First Order Logic
Next Chapter:

Null Quantification Rule

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GATE CSE AIR 53; AIR 67;
AIR 107; AIR 206; AIR 256

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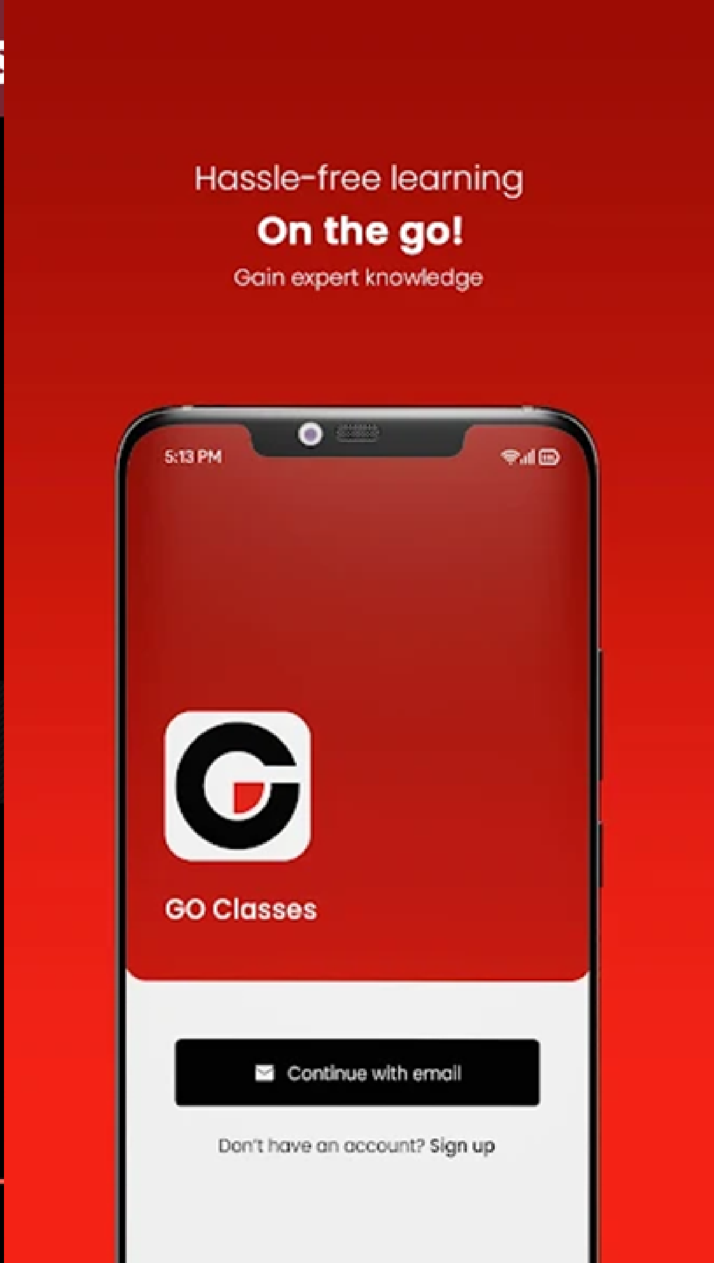

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Next Topic:

Practice

Distributive Properties of Quantifiers

Distribution of Quantifiers over logical Connective :

Quantifier : $\Pi \to \forall$ OR $\exists$

Logical Connective : # : $\vee, \wedge, \rightarrow, \leftrightarrow, \oplus, \uparrow, \downarrow$

$$\Pi_x \left(P(x) \# Q(x) \right) \underset{?}{\overset{?}{\rightleftarrows}} \Pi_x P(x) \# \Pi_x Q(x)$$

Distributive Properties of Quantifiers:

Method to check for Validity :

(1) Take Some Non-Empty Abstract Domain : { a , b , c , - - - - - }

(2) Try to make false → You can ⇒ Invalid

→ Never ⇒ Valid

Valid $\equiv$ Always True

Expression α

I try to make α false :

If I can ; α is Invalid

Valid $\equiv$ Always True

Expression α

I try to make α false :

If I Can Never : α valid

NOTE:

By default, For ENTIRE expression, we take a Domain, So, Every variable has the same domain.

& Domain is ALWAYS Non-Empty.

FOL Expression: G

G : Domain D

All variables in G have Domain D.

NOTE:

While checking for Validity of a FOL expression:

For ENTIRE expression, Take a abstract Non-empty Domain, So, Every variable has the same domain.

Daily Practice

Standard Books’ Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 43

Q 43: Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

43. Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

$$\underbrace{\forall x \left(P(x) \rightarrow Q(x) \right)}_{\alpha} \overset{?}{\equiv} \underbrace{\forall x\, P(x) \rightarrow \forall x\, Q(x)}_{\beta}$$

$\alpha \equiv \beta$ iff

$$\boxed{\begin{array}{l} \alpha \rightarrow \beta \text{ valid} \\ \underline{\text{and}} \quad \beta \rightarrow \alpha \text{ valid} \end{array}}$$

$\forall x \left(P(x) \rightarrow Q(x) \right)$

Scope

Same

$\left(\forall x\, P(x)\right) \rightarrow \left(\forall x\, Q(x)\right)$

m

n

Nothing to do with each other

$$\underbrace{\forall x \left(P(x) \rightarrow Q(x)\right)}_{\alpha} \overset{?}{\longrightarrow} \underbrace{\overset{M}{\forall x\, P(x)} \rightarrow \overset{N}{\forall x\, Q(x)}}_{\beta}$$

Method: Try to make

α = True

&

β = false

Together

$$\underbrace{\forall x\,(P(x) \rightarrow Q(x))}_{\alpha} \overset{?}{\longrightarrow} \underbrace{\overset{M}{\forall x\, P(x)} \rightarrow \overset{N}{\forall x\, Q(x)}}_{\beta}$$

Can I make α True?

↳ No

Make β false: 1

$M \rightarrow N$: false

M: True ; N: false

Domain: a b c - - - - -

$P = T$ $P = T$ $P = T$ - - - - -

$Q = F$

$$\underbrace{\forall x \left(P(x) \rightarrow Q(x)\right) \rightarrow \left[\forall x\, P(x) \rightarrow \forall x\, Q(x)\right]}$$

→ Valid

$$\underbrace{\forall x\,(P(x) \to Q(x))}_{\beta} \longleftarrow \underbrace{\overbrace{\forall x\, P(x)}^{M} \to \overbrace{\forall x\, Q(x)}^{N}}_{\alpha}$$

① make β false;

for some element: $\boxed{P \to Q}$ false

② Can I make α : True?

yes

α : $\underset{F}{M} \to N \equiv$ True

Domain: a b c d - - - - - - -

$P(a) = F$

$\underset{T}{P_c} \to \underset{F}{Q_c}$

false

$$\underbrace{\forall x \left(P(x) \rightarrow Q(x) \right) \leftarrow \forall x\, P(x) \rightarrow \forall x\, Q(x)}_{\text{Invalid}}$$

$$\underbrace{\forall x\,(P(x) \rightarrow Q(x)) \equiv \forall x\, P(x) \rightarrow \forall x\, Q(x)}$$

Invalid

Conclusion:

$$\underbrace{\forall x\,(P(x) \to Q(x))}_{\text{Compact}} \longrightarrow \underbrace{(\forall x\, P(x) \to \forall x\, Q(x))}_{\text{Expanded}}$$

$$\not\longleftarrow$$

	$\forall$
$\to$	$\to$

The Table

The Table

Prove it, then Remember

Don't blindly by-heart.

Daily Practice

Standard Books' Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 44

Q 44: Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\, P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

44. Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall x\ P(x) \leftrightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

$$\underbrace{\forall x \left(P(x) \leftrightarrow Q(x) \right)}_{\alpha} \xrightarrow{?} \underbrace{\overset{M}{\forall x P(x)} \leftrightarrow \overset{N}{\forall x Q(x)}}_{\beta}$$

① make α True:
for every element, $P = Q$

② Try to make β = false;
β: $M \leftrightarrow N$; $\left.\begin{array}{l} M = T \\ N = F \end{array}\right\}$ Never

No

Domain: a | b | c | - - - -

$P(a) = Q(a)$
$= T$

$P(b) = Q(b)$
$= T$

$P(c) = Q(c)$
$= T$

- - -

$$\underbrace{\forall x \left(P(x) \leftrightarrow Q(x) \right) \xrightarrow{\checkmark} \left(\forall x P(x) \leftrightarrow \forall x Q(x) \right)}$$

→ Valid

$$\underbrace{\forall x \left(P(x) \leftrightarrow Q(x) \right)}_{\beta} \leftarrow \underbrace{\overset{M}{\forall x P(x)} \leftrightarrow \overset{N}{\forall x Q(x)}}_{\alpha}$$

① make β = false ;

for some ; $P \neq Q$

② Can we make α = True ? → YES

α : $M \leftrightarrow N$: True

F F

Domain : a | b | c - - - - -

$P(a) = F$

T $P(b) \neq Q(b)$ F

$$\underbrace{\forall x\,(P(x) \leftrightarrow Q(x))}_{\beta} \;\cancel{\leftarrow}\; \underbrace{\overbrace{\forall x\, P(x)}^{M} \leftrightarrow \overbrace{\forall x\, Q(x)}^{N}}_{\alpha}$$

① make β = false;

② Can we make True?

α: $M \leftrightarrow N$

M: F → Can I make M false?

N: F → Already false

α: True

Domain: { ⓐ: $P(a)$: T; $Q(a) = F$
b: $P(b) = F$
⋮

$$\underbrace{\forall x \left(P(x) \leftrightarrow Q(x) \right) \leftarrow \forall x P(x) \leftrightarrow \forall x Q(x)}$$

$\rightarrow$ Invalid

$$\forall x \left(P(x) \leftrightarrow Q(x) \right) \equiv \forall x P(x) \leftrightarrow \forall x Q(x)$$

Invalid

Conclusion:

$$\underbrace{\forall x\,\big(P(x) \leftrightarrow Q(x)\big)}_{\text{Compact}} \;\xrightarrow{\;\checkmark\;}\; \underbrace{\forall x\, P(x) \leftrightarrow \forall x\, Q(x)}_{\text{Expanded}}$$

$$\forall x\,\big(P(x) \leftrightarrow Q(x)\big) \;\cancel{\longleftarrow}\; \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$$

	$\forall$
$\leftrightarrow$	$\rightarrow$

The Table ✓

More logical equivalences and non-equivalences

- $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
- $\forall x(P(x) \vee Q(x)) \not\equiv \forall x P(x) \vee \forall x Q(x)$
- $\forall x(P(x) \rightarrow Q(x)) \not\equiv \forall x P(x) \rightarrow \forall x Q(x)$
- $\exists x(P(x) \vee Q(x)) \equiv \exists x P(x) \vee \exists x Q(x)$
- $\exists x(P(x) \wedge Q(x)) \not\equiv \exists x P(x) \wedge \exists x Q(x)$
- $\exists x(P(x) \rightarrow Q(x)) \stackrel{?}{\equiv} \exists x P(x) \rightarrow \exists x Q(x)$

More logical equivalences and non-equivalences

- $\forall x(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$
- $\forall x(P(x) \lor Q(x)) \not\equiv \forall x P(x) \lor \forall x Q(x)$
- $\forall x(P(x) \rightarrow Q(x)) \not\equiv \forall x P(x) \rightarrow \forall x Q(x)$
- $\exists x(P(x) \lor Q(x)) \equiv \exists x P(x) \lor \exists x Q(x)$
- $\exists x(P(x) \land Q(x)) \not\equiv \exists x P(x) \land \exists x Q(x)$
- $\exists x(P(x) \rightarrow Q(x)) \stackrel{?}{\equiv} \exists x P(x) \rightarrow \exists x Q(x)$

Already proven

$$\underbrace{\exists x\,(P(x) \rightarrow Q(x))}_{\alpha} \rightarrow \underbrace{\overbrace{\exists x\, P(x)}^{M} \rightarrow \overbrace{\exists x\, Q(x)}^{N}}_{\beta}$$

(2) Can we make α: True

→ YES

(1) make β false: β: $M \rightarrow N$ (M: T, N: F)

Domain: a b c - - - -

$Q(a) = F$ $P(b) = T$ $Q(c) = F$ - - - -

$P(a) = F$ $Q(b) = F$

$$\exists x\,(P(x) \rightarrow Q(x)) \rightarrow \exists x\, P(x) \rightarrow \exists x\, Q(x)$$

$$\exists x\,(P(x) \rightarrow Q(x)) \not\rightarrow \exists x\, Q(x)$$

~~Invalid~~

$\underbrace{\exists x\,(P(x) \rightarrow Q(x))}_{\beta} \Leftarrow \underbrace{\overset{M}{\exists x\, P(x)} \rightarrow \overset{N}{\exists x\, Q(x)}}_{\alpha}$

1. make β false;

2. Can I make α: True; $\rightarrow$ NO

$M: T$; $N: F$

Domain:

a	b	c	...
$\overset{T}{P(a)} \rightarrow \overset{F}{Q(a)}$	$\overset{T}{P(b)} \rightarrow \overset{F}{Q(b)}$	$\overset{T}{P(c)} \rightarrow \overset{F}{Q(c)}$	...
false	false	false	

$$\exists x \left(P(x) \rightarrow Q(x) \right) \leftarrow \exists x\, P(x) \rightarrow \exists x\, Q(x)$$

Valid

$$\underbrace{\exists x \left(P(x) \rightarrow Q(x) \right)}_{\text{Compact}} \;\; \overset{\times}{\longrightarrow} \;\; \underbrace{\exists x\, P(x) \rightarrow \exists x\, Q(x)}_{\text{Expanded}}$$

$$\exists x \left(P(x) \rightarrow Q(x) \right) \;\; \underset{\checkmark}{\longleftarrow} \;\; \exists x\, P(x) \rightarrow \exists x\, Q(x)$$

	$\forall$	$\exists$
$\rightarrow$	$\rightarrow$	$\leftarrow$

The Table

Q:

$\exists x[\, P(x) \leftrightarrow Q(x)\,]$

and

$\exists x P(x) \leftrightarrow \exists x Q(x)$

Q:

α

$\exists x[\, P(x) \leftrightarrow Q(x) \,]$

① make α True:

Domain:

a $P(a) = F$, $Q(a) = T$

b $\boxed{P(b) = Q(b)}$ F

c $P(c) = F$

$\exists x P(x) \leftrightarrow \exists x Q(x)$

F T

β ② Can we make β = false → YES

Q:

$$\exists x [\, P(x) \leftrightarrow Q(x) \,]$$

$$\exists x P(x) \leftrightarrow \exists x Q(x)$$

Invalid

Q:

β

$\exists x[\, P(x) \leftrightarrow Q(x)\,]$

① make β = False

Domain:

a : P(a) ≠ Q(a)

b : P(b) ≠ Q(b)

c : P(c) ≠ P(c)

$\exists x P(x) \leftrightarrow \exists x Q(x)$

α

② Can we make α : True? → YES

Q:

$\exists x[\, P(x) \leftrightarrow Q(x)\,]$

$\exists x P(x) \leftrightarrow \exists x Q(x)$

Invalid

Q:

Compact

$$\exists x[\ P(x) \leftrightarrow Q(x)\ ]$$

$$\exists x P(x) \leftrightarrow \exists x Q(x)$$

Expanded

	$\forall$	$\exists$
$\rightarrow$	$\rightarrow$	$\leftarrow$
$\leftrightarrow$	$\rightarrow$	✗
✗	✗	$\leftarrow$

The Table

In 2017: I: GATE Aspirant

$\forall$ $\exists$

Next Topic:

Distributive Properties of Quantifiers

The Table

Note: In the Table :

Compact Exp — LHS

Expanded Exp — RHS

② $\forall$, for all Arrow-type Connectives ($\rightarrow$, $\leftrightarrow$, $\uparrow$, $\downarrow$) goes from Compact to Expanded Only.

	$\forall$	$\exists$
$\wedge$	$\longleftrightarrow$	$\longrightarrow$
$\vee$	$\longleftarrow$	$\longleftrightarrow$
$\rightarrow$	$\longrightarrow$	$\longleftarrow$
$\leftrightarrow$	$\longrightarrow$	$\times$
$\oplus$	$\times$	$\longleftarrow$
$\uparrow$	$\longrightarrow$	$\longleftarrow$
$\downarrow$	$\longrightarrow$	$\longleftarrow$

Q:

A. $\alpha \equiv \beta$

B. $\alpha \equiv \beta$

C. $\alpha \equiv \beta$

D. $\alpha \equiv \beta$

8 Questions

Q: $\forall x \left(P(x) \oplus Q(x) \right) \overset{?}{\rightarrow} \exists x P(x) \rightarrow \forall x Q(x)$

$\overset{?}{\leftarrow}$

HW ✓

Proper understanding needed

Which of the following semantic entailments are valid in predicate logic?

1. $\forall x\,(P(x) \lor Q(x)) \models \forall x\, P(x) \lor \forall x\, Q(x)$
2. $\forall x\,(P(x) \rightarrow Q(x)) \models \forall x\, P(x) \rightarrow \forall x\, Q(x)$
3. $\forall x\, P(x) \rightarrow \forall x\, Q(x) \models \forall x\,(P(x) \rightarrow Q(x))$
4. $\neg\forall x\,(P(x) \land Q(x)) \models \exists x\, \neg P(x) \land \exists x\, \neg Q(x)$
5. $\exists x\, P(x) \land \exists x\, Q(x) \models \exists x\,(P(x) \land Q(x))$

Which of the following semantic entailments are valid in predicate logic?

Compact ← Expanded

1. $\underbrace{\forall x\,(P(x) \vee Q(x))}_{\text{Compact}} \models \underbrace{\forall x\, P(x) \vee \forall x\, Q(x)}_{\text{Expanded}}$
2. $\forall x\,(P(x) \to Q(x)) \models \forall x\, P(x) \to \forall x\, Q(x)$
3. $\forall x\, P(x) \to \forall x\, Q(x) \models \forall x\,(P(x) \to Q(x))$ → Compact
4. $\neg\forall x\,(P(x) \wedge Q(x)) \models \exists x\, \neg P(x) \wedge \exists x\, \neg Q(x)$
5. $\underbrace{\exists x\, P(x) \wedge \exists x\, Q(x)}_{\text{Expanded}} \models \underbrace{\exists x\,(P(x) \wedge Q(x))}_{\text{Compact}}$

$\models$: logical implication

$$\neg \forall x \, (P \land Q)$$

$$\equiv \exists x \, (\overset{A}{\overline{P}} \lor \overset{B}{\overline{Q}})$$

$$\exists x \, \overline{P} \land \exists x \, \overline{Q}$$

$$\underbrace{\exists x\,(A(x) \vee B(x))}_{\alpha} \rightarrow \underbrace{\exists x\, A(x) \wedge \exists x\, B(x)}_{\beta}$$

① make α : True ② Can make β = false? YES

Domain: a b c

$A(b)$: T

$B(a) = F$ $B(c) = F$

$$S: \boxed{\underbrace{\exists_x P(x)}_{M} \wedge \underbrace{\exists_x Q(x)}_{N}} = \underline{\text{false}}$$

$$S: \underline{M \wedge N} = F$$

<u>one possibility</u>: $M: T$, $N = F$

$$\neg \forall x (P \land Q)$$

$$\equiv \exists x (\overset{A}{\overline{P}} \lor \overset{B}{\overline{Q}})$$

$$\exists x \overline{P} \lor \exists x \overline{Q}$$

$$\boxed{\exists x \left(A(x) \lor B(x) \right) \equiv \exists x\, A(x) \lor \exists x\, B(x)}$$

(4) $\neg \forall_x \left(P(x) \wedge Q(x) \right)$

$\equiv \exists x \left(\overline{P(x)} \vee \overline{Q(x)} \right)$

$\equiv \exists x \overline{P(x)} \vee \exists x \overline{Q(x)}$

$\equiv$

$\exists x \left(A(x) \vee B(x) \right) \equiv \exists x A(x) \vee \exists x B(x)$

Next Topic:

Null Quantification Rules

Distributive prop. of Quantifiers:

$\forall x\,(P(x) \vee Q(x))$ → True

Affect

Affect

$\forall x\, P(x) \vee \forall x\, Q(x)$ → False

$P(x)$: x is male

$Q(x)$: x ,, femal

free Var.

Domain:

Set of all People

Distributive prop. of Quantifiers:

$$\forall x \, (P(x) \vee Q(x)) \rightarrow \forall x \, P(x) \vee \forall x \, Q(x)$$

Invalid

x free

x free

Affect

Affect

$P(x)$: x is prime

Domain: N

x is free

$\boxed{\forall_x P(x) = P(x)}$ ✗

$\forall_x P(x) = \text{False}$

proposition

A : $\boxed{2+2=10}$ ✓

Domain: $\mathbb{N}$

No free x

Does not depend on x

$$\boxed{\forall_x A \equiv A}$$

$$\boxed{\exists_x A \equiv A}$$

$$\boxed{\forall_x A \equiv A}$$ ✓

Distributive prop. of Quantifiers:

$\forall x\,(P(x) \vee Q(x))$

Affects

Affects

$\forall x\, P(x) \vee \forall x\, Q(x)$

Affects

Affects

Distributive prop. of Quantifiers:

$$\forall x \left(P(x) \vee A \right)$$

$$\forall x\, P(x) \vee \forall x\, A$$

Null Quantification

Unaffectes by $\forall x$

Null Quantification;

$\forall_x \left(P(x) \to A \right)$

this Part is unaffected by $\forall x$

Null Quantification

Unaffected by $\forall x$

A is Unaffected by $\forall x, \exists x$

A has No free x

Quantifier : $\Pi \rightarrow \forall$ OR $\rightarrow \exists$

Operator : $\#$: $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus, \uparrow, \downarrow$

Null Quantification :

Some Part is unaffected by Quantifier Π_x

Quantifier : $\Pi \rightarrow \forall$, $\rightarrow \exists$

Operator : # : $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus, \uparrow, \downarrow$

Null Quantification : ✓

$\boxed{\text{Some Part has no free var. } x}$

~~x x x x~~

Quantifier : $\Pi \to \forall$, $\exists$

Operator : # : $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus, \uparrow, \downarrow$

$\forall x \, (P(x) \wedge A)$ ✓

A has No free x

$\exists x \, (A \rightarrow P(x))$

Null Quantification

Null Quantification Rules

(Distribution of Quantifiers over logical connectives, when some expression isn't affected by Quantifier)

$\equiv$ Some Part has no free var. x

- **Binding variables and scope**

When a quantifier is used on the variable x we say that this occurrence of x is **bound**. When the occurrence of a variable is not bound by a quantifier or set to a particular value, the variable is said to be **free**.

The part of a logical expression to which a quantifier is applied is the **scope** of the quantifier. A variable is free if it is outside the scope of all quantifiers.

In the example above, $(\forall x \underline{P(x)}) \vee Q(x)$, the x in $P(x)$ is bound by the existencial quantifier, while the x in $Q(x)$ is free. The scope of the universal quantifier is underlined.

In the expressions $\forall x\ \varphi$, the wff φ is called the **scope** of the quantifier $\forall x$

In the expressions $\exists x\ \varphi$, the wff φ is called the **scope** of the quantifier $\exists x$

A wff φ is a **proposition** iff it has no free variables in it.

A: has No free x.

$$\boxed{\begin{aligned} \forall_x A &\equiv A \\ \exists_x A &\equiv A \end{aligned}}$$

$$\forall x\,(P(x) \vee A(x)) \not\equiv (\forall x\, P(x)) \vee (\forall x\, A(x))$$

If A has No free x :

A ✓

$A(x)$ ✗

If "A" does Not have any Free Variable x

then:

$$\forall x \left(P(x) \vee \cancel{A} \right) \stackrel{?}{\equiv} \left(\forall x\, P(x) \right) \vee A$$

Independent of $\forall x$

No free x

If "A" does Not have any Free Variable x

then:

$$\forall x \left(P(x) \vee A \right) \overset{?}{\equiv} \left(\forall x \, P(x) \right) \vee A$$

$\forall x$ can not affect A : So

Simply create two cases: $\left.\begin{array}{l} A : \text{True} \\ A : \text{false} \end{array}\right\}$

Valid $\equiv$ $\boxed{\text{Always True}}$

$\boxed{\text{In All Cases, True}}$

If "A" does Not have any Free Variable x

then:

$$\underbrace{\forall x \, (P(x) \vee A)}_{\text{LHS}} \overset{?}{\equiv} \underbrace{(\forall x \, P(x)) \vee A}_{\text{RHS}}$$

Case 1: A: True

LHS: True
RHS: True

$\equiv$

Case 2: A: false

LHS: $\forall x \, P(x)$
RHS: $\forall x \, P(x)$

$\equiv$

$\boxed{P(x) \vee \text{True}}$: $\boxed{\text{Always True for every element in Domain}}$

Conclusion:

$$\forall x\,(P(x) \lor Q(x)) \not\equiv \forall x\,P(x) \lor \forall x\,Q(x)$$

$$\forall x\,(P(x) \lor A) \equiv \forall x\,P(x) \lor A$$

has No free x

Checking Validity:

If "A" does Not have any Free Variable x then:

Create two cases: A = True ; A = False.

Expression is Valid iff Valid in both cases.

If in some case invalid, then expression Invalid.

Daily Practice

Standard Books' Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 46

Exercises 46–49 establish rules for **null quantification** that we can use when a quantified variable does not appear in part of a statement.

46. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \vee A \equiv \forall x (P(x) \vee A)$

b) $(\exists x P(x)) \vee A \equiv \exists x (P(x) \vee A)$

(b) $\underbrace{\exists x \left(P(x) \vee A \right)}_{LHS} \overset{?}{\equiv} \left(\exists x\, P(x) \right) \vee A$

A has No free x.

Case 1: A: True

LHS: True

RHS: True

$\equiv$

Case 2: A = false

LHS: $\exists x\, P(x)$

RHS: $\exists x\, P(x)$

$\equiv$

Daily Practice

Standard Books' Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 47

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $(\forall x P(x)) \land A \equiv \forall x(P(x) \land A)$

b) $(\exists x P(x)) \land A \equiv \exists x(P(x) \land A)$

47. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

Null Quantification

a) $(\forall x P(x)) \land A \equiv \forall x(P(x) \land A)$

b) $(\exists x P(x)) \land A \equiv \exists x(P(x) \land A)$

(a) $\underbrace{\forall x\,(P(x) \wedge A)}_{\text{LHS}} \equiv \underbrace{(\forall x\, P(x)) \wedge A}_{\text{RHS}}$

No free x

Case 1: A: True

LHS: $\forall x\, P(x)$

RHS: $\forall x\, P(x)$

$\equiv$

Case 2: A: False

LHS: False

RHS: False

$\equiv$

(b) $\exists x\,(P(x) \wedge A) \equiv (\exists x\, P(x)) \wedge A$?

No free x

Case 1: A : True

LHS : $\exists x\, P(x)$

RHS : $\exists x\, P(x)$

$\equiv$

Case 2 : A : False

LHS : False

RHS : False

$\equiv$

$$\forall x \left(P(x) \# A \right) \equiv \left(\forall x P(x) \right) \# A$$

$\#$: $\wedge, \vee$

$$\exists x \left(P(x) \# A \right) \equiv \left(\exists x P(x) \right) \# A$$

$\#$: $\wedge, \vee$

Daily Practice

Standard Books' Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 48

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \rightarrow P(x)) \equiv A \rightarrow \forall x P(x)$

b) $\exists x(A \rightarrow P(x)) \equiv A \rightarrow \exists x P(x)$

48. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(A \to P(x)) \equiv A \to \forall x P(x)$

b) $\exists x(A \to P(x)) \equiv A \to \exists x P(x)$

$\forall x A \equiv A$

$\exists x A \equiv A$

(a) $\forall x\,(A \rightarrow P(x)) \overset{?}{\equiv} A \rightarrow \forall x\, P(x)$

Case 1: A: True

LHS: $\forall x\, P(x)$

RHS: $\forall x\, P(x)$

$\equiv$

Case 2: A: false

LHS: True

RHS: True

$\equiv$

$$\boxed{T \to \mathcal{M} \equiv M}$$

(b) $\exists x\,(A \rightarrow P(x)) \stackrel{?}{\equiv} A \rightarrow \exists x\, P(x))$ ✓

(1) A : false

LHS : True

RHS : True

$\equiv$

(2) A : True

LHS : $\exists x\, P(x)$

RHS : $\exists x\, P(x)$

$\equiv$

$$\boxed{F \longrightarrow m \equiv T}$$

$$\boxed{T \longrightarrow m \equiv m}$$

① $\exists x \left(A \rightarrow P(x)\right) \equiv A \rightarrow \exists x P(x)$ ✓

② $\forall x \left(A \rightarrow P(x)\right) \equiv A \rightarrow \forall x P(x)$ ✓

Daily Practice

Standard Books' Questions

Topic: First Order Logic

Source:

Kenneth H. Rosen,

Discrete Mathematics and Its Applications,

Seventh Edition, Exercise 1.4 Question 49

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$

b) $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$

46 ✓

47 ✓

48 ✓

49 ✓

49. Establish these logical equivalences, where x does not occur as a free variable in A. Assume that the domain is nonempty.

a) $\forall x(P(x) \rightarrow A) \equiv \exists x P(x) \rightarrow A$

b) $\exists x(P(x) \rightarrow A) \equiv \forall x P(x) \rightarrow A$

Null Quantification

A: T
Case 1

A = F
Case 2

(a) $\forall x\,(P(x) \rightarrow A) \stackrel{?}{\equiv} (\forall x\, P(x)) \rightarrow A$

Case 1: A : True

LHS : True
RHS : True
$\equiv$

Case 2: A : false

LHS: $\forall x\, \neg P(x)$
RHS: $\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\not\equiv$

$$\boxed{m \to T \equiv T}$$

$$m \to F \equiv \overline{m}$$

(a) $\forall x \left(P(x) \rightarrow A \right) \xrightarrow{\checkmark} \left(\forall x P(x) \right) \rightarrow A$

$\forall x \left(P(x) \rightarrow A \right) \xleftarrow{\times} \left(\forall x P(x) \right) \rightarrow A$

(b) $\exists x \left(P(x) \rightarrow A \right) \overset{?}{\equiv} \left(\exists x P(x) \right) \rightarrow A$

Case 1 : A : True

LHS : True
RHS : True
$\equiv$

Case 2 : A : false

LHS : $\exists x \neg P(x)$ $\not\equiv$

RHS : $\neg \exists x P(x) \equiv \forall x \neg P(x)$

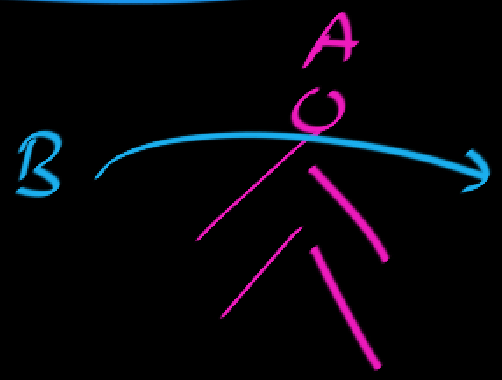
(b) $\exists x \left(P(x) \rightarrow A \right) \underset{\checkmark}{\overset{\times}{\rightleftarrows}} \left(\exists x P(x) \right) \rightarrow A$

Interesting point:

① $\forall x\,(A \rightarrow P(x)) \equiv A \rightarrow \forall x\, P(x)$

② $\forall x\,(P(x) \rightarrow A) \not\equiv \forall x\, P(x) \rightarrow A$

Superstition:



Mother:

Don't cross over a person, else their height will not increase.

A, B: Siblings

Bollywood Dialogue:

$\boxed{\text{If you want to go, go over my dead body.}}$

Interesting point :

A is dead for $\forall x$.

① $\forall x \, (A \rightarrow P(x)) \equiv A \rightarrow \forall x \, P(x)$

Dead

② $\forall x \, (P(x) \rightarrow A) \not\equiv \forall x \, P(x) \rightarrow A$

Alive for $\forall x$.

Interesting point:

① $\exists x \left(A \rightarrow P(x) \right) \equiv A \rightarrow \exists x P(x)$

② $\exists x \left(P(x) \rightarrow A \right) \not\equiv \left(\exists x P(x) \right) \rightarrow A$

Interesting point:

$$\forall x\,(P(x) \rightarrow A) \not\equiv (\forall x\, P(x)) \rightarrow A$$

$$\boxed{\forall x\,(P(x) \rightarrow A) \equiv \exists x\, P(x) \rightarrow A}$$

$$\forall x\,(P(x) \to A) \overset{?}{\equiv} (\exists x\, P(x)) \to A$$

Case1: A: True

LHS: True

RHS: True

$\equiv$

Case2: A: false

LHS: $\forall x\, \neg P(x)$

RHS: $\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

$\equiv$

$$\forall x \left(P(x) \rightarrow A \right) \stackrel{?}{\equiv} \underline{\underline{\left(\exists x P(x) \right) \rightarrow A}}$$

$$\equiv \forall x \left(\overline{P(x)} \vee A \right)$$

$$\equiv \forall x \overline{P(x)} \vee A$$

$$\equiv \neg \exists x P(x) \vee A \equiv \boxed{\exists x P(x) \rightarrow A}$$

$$\exists x\,(P(x) \rightarrow A) \not\equiv (\exists x\,P(x)) \rightarrow A$$

$$\boxed{\exists x\,(P(x) \rightarrow A) \equiv \forall x\,P(x) \rightarrow A}$$ ✓

prove it

Topic: First Order Logic

GATE CSE 2020 Question

on Null Quantification

GATE CSE 2020 | Question: 39

33

Which one of the following predicate formulae is NOT logically valid?

Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x\ (px) \vee W$
B. $\exists x(p(x) \wedge W) \equiv \exists x\ p(x) \wedge W$
C. $\forall x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$
D. $\exists x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$

gatecse-2020 first-order-logic mathematical-logic 2-marks

GATE CSE 2020 | Question: 39

asked in Mathematical Logic Feb 12, 2020 • retagged Dec 1, 2022 by Lakshman Bhaiya 11,677 views

33

Which one of the following predicate formulae is NOT logically valid?

Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x\,(px) \vee W$
B. $\exists x(p(x) \wedge W) \equiv \exists x\, p(x) \wedge W$
C. $\forall x(p(x) \rightarrow W) \equiv \forall x\, p(x) \rightarrow W$
D. $\exists x(p(x) \rightarrow W) \equiv \forall x\, p(x) \rightarrow W$

Q 46 of Keneth Rosen
Q 47 " " "
Q 48 " " "
Q 49 " " "

gatecse-2020 first-order-logic mathematical-logic 2-marks

GATE CSE 2020 | Question: 39

I got AIR 53

$W = F$

A: LHS: $\forall x\, p(x)$
RHS: $\forall x\, p(x)$

C: LHS: $\forall x \neg p(x)$
RHS $\exists x \neg p(x)$

$W = T$

A. LHS: True
RHS: True

asked in **Mathematical Logic** Feb 12, 2020 • retagged Dec 1, 2022 by **Lakshman Bhaiya**

33

Which one of the following predicate formulae is NOT logically valid?

Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x\ (px) \vee W$ — valid
B. $\exists x(p(x) \wedge W) \equiv \exists x\ p(x) \wedge W$
C. $\forall x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$ — invalid
D. $\exists x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$

gatecse-2020 first-order-logic mathematical-logic 2-marks → easy 2 marks

GATE CSE 2020 | Question: 39

Only one Q. in GATE on Null Quantification

asked in Mathematical Logic Feb 12, 2020 • retagged Dec 1, 2022 by Lakshman Bhaiya 11,677 views

33

Which one of the following predicate formulae is NOT logically valid?

Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x\ (px) \vee W$
B. $\exists x(p(x) \wedge W) \equiv \exists x\ p(x) \wedge W$
C. $\forall x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$
D. $\exists x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$

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GATE CSE 2020 | Question: 39

asked in Mathematical Logic Feb 12, 2020 • retagged Dec 1, 2022 by Lakshman Bhaiya

11,677 views

33

Which one of the following predicate formulae is NOT logically valid?

Note that W is a predicate formula without any free occurrence of x.

A. $\forall x(p(x) \vee W) \equiv \forall x\ (px) \vee W$ valid
B. $\exists x(p(x) \wedge W) \equiv \exists x\ p(x) \wedge W$
C. $\forall x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$ invalid
D. $\exists x(p(x) \rightarrow W) \equiv \forall x\ p(x) \rightarrow W$ valid

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Null quantification

In the following examples, P represents any wff in which x does not occur free.

$$\forall x\, P \Leftrightarrow P$$

$$\exists x\, P \Leftrightarrow P$$

$$\forall x\, (P \vee Q(x)) \Leftrightarrow P \vee \forall x\, Q(x)$$

$$\exists x\, (P \wedge Q(x)) \Leftrightarrow P \wedge \exists x\, Q(x)$$

Null quantification over $\rightarrow$

$$\forall x\,(P \rightarrow Q(x)) \quad \Leftrightarrow \quad P \rightarrow \forall x\, Q(x)$$

$$\exists x\,(P \rightarrow Q(x)) \quad \Leftrightarrow \quad P \rightarrow \exists x\, Q(x)$$

$$\forall x\,(Q(x) \rightarrow P) \quad \Leftrightarrow \quad \exists x\, Q(x) \rightarrow P$$

$$\exists x\,(Q(x) \rightarrow P) \quad \Leftrightarrow \quad \forall x\, Q(x) \rightarrow P$$

More non-equivalences to beware of

$$\forall x\,(Q(x) \rightarrow P) \quad \not\Leftrightarrow \quad \forall x\, Q(x) \rightarrow P$$

$$\exists x\,(Q(x) \rightarrow P) \quad \not\Leftrightarrow \quad \exists x\, Q(x) \rightarrow P$$

$$(1)\ \forall x\,(P(x) \land A(x)) \equiv \forall x\, P(x) \land \forall x\, A(x)$$

$$(2)\ \forall x\,(P(x) \lor A(x)) \not\equiv \forall x\, P(x) \lor \forall x\, A(x)$$

$$(3)\ \forall x\,(P(x) \land A) \equiv \forall x\, P(x) \land A$$

$$(4)\ \forall x\,(P(x) \lor A) \equiv \forall x\, P(x) \lor A$$

FOL is Complete ;

Optional

Argument in FOL ; 1 Q in GATE

Model in FOL ; 2 Q " "

Tautology " " ; 1 " " "

Upload Today

FOL : Valid $\neq$ Tautology

Fol GATE PYQs

Solve All

DM Test 1 in test Serise

GoClasses

Attempt

WQ 10

Next Topic:

Arguments in FOL

Some logical equivalences involving quantifiers

- $\neg\forall x A \equiv \exists x \neg A$.
- $\neg\exists x A \equiv \forall x \neg A$.
- $\forall x A \equiv \neg\exists x \neg A$.
- $\exists x A \equiv \neg\forall x \neg A$.
- $\exists x \exists y A \equiv \exists y \exists x A$.
- $\forall x \forall y A \equiv \forall y \forall x A$.

NB: $\forall x \exists y A \not\equiv \exists y \forall x A$. Why?

For instance, "For every integer x there is an integer y such that $x < y$" is true, but it does not imply "There is an integer y such that for every integer x, $x < y$.", which is false.

1. $\forall x \ \forall y \ P(x,y) \equiv \forall y \ \forall x \ P(x,y)$
2. $\forall x \ \exists y \ P(x,y) \not\equiv \exists y \ \forall x \ P(x,y)$

Logical Equivalences Involving Quantifiers

Definition

Two statements S and T involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value regardless of the **interpretation**, i.e. regardless of

- the meaning that is attributed to each propositional function,
- the domain of discourse.

We denote $S \equiv T$.

Is $\forall x(P(x) \wedge Q(x))$ **logically equivalent** to $\forall x P(x) \wedge \forall x Q(x)$?

Is $\forall x(P(x) \vee Q(x))$ **logically equivalent** to $\forall x P(x) \vee \forall x Q(x)$?

- Prove that $\forall x(P(x) \vee Q(x))$ **is not logically equivalent** to $\forall x P(x) \vee \forall x Q(x)$.
- It is enough to give a counterexample to the assertion that they have the same truth value for all possible interpretations.
- Under the following interpretation:
  domain: set of people in the world
  $P(x) =$ "x is male".
  $Q(x) =$ "x is female".

  We have:
  $\forall x(P(x) \vee Q(x))$ (every person is a male or a female) is true;
  while $\forall x P(x) \vee \forall x Q(x)$ (every person is a male or every person is a female) is false.

Distributing ∀ through ∧

$$\forall x\,(P(x) \wedge Q(x)) \quad \Leftrightarrow \quad \forall x\, P(x) \wedge \forall x\, Q(x)$$

Distributing ∃ through ∨

$$\exists x\,(P(x) \vee Q(x)) \quad \Leftrightarrow \quad \exists x\, P(x) \vee \exists x\, Q(x)$$

Non-equivalences to beware of

Beware of the following non-equivalences:

$$\forall x\,(P(x) \vee Q(x)) \quad \not\Leftrightarrow \quad \forall x\, P(x) \vee \forall x\, Q(x)$$

$$\exists x\,(P(x) \wedge Q(x)) \quad \not\Leftrightarrow \quad \exists x\, P(x) \wedge \exists x\, Q(x)$$

Notice that you **can** distribute ∀ through ∧, and you can distribute ∃ through ∨, but you **cannot** distribute ∀ through ∨ or ∃ through ∧. If you are in any doubt about these last two