Set Theory

Next Chapter:

Cartesian Product of Sets

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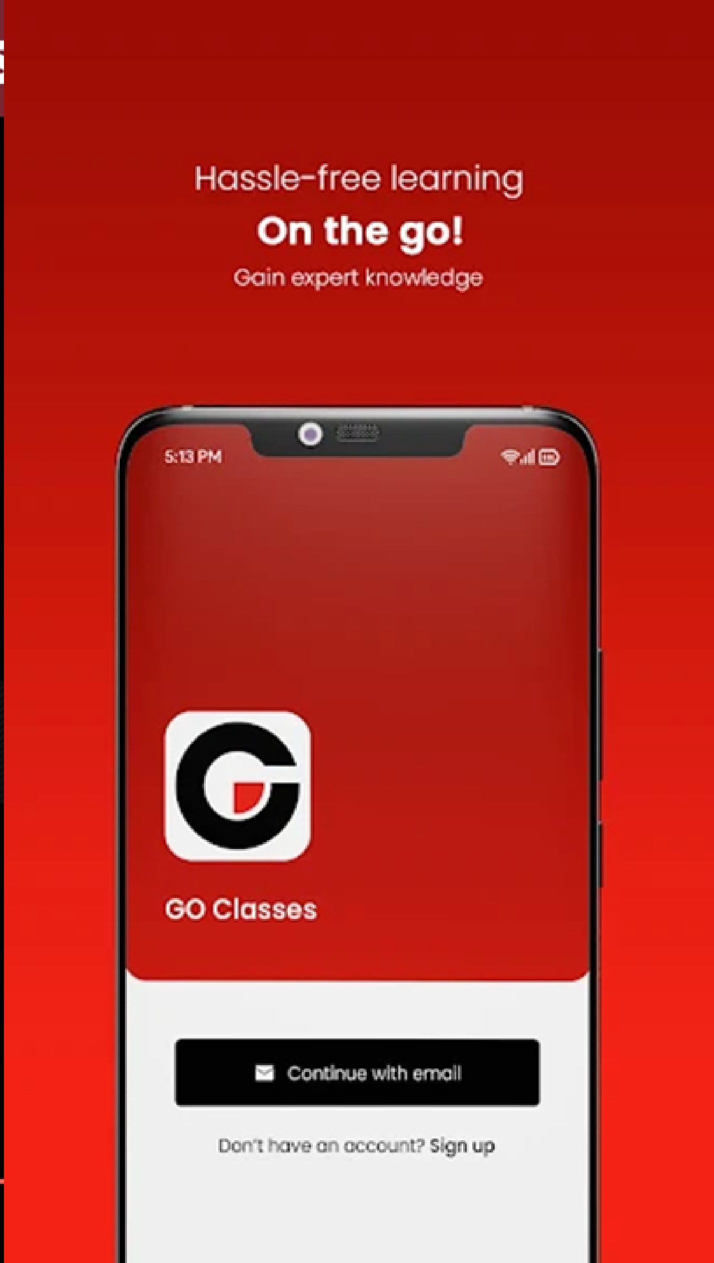

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Next Topic:

Ordered Pairs

(a new mathematical structure)

Ordered Pair:

Sequence of two elements a, b

{ Order matters.
&
Repetition matters.

Notation: (a, b)

$(1,2) \neq (2,1)$ Order matters.

Ordered Pair

(a,b)

$(1,1) \neq (1)$

Repetition matters.

Two Mathematical Structures:

1. **Set:**
Order Doesn't Matter, Repetition Doesn't Matter.

2. **Ordered Pairs:**
Order Matters, Repetition Matters.

Set

$\{2,3\} = \{3,2\}$

$\{2,2\} = \{2\}$

Notation: $\{\quad\}$

Ordered Pair

$(2,3) \neq (3,2)$

$(2,2) \neq (2)$

(a,b)

Have you seen/used this structure, Ordered Pairs, in your study of mathematics??

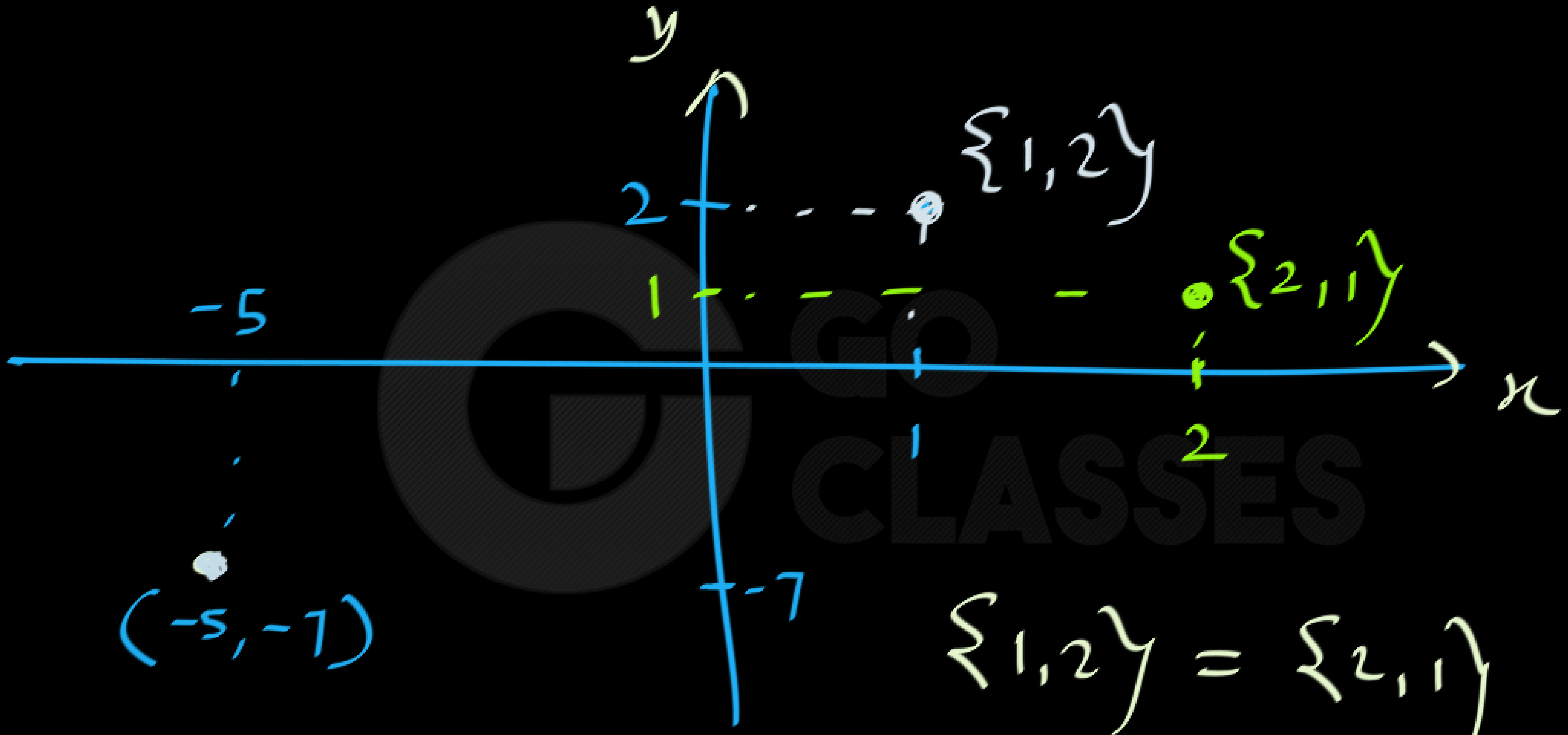
(a, b)

Have you seen/used this structure, Ordered Pairs, in your study of mathematics??

(a, b)

Cartesian plane
(x-y plane)

$(1,2) \neq (2,1)$



$$\{1,2\} = \{2,1\}$$

Set

$\{a, b\} = \{b, a\}$

Ordered Pair

$(a, b) \neq (b, a)$

Next Topic:

Ordered n-tuple

Two Mathematical Structures:

1. Set:

Order Doesn’t Matter, Repetition Doesn’t Matter.

2. Ordered n-tuple:

Order Matters, Repetition Matters.

n-Tuple: Sequence of n elements

$$(a_1, a_2, \text{-----}, a_n)$$

first element

second element

nth element

order matters.
Repetition matters.

3-Tuple :

$$(1, 3, 5) \neq (2, 1, 5)$$

$$\underbrace{(1, 1, 3)}_{\text{3-Tuple}} \neq \underbrace{(1, 3)}_{\text{2-Tuple (ordered pair)}}$$

$(1,1,3)$: Point on xyz plane

$(1,3)$: Point on xy plane.

$(1,3) \neq (1,1,3)$

$$(1,1,3) \neq (1,3,1)$$

$$(1,3,1) \neq (3,1,1)$$

$$(1,1) \neq (1) \neq (1,1,1)$$

2D plane

Point on 3D plane

when does $(a,b) = (c,d)$?

$(1,2) \neq (2,1)$

1st element

1st element

$(a,b) = (c,d)$

iff $a = c$ & $d = b$

$(2,3) = (2,3)$; $(2,2) = (2,2)$

When $(a, b) = (b, a)$?

iff $a = b$

when will two n-tuples be same?

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$$

iff $\forall_i \; a_i = b_i$

Next Topic:

Set

Vs Sequence

Vs Ordered n-tuple

Two Mathematical Structures:

1. **Set:** (Finite or Infinite)
Order Doesn't Matter, Repetition Doesn't Matter.
2. **Ordered n-tuple:** (Finite)
Order Matters, Repetition Matters.
3. **Sequence:** (Finite or Infinite)
Order Matters, Repetition Matters.

$\langle 1, 2, 3, 4, \ldots \ldots \rangle$: Sequence

$\langle 1, 1, 1, 1, \ldots \ldots \rangle$: "

$(1, 2, 3) \neq (3, 2, 1)$

$\langle 1, 2, 3 \rangle \neq \langle 3, 2, 1 \rangle$

$\langle 1, 1 \rangle \neq \langle 1 \rangle$

Notation for sequence:

$$< a_1, a_2, \ldots\ldots >$$

Ordered tuple

- Recall that a set does not consider its elements order.
- But sometimes, we need to consider a sequence of elements, where the order is important.
- An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ has a_1 as its first element, a_2 as its second element, ..., a_n as its n^{th} element.
- The order of elements is important in such a tuple.
- Note that $(a_1, a_2) \neq (a_2, a_1)$ but $\{a_1, a_2\} = \{a_2, a_1\}$.

Tuples

- The ordered n-tuple $(a_1, a_2, \ldots, a_n)$ is the ordered collection of n elements, where a_1 is the first, a_2 the second, etc., and a_n the n-th (i.e., the last).
- Two n-tuples are equal iff their corresponding elements are equal.

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \leftrightarrow a_1 = b_1 \wedge a_2 = b_2 \wedge \cdots \wedge a_n = b_n$$

- 2-tuples are called ordered pairs.

Definitions

A **sequence** of objects is a list of these objects in some order. Sequences may be finite or infinite.

A finite sequence is called a **tuple**. A sequence with **k** objects is called a **k-tuple**.

An **ordered pair** is a 2-tuple; that is, an ordered sequence of two elements. We write ordered pairs in parentheses, for example **(a, b)**, and we call **a** the first element and **b** the second element of the pair.

The **Cartesian product** or **cross product** of two sets A and B, written **A × B**, is the set of all ordered pairs wherein the first element is a member of A and the second element is a member of B.

Next Topic:

Cartesian Product of Sets

Set Operations:

Union

Intersection

Complement

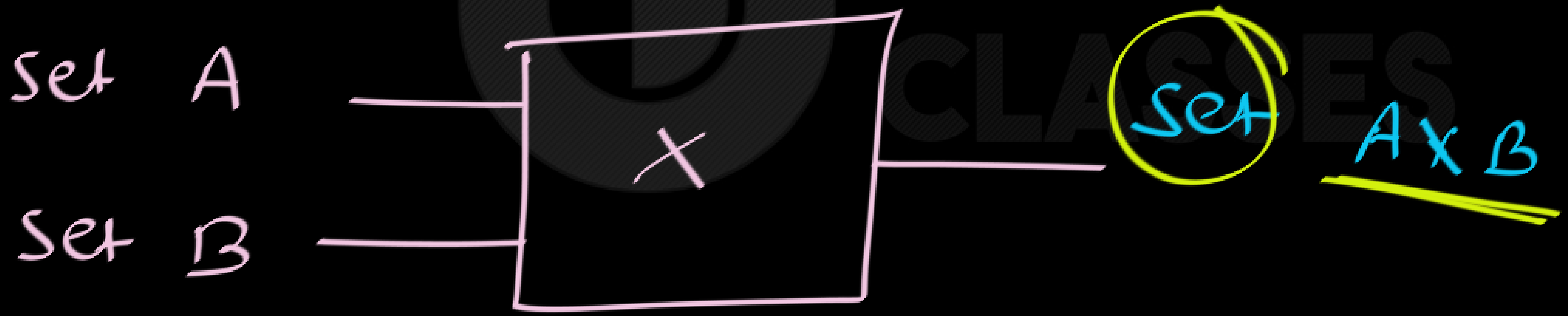
Set Difference

Symmetric Difference

Cartesian(Cross) Product

Cross product : $\times \rightarrow$ Cross

$\rightarrow$ set operation



$A : \{a, b\}$ $B = \{1, 2, 3\}$

$A \times B = \underbrace{\{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}}_{\text{Set}}$

$A \ \{a, b\}$

$B = \{1, 2, 3\}$

$|A \times B| = 6$

$A : \{a, b\}$; $B = \{1, 2, 3\}$

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$\underbrace{A \times B = \{ (\overset{A}{x}, \overset{B}{y}) \mid x \in A ; y \in B \}}_{\text{set}}$

$A : \{a, b\}$; $B = \{1, 2, 3\}$

$\underline{B} \times \underline{A} = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \quad \{1, 2, 3\}$

$A \quad \{a, b\}$

$B \times A = \{(x, y) \mid x \in B ; y \in A\}$

$A : \{a, b\}$; $B = \{1, 2, 3\}$

$A \times B = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

$\underline{B \times A} = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Diff.

$\boxed{(1,a) \neq (a,1)}$

Set A, B

$$\underbrace{A \times B = \left\{ (x, y) \mid x \in A ; \ y \in B \right\}}_{\text{Set}}$$

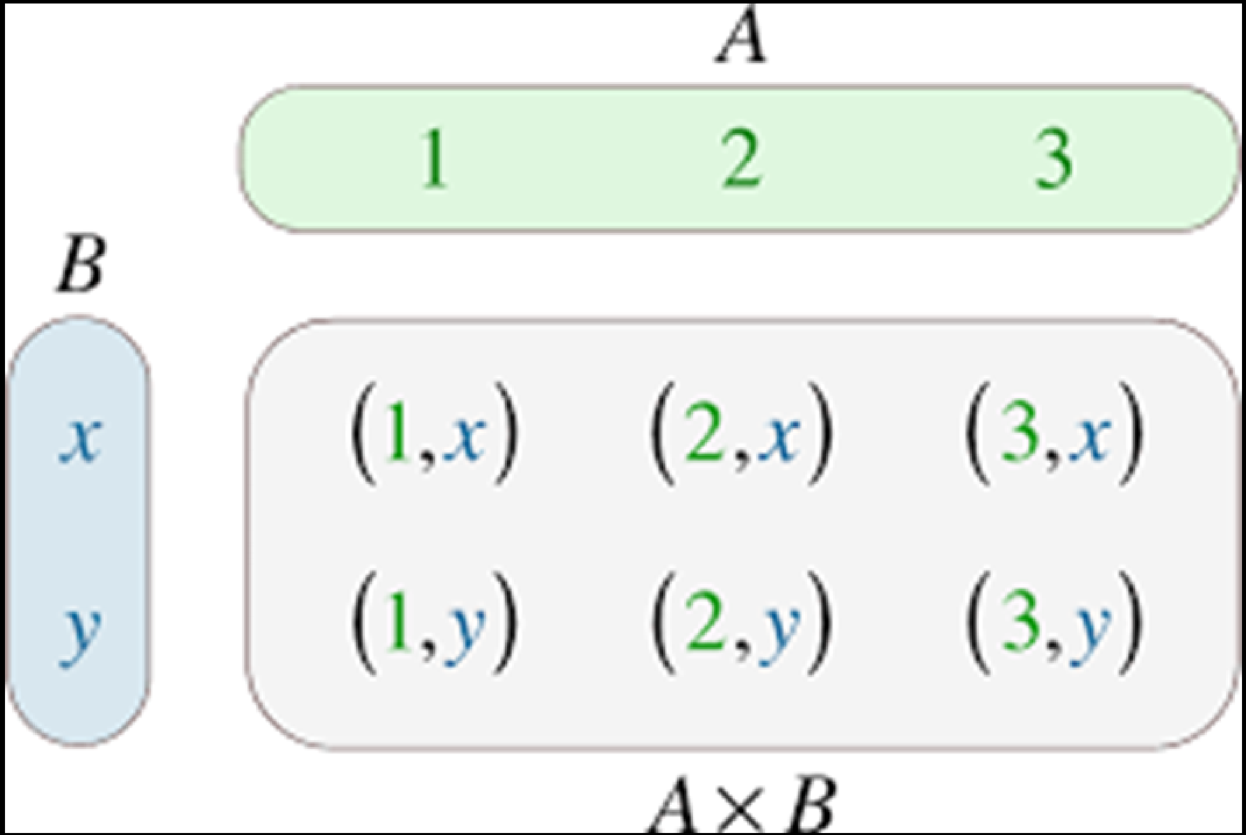
$$B \times A = \left\{ (x, y) \mid x \in B ; \ y \in A \right\}$$

The Cartesian Product

- Recall: The **power set** $\wp(S)$ of a set is the set of all its subsets.
- The **Cartesian Product** of $A \times B$ of two sets is defined as

$$A \times B \equiv \{ \ (a, b) \mid a \in A \text{ and } b \in B \ \}$$

$$\underbrace{\{0, 1, 2\}}_{A} \times \underbrace{\{a, b, c\}}_{B} = \left\{ \begin{array}{l} (0, a),(0, b),(0, c), \\ (1, a),(1, b),(1, c), \\ (2, a),(2, b),(2, c) \end{array} \right\}$$


A
1
2
3
B
x
y
(1,x)
(2,x)
(3,x)
(1,y)
(2,y)
(3,y)
A×B

Cartesian Product

A \ B	a	b	c
1	(1,a)	(1,b)	(1,c)
2	(2,a)	(2,b)	(2,c)

A x B

$$A \times B \times C = \underbrace{\{ \underbrace{(x, y, z)}_{\text{3-tuple}} \mid x \in A; y \in B; z \in C \}}_{\text{set}}$$

$A = \{2, 3\}$; $B = \{x, y\}$; $C = \{m\}$

$A \times B \times C = \{(2, x, m), (2, y, m), (3, x, m), (3, y, m)\}$

$(2, y, m) \rightarrow$ 3-tuple.

$|A \times B \times C| = 4$

$A \times C \times B = \{(2, m, x), (2, m, y), (3, m, x), (3, m, y)\}$

$A = \{2, 3\}$; $B = \{x, y\}$; $C = \{m\}$

$A \times A \times C = \{(2,2,m), (2,3,m), (3,2,m), (3,3,m)\}$

$A \times A = \{(2,2), (2,3), (3,2), (3,3)\}$

$C \times C = \{(m,m)\}$

2 Tuple.

$A = \{2, 3\}$; $B = \{x, y\}$; $C = \{m\}$

$A \times C \times A \times C = \{ (2, m, 2, m), (2, m, 3, m), (3, m, 2, m), (3, m, 3, m) \}$

$|A \times C \times A \times C| = 4$

4-Tuple

$C \times C \times C \times C = \{ (m, m, m, m) \}$

$|C \times C \times C \times C| = 1$

General:

Set $A_1, A_2, A_3, \ldots, A_n$

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n$$

$$= \left\{ \underbrace{(x_1, x_2, x_3, \ldots, x_n)}_{\text{n-Tuple}} \;\middle|\; x_i \in A_i \right\}$$

n-Tuple

one element

Cartesian product

Definition:

The Cartesian product of $A_1, A_2, \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is defined as the set of ordered tuples $(a_1, a_2, \ldots, a_n)$ where $a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n$. That is:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$$

Example Cartesian products

Examples:

- $\{1,2\} \times \{3,4,5\} = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$.
- $\{$Male, Female$\} \times \{$Married, Single$\} \times \{$Student, Faculty$\}$ = $\{$(Male, Married, Student), (Male, Married, Faculty), (Male, Single, Student), (Male, Single, Faculty), (Female, Married, Student), (Female, Married, Faculty), (Female, Single, Student), (Female, Single, Faculty)$\}$.
- $R \times R = \{(x,y) \mid x \in R, y \in R\}$ is the set of point coordinates in the 2D plane.

Cardinality of $A \times B$

Cardinality of $A \times B$:

$|A| = 3$; $|B| = 2$

$A = \{1, 2, 3\}$; $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

$|A \times B| = 6$

$A \times B \ni$ $(1, -)$ $(2, -)$ $(3, -)$

$2 + 2 + 2 = 6$

Set A ; B $\quad |A| = m$; $|B| = n$

$$A \times B = \{ (x, y) \mid x \in A ; y \in B \}$$

x: m choices $\quad$ y: n choices

$$|A \times B| = m \times n = \underline{mn}$$

$|A| = m \; ; \; |B| = n$

$A = \{a_1, a_2, \ldots, a_m\}$

$B = \{b_1, b_2, \ldots, b_n\}$

$|A \times B| = \underbrace{n + n + \ldots + n}_{m \text{ times}}$

$|A \times B| = mn$ ✓

$A \times B = \{ \boxed{(a_1, b_i)}, \boxed{(a_2, b_i)}, \ldots, (a_n, b_i) \}$

n elements

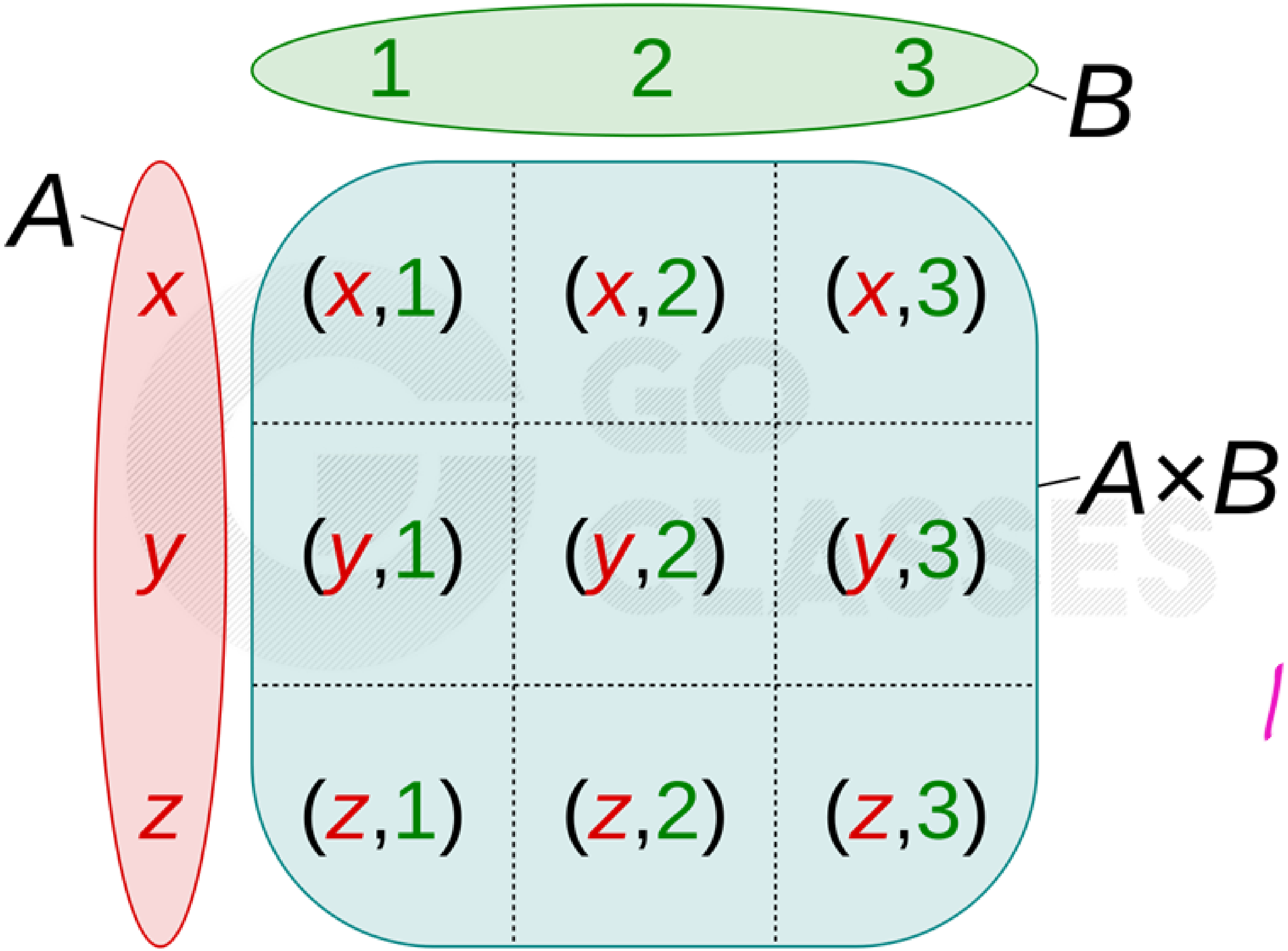
n elements

n elements

$$\boxed{|A \times B| = |A| \cdot |B|}$$

$$|A \times B \times C| = |A| \cdot |B| \cdot |C|$$

$$|A \times A \times B| = |A| \cdot |A| \cdot |B|$$



$|A \times B| = 3 \times 3$
$= 9$

Cardinality of Cartesian product

Fact:

In general, if A_i's are finite sets, we have:

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \times |A_2| \times \cdots \times |A_n|$$

What is $A \times \Phi$?

What is $A \times \Phi$?

$A : \{1, 2\}$; $\phi = \{ \}$

$A \times \phi = \{ (x, y) \mid x \in A ; y \in \phi \}$

$A \times \phi = \{ \quad \} = \phi$

$$A \times \phi = \phi$$

$$\phi \times A = \phi$$

$$\phi \times \phi = \phi$$

$$\boxed{A \times B \times \phi \times C = \phi}$$

$A \times B$ Vs $B \times A$

$A \times B$ Vs $B \times A$

$A : \{1\}$; $B = \{x, y\}$

$A \times B = \{(1, x), (1, y)\}$

$B \times A = \{(x, 1), (y, 1)\}$

Different

$\boxed{(1, x) \neq (x, 1)}$

$A \times B$ Vs $B \times A$

$A : \{1\}$; $B = \{x, y\}$

$$\boxed{A \times B \neq B \times A}$$

When $A \times B = B \times A$?

When $A \times B = B \times A$?

1. $A = B$ ✓ $A \times A = A \times A$

2. If $A = \phi$ OR $B = \phi$

$$\boxed{S \times \phi = \phi}$$

$\boxed{A \times B = B \times A}$ iff

$\boxed{\begin{array}{l} A = B \ \underline{\underline{OR}} \ A = \phi \\ \quad or \ B = \phi \end{array}}$

Is Cartesian Product Associative????

$$(A \times B) \times C \overset{?}{=} A \times (B \times C)$$

Cartesian Product is NOT Associative:

$A = \{1\}$; $B = \{2\}$; $C = \{3\}$

$(A \times B) \times C = \{ ((1,2), 3) \}$

$A \times B = \{ (1,2) \}$

Cartesian Product is NOT Associative:

$A = \{1\}$; $B = \{2\}$; $C = \{3\}$

$(A \times B) \times C = \{ ((1,2), 3) \}$

$A \times (B \times C) = \{ (1, (2,3)) \}$

$A \times B \times C = \{ (1,2,3) \}$

$$A \times B \times C = \{ (x, y, z) \}$$

a 3-Tuple

$$(A \times B) \times C = \{ ((x, y), z) \}$$

$$(A \times B) \times C \neq A \times (B \times C)$$

$$A \times B \times C \neq (A \times B) \times C$$

$\times$: Not Asso. ✓

Not Comm. ✓

$$\boxed{A \times B \neq B \times A}$$

$$(A \times B) \times C \neq A \times (B \times C)$$

Definitions

A **sequence** of objects is a list of these objects in some order. Sequences may be finite or infinite.

A finite sequence is called a **tuple**. A sequence with **k** objects is called a **k-tuple**.

An **ordered pair** is a 2-tuple; that is, an ordered sequence of two elements. We write ordered pairs in parentheses, for example **(a, b)**, and we call **a** the first element and **b** the second element of the pair.

The **Cartesian product** or **cross product** of two sets A and B, written **A × B**, is the set of all ordered pairs wherein the first element is a member of A and the second element is a member of B.

6.1 Cartesian Products

In the Cartesian plane (or x-y plane), we associate the set of points in the plane with the set of all ordered points (x, y), where x and y are both real numbers. The idea of a Cartesian product of sets replaces $\mathbb{R}$ in the description by some other set(s), and drops the geometric interpretation.

If A and B are sets, the *Cartesian product* of A and B is the set

$$A \times B = \{(a, b) : (a \in A) \text{ and } (b \in B)\}.$$

The following points are worth special attention:

- The Cartesian product of two sets is a set.
- The elements of that set are ordered pairs.
- In each ordered pair, the first component is an element of A, and the second component is an element of B.

The points in the x-y plane correspond to the elements of the set $\mathbb{R} \times \mathbb{R}$.

For example, if $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$ and $B \times A = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.

Suppose A has m elements and B has n elements. Then, each element of A is the first component of n ordered pairs in $A \times B$: one for each element of B. Thus the number of elements in $A \times B$ equals $m \times n$, the number of elements in A times the number of elements in B. This is one way in which the "$\times$" symbol is suggestive notation for the Cartesian product.

What should $A \times \emptyset$ be? By definition, it is the set of all ordered pairs (a, b) where $a \in A$ and $b \in \emptyset$. There are no such pairs, as there are no elements $b \in \emptyset$. Hence $A \times \emptyset = \emptyset$. Similarly, $\emptyset \times B = \emptyset$.

We says that two ordered pairs are *equal* if the first components are identical and so are the second components. That is, $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. This corresponds to (and generalizes) our idea of equality for ordered pairs of real numbers.

The example above shows that $A \times B \neq B \times A$ in general. This leads to the question of when they are equal. Certainly they are equal if $A = B$ because then $A \times B = A \times A = B \times A$. They are also equal when $A = \emptyset$ or $B = \emptyset$ because, then $A \times B = \emptyset = B \times A$.

Cartesian Products and Relations

Definition (Cartesian product) If A and B are sets, the *Cartesian product* of A and B is the set

$$A \times B = \{(a, b) : (a \in A) \text{ and } (b \in B)\}.$$

The following points are worth special attention: The Cartesian product of two sets is a set, and the elements of that set are ordered pairs. In each ordered pair, the first component is an element of A, and the second component is an element of B.

Example (Cartesian product) If $A = \{\{1, 2\}, \{3\}\}$ and $B = \{(a, b), (c, d)\}$, then

$$A \times B = \{(\{1, 2\}, (a, b)), (\{1, 2\}, (c, d)), (\{3\}, (a, b)), (\{3\}, (c, d))\}.$$

Determining $|A \times B|$. If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$ because there are $|A|$ choices for the first component of each ordered pair and, for each of these, $|B|$ choices for the second component of the ordered pair.

Cartesian Product

Definition

The Cartesian product of two sets A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.
$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$

Definition

The Cartesian product of n sets $A_1, A_2 \ldots, A_n$, denoted by $A_1 \times A_2 \times \cdots \times A_n$, is the set of all tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots, n$.
$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } i = 1, 2, \ldots, n\}$

Example: What is $A \times B \times C$ where $A = \{0, 1\}, B = \{1, 2\}$ and $C = \{0, 1, 2\}$.
Solution: $A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 1, 2)\}$

Next Topic:

Relation