Set Theory

Practice Set - 3

Equivalence Relations

30 Standard Questions

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GATE CSE AIR 53; AIR 67;
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Q 1:

15. Determine which of the following congruence relations are true and which are false.

a. $17 \equiv 2 \pmod 5$

b. $4 \equiv -5 \pmod 7$

c. $-2 \equiv -8 \pmod 3$

d. $-6 \equiv 22 \pmod 2$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Congruence.

Definition. Let a and b be integers and m be a natural number. Then a is *congruent to* b *modulo* m:

$$a \equiv b \pmod{m}$$

if $m|(a-b)$.

The number m is called the *modulus* of the congruence. Congruence modulo m divides the set $\mathbb{Z}$ of all integers into m subsets called *residue classes*. For example, if $m = 2$, then the two residue classes are the *even integers* and the *odd integers*. Integers a and b are in the same class if and only if $a \equiv b \pmod{m}$. The following basic properties follow from the definition of congruence.

Property 1. Congruence if *reflexive*, i.e., $a \equiv a \pmod{m}$ for every integer a and natural number m.

Property 2. Congruence is *symmetric*, i.e., if $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$.

Property 3. Congruence is *transitive*, i.e., if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$.

Q 2:

7. Equivalence Classes. Let $A = \{0, 1, 2, 3\}$ and let

$$r = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,1), (3,2), (2,3), (3,1), (1,3)\}$$

(a) Verify that r is an equivalence relation on A.

(b) Let $a \in A$ and define $c(a) = \{b \in A \mid arb\}$. $c(a)$ is called the **equivalence class of a under r**. Find $c(a)$ for each element $a \in A$.

(c) Show that $\{c(a) \mid a \in A\}$ forms a partition of A for this set A.

Q 3,4:

In each of 3–14, the relation R is an equivalence relation on the set A. Find the distinct equivalence classes of R.

3. $A = \{0, 1, 2, 3, 4\}$
$R = \{(0, 0), (0, 4), (1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 0), (4, 4)\}$

4. $A = \{a, b, c, d\}$
$R = \{(a, a), (b, b), (b, d), (c, c), (d, b), (d, d)\}$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 5:

5. $A = \{1, 2, 3, 4, \ldots, 20\}$. R is defined on A as follows:

$$\text{For all } x, y \in A, \quad x\,R\,y \quad \Leftrightarrow \quad 4 \mid (x - y).$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 6:

6. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows:

$$\text{For all } x, y \in A, \quad x \; R \; y \quad \Leftrightarrow \quad 3 \mid (x - y).$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 7:

7. $A = \{(1, 3), (2, 4), (-4, -8), (3, 9), (1, 5), (3, 6)\}$. R is defined on A as follows: For all $(a, b), (c, d) \in A$,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad ad = bc.$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 8:

8. $X = \{a, b, c\}$ and $A = \mathscr{P}(X)$. R is defined on A as follows: For all sets u and v in $\mathscr{P}(X)$,

$$u\ R\ v \quad \Leftrightarrow \quad N(u) = N(v).$$

(That is, the number of elements in u equals the number of elements in v.)

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Q 9:

9. $X = \{-1, 0, 1\}$ and $A = \mathscr{P}(X)$. R is defined on $\mathscr{P}(X)$ as follows: For all sets s and t in $\mathscr{P}(X)$,

$s \; R \; t \quad \Leftrightarrow \quad$ the sum of the elements in s equals the sum of the elements in t.

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 10:

10. $A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. R is defined on A as follows: For all $m, n \in \mathbf{Z}$,

$$m \; R \; n \quad \Leftrightarrow \quad 3 \mid (m^2 - n^2).$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 11:

11. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For all $(m, n) \in A$,

$$m \; R \; n \Leftrightarrow 4 \mid (m^2 - n^2).$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 12:

12. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$. R is defined on A as follows: For all $(m, n) \in A$,

$$m \; R \; n \Leftrightarrow 5 \mid (m^2 - n^2).$$

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Q 13:

13. A is the set of all strings of length 4 in a's and b's. R is defined on A as follows: For all strings s and t in A,

$$s\ R\ t \quad \Leftrightarrow \quad s \text{ has the same first two characters as } t.$$

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 14:

14. A is the set of all strings of length 2 in 0's, 1's, and 2's. R is defined on A as follows: For all strings s and t in A,

$s\ R\ t \quad \Leftrightarrow \quad$ the sum of the characters in s equals the sum of the characters in t.

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44. Let $A = \mathbf{Z}^+ \times \mathbf{Z}^+$. Define a relation R on A as follows: For all (a, b) and (c, d) in A,

$$(a, b)\ R\ (c, d) \quad \Leftrightarrow \quad a + d = c + b.$$

a. Prove that R is reflexive.

b. Prove that R is symmetric.

H **c.** Prove that R is transitive.

d. List five elements in $[(1, 1)]$.

e. List five elements in $[(3, 1)]$.

f. List five elements in $[(1, 2)]$.

g. Describe the distinct equivalence classes of R.

Q 16:

16. **a.** Let R be the relation of congruence modulo 3. Which of the following equivalence classes are equal?

$$[7], [-4], [-6], [17], [4], [27], [19]$$

b. Let R be the relation of congruence modulo 7. Which of the following equivalence classes are equal?

$$[35], [3], [-7], [12], [0], [-2], [17]$$

Q 17:

2. Each of the following partitions of $\{0, 1, 2, 3, 4\}$ induces a relation R on $\{0, 1, 2, 3, 4\}$. In each case, find the ordered pairs in R.

a. $\{0, 2\}, \{1\}, \{3, 4\}$
b. $\{0\}, \{1, 3, 4\}, \{2\}$
c. $\{0\}, \{1, 2, 3, 4\}$

Q 18:

Test Yourself

1. For a relation on a set to be an equivalence relation, it must be _____.

2. The notation $m \equiv n \pmod{d}$ is read "_____" and means that _____.

3. Given an equivalence relation R on a set A and given an element a in A, the equivalence class of a is denoted _____ and is defined to be _____.

4. If A is a set, R is an equivalence relation on A, and a and b are elements of A, then either $[a] = [b]$ or _____.

5. If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form _____.

6. Let $A = \mathbf{Z} \times (\mathbf{Z} - \{0\})$, and define a relation R on A by specifying that for all (a, b) and (c, d) in A, $(a, b)\, R\, (c, d)$ if, and only if, $ad = bc$. Then there is exactly one equivalence class of R for each _____.

Source: SUSANNA S. EPP, DISCRETE MATHEMATICS WITH APPLICATIONS FOURTH EDITION

Q 19:

41. For the set $A = \{1, 2, 3, \ldots, 8\}$. consider

$$P = \{\{1, 3\}, \{4\}, \{2, 5, 6\}, \{7, 8\}\}.$$

Show P is a partition of A and define the induced equivalence relation (ala the Equivalence Class Theorem). Find $\bar{1}$, $\bar{2}$, $\bar{3}$, ..., $\bar{8}$.

3. Consider the following relations R. *(36 + 10 points)*

a) xRy iff $x-y$ is even	e) xRy iff $4 \mid x+y$	i) xRy iff $x^2-y^2=0$
b) xRy iff $x+y$ is even	f) xRy iff $3 \mid 2x-y$	j) xRy iff $(x-1)^2-(y-1)^2=0$
c) xRy iff $x-y$ is odd	g) xRy iff $3 \mid 4x-y$	k) xRy iff $x^2-4x=y^2-4y$
d) xRy iff $4 \mid x-y$	h) xRy iff $3 \mid x^2-y^2$	l) xRy iff $x \mid y$ and $y \mid x$

(a) In each case, decide whether the relation R on $\mathbb{Z}$ is reflexive, symmetric and transitive (three answers in each case). **If you find that certain property fails, give concrete numbers showing that the given property fails.** E.g.: The relation divides $'|'$ is not symmetric because $3 \mid 6$ but $\neg 6 \mid 3$.

(b) Which ones are equivalence relations? For each equivalence relation above, **partition** the set $X=\{-3,-2,\ldots,4,5\}$ into the corresponding equivalence classes.

Q 21:

1. Define the relation R on $\mathbb{N}_0 \times \mathbb{N}_0$ by

$$R = \{((m,n),(p,q)) : m + q = n + p\}$$

(a) Prove that R is an equivalence relation.

(b) Determine the following equivalence classes:

i. $[(0,0)]$

ii. $[(1,0)]$

iii. $[(0,1)]$

Q 22:

4. Define a relation R on $\mathbb{Z} \times \mathbb{Z}$ by

$$(a, b)\ R\ (c, d) \ \text{ iff } \ a^2 = c^2.$$

(a) (10 points) Verify that R is an equivalence relation.

(b) (10 points) Find the equivalence class $[(2, 4)]$, and draw the set in the xy-plane.

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Q 23:

Consider the following relation $\mathcal{R}$ on the set $A = \{1, 2, 3, 4, 5\}$.

$$\mathcal{R} = \{(1,1),\ (1,2),\ (2,1),\ (2,2),\ (3,3),\ (4,4),\ (4,5),\ (5,4),\ (5,5)\}$$

Given that $\mathcal{R}$ is an equivalence relation on A, which of the following is the partition of A into equivalence classes?

Select the correct response.

A. $\mathcal{P} = \Big\{\{1\}, \{1,2\}, \{3\}, \{3,4\}, \{4\}, \{5\}\Big\}$

B. $\mathcal{P} = \Big\{\{1,2,3,4,5\}\Big\}$

C. $\mathcal{P} = \Big\{\{1,2\}, \{3,4\}, \{5\}\Big\}$

D. $\mathcal{P} = \Big\{\{1\}, \{2,3\}, \{4,5\}\Big\}$

E. $\mathcal{P} = \Big\{\{1,2,3\}, \{4,5\}\Big\}$

F. $\mathcal{P} = \Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\Big\}$

G. $\mathcal{P} = \Big\{\{1,2\}, \{3\}, \{4,5\}\Big\}$

Q 24:

5 2 Relations-Equivalence Classes: Problem 3

(2 pts) Consider the equivalence relation on $A = \{1, 2, 3, 4, 5\}$ given by,

$R = \{(1,1),(1,2),(1,3),(2,2),(2,3),(2,1),(3,3),(3,1),(3,2),(4,4),(4,5),(5,5),(5,4)\}$

Clearly, $[1] = [2]$

Part 1
How many different equivalence classes does this relation have?

Part 2
Decide if the following statements are true or false (Enter 'true' or 'false'):

(a) $[1] = [3]$.

(b) $[4] = [3]$.

(c) $[5] = [3]$.

(d) $[4] = [5]$.

(e) $[2] = [3]$.

(f) $[2] = [4]$.

Q 25:

(4) Let A be the set of all ordered pairs of positive integers. So some members of A are $(3,6),(7,7),(11,4),(1,2981)$. A relation on A is defined by the rule $(a,b)R(c,d)$ if and only if $ad = bc$. For example $(3,5)R(6,10)$ is true since $(3)(10) = (5)(6)$.

(a) Explain why R is an equivalence relation on A.

(b) List four ordered pairs in the equivalence class of $(2,3)$.

(5) Let $A = \{1,2,3,4,5,6\}$. The sets $\{1,2\}, \{3,4,5\}$, and $\{6\}$ form a partition of A. These are the equivalence classes for an equivalence relation, E, on A. Draw the **digraph** of E.

Q 26:

Question 5.(4 points each)

Let R be the relation on the set of ordered pairs of positive integers such that

$$((a, b), (c, d)) \in R \text{ if and only if } ad = bc.$$

(1) Show that R is an equivalence relation

(2) What is the equivalence class of of $(1, 2)$, i.e. $[(1, 2)]$?

Q 27:

Definition: Let S be a set, and let R be an equivalence relation on S. For $a \in S$, the **equivalence class** of a with respect to R, denoted by [a], is defined by

$$[a] = \{ x \in S \mid (x,a) \in R \}.$$

EXAMPLE 8. Let S = {2,3,5,7,11,13,17}, and let R be the equivalence relation on S defined by

R = { (2,2), (2,7), (2,17), (3,3), (5,5), (5,11), (7,2), (7,7), (7,17), (11,5), (11,11), (13,13), (17,2), (17,7), (17,17) }.

a. Find [11].

b. Find [3].

c. Find [7].

d. Find [17].

e. Find the set of all equivalence classes of R.

Q 28:

[4] Suppose that R is an equivalence relation on the set $\{1, 2, 3, 4, 5\}$ such that

$$(2, 5) \in R \text{ and } (1, 3) \in R,$$

$$(2, 3) \notin R \text{ and } (4, 5) \notin R$$

(a) [1] Explain why $(3, 5)$ cannot be in R.

(b) [1] List all of the elements of the equivalence class $[2]$. Justify your list.

(c) [2] Is it possible for R to have exactly three equivalence classes? If so, give an example of one such R, expressed as a set of ordered pairs. If not, explain why not.

Q 29:

Problem 5. For the given set A, determine whether $\mathcal{P}$ is a partition of A.

(a) $A = \{1, 2, 3, 4\}, \quad \mathcal{P} = \{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$
(b) $A = \{1, 2, 3, 4, 5, 6, 7\}, \quad \mathcal{P} = \{\{1, 2\}, \{3\}, \{4, 5\}\}$
(c) $A = \{1, 2, 3, 4, 5, 6, 7\}, \quad \mathcal{P} = \{\{1, 3\}, \{5, 6\}, \{2, 4\}, \{7\}\}$
(d) $A = \mathbb{N}, \quad \mathcal{P} = \{1, 2, 3, 4, 5\} \cup \{n \in \mathbb{N} : n > 5\}$
(e) $A = \mathbb{R}, \quad \mathcal{P} = (-\infty, 1) \cup [-1, 1] \cup (1, \infty)$

Q 30:

10. Given that $A = \{0, 1, 2, 3, 4\}$

(a) Find the smallest equivalence relation on A containing the ordered pairs $\{(1, 1), (1, 2), (3, 4), (4, 0)\}$

(b) Draw the graph of the equivalence relation for found in part (a).

(c) List the equivalence classes of the relation for found in (a).

|R|=13 , equivalent class =2