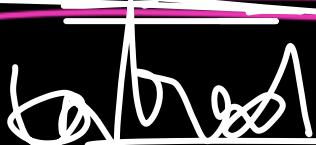
Q: True or False?

Let  $G$  be a graph with at least two vertices,  
with NO vertex of degree 1.  
 $G$  cannot be a tree.

Q: True or False?

Let G be a graph with at least two vertices,  
with NO vertex of degree 1.

G cannot be a tree.



Assume  $G$  is Tree:  $\rightarrow \text{Deg}(v) \geq 1 \quad \forall v$

$G$ :

$a_1, a_2, a_3, \dots, a_n$

$\text{Deg} = \geq 2$

$\text{Total Degree} \geq 2n$

Not Tree

Contradiction

BUT in Tree, Total Def =  $2n - 2$



Q: True or False?

Let  $G$  be a graph with at least two vertices,

with exactly one vertex of degree 1.

$G$  cannot be a tree.



Q: True or False?

Let  $G$  be a graph with at least two vertices,  
with exactly one vertex of degree 1.

$G$  cannot be a tree.

Assume  $G = \text{Tree} \rightarrow \text{Connected} \rightarrow \deg(v) > 1$

$G : a_1, a_2, a_3, \dots, a_n \quad \forall v$

Def:  $\begin{cases} \geq 2 & \geq 2 \\ \geq 2 & \geq 2 \end{cases}$

$$\text{Total Deg} \geq 2(n-1) + 1 = 2n - 1$$

But in Tree, Total Deg =  $2n - 2$

So G  
NOT Tree  
Connected  
Chain

NOTE:

Every Tree with  $n \geq 2$

has at least two leaves.

leaf: deg 1 vertex.

Pendant vertex.



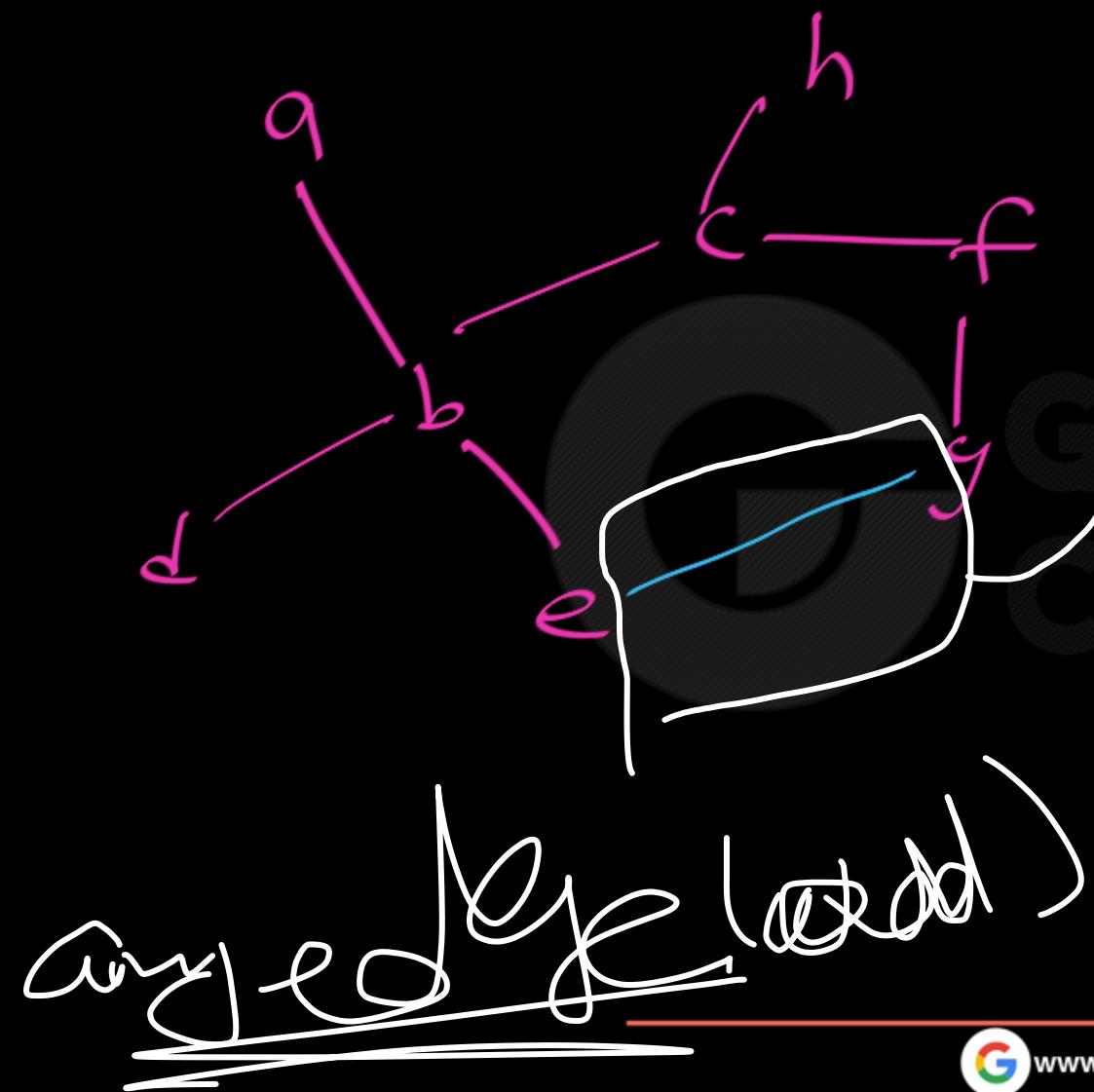
Q: True or False?

To a tree if we add an edge between two existing vertices of T, it creates exactly one cycle.



Q: True or False?

To a tree if we add an edge between two existing vertices of T, it creates exactly one cycle.



Tree + one Edge

Cyclic Graph  
with EXACTLY  
one cycle.



Q: True or False?

To a tree if we add a new vertex with one edge, it creates a cycle. (not Possible)

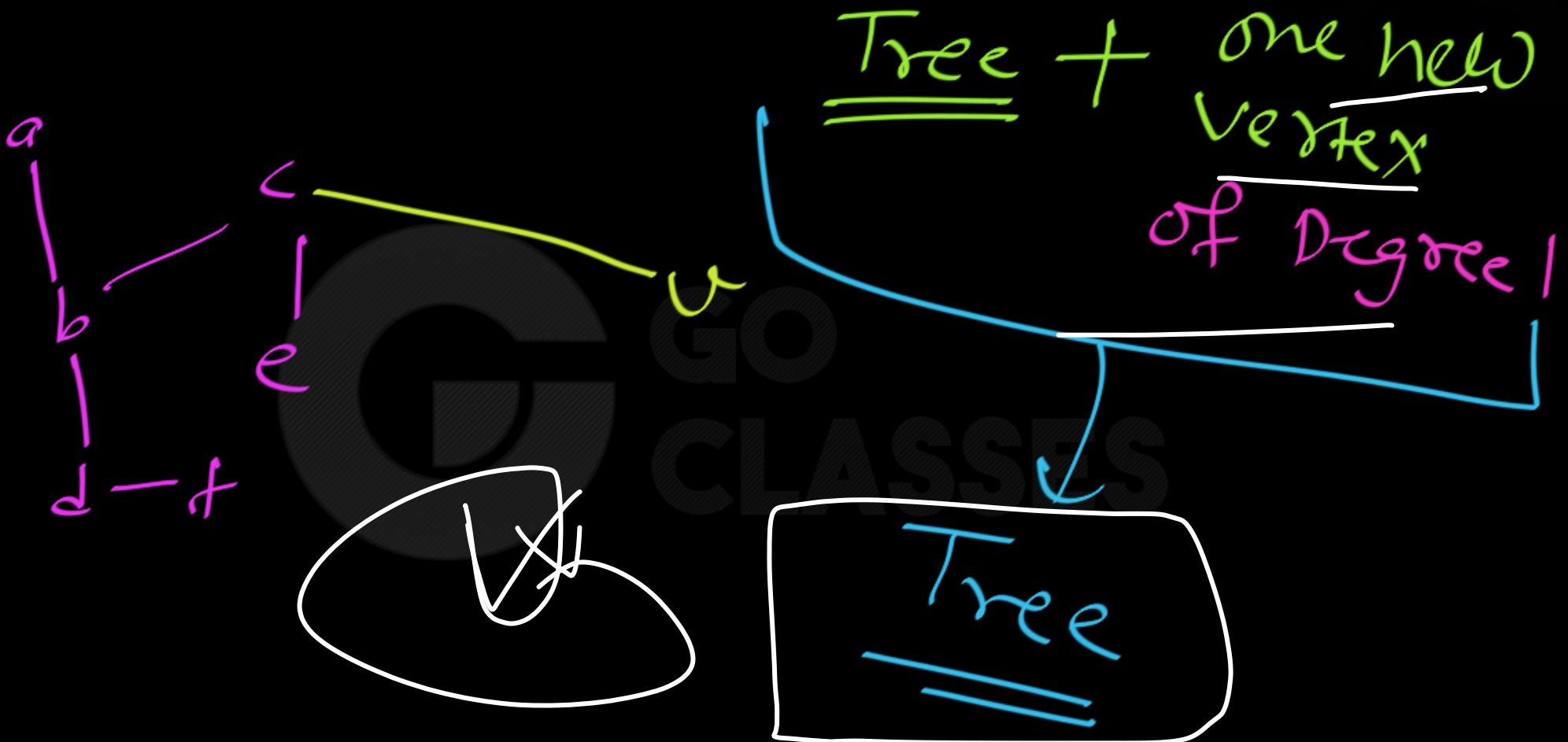
O + V



Q: True or False?

To a tree if we add a new vertex with one edge, it creates a cycle.

↓  
Can

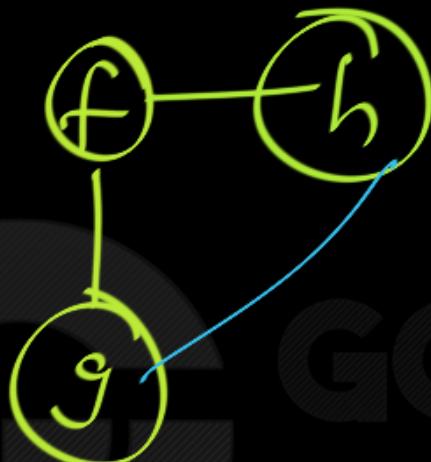


Algorithms : MST → cycle intermediate Tree +  
Prim's ⇒ one new vertex  
with one Edge

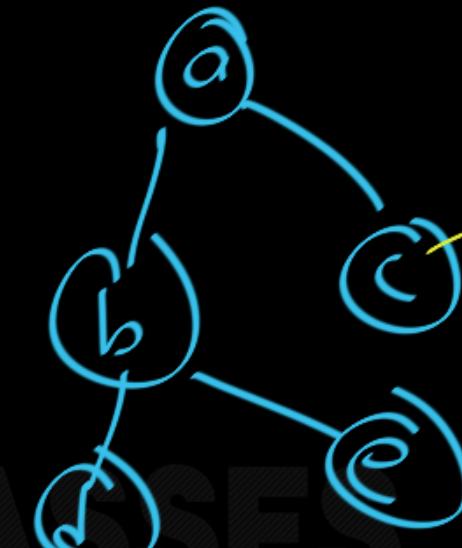
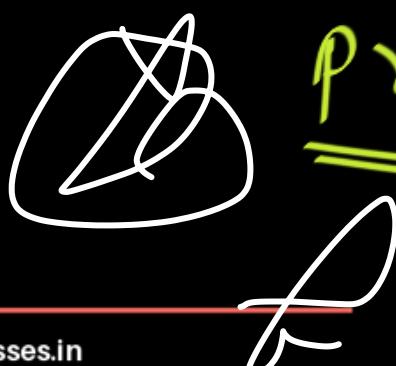
Kruskal's → cycle can  
occur.

Tree +

Intermediate forest + Add one more  
Edge



Kruskal



Prim

Cycle  
can  
NEVER  
Come.



Q: True or False?

If  $G$  has no cycles but by adding one edge between any two vertices will create a cycle then  $G$  is a tree.

Q: True or False?

forest

= no cycle

If G has no cycles but by adding one edge  
between any two vertices will create a cycle  
then G is a tree. True



Q: True or False?

Let  $G$  be a connected graph, then  $G$  is a tree  
iff  $G$  has no cycles



Q: True or False?

Let G be a connected graph, then G is a tree  
iff G has no cycles

*True Tree*



Q: True or False?

Let T be a tree. G is a new graph by adding vertex x and connect it to one vertex through an edge in T. Then G is a Tree.



Q: True or False?

Let T be a tree. G is a new graph by adding vertex x and connect it to one vertex through an edge in T. Then G is a Tree.

Tree + new vertex with one edge  $\Rightarrow$  New Tree



Q: True or False?

Adding one edge to a tree creates exactly one cycle.





Q: True or False?

Every tree with  $n \geq 2$  vertices has at least  
two vertices of degree one.



Q: True or False?

A finite graph in which every vertex has  
degree  $\geq 2$  has a cycle



Tree with  $n \geq 2$   $\Rightarrow$  At least two vertices have degree 1,





Q:

Removing any one edge from a tree results in  
2 number of trees.

Tree A  Tree B   
Remove Parent Tree form.



Q:

Removing any one edge from a tree results in 2 number of trees.





## 2.11.2 Trees: TIFR2011-B-33

Which of the following is NOT a sufficient and necessary condition for an undirected graph  $G$  to be a tree?

- A.  $G$  is connected and has  $n - 1$  edges.
- B.  $G$  is acyclic and connected.
- C.  $G$  is acyclic and has  $n - 1$  edges.
- D.  $G$  is acyclic, connected and has  $n - 1$  edges.
- E.  $G$  has  $n - 1$  edges.

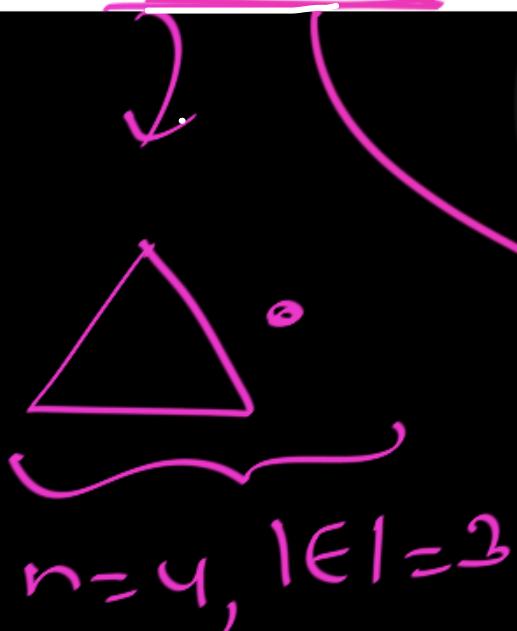




## 2.11.2 Trees: TIFR2011-B-33

Which of the following is NOT a sufficient and necessary condition for an undirected graph  $G$  to be a tree?

- A.  $G$  is connected and has  $n - 1$  edges.
  - C.  $G$  is acyclic and has  $n - 1$  edges.
  - E.  $G$  has  $n - 1$  edges.
- B.  $G$  is acyclic and connected.**
- D.  $G$  is acyclic, connected and has  $n - 1$  edges.**



Definition (iff)

Necessary for Tree,

NOT sufficient



## 2.4.22 Graph Connectivity: GATE2014-3-51

If  $G$  is the forest with  $n$  vertices and  $k$  connected components, how many edges does  $G$  have?

- A.  $\lfloor \frac{n}{k} \rfloor$   
C.  $n - k$

- B.  $\lceil \frac{n}{k} \rceil$   
D.  $n - k + 1$

Every component } = Tree       $k$  Trees

forest + G

Tree  
 $G_1$        $G_2$   
 $|V| = n_1$        $n_2$

$|E| = n_1 - 1$        $n_2 - 1$

Tree  
 $G_3$        $\dots$        $G_k$   
 $n_3$        $\dots$        $n_k$

Total  $|E|$  in forest + G =  $(n_1 + n_2 + \dots + n_k) - k$   
=  $n - k$  ✓

In forest, Every Component is a Tree.

If  $k = 1$  then  $\text{forest} \equiv \text{Tree}$

$|E| = n - 1$

$k$  component no of edge  $= n - k$



## 2.4.7 Graph Connectivity: GATE1995-1.25

The minimum number of edges in a connected cyclic graph on  $n$  vertices is:

- A.  $n - 1$
- B.  $n$
- C.  $n + 1$
- D. None of the above



## 2.4.5 Graph Connectivity: GATE1993-8.1

Consider a simple connected graph  $G$  with  $n$  vertices and  $n$  edges ( $n > 2$ ). Then, which of the following statements are true?

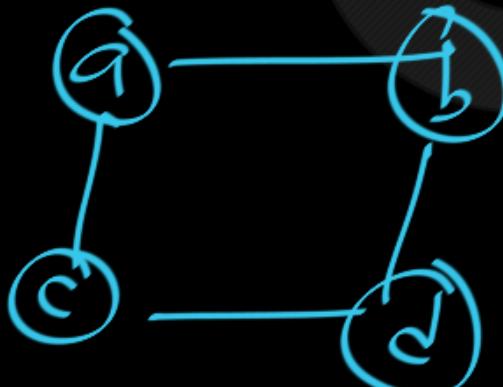
- A.  $G$  has no cycles
- B. The graph obtained by removing any edge from  $G$  is not connected
- C.  $G$  has at least one cycle
- D. The graph obtained by removing any two edges from  $G$  is not connected
- E. None of the above



## 2.4.5 Graph Connectivity: GATE1993-8.1

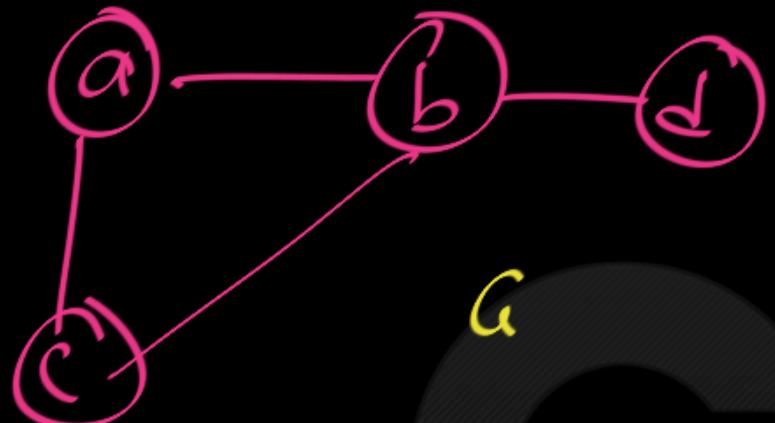
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- A.  $G$  has no cycles X
- B. The graph obtained by removing any edge from  $G$  is not connected X
- C.  $G$  has at least one cycle ✓
- D. The graph obtained by removing any two edges from  $G$  is not connected ✓
- E. None of the above



Remove any  
one edge

Connected



G has at least one cycle!. Yes.

Exactly one cycle  $\rightarrow$  At least one cycle



Connected  
 $n$  vertices  
 $n$  edges

remove 2 edges

$n$  vertices  
 $n-2$  edges

minimally Connected  
Graph has  $n-1$  edges.

Can it be Connected?



2.4.13 Graph Connectivity: GATE2004-IT-5

What is the maximum number of edges in an acyclic undirected graph with  $n$  vertices?

- A.  $n - 1$
- B.  $n$
- C.  $n + 1$
- D.  $2n - 1$



2.4.13 Graph Connectivity: GATE2004-IT-5

What is the maximum number of edges in an acyclic undirected graph with  $n$  vertices?

A.  $n - 1$

B.  $n$

C.  $n + 1$

D.  $2n - 1$





**Theorem 30.** *The following statements are equivalent for a graph  $T$ :*

- (1)  $T$  is a tree.
- (2) Any two vertices in  $T$  are connected by a unique path.
- (3)  $T$  is minimally connected, i.e.  $T$  is connected but  $T - e$  is disconnected for any edge  $e$  of  $T$ .
- (4)  $T$  is maximally acyclic, i.e.  $T$  is acyclic but  $T + uv$  contains a cycle for any two non-adjacent vertices  $u, v$  of  $T$ .
- (5)  $T$  is connected and  $|E(T)| = |V(T)| - 1$ .
- (6)  $T$  is acyclic and  $|E(T)| = |V(T)| - 1$ .



# Graph Theory :

Next Topic :

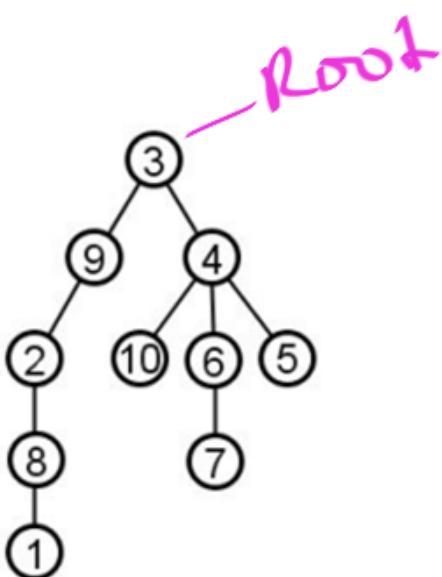
Rooted Trees

(Special Types of DAGs) Binary Trees

Website : <https://www.goclasses.in/>



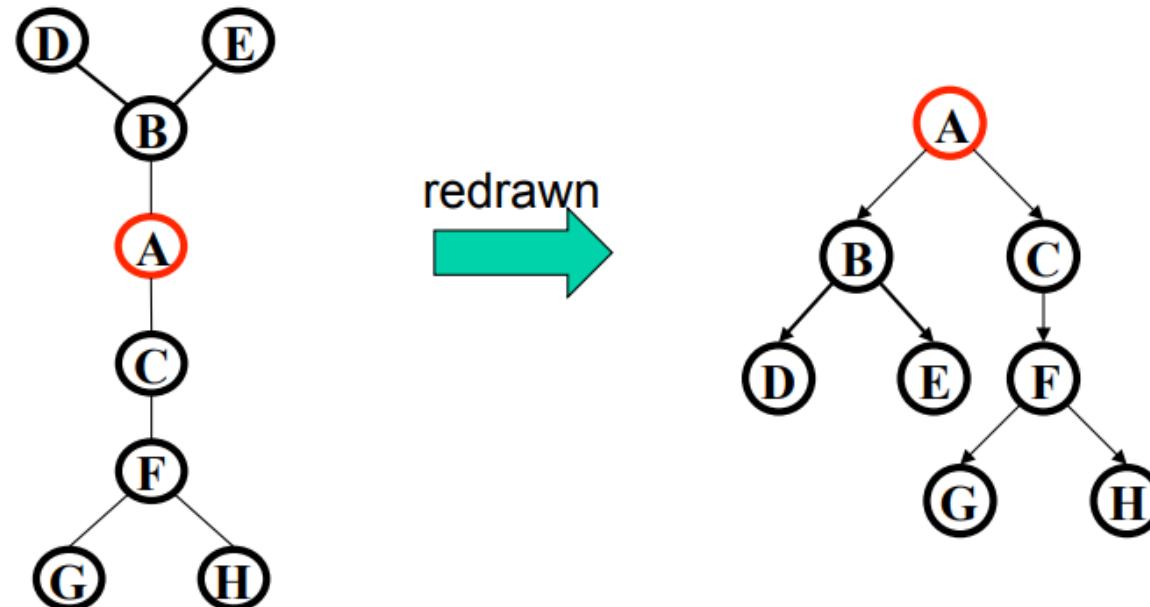
## Rooted Trees



- ▶ A **rooted tree** has a designated root vertex and every edge is directed away from the root.
- ▶ Vertex  $v$  is a **parent** of vertex  $u$  if there is an edge from  $v$  to  $u$ ; and  $u$  is called a **child** of  $v$ .
- ▶ Vertices with the same parent are called **siblings**.
- ▶ Vertex  $v$  is an **ancestor** of  $u$  if  $v$  is  $u$ 's parent or an ancestor of  $u$ 's parent.
- ▶ Vertex  $v$  is a **descendant** of  $u$  if  $u$  is  $v$ 's ancestor.

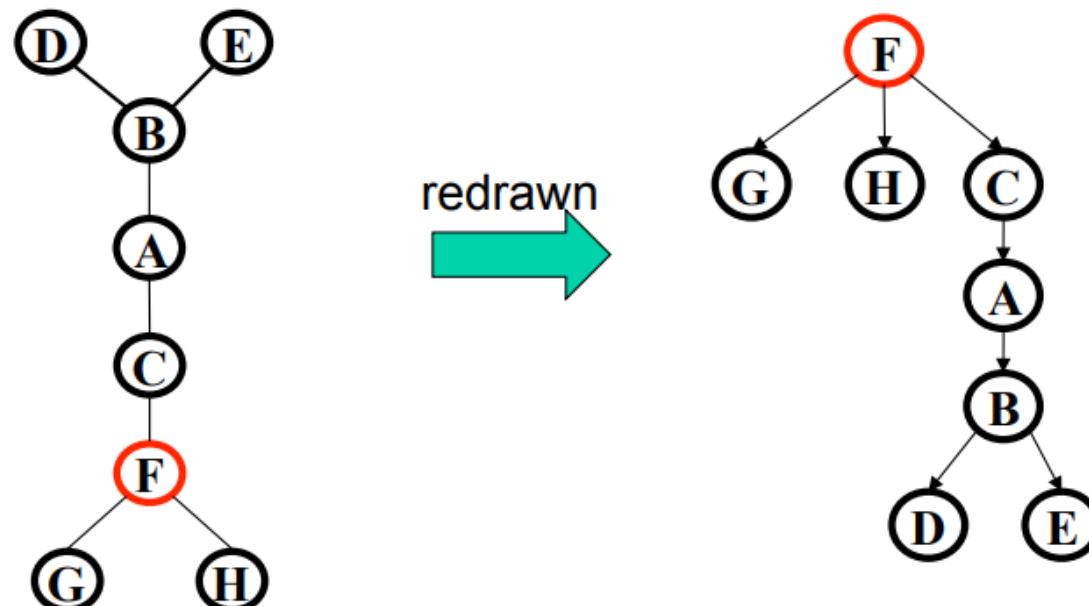
# Rooted Trees

- We are more accustomed to **rooted trees** where:
  - We identify a unique root
  - We think of edges as directed: parent to children
- Given a tree, picking a root gives a unique rooted tree
  - The tree is just drawn differently



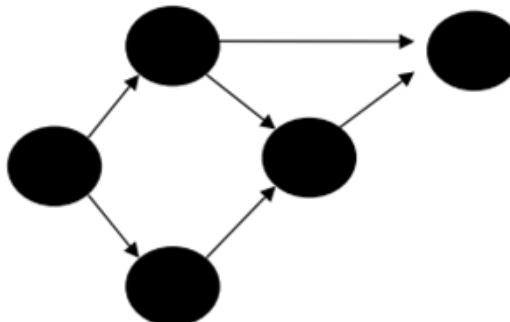
# Rooted Trees

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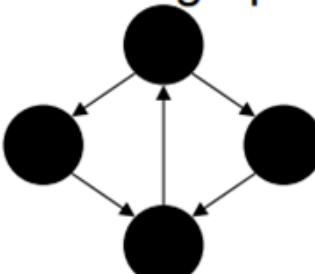


# Directed Acyclic Graphs (DAGs)

- A DAG is a directed graph with no (directed) cycles
  - Every rooted directed tree is a DAG
  - But not every DAG is a rooted directed tree



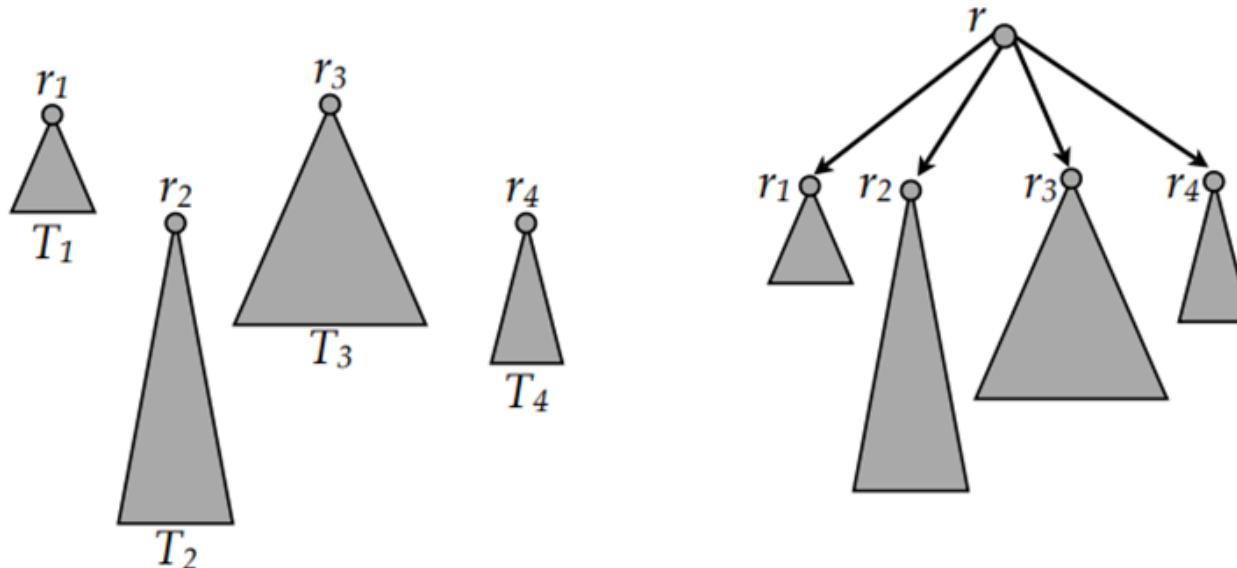
- Every DAG is a directed graph
- But not every directed graph is a DAG





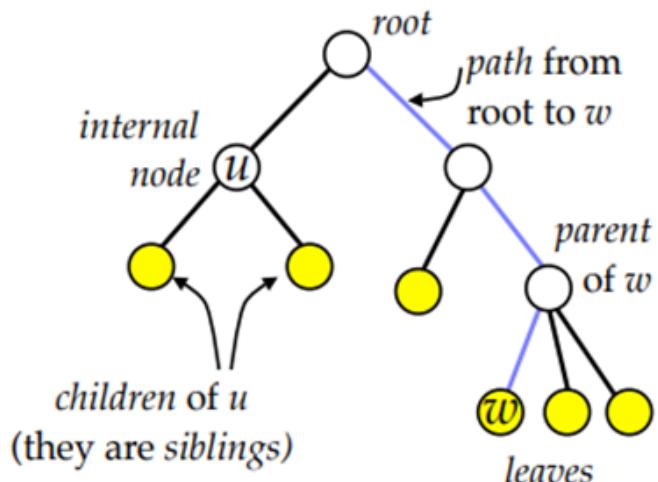
## Definition – Rooted Tree

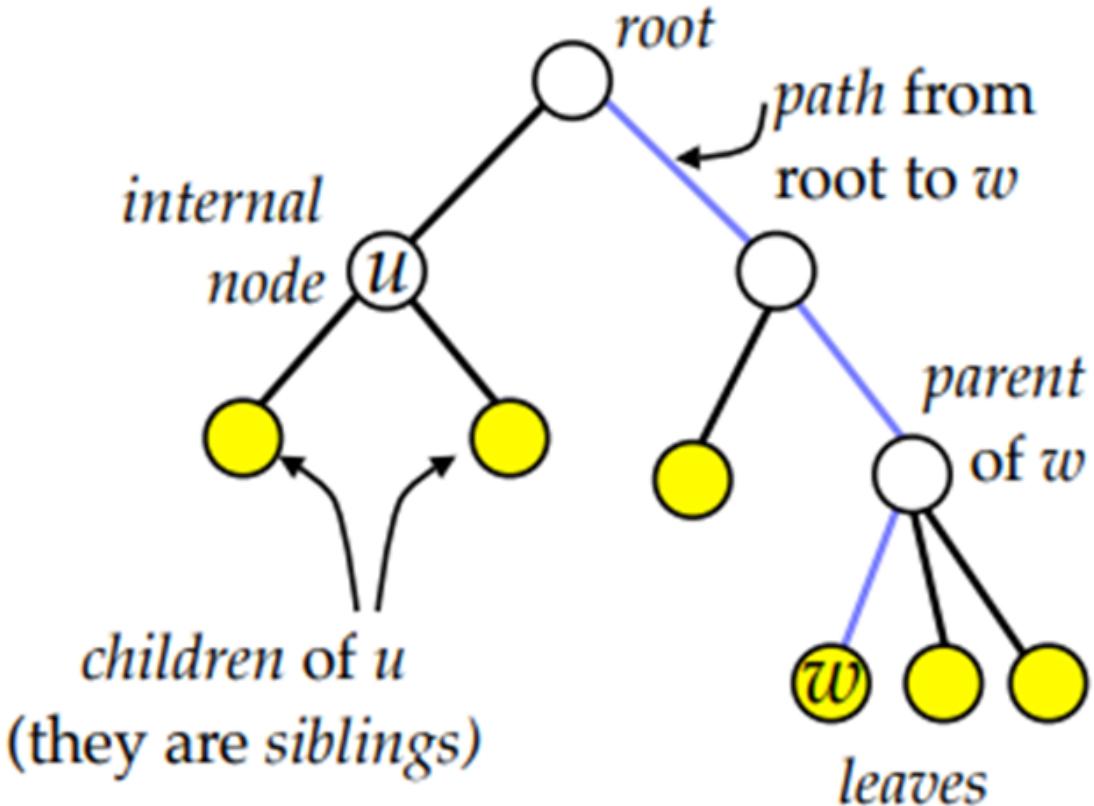
- $\Lambda$  is a tree
- If  $T_1, T_2, \dots, T_k$  are trees with roots  $r_1, r_2, \dots, r_k$  and  $r$  is a node  $\notin$  any  $T_i$ , then the structure that consists of the  $T_i$ , node  $r$ , and edges  $(r, r_i)$  is also a tree.



## Terminology

- $r$  is the parent of its children  $r_1, r_2, \dots, r_k$ .
- $r_1, r_2, \dots, r_k$  are siblings.
- root = distinguished node, usually drawn at top. Has no parent.
- If all children of a node are  $\Lambda$ , the node is a leaf. Otherwise, the node is a internal node.
- A path in the tree is a sequence of nodes  $u_1, u_2, \dots, u_m$  such that each of the edges  $(u_i, u_{i+1})$  exists.
- A node  $u$  is an ancestor of  $v$  if there is a path from  $u$  to  $v$ .
- A node  $u$  is a descendant of  $v$  if there is a path from  $v$  to  $u$ .







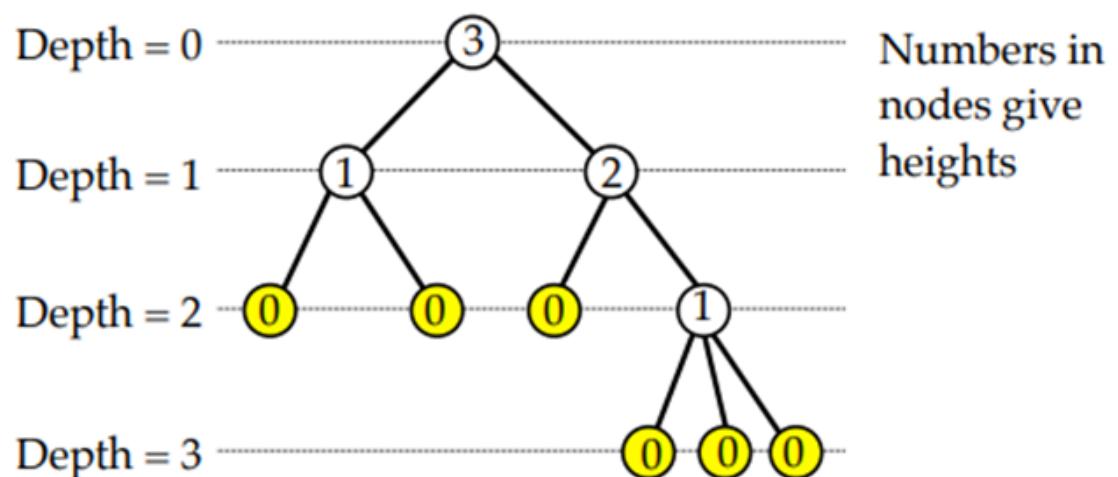
## Rooted trees

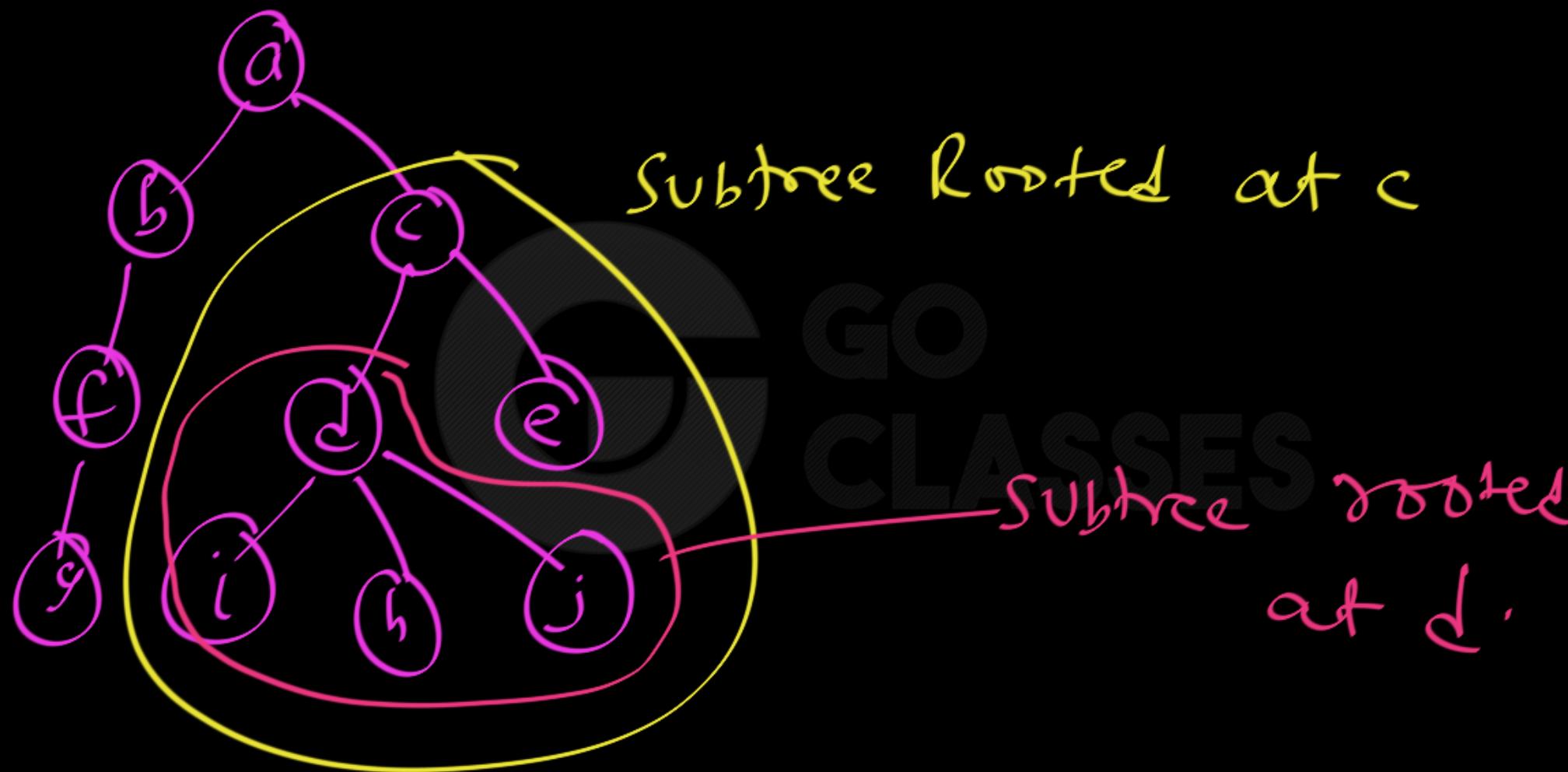
- A *rooted tree* is a tree with a designated vertex  $v_0$  called the *root*. According to Theorem 30, each vertex  $v_i$  is connected to the root by a unique path  $v_i, v_{i-1}, \dots, v_1, v_0$ , in which case  $v_{i-1}$  is called *father* (or *parent*) of  $v_i$ , and  $v_i$  is a *child* (or *son*) of  $v_{i-1}$ .
- The *height* of a rooted tree is the distance from the root to the farthest leaf.
- A *binary tree* is a rooted tree in which every vertex has at most two children.
- A *full binary tree* is a binary tree in which every non-leaf vertex has exactly two children.

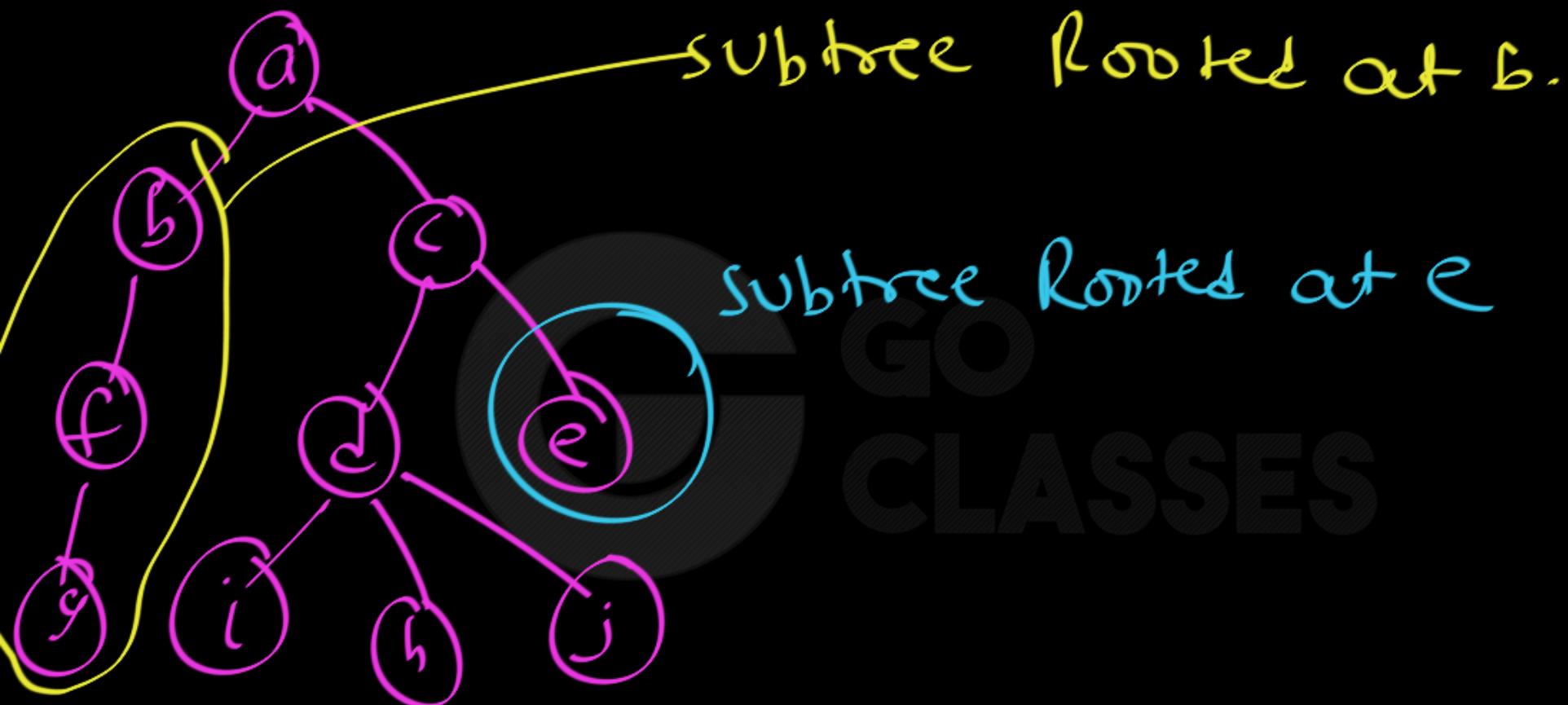


# Height & Depth

- The height of node  $u$  is the length of the longest path from  $u$  to a leaf.
- The depth of node  $u$  is the length of the path from the root to  $u$ .
- Height of the tree = maximum depth of its nodes.
- A level is the set of all nodes at the same depth.



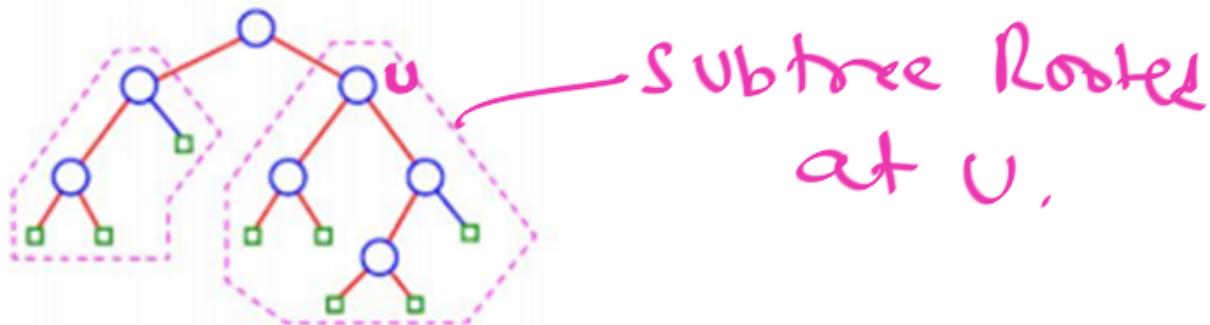


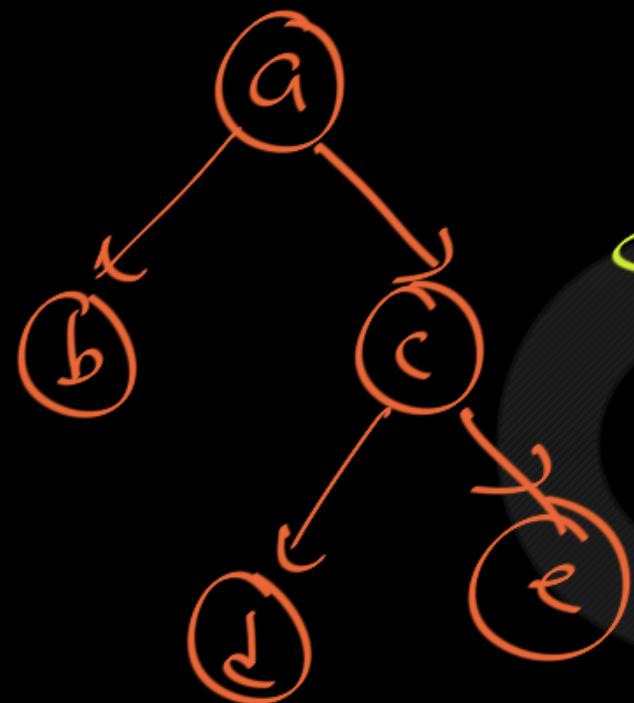




## Subtrees

- Given a rooted tree and a node  $v$ , the **subtree** rooted at  $v$  includes  $v$  and its descendants.





Rooted Tree

underlying  
structure



Tree



## Basic Properties

- Every node except the root has exactly one parent.
- A tree with  $n$  nodes has  $n-1$  edges  
(every node except the root has an edge to its parent).
- There is exactly one path from the root to each node.

"Rooted Tree" → Directed Graph

↓  
very special

Term "Degree" is defined (but  
in another way)



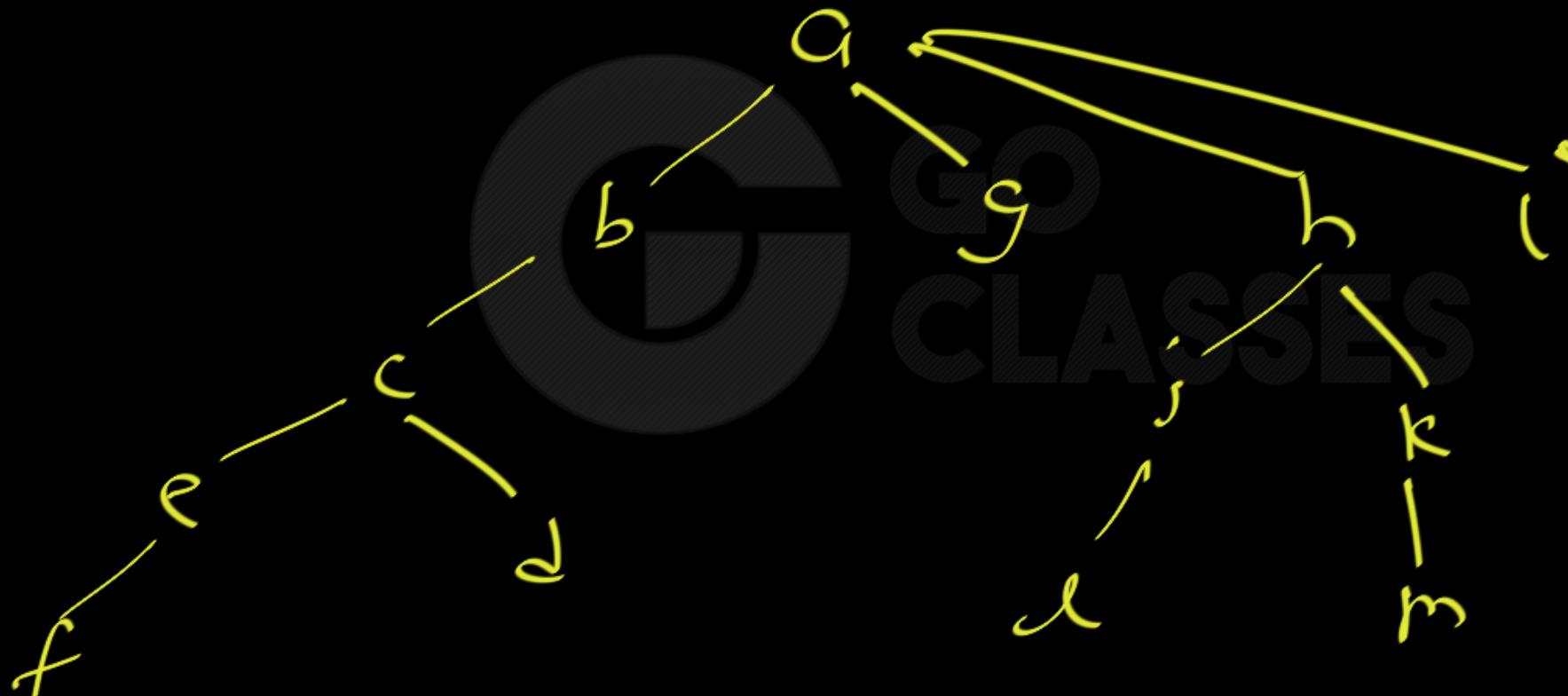
Degree of a node in a Rooted Tree:

= number of children.



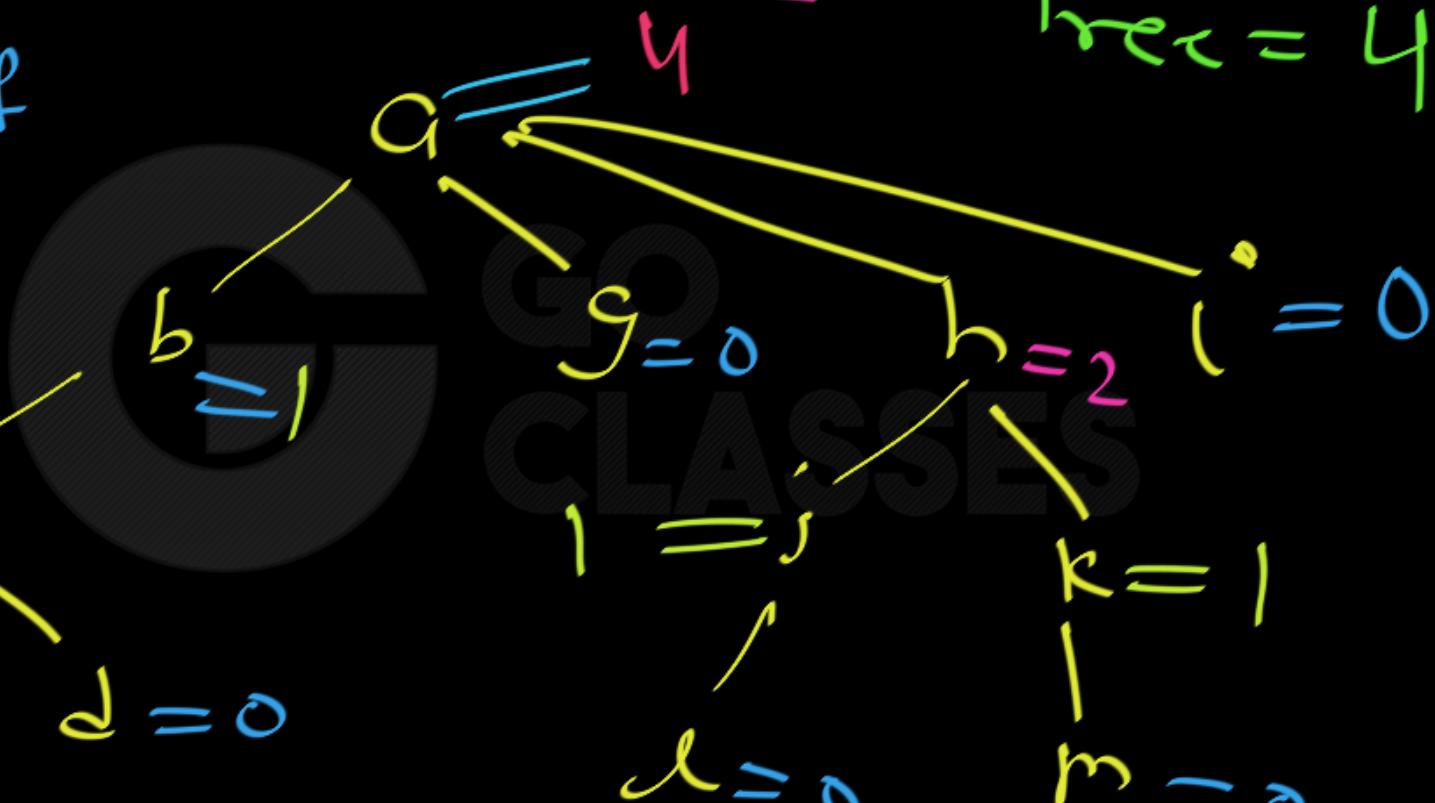
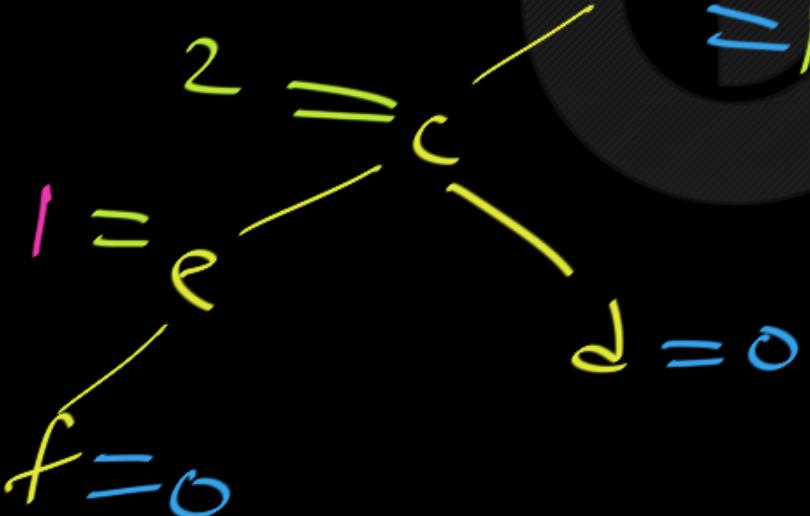


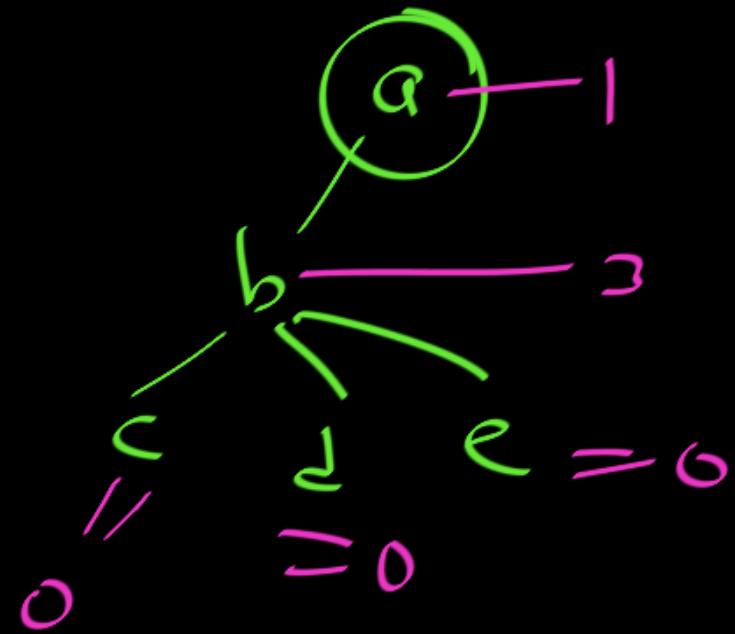
find degree of all nodes;



find Degree of all nodes; Degree of Tree = 4

Number of children





Degree of Tree = 3 ✓



- **DEFINITION:** Nodes with no children are called *leaf nodes*. All other nodes are called *interior nodes*.
- **DEFINITION:** The degree of a node is the number of its children.  
The degree of a tree is the maximum degree of any of its nodes.
- **DEFINITION:** Nodes with the same parent are called *siblings*.

- Note that the edges are directed edges. This means there is a direction associated with them.  
Nobody draws the edges as directed edges, it's just understood.



# Graph Theory :

Next Topic :

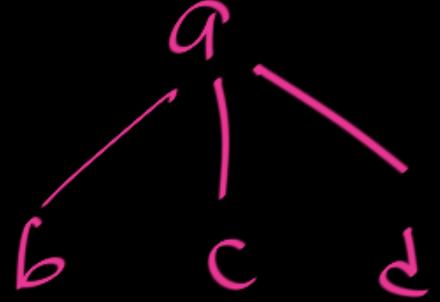
Binary Trees

Website : <https://www.goclasses.in/>

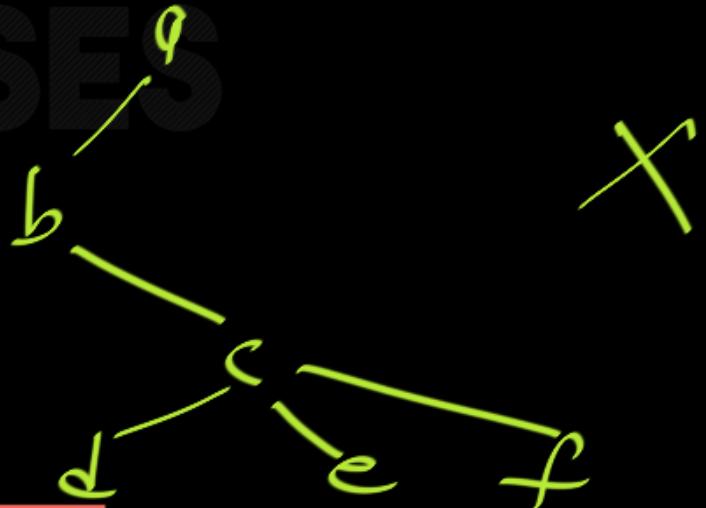


- The *height* of a rooted tree is the distance from the root to the farthest leaf.
- A binary tree is a rooted tree in which every vertex has at most two children.
- A *full binary tree* is a binary tree in which every non-leaf vertex has exactly two children.





Not binary  
Tree





leaf: node without children.

Deg 0 node.

Degree 2 node:

exactly 2 child node



$\varphi: L = \text{no. of Leaves} (\text{deg 0 nodes})$

$D'' = \text{no. of Degree 2 nodes.}$

find Relation b/w them.



Proof 1: Binary Tree  $\rightarrow$  Rooted Tree

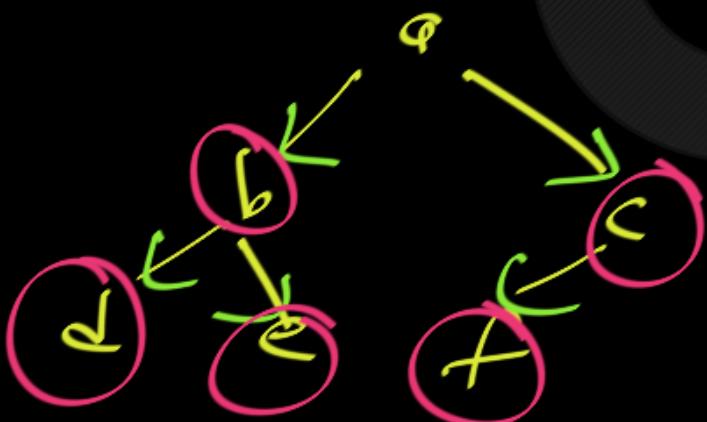
$$\begin{aligned} \text{Total InDef} &= \text{Total outDef} \xleftarrow{\text{Directed Graph}} \\ &= |E| = n - 1 \end{aligned}$$

$$\frac{\text{Total InDeg}}{\checkmark} = \underline{\text{Total outDeg}} = |E| = n - 1 \quad \checkmark$$

Total In Deg :

In Deg of every node Except Root = 1

In Deg (Root) = 0



$$\boxed{\text{Total In Deg} = n - 1}$$



In a Binary Tree:

How many type of Nodes?

Dep 0 nodes = Leaves  
Dep 1  
Dep 2

$L = \frac{\text{Number of Leaves}(\text{Depth})}{\text{Depth}}$

$D^1 = \frac{\text{" " Depth - 1 nodes}}{\text{" "}}$

$D'' = \frac{\text{" " Depth - 2 " "}}{\text{" "}}$



Total Out-Deg :

Out-Deg of leaves = 0

Out-Deg of Dep-1 node = 1  
" 1, " Dep-2 = 2

Total OutDegree = L(0) + D'(1) + 2 D'



So,

$$\underline{\underline{n = L + D' + D''}}$$

$$D' + 2D'' = n - 1 = L + D' + D'' - 1$$

$$= \boxed{D' = L - 1}$$

$$L = D'' + 1$$

$$\boxed{L = D'' + 1}$$

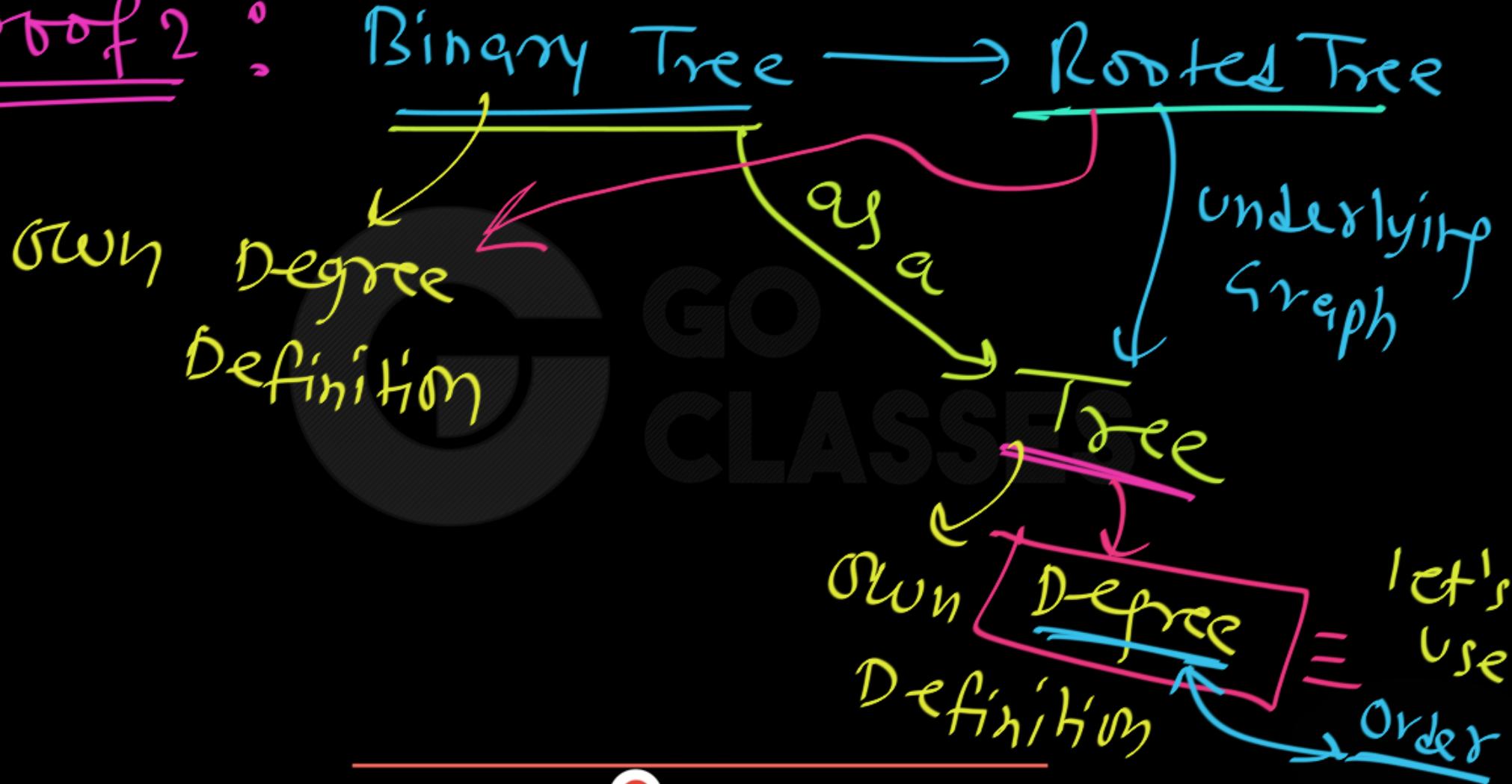


In Binary Tree :

$$L = D + 1$$

#Leaves = number of 2 - Deg nodes  
+ 1

Proof 2 :



Let's use order won for Tree instead  
of Degree.

$$\boxed{n \geq 2}$$

Order of Leaf = 1

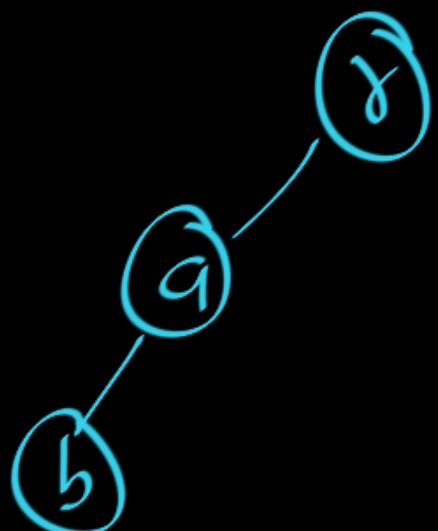
Order of Dep-1 Node = 2

" " Dep-2 " = 3

If Root is Dep-1 Node

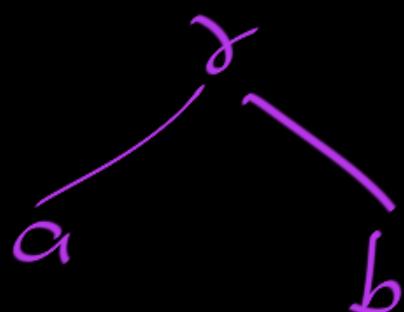


$$\text{order}(\text{Root}) = 1$$



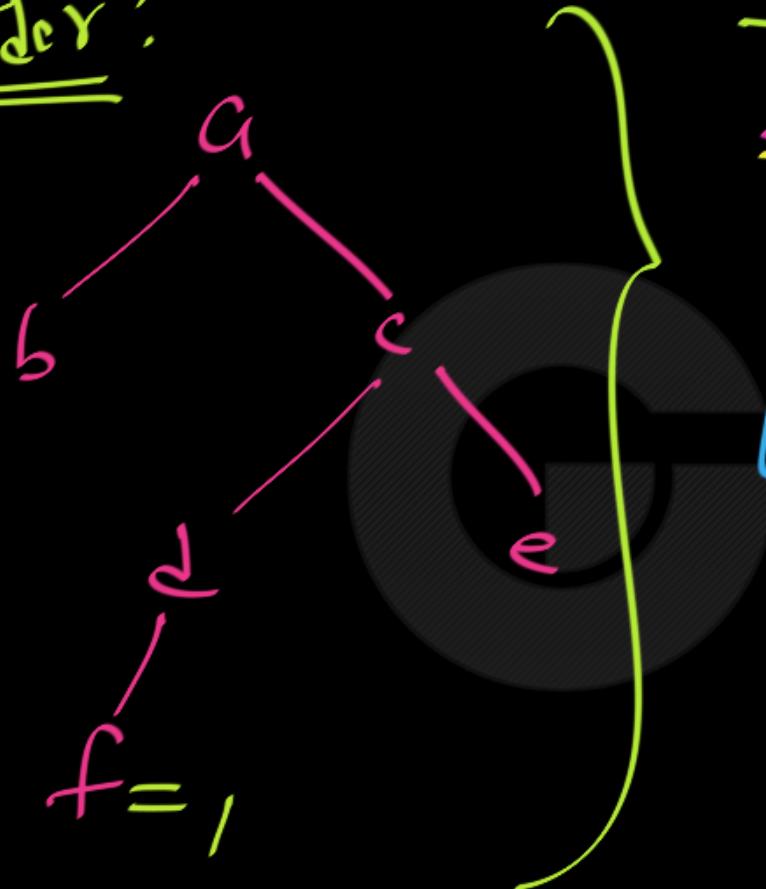
If Root is  
Dep-2 node:

$$\text{order}(\text{Root}) = 2$$





Order:



Total Order =  $\underline{\underline{2(n-1)}}$

$L(1) + D'(2) + D''(3) - 1$   
 $= \underline{\underline{2(n-1)}}$



$$L + 2D' + 3D'' - 1 = 2(L + D' + D'' - 1)$$

$$D'' = L -$$

$$L = D' + 1$$



In A binary tree if there are  $N$  leaf nodes then the number of nodes having 2 children will be  $N-1$





GATE CSE 2015 Set 2

A binary tree T has 20 leaves. The number of nodes in T having two children is

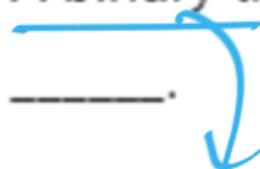
\_\_\_\_\_.





GATE CSE 2015 Set 2

A binary tree T has 20 leaves. The number of nodes in T having two children is



19

Rooted  
Tree





## GATE CSE 2015 Set 3

Consider a binary tree T that has 200 leaf nodes. Then the number of nodes in T that have exactly two children are \_\_\_\_\_.

199





## Graph Theory :

Next Topic :

Full Binary Trees

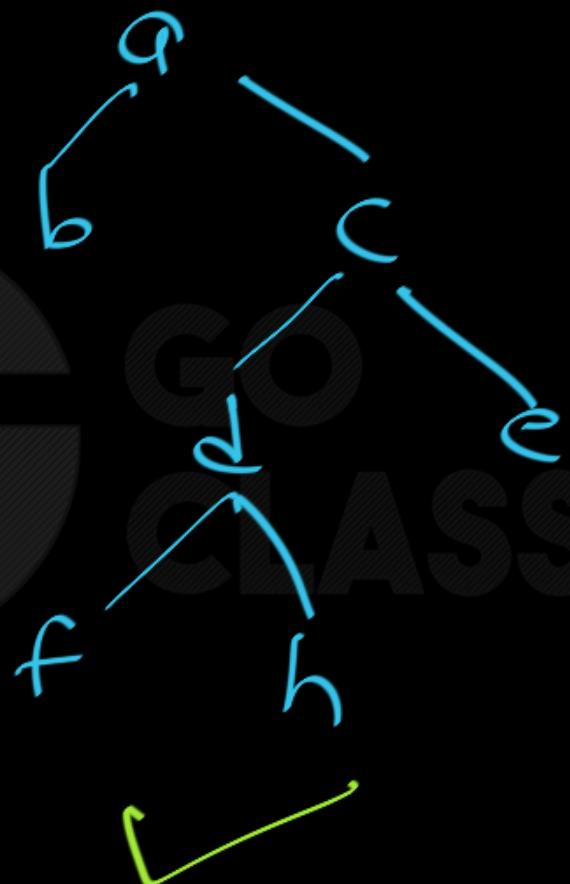
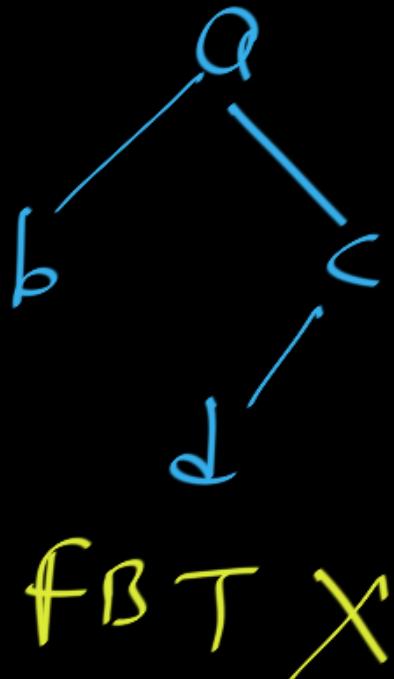
Website : <https://www.goclasses.in/>



full Binary Tree : (FBT)

No node has exactly one  
child.

Every internal node has 2  
children.



In FBT,

$$D' = 0$$

$$n = L + D''$$



In FBT,

$D' = \text{Number of 1-child nodes} = 0$

$L = "0 - " = L$

$D'' = "2 - 1, " = D''$

$$n = L + D'$$

In FBT,

If L is known then

$$\mathcal{D}'' = ? = L - 1$$

$$n = ? = \mathcal{D}'' + L = (L - 1) + L = \underline{\underline{2L - 1}}$$

$$n = 2L - 1$$

In Binary Tree, If L is known then

$$D'' = ? = L - 1$$

$$n = ? = ? = L + D'' + D'''$$

We can not find n.

In FBT,

If  $n$  is known then

$$D' = ? = L - 1 = \frac{n+1}{2} - 1 = \frac{n-1}{2}$$

$$L = ? = \frac{n+1}{2} \checkmark$$

$$\boxed{n = 2L - 1}$$

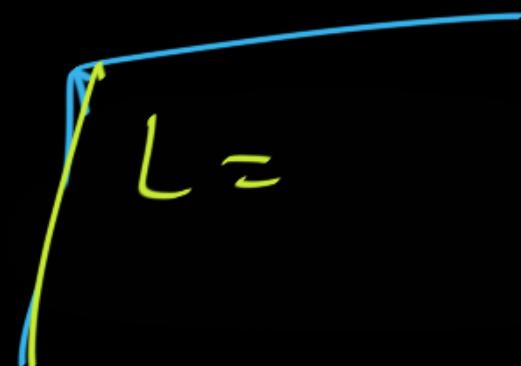
In FBT,

If  $D''$  is known then

$$\underline{n} = ? = 2L - 1 = 2(D'' + 1) - 1$$

$$= 2D'' + 1 \checkmark$$

$$L = ? = D'' + 1 \checkmark$$





In FBT,

Number of internal nodes =  $D$  ✓  
 $= L - 1$



In BT,

Number of internal nodes =  $D'' + D'$

internal nodes

- A full binary tree is a binary tree in which every non-leaf vertex has exactly two children.

Theorem: Let  $T$  be a nonempty, full binary tree Then:

- 
- (a) If  $T$  has  $I$  internal nodes, the number of leaves is  $L = I + 1$ .
  - (b) If  $T$  has  $I$  internal nodes, the total number of nodes is  $N = 2I + 1$ .
  - (c) If  $T$  has a total of  $N$  nodes, the number of internal nodes is  $I = (N - 1)/2$ .
  - (d) If  $T$  has a total of  $N$  nodes, the number of leaves is  $L = (N + 1)/2$ .
  - (e) If  $T$  has  $L$  leaves, the total number of nodes is  $N = 2L - 1$ .
  - (f) If  $T$  has  $L$  leaves, the number of internal nodes is  $I = L - 1$ .

Basically, this theorem says that the number of nodes  $N$ , the number of leaves  $L$ , and the number of internal nodes  $I$  are related in such a way that if you know any one of them, you can determine the other two.



## GATE IT 2006

In a binary tree, the number of internal nodes of degree 1 is 5, and the number of internal nodes of degree 2 is 10. The number of leaf nodes in the binary tree is

- A. 10
- B. 11
- C. 12
- D. 15



GATE IT 2006

In a binary tree, the number of internal nodes of degree 1 is 5, and the number of internal nodes of degree 2 is 10. The number of leaf nodes in the binary tree is

- A. 10
- B. 11
- C. 12
- D. 15

$$L = D'' + 1$$

$$= 10 + 1 = 11$$

$$n = L + D^1 + D'' = 11 + 5 + 10 = 26$$



## 2.11.1 Trees: GATE2010-1

Let  $G = (V, E)$  be a graph. Define  $\xi(G) = \sum_d i_d * d$ , where  $i_d$  is the number of vertices of degree  $d$  in  $G$ .

If  $S$  and  $T$  are two different trees with  $\xi(S) = \xi(T)$ , then

- A.  $|S| = 2|T|$
- B.  $|S| = |T| - 1$
- C.  $|S| = |T|$
- D.  $|S| = |T| + 1$





## 2.11.1 Trees: GATE2010-1

Let  $G = (V, E)$  be a graph. Define  $\xi(G) = \sum_d i_d * d$ , where  $i_d$  is the number of vertices of degree  $d$  in  $G$ .

If  $S$  and  $T$  are two different trees with  $\xi(S) = \xi(T)$ , then

- A.  $|S| = 2|T|$
- C.  $|S| = |T|$
- B.  $|S| = |T| - 1$
- D.  $|S| = |T| + 1$

Total degree

Tree  $S \Rightarrow n_1$  vertices

Tree  $T \Rightarrow n_2$  vertices

$$\text{Total deg} = 2(n_1 - 1) = 2(n_2 - 1) = \text{Total deg}$$

$n_1 = n_2$

$; |E(S)| = |E(T)|$



$$\begin{aligned} \checkmark |S| &= \frac{\text{number of vertices}}{n} \\ |T| &= \dots \end{aligned} \quad } \quad |S|=|T| \quad \checkmark$$

---

$$\begin{aligned} \checkmark |S| &= \frac{\text{number of Edges}}{n} \\ |T| &= \dots \end{aligned} \quad } \quad |S|=|T| \quad \checkmark$$



# Graph Theory :

Next Topic :

Questions related to Components

Website : <https://www.goclasses.in/>



Give the minimum and the maximum number of edges in an undirected connected graph of  $n$  vertices?

Connected Graph

$$\begin{cases} \min |E| = n-1 & (\text{Tree}) \\ \max |E| = n_{\Sigma} = \frac{n(n-1)}{2} & (\underline{\underline{k_n}}) \end{cases}$$

minimally  
connected



Maximum number of edges in unconnected graph





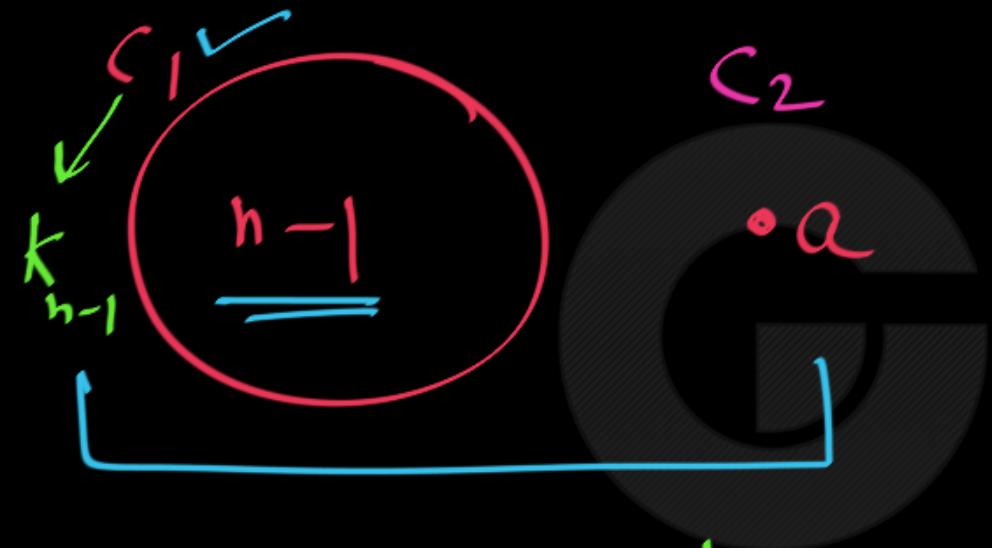
Maximum number of edges in unconnected graph

Disconnected Graph  $\Rightarrow$  At least two Components.

two Components.

to get max. number  
of Edges.

2 Components → Want  $\max |E|$  ?



$$\max |E| = \frac{n-1}{2} C_2$$

$$= \frac{(n-1)(n-2)}{2}$$

$C_1$

$n/2$

$k_{n/2}$

$C_2$

$n/2$

$k_{n/2}$

$$\begin{aligned} \max |E| &= \frac{n/2}{2} C_2 + \frac{n/2}{2} C_2 \\ &= 2 \frac{\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)}{2} = \frac{n(n-2)}{4} \end{aligned}$$

DisConnected Graph  $\Rightarrow \# \text{Components} \geq 2$

we want "max |E|"  $\Rightarrow \# \text{Components} = 2$

To get max |E| :

$$|E| = \frac{(n-1)(n-2)}{2}$$

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Minimum number of Edges in a Connected Graph??

$$\text{---} \quad n-1 \quad \text{---}$$

G GO CLASSES

minimum

How many edges must a graph with N vertices have  
in order to guarantee that it is connected?

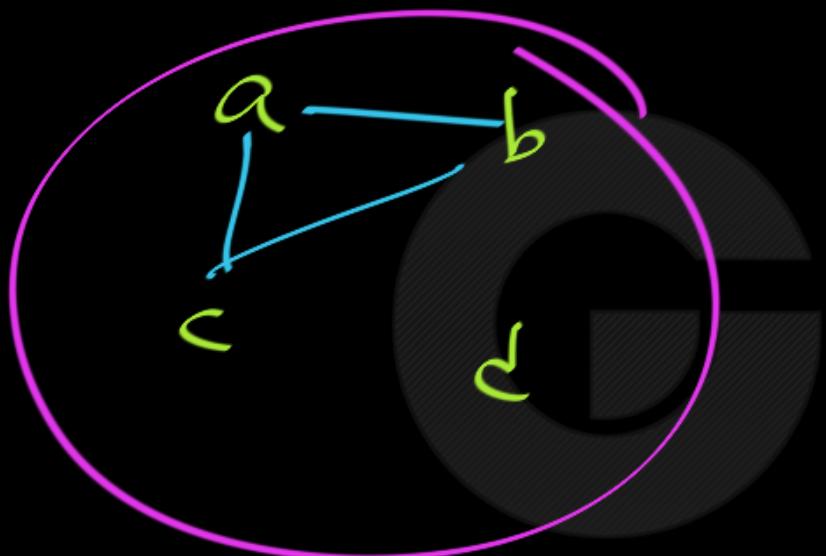


minimum

How many edges must a graph with N vertices have in order to guarantee that it is connected?

$$= (n - 1)C_2 + 1$$

4 vertices  $\rightarrow$  will 3 Edges guarantee that it will be Connected .





"DisConnected Graph"  $\rightarrow \max |E| = \frac{n-1}{2} C_2$

G with  $> \frac{n-1}{2} C_2$  edges  
Connected ✓



How many edges must a graph with N vertices have in order to guarantee that it is connected?

$$= \frac{n-1}{2} + 1$$

Every simple undirected graph with more than  $(n - 1)(n - 2)/2$  edges is connected





1. There's a **connected** graph with 40 vertices, what is the minimum amount of edges there could be?
2. There's a graph with 40 vertices, what is the minimum amount of edges that **guarantees** a connected graph?



1. There's a connected graph with 40 vertices, what is the minimum amount of edges there could be?

$$40 - 1 = 39$$

2. There's a graph with 40 vertices, what is the minimum amount of edges that guarantees a connected graph?

$$29_{C_2} + 1 = \frac{39 \times 38}{2} + 1 =$$

① Already have : Connected Graph

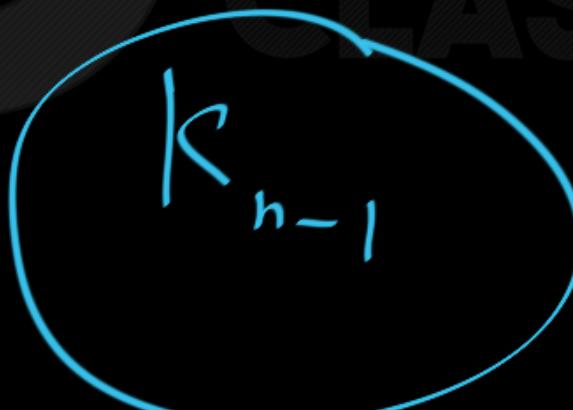
② Already have : 40 vertices



Graph with 2 Components  $\Rightarrow$

$$\max |E| = n-1$$

To get this;



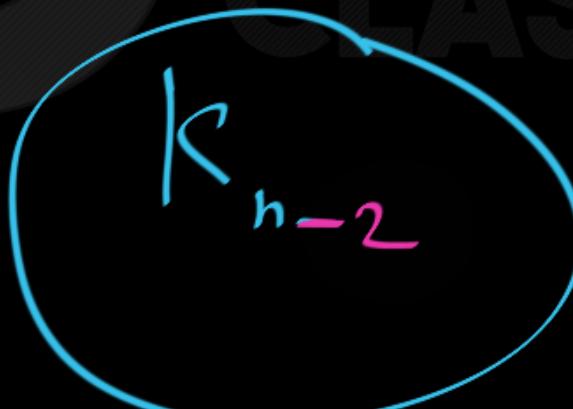
a



Graph with 3 Components  $\Rightarrow$

$$\max |E| = {}^{n-2}C_2$$

To get this;



a

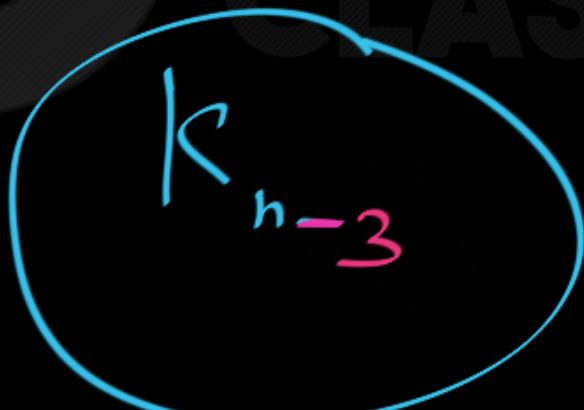
b



Graph with 4 Components  $\Rightarrow$

$$\max |E| = h-3 C_2$$

To get this;



a b c



Graph with  $k$  Components  $\Rightarrow$

$$\max |E| = n-k+1 C_2$$

To get this;





Graph is Connected iff one Component.

GO  
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## 2.4.4 Graph Connectivity: GATE1991-01,xv

The maximum number of possible edges in an undirected graph with  $n$  vertices and  $k$  components is \_\_\_\_\_.





## 2.4.4 Graph Connectivity: GATE1991-01,xv

The maximum number of possible edges in an undirected graph with  $n$  vertices and  $k$  components is \_\_\_\_\_.

$$\frac{(n-k+1)(n-k)}{2}$$



- Show that if a graph with  $n$  vertices has more than  $\binom{n-1}{2}$  edges, then it is connected.





## 2.4.11 Graph Connectivity: GATE2003-8, ISRO2009-53

HW

Let  $G$  be an arbitrary graph with  $n$  nodes and  $k$  components. If a vertex is removed from  $G$ , the number of components in the resultant graph must necessarily lie down between

- A.  $k$  and  $n$
- B.  $k - 1$  and  $k + 1$
- C.  $k - 1$  and  $n - 1$
- D.  $k + 1$  and  $n - k$





## 2.4.4 Graph Connectivity: GATE1991-01,xv

The maximum number of possible edges in an undirected graph with  $n$  vertices and  $k$  components is \_\_\_\_\_.





## 2.4.4 Graph Connectivity: GATE1991-01,xv

The maximum number of possible edges in an undirected graph with  $n$  vertices and  $k$  components is \_\_\_\_\_.

$$\frac{n-k+1}{2} C_2 = \frac{(n-k+1)(n-k)}{2}$$



- Show that if a graph with  $n$  vertices has more than  $\binom{n-1}{2}$  edges, then it is connected.





- Prove that a graph is connected if and only if for every partition of its vertex set into two non-empty sets  $A$  and  $B$  there is an edge  $ab \in E(G)$  such that  $a \in A$  and  $b \in B$ .

