Graph Theory

Graph Isomorphism, Components, Graph Complement

Q: The number of different labeled undirected graphs (no self loops and no parallel edges) on n vertices is ?

Q: The number of different labeled undirected graphs (no self loops and no parallel edges) on n vertices is ?

a_1 — — — a_n

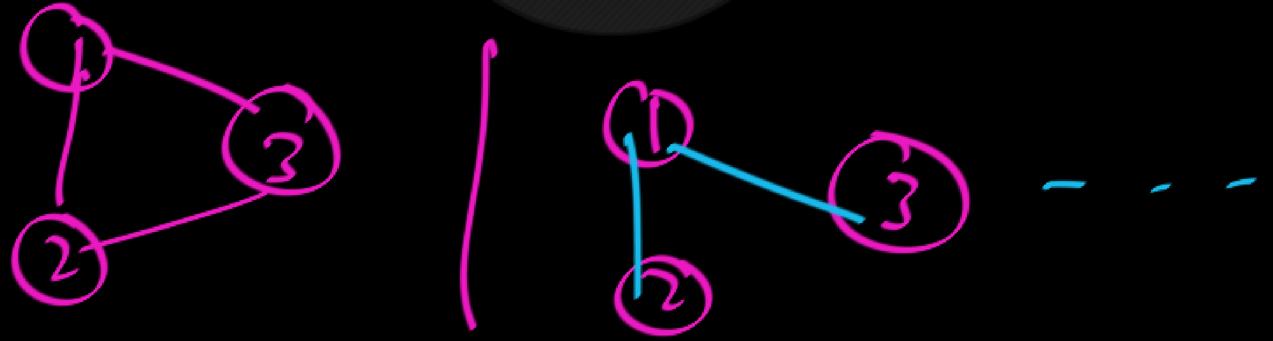
a_2 — — — —

for every Distinct Pair of vertices, we have 2 choices

$$\#\text{Graphs} = 2^{\binom{n}{2}} = 2^{{}^nC_2}$$

Put Edge
or
Don't Put Edge

$n=3 \longrightarrow$ max #edges possible $= {}^3C_2 = 3$

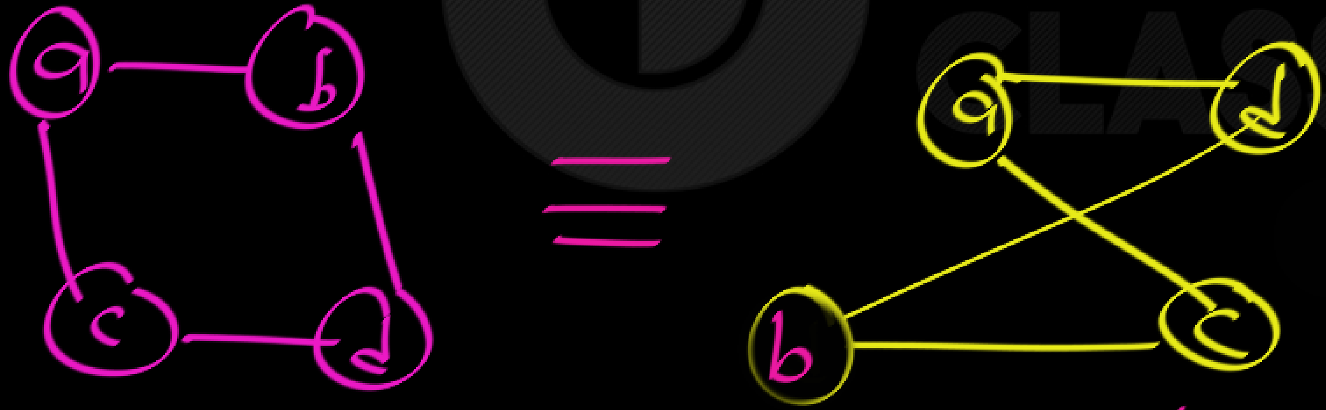


$$\begin{array}{ccccc} e_1 & & e_2 & & e_3 \\ \downarrow & & \downarrow & & \downarrow \\ 2 & \times & 2 & \times & 2 \\ \text{choices} & & \text{choices} & & \text{choi.} \end{array}$$

n vertices Simple Graph:

max. no. of Edges $= {}^{n}C_2 = \dfrac{n(n-1)}{2}$

number of Simple Graphs: $2^{\left({}^{n}C_2\right)}$



G

Twisted G

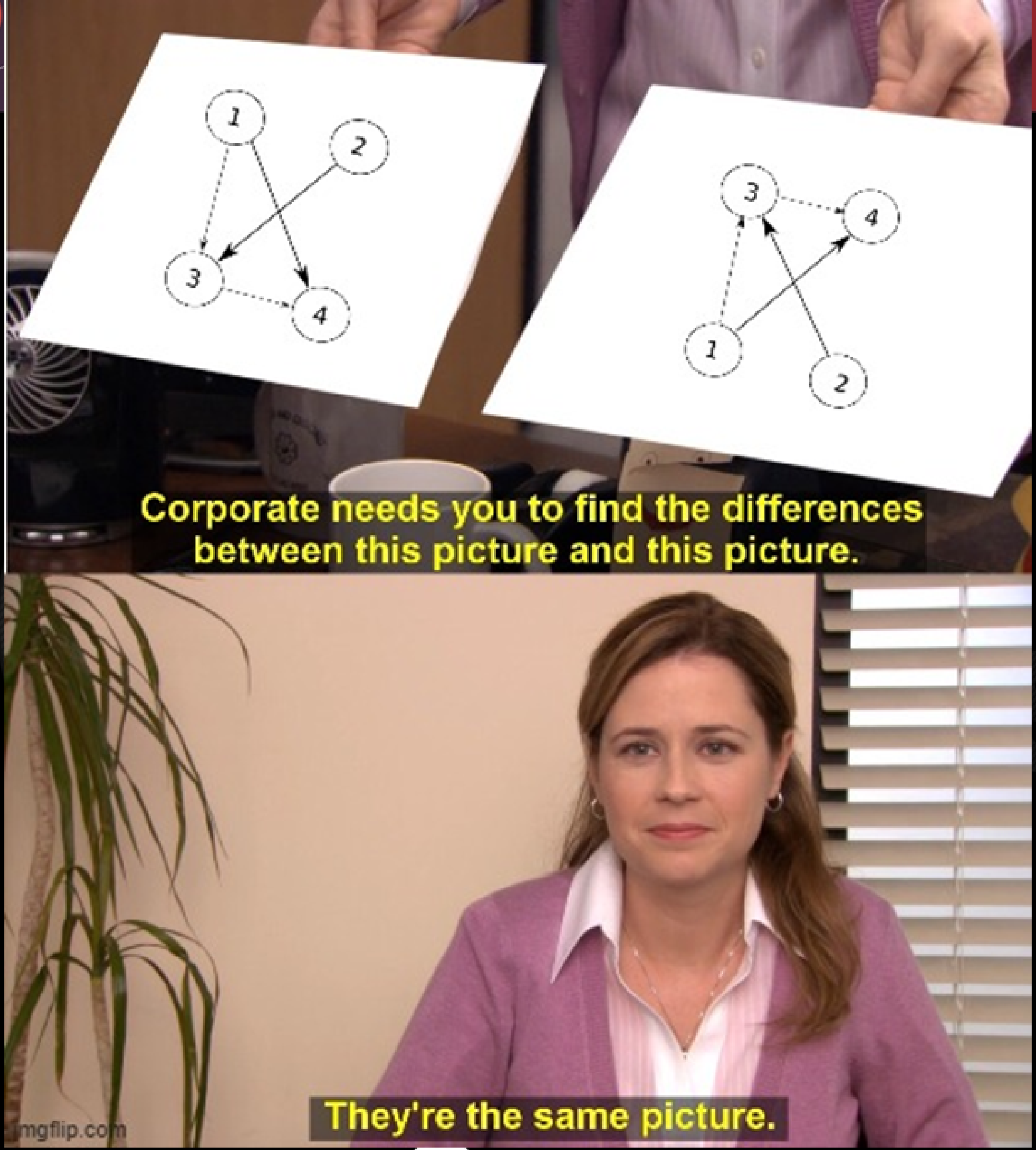
Graph Theory :

Next Topic :

Graph Isomorphism

Same form

Same Graphs in Abstract View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Corporate needs you to find the differences between this picture and this picture.
They're the same picture.
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Graph Isomorphism

Most properties of a graph do not depend on the particular names of the vertices. For example, although graphs A and B is Figure 10 are technically different (as their vertex sets are distinct), in some very important sense they are the "same"

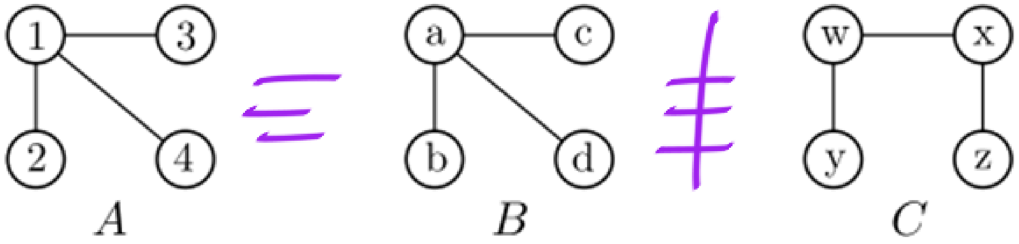


Figure 10: Two isomorphic graphs A and B and a non-isomorphic graph C; each have four vertices and three edges.

graph. For example, both graphs are connected, have four vertices and three edges. However, notice that graph C also has four vertices and three edges, and yet as a graph it seems different from the first two. Isomorphism is the idea that captures the kind of sameness that we recognize between A and B, and which distinguishes both of them from C.

Graph Isomorphism:

Labelling Doesn't matter.

Drawing " " .

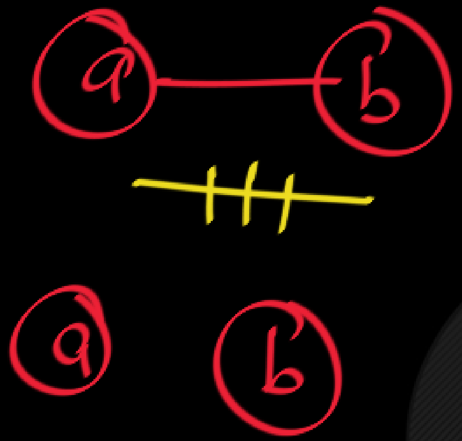
Twist " " .

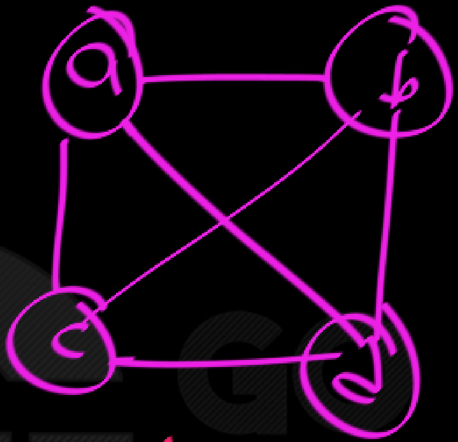
Shape " " .

Design " " .

G_1

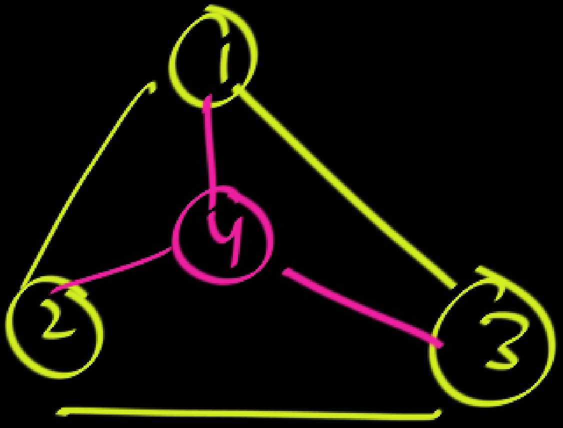
G_2



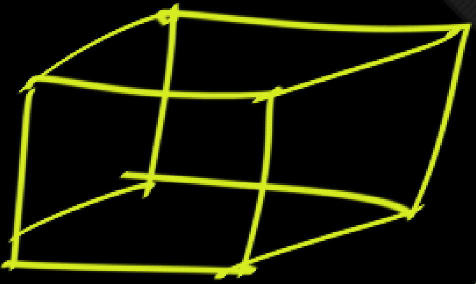


K_4

$\equiv$

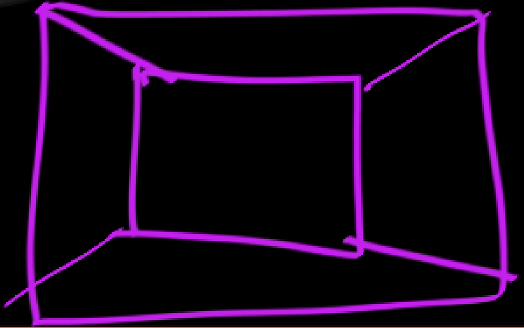


K_4



Q_3 (3D drawing)

$\equiv$



Q_3

(2D drawing)

Formal Definition: Idea:

South		Bollywood
Graph: Drishyam	=	Drishyam: Graph
Dialogue { Mohanlal	—	Ajay } Dialogue
wife	—	Shriya
* female police	—	Tabbu *

moview Analogy:

Graph $G(V_1, E_1)$

Graph $H(V_2, E_2)$

Drishayam (South)

Drishaya (Bolly)

Characters $= V_1$

Characters $= V_2$

Dialogue(a,b) = Edge(a,b)

Two movies Isomorphic iff

Character —— Bijection —— Character

a —— b

Should preserve Dialogues

$f(a) — f(b)$

c $\not—$ d

$f(c) \not— f(d)$

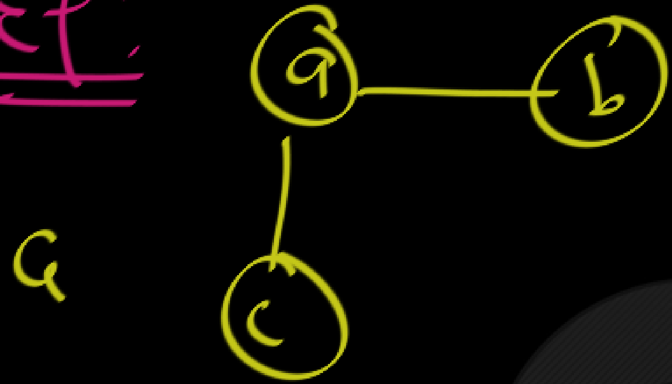
formal Definition: $G(V_1, E_1)$ $H(V_2, E_2)$

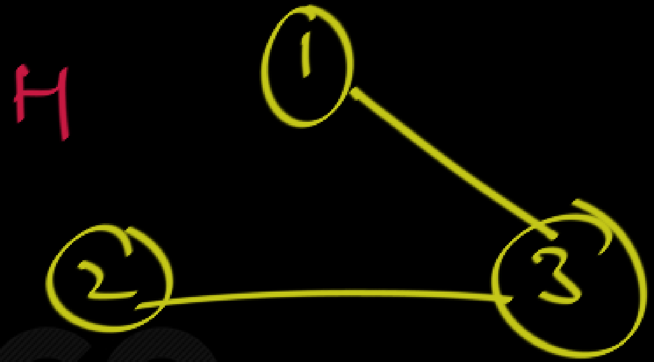
$G \equiv H$ iff "$\exists$ bijection" b/w V_1, V_2

which "preserves Edges"

i.e. $\underbrace{(a,b) \in E_1}$ iff $(f(a), f(b)) \in E_2$

Eg:

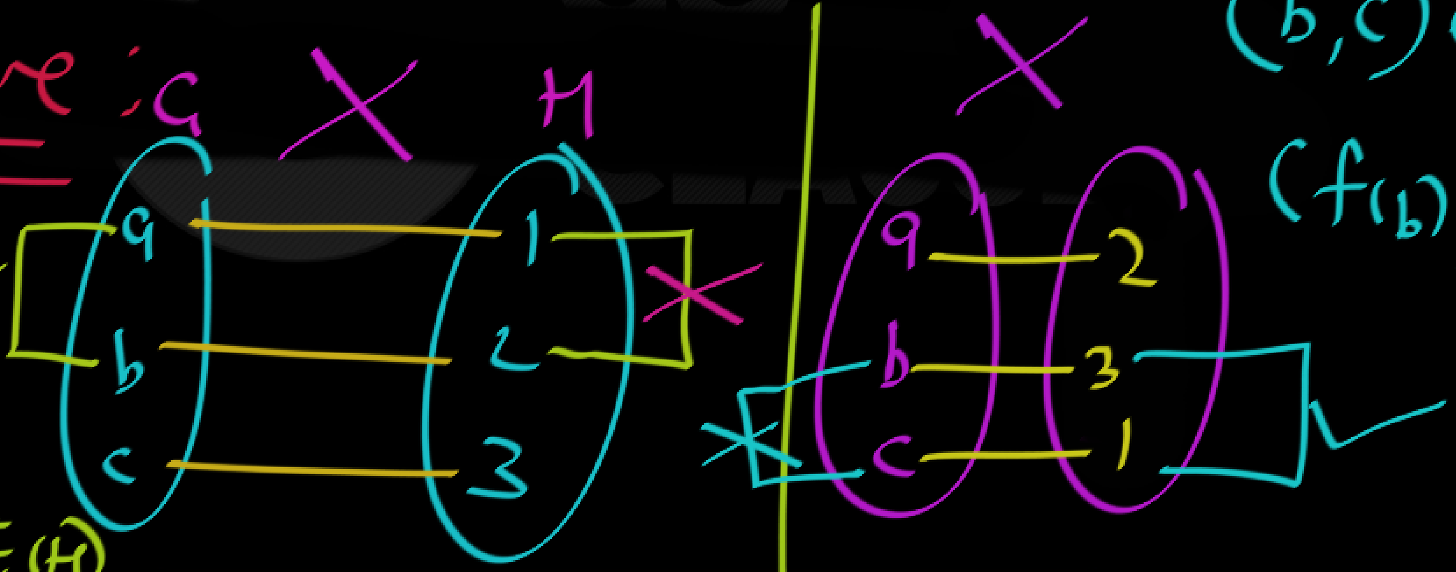




Prove they are Isomorphic?

Eg:

To prove: $G \not\cong H$



$(a,b) \in E(G)$

But

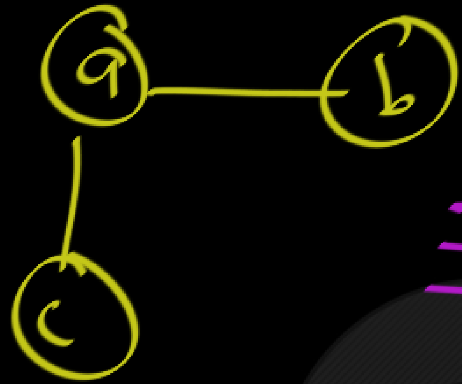
$(f(a), f(b)) \notin E(H)$

$(b,c) \notin E(G)$

$(f(b), f(c)) \in E(H)$

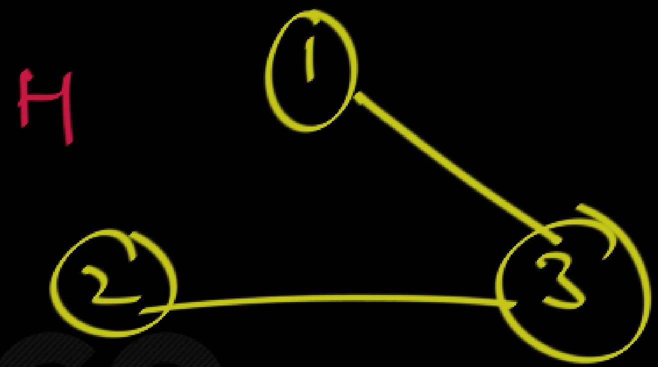
Eg:

G

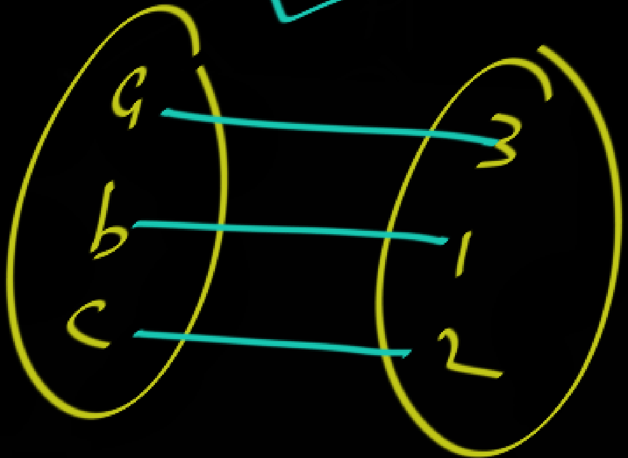


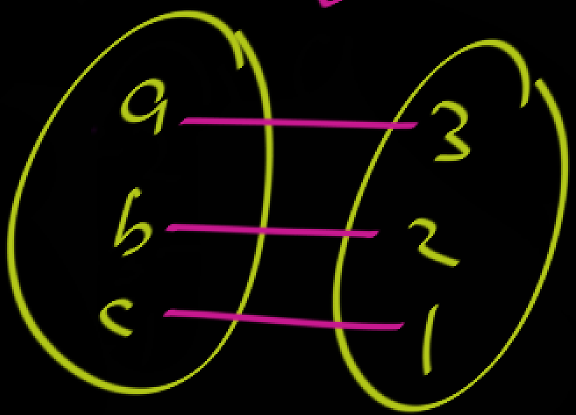
≡

H



To prove:

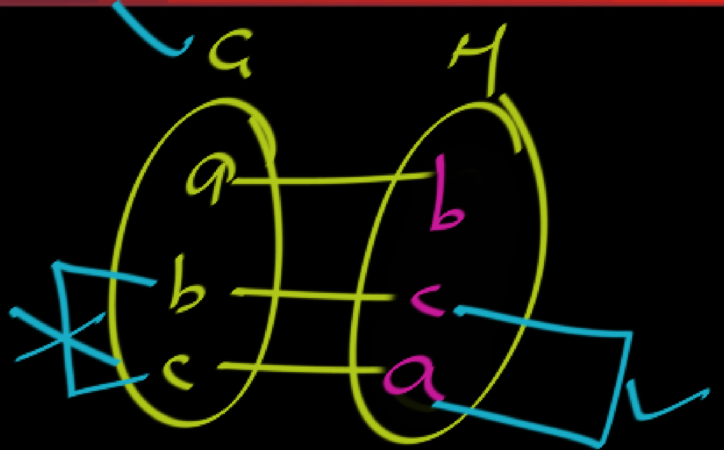




Q:

Prove they are NOT Isomorphic ?

Q:



3! bijections b/w V(G), V(H) But None of them will preserve Edges.

Q: Number of Bijections:

$|A| = 3$; $|B| = 2$

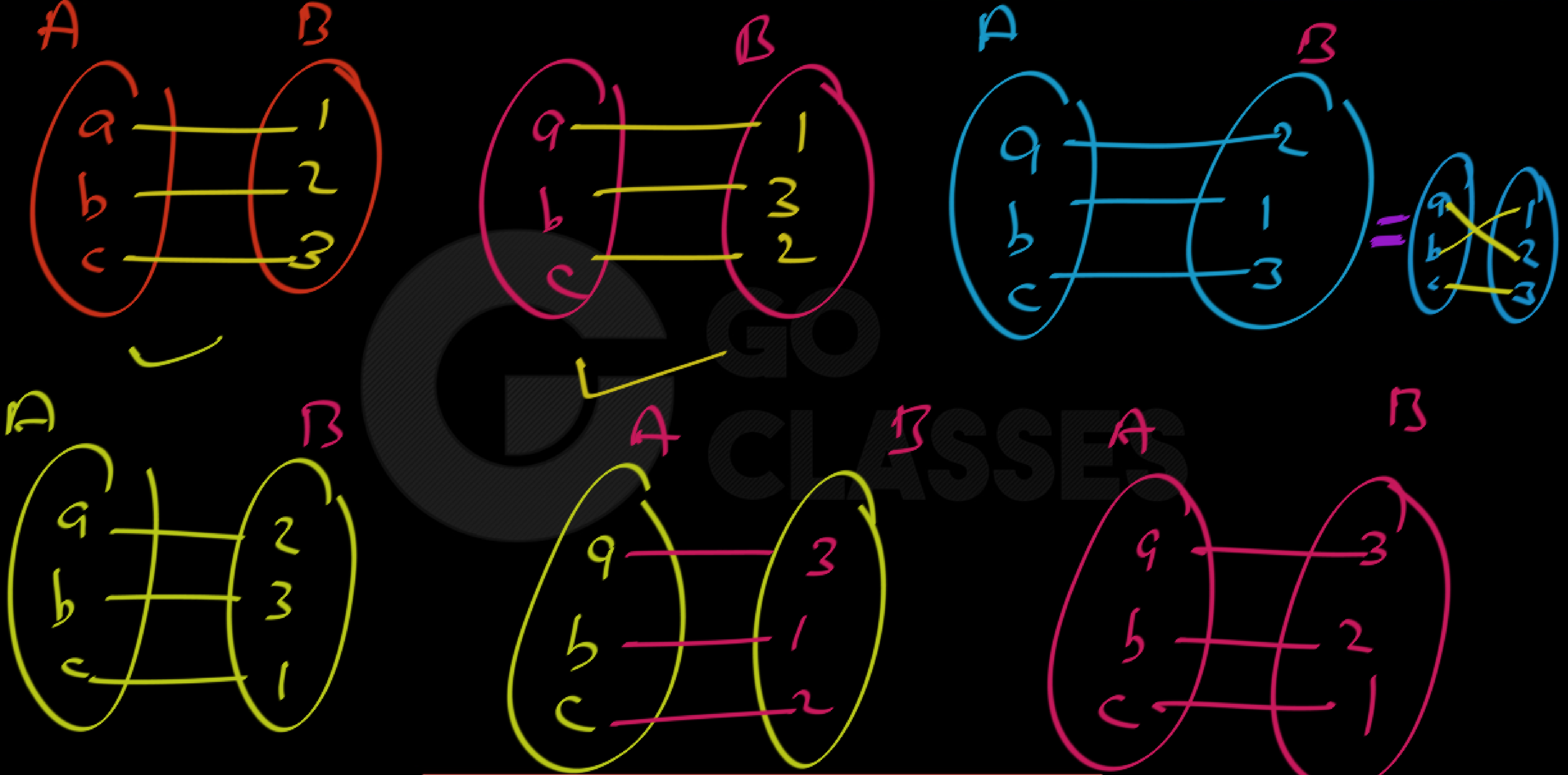
Bijections from $A \to B$? $= 0$

Bijection exists iff $|A| = |B|$.

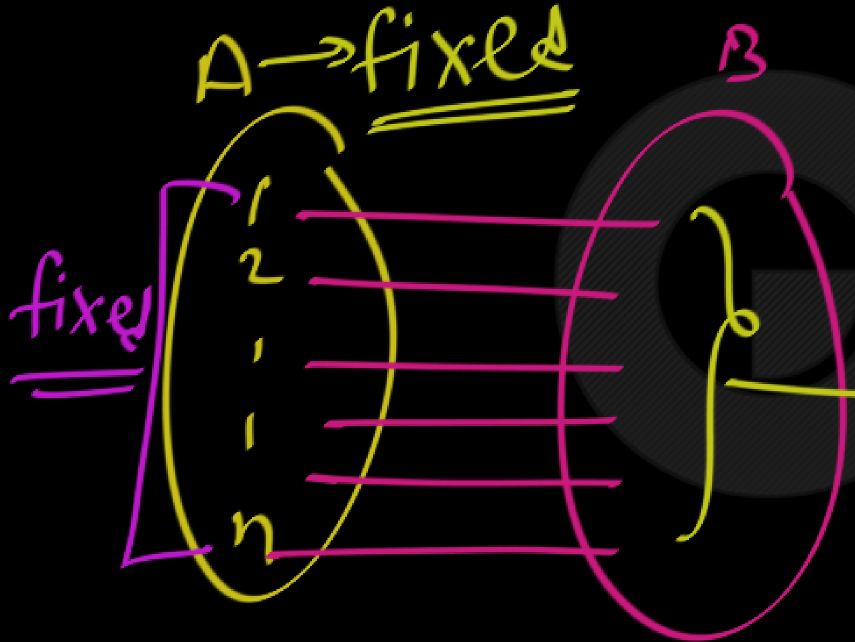
Q: Number of Bijections:

$|A| = 3 \quad ; \quad |B| = 3$

#Bijections $A \to B$?



Bijections $\underbrace{A \longrightarrow B}$; $\underbrace{|A| = |B| = n}$



Any permutation of B

$\#\text{Bijections} = n!$

Conclusion: Definition of Isomorphism:

$G(V_1, E_1)$ $\quad$ $H(V_2, E_2)$

$G \equiv H$ iff $\exists$ bijection $f : V_1 \longrightarrow V_2$

Such that $(a,b) \in E_1$ iff $(f(a), f(b)) \in E_2$

$\forall a,b$

Q: G = 10 vertices ; H = 10 vertices

To check Isomorphism if you Apply Definition: $10!$ bijections need to check.

NEVER apply Isomorphism Definition to check if two Graphs are Isomorphic.

Proving "Graph Isomorphism"

$\underbrace{\text{Proving "Graph Isomorphism"}}_{\text{problem}}$

$\boxed{\text{Very Hard}}$ $\left(\underbrace{\text{NP-Intermediate}}\right)$

Exponential Time Comp.

Showing "NOT Isomorphic" Can be Done using "Graph Invariants"

"Properties which should be Same in Isomorphic Graphs.

Graph Invariants $\longrightarrow$ Necessary BUT NOT sufficient.

1. Order of Graph $|V|$
2. Size of Graph $|E|$
3. Degree Sequence
4. Δ, δ
5. Number of 3-length cycles
6. Number of k-length cycles
7. Diameter
8. Connectedness

NOTE :

If Some Graph Invariant is NOT Same for two Graphs then they are NOT Iso-morphic.

NOTE:

previous

If ALL Graph Invariant are Same for two Graphs then It Does NOT mean they are Iso-morphic.

Definition 19. *Two graphs G_1 and G_2 are* **isomorphic** *if there exists a matching between their vertices so that two vertices are connected by an edge in G_1 if and only if corresponding vertices are connected by an edge in G_2.*

Definition 5. [ISOMORPHISM] Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a bijection $f : V_1 \to V_2$ that preserves the adjacency, i.e. $uv \in E_1$ if and only if $f(u)f(v) \in E_2$.

*The word *isomorphism* comes from the Greek roots *isos* for "equal" and *morphe* for "form."

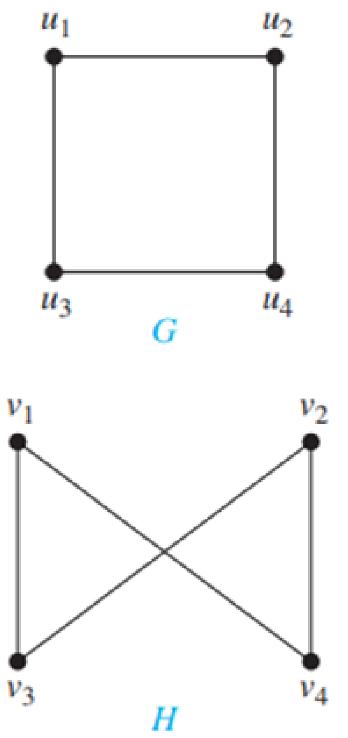
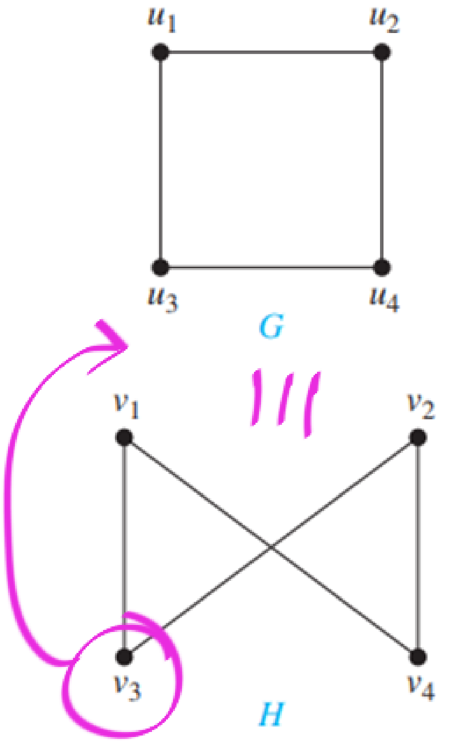


FIGURE 8 The Graphs G and H.



G

$\cong$

Twisted G

FIGURE 8 The Graphs G and H.

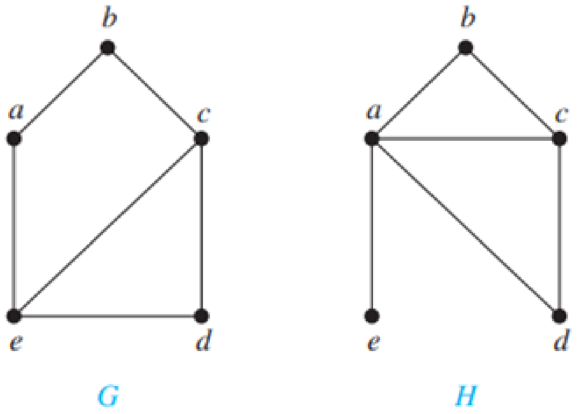


FIGURE 9 The Graphs G and H.

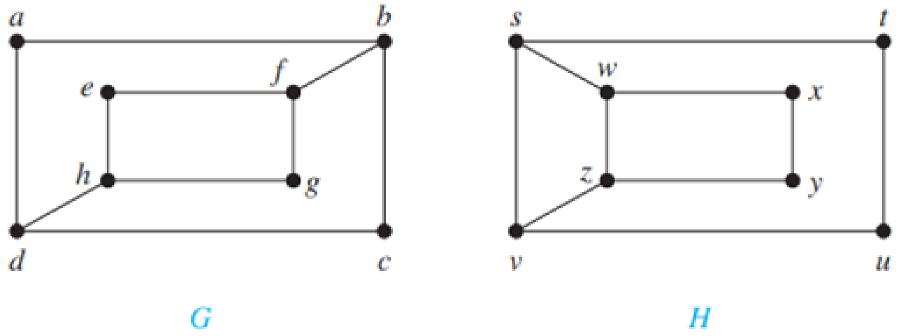
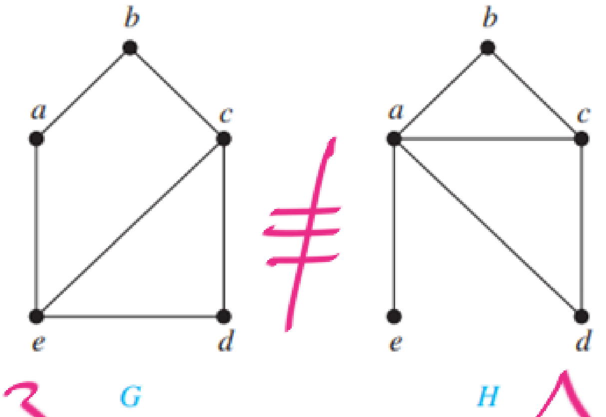


FIGURE 10 The Graphs G and H.



$\Delta = 3$ $\Delta = 4$ ✓

FIGURE 9 The Graphs G and H.

$H = 4, 3, 2, 2, 1$

$G = 3, 3, 2, 2, 2$

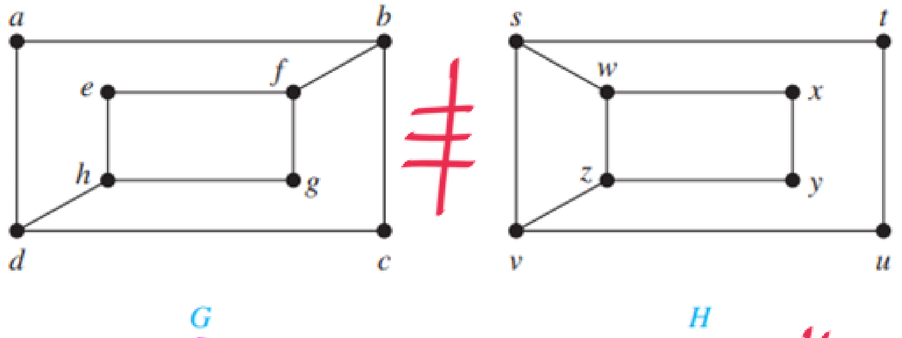


FIGURE 10 The Graphs G and H.

#4-length cycles = 2

#4-length cycles = 3

A *graph invariant* is a property of a graph that is preserved by isomorphisms. (If graphs G_1 and G_2 are isomorphic, and G_1 has some invariant property, then G_2 must have the same property.) Common examples of graph invariants are the number of edges, the number of vertices, the degree of a vertex, and there are many others.

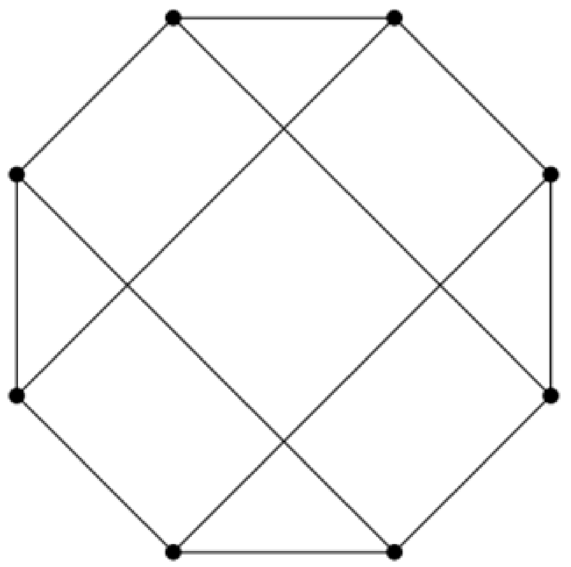
Definition
A property P is called an *isomorphic invariant* iff given any graphs G and G_1, if G has property P and G_1 is isomorphic to G, then G_1 has property P.

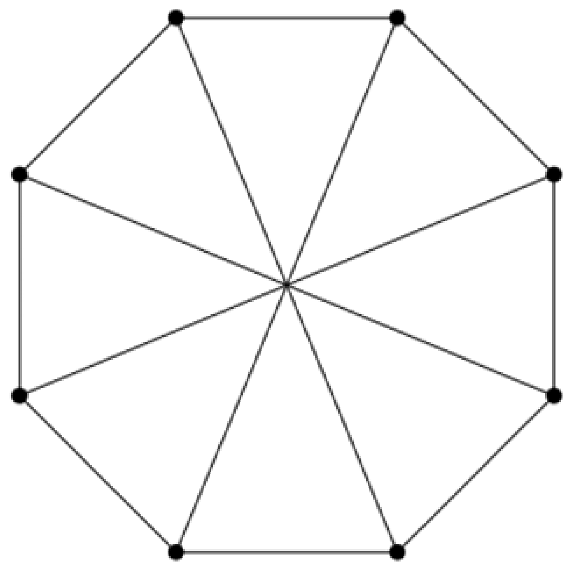
Theorem 11.4.1
Each of the following properties is an invariant for graph isomorphism, where n, m, and k are all nonnegative integers:

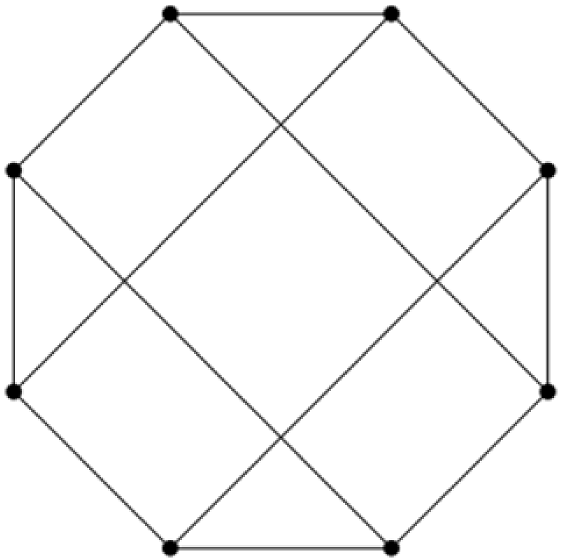
1. has n vertices;
2. has m edges;
3. has a vertex of degree k;
4. has m vertices of degree k;
5. has a circuit of length k;
6. has a simple circuit of length k;
7. has m simple circuits of length k;
8. is connected;
9. has an Euler circuit;
10. has a Hamiltonian circuit.

Simple circuit $\equiv$ Cycle

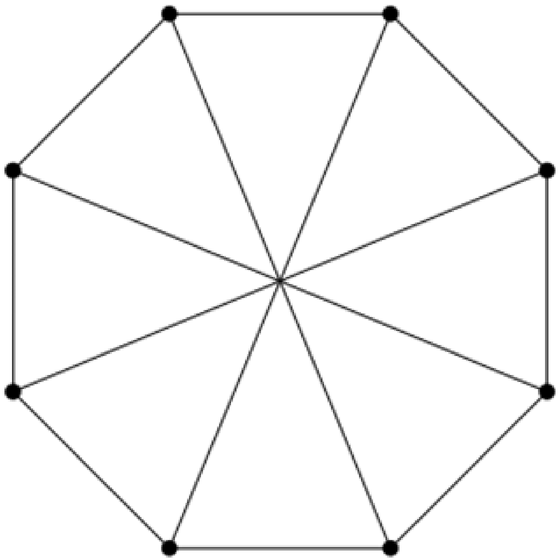
$\underbrace{\chi(G),\ \omega(G),\ \alpha(G), \ldots}_{\text{In future.}}$





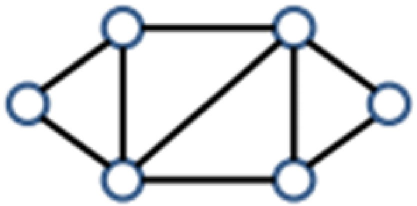


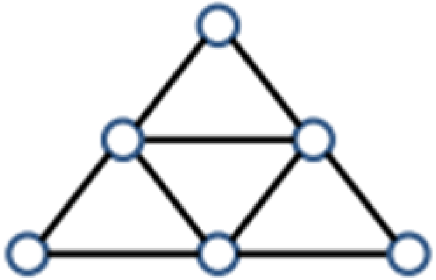
$\not\cong$

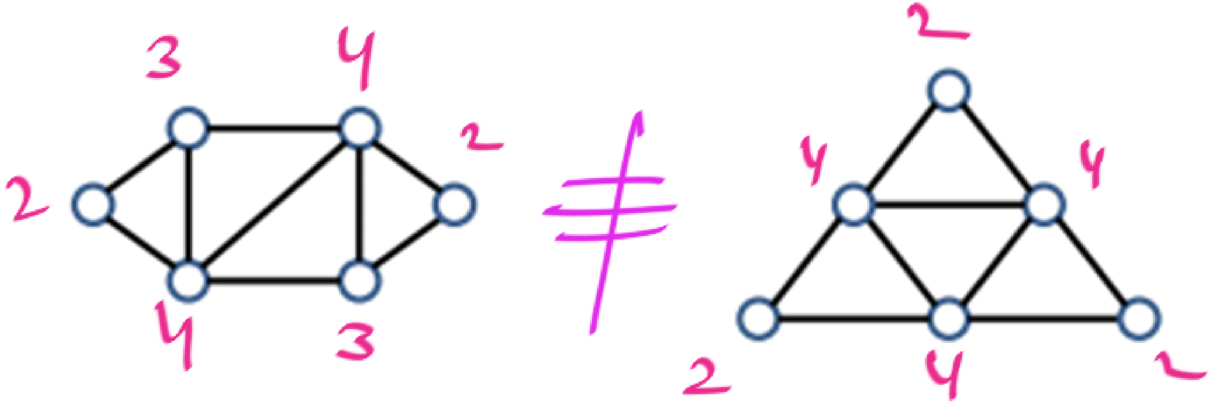


No 5 length cycle

$\exists$ 5 length cycle



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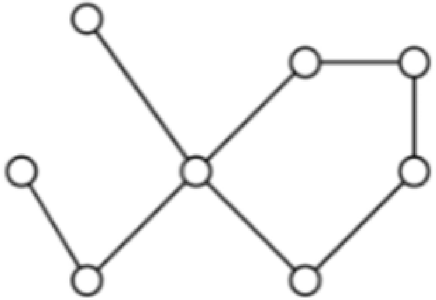
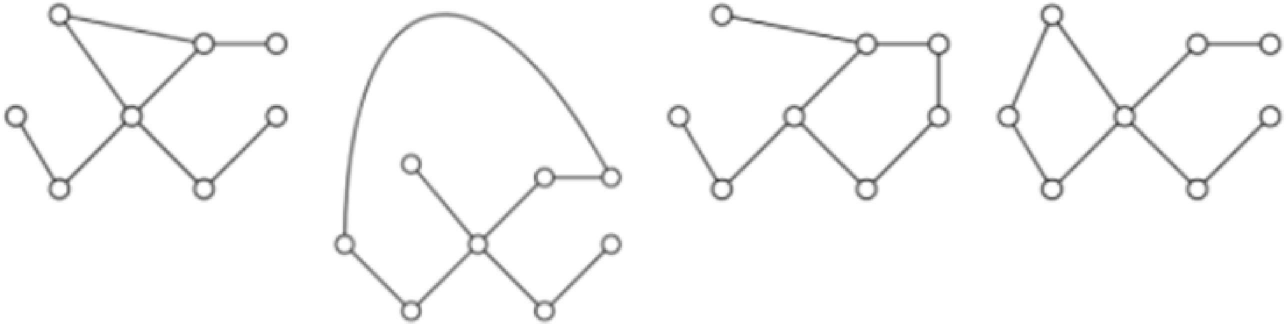
GATE CSE 2012 |

Which of the following graphs is isomorphic to



A. B. C. D.



GATE CSE 2012 |

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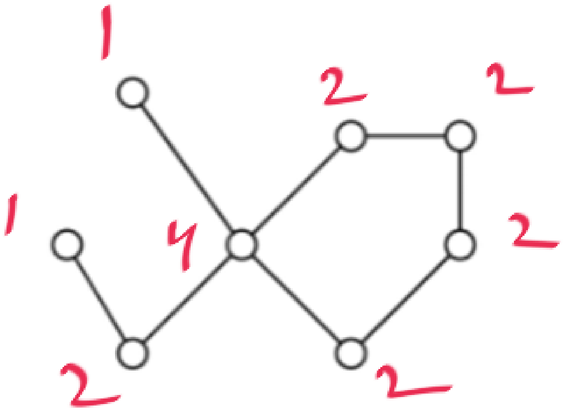
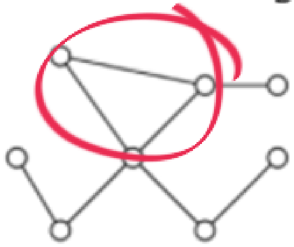
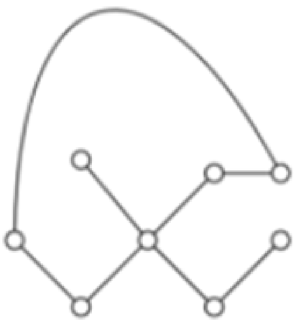


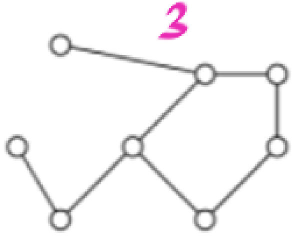
A.



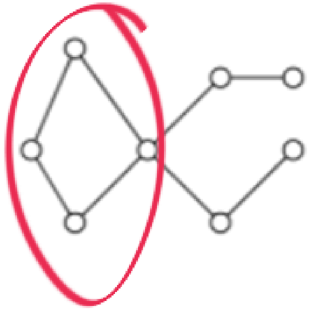
B.



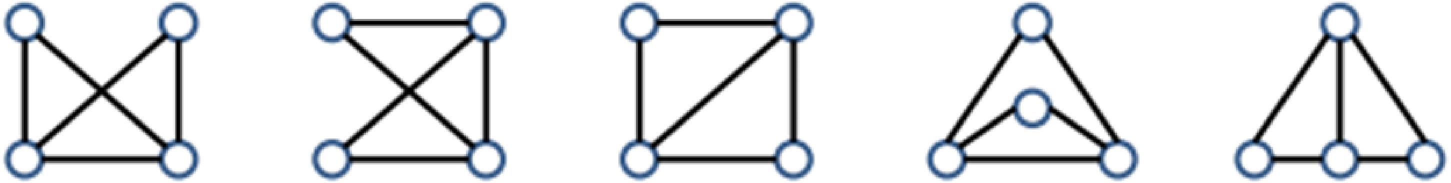
C.



D.



Ex : The following are isomorphic to each other :



Order of Graph Invariants to check:

1. |V|
2. |E|
3. Connectedness
4. Degree sequence
5. no. of 3 length cycles
6. " " 4 " "
7. " " 5 " "
8. Diameter

Q : washigton

Let A be the set of all undirected, simple graphs on n nodes. Define a relation R on A as follows: Two graphs G and G' in A are related by R if there is a bijection f from the vertices of G to the vertices of G' such that (u, v) is an edge in G if and only if $(f(u), f(v))$ is an edge in G'. *True or false:* R is an equivalence relation.

Q :

Let A be the set of all undirected, simple graphs on n nodes. Define a relation R on A as follows: Two graphs G and G' in A are related by R if there is a bijection f from the vertices of G to the vertices of G' such that (u, v) is an edge in G if and only if $(f(u), f(v))$ is an edge in G'. *True or false:* R is an equivalence relation.

$A = \{G_1, G_2, G_3, ---, K_3, K_4, W_3, C_3, ---, Q_2 --\}$

R on A

$G_1 R G_2$ iff G_1, G_2 are Isomorphic.

$R : A \rightarrow A$

Graph Isomorphism: ✓

Reflexive ✓ $G \cong G$

Symmetric ✓ $G \cong H \Rightarrow H \cong G$

Transitive ✓ $G \cong H \cong P \Rightarrow G \cong P$

Graph Theory :

Next Topic :

Complement of a Graph

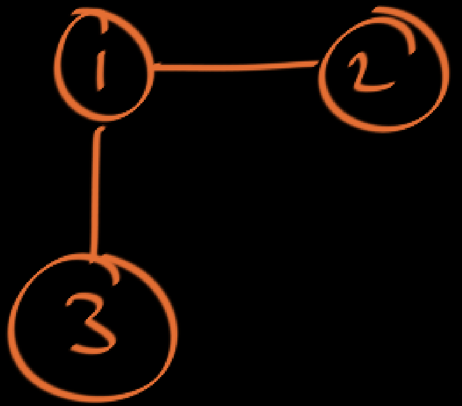
Website : https://www.goclasses.in/

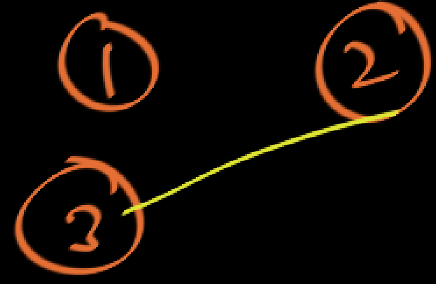
"Every Graph" on "n-vertices" is a SubGraph of K_n.

$G(V, E)$

$\overline{G}(V, \overline{E})$ $\boxed{\overline{E} = E(K_n) - E}$

G

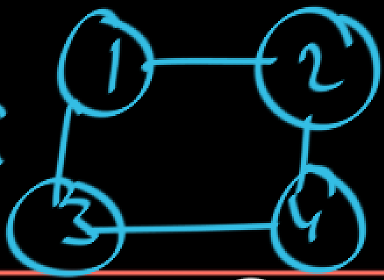




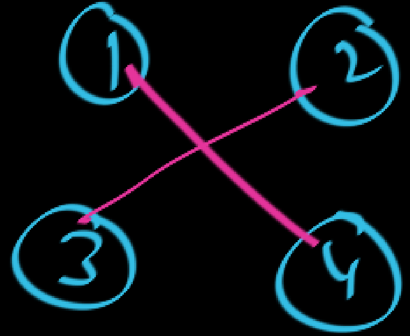
$\overline{G}$

$K_4 \longrightarrow \overline{K_4} = E_4$ → Edgeless

$\overline{K_n} = E_n$; G



$\overline{G}$



NOTE: $G(V_1, E_1)$ $\quad$ $\underbrace{\overline{G}}(V_2, E_2)$

① $V_1 = V_2 = n$

② $E_2 = \overline{E_1}$

$|E_2| = {}^nC_2 - |E_1|$

$\underbrace{|E_1| + |E_2|} = {}^nC_2$

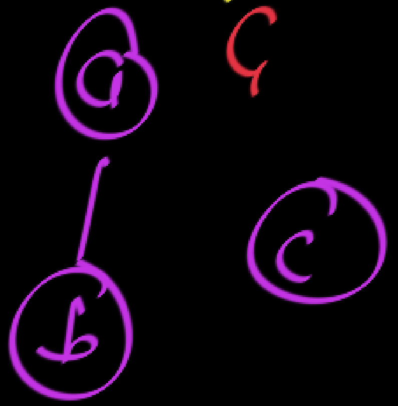
#Edges in K_n

Self-Complementary Graph: (SCG)

$$\boxed{G \equiv \overline{G}}$$

Isomorphic

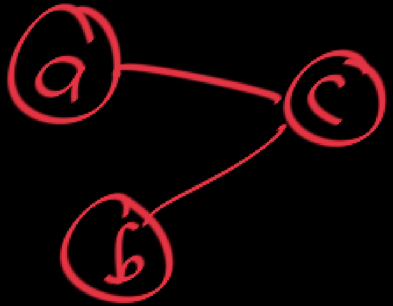
NOT SCG



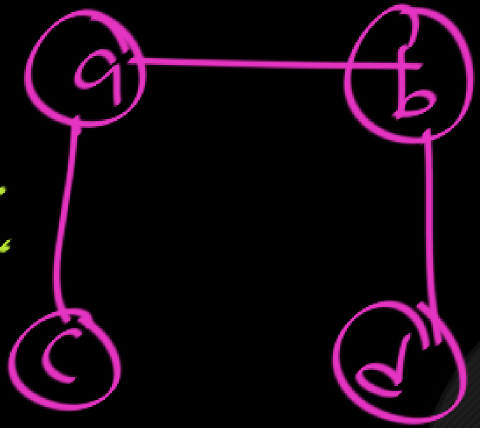
G

$\not\equiv$

G'

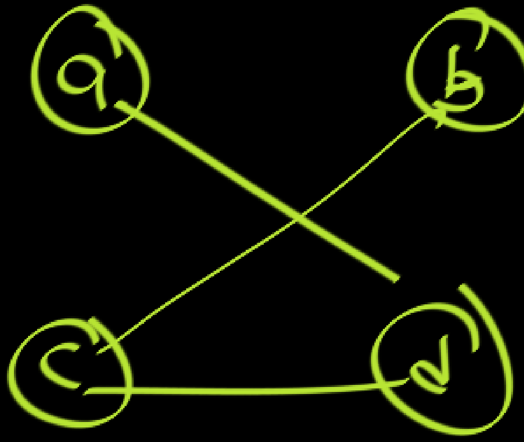


G :

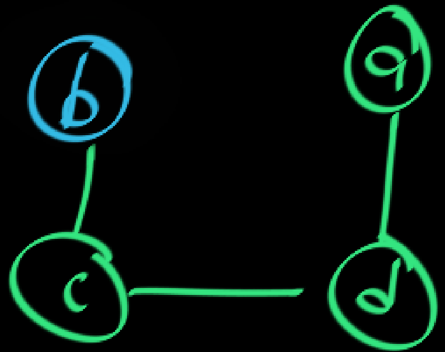


$\equiv$

$\overline{G}$



self Complementary



Definition 6. [Graph complement] The *complement* of a graph $G = (V, E)$ is a graph with vertex set V and edge set E' such that $e \in E'$ if and only if $e \notin E$. The complement of a graph G is denoted $\overline{G}$ and sometimes is called co-G.

Definition 7. [Self-complementary graphs] A graph G is *self-complementary* if G is isomorphic to its complement.

Self-Complementery Graph:

$$G \equiv \overline{G}$$

for Self-Comp Graph

$$\boxed{e = {}^{n}C_2 - e}$$

$$\boxed{2e = {}^{n}C_2} \Rightarrow \boxed{\frac{n(n-1)}{2} = 2e}$$

GATE CSE 2014 Set 2

A cycle on n vertices is isomorphic to its complement. The value of n is _____.

GATE CSE 2014 Set 2

A cycle on n vertices is isomorphic to its complement. The value of n is $\underline{5}$.

C_n

$\overline{C_n}$

$$\underbrace{C_n \equiv \overline{C_n}}_{} \Rightarrow$$

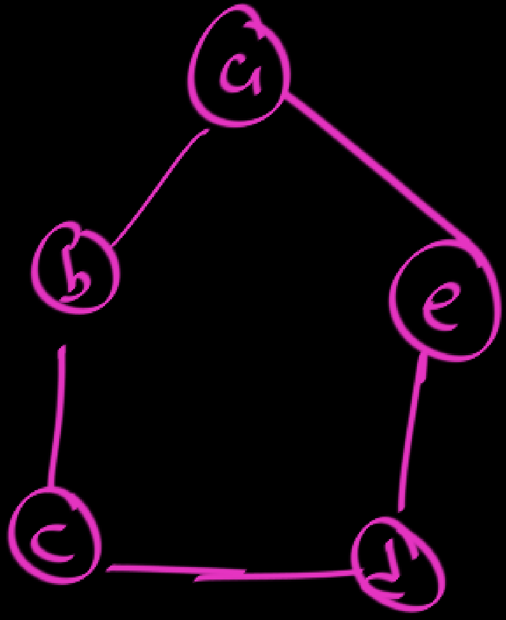
#Edges = Same

$$n = {}^nC_2 - n$$

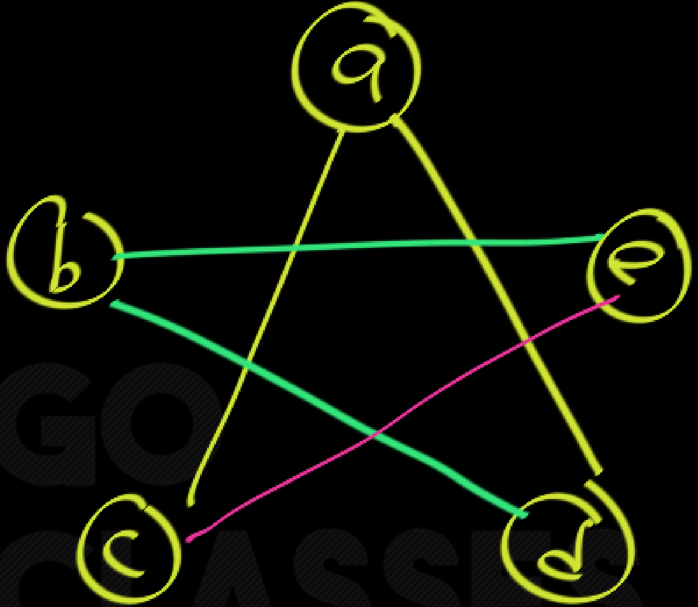
$$\Leftarrow 2n = \frac{n(n-1)}{2}$$

$$4n = n^2 - n$$

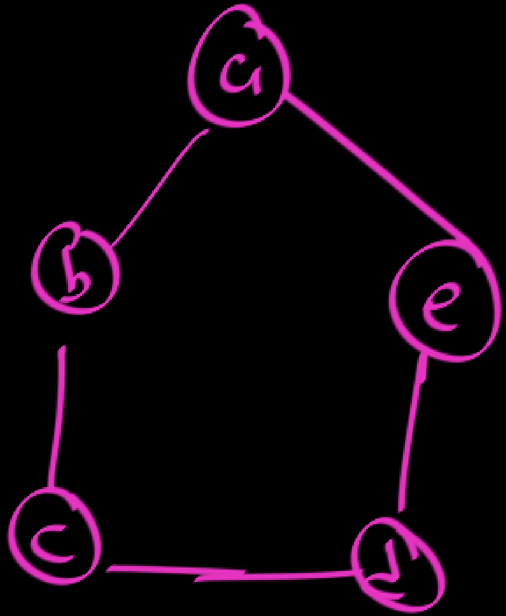
$$n^2 = 5n \Rightarrow \boxed{n = 5}$$



C_5



$\overline{C_5}$



C_5

C_5

Self Comple-mentary.

$\overline{C_5}$

Graph Theory :

Next Topic :

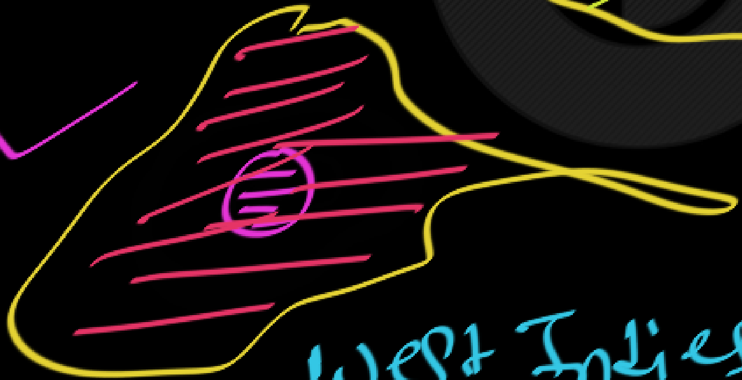
Connected Components

Components in a Graph

Website : https://www.goclasses.in/

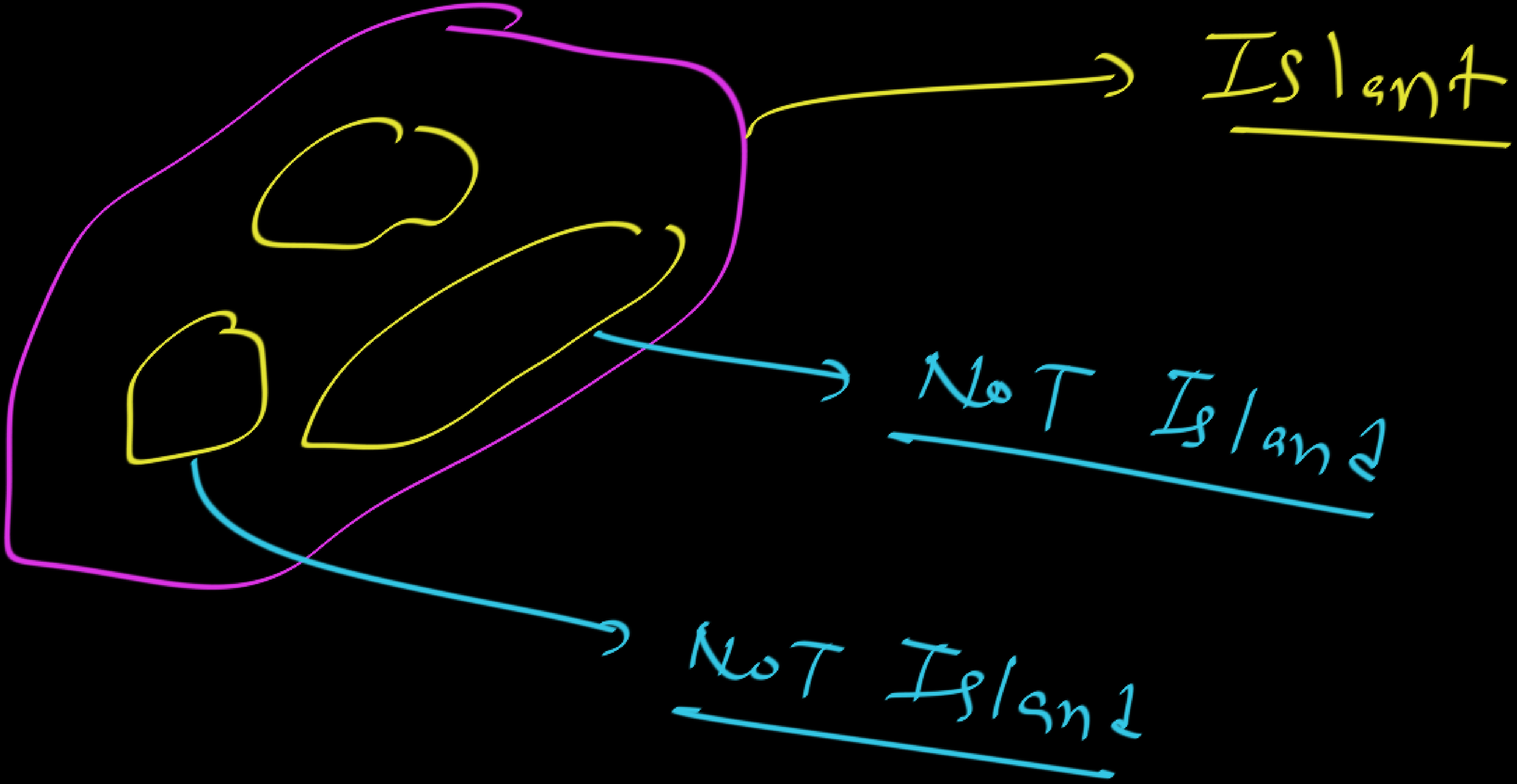
Component ≡ Island = "maximal" piece of land surrounded by water.

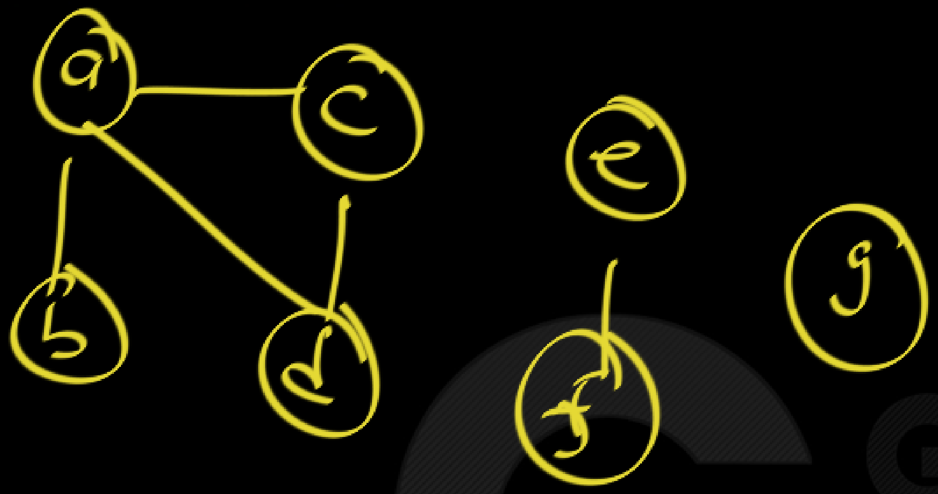
3 Islands



West Indies

(Black Water)


Islant
NoT Island
NoT Island



{b,d} → Connected

{b,d,a,c} ✓

Components = 3

Component = "maximal" "Connected" "Subgraph"

Graph

{c,d,a,b}

{c,d} → Connected But Not maximal

"Maximal" Connected ;

You cannot Add Something more keeping it Connected.

"Child" → I'm full ← ~~more food~~

Don't Add Extra Tummy.

Connected Vs Adjacent

a ~ b

Path

a — b

Edge

CONNECTED COMPONENTS A **connected component** of a graph G is a connected subgraph of G that is not a proper subgraph of another connected subgraph of G. That is, a connected component of a graph G is a maximal connected subgraph of G. A graph G that is not connected has two or more connected components that are disjoint and have G as their union.

An undirected graph is *connected* if there is a path between every pair of distinct vertices. A *connected component* of a graph G is a connected subgraph of G that is not contained in any other connected subgraph of G. (In other words, it is a maximal connected subgraph.)

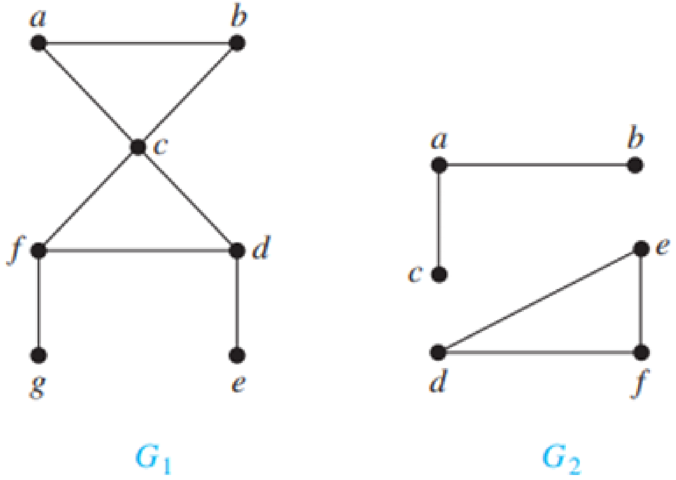


FIGURE 2 **The Graphs G_1 and G_2.**

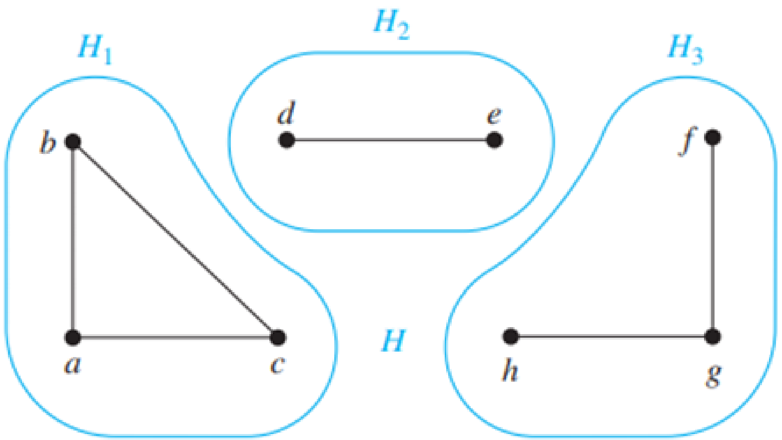


FIGURE 3 **The Graph H and Its Connected Components H_1, H_2, and H_3.**

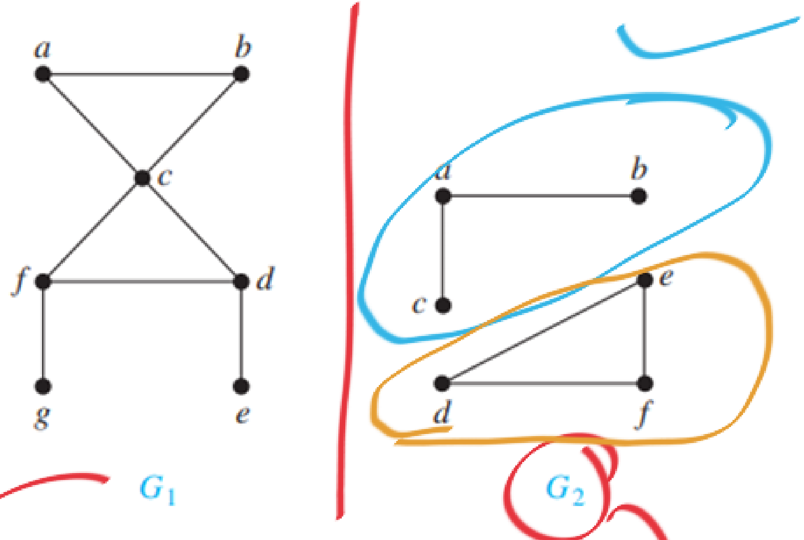


FIGURE 2 The Graphs G_1 and G_2.

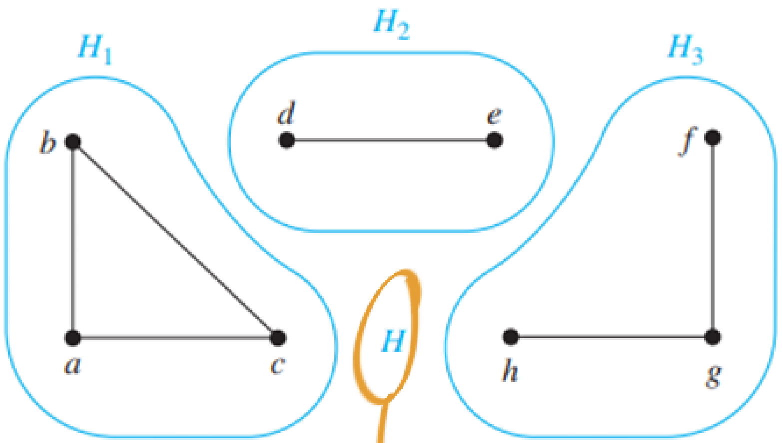


FIGURE 3 The Graph H and Its Connected Components H_1, H_2, and H_3.

#Components = 1

#Components = 2

3 Components

$G(V, E)$ $\quad a, b \in V$

a, b belong to same Component **iff**

there is a path b/w a, b

$G(V, E)$ $\quad a, b \in V$

a, b belong to Different Component iff

there is NO Path b/w a, b

a ~ No Path ~ b

Q: Graph $G(V, E)$ → #Components = y

on V, Relation R, aRb iff ∃ Path b/w a, b.

R = Eq. Relation?

If yes, #Eq. Classes?

Q: Graph $G(V, E) \rightarrow$ # Components $= y$

On V, Relation R, aRb iff $\exists$ Path b/w a, b. $\rightarrow$ $a, b \in$ Same Component

$R =$ Eq. Relation ?

If yes, # Eq. Classes ? y $\rightarrow$ Every Eq. Class is a Component.

Eg: G : (V, E)

R on V ← Base set ; $v_1 R v_2$ iff $\exists$ path $v_1 \sim v_2$

aRb ✓ dRe ✓ fRf ✓ bRc ✗

cRe ✓ fRg ✗ gRg ✓ aRa ✓

Symmetric : $a \overset{\text{Path}}{\sim} b \Longrightarrow b \overset{\text{Path}}{\sim} a$

Transitive : $a \overset{\text{Path1}}{\sim} b \overset{\text{Path2}}{\sim} c \Longrightarrow a \overset{\text{Path3}}{\sim} c$

Reflexive : $a - a$

Trivial Path (length 0)

R = Eq. Relation

So R = Eq. Relation

Base set = V

$[a] = \{a, b\}$ $[c] = \{c, d, e\}$ $[g] = \{g\}$ $[f] = \{f\}$

Component Component Component

4 Eq. Classes