Group Theory

Next Topic

Groups of Small Orders

Groups of Order 1,2,3,4

Website : https://www.goclasses.in/

How many Different binary operations we can have with Order 1 Structure.

function

$(\{a\}, *)$

$a * a = a$

$(a, a) \rightarrow a$

$\Rightarrow$ 1 binary op^n.

Groups of Order 1:

$(\{T\}, \wedge)$ ✓ $\quad (\{0\}, +)$ ✓ $\quad (\{a\}, *)$

$(\{T\}, \vee)$ ✓ $\quad (\{T\}, \leftrightarrow)$ $\quad a * a = a$

$(\{1\}, \times)$ ✓ $\quad$ - - - - - - ✓

$\Rightarrow$ All are Isomorphic

Isomorphic = Same Structure

How many "Non-Isomorphic" Groups of order 1 are there? $\Rightarrow$ 1 ✓

*	a
a	a

closed ✓
$e = a$
Comm ✓
$a^{-1} = a$
Asso ✓

Order 1 Group:

$*$	e
e	e

Template for all Groups of order 1

Only 1 Template

Template of order 1 Group:

	a
a	a

$\{T\}, \wedge$

	T
T	T

$\{1\}, \times$

	1
1	1

$\{0\}, +$

	0
0	0

How many Non-Isomorphic Groups of order n

$\equiv$ How many Abstract templates of Groups of order n

Order 2 Group:

	e	a
e	e	a
a	a	e

Only 1 Template for Group of Order 1

↳ Abelian

Order 3 Group :

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Only 1 Group of order 3

$\Rightarrow$ Abelian

Order 4 Groups —— 2 templates of Group

2 Non-Isomorphic Groups

both are Abelian

Groups of Order 2

Ex1 $\{1, 0\}$

operation $+$

$+$	0	1
0	0	1
1	1	0

Identity $= 0$

$0^{-1} = 0$

$1^{-1} = 1$

$0+0 = 0$

$1+1 = 0$

Ex2 $\{1, -1\}$

operation $\times$

$\times$	1	-1
1	1	-1
-1	-1	1

Identity $= 1$

$1^{-1} = 1$

$(-1)^{-1} = -1$

$1 \times 1 = 1$

$(-1) \times (-1) = 1$

Ex3 $\{e, a\}$

operation $*$

$*$	e	a
e	e	a
a	a	e

Identity $= e$

$e^{-1} = e$

$a^{-1} = a$

$e * e = e$

$a * a = e$

Ex4 $\{(0,0), (1,1)\}$

operation $\oplus$

$\oplus$	(0,0)	(1,1)
(0,0)	(0,0)	(1,1)
(1,1)	(1,1)	(0,0)

Identity $= (0,0)$

$(0,0)^{-1} = (0,0)$

$(1,1)^{-1} = (1,1)$

$(0,0) \oplus (0,0) = (0,0)$

$(1,1) \oplus (1,1) = (0,0)$

Groups of Order 2

Ex1 $\{1, 0\}$

operation $+$

$+$	0	1
0	0	1
1	1	0

Identity $= 0$

$0^{-1} = 0$

$1^{-1} = 1$

$0+0 = 0$

$1+1 = 0$

Ex2 $\{1, -1\}$

operation $\times$

$\times$	1	-1
1	1	-1
-1	-1	1

Identity $= 1$

$1^{-1} = 1$

$(-1)^{-1} = -1$

$1 \times 1 = 1$

$(-1) \times (-1) = 1$

Ex3 $\{e, a\}$

operation $*$

$*$	e	a
e	e	a
a	a	e

Identity $= e$

$e^{-1} = e$

$a^{-1} = a$

$e * e = e$

$a * a = e$

Non-Isomorphic Group

$\equiv$

Different Templates → for understanding

$\equiv$

(Up to Isomorphism) means Consider Non-Isomorphic

Order 2 "Group" Template; $\Rightarrow$ 1

	e	a
e	e	a
a	a	e

Abelian Group

Every order 2 Group is Abelian Group.

Order 4 "Group" Templates :

Non-Iso-morphic

Only 2 Non-Isomorphic Groups :

① $\{e, x, y, z\} \Rightarrow x^{-1} = x, y^{-1} = z$

② $\{e, x, y, z\} \Rightarrow x^{-1} = x, y^{-1} = y, z^{-1} = z$

both Abelian

St1

	e	a	b	c
e	e	a	b	c
a	a	c	e	b
b	b	e	c	a
c	c	b	a	e

$\equiv$

St2

	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$\{e, x, y, z\}$

$\bar{y}^1 = z$

$\bar{x}^1 = x$

$\{e, x, y, z\}$

$\bar{x}^1 = x$ $\bar{y}^1 = z$

	e	a	b	c
e	e	a	b	c
a	a	a		
b	b			
c	c			

Not a Group

St3

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$\{e, x, y, z\}$

$x^{-1} = x$ $\quad y^{-1} = y$ $\quad z^{-1} = z$

Abelian

St4

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

Abelian ✓

$\{e, x, y, z\}$

$x^{-1} = x$

$y^{-1} = z$

$$\boxed{St1 \equiv St2 \equiv St4}$$

Iso-morphism of Groups $\Rightarrow$ Not Required for GATE.

Group of order 4

$\{e, a, b, c\}$

$\boxed{St1 \equiv St2 \equiv St4}$ $\{e, x, y, z\}$

$x^{-1} = x$ $y^{-1} = z$

$\boxed{St3}$ $\{e, x, y, z\}$

$e^{-1} = e, x^{-1} = x, y^{-1} = y, z^{-1} = z$

4.4.20 Group Theory: GATE CSE 2007 | Question: 21 top

How many different non-isomorphic Abelian groups of order 4 are there?

A. 2
B. 3
C. 4
D. 5

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How many different non-isomorphic Abelian groups of order 4 are there?

A. 2
B. 3
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D. 5

4.4.20 Group Theory: GATE CSE 2007 | Question: 21 top

How many different non-isomorphic Abelian groups of order 4 are there?

Non

A. 2
B. 3
C. 4
D. 5
E. 0

4.4.20 Group Theory: GATE CSE 2007 | Question: 21 top

How many different non-isomorphic [redacted] groups of order 4 are there?

A. 2
B. 3
C. 4
D. 5

2 Templates

Note:

Group order	Non-Isomorphic Groups	Non-Isomorphic Abelian Group
1	1	1
2	1	1
3	1	1
4	2	2

Q1. Consider the Cayley tables of 3 groups: $(F, +)$, $(G, *)$ and $(H, \cdot)$

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	x	y	z
x	x	y	z
y	z	y	x
z	y	x	z

(a) Which two groups above are isomorphic to each other why?

(b) Are the two Abelian? Why or why not?

structures

not group

Q1. Consider the Cayley tables of 3 groups: $(F, +)$, $(G, *)$ and $(H, \cdot)$

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	x	y	z
x	x	y	z
y	z	y	x
z	y	x	z

No identity element

(a) Which two groups above are isomorphic to each other why?

(b) Are the two Abelian? Why or why not?

structure

Q1. Consider the Cayley tables of 3 groups: (F, +) , (G, *) and (H, •)

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\equiv$

$*$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	x	y	z
x	x	y	z
y	z	y	x
z	y	x	z

$e = 0$

$1^{-1} = 2$

$2^{-1} = 1$

NOT even a Group

(a) Which two groups above are isomorphic to each other why?

(b) Are the two Abelian? Why or why not?

$e = a$

$b^{-1} = c$; $c^{-1} = b$

Only 1 Template for a Group of order 3 :

Group $(\{e, a, b\}, \#)$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

id

$\{e, x, y\}$

$x^{-1} = y$; $y^{-1} = x$

Order 3 Group:

One Template Possible

one Structure "

One Non-Isomorphic Group

Only Template: $\{e, x, y\}$ id

$x^{-1} = y$; $y^{-1} = x$

$(\{1, \omega, \omega^2\}, \times) \rightarrow$ Group(order 3)

Cube Roots of 1.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\rightarrow \{\overset{e}{1}, \omega, \omega^2\}$

x, y

$\omega^{-1} = \omega^2$;

$(\omega^2)^{-1} = \omega$

$(\{1, \omega, \omega^2\}, \times) \rightarrow$ Group (order 3)

Cube Roots of 1.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\equiv$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$e \rightarrow 1$, $a \rightarrow \omega$, $b \rightarrow \omega^2$

$(\{1, \omega, \omega^2\}, \times) \rightarrow$ Group(order 3)

Cube Roots of 1.

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$\equiv$

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\{0, 1, 2\}, \oplus_3$ → Group (order 3)

Z_3

Addition modulo 3

$\{ \underset{e}{0}, \underset{x}{1}, \underset{y}{2} \}$

$x^{-1} = y$; $y^{-1} = x$

$1^{-1} = 2$

$2^{-1} = 1$

Group of order 3

Only one Template:

$$\{ e, x, y \}$$

$e \rightarrow$ identity

$x^{-1} = y; \quad y^{-1} = x$

$(\{1, \omega, \omega^2\}, \times)$

$(\{0, 1, 2\}, \oplus_3)$

$(\{e, a, b\}, \#)$

id $\quad a^{-1} = b, \; b^{-1} = a$

$\rightarrow$ Same Structure

$\equiv$ Isomorphic

Group of order 3 :

Only one Template :

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Abelian

$\{e, a, b\}$

$a^{-1} = b$