Partial Order Relations

Next Topic:

Lattices

Website : https://www.goclasses.in/

Lattices

Definition

A lattice is a poset $\langle A, \leq \rangle$ in which any two elements a, b have an LUB(a, b) and a GLB(a, b).

We write $a \cup b = \text{LUB}(a, b)$ and $a \cap b = \text{GLB}(a, b)$. We also call them join and meet, respectively.

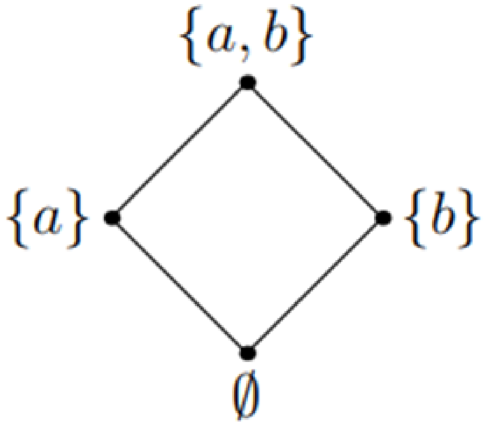
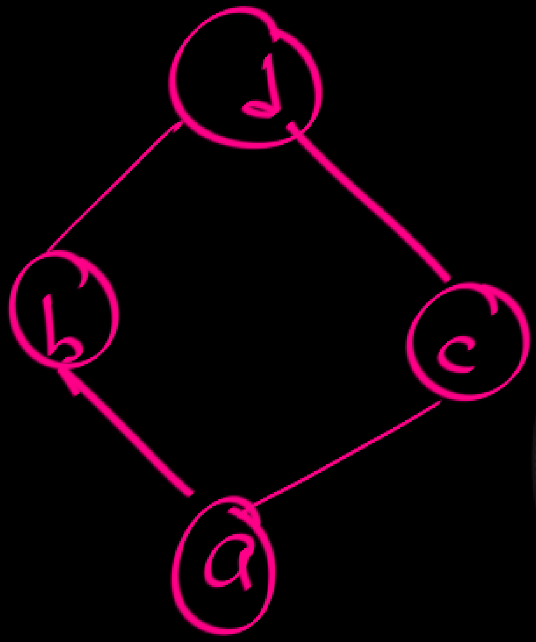


Figure: Hasse diagram of $\langle \text{pow}(\{a, b\}), \subseteq \rangle$

Lattice: a poset in which Every Pair of elements have GLB, LUB.

$\forall a, b$

$$\left.\begin{array}{l} a \vee b \text{ — exists} \\ a \wedge b \text{ — "} \end{array}\right\} \text{lattice}$$



$$a \vee a = a, \quad a \wedge a = a$$

$$a \vee d = d, \quad a \wedge d = a$$

$$a \vee b = b, \quad a \wedge b = a$$

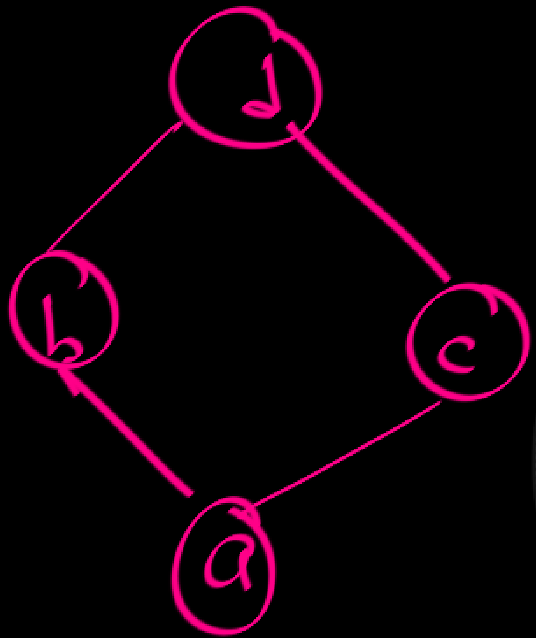
$$b \vee c = d, \quad b \wedge c = a$$

$$a \vee c = c, \quad a \wedge c = a$$

$$b \vee b = b, \quad b \wedge b = b$$

HD

Poset



$\cancel{\left.\begin{array}{l} a \vee a = a \\ a \wedge a = a \end{array}\right\}}$ $\cancel{\left.\begin{array}{l} a \vee d = d \\ a \wedge d = a \end{array}\right\}}$

$\cancel{\left.\begin{array}{l} a \vee b = b \\ a \wedge b = a \end{array}\right\}}$ $\left.\begin{array}{l} b \vee c = d \\ b \wedge c = a \end{array}\right\}$

$\cancel{\left.\begin{array}{l} a \vee c = c \\ a \wedge c = a \end{array}\right\}}$ $\cancel{\left.\begin{array}{l} b \vee b = b \\ b \wedge b = b \end{array}\right\}}$

HD

Poset

_ _ _ _ _ _

$a \vee a = a$

$a \wedge a = a$

If aRb then $a \vee b = b$

$a \wedge b = a$

Always,

In EVERY Poset,

In Every Hasse Diag

If a,b are Comparable then

Definitely $a \vee b$, $a \wedge b$ Exists.

If aRb then $a \vee b = b$, $a \wedge b = a$

OR

If bRa then $a \vee b = a$, $a \wedge b = b$

Q: To check if a given POSET (of n elements) is lattice or not, we have to check for EVERY Pair of elements to see if every pair of elements has LUB, GLB. It will take a lot of time as number of pairs of elements is n^2. Can we do better than this??

Q: To check if a given POSET (of n elements) is lattice or not, we have to check for EVERY Pair of elements to see if every pair of elements has LUB, GLB. It will take a lot of time as number of pairs of elements is n^2. Can we do better than this??

Ans: Check only pairs of incomparable elements.

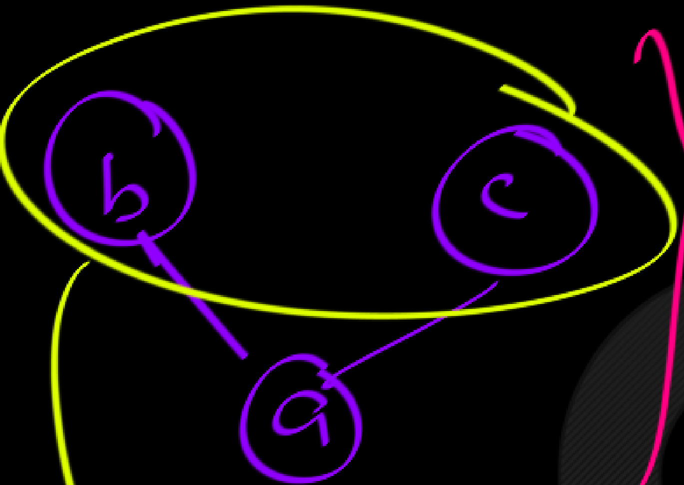
Because Pair of Comparable elements ALWAYS have GLB, LUB in every POSET. So, there is No need of checking for them.

"Lattice":

A Poset in which "Every two" elements have LUB, GLB.

Lattice: a Poset in which Every two Non-Comparable elements have GLB, LUB.

(i.e. GLB, LUB exists for Every two Non-Comparable elements)



$b \wedge c = a$

$b \vee c = \text{DNE}$

So, Not lattice

non-Comparable elements

To check for lattice :

① Don't check $\{a, a\}$ type of Pairs : $\underbrace{a \vee a = a \; ; \; a \wedge a = a}$

② Don't check Comparable elements

$a \wedge b = a \; ; \; a \vee b = b$

③ Just check Non-Comparable elements.

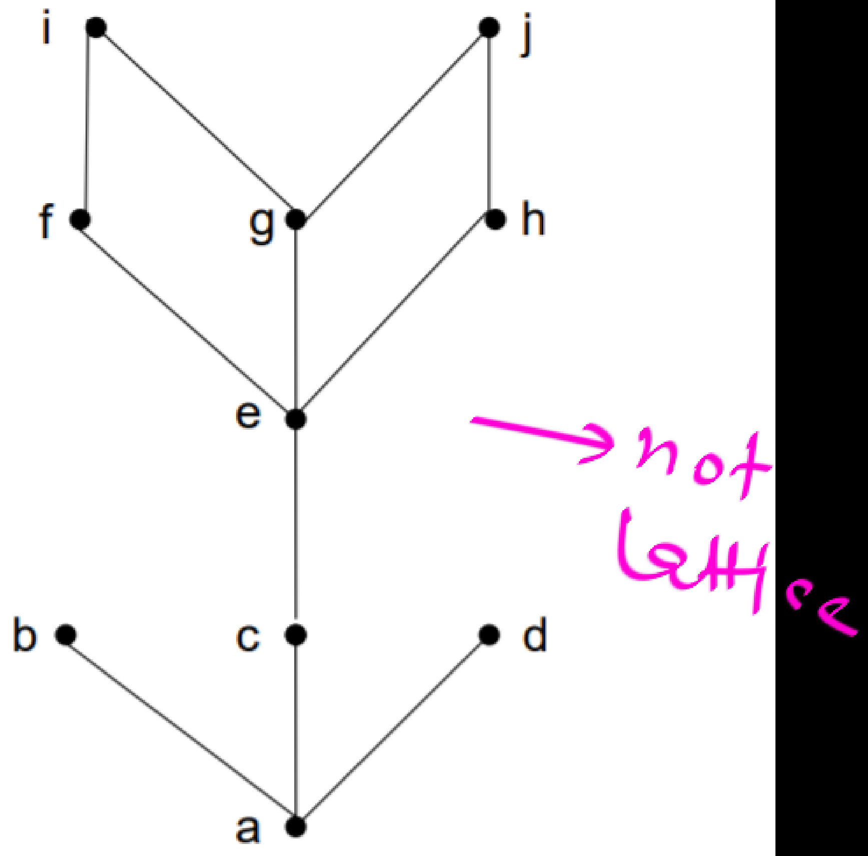
Lattices: Example 1

- Is the example from before a lattice?

$i \vee j = DNE$

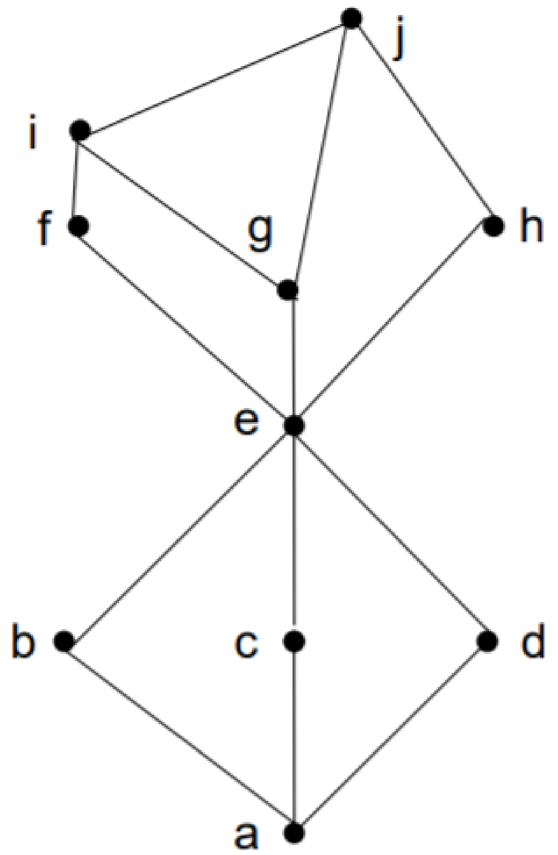
- **No, because the pair {b,c} does not have a least upper bound**

$b \vee d = DNE$



Lattices: Example 2

- What if we modified it as shown here?

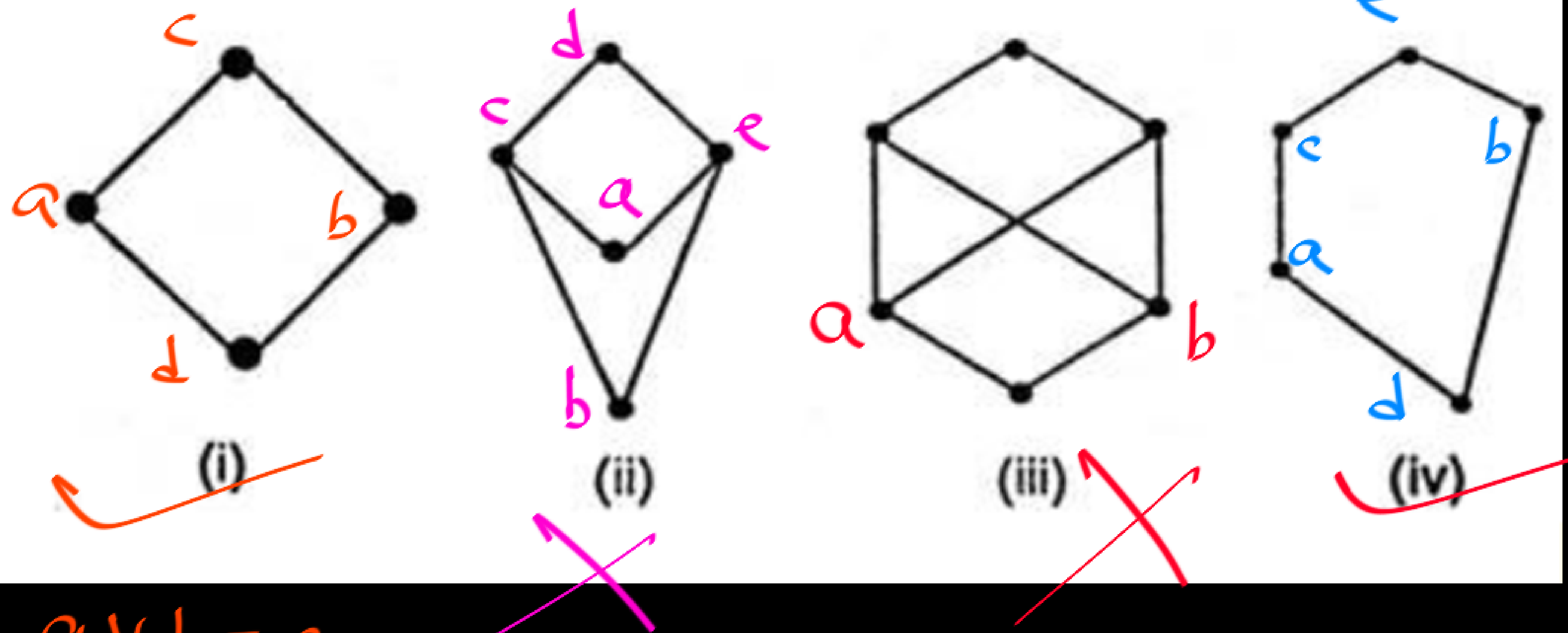


- **Yes, because for any pair, there is an lub & a glb**

lattice

b, c
b, d
c, d
g, h
i, h
⋮

Which is a Lattice?

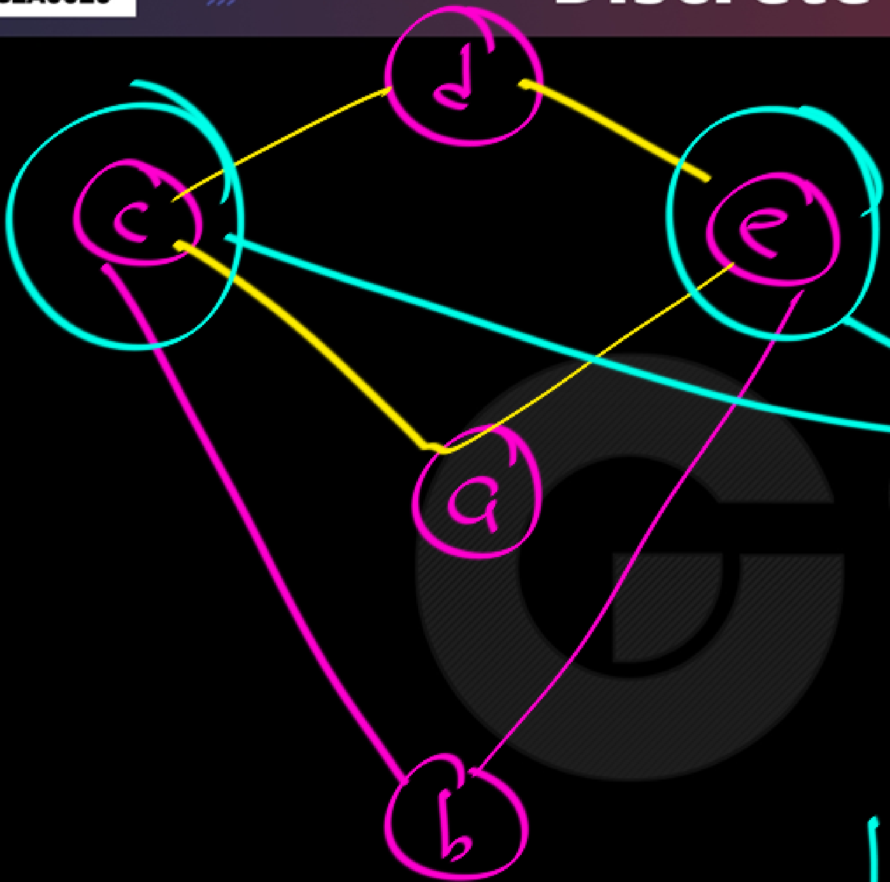


(i): $a \vee b = c$, $a \wedge b = d$

(ii): $a \vee b = \text{DNE}$

(iii): $a \vee b = \text{DNE}$

(iv): $a \vee b = e$, $a \wedge b = d$, $c \vee b = e$, $c \wedge b = d$

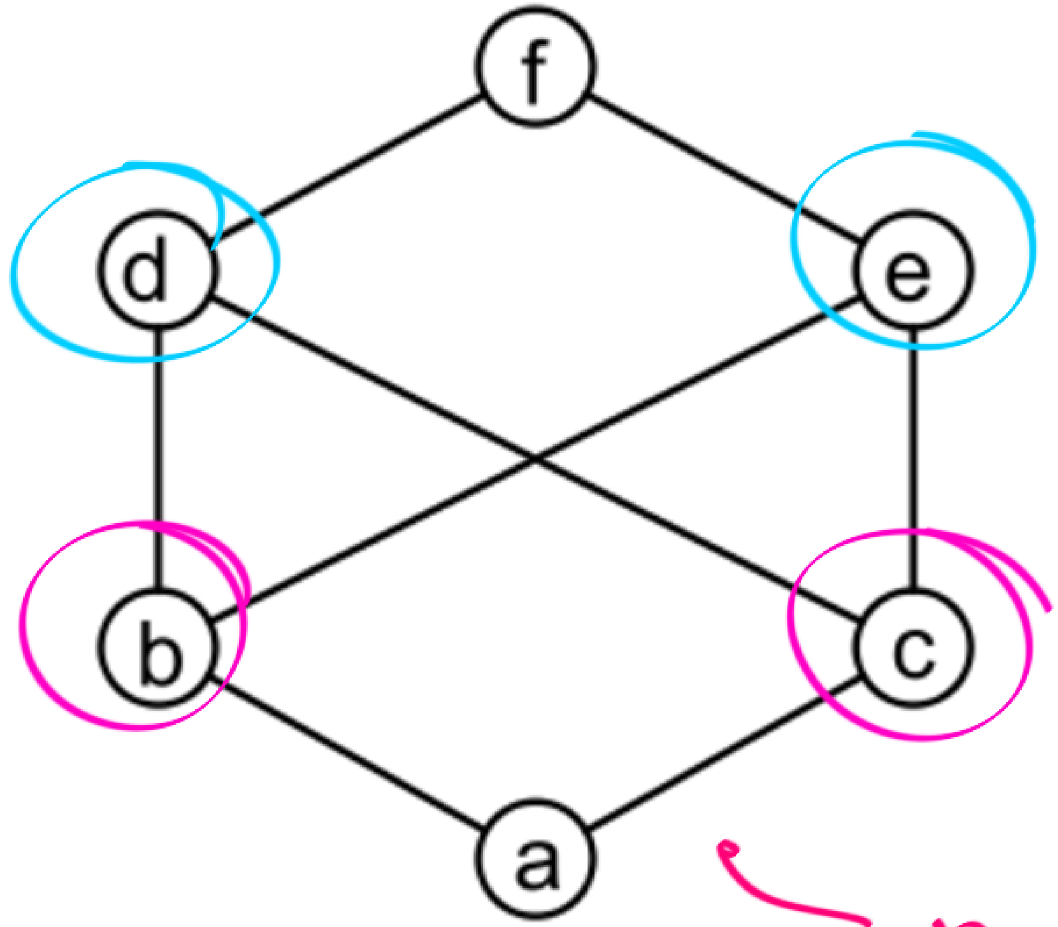


$b \vee a = DNE$

1st Joining points not unique

$b \wedge a = DNE$

$b \vee c =$ DNE

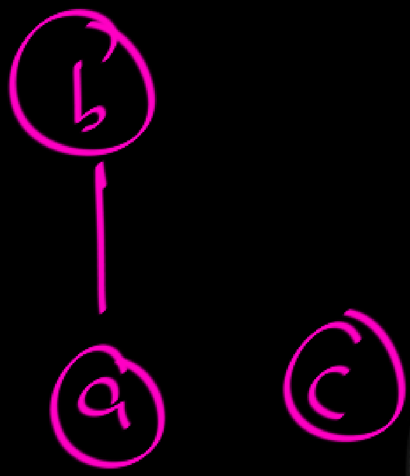


$d \wedge e$ = DNE

not Lattice

$|\text{Base set}| = 3$

$|\text{POR}| = 4$ ✓



$\text{POR} = \{(a,a), (b,b), (c,c), (a,b)\}$

$\neq$

HD

Representation

$\#\text{Edges} = 1$