Graph Theory :

Next Topic :

Some Questions and Recap

Website : https://www.goclasses.in/

Isomorphic Graphs: (Intuitive/Informal Definition)

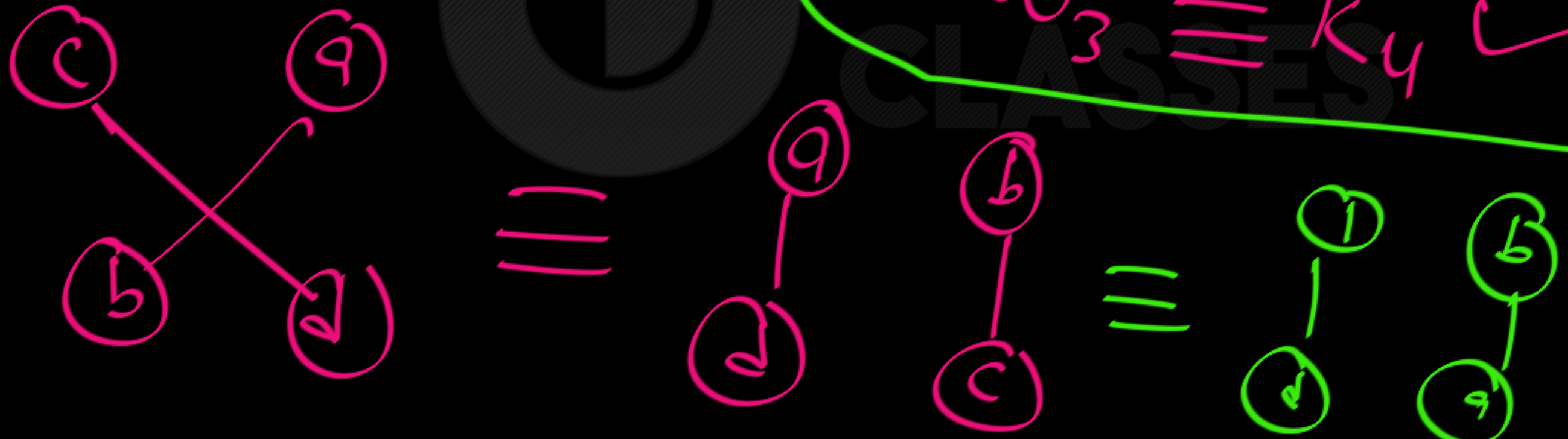
H, G are Isomorphic If they are basically same graph But Drawn/labelled in Different ways.

$C_4 \not\equiv C_3$

$W_3 \not\equiv K_3$

$C_3 \equiv K_3$ ✓

$W_3 \equiv K_4$ ✓



At this moment, there is No efficient way to "Check Isomorphism"

we can Just "eliminate options" by seeing violation of some properties.

Graph InVariant :

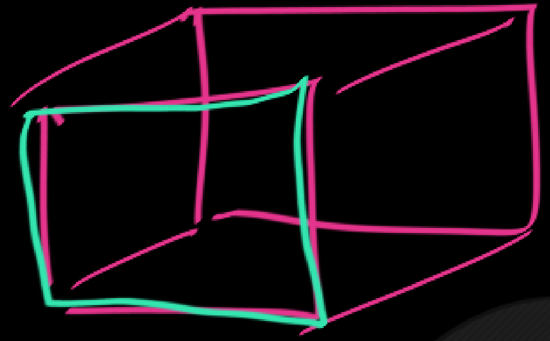
Graph properties which are Preserved by Isomorphism.

i.e. Isomorphic Graph will Satisfy these Properties.

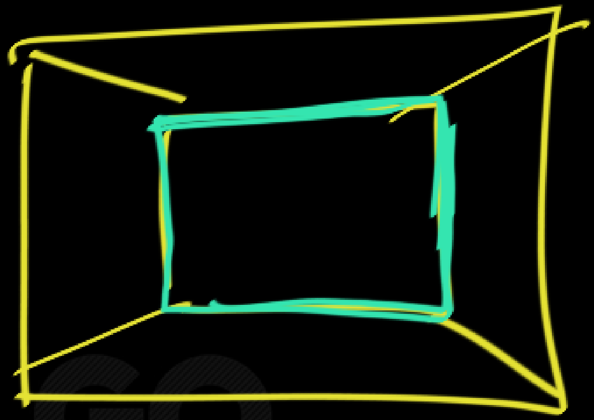
Even If 100000 properties are Same, Does not prove Isomorphism.

In Exam, we can only Eliminate,

or If "Obvious things", then no problem

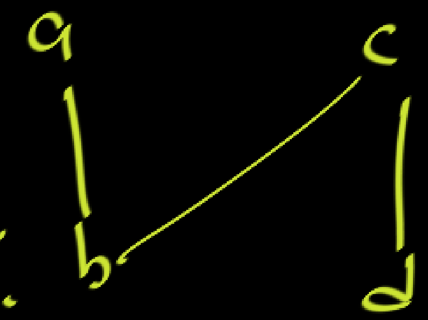


$\equiv$

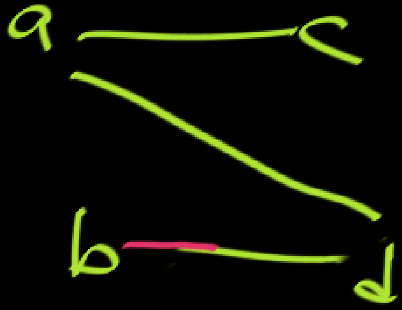


P_4 = Self-Complementary ? yes $P_4 \equiv \overline{P_4}$

P_4:



$\longrightarrow$ $\overline{P_4}$



P_4

Find some Isomorphic Drawings of K4?

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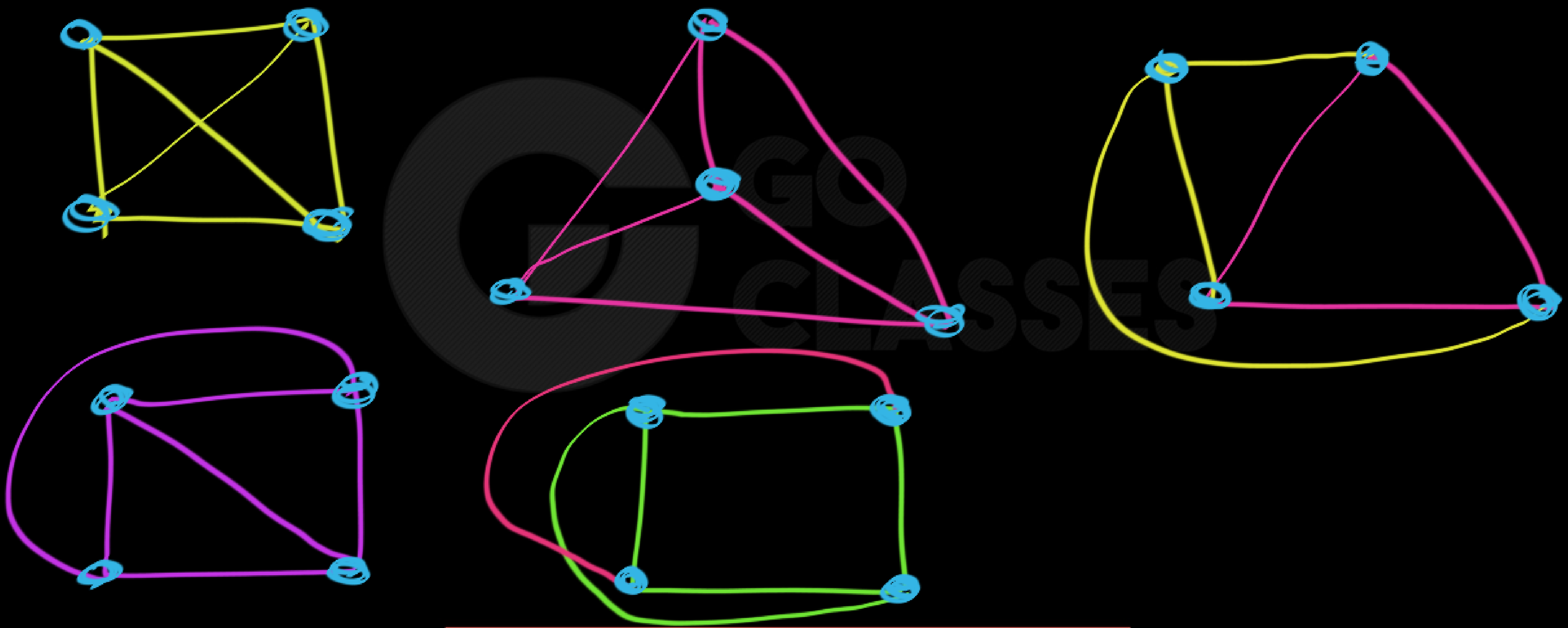

Define Connected Graph in terms of number of components.

$\forall a, b \in V$

a ~ b

Path

iff $\boxed{\#\text{Components} = 1}$

$K_1 = $ a — Connected

1 Component

Not Disconnected

Graph Theory :

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Some Theorems

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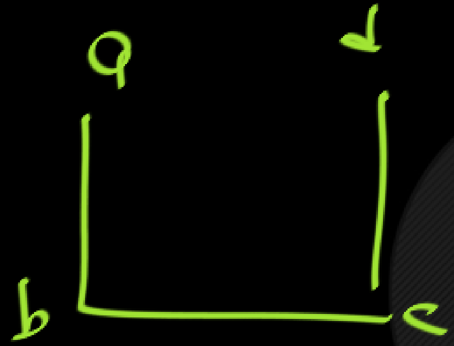
Exercises

- Is it true that the complement of a connected graph is necessarily disconnected? ✗
- Prove that the complement of a disconnected graph is necessarily connected.

Given a simple graph and its complement, prove that either of them is always connected.

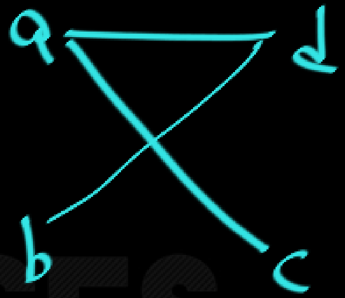
Connected $\xrightarrow{\text{Complement}}$ Disconnected ? **No**

G :



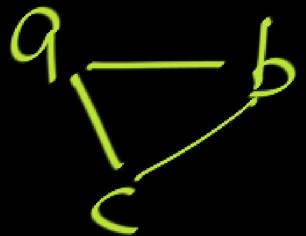
Connected

$\longrightarrow \overline{G}$



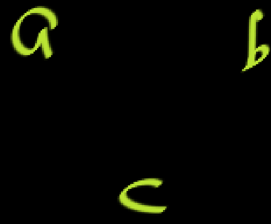
Connected ✓

G :



Conn.

$\longrightarrow \overline{G}$



Disconne.

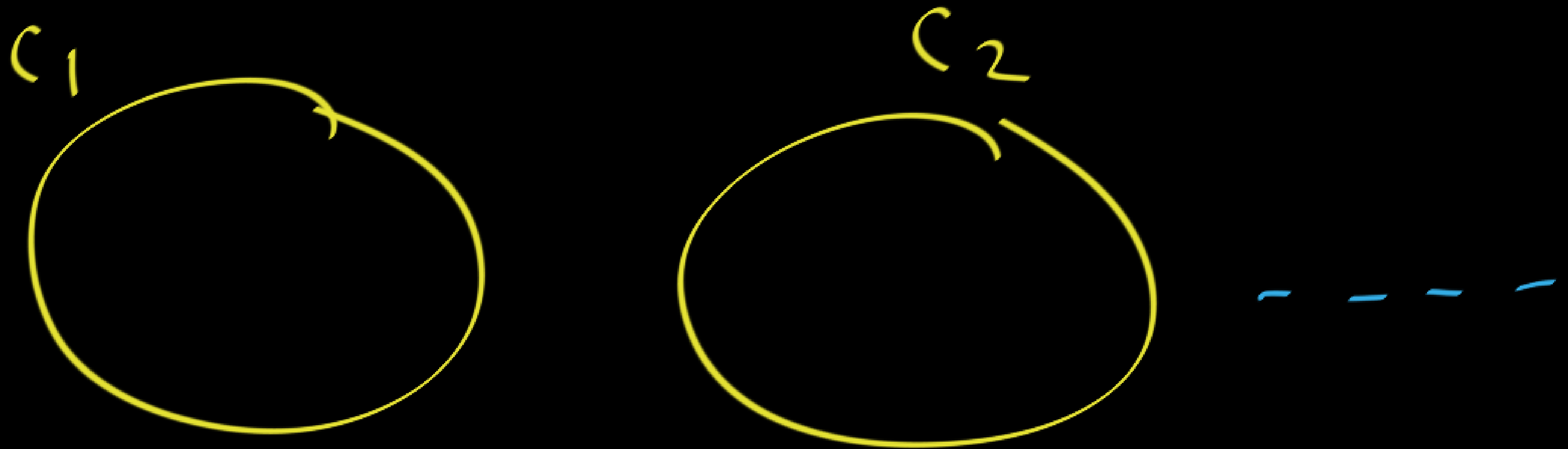
Q: Give a Disconnected Graph whose Complement is Also Disconnected?

NOT Possible.

$\text{Discon} \xrightarrow{\text{Complement}} \text{Connected}$ ✓

Disconnected Graph G :

→ At least two Components in G.



To prove: If G is Disconnected then $\overline{G}$ is Connected. $G(V, E)$ $\overline{G}(V, \overline{E})$

same

Proof: $\underbrace{a, b \in \overline{G}}$

"To prove": a $\underset{\text{path}}{\sim\sim\sim}$ b

Case 1: a, b are NOT Adjacent in G

$\downarrow$

a, b are Adj. in $\overline{G}$ so

$a \underset{\text{Path}}{\longrightarrow} b$.

Case 2: a, b are Adjacent in $G \longrightarrow$ "Disconnected"

$\overline{G}$:

a — z

b — z

$a \underset{\text{Path}}{\sim} b$

Disconnected $\xrightarrow{\text{Complement}}$ Connected ✓

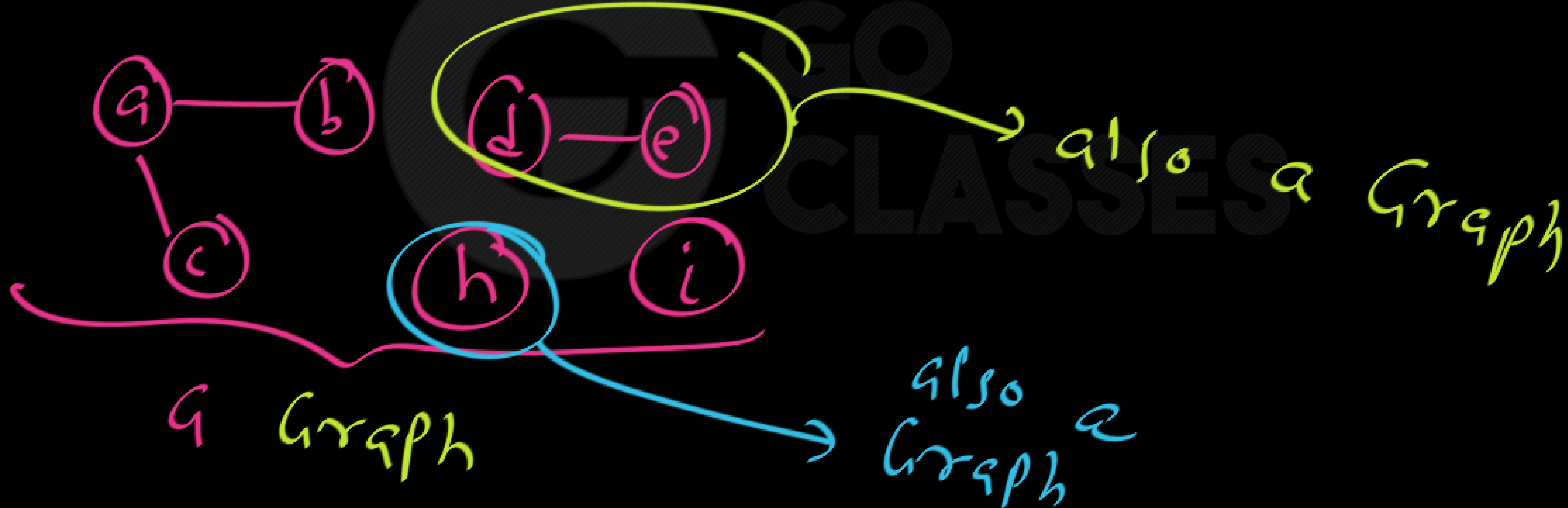
Connected $\xrightarrow{\text{Complement}}$ Conn OR Discon.

Prove that the complement of a disconnected graph is connected.

We begin by assuming we have a disconnected graph G. Now consider two vertices x and y in the complement. If x and y are not adjacent in G, then they will be adjacent in $\overline{G}$ and we can find a trivial x-y path. If x and y are adjacent in G then they must have been in the same component. Let z be some vertex in another component of G. This means that the edges xz and yz were not in G. This implies that they both must be edges in $\overline{G}$. This gives us the path x-z-y. Therefore, in $\overline{G}$ we have that there exists a path between any two vertices and hence it is connected.

Every subgraph is also a Graph ✓

Every Component is " " " ✓



Q: [redacted] if a graph has exactly two vertices of odd degrees, then they are connected by a path. ?

Q : [redacted] if a graph has exactly two vertices of odd degrees, then they are connected by a path?

a, b

Is it possible that a, b are in Diff Component?

No

C_1 → 1 odd Degree vertex

a

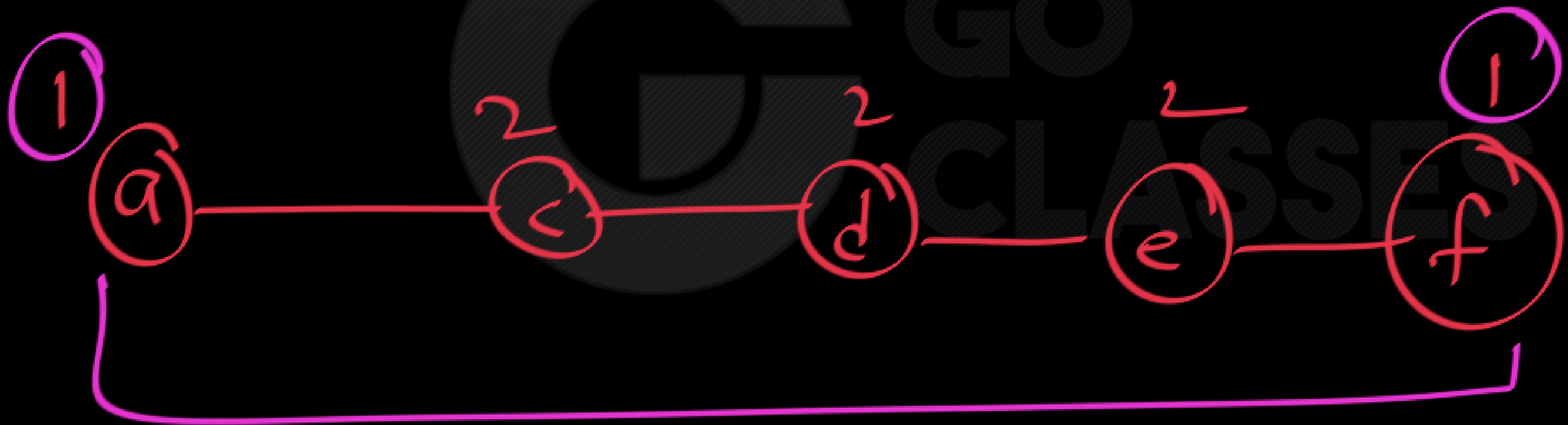
b

C_2

This Graph has odd number of odd-degree vertices.

So, $\underbrace{a, b \in \text{Same Component}}$

So, a $\overset{\text{path}}{\sim\sim}$ b



Exactly two vertices of odd Degree.

- Prove that if a graph has exactly two vertices of odd degrees, then they are connected by a path.

Show that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

If a = odd-Deg vertex

Some other

a path b → odd Deg.

2.2.14 Degree Of Graph: TIFR2018-B-8

In an undirected graph G with n vertices, vertex 1 has degree 1, while each vertex $2, \ldots, n-1$ has degree 10 and the degree of vertex n is unknown, Which of the following statement must be TRUE on the graph G?

a. There is a path from vertex 1 to vertex n.
b. There is a path from vertex 1 to each vertex $2, \ldots, n-1$.
c. Vertex n has degree 1.
d. The diameter of the graph is at most $\frac{n}{10}$
e. All of the above choices must be TRUE

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n = Degree odd

1 2 3 4 - - - - n-1 n

Deg = 1 | 10 10 10 10 | ?

odd | even | odd Degree

Path ✓

1 n

odd odd

3 - Idiot :

"Humne padha tha", usne Apply kiya.

Almost 10k people → who cover concepts

Almost 1k people → who understand the concepts.

- Suppose we have a simple graph G with n vertices What is the maximum number of edges G can contain, if

  (i) G is an undirected graph ?

  (ii) G is a directed graph ?

- Suppose we have a simple graph G with n vertices What is the maximum number of edges G can contain, if

(i) G is an undirected graph ? → ${}^{n}C_{2}$

(ii) G is a directed graph ?

→ $n(n-1)$

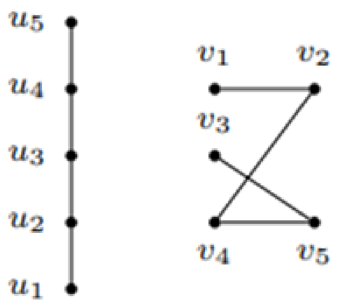
Edge = (,)

n

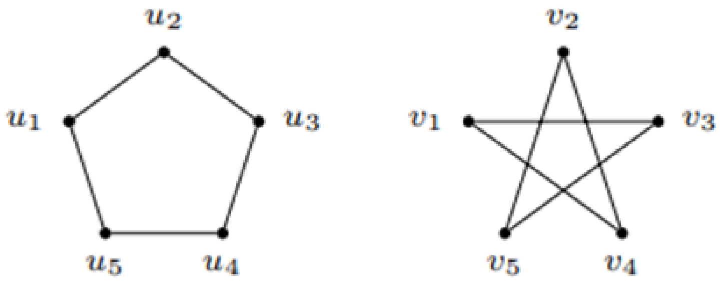
n-1 choices

For each of the following pairs, list their degree sequences. Then are they isomorphic?. If not, why? If yes, give an isomorphism.

(i)



(ii)



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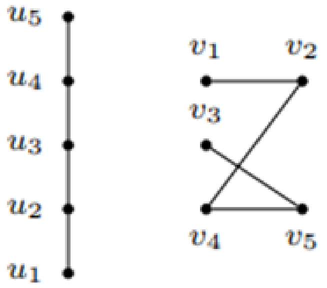


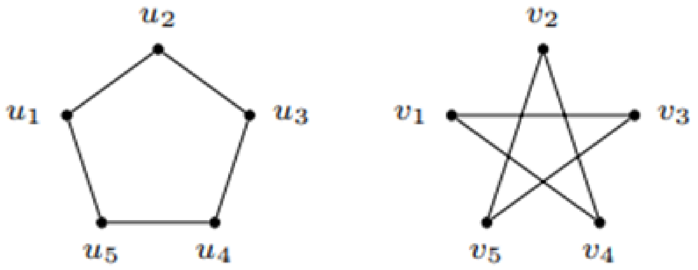
deg seq: 2,2,2,1,1

Isomorphic:

$u_1 \mapsto v_1,\ u_2 \mapsto v_2,\ u_3 \mapsto v_4,$

$u_4 \mapsto v_5,\ u_5 \mapsto v_3$

(ii)



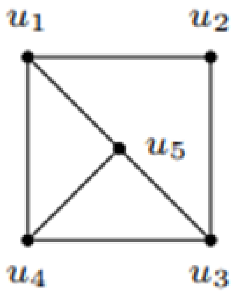
deg seq: 2,2,2,2,2

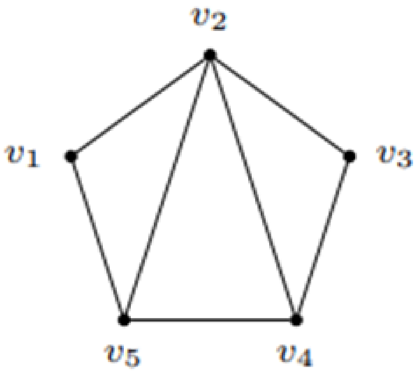
Isomorphic:

$u_1 \mapsto v_1,\ u_2 \mapsto v_3,\ u_3 \mapsto v_5$

$u_4 \mapsto v_2,\ u_5 \mapsto 4.$

(iii)



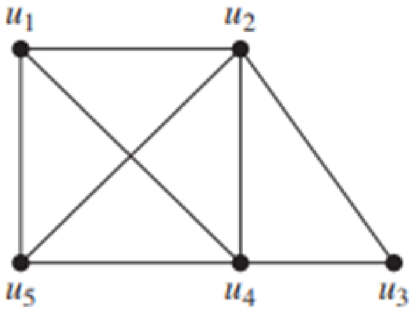


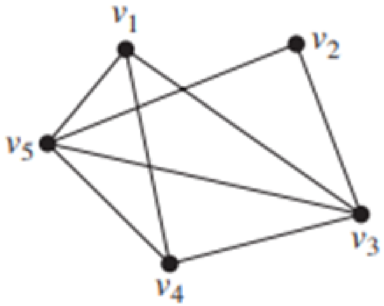
Left deg seq: 3,3,3,3,2;

Right deg seq: 4,3,3,2,2.

Not isomorphic: different deg seq's.

(iv)



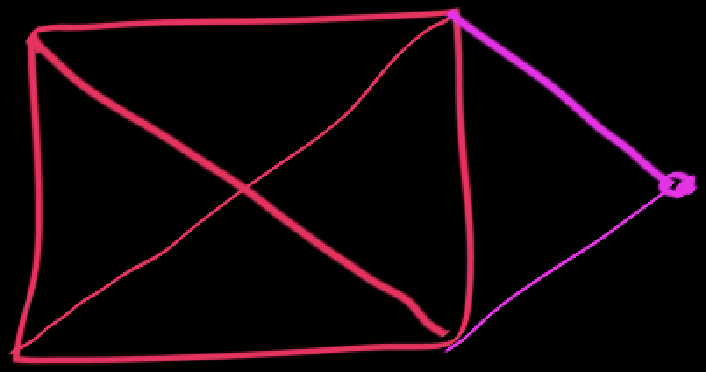


Deg seq: 4,4,3,3,2

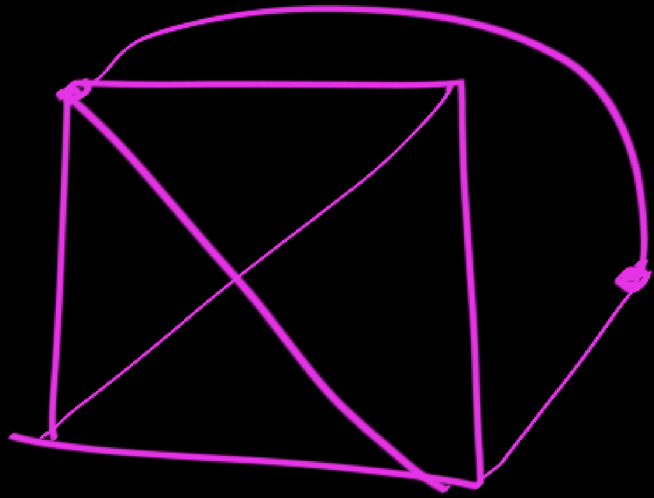
Isomorphic:

$u_3 \mapsto v_2,\ u_2 \mapsto v_3,\ u_4 \mapsto v_5$

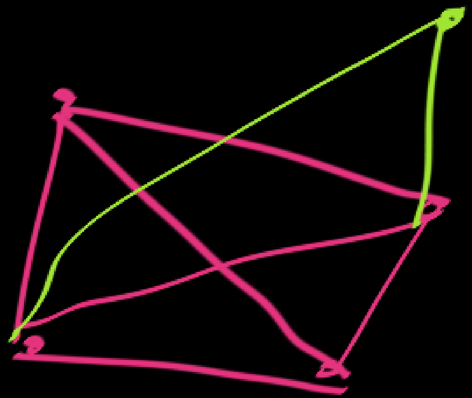
$u_1 \mapsto v_1,\ u_5 \mapsto v_4$



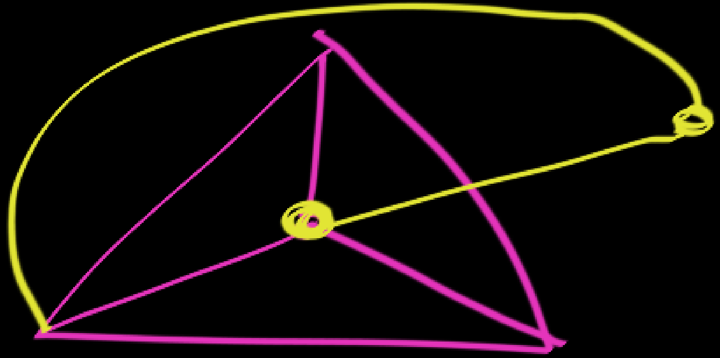
$\equiv$



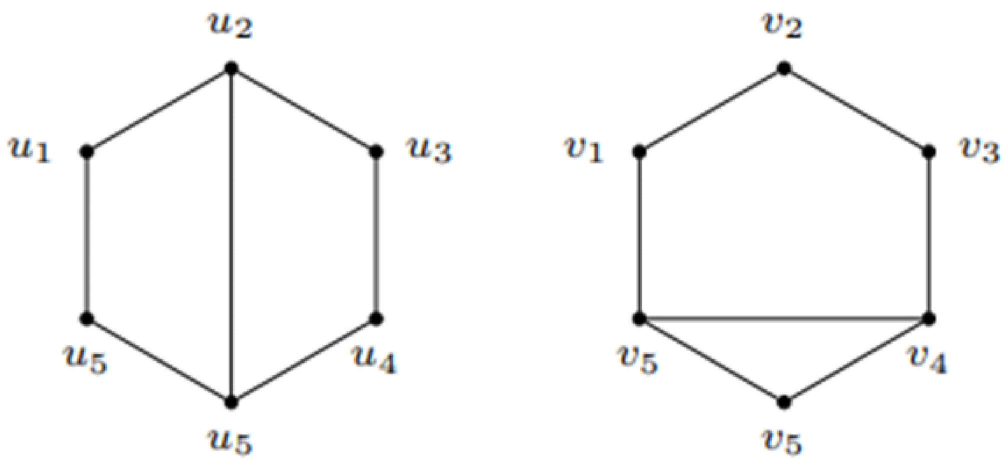
$\equiv$



$\equiv$



(v)



Deg seq: 3,3,2,2,2,2

Not isomorphic:

Right has a 3-cycle; Left doesn't.

Graph Theory :

Next Topic :

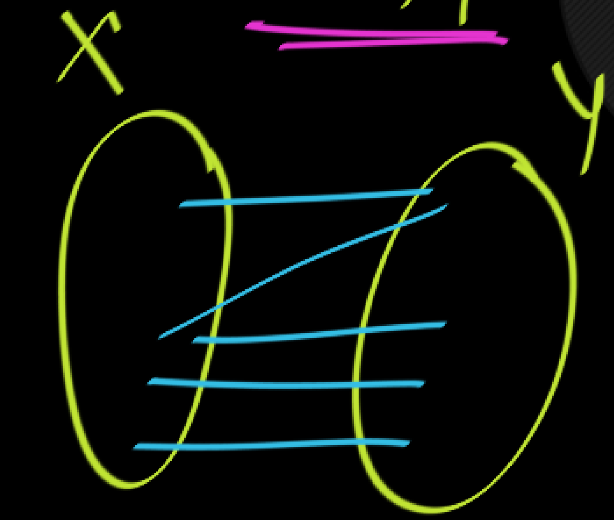
Bipartite Graphs

Website : https://www.goclasses.in/

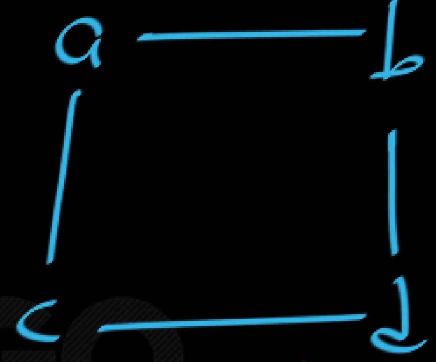
Bipartite:

Two Partition

X , Y



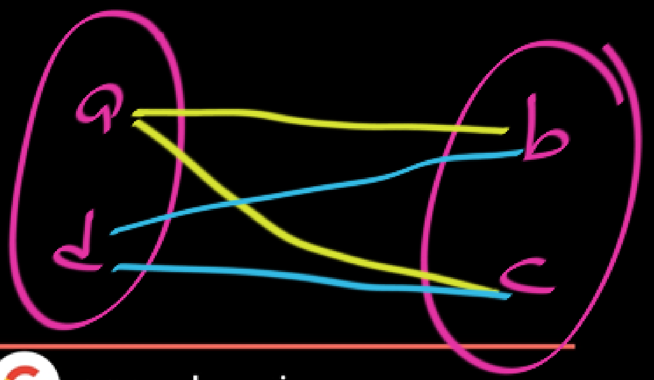
Eg:



G

$V = \{a, b, c, d\}$

$X = \{a, d\}$ $Y = \{b, c\}$



Bipartite

Graph $G(V,E)$ is bipartite iff

$\exists$ a bipartition X, Y of $V \rightarrow$ set

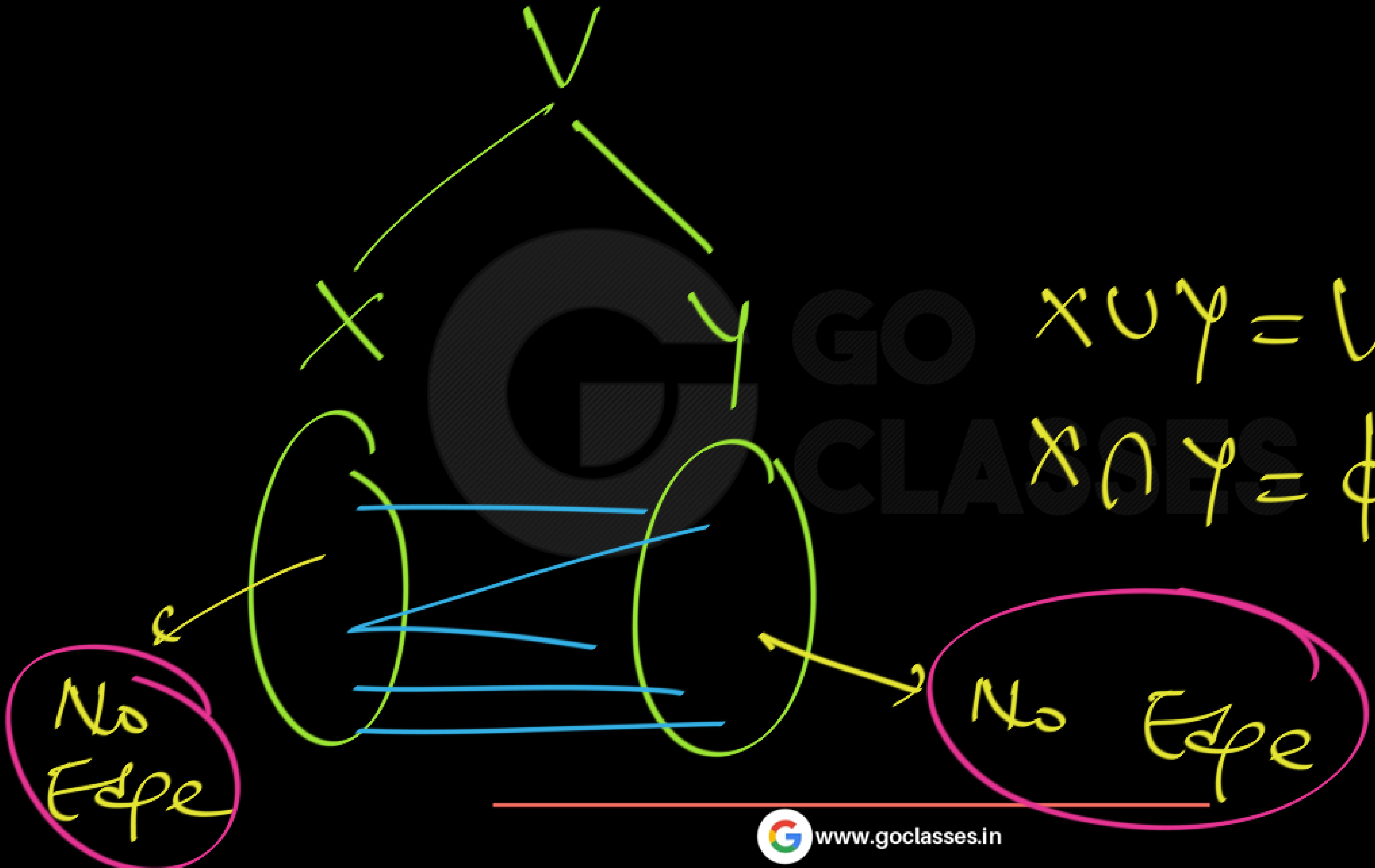
① $X \cap Y = \phi$

② $\forall a,b \in X \quad (a,b) \notin E(G)$

$\forall a,b \in Y \quad (a,b) \notin E(G)$

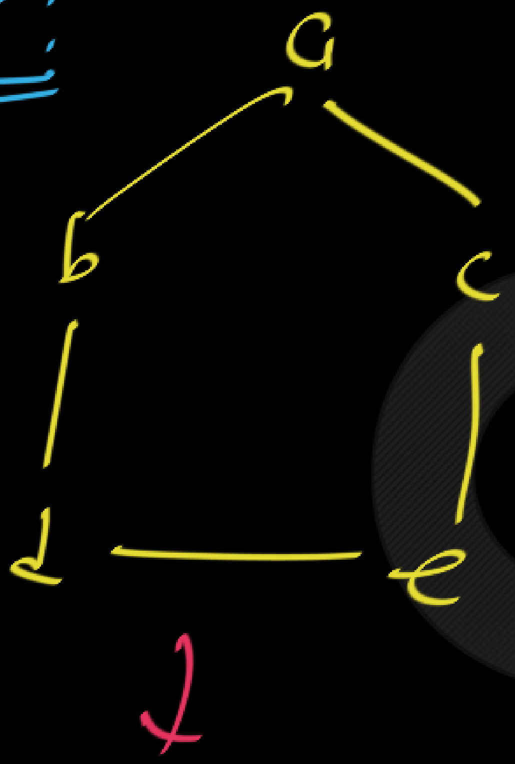
③ $X \cup Y = V$

④ X, Y can be empty.



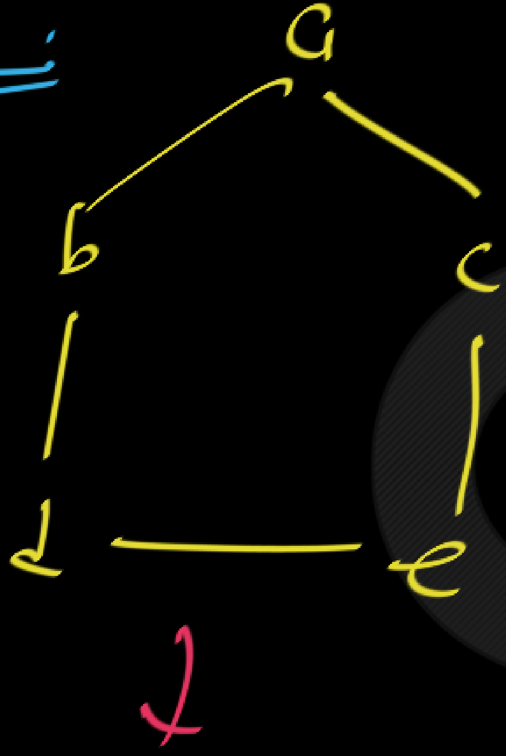
$$X \cup Y = V$$
$$X \cap Y = \phi$$

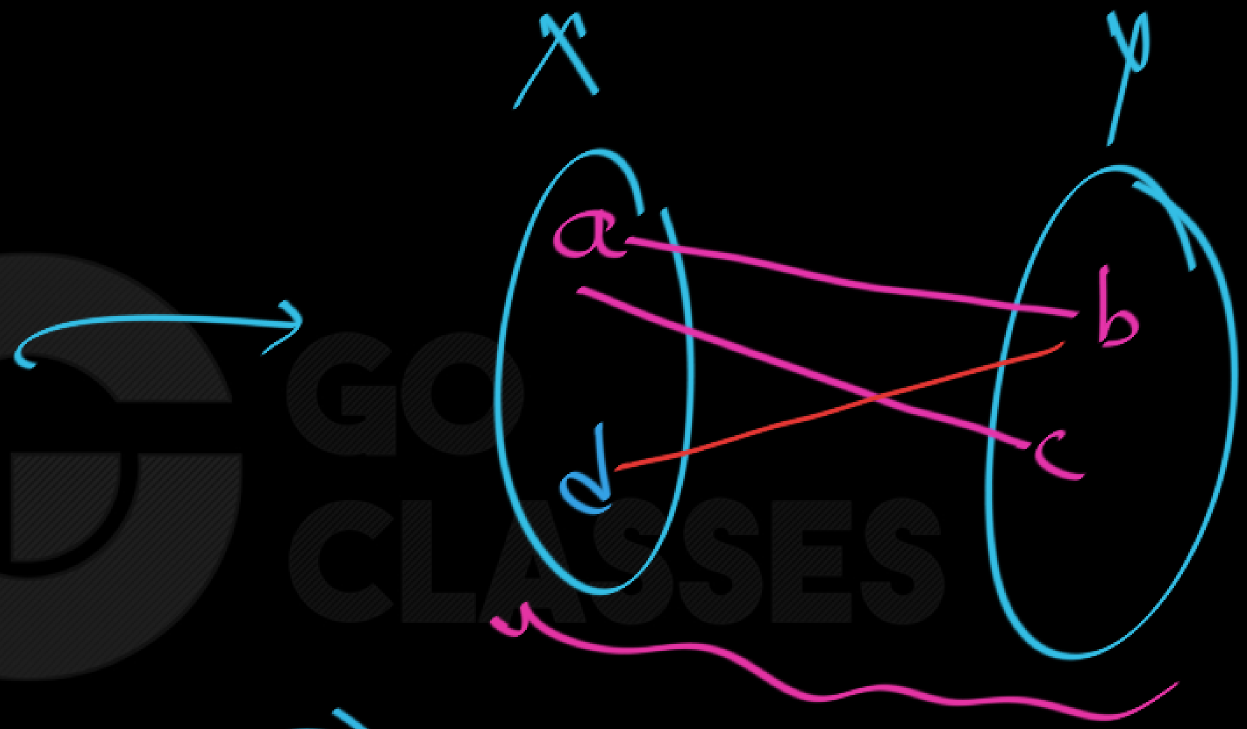
Eg:



Is Bipartite?

Eg:

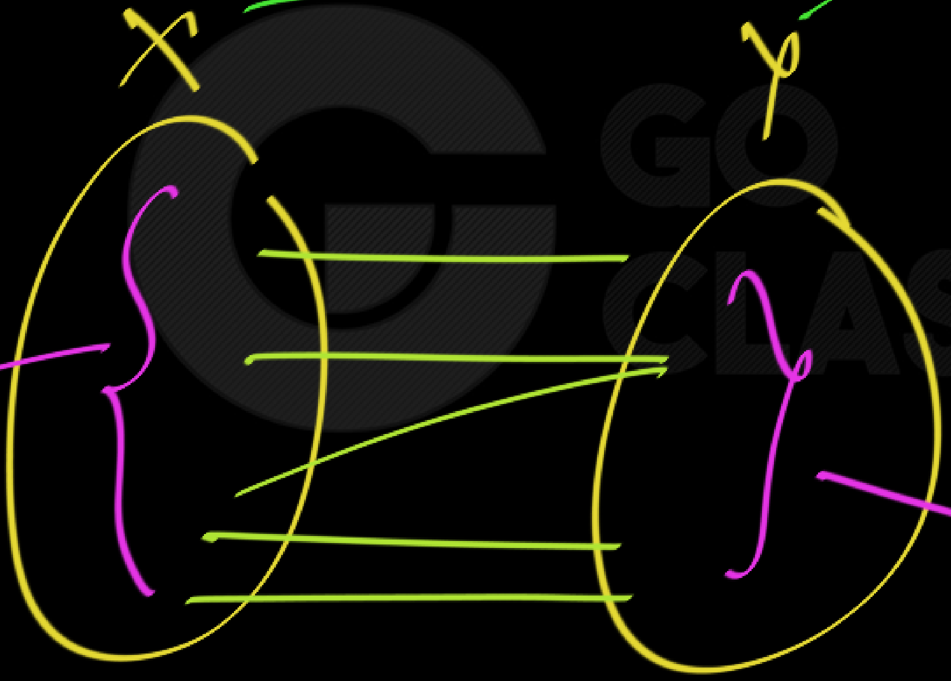




Is Bipartite? $\Rightarrow$ No

$G(V, E)$ is bipartite

V



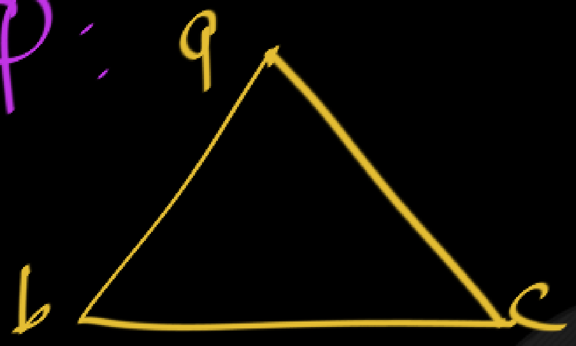
Non-Adjacent

Non-Adjacent

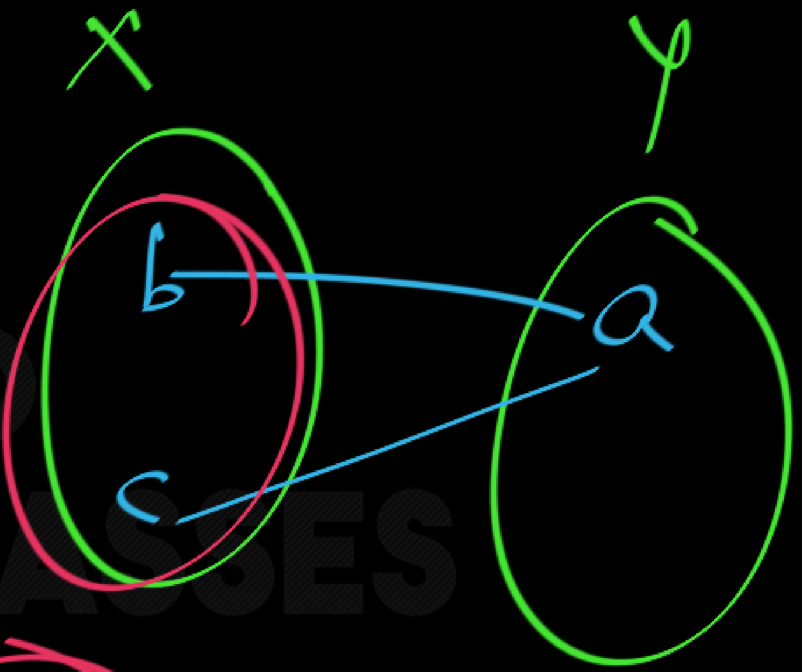
① $X \cup Y = V$

② $X \cap Y = \phi$

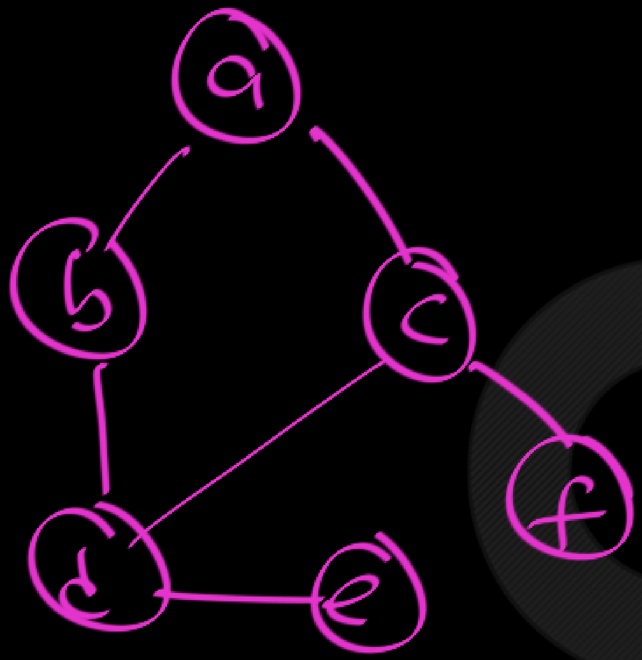
φ :



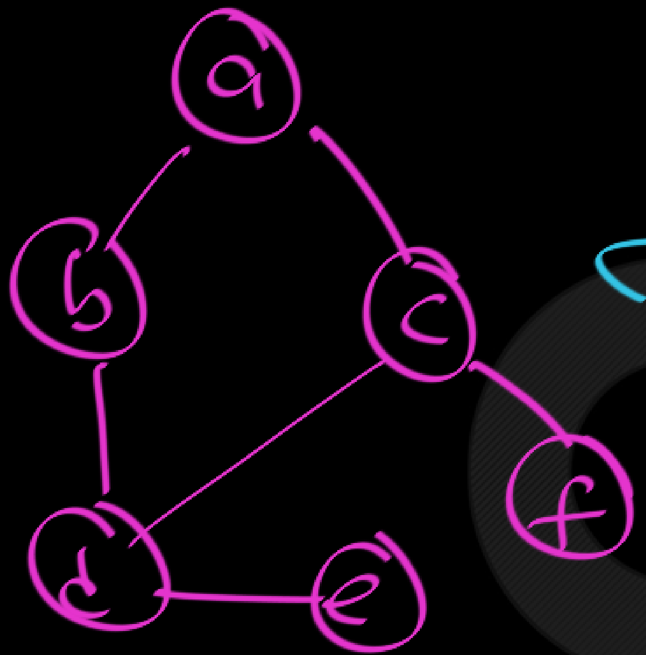
Adjacent to

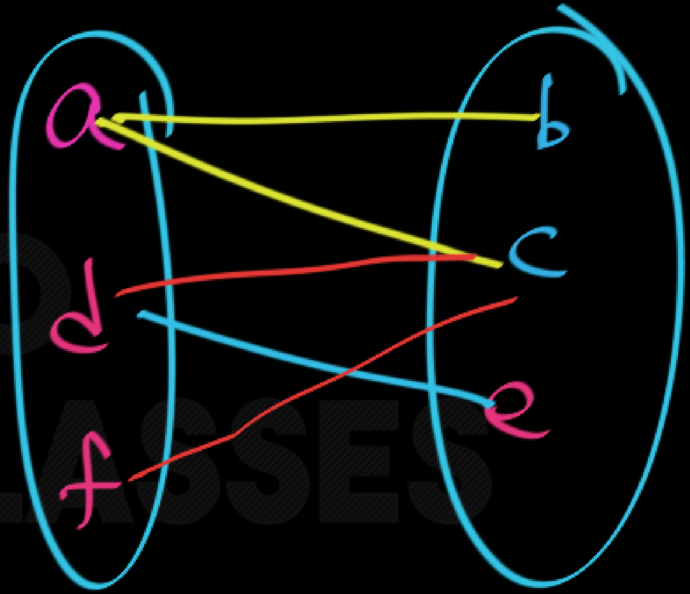


Is bipartite? → NO



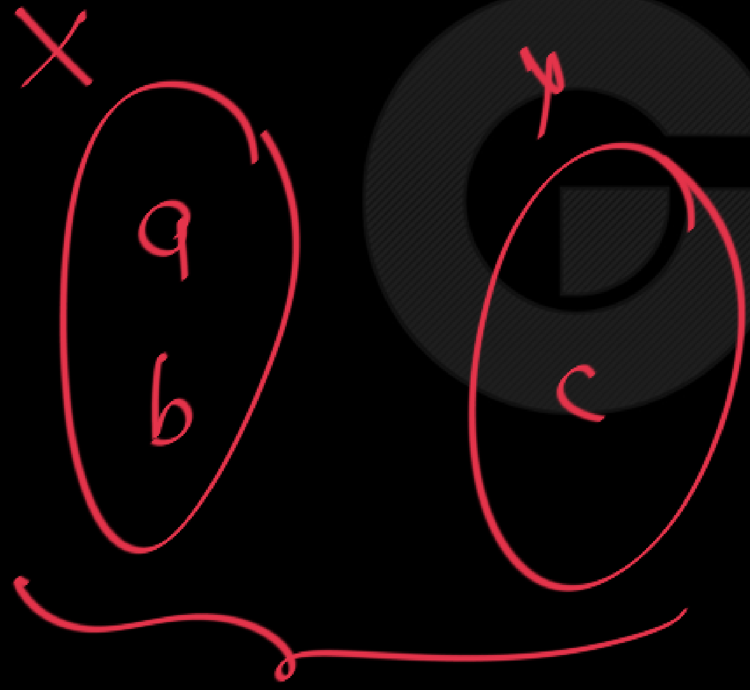
Is bipartite?





Is bipartite? → Yes

Q: (a) (b) (c) — bipartite? ✓



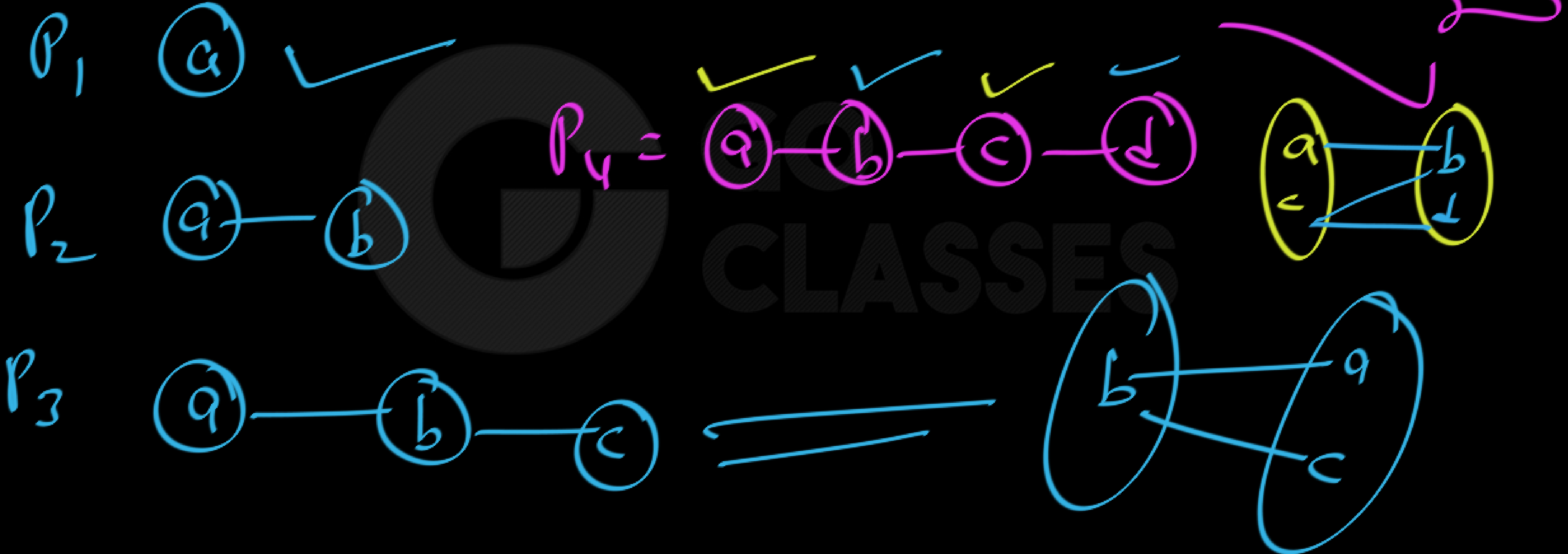
Q: $\underbrace{a = K_1}$ —— bipartite? Yes

Bipartite Graphs

Sometimes a graph has the property that its vertex set can be divided into two disjoint subsets such that each edge connects a vertex in one of these subsets to a vertex in the other subset. For example, consider the graph representing marriages between men and women in a village, where each person is represented by a vertex and a marriage is represented by an edge. In this graph, each edge connects a vertex in the subset of vertices representing males and a vertex in the subset of vertices representing females. This leads us to Definition 5.

A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

Q: P_n is bipartite ; $n = ? \Rightarrow n \geq 1$



Q: C_n is bipartite ; n? —— even

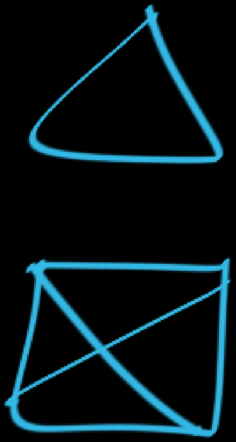
C_3, C_5 —— NOT bipartite

C_4, C_6 —— bipartite

Q: K_n is bipartite ; $n = ? = 1, 2$

K_3

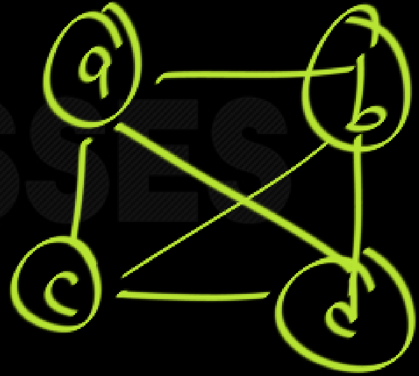
K_4



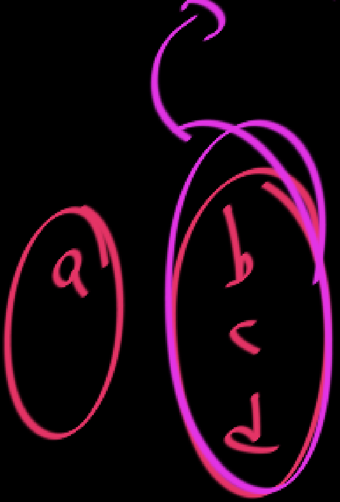
K_5 ----

NOT bipartite

K_4



Adjacent



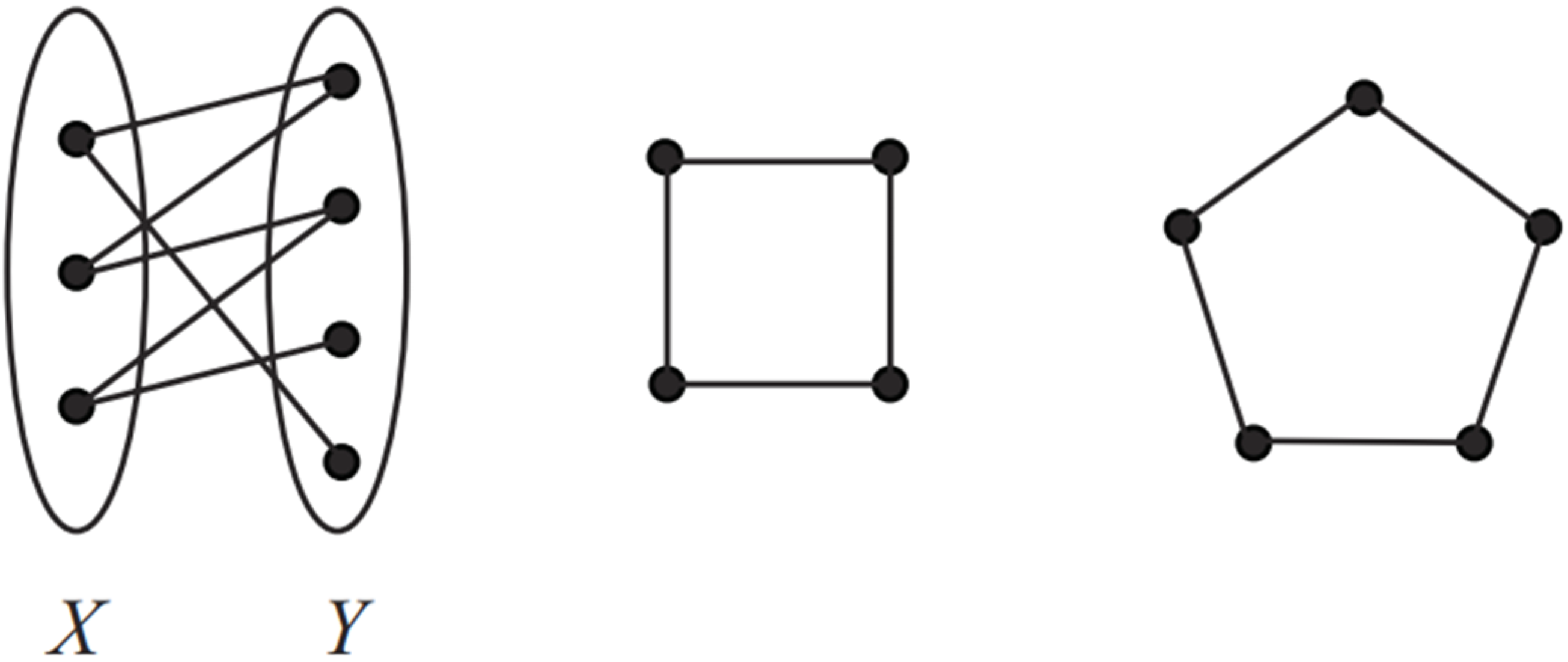


FIGURE 1.19. Two bipartite graphs and one non-bipartite graph.

we now consider *bipartite graphs*. A bipartite graph is a simple graph in which $V(G)$ can be partitioned into two sets, V_1 and V_2 with the following properties:

1. If $v \in V_1$ then it may only be adjacent to vertices in V_2.
2. If $v \in V_2$ then it may only be adjacent to vertices in V_1.
3. $V_1 \cap V_2 = \emptyset$
4. $V_1 \cup V_2 = V(G)$

1. Are paths bipartite graphs? What about cycles?

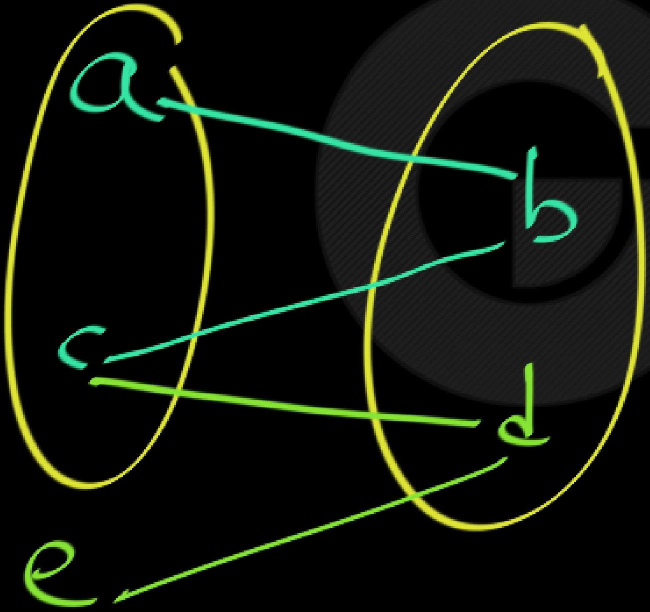
Theorem: → Extremely Important

a graph is bipartite if and only if it doesn't have an odd cycle.

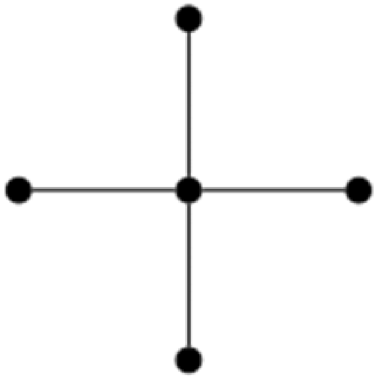
bipartite ⟷ No odd Cycle

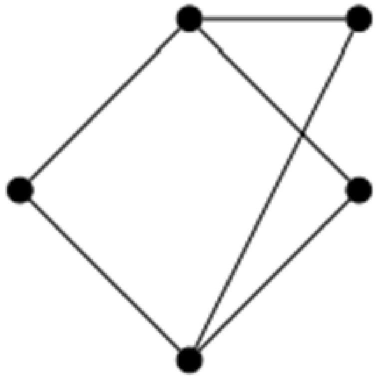
bipartite ⟷ All cycles are Even length

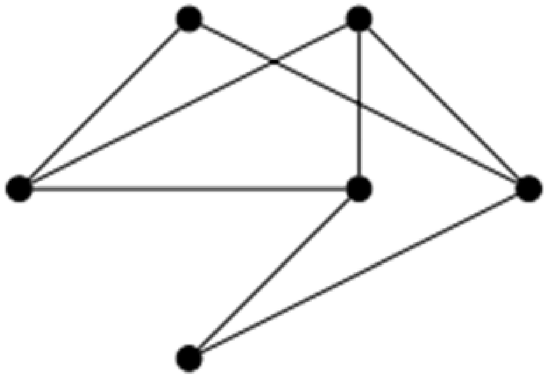
bipartite Graph: → $\boxed{\text{No odd cycle}}$



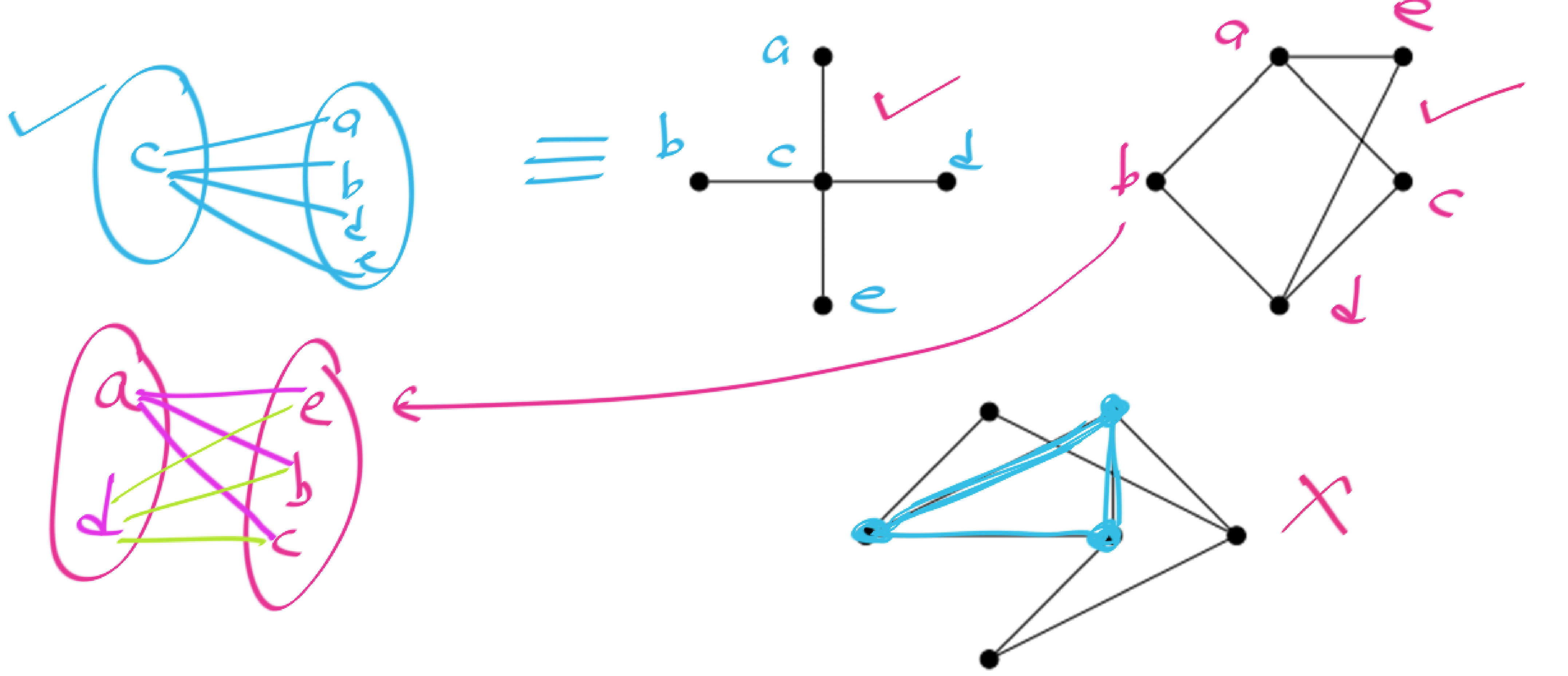
Which of the following are bipartite? Justify your answer.



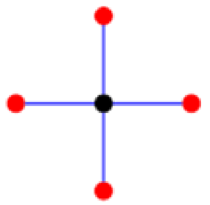




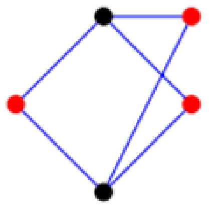
Which of the following are bipartite? Justify your answer.



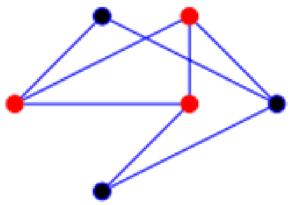
Solution:



Bipartite: put the red vertices in V_1 and the black in V_2.



Bipartite: put the red vertices in V_1 and the black in V_2.



Not bipartite!

Consider the three vertices colored red. For the sake of contradiction, assume that it is bipartite. Pick any one of them to be in V_1. That would force the other two to be in V_2. But they are adjacent, which is a contradiction.

Are the graphs G and H displayed in Figure 8 bipartite?

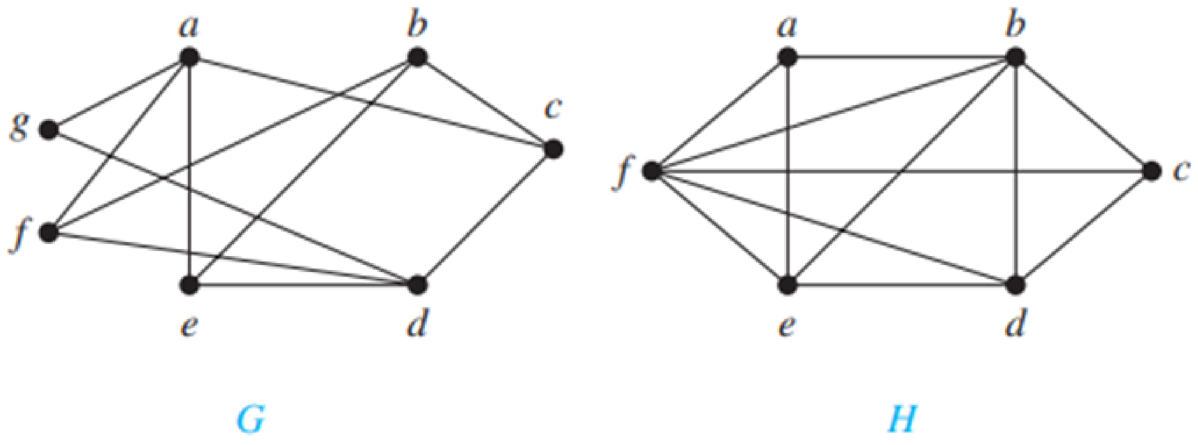
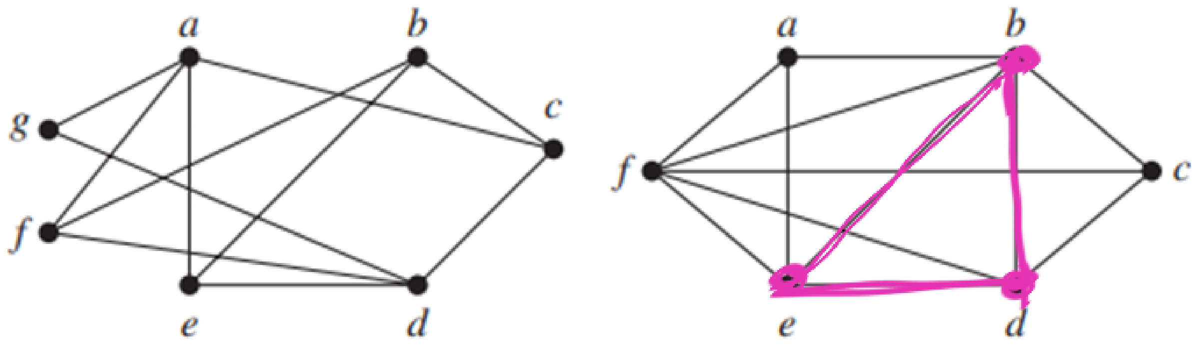


FIGURE 8 **The Undirected Graphs G and H.**

Are the graphs G and H displayed in Figure 8 bipartite?



G → bipartite

H ✗

FIGURE 8 **The Undirected Graphs G and H.**

g c f e

a d b

Lemma 2.3 *If G is a bipartite graph and the bipartition of G is X and Y, then*
$$\underbrace{\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)}.$$

Total Deg of X = Total Deg of Y

X +1, +1, +1, +1 $\quad$ Y +1, +1, +1, +1

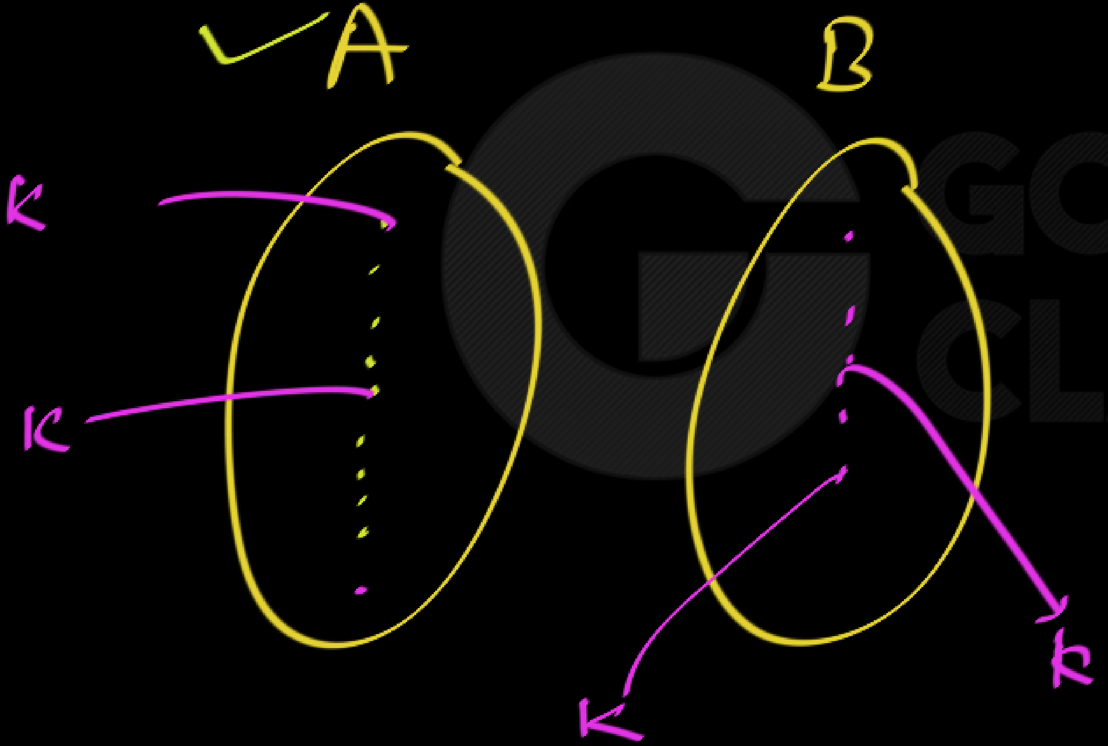
$$\sum_{v \in X} deg(v) = \sum_{v \in Y} deg(v)$$

Lemma 2.3 *If G is a bipartite graph and the bipartition of G is X and Y, then* $\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v)$.

Proof. By induction on the number of edges. We denote the number of elements of X as $|X|$. Suppose $|X| = r$ and $|Y| = s$ for some integers $r, s > 1$. Note that the case where X or Y has one vertex is trivial, as only one edge can be drawn. Take the subgraph of G which consists of only the vertices of G. Now we begin inducting: add one edge from any vertex in X to any vertex in Y. Then, $\sum_{v \in X} \deg(v) = \sum_{v \in Y} \deg(v) = 1$. Now, suppose this is true for n-1 edges and add one more edge. Since this edge adds exactly 1 to both $\sum_{v \in X} \deg(v)$ and $\sum_{v \in Y} \deg(v)$, we have that this is true for all n$\in N$.

A graph is called k-regular if all vertices have degree k. Prove that if a bipartite G is also k-regular with $k \geq 1$ then $|A| = |B|$.

k-Regular Bipartite Graph :



We know

$$\text{Total Deg}(A) = k|A|$$

$$\| \qquad \qquad \|$$

$$\text{Total Deg}(B) = k|B|$$

$$\Rightarrow |A| = |B|$$

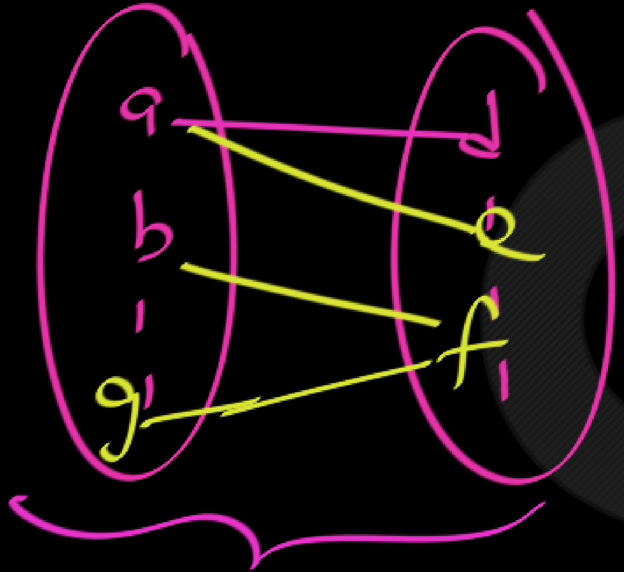
A *k-regular* graph G is one such that $\deg(v) = k$ for all $v \in G$.

Theorem 2.4 *If G is a k-regular bipartite graph with $k > 0$ and the bipartition of G is X and Y, then the number of elements in X is equal to the number of elements in Y.*

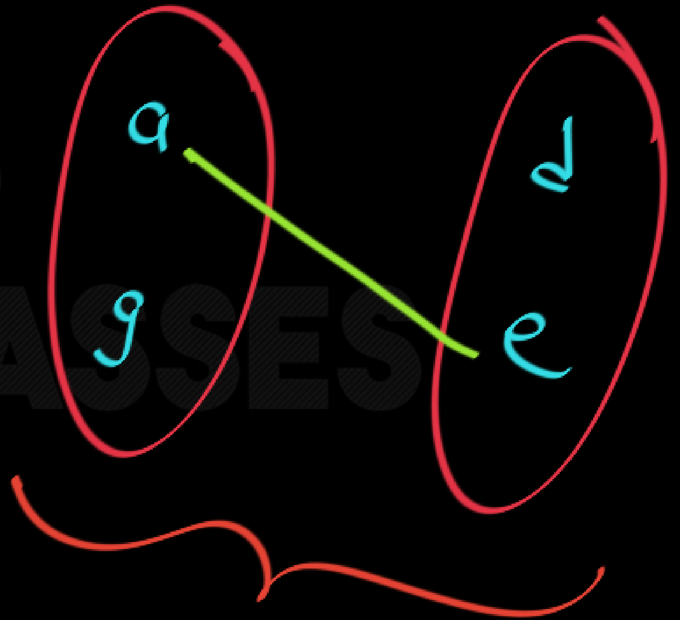
Proof. We observe $\sum_{v \in X} \deg(v) = k\,|X|$ and similarly, $\sum_{v \in Y} \deg(v) = k\,|Y|$. By the previous lemma, this means that $k\,|X| = k\,|Y| \implies |X| = |Y|$.

Theorem 2.6 (Subgraph of a Bipartite Graph) *Every subgraph H of a bipartite graph G is, itself, bipartite.*

G = Bip Graph



Subgraph



[redacted] (Subgraph [redacted]) *Every subgraph H of a* [redacted] Complete *graph G is, itself,* [redacted] Complete

false

[redacted] **(Subgraph** [redacted]**)** *Every subgraph H of a* [redacted] cycle *graph G is, itself,* [redacted] cycle

→ false

[redacted] **(Subgraph** [redacted]**)** *Every subgraph H of a* [redacted] Edgeless *graph G is, itself,* [redacted] Edgeless

→ True ✓

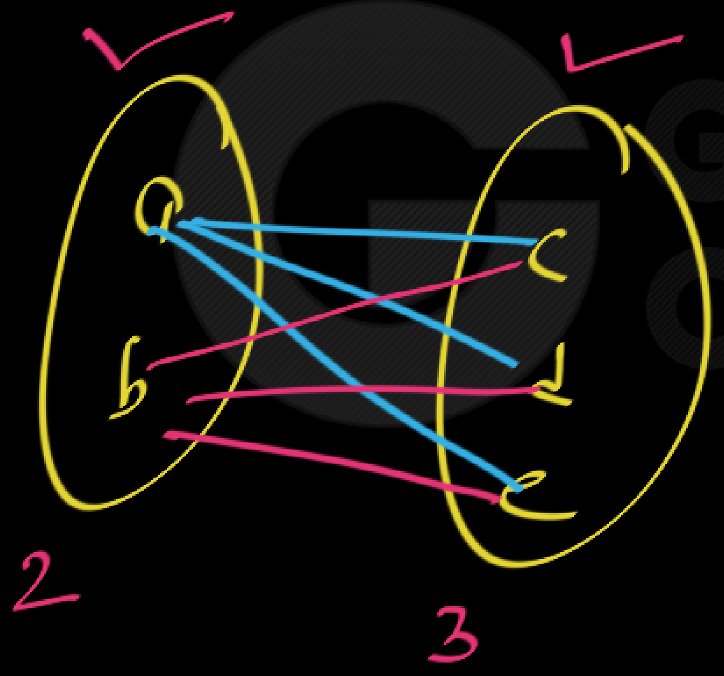
Graph Theory :

Next Topic :

Complete Bipartite Graph

Website : https://www.goclasses.in/

Complete Bipartite Graph (CBG)

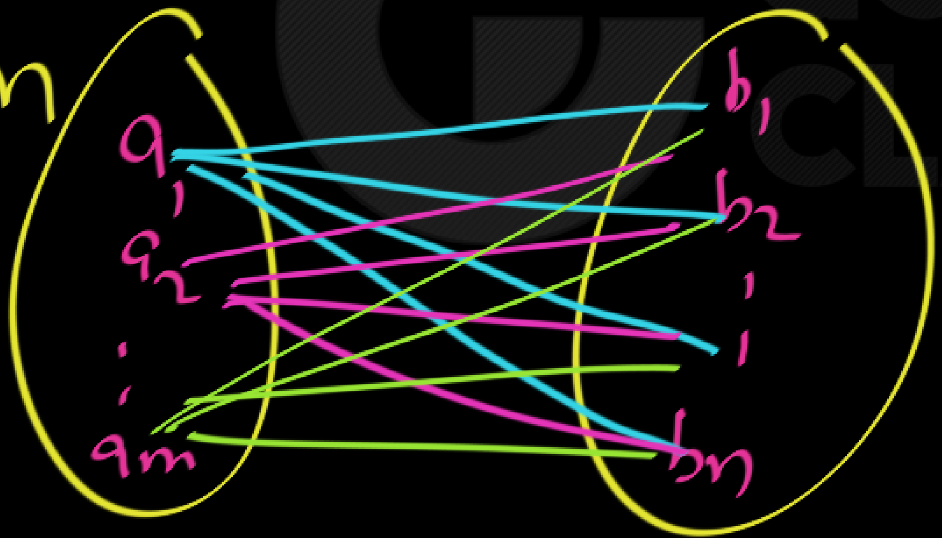


$\boxed{K_{2,3}}$

$K_{m,n}$ = "Complete" Bipartite Graph

X ✓   ✓ Y

$|X| = m$ $|Y| = n$



$\forall a \in X, \forall b \in Y$

ab is Edge ✓

Definition 62. A bipartite graph $G = (A, B, E)$ is *complete bipartite* if every vertex of A is adjacent to every vertex of B. A compete bipartite graph with parts of size n and m is denoted $K_{n,m}$.

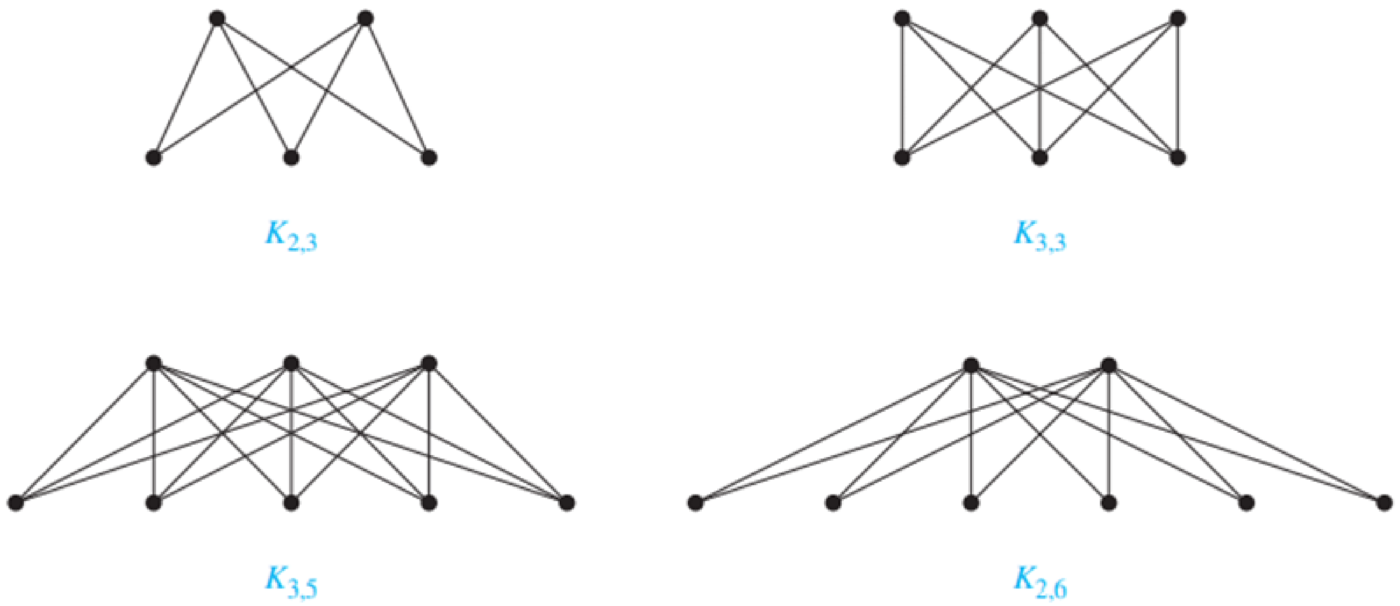
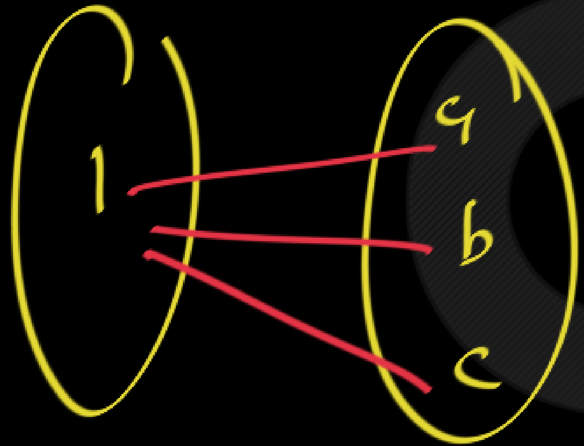
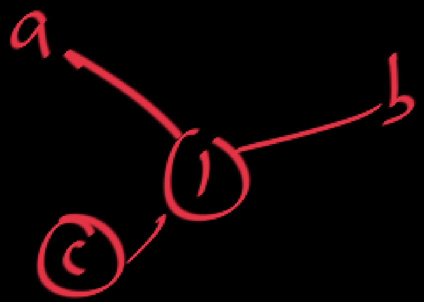


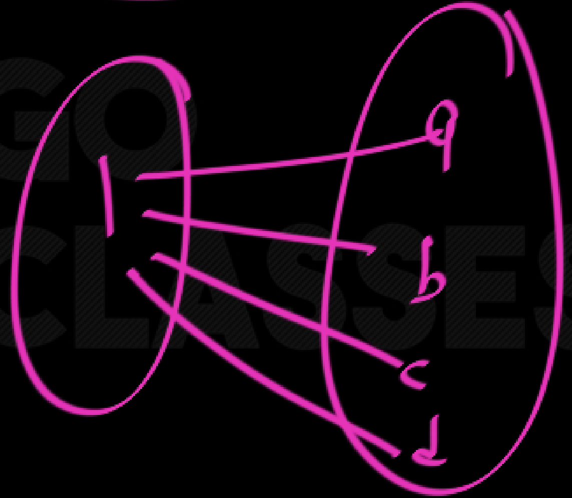
FIGURE 9 **Some Complete Bipartite Graphs.**

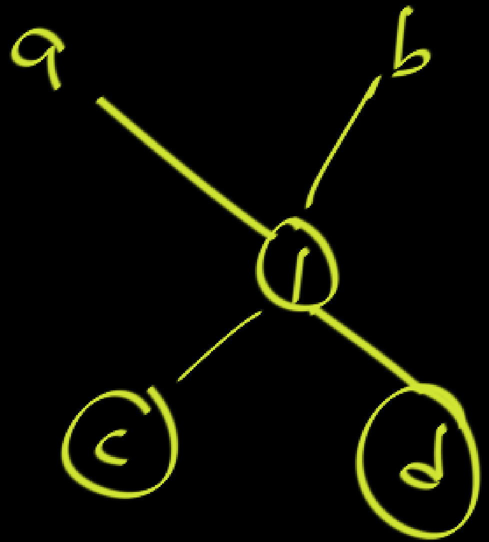
$K_{1,3}$





$K_{1,4}$





$K_{1,4}$

Star Graph: $K_{1,n}$ ✓

$\equiv K_{1,8}$

$K_{m,n}$; $m \geq 1, n \geq 1$ → Complete Bipartite Graph

① $|V| = m+n$

② $|E| = mn$

③ Degree seq: $\underbrace{n, n, n, \ldots, n}_{m \text{ times}}, \underbrace{m, m, m, \ldots, m}_{n \text{ times}}$

④ $\Delta = \max(m,n)$

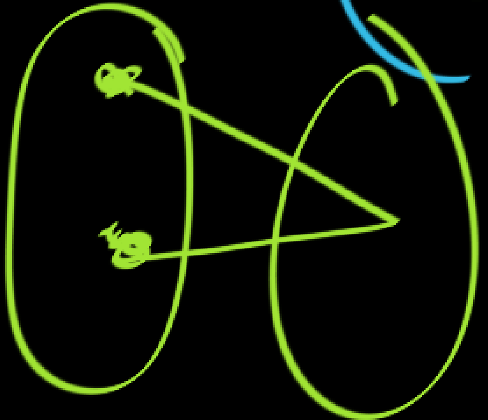
$\delta = \min(m,n)$

Diameter:

1 $K_{1,1}$

2 $K_{m,n}$ $\underbrace{m \geq 1, n \geq 1}$

But not $K_{1,1}$



$E_n \longrightarrow$ Complete bipartite ? Yes

a b
c d E_4

a
b
c
d