Basic Proof Techniques

Homework 2

Proof by Contraposition(indirect proof)

Proof by Contradiction

Complete Discrete Mathematics Course Link:

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Q1. (Even/odd squares:)

Prove the following:

(a) Let n be an integer. If n^2 is odd, then n is odd.

(b) Let n be an integer. If n^2 is even, then n is even.

Solution.

Proof of (a): *We proof the statement in (a) by contraposition. Since the negation of "even" is "odd", the contrapositive of the statement (a) is is:*

(a)' *Let n be an integer. If n is even, then n^2 is even.*

Proof of (a)':

- *Assume n is an even integer.*
- *Then $n = 2k$ for some $k \in \mathbb{Z}$, by the definition of an even integer.*
- *Squaring both sides, we get $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$.*
- *Since k is an integer, so is $2k^2$.*
- *Hence $n^2 = 2p$ with $p = 2k^2 \in \mathbb{Z}$.*
- *Therefore n^2 is even, by the definition of an even integer.* ■

(b) can be proved in the same way as (a).

Q 2.

Consider the following statement: for every prime number p, either $p = 2$ or p is odd. We can rephrase this: for every prime number p, if $p \neq 2$, then p is odd. Now try to prove it.

Hint: Prove using contrapositive.

Ans 2.

Consider the following statement: for every prime number p, either $p = 2$ or p is odd. We can rephrase this: for every prime number p, if $p \neq 2$, then p is odd. Now try to prove it.

Solution.

Proof. Let p be an arbitrary prime number. Assume p is not odd. So p is divisible by 2. Since p is prime, it must have exactly two divisors, and it has 2 as a divisor, so p must be divisible by only 1 and 2. Therefore $p = 2$. This completes the proof (by contrapositive). ∎

Q3.

If x and y are integers and $x^2 + y^2$ is even, prove that x+y is even.

Hint : Prove by Contraposition.

Q4.

Prove the following proposition :

Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

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Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

Proof. Suppose for the sake of contradiction there exist $a, b \in \mathbb{Z}$ with $a \geq 2$, and for which it is not true that $a \nmid b$ or $a \nmid (b+1)$.
By DeMorgan's Law, we have $a \mid b$ and $a \mid (b+1)$.
The definition of divisibility says there are $c, d \in \mathbb{Z}$ with $b = ac$ and $b+1 = ad$.
Subtracting one equation from the other gives $ad - ac = 1$, or $a(d-c) = 1$.
Since a is positive, $d-c$ is also positive (otherwise $a(d-c)$ would be negative).
Then $d-c$ is a positive integer and $a(d-c) = 1$, so $a = 1/(d-c) < 2$.
Thus we have $a \geq 2$ and $a < 2$, a contradiction. ■

Q 5. Prove the following propositions :

Prove that $\sqrt[3]{2}$ is irrational.

Prove that $\sqrt{6}$ is irrational.

Hint: Just like we proved that sqroot(2) is irrational in the lecture 2, these two can be proved in exactly similar way.

Q 6. Prove the following propositions :

$$\text{If } a, b \in \mathbb{Z}, \text{ then } a^2 - 4b - 2 \neq 0.$$

$$\text{If } a, b \in \mathbb{Z}, \text{ then } a^2 - 4b - 3 \neq 0.$$

Video Solution: https://youtu.be/69xn4b1I8LA

$a^2 - 4b \neq 3$ can be proved in exactly similar way.

Q7. Prove the following proposition :

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.

Hint : Proof by Contraposition OR Proof by Contradiction.

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

This result can be re-phrased as following :

Cool application I: Sums of odd perfect squares. Can a sum of two perfect squares be another perfect square? Sure; for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, no matter how much you try, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

Prove that a sum of two odd perfect squares is never a perfect square.

(An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

Q 8. Prove the following proposition :

Suppose $a,b \in R$. If a is rational and ab is irrational, then b is irrational.

Hint : Proof by Contradiction.

Q9. Prove the following propositions :

There exist no integers a and b for which $21a + 30b = 1$.

There exist no integers a and b for which $18a + 6b = 1$.

Hint : Proof by Contradiction.

Q10. Prove the following proposition :

$$\text{If } b \in \mathbb{Z} \text{ and } b \nmid k \text{ for every } k \in \mathbb{N}, \text{ then } b = 0.$$

Hint : Proof by Contradiction.

Q 11. Prove the following proposition :

If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Hint : Proof by Contradiction.

Q 12. Prove the following proposition :

$$\text{For every } n \in \mathbb{Z},\ 4 \nmid (n^2 + 2).$$

Hint : Proof by Contradiction.

Q13. Prove the following proposition :

Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

Hint : Proof by Contradiction.

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