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GATE AIR 33

ML

10 - 15

Summary So Far...

Sigma Notation

Last value of *i*

$$\sum_{i=1}^{n} x_i$$

Formula for the terms

First value of *i*

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + (n-1) + n$$

Shifting indices of Summation

$$\sum_{j=0}^{3} 2j$$

Shift summation to 1 as start

Solution:

$$\sum_{j=0}^{3} 2j$$

Relation

$j=0,1,2,3$ $\quad$ $k=1,2,3,4$ $\quad$ $k = j+1$

$$\sum_{j=0}^{3} 2j = \sum_{k=1}^{4} 2(k-1)$$

Will this help me in GATE ?

GATE CSE 2003 | Question: 64

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Let **S** be a stack of size $n \geq 1$. Starting with the empty stack, suppose we push the first n natural numbers in sequence, and then perform n pop operations. Assume that Push and Pop operations take X seconds each, and Y seconds elapse between the end of one such stack operation and the start of the next operation. For $m \geq 1$, define the stack-life of m as the time elapsed from the end of $Push(m)$ to the start of the pop operation that removes m from **S**. The average stack-life of an element of this stack is

A. $n(X+Y)$
B. $3Y+2X$
C. $n(X+Y)-X$
D. $Y+2X$

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Best answer

Let us represent stack-life of i^{th} element as $S(i)$. The i^{th} element will be in stack till $(n-i)$ elements are pushed and popped. Plus one more Y for the time interval between the push of i^{th} element and the $i+1^{th}$ element. So,

$$S(i) = Y + 2.(n-i)(Y+X) = Y + 2.(n-i)Z$$

average stack-life will, $A = \Sigma \frac{S(i)}{n}$

Sequence and Series

Arithmetic Progression

+d +d

2, 4, 6, 8, 10, 12, 14, . . .

General Term and Summation of n terms ?

Gauss

$1 + 2 + - - - 100$

$100 + 99 + \quad - - - 1$

General Term and Summation of n terms ?

$$T_n = a + (n-1)d$$

$$S_n = \frac{n}{2}(a + l) = \frac{n}{2}\left(2a + (n-1)d\right)$$

Few Formulas

$$1 + 2 + 3 + \cdots n = \frac{n(n+1)}{2}$$

$$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} (i+1)^2$$

$$= \sum_{i=1}^{n} (i^2 + 1 + 2i)$$

$$= \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i$$

$$\frac{n(n+1)(2n+1)}{6} \qquad n \qquad \cancel{2} \cdot \frac{n(n+1)}{\cancel{2}}$$

$$(a+b)^2$$

$$= (a)^2 + (b)^2$$

$$+ 2(a)(b)$$

$$\sum_{i=1}^{n} (i+1)^2$$

$i+1 = j$

$$\sum_{j=2}^{n+1} j^2 = 2^2 + 3^2 + \cdots + (n+1)^2$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} - 1$$

$$2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6} - 1^2$$

$(n+1)$

$$\sum_{i=1}^{n} i(n-i)$$

$$= \sum_{i=1}^{n} ni - i^2$$

$$= \sum_{i=1}^{n} ni - \sum_{i=1}^{n} i^2$$

$$\sum_{i=1}^{n} i(n-i)$$

$$= \sum_{i=1}^{n} ni - i^2$$

$$= n\sum_{i=1}^{n} i - \sum_{i=1}^{n} i^2$$

$$\frac{n(n+1)}{2}$$

$$\frac{n(n+1)(2n+1)}{6}$$

Geometric Progressions

$$10 \xrightarrow{\times 10} 100 \xrightarrow{\times 10} 1000 \xrightarrow{\times 10} 10000, \ldots$$

1 2 4 8 16 32 64

a_1 a_2

$$1 \xrightarrow{\times 2} 2 \xrightarrow{\times 2} 4 \xrightarrow{\times 2} 8 \xrightarrow{\times 2} 16 \xrightarrow{\times 2} 32 \xrightarrow{\times 2} 64$$

Common ratio (r)

$$a, \xrightarrow{\times r} ar, \xrightarrow{\times r} ar^2, \quad ar^3, \quad \ldots .$$

$T_n = a \cdot r^{n-1}$

Yesterday

$a \quad a+d \quad a+2d \quad \ldots$

$T_n = a+(n-1)d$

Given the first two terms in a geometric progression as 5 and 1 2 , what is the 7th term?

$$5 \xrightarrow{\times r} 12, \ldots$$

$$r = \frac{12}{5}$$

$$5r = 12$$

$$T_7 = 5\left(\frac{12}{5}\right)^{7-1}$$

$$= 5\left(\frac{12}{5}\right)^{6}$$

$$r = \frac{a_{k+1}}{a_k}$$

Example

How many terms are there in the geometric progression

$$2, \quad 4, \quad 8, \quad \ldots, \quad 128\,?$$

2^7

nth term

let there are n terms

$T_n = a r^{n-1}$

2^6

$128 = 2\, r^{n-1} \Rightarrow 2^{n-1} = 64 \Rightarrow n-1 = 6$

$n = 7$

Example

How many terms are there in the geometric progression

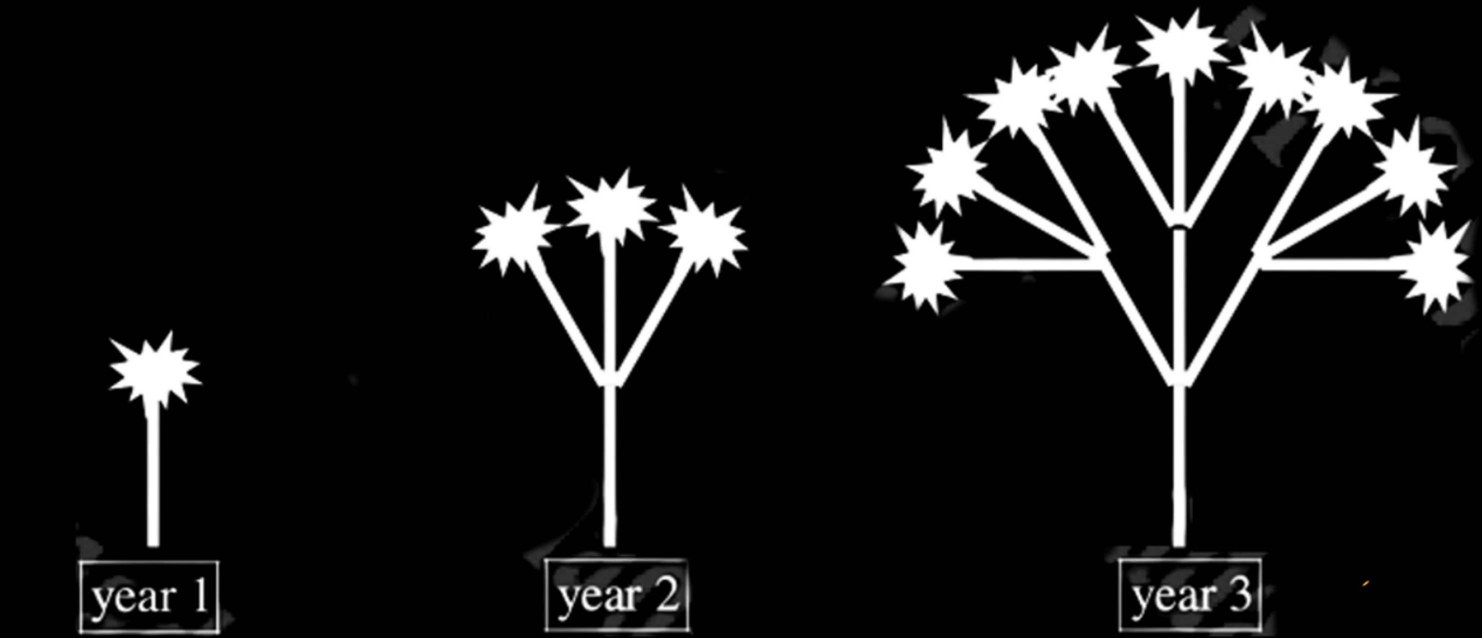
$$2, \quad 4, \quad 8, \quad \ldots, \quad 128\,?$$

Solution

In this sequence $a = 2$ and $r = 2$. We also know that the n-th term is 128. But the formula for the n-th term is ar^{n-1}. So

$$\begin{aligned} 128 &= 2 \times 2^{n-1} \\ 64 &= 2^{n-1} \\ 2^6 &= 2^{n-1} \\ 6 &= n - 1 \\ n &= 7\,. \end{aligned}$$

So there are 7 terms in this geometric progression.


year 1
year 2
year 3

The figure shows a flowering plant. In year 1 it produces a single stem with a flower at the end.

In year 2 , the flower withers and in its place three more stems are produced, with each new stem having a new flower at its end, i.e. 4 stems in total.

In year 3 , the flowers wither again and in each of their places a new stems is produced, with each new stem having a new flower at its end, i.e. 13 stems in total.

This flowering pattern continues every year.

Find an expression for

i. The number of flowers in the n^{th} year

ii. The number of stems in the n^{th} year.

No. of flowers

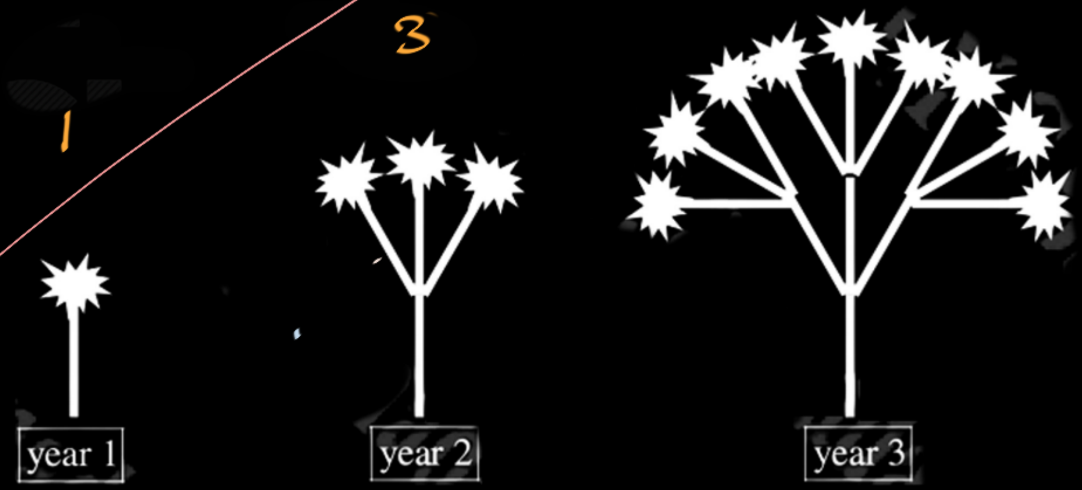
$1+3+3^2+ \cdots 3^{n-1}$

3^0 $\quad$ 3^1 $\quad$ 3^2

3

1

3^{n-1}



No. of stems

1 $\quad$ $1+3$ $\quad$ $1+3+3^2$

$0 \quad 0$

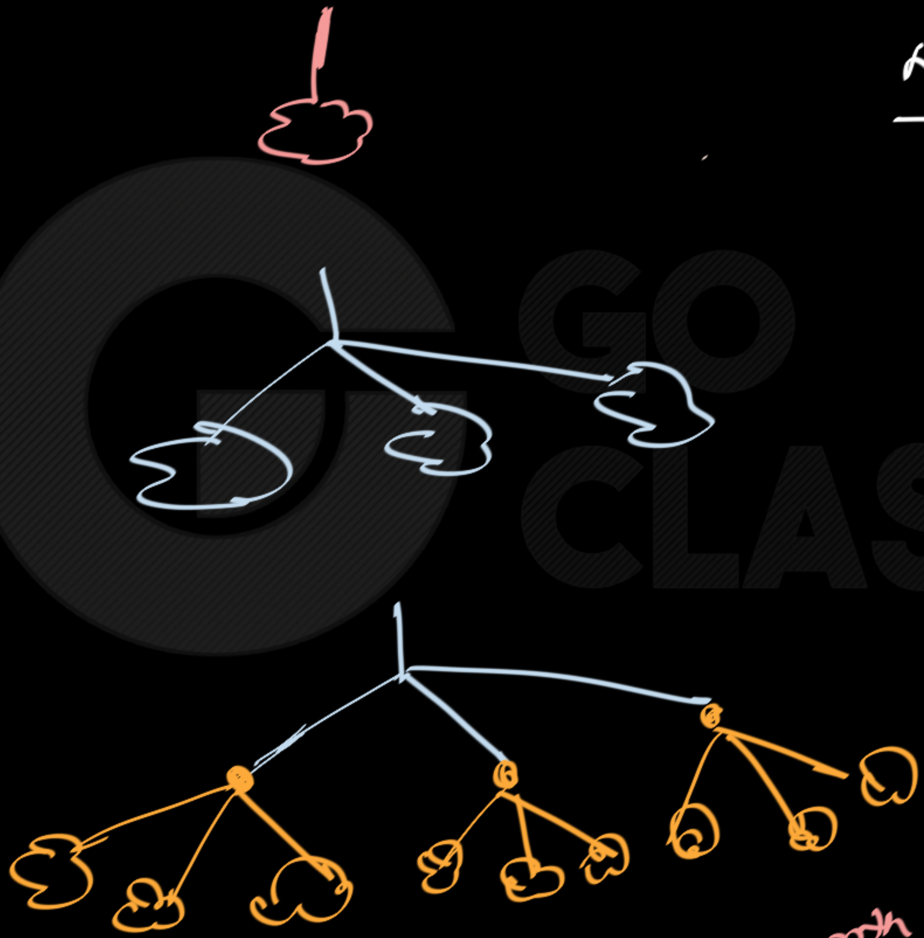
$0 \quad 0 \quad 0$

$a \;,\; ar \;,\; ar^2$

$r=0$

$a=0$

$0, \; 0, \; 0$



No. of flowers		No. of Stems
1	3^0	1
3	3^1	$1+3$
3^2	3^2	$1+3+3^2$

nth Year : 3^{n-1}

YEAR	FLOWERS	STEMS
1	1	1
2	3	1+3 = 4
3	9	4+9 = 13
4	27	13+27 = 40
⋮	⋮	⋮
	GP	SUM OF
	$a=1$	ITS TERMS
	$r=3$	

∴ Flowers $\boxed{f_n = 3^{n-1}}$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$= \frac{1(3^n-1)}{3-1}$$

$$\boxed{S_n = \frac{3^n-1}{2}}$$

We can take the sum of the first n terms of a geometric series and this is denoted by S_n:

$$S_n = a + ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + 0$$

$$rS_n = 0 + ar + ar^2 + ar^3 + \cdots\cdots ar^{n-1} + ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n) \Rightarrow S_n = \frac{a(1-r^n)}{1-r} \quad r \neq 1$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad r \neq 1$$

$$S_n = a + a + \cdots + a = na \quad (\text{if } r = 1)$$

$2x = x^2$

$x \neq 0$

$\Rightarrow \quad x = 2$

We can take the sum of the first n terms of a geometric series and this is denoted by S_n:

$$\boxed{S_n = \frac{a(1-r^n)}{1-r}}$$

$|r|<1$ $\quad n \to \infty \quad r^n \to 0$

$$S = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

$$\begin{aligned} S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\ rSn &= \phantom{a + {}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\ S_n - rS_n &= a \phantom{{} + ar + ar^2 + ar^3 + \ldots + ar^{n-1}} - ar^n \end{aligned}$$

$$S_n(1 - r) = a(1 - r^n)\,.$$

$$
\begin{aligned}
S_n &= a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \\
rSn &= \phantom{a +{}} ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n \\
S_n - rS_n &= a \qquad\qquad\qquad\qquad\qquad\qquad\qquad - ar^n
\end{aligned}
$$

$$S_n(1 - r) = a(1 - r^n)\,.$$

Now divide by $1 - r$ (as long as $r \neq 1$) to give

$$S_n = \frac{a(1 - r^n)}{1 - r}\,.$$

$$S_n = a + ar + ar^2 + \ldots\ldots\ldots\ldots + ar^{n-2} + ar^{n-1} \qquad (1)$$

$$rS_n = ar + ar^2 + ar^3 + \ldots\ldots\ldots\ldots\ ar^{n-2} + ar^{n-1} + ar^n \qquad (2)$$

$$\Rightarrow (1-r)\, S_n = a(1-r^n)$$

Now, if $r \neq 1$ then $S_n = \frac{a(1-r^n)}{1-r}$

Thus, the sum to n terms of a GP is $S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$

Clearly $S_n = na$ where $r = 1$.

Sum of infinite GP ?

$|r| < 1$

$r = 10$

$a \quad ar \quad ar^2 \quad ar^3 \quad ar^4 \quad - - - - - -$

$S = a + ar + ar^2 + - - - - - - - -$

$rS = \quad ar + ar^2 + ar^3 + - - - - - -$

$S - rS = a \Rightarrow S(1-r) = a \Rightarrow S = \frac{a}{1-r}$

$|r| < 1$

$-1 < r < 1$

$\to r = 1/2 \qquad |r| = 1/2 < 1$

$r = -1/2 \qquad |r| = 1/2 < 1$

$|x| > 1$

$r > 1$

$r < -1$

$\sqrt{4} = 2$

$x = \sqrt{4}$

$= +2$

$= -2$ (wrong)

$+2, -2$

$x^2 = 4$

$\Rightarrow x = \pm\sqrt{4} = \pm 2$

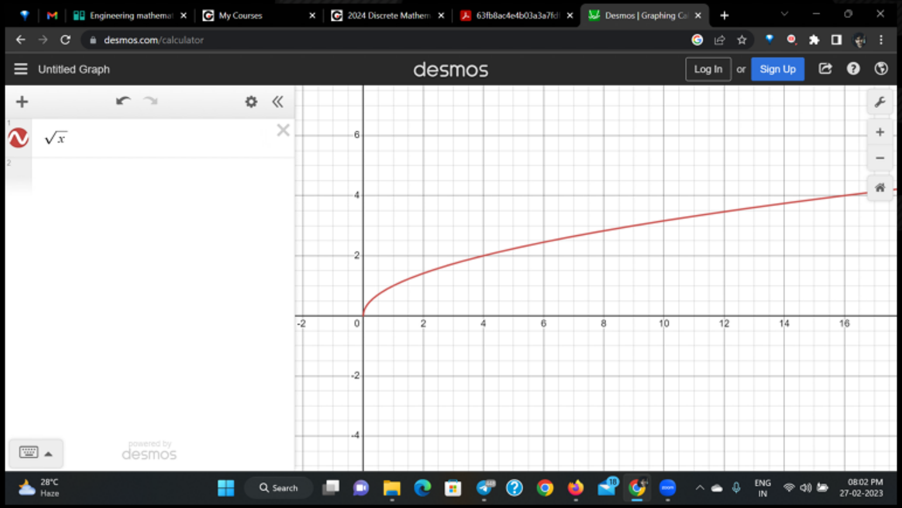
$x^2 = 4$

tell me the possible values of x?

$x = \pm \boxed{\sqrt{4}}$

± 2

$\sqrt{x}$ is ALWAYS positive


desmos.com/calculator
Untitled Graph
desmos
Log In
or
Sign Up
$\sqrt{x}$
6
4
2
-2
-4
-2
0
2
4
6
8
10
12
14
16
powered by
desmos

equation

given

$x = 2$

$x \geq 0$

1

$x^2 = 4$

2

$x = \pm\sqrt{4}$

$= \pm 2$

$\boxed{x = 2 \text{ or } \cancel{-2}}$

$\sqrt{-1} = i$

- $i^2 = (\sqrt{-1})^2 = -1$

- $\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$

$= \sqrt{1}$

$= 1$

$\sqrt{-1} = i$

- $i^2 = (\sqrt{-1})^2 = -1$
- $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1}$
- $\sqrt{-1} \times \sqrt{-1} = \sqrt{-1 \times -1}$
  $= \sqrt{1}$
  $= 1$

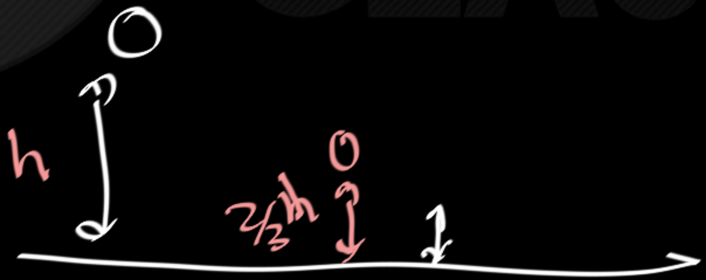
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

true only when at least one of the a or b is positive

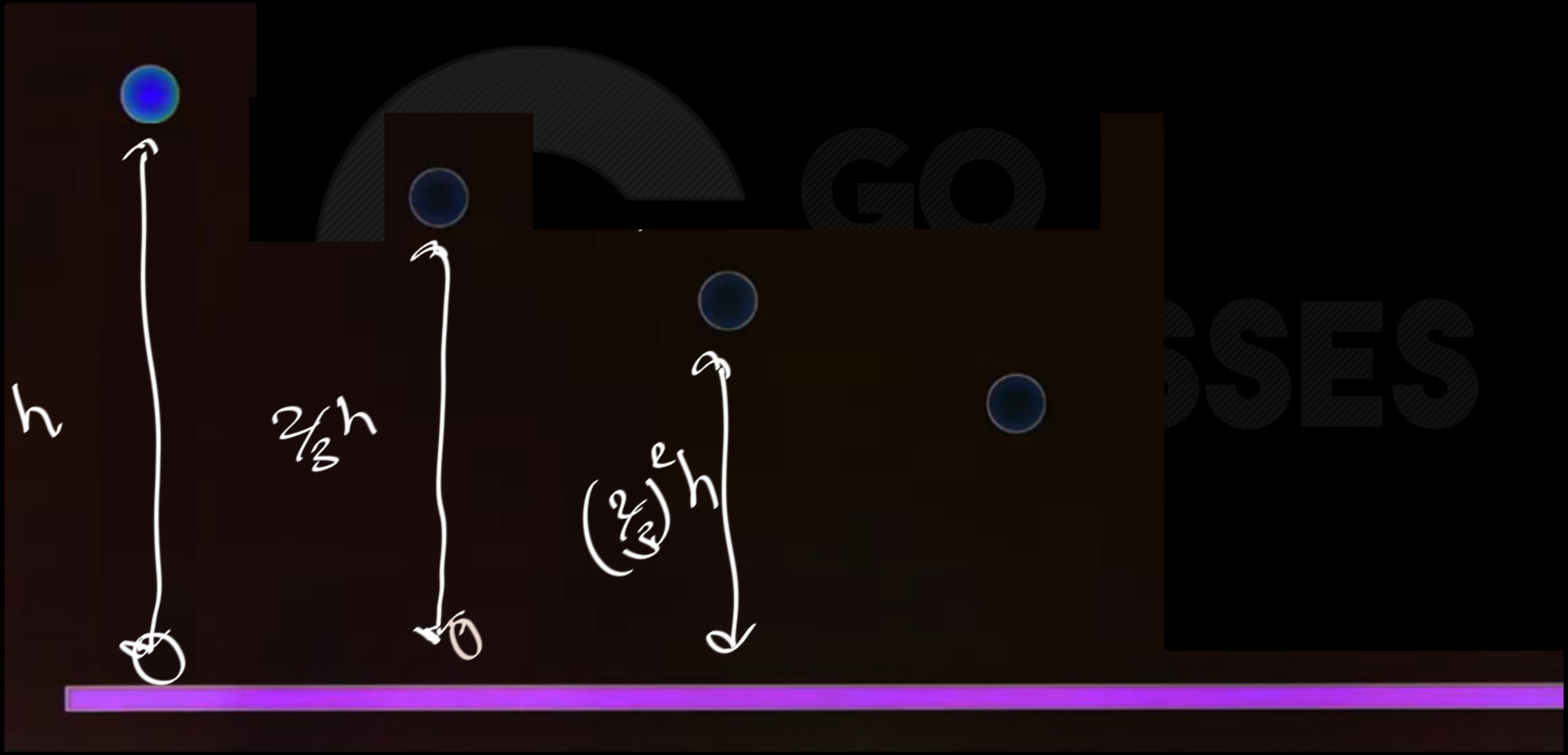
Question:

A football is dropped from 600cm each time it rebounce it rises 2/3rd of the height it has fallen through. Find the total distance travelled by the ball before it comes to rest.?

$h + 2 \times \frac{2}{3} h +$



$$h + 2 \times \frac{2}{3}h + 2 \times \left(\frac{2}{3}\right)^2 h +$$

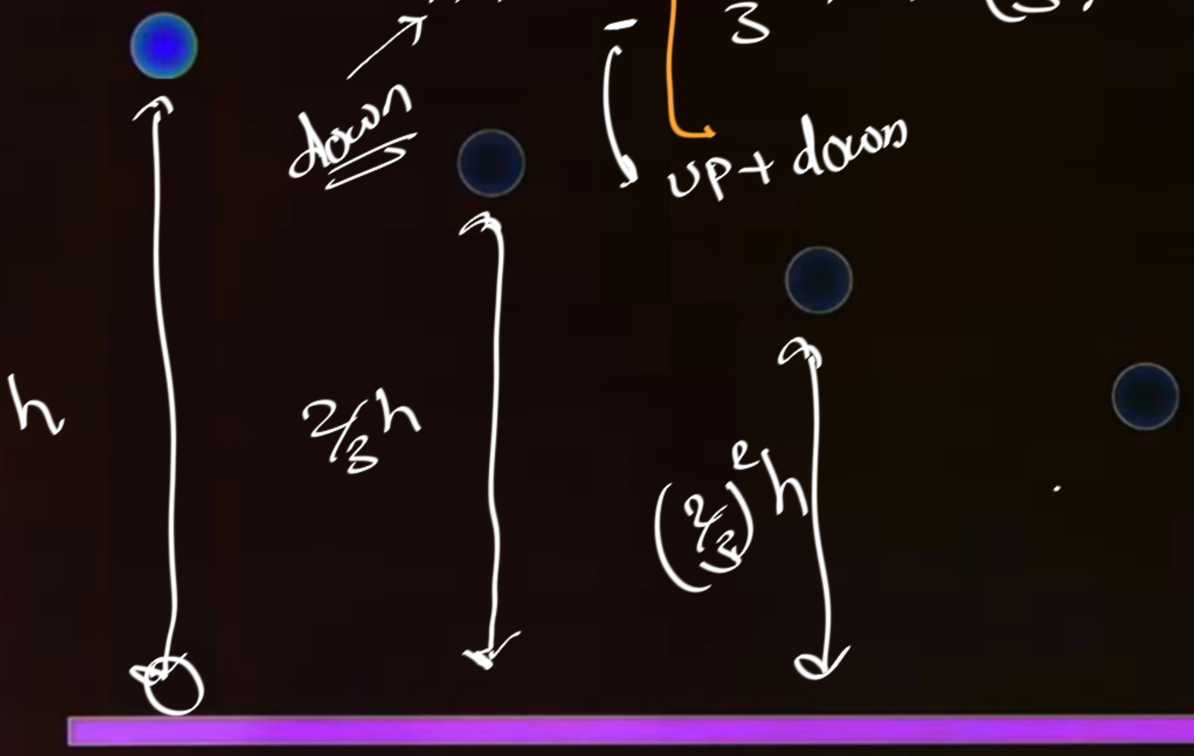


$$h + 2\times\frac{2}{3}h + 2\times\left(\frac{2}{3}\right)^2 h + 2\times\left(\frac{2}{3}\right)^3 h$$

$$h + 2\left[\frac{2}{3}h + \left(\frac{2}{3}\right)^2 h + \left(\frac{2}{3}\right)^3 h + \cdots\right]$$

down

up + down



$$2\left(\frac{\frac{2}{3}h}{1-2/3}\right) + h = \underline{\underline{5h}}$$

Answer:

3000 cm answer

football dropped from 600 cm and each time it renounce and rises to 2/3 of the height it has fallen

so first time 600 cm travel then 2/3 of it approx 400 cm twice (bounce and then down) and so on,

use the concept of sum of term with the help of infinite GP

like

$s = 600 + 600\times2\times2/3 + 600\times2\times2/3\times2/3 + 600x2x3/3\times2/3 +......$

$s = 600 + 600\times2\times2/3(1 + 2/3 +2/3\times2/3.....)$

$s = 600 +600\times2\times2/3(1 / (1 -2/3))$

$s = \underline{3000} \text{ cm answer}$

Question:

Find Summation of the following infinite series -

$$n, \frac{n}{2}, \frac{n}{4}, \dots$$

$\times \frac{1}{2}$

$$\frac{n}{1 - \frac{1}{2}} = 2n$$

GATE CSE 2023 | Question: 44

Q.44 Consider functions **Function_1** and **Function_2** expressed in pseudocode as follows:

Function_1
```
while n > 1 do
    for i = 1 to n do
        x = x + 1;
    end for
    n = ⌊n/2⌋;
end while
```

Function_2
```
for i = 1 to 100 * n do
    x = x + 1;
end for
```

Let $f_1(n)$ and $f_2(n)$ denote the number of times the statement "$x = x + 1$" is executed in **Function_1** and **Function_2**, respectively.

Which of the following statements is/are TRUE?

(A)	$f_1(n) \in \Theta(f_2(n))$
(B)	$f_1(n) \in o(f_2(n))$
(C)	$f_1(n) \in \omega(f_2(n))$
(D)	$f_1(n) \in O(n)$

GATE 2023 Question

Geometric Progression **Application** in Algorithms

Will solve later in Algorithms Course

ARITHEMETIC – GEOMETRIC PROGRESSION

$1, 2, 3, \ldots n$ A.P.

$x, x^2, x^3, \ldots x^n$ G.P.

$1x, \; 2x^2, \; 3x^3, \ldots n\,x^n$ A. G.P.

Question:

$1 + 2x + 3x^2 + 4x^3 + \cdots$ $\qquad$ $\underline{\underline{|x|<1}}$

$$S = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$xS = x + 2x^2 + 3x^3 + \cdots$$

$$S - xS = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

G.P.

$$S(1-x) = \frac{1}{1-x} \Rightarrow S = \frac{1}{(1-x)^2}$$

Question:

1/2 + 2/4 + 3/8 + 4/16 + ...

$$S = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots$$

$$\frac{1}{2}S = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots$$

$$\frac{1}{2}S = \underbrace{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots}_{} \rightarrow GP = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

$$\Rightarrow \frac{1}{2}S = 1 \Rightarrow \underline{\underline{S = 2}}$$

Graphical View –

```
  1/2 + 2/4 + 3/8 + 4/16 + ...

= 1/2 + 1/4 + 1/8 + 1/16 + ...     (= 1)
      + 1/4 + 1/8 + 1/16 + ...     (= 1/2)
```

Sai teja to Everyone 8:39 PM

previous question as
1/2^1+2/2^2+...+k/2^k
can we solve like this ?

$$\sum_{k=1}^{\infty} \frac{k}{2^k}$$

Graphical View –

```
  1/2 + 2/4 + 3/8 + 4/16 + ...

= 1/2 + 1/4 + 1/8 + 1/16 + ...     (= 1)
      + 1/4 + 1/8 + 1/16 + ...     (= 1/2)
            + 1/8 + 1/16 + ...     (= 1/4)
                  + 1/16 + ...     (= 1/8)
                        ....        ....
```

$\frac{1}{1-\frac{1}{2}} = 2$

Question:

$$2 + 5x + 8x^2 + 11x^3 + \cdots \infty$$

$\frac{a}{1-r}$

$S = \frac{2}{1-x} + \frac{3x}{(1-x)^2}$

$S = 2 + 5x + 8x^2 + \cdots$

$xS = 2x + 5x^2 + \cdots$

$S(1-x) = 2 + 3x + 3x^2 + 3x^3 + \cdots$

$2 + 3(x + x^2 + \cdots) \rightarrow 2 + \frac{3x}{1-x}$

Question:

$$1 - 3x + 5x^2 - 7x^3 + \cdots \infty$$

H.W.

$r = -x$

AMP $\rightarrow$ generating functions

[GATE 2022 question

was just AMP]