Partial Orders/

Partial order Relation (POR)

We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary.

We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins.

We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

Set = { TOC, DM, CD, DL, CO } = S

Unordered

new Aspirant m

m

In which order to study?

Go Classes

Try to put "ordering" on this set S.

To put order on S, we define a Relation R "Pre-requisite" on S.

xRy iff x is Pre-requisite of y.

DM R TOC   DL R CO

TOC R CD

DM R CD

mo
x

DM R TOC R CD

$$\left\{\begin{array}{l} DM \rightarrow TOC \rightarrow CD \\ DL \rightarrow CO \end{array}\right\}$$ Partial order

(Not Total order)

DM → TOC → CD
DL → CO
→ TO 3

DM → DL → TOC → CO → CD ✓
DL → CO → DM → TOC → CD ✓
DL → DM → CD → CO → TOC ✗

P Q

Child "a" $A = \{12, 11, 1, 2, 9, 3, 5, 6, 4, 8 \ldots\}$

Parents → School → Classes

Define a relation R on A.

xRy iff x to be done before y.

R gives Total order on A.

$1 \to 2 \to 3 \to 4 \to 5 \to \text{--} \to 12$

Total order

Set A → Unordered

To get Order on elements of A then we Define a Relation on A.

which properties this Relation should satisfy ?

"Order" of elements

~~1 Symmetric~~ :

TOC → CD
CD → TOC

mess

2 Antisym : b/w Diff elements a, b
we Don't want

a → b
b → a

Sit-
-uation

③ Transitive :

DM → TOC → CD

④ Reflexive :

Why we want "Reflexive" property?

$\{Apti, TOC, CN\}$

GO — pre-req relation

new Aspirant

$\{TOC, Apti, CN\}$

$Apti \not R TOC$

$TOC \not R Apti$

$TOC \not R CN$

$CN \not R TOC$

$Apti \not R CN$

$CN \not R Apti$

Partial order Relation :

- Ref ✓
- Antisym ✓
- Transitive ✓

RAT Properties

Partial Orders

- Many relations are equivalence relations:

$$x = y \qquad x \equiv_k y \qquad u \leftrightarrow v$$

- What about these sorts of relations?

$$x \leq y \qquad x \subseteq y$$

- These relations are called ***partial orders***, and we'll explore their properties next.

Properties of Partial Orders

$$x \leq y$$

$$x \leq y \quad \textbf{and} \quad y \leq z$$

$$x \leq z$$

Transitivity

Properties of Partial Orders

$$x \leq y$$

$$x \leq x$$

Reflexivity

Antisymmetry

- A binary relation R over a set A is called ***antisymmetric*** if the following is true:

$$\forall a \in A.\ \forall b \in A.\ (a \neq b \land aRb \rightarrow \neg(bRa))$$

("If a is related to b and a ≠ b, then b is not related back to a.")

- Equivalently:

$$\forall a \in A.\ \forall b \in A.\ (aRb \land bRa \rightarrow a = b)$$

("If a is related to b and b is related back to a, then a = b.")

Partial Orderings

Definitions

A relation R that is reflexive, antisymmetric, and transitive on a set S is called a **partial ordering** on S.

A set S together with a partial ordering R is called a **partially ordered set** or **poset**.

Partial Orders

Definition: A relation R on a set A is a ***partial order*** (or ***partial ordering***) for A if R is *reflexive, antisymmetric* and *transitive*.
A set A with a partial order is called a ***partially ordered set***, or ***poset.***

Examples:
The natural ordering " $\leq$ "on the set of real numbers $\mathbb{R}$.

For any set A, the subset relation $\subseteq$ defined on the power set P (A).

Integer division on the set of natural numbers $\mathbb{N}$.

If R on set A is POR then

(A, R) is called poset.

(Base set, POR) is called poset.

POR / PO

Some Standard POR :

① $\leq$ Relation :

On $\mathbb{N}$, $\leq$ Relation

$x R y$ iff $x \leq y$.

- Refl ✓ $a \leq a$
- Antisym ✓ $a \leq b$ and $b \leq a$ then $a = b$
- Trans ✓ $a \leq b \leq c$

$(N, \leq)$ Poset

"$\leq$" is a POR on N.

"$\leq$" " " " " Z.

"$\leq$" " " " " R.

Set $A = \{1, 2, 3, 4\}$

R on A

"$\leq$" Relation

xRy iff $x \leq y$

$1R2$ $1 \leq 2$

$2R3$ $2 \leq 3$

$3R4$ $3 \leq 4$

$1R3$ $1 \leq 3$

$1R4$ $1 \leq 4$

Ref

Antisym

Trans

POR

for any $a, b \in A$

aRb OR bRa.

$2, 4 \in A$

$2R4$ ✓

$2 \leq 4$

$A = \{1, 2, 3, 4\}$

Relation $R =$ "$\leq$"

$\boxed{1 \leq 2 \leq 3 \leq 4}$

ordering of A

(2) $\geq$ → Partial order relation

Ref, Antisym, Trans

$A = \{1, 2, 3, 4\}$

R on A ; xRy iff $x \geq y$

$2R1$; $3R2$; $4R3$;

$1 \cancel{R} 2$ $3R1$, $4R1$, $4R2$

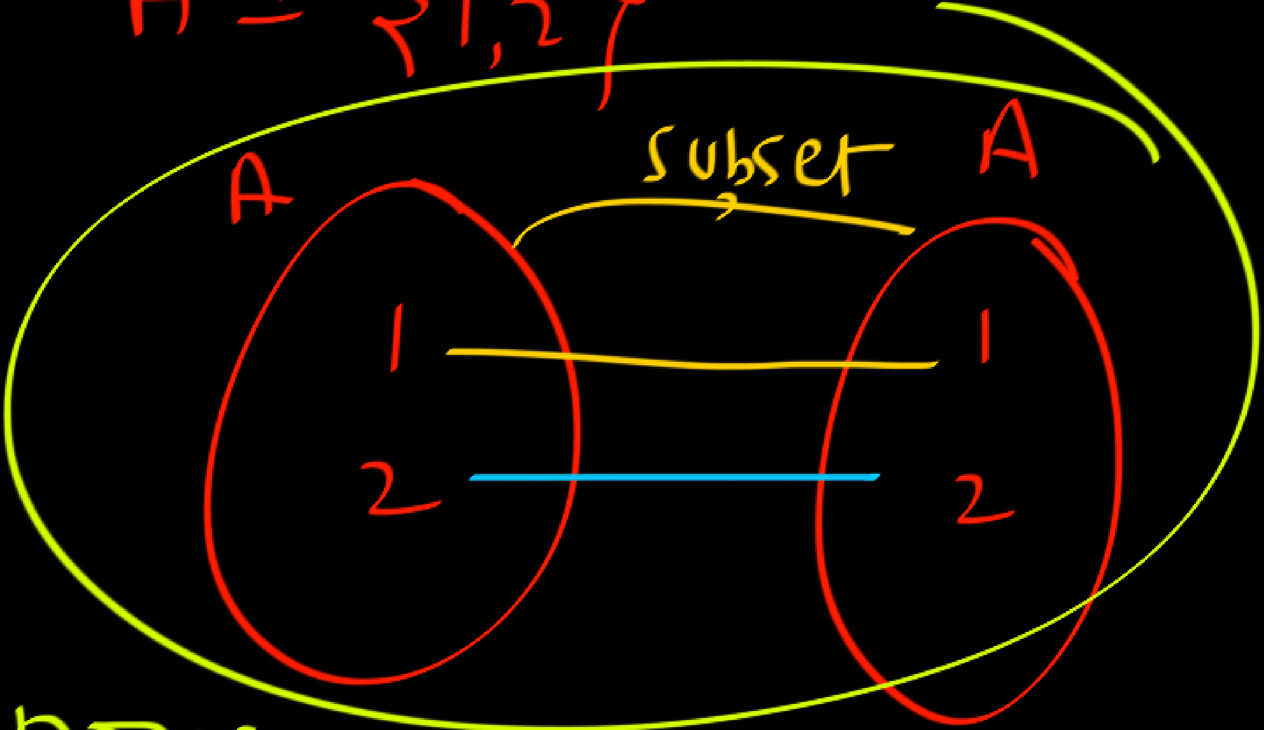
ordering

$4 \geq 3 \geq 2 \geq 1$

Q: on $\{1, 2\}$, Can we Define Subset Relation?

→ NO

$A = \{1,2\}$



nonsense

$\cancel{1 \subseteq 1}$

$\cancel{2 \subseteq 2}$

Q: $S = \{ \text{set}, \text{set}, \text{set}, \text{set} \}$

On S, "$\subseteq$" Relation.

$\text{set} \subseteq \text{set}$

$\in S \qquad \in S$

Q:

$S = \{ \{1\}, \{2\} \}$

On S, R

xRy iff $x \subseteq y$

$|R| = ? = 2$ ✓

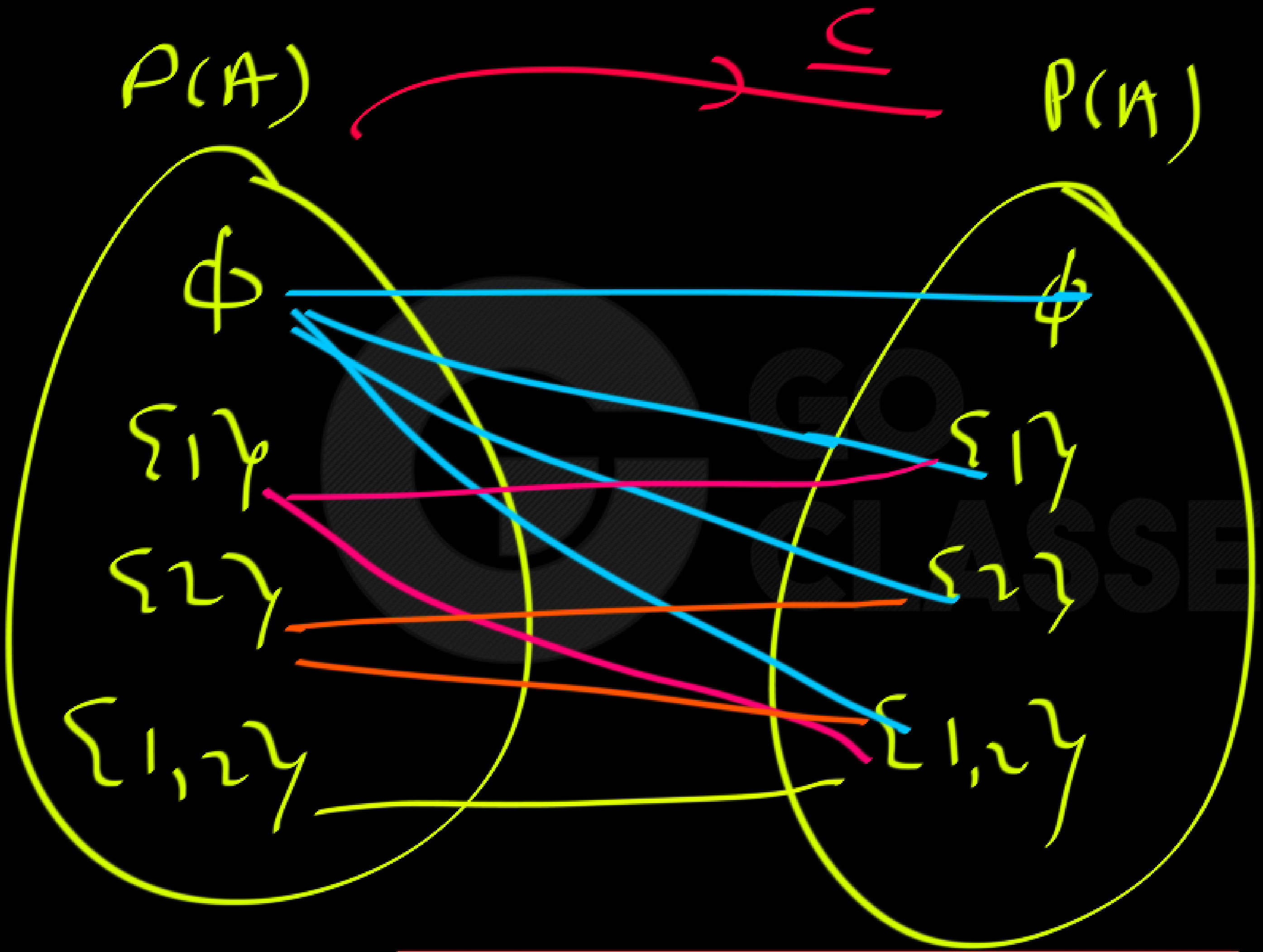
$R = \{ (\{1\}, \{1\}), (\{2\}, \{2\}) \}$

3 $\subseteq$ Relation on $P(A)$.

$A = \{1, 2\}$

$P(A) = \{ \phi, \{1\}, \{2\}, \{1,2\} \}$

R on $P(A)$; xRy iff $x \subseteq y$.



$\{1\} \subseteq \{1\}$

$\phi \subseteq \phi$

$\phi \subseteq \{1\}$

$\phi \subseteq \{2\}$

$\{1\} \subseteq \{1,2\}$

POR → Ref, Antisym, Trans

$(P(A), \subseteq)$ — Poset — POR

① Ref ✓ set $X \subseteq X$

② Antisym: $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$

③ Trans: $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$

(4) $\geq$ — POR (RAT)

(5) Divisibility Relation:

R on N, xRy iff $x \mid y$.

Divides

$x R y$ iff $x | y$ → POR ✓

① Refl ✓ $x | x$

② Antisym: If $x | y$ and $y | x$ then $x = y$

③ Trans: $x | y$, $y | z$ then $x | z$

$\leq$ $\geq$

$\subseteq$ $\supseteq$

Divides multiple

R on N, xRy iff x is multiple of y.

POR ✓

$2 \mid 4 = T$

$4 / 2 = 2$

operation

$4 \mid 2 = F$

Divides

Relation

A binary relation on a set S is called an order relation if it satisfies the following three conditions and then it is usually written $x \preceq y$ instead of $x\ R\ y$.

(i) (Reflexive) For all $s \in S$ we have $s \preceq s$.

(ii) (Antisymmetric) For all $s, t \in S$ such that $s \neq t$, if $s \preceq t$ then $t \not\preceq s$.

(iii) (Transitive) For all $r, s, t \in S$ such that $r \preceq s$ and $s \preceq t$ we have $r \preceq t$.

A set S together with an order relation $\preceq$ is called a *partially ordered set* or *poset*. Formally, a poset is a pair $(S, \preceq)$. We shall, once the binary relation is defined, refer to the poset by the set S alone, not the pair.

If we use the alternative notation R for the relation, then the three conditions for an order relation are written as follows.

(i) For all $s \in S$ we have $(s, s) \in R$.

(ii) For all $s, t \in S$ such that $s \neq t$, if $(s, t) \in R$ then $(t, s) \notin R$.

(iii) For all $r, s, t \in S$ such that $(r, s) \in R$ and $(s, t) \in R$ we have $(r, t) \in R$.

Examples

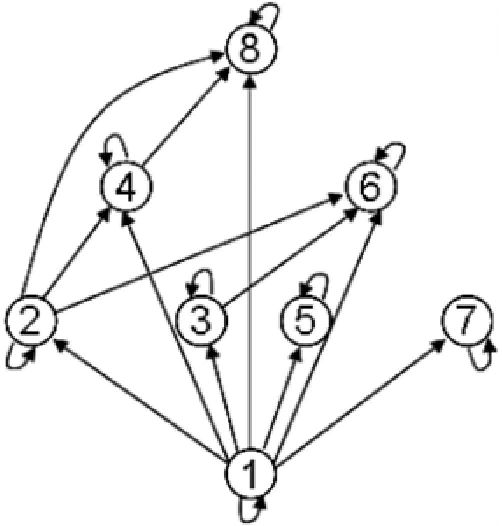
1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \preceq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\preceq = \subseteq$.

Partial Orders

- A binary relation R is a ***partial order*** over a set A if it is
  - ***reflexive***,
  - ***antisymmetric***, and
  - ***transitive***.

Why "partial"?

As a small example, let S = {1, 2, 3, 4, 5, 6, 7, 8}, and let R be the binary relation "divides." So (2,4) ∈ R, (2, 6) ∈ R, etc. Using | as the symbol for "divides", we see that R is reflexive, since x | x; R is transitive since x | y and y | z implies x | z; and R is antisymmetric since we never have x | y and y | x where x ≠ y. Thus S and R form a poset. The following graph represents R:



The loops on each vertex show reflexivity; the arrows between nodes like 2, 4, and 8 show transitivity; and that fact that no two nodes have arrows to and from each other shows antisymmetry.