The Fundamental Course

Chapter:

Proof Techniques

(Part 2 – Proof by Contrapositive & Contradiction)

- By Deepak Poonia(IISc Bangalore)

Instructor:

Deepak Poonia
IISc Bangalore
GATE CSE AIR 53; AIR 67; AIR 206; AIR 256;

Discrete Mathematics Complete Course:

https://www.goclasses.in/courses/Discrete-Mathematics-Course

Join GO Classes **GATE CSE Complete Course** now:

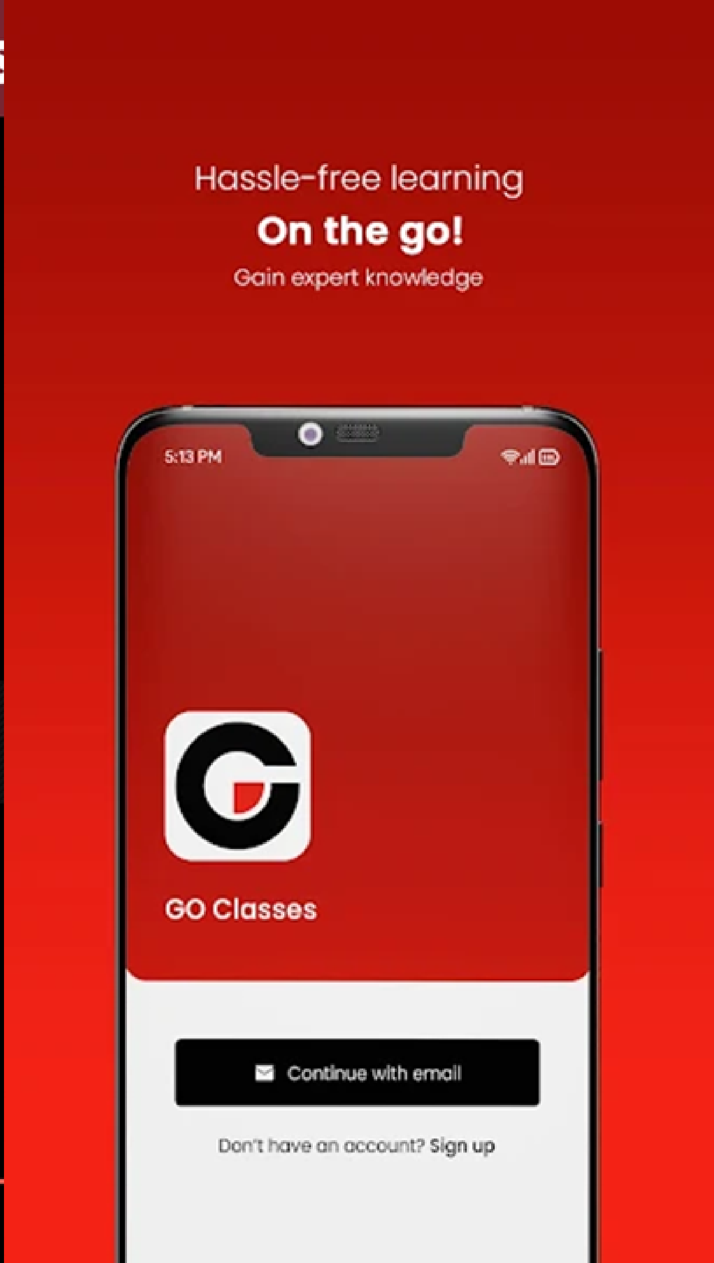
https://www.goclasses.in/s/pages/gatecompletecourse

1. Quality Learning: No Rote-Learning. Understand Everything, from basics, In-depth, with variations.
2. Daily Homeworks, Quality Practice Sets, Weekly Quizzes.
3. Summary Lectures for Quick Revision.
4. Detailed Video Solutions of Previous ALL GATE Questions.
5. Doubt Resolution, Revision, Practice, a lot more.

Download the GO Classes Android App:

https://play.google.com/store/apps/details?id=com.goclasses.courses

Search "GO Classes" on Play Store.



NOTE :

Complete Discrete Mathematics & Complete Engineering Mathematics Courses, by GO Classes, are FREE for ALL learners.

Visit here to watch : https://www.goclasses.in/s/store/

SignUp/Login on Goclasses website for free and start learning.

We are on Telegram. **Contact us** for any help.

Link in the Description!!

Join GO Classes **Doubt Discussion** Telegram Group :

Username:

@GATECSE_Goclasses

We are on Telegram. **Contact us** for any help.

Join GO Classes Telegram Channel, Username: @GOCLASSES_CSE

Join GO Classes **Doubt Discussion** Telegram Group :

Username: @GATECSE_Goclasses

(Any doubt related to Goclasses Courses can also be asked here.)

Join GATEOverflow Doubt Discussion Telegram Group :

Username: @GateOverflow_CSE

Recap:

We have seen “Direct Proof” proof technique in the last lecture.

This Lecture:

We will study:

- Proof by Contrapositive
- Proof by Contradiction

Methods of Proof:

2 Proof by Contraposition

To prove: $\boxed{\text{If } P \text{ then } Q}$

Direct Proof:

$\boxed{P \text{ is True}}$

facts that you know

$\boxed{Q \text{ is True}}$ (Derive)

Indirect Proof (Proof by Contraposition)

Q is False

facts we know

P is false

Q :

If you are India's PM then you are an Indian.

Which of the following is same as above statement?

- If you are an Indian then you are India's PM.
- If you are Not an Indian then you are Not India's PM.

Q :

P Q

If you are India's PM then you are an Indian.

Which of the following is same as above statement?

- If you are an Indian then you are India's PM. ✗

Q P

- ✓ If you are Not an Indian then you are Not India's PM.

$\boxed{\text{If } p \text{ then } q}$: Given

(a) If q then p ✗

(b) If $\neg q$ then $\neg p$ ✓

negation

Q :

If P happens then Q happens.

Which of the following is same as above statement?

If Q happens then P happens.

If Q doesn't happen then P doesn't happen.

Q :

If P happens then Q happens.

Which of the following is same as above statement?

If Q happens then P happens.

If Q doesn't happen then P doesn't happen.

Given: If P happens then Q happens.

Q: If Q Does not happen,

Can P happen?

Given: If P happens then Q happens.

Q: If Q Does not happen,

Can P happen? NO

Same as

If Q Does not happen then P Does not happen.

Q : If P happens then Q happens. : True

Which of the following is same as above statement?

If Q happens then P happens. ? No

If Q doesn't happen then P doesn't happen. : True

P: Top GATE exam ; Q : Appear in GATE exam

Q : If P happens then Q happens. True

Which of the following is same as above statement?

If Q happens then P happens. False

If Q doesn't happen then P doesn't happen. True

P: I'm in Rajasthan ; Q : I'm in India.

If P then Q $\equiv$ If $\neg$Q then $\neg$P.

Same

To prove: If p then q

If p then q

p is True

facts

q is True

If ¬q then ¬p

q is false

facts

p is false

To prove: $\boxed{\text{If } p \text{ then } q}$

Direct proof

p is True
facts
q is True

Indirect proof

Contraposition

q is false
facts
p is false

Q :

If P happens then Q happens.

Which of the following is same as above statement?

If Q happens then P happens.

If Q doesn't happen then P doesn't happen.

Elementary methods of proving $P \Rightarrow Q$.

Basic methods to prove "If P then Q" type of statements :

Direct Proof: Assume P, follow logical deductions, conclude Q.
Contrapositive: Assume $\neg Q$, follow deductions, conclude $\neg P$.

Contraposition

A type of indirect proof that makes use of the fact that $p \to q$ is equivalent to its contrapositive $\neg q \to \neg p$. So we assume $\neg q$ is true, then work to prove $\neg p$ is true.

Outline for Contrapositive Proof

Proposition If P, then Q.

Proof. Suppose $\sim Q$.

$\vdots$

Therefore $\sim P$. ■

Statement: If n^2 is even, then n is even

n : Integer

Statement: If $\underbrace{n^2 \text{ is even}}_{P}$, then $\underbrace{n \text{ is even}}_{Q}$

n : Integer

Direct Proof:

P is True

facts

Q is True

$\boxed{n^2 \text{ is even}}$ P

$n^2 = 2K$ → int

$\boxed{n = \sqrt{2K}}$ Stuck!!

$\boxed{n \text{ is even}}$ Q

Statement: If $\underbrace{n^2 \text{ is even}}_{P}$, then $\underbrace{n \text{ is even}}_{Q}$

n : Integer ✓

Proof by Contrapositive :

Q is false

facts

P is false

$\boxed{Q \text{ is false}}$ $\neg Q$

n is odd

$n = 2k+1$ → int

$n^2 = (2k+1)^2 = \boxed{4k^2+4k}+1 = 2(\text{int})+1$ → Odd

$\boxed{P \text{ is false} \equiv n^2 \text{ is odd}}$ $\neg P$

Statement: If n^2 is even, then n is even

Proof:

$$n^2 = 2k$$

$$n = \sqrt{(2k)}$$

??

Statement: If n^2 is even, then n is even

Contrapositive: If n is odd, then n^2 is odd.

Proof (the contrapositive):

Since n is an odd number, $n = 2k+1$ for some integer k.

So $n^2 = (2k+1)^2$

$$= (2k)^2 + 2(2k) + 1 \quad = 2(2k^2 + 2k) + 1$$

So n^2 is an odd number.

EXAMPLE 8 Prove that if n is an integer and n^2 is odd, then n is odd.

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

EXAMPLE 8 Prove that if n is an integer and n^2 is odd, then n is odd.

P Q

Direct Proof :

$\boxed{n^2 \text{ is odd}}$ P

$n^2 = 2k+1$

$\boxed{n = \sqrt{2k+1}}$ Stuck !!

$\boxed{n \text{ is odd}}$ Q

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

EXAMPLE 8 Prove that if n is an integer and n^2 is odd, then n is odd.

p q

Proof by Contraposition:

$\neg q$

facts

$\neg p$

$\boxed{n \text{ is even}}$ $\neg q$

$n = 2k$ int

$n^2 = 4k^2 = 2(2k^2) = 2(\text{int})$

$\boxed{n^2 \text{ is even}}$ $\neg p$

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

Prove " If n is an integer and $3n+2$ is odd, then n is odd.

EXAMPLE 3 Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Source: Discrete Mathematics and Its Applications

Seventh Edition Kenneth H. Rosen

n : int

To prove: If $\underbrace{3n+2 = \text{odd}}_{p}$ then $\underbrace{n = \text{odd}}_{q}$

Indirect proof: $\boxed{n = \text{even}}$ $\neg q$

$n = 2k \rightarrow$ int

$3n+2 = 3 \cdot 2k + 2 = 2(3k+1)$ int

$\boxed{3n+2 = \text{even}}$ $\neg p$

$$\left\{\begin{array}{l} \text{even} * \text{any int} = \text{even} \\ \\ \text{odd} * \text{odd} = \text{odd} \end{array}\right\} \underline{\text{facts}}$$

we have proven in HW1.

Odd + even = odd

even + even = even

$\boxed{\text{odd} + \text{odd} = \text{even}}$

Quick HW ✓

Proof:
$$(2a+1) + (2b+1) = 2\overbrace{(a+b+1)}^{\text{int}}$$

$= \text{even}$

n : int

To prove: If $3n+2 = \text{odd}$ then $n = \text{odd}$

Direct proof:

$3n+2 = \text{odd}$

$3n+2 = 2k+1$ → int

$3n+1 = 2k$ even

odd + odd

odd

$\boxed{3n = \text{odd}}$ → $3n = \text{odd}$

odd

Prove " If n is an integer and $3n+2$ is odd, then n is odd.

Assume n is even. So $n = 2k$, $k \in \mathbb{Z}$ by defn. of an even integer.

$$3n+2 = 3(2k)+2$$
$$= 6k+2$$
$$= 2(3k+1)$$
$$= \underline{2r} \text{ where } r = 3k+1, r \in \mathbb{Z}$$

Prove " If n is an integer and 3n+2 is even, then n is even.

Q HW

Prove " If n is an integer and $3n+2$ is even, then n is even.

Assume n is odd. Then $n = \boxed{2k+1}$, $k \in \mathbb{Z}$.

$$
\begin{aligned}
3n+2 &= 3(2k+1)+2 \\
&= 6k + \underline{3+2} \\
&= \underline{2}(3k+2)+1 \\
&= 2\underline{r+1}, \quad r = 3k+2, \ r \in \mathbb{Z}
\end{aligned}
$$

Q :

What is the Negation of "P or Q"

$\neg\neg$

$(\neg P$ and $\neg Q)$

not

not

Q :

What is the Negation of "P and Q"

$\neg\neg$

$\wedge$

$\boxed{\neg P \text{ OR } \neg Q}$

- Show that

  if $n = a\,b$, where a and b are positive,

  then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

EXAMPLE 4 Prove that if $n = ab$, where a and b are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

- Show that

  if n = a b, where a and b are positive,

  then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

EXAMPLE 4 Prove that if $n = ab$, where a and b are positive integers, then $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Direct Proof: $\boxed{n = ab}$

Stuck!!

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

- Show that

  if n = a b, where a and b are positive,

  then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

EXAMPLE 4 Prove that if $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

Try Proof by Contraposition

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen a, b : Positive integers → {1, 2, 3, 4, ...}

Note 0 – Neither Positive, Nor Negative

Negation of "X OR Y" ≡ NOT X and NOT Y

$\underbrace{5 \geq 4}_{T} \xrightarrow{\text{Negation}} 5 < 4$

a, b : Positive integers

To prove: If $\underbrace{n = ab}_{p}$ then $\underbrace{\overset{x}{a \le \sqrt{n}} \text{ OR } \overset{y}{b \le \sqrt{n}}}_{q}$

Proof by CP.

$\neg q$

$\neg p$

$\neg x$ and $\neg y$: $a > \sqrt{n}$ and $b > \sqrt{n}$

$\neg q$ $\neg x$ $\neg y$

$a > \sqrt{n}$
$b > \sqrt{n}$ $\rightarrow$ $\boxed{ab \neq n}$ $\neg p$

$x > y$ → $x > 1$

$a > 4$
$b > 5$
→ $ab \neq 20$

info → info

$a > \sqrt{n}$
$b > \sqrt{n}$
→ $ab \neq n$

info

info

- Show that

  if $n = ab$, where a and b are positive,

  then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$.

- Proof :

  Assume that both a and b are larger than $\sqrt{n}$. Thus, $ab > n$ so that $n \neq ab$. Since the negation of conclusion implies the negation of hypothesis, the original conditional statement is true

Rational Numbers:

$$\frac{\text{int}}{\text{int} \neq 0} \rightarrow \text{Rational Number}$$

$2 \rightarrow \frac{2}{1}$

$4.5 \rightarrow \frac{45}{10}$

$0 \rightarrow \frac{0}{1}$

$\frac{0}{3}$

Rational Number

R is rational ⇔ there are integers a and b such that

numerator

$$r = \frac{a}{b}$$ and b ≠ 0.

denominator

Is 0.281 a rational number? Yes, 281/1000

Is 0 a rational number? Yes, 0/1

If m and n are non-zero integers, is (m+n)/mn a rational number? Yes

Is the sum of two rational numbers a rational number? Yes, a/b+c/d=(ad+bc)/bd

Rational Numbers

A real number is called a ***rational number*** if it can be expressed in the form p/q, where p and q are integers and $q \neq 0$. The set of all rational numbers is denoted by $\mathbb{Q}$. A real number is said to be ***irrational*** if it is not rational.

Theorem 88. *0 and 1 are rational numbers.*

Proof. Just note that $0 = 0/1$ and $1 = 1/1$. $\square$

$x = 0.12121212\ldots$

Is $\underline{x \text{ Rational}}$?

$$x = 0.12121212\ldots$$

$$100x = 12.121212\ldots$$

$$100x - x = 12$$

$$\boxed{x = \frac{12}{99}}$$

Rational → logical (Don't behave Randomly) → Pattern

$x = 0.12121212\ldots$

→ ir-rational ✓

$y = 0.129872334521100122344556709 23\ldots$

→ Psycho

$x : 0.\underline{54}\,\underline{54}\,\underline{54}\ldots$

$100x = 54.5454\ldots$

$100x - x = 54$

$x = \frac{54}{99}$ ✓

Recurring two digits

$x = \frac{55}{99}$

Trick

$$x = 0.\underline{123}\,\underline{123}\,\underline{123}\ldots$$

$$1000x = 123.123123\ldots$$

$$x = \frac{123}{999}$$

$$x = \frac{123}{999}$$

$x: 0.1234\,98\,98\,98\,98 \ldots$

Rational

$10000x = 1234.\,98\,98\,98 \ldots$

$= 1234 + .989898 \ldots$

$10000x = \left(1234 + \frac{98}{99}\right) \rightarrow x = ?$

Irrational $:\equiv$ Not Rational

Some famous Irrationals : $\sqrt{2}, \sqrt{3}, \sqrt{\text{prime}}, \pi, e, \ldots$

$\pi = \frac{22}{7}$ → Rational

Irrational

$\pi = \frac{22}{7}$

WRONG

$\pi \approx \frac{22}{7}$

Approx

Q : True/False??

If m,n are non-zero integers then (m+n)/mn is a rational.

m,n : non-zero int

$mn \neq 0$ ✓

$m+n \neq 0$ ✗

$\frac{m+n}{mn}$ int, int, $\neq 0$

Q : True/False??

Sum of two rational numbers is a rational number.

$\frac{p}{q}, \frac{a}{b}$

$q \neq 0$ $b \neq 0$

$qb \neq 0$

$\frac{p}{q} + \frac{a}{b} = \frac{pb + qa}{qb}$

int

int

$\neq 0$

Q : True/False??

If r is rational then sqroot(r) is rational.

Q : True/False??

If r is rational then sqroot(r) is rational.

$r = 2$ $\qquad \sqrt{2}$: Irrational

$r = 3$ $\qquad \sqrt{3}$: "

Counter examples

Claim: If r is [redacted]rational, then $\sqrt{r}$ is [redacted]rational.

false

Claim: If r is irrational, then $\sqrt{r}$ is irrational.

True or false?

prove it by Contraposition.

Claim: If r is irrational, then √r is irrational.

p q

Proof by Contraposition:

$\neg q$: $\boxed{\sqrt{r} \text{ Rational}}$

$\neg q$

$\neg p$

$\sqrt{r} = \frac{a}{b}$ int

$b \neq 0$

$r = \frac{a^2}{b^2}$

a^2 int

int b^2 $\neq 0$

$r = \frac{\text{int}}{\text{int}}$ $\neq 0$

Rational

Claim: If r is irrational, then $\sqrt{r}$ is irrational.

Proof:

We shall prove the contrapositive –

"if $\sqrt{r}$ is rational, then r is rational."

Since $\sqrt{r}$ is rational, $\sqrt{r} = a/b$ for some integers a,b.

So $r = a^2/b^2$. Since a,b are integers, a^2, b^2 are integers.

Therefore, r is rational. □ Q.E.D.

(Q.E.D.) "which was to be demonstrated", or "quite easily done". ☺

Q : Prove using Direct Proof as well as using Proof by Contraposition :

Suppose $x \in Z$. If $7x+9$ is even, then x is odd.

HW

Proposition Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then x is odd.

Proof. (Direct) Suppose $7x+9$ is even.
Thus $7x+9=2a$ for some integer a.
Subtracting $6x+9$ from both sides, we get $x=2a-6x-9$.
Thus $x=2a-6x-9=2a-6x-10+1=2(a-3x-5)+1$.
Consequently $x=2b+1$, where $b=a-3x-5 \in \mathbb{Z}$.
Therefore x is odd.

Here is a contrapositive proof of the same statement:

Proposition Suppose $x \in \mathbb{Z}$. If $7x+9$ is even, then x is odd.

Proof. (Contrapositive) Suppose x is not odd.
Thus x is even, so $x = 2a$ for some integer a.
Then $7x+9 = 7(2a)+9 = 14a+8+1 = 2(7a+4)+1$.
Therefore $7x+9 = 2b+1$, where b is the integer $7a+4$.
Consequently $7x+9$ is odd.
Therefore $7x+9$ is not even.

Basic methods to prove "If P then Q" type of statements :

Direct Proof

Assume P. Explain, explain, $\ldots$, explain. Therefore Q.

Proof by Contrapositive

Assume $\neg Q$. Explain, explain, $\ldots$ explain. Therefore $\neg P$.

Methods of Proof: Proof by Contradiction

Proof by Contradiction:

As Sherlock Holmes said, "When you have eliminated the impossible, whatever remains, however improbable, must be the truth".

If you can prove that “P is false” is impossible, then you conclude that P must be true.

11

As Sherlock Holmes said, "When you have eliminated the impossible, whatever remains, however improbable, must be the truth".

If you can prove that not-P is impossible, then you conclude that P must be true.

Share Cite Edit Follow

answered Dec 4, 2015 at 5:05

Robert Israel
429k 26 315 625

To prove P, you prove that not P would lead to ridiculous result, and so P must be true.

To prove: p

Proof by Contradiction:

"p is false" is Impossible.

$\equiv$ "NOT p" is NOT possible.

Proof by Contradiction

In this method of proof, we assume a proposition is <u>not true</u>, then through that premise and logic find a contradiction that shows our original premise must have been incorrect, and therefore, the proposition was true.

- For <u>one</u> proposition, p, assume $\neg p$ is true, then find a contradiction that shows $\neg p$ is false, so p is true.
- For an implication, $p \to q$, assume p and $\neg q$ are true, then find a contradiction that shows either $p \to q$ or $\neg q \to \neg p$.

We defined a rational number to be a real number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and b is not zero. If a number can be written as such a fraction, it can be written as a fraction in lowest terms, i.e. where a and b have no common factors. If a and b have common factors, it's easy to remove them.

A fraction p/q is said to be in ***lowest terms*** if the largest integer that evenly divides both p and q is 1.

Theorem 90. *Every rational number can be expressed as a fraction in lowest terms.*

Rational Numbers:

int ← 4

$$\frac{4}{10} = \frac{2}{5}$$ → lowest form

int

$$\frac{18}{12} = \frac{9}{6} = \frac{3}{2}$$

$\frac{9}{6}$ → Not lowest form

$$\frac{20}{8} = \frac{5}{2}$$

Numerator (N)

Denominator (D)

lowest form

Only Common factor of N, D is 1.

Rational

$\frac{int}{int}$

Lowest form of a Rational No:

$$\frac{N}{D}$$

$N \rightarrow$ int, $D \rightarrow$ int

$D \neq 0$

and $GCD(N, D) = 1$

(N, D have No Common factor other than 1)

$\frac{0}{3}$ lowest form? No

$\mathrm{GCD}(0,3) = 3$

$3 \mid 3$; $3 \mid 0$

$0 = \frac{0}{1}$ lowest form

Prove by contradiction "$\sqrt{2}$ is irrational".

Prove by contradiction "$\sqrt{2}$ is irrational".

To prove: $\sqrt{2}$ is Irrational.

Proof by Contradiction $\Rightarrow$

Assume $\sqrt{2}$ is Rational

facts → No mistake

bring some Contradiction

To prove: P : $\sqrt{2}$ is Irrational.

Proof by Contradiction $\Rightarrow$ Assume $\sqrt{2}$ is Rational

$\sqrt{2} = \frac{a}{b}$ $b \neq 0$

$\frac{a}{b} \rightarrow$ a, b have No common factor other 1.

Lowest form ✓

$\sqrt{2} = \frac{a}{b} \Rightarrow a^2 = 2b^2$ even

$a^2 = \text{even} \longrightarrow \boxed{a = \text{even}}$

$a = 2k \longrightarrow$ int

$a^2 = 2b^2$

$4k^2 = 2b^2 \Rightarrow b^2 = 2k^2$ even

b^2 = even $\longrightarrow$ b = even

$\left.\begin{cases} a = \text{even} \\ b = \text{even} \end{cases}\right\}$ $\longrightarrow$ Contradiction $\longrightarrow$ GCD(a,b) $\geq$ 2

$\frac{a}{b}$ NOT in lowest form

So Initial Assumption WRONG $\longleftarrow$ Contradiction

P

$\boxed{\sqrt{2} \text{ is rational.}}$

You

Assume $\boxed{\sqrt{2} = \text{Rational.}}$

$\boxed{\text{You} \rightarrow P}$

$\sqrt{2} = \frac{a}{b}$ → lowest form

a = even
b = even

→ $\boxed{\text{P is wrong}}$

Proof by Contradiction

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

- Suppose $\sqrt{2}$ was rational.
- Choose m, n integers without common prime factors (always possible)

  such that $\sqrt{2} = \frac{m}{n}$
- Show that m and n are both even, thus having a common factor 2, a **contradiction**!

Theorem: $\sqrt{2}$ is irrational.

Proof (by contradiction):

Want to prove both m and n are even.

$$\sqrt{2} = \frac{m}{n}$$

$$\sqrt{2}n = m$$

$$2n^2 = m^2$$

so m is even.

so can assume $m = 2l$

$$m^2 = 4l^2$$

$$2n^2 = 4l^2$$

$$n^2 = 2l^2$$

so n is even.

Recall that m is even if and only if m^2 is even.

Prove by contradiction "$\sqrt{2}$ is irrational".

Assume $\neg P$.

P

$\sqrt{2}$ is rational. Then there exists 2 integers a and b such that $\boxed{\sqrt{2} = \frac{a}{b}}$, $b \neq 0$ and a and b have no common factors.

$$\sqrt{2} = \frac{a}{b}$$

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

Therefore a must be even. so $a = 2c$ for some $c \in \mathbb{Z}$.

$$2b^2 = (2c)^2$$

$$2b^2 = 4c^2$$

$$b^2 = 2c^2$$

Therefore b must be even.

Since a and b are both even, they have a common factor. Therefore $\sqrt{2}$ is irrational by contradiction. $\square$

∞ is Largest Integer?

∞ is Largest Integer? No

∞ → NOT Integer

∞ → Not a Complex No.

∞ → NOT a Number

∞ → NOT Real Number

∞ → Concept

Prove that :

There is no largest integer.

Prove that :

There is no largest integer.

proof by Contradiction

Assume "a" is Largest int.

a+1 ?? → int

a+1 > a

int

Contradiction

Prove that :

There is no largest even integer.

Assume "a" is Largest even int

$a+2$

bring a Contradiction

$a+2 > a$

int

Prove that :

There is no largest even integer.

Proof :

Prove by Contradiction. That is, suppose that there were a largest even integer. Let's call it k.

Since k is even, it has the form 2n, where n is an integer.

Consider k + 2.

k + 2 = (2n) + 2 = 2(n + 1). So k + 2 is even. But k + 2 is larger than k. This contradicts our assumption that k was the largest even integer. So our original claim must have been true.

Proposition If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

HW

integer

Hint: Proof by Contradiction

EXAMPLE 11 Give a proof by contradiction of the theorem "If $3n + 2$ is odd, then n is odd."

HW

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

EXAMPLE 9 Show that at least four of any 22 days must fall on the same day of the week.

Source: Discrete Mathematics and Its Applications Seventh Edition Kenneth H. Rosen

EXAMPLE 9 Show that at least four of any 22 days must fall on the same day of the week.

Take any 22 Days

M
T
W
T
F
S
Sun

→ at least one of them Repeats at least 4 times

→ Assume
→ each of them Repeats at most 3 times

max 7x3 = 21 Days

Source: Discrete Mathematics and Its Applications Seventh Edi

EXAMPLE 9 Show that at least four of any 22 days must fall on the same day of the week.

Extra Examples

Solution: Let p be the proposition "At least four of 22 chosen days fall on the same day of the week." Suppose that $\neg p$ is true. This means that at most three of the 22 days fall on the same day of the week. Because there are seven days of the week, this implies that at most 21 days could have been chosen, as for each of the days of the week, at most three of the chosen days could fall on that day. This contradicts the premise that we have 22 days under consideration.

Proposition If $a, b \in \mathbb{Z}$, then $a^2 - 4b \neq 2$.

Proof. Suppose this proposition is **false**.
This conditional statement being false means there exist numbers a and b for which $a, b \in \mathbb{Z}$ is true but $a^2 - 4b \neq 2$ is false.
Thus there exist integers $a, b \in \mathbb{Z}$ for which $\boxed{a^2 - 4b = 2.}$
From this equation we get $a^2 = 4b + 2 = 2(2b+1)$, so a^2 is even.
Since a^2 is even, it follows that a is even, so $a = 2c$ for some integer c.
Now plug $a = 2c$ back into the boxed equation $a^2 - 4b = 2$.
We get $(2c)^2 - 4b = 2$, so $4c^2 - 4b = 2$. Dividing by 2, we get $2c^2 - 2b = 1$.
Therefore $1 = 2(c^2 - b)$, and since $c^2 - b \in \mathbb{Z}$, it follows that 1 is even.
Since we know 1 is **not** even, something went wrong.
But all the logic after the first line of the proof is correct, so it must be that the first line was incorrect. In other words, we were wrong to assume the proposition was false. Thus the proposition is true. ∎

So far:

P: - - - - - - - -

we have Tries to prove p.

Now :

P : - - - - - - - -

we will Try to DisProve P

by Counterexample

Next Proof Technique:

Disproof

by Counterexample

Disproof By Counterexample

- Example – Disprove the following conjectures:
  - All animals living in the ocean are fish.
    - Pf. For example, whales live in the ocean and are not fish.
  - All input to a computer is provided by the keyboard.
    - Pf. For example, mice provide input to a computer and are not keyboards.

Crab
Turtle

- For a positive integer n, n factorial is defined as n(n-1)(n-2) … 1 and is denoted by n!. Prove or disprove the conjecture "For every positive integer n, n! <= n^2."

$n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$

$6! = 6(5)(4)(3)(2)(1)$

- For a positive integer n, n factorial is defined as n(n-1)(n-2) … 1 and is denoted by n!. Prove or disprove the conjecture "For every positive integer n, n! <= n^2."

$n! = n(n-1) \cdots 3 \cdot 2 \cdot 1$

$6! = 6(5)(4)(3)(2)(1)$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 \nleq 4^2$

Disprove using Counter example

$n=4$

Q: True/False?

For integers a; b; c, If a divides bc, then either a divides b or a divides c.

Q: True/False?

For integers a; b; c, If a divides bc, then either a divides b or a divides c.

$6 \mid 2 \cdot 3$ but $6 \nmid 2$

$6 \nmid 3$

$a \mid bc \longrightarrow a \mid b$ OR $a \mid c$?

False

$12 \mid 6 * 4$ But $12 \nmid 6$, $12 \nmid 4$

Disproof by Counterexample

NOTE:

Euclid's Lemma:

If a prime number P divides bc, then either P divides b or P divides c.

(Proof is complicated, So Skip)

$p \mid xy \longrightarrow p \mid x$ OR $p \mid y$

prime

True

An Interesting Fact:

One of the world's smallest research paper is a disproof by counterexample.

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n > 2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

Proof Techniques:

Should I do More Practice of Proof Techniques??

Proof Techniques:

Should I do More Practice of Proof Techniques??

Ans: NO. What we have seen in Goclasses lectures, HWs, Weekly Quizzes & then throughout the GATE 2024 course we will keep proving things...

Proof Techniques:

Throughout the GATE 2024 course we will keep proving things... All concepts we will study with Proofs.. By the end of GATE24 course, You will NOT Fear Proofs... You will Love them.

Next Proof Technique:

"Proof by Induction" lecture has been uploaded in Recorded form.

More practice questions will be recorded & uploaded by tomorrow.