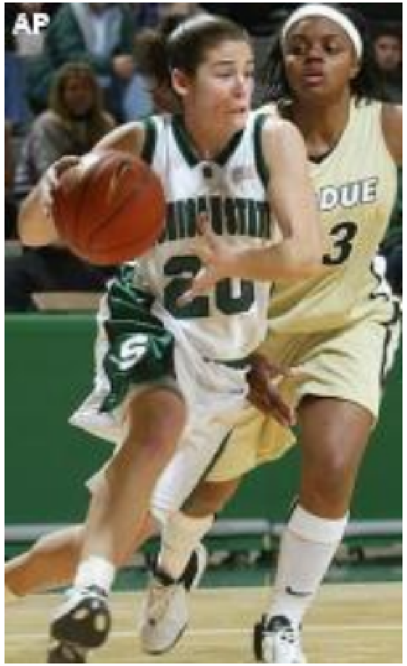
# Combinatorics

## Recap:

## Permutation, Combination

Website : https://www.goclasses.in/

# Player Numbers




Shirts have 2-digit numbers

Six possible digits: 0, 1, 2, 3, 4, 5

How many different numbers?

$$\boxed{6}\ \boxed{6} = 36$$

6 possibilities for first digit; 6 for second

Suppose double digits are not allowed? You have one of each the numbers; you select two and iron them on. How many choices have you for one shirt?

$6 * 5 = 30$

# Permutations vs. Combinations

It is **very** important to make the distinction between permutations and combinations. In permutations, order matters and in combinations order does **not** matter. The important information can be summarized by:

| | Order | Number |
|---|---|---|
| Permutation | matters | $P(n,k) = \frac{n!}{(n-k)!}$ |
| Combination | does not matter | $C(n,k) = \frac{n!}{(n-k)!k!}$ |

# Permutations and Combinations



$S = \{1, 2, 3, 4\}$,

the 2-orderings are:
$\{1\,2, 1\,3, 1\,4, 2\,1, 2\,3, 2\,4, 3\,1, 3\,2, 3\,4, 4\,1, 4\,2, 4\,3\}$
(permutations)

the 2-subsets are:
$\{1\,2, 1\,3, 1\,4, 2\,3, 2\,4, 3\,4\}$
(combinations)

With $n$ elements, by the product rule, the number of $k$-orderings is

$$\text{number of } k\text{-orderings} = n \times (n-1) \times (n-2) \times \cdots \times (n-(k-1)) = \frac{n!}{(n-k)!}.$$

e.g. number of top-3 finishes in 10-person race is $10 \times 9 \times 8 = 10!/7!$.

Pick a $k$-subset ($\binom{n}{k}$ ways) and reorder it in $k!$ ways to get a $k$-ordering.

$$\begin{aligned} \text{number of } k\text{-orderings} &= \text{number of } k\text{-subsets} \times k! && \leftarrow \text{product rule} \\ &= \binom{n}{k} \times k! && \leftarrow \text{bijection to sequences with } k \text{ 1's} \end{aligned}$$

$$\boxed{\text{number of } k\text{-subsets} = \binom{n}{k} = \frac{n!}{k!(n-k)!}}$$

**Exercise.** How many 10-bit binary sequences with four 1's?

# Permutations - Order Matters

The number of ways one can select 2 items from a set of 6, with order mattering, is called

the number of permutations of 2 items selected from 6

$$6\times5 = 30 = {}_6P_2$$

Example: The final night of the Folklore Festival will feature 3 different bands. There are 7 bands to choose from. How many different **programs** are possible?

# Calculating ${}_nP_r$



Solution to band problem: ${}_7P_3 = 7{\cdot}6{\cdot}5 = 210$

This is not the same as asking "How many ways are there to choose 3 bands from 7?"

Write out expressions for

${}_{52}P_4 \quad = 52 \cdot 51 \cdot 50 \cdot 49$

${}_7P_6 \quad = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 7!$

Number of factors

Starting factor

# Factorial Notation for ${}_nP_r$

$$ {}_7P_3 = 7\cdot 6\cdot 5 = \frac{7\cdot 6\cdot 5\cdot}{} $$

$$ {}_{52}P_4 = 52\cdot 51\cdot 50\cdot 49 = \frac{52\cdot 51\cdot 50\cdot 49}{} $$

$$ {}_{20}P_3 = 20\cdot 19\cdot 18 = \frac{20!}{?} = \frac{20!}{17!} $$

Formula for ${}_nP_r$ : $\quad {}_nP_r = \frac{n!}{(n-r)!}$

# Combinations - Order Does Not Matter

- The Classical Studies Department has 7 faculty members. Three must attend the graduation ceremonies. How many different groups of 3 can be chosen?
- If order mattered, the answer would be $7 \cdot 6 \cdot 5 = 210$
- Let's look at one set of three professors: A, B, C:
  A B C   A C B   B C A   B A C   C A B   C B A
- Why are there 6 listings for the same set of 3 profs?

There are 3! = 6 possible arrangements of three objects.

# ${}_nP_n$

Recall the convention: 0! =1

There are $3 * 2 * 1 = 3!$

arrangements of 3 objects.

Using the nPr notation, from a set of 3 objects we are choosing 3.

$${}_3P_3 = \frac{3!}{(3-3)!} = \frac{3!}{0!} = \frac{3!}{1} = 3!$$

# Combinations: $_7C_3$



- In our list of 210 sets of 3 professors, with order mattering, each set of three profs is counted 3! = 6 times. The number of distinct combinations of 3 professors is

$$_7C_3 = \frac{_7P_3}{6} = \frac{_7P_3}{3!}$$

$$= \frac{7!}{(7-3)!3!} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = \frac{210}{6} = 35$$

$_7C_3$ is the number **combinations** of 3 objects chosen from a set of 7. “Of seven, take three”

# ${}_nC_r$

- Factorial formula is: $_nC_r = \frac{nPr}{r!} = \frac{n!}{r!(n-r)!}$

- Practice:

$_8C_2$

$_8C_6$

$_{10}C_4$

# $_nC_r$ (of n, pick r)

- Factorial formula is: $$_nC_r = \frac{n!}{r!(n-r)!}$$

- Practice:

$$_8C_2 = \frac{8!}{2!6!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{8\cdot 7}{2\cdot 1} = 28$$

$$_8C_6 = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 28$$

$$_{10}C_4 = \frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2\cdot 1} = 210$$

## What about xP3 and xC3?

$$ {}_{x}P_{3} = x(x-1)(x-2) $$

$$ {}_{x}C_{3} = \frac{x(x-1)(x-2)}{3!} $$

# Combinatorics

## Next Sub-Topic:

## Why $0! = 1$ ? Why $nC(n+1) = 0$?

Website : https://www.goclasses.in/

Q:

Why

$0! = 1$ ?

CORRECT Answer: By Convention. By Definition.

So, the right question that should be asked is “Why this convention”?

Q.

order Does not matter

Number of subsets of size r of a set of size n?

$= {}^{n}C_{r}$

**Q.**

**Number of subsets of size 0 of a set of size n?** $\rightarrow$ 1

$\phi$ ✓

So, ${}^nC_0 = 1$

$$\frac{\cancel{n!}}{\cancel{n!}\,0!} = 1 \Rightarrow 0! = 1$$

Not a proof of $0! = 1$.

$\rightarrow$ one of the Reasons behind Convention $0! = 1$

Q.

Number of subsets of size n+1 of a set of size n? $\Rightarrow 0$

so, $\boxed{{}^{n}C_{n+1} = 0}$

Q.

Number of subsets of size r (>n) of a set of size n? = 0

$\boxed{{}^{n}C_{r}} = 0 \quad ; \quad r > n$

no. of subsets of size r from n elements.

Why we Define $0! = 1$ ?

$^6C_3$ = selection of 3 out 6

$\{a, b\} \rightarrow$ select 2 of them $\Rightarrow$ 1 way

$\rightarrow$ " 1 of them $\Rightarrow$ 2 ways

$\rightarrow$ " 0 " " $\Rightarrow$ 1 way

$\rightarrow$ select 3 of them $\Rightarrow$ 0 way

Selecting n people from n people :

$$^{n}C_{n} = 1$$

Selecting 0 people from n = ${}^{n}C_{0}$

= 1 ways =

$$1 = {}^{n}C_{0}$$

Selecting 0 from 0 elements = ${}^{0}C_{0} = 1$

Q: $A = \phi = \{\ \}$

How many subsets of A of zero cardinality? ]

$\phi \subseteq \phi$

from $\overset{n}{0}$ elements, selecting $\overset{n}{0}$ elements $= {}^nC_n$

[redacted]

Q: $A = \{1\}$

How many subsets of A of cardinality 3?

$$\boxed{{}^{1}C_3 = 0}$$

No subset of cardinality 3

$= 0$

Note: ${}^nC_r$ binomial Coefficient

$${}^nC_r = \frac{n!}{(n-r)!\,r!}$$ ; Algebraic meaning

${}^nC_r$ = Selecting $r$ people from $n$ people ; Combinatorial meaning

# Combinatorics

## Next Topic:

## Standard Questions(Templates)

## in P & C

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

9 maths

11 CS

AND

Select 3

select 4

$$\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$$

$${}^{9}C_{3} * {}^{11}C_{4} = {}^{9}C_{6} * {}^{11}C_{4} = {}^{9}C_{6} * {}^{11}C_{7}$$

Q 1.

Mr. Smith is the chair of a committee. How many ways can a committee of 4 be chosen from 9 people given that Mr. Smith must be one of the people selected?

Q 1.

Mr. Smith is the chair of a committee. How many ways can a committee of 4 be chosen from 9 people given that Mr. Smith must be one of the people selected?

$9 \xrightarrow{\text{select}} 4$

But Mr. Smith is Already Selected.

$8 \xrightarrow{\text{select}} 3$

Ans: ${}^8C_3 = \frac{8 \times 7 \times 6}{6}$

1) **Mr. Smith is the chair of a committee. How many ways can a committee of 4 be chosen from 9 people given that Mr. Smith must be one of the people selected?**

Mr. Smith is already chosen, so we need to choose another 3 from 8 people. In choosing a committee, order doesn't matter, so we need the combination without repetition formula.

$\frac{n!}{r!(n-r)!} = \frac{8!}{3!(8-3)!} = 56$ ways

Q 2.

A certain password consists of 3 different letters of the alphabet where each letter is used only once. How many different possible passwords are there?

## Q 2.

order matter

A certain password consists of 3 different letters of the alphabet where each letter is used only once. How many different possible passwords are there?

$^{26}P_3$ ✓

$= (26)(25)(24)$

$26 \times 25 \times 24$

2) **A certain password consists of 3 different letters of the alphabet where each letter is used only once. How many different possible passwords are there?**

Order does matter in a password, and the problem specifies that you cannot repeat letters. So, you need a permutations without repetitions formula. The number of permutations of 3 letters chosen from 26 is

$\frac{n!}{(n-r)!} = \frac{26!}{(26-3)!} = 15{,}600$ passwords

Q 3.

A password consists of 3 letters of the alphabet followed by 3 digits chosen from 0 to 9. Repeats are allowed. How many different possible passwords are there?

# Q 3.

A password consists of 3 letters of the alphabet followed by 3 digits chosen from 0 to 9. Repeats are allowed. How many different possible passwords are there?

| | | | | | |
|---|---|---|---|---|---|
| a | b | a | 0 | 0 | 2 |
| 26 | 26 | 26 | 10 | 10 | 10 |

$$\Rightarrow (26)^3 \times (10)^3$$

**Note:**

$^nP_r$

$n \rightarrow$ n Distinct elements

$r \rightarrow$ r Distinct elements

Permute (ordered Arrangement)

# Q 4.

## An encyclopedia has 6 volumes. In how many ways can the 6 volumes be placed on the shelf?

**4) An encyclopedia has 6 volumes. In how many ways can the 6 volumes be placed on the shelf?**

This problem doesn't require a formula from the chart. Imagine that there are 6 spots on the shelf. Place the volumes one by one.

The first volume to be placed could go in any 1 of the 6 spots. The second volume to be placed could then go in any 1 of the 5 remaining spots, and so on. So the total number of ways the 6 volumes could be placed is

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways

Cards: → 4 suits/Types

→ 13 Values of each suit

13 values : 2, 3, 4, - - , 10, J, Q, K, A

4 suits : $2\heartsuit$, $2\diamondsuit$, $2\clubsuit$, $2\spadesuit$

Q.

Hand in Poker = A poker hands consists of a 5 randomly chosen cards out of a deck of 52 cards.

A hand is called “Full House”:

Full house: This hands consists three cards of the same value and two cards of an another value (e.g. 3 kings and 2 eights). How many different Full Houses are possible?

$$5\clubsuit, 5\spadesuit, 5\diamondsuit, 4\heartsuit, 4\clubsuit.$$

full House : 2 Different values a, b

$\boxed{a, a, a, b, b}$

method 1:

$$\underbrace{{}^{13}C_1 * {}^{4}C_3}_{\text{3 Cards of a value}} * \underbrace{{}^{12}C_1 * {}^{4}C_2}_{\text{2 Cards of another value}}$$

full House :

select a value which has 3 cards → $^{13}C_1$

for this value, we need 3 suits → $^{4}C_3$

select a Value which has 2 cards → $^{12}C_1$

for this Value need 2 suits → $^{4}C_2$

## Full House: value with 3 cards

$^{13}C_1$

7

7 ♥   7 ♠   7 ♦

$^4C_3$

Another value with 2 cards

$^{12}C_1$

9

9 ♥   9 ♠

$^4C_2$

*Full house*: This hands consists three cards of the same value and two cards of an another value (e.g. 3 kings and 2 eights). There are 13 ways to choose the value of three of a kind and once this value is chose there is $\binom{4}{3}$ to select the three cards out of the four of same value. There are then 12 values left to choose from for the pair and there $\binom{4}{2}$ to select the the pair. So we have

method 2:

$$\underbrace{{}^{13}C_2}_{4,\,k} * \underbrace{{}^{2}C_1}_{3 \text{ cards}} * \underbrace{{}^{4}C_3}_{\text{for } 4} * \underbrace{{}^{4}C_2}_{\text{for } k}$$

4

How many bit strings of length $n$ contain exactly $r$ 1s?

How many bit strings of length $n$ contain exactly $r$ 1s?

length 4 ; Exactly 2 1's

| 1 | 0 | 0 | 1 |
|---|---|---|---|

$\rightarrow \underbrace{{}^4C_2}_{\text{for 1's}}$

$\underbrace{{}^nC_r}$

**Example 7.** *How many bit strings of length 10 have*

1. *exactly three 0s?*
2. *more 0s than 1s?*
3. *at least seven 1s?*
4. *at least three 1s?*

**Example 7.** *How many bit strings of length 10 have*

1. *exactly three 0s?* $\longrightarrow$ ${}^{10}C_3$
2. *more 0s than 1s?* $\longrightarrow$ by Case
3. *at least seven 1s?*
4. *at least three 1s?*

(b) more 0's than 1's :

$\underbrace{10\ 0's,\ 0\ 1's}_{1\ \text{way}}$ OR $\underbrace{9\ 0's,\ 1\ 1's}_{{}^{10}C_1}$ OR $\underbrace{8\ 0's,\ 2\ 1's}_{{}^{10}C_2}$ OR $\underbrace{7,\ 3}_{{}^{10}C_3}$

OR $\underbrace{6,\ 4}_{{}^{10}C_4}$

Ans: ${}^{10}C_{10} + {}^{10}C_9 + {}^{10}C_8 + {}^{10}C_7 + {}^{10}C_6$

$= {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4$

(c) At least 7 one's ;

$^{10}C_7 \times 2^3 \Rightarrow$ WRong (over counting)

for 1's (Reserved)

Anything (1's, 0's)

MOST Common MisTake

Eg: length 4 ; At least 3 ones ;

$\underbrace{{}^4C_3}_{\text{Reserved for 1's}} * \underbrace{2^1}_{\text{Anything}} = 8$

Again 1 is Allowed

8 → wrong

Actual ⇒ 5

1111
1110
1101
1011
0111

$$\underbrace{{}^4C_3}_{\text{for 1's}} * \underbrace{2^1}$$

1111

has been Counted 4 times

Select 1, 2, 3 positions for 1

Remaining position ⇒ 1

| 1 | 1 | 1 | 1 |
|---|---|---|---|

Select 1, 2, 4 positions for 1

Remaining position ⇒ 1

| 1 | 1 | 1 | 1 |
|---|---|---|---|

$\underbrace{{}^4C_3}_{\text{for 1's}} * \underbrace{2^1}$

1111

has been Counted 4 times → 3 is Extra

Select 2,3,4 positions for 1

Remaining position ⇒ 1

| 1 | 1 | 1 | 1 |
|---|---|---|---|

✓

Select 1,3,4 positions for 1

Remaining position ⇒ 1

| 1 | 1 | 1 | 1 |
|---|---|---|---|

✓

NOTE:


$\boxed{\text{Reserves}}$ by type 1 people

Don't Allow type 1 people

Most Common mistake : ⇒ $\boxed{\text{overcounting}}$

×

$\boxed{\text{Reserves}}$ by type 1 people

Allowing type 1 people

**Overcounting** : All Valid Cases Counted.

No Invalid Cases Counted.

Some Valid Cases Counted more than one time.

Complete Analysis of the Previous Most Common Mistake :

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ How many times 1001 Counted ?

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ wrong (overcounting)

→ How many times 1001

Counted ⇒ once

| 1 | 0 | 0 | 1 |
|---|---|---|---|

Q: 4 length bit string; At least 2 ones.

$$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$$

→ How many times 1110 Counted ?

→ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ How many times 1110 Counted? ⇒ 3 times

| 1 | 1 | 1 | 0 |
|---|---|---|---|
| 1 | 1 | 1 | 0 |
| 1 | 1 | 1 | 0 |

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$ → How many times 1011 Counted?

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ How many times 1011

Counted 1 ⇒ 3 times (2 Extra)

| 1 | 0 | 1 | 1 |
|---|---|---|---|

| 1 | 0 | 1 | 1 |
|---|---|---|---|

| 1 | 0 | 1 | 1 |
|---|---|---|---|

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ How many times 0001

Counted ?

→ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$ → How many times 0001 counted? ⇒ 0 times

invalid

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

→ How many times 1111 Counted?

↳ wrong (overcounting)

Q: 4 length bit string; At least 2 ones.

$\underbrace{{}^4C_2}_{\text{Reserved for 1's}} \times \underbrace{2^2}_{\text{Anything (0's, 1's)}}$

Reserved for 1's

Anything (0's, 1's)

→ How many times 1111 Counted? ⇒ 6 times

${}^4C_2$

1 1 1 1

1 1 1 1

1 1 1 1

1 1 1 1

1 1 1 1

1 1 1 1

↳ wrong (overcounting)

Q: 10 length bit-string ; at least 6 ones.

One student:

$$\underbrace{{}^{10}C_6}_{\text{Reserves for 1's}} * \underbrace{2^4}_{\text{Anything (0's, 1's)}}$$

Y has Counted :

$\boxed{11111111}00 \longrightarrow {}^{8}C_{6}$ times

from 8 1's, Decide which 6 are Reserved Seats for 1's.

$0110110110 \longrightarrow {}^{6}C_{6} = 1$ times

$11111\ 11111 \longrightarrow {}^{10}C_{6}$ times

(c) length 10; At least 7 ones:

Exactly 7 1's OR Exactly 8 1's OR Exactly 9 1's OR 10 1's

$^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}$

(d) length 10; At least 3 ones:

$${}^{10}C_3 + {}^{10}C_4 + {}^{10}C_5 + {}^{10}C_6 + \cdots + {}^{10}C_{10}$$

(d) length 10; At least 3 ones:

$$2^{10} - \left( \underline{\text{At most 2 1's}} \right)$$

$$2^{10} - \left( {}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 \right)$$

At least 3 ones' (length 10)

Complement

At most 2 1's ✓

" " 3 " ✗

At most 7 0's ✗

" least 8 0's ✓

Same

length 10

At least 7 1's $\equiv$ At most 3 0's

# Combinatorics

## Next Template:

## Some elements always together

Put them in a box. (Tie them together)

↳ Consider a single unit.

How many permutations of the letters $ABCDEFGH$ contain the string $ABC$? (Substring ABC)

$\boxed{ABC}$ D E F G H

$= 6!$

How many permutations of the letters $ABCDEFGH$ contain the string $ABC$?

*Solution:* Because the letters $ABC$ must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block $ABC$ and the individual letters $D$, $E$, $F$, $G$, and $H$. Because these six objects can occur in any order, there are $6! = 720$ permutations of the letters $ABCDEFGH$ in which $ABC$ occurs as a block. ◀

Q: How many "Permutations" of ABC has substring CA? ⇒ 2

Permutations of ABC

{ ABC, BAC, CAB,
ACB, BCA, CBA }

. How many permutations of the letters $ABCDEFG$ contain

a) the string $BCD$?
b) the string $CFGA$?
c) the strings $BA$ and $GF$?
d) the strings $ABC$ and $DE$?
e) the strings $ABC$ and $CDE$?
f) the strings $CBA$ and $BED$?

. How many permutations of the letters $ABCDEFG$ contain

a) the string $BCD$? → 5!
b) the string $CFGA$? → 4!
c) the strings $BA$ and $GF$? → 5!
d) the strings $ABC$ and $DE$? → 4!
e) the strings $ABC$ and $CDE$? → 3!
f) the strings $CBA$ and $BED$? → 0

ABCDE F G

(C) $\boxed{BA}$ CDE $\boxed{GF}$ = 5!

(D) $\boxed{ABC}$ $\boxed{DE}$ G F = 4!

(E) Substring $\boxed{ABC}$ AND $\boxed{CDE}$

$\boxed{ABCDE}$

# Combinatorics

## Next Template:

## Some elements NEVER together

= first Arrange Remaining elements ;
then in Gaps, put them.

How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

8 m , 5 w → no two together

| $m_8$ | $m_1$ | $m_2$ | $m_3$ | $m_4$ | $m_5$ | $m_6$ | $m_7$ |

$$\boxed{8! * {}^{9}C_{5} * 5!} = 8! * {}^{9}P_{5}$$

8m, 5w :

No women Together = $\boxed{\text{Total} - \text{All women Together}}$

should not count

mwmmwwww mm --

will be counted

How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]

$$\underbrace{10!}_{\text{for women}} \times {}^{11}P_6$$

Seven women and nine men are on the faculty in the mathematics department at a school.

a) How many ways are there to select a committee of five members of the department if at least one woman must be on the committee?

b) How many ways are there to select a committee of five members of the department if at least one woman and at least one man must be on the committee?

7w, 9m

(a) Select 5 ; At least one woman

$^7C_1 * {}^{15}C_4$

Reserved for women

Allowing women

Same most Common Mistake

Over Counting

7w, 9M = Total 16 People

(a) Select 5; At least one woman

Total − No women

$^{16}C_5 - {}^{9}C_5$

(b) 7w, 9m ; At least 1 man ; At least 1 woman

select 5

$$\underbrace{{}^{7}C_{1}}_{\text{Reserved for woman}} * \underbrace{{}^{9}C_{1}}_{\text{Reserved for man}} * \underbrace{{}^{14}C_{3}}_{\text{Allowing men, women}}$$

wrong

Same most Common mistake

over counting

(b) 7w, 9m ; At least 1 man ; At least 1 woman

Total — No men — No women

$$\underbrace{{}^{16}C_5 - {}^{7}C_5 - {}^{9}C_5}$$

# Stock Market Example

Suppose 12 stocks have been traded, and
7 increased, 3 decreased, 2 stayed the same.

In how many ways could this happen?

# Stock Market Example

Suppose 12 stocks have been traded, and
7 increased, 3 decreased, 2 stayed the same.

In how many ways could this happen?

increase     Decrease     same

$^{12}C_7 \times {}^{5}C_3 \times {}^{2}C_2$

# Stock Market Example

Suppose 12 stocks have been traded, and
7 increased, 3 decreased, 2 stayed the same.

In how many ways could this happen?

Think: 3 separate selections: “three-box problem”

| Of the 12, 7 increased | Of the remaining 5, 3 decreased | Of the remaining 2, 2 stayed the same |
|---|---|---|
| ${}_{12}C_7$ | ${}_{5}C_3$ | ${}_{2}C_2 = 1$ |

$$\frac{12!}{7!5!}\frac{5!}{3!2!}1 = \frac{12!}{7!3!2!} = 7920$$

# Combinatorics

## Next Topic:

# Combinatorial Identities

Always listen to the both sides of a story.

(Combinatorial Arguments: Make a story, think of it in two ways.)

① $^nC_r = {^nC_{n-r}}$

$n-(n-r) = r$

Algebraic (Boring) Proof:

$$\frac{n!}{(n-r)!\,r!} = \frac{n!}{(r)!\,(n-r)!}$$

1) $^nC_r = {}^nC_{n-r}$

Combinatorial Argument:

keep $r$ $\equiv$ rejecting $n-r$

$$\boxed{^nC_r = {}^nC_{n-r}}$$

Question: What is the relationship between $C(n, r)$ and $C(n, n-r)$?

**Corollary.** *Let $n$ and $r$ be nonnegative integers with $r \leq n$. Then $C(n, r) = C(n, n-r)$.*

*Proof.*

$$C(n, n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{(n-r)!r!} = C(n, r). \qquad \square$$

**PASCAL'S IDENTITY** Let $n$ and $k$ be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Algebraic Proof: (Boring, no intuition)

$$\frac{(n+1)!}{k!\,(n+1-k)!} \stackrel{?}{=} \frac{n!}{(k-1)!\,(n-k+1)!} + \frac{n!}{(n-k)!\,k!}$$

**PASCAL'S IDENTITY** Let $n$ and $k$ be positive integers with $n \geq k$. Then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Story: Class → n+1 students; select k student

Side 1: $^{n+1}C_k$

Side2: a particular student ⇒ $x$

take $x$ OR Don't take $x$

Side 2:

$$\underbrace{{}^{n}C_{k-1}}_{\text{taken } x} + \underbrace{{}^{n}C_{k}}_{\text{not taking } x}$$

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

$$\boxed{{}^{n}C_{k} = {}^{n-1}C_{k-1} + {}^{n-1}C_{k}}$$

## Proof 1 (algebraic)

Show that $\frac{n!}{k!(n-k)!} = \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \ldots$ □

## Proof 2 (combinatorial)

Let's count, using two different methods, the number of ways to elect $k$ candidates from a pool of $n$.

For the second method, assume that there is one "distinguished" candidate...

□

**Pascal's Identity**. For integers $n$ and $k$, $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

**Proof**.

The LHS counts the number of ways to select $k$ out of $n$ children to have their face painted. The RHS counts the same thing according to two cases: either a specific child of the $n$, say Gary, is in the group selected, or he is not selected. In the first case the remaining $k-1$ children in the group must be selected from the remaining $n-1$ children. The number of ways to do this is $\binom{n-1}{k-1}$. In the second case all $k$ children in the group must be selected from the remaining $n-1$ children. The number of ways to do this is $\binom{n-1}{k}$. By the Rule of Sum, the number of selections is $\binom{n-1}{k-1} + \binom{n-1}{k}$. Hence, the result. ∎

Variations:

$$ {}^{n}C_{k} = {}^{n-2}C_{k-2} + {}^{n-2}C_{k} + 2\,{}^{n-2}C_{k-1} $$

Story: Side1 = ${}^{n}C_{k}$

Side2: Take two particular student $x, y$.

Side 2 :

$$\underbrace{{}^{n-2}C_{k-2}}_{\text{Select } x, y} + \underbrace{{}^{n-2}C_{k}}_{\text{Neither } x, \text{ nor } y} + \underbrace{2\,{}^{n-2}C_{k-1}}_{\text{one of } x,y \text{ taken, another not taken}}$$

${}^{n}C_{r} \Rightarrow$ binomial Coefficient

$$\boxed{{}^{n}C_{r} = {}^{n-1}C_{r-1} + {}^{n-1}C_{r}}$$

Recursive Definition of Binomial Coefficient

# ON A RECURSIVE FORMULA FOR BINOMIAL COEFFICIENTS

ABSTRACT. We demonstrate that the binomial coefficients $B(n, k)$ satisfy the recursive formula

$$B(n, k) = B(n-1, k-1) + B(n-1, k).$$

Moreover, we prove that together with the initial conditions $B(n, 0) = B(n, n) = 1$ our formula can be used to calculate all binomial coefficients.

main{
}

int k=17, n=20

fun(n,k);

```
[redacted]
int fun(int n, int k)
{
    // Base Cases
    if (k > n)
        return 0;
    if (k == 0 || k == n)
        return 1;

    // Recur
    return fun(n - 1, k - 1)
            + fun(n - 1, k);
}
```

main {

int k=17, n=20

fun(n,k);

Ans: $^{20}C_{17}$

$= {}^{20}C_3 = \frac{20\times19\times18}{6}$

```
[redacted]
int [redacted] fun (int n, int k)
{
    // Base Cases
    if (k > n)
        return 0;
    if (k == 0 || k == n)
        return 1;

    // Recur
    return [redacted] fun (n - 1, k - 1)
           + [redacted] fun (n - 1, k);
}
```

Recursive Def of $^nC_k$

```
// Returns value of Binomial Coefficient C(n, k)
int binomialCoeff(int n, int k)
{
    // Base Cases
    if (k > n)
        return 0;
    if (k == 0 || k == n)
        return 1;

    // Recur
    return binomialCoeff(n - 1, k - 1)
           + binomialCoeff(n - 1, k);
}
```

a. Solve the recurrence relation for computing the binomial coefficients (seen below).

$$\begin{aligned} C(n,k) &= C(n-1,k-1)+C(n-1,k) \quad \text{for } n>k>0, \\ C(n,0) &= C(n,n)=1. \end{aligned}$$

**Algorithm** $Binomial(n,k)$

```
//Computes C(n, k) by the dynamic programming algorithm
//Input: A pair of nonnegative integers n ≥ k ≥ 0
//Output: The value of C(n, k)
for i ← 0 to n do
    for j ← 0 to min(i, k) do
        if  j = 0 or j = i
            C[i, j] ← 1
        else C[i, j] ← C[i − 1, j − 1] + C[i − 1, j]
return C[n, k]
```

**Hint:** Think about the "shape" of a table that records values for binomial coefficient, that has n+1 rows and k+1 columns.

## Advice (using combinatorial arguments to prove identities) :

You need to argue that LHS and RHS are just different expressions for the same number.
This is often done by counting something and obtaining the LHS, and then counting it in a different way and obtaining the RHS.
If one side involves a summation, take that as a suggestion to proceed by organizing the counting into cases and using the Rule of Sum.
A good way to break into cases is by considering what happens with some special object (Gary, in the above example). If products are involved somewhere, then the counting likely involves a sequence of steps and the Rule of Product. Also, there is no "right" thing to count, so you might as well pick something to make it entertaining.

Remember that we could define $\binom{n}{k}$ *combinatorially* (meaning as an expression that counts the number of objects of some kind) as the number of ways of selecting $k$ distinct objects from $n$ distinct objects without regard for order. Then, since $\binom{n}{k}$ counts selections, it must be that $\binom{n}{k} = 0$ if $k < 0$, or $k < n$, or $n < 0$. (Note: the previous sentence is a combinatorial argument.)

**Example (easy combinatorial argument).** We show that $\binom{n}{k} = \binom{n}{n-k}$: The LHS counts the number of ways to select $k$ people from a group of $n$ people to receive a candy. The RHS counts the same thing by counting the number of ways to select the $n-k$ people who will not receive a candy.

## Combinatorial Arguments

A *combinatorial argument*, or *combinatorial proof*, is an argument that involves counting. We have already seen this type of argument, for example in the section on Stirling numbers of the second kind.

Remember that we could define $\binom{n}{k}$ *combinatorially* (meaning as an expression that counts the number of objects of some kind) as the number of ways of selecting $k$ distinct objects from $n$ distinct objects without regard for order. Then, since $\binom{n}{k}$ counts selections, it must be that $\binom{n}{k} = 0$ if $k < 0$, or $k < n$, or $n < 0$. (Note: the previous sentence is a combinatorial argument.)

**Example (easy combinatorial argument).** We show that $\binom{n}{k} = \binom{n}{n-k}$: The LHS counts the number of ways to select $k$ people from a group of $n$ people to receive a candy. The RHS counts the same thing by counting the number of ways to select the $n-k$ people who will not receive a candy.

Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots \binom{n}{r}\binom{m}{0}.$$

Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots \binom{n}{r}\binom{m}{0}.$$

Story: n women, m men; select r People

Side 1 = ${}^{n+m}C_r$

Side 2: 0 w, r m OR 1 w, r-1 m OR - - - OR r w, 0 m

$$= \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \binom{n}{2}\binom{m}{r-2} + \cdots + \binom{n}{r}\binom{m}{0}$$

$$\boxed{\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}}$$

Vandermonde's Identity

**VANDERMONDE'S IDENTITY** Let $m$, $n$, and $r$ be nonnegative integers with $r$ not exceeding either $m$ or $n$. Then

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

**Vandermonde's Identity.** $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$.

**Proof.**

The LHS counts the number of ways to choose a committee of $r$ people from a group of $m$ men and $n$ women. The RHS counts the same thing according to cases depending on the number of men on the committee, which can range from 0 to $r$. If there are $t$ men, then there must be $r-t$ women. Since in such a case there are $\binom{m}{t}$ ways to select the men and $\binom{n}{r-t}$ ways to select the women, the number of such committees is $\binom{m}{t}\binom{n}{r-t}$. The result now follows from the Rule of Sum. ∎

Q: $\binom{30}{12} \neq \binom{16}{4}\binom{14}{8}$

$$\binom{30}{12} = \sum_{k=0}^{12} \binom{16}{k}\binom{14}{12-k}$$

Q: $\binom{30}{12} = \sum_{k=0}^{12} \binom{16}{k}\binom{14}{12-k}$ ✓

$= \sum_{k=0}^{12} \binom{20}{12-k}\binom{10}{k}$ ✗

$= \sum_{k=0}^{12} \binom{15}{12-k}\binom{15}{k}$ ✓

Q:

$$\binom{52}{10} = \sum_{k=0}^{10} \binom{40}{k}\binom{12}{10-k}$$

$$= \sum_{k=0}^{10} \binom{30}{10-k}\binom{22}{k}$$

$$= \sum_{k=0}^{10} \binom{20}{k}\binom{32}{10-k}$$

**Q:** which is True?

① $$\binom{12}{4} = \sum_{k=0}^{4} \binom{6}{6-k}\binom{6}{4-k}$$

② $$\binom{12}{4} = \sum_{k=0}^{4} \binom{6}{k}\binom{6}{4-k}$$

③ $$\binom{12}{4} = \sum_{k=0}^{4} \binom{6}{k}\binom{6}{k+2}$$

$$\boxed{{}^{n}C_{r} = {}^{n}C_{n-r}}$$

$$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

$$= \sum_{k=0}^{r} \binom{n}{n-k}\binom{m}{r-k}$$

$$= \sum_{k=0}^{r} \binom{n}{k}\binom{m}{m-r+k}$$

$$\binom{12}{4} = \sum_{k=0}^{4} \binom{6}{k}\binom{6}{4-k}$$

$$= \sum_{k=0}^{4} \binom{6}{k}\binom{6}{2+k}$$

$$= \sum_{k=0}^{4} \binom{6}{6-k}\binom{6}{2+k}$$

Note:

$$\binom{2n}{n} = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\text{men}} \underbrace{\binom{n}{n-k}}_{\text{women}}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \checkmark$$

**Exercise 3:** Explain why the identity

$$\binom{2n}{n} = \sum_{j=0}^{n} \binom{n}{j}^2$$

holds.

*Hint:* Think of a group of consisting of $n$ boys and $n$ girls.

A good time to try a committee selection argument is when proving an identity involving binonial coeffocients and the "top" of one side of the equality involves a sum like $m+n$. This suggests one might try counting selections of people from a group of $m$ men and $n$ women. (Note: $2n = n + n$.)

**Corollary**. $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

**Exercise.** Use a committee selection argument to show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

**Exercise.** Use a committee selection argument to show that $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Story: n boys ; n girls ; select 2

Side1: ${}^{2n}C_2$

Side2:

$\underbrace{{}^{n}C_2}_{\text{both boys}} + \underbrace{{}^{n}C_2}_{\text{both girls}} + \underbrace{{}^{n}C_1 * {}^{n}C_1}_{\text{1 boy 1 girl}}$

n

$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$