Boolean Expressions
&
Boolean Algebra Laws

$AB + C = (AB) + C$ Expression

$\boxed{\text{AND} > \text{OR}}$ priority

Equation:

$\underbrace{AB + C} = \underbrace{(AB) + C}$

OR

rph

$\underbrace{A + A = A}$

- **Expression**: a set of literals (possibly with repeats) combined with logic operations (and possibly ordered by parentheses)

  - e.g., 4 expressions: $AB + C, (AB) + C, (\overline{A} + B)C, ((\overline{A}) + B)C$

  - Note: can compliment expressions, too, e.g., $\overline{((\overline{A}) + B)C}$

- **Equation**: expression1 = expression2

  - e.g., $(\overline{A} + B)C = ((\overline{A}) + B)C$

- **Function** of (possibly several) variables: an equation where the lefthand side is defined by the righthand side $F(A,B,C) = ((\overline{A}) + B)C$

Boolean Expression = Boolean function

$f(a,b,c) = \underbrace{(ab) + c}_{\text{Expression}}$

$f(a,b) = a + \bar{b}$

Truth Table: all combinations of input variables

k variables ➜ 2^k input combinations

D	X	A	$D\overline{X} + A$ = f
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

Input Combinations

Truth Table in Standard form

If A=1 then f=1

If A=0 then $f = D\overline{X}$

f=1 iff D=1, X=0

Boolean function $\underbrace{f(a,b,c,d)}$

In Truth Table ; #Rows ? $= 2^4 = \underline{2 \times 2 \times 2 \times 2}$

a	b	c	d	f
0	0	0	0	
0	0	0	1	
0	0	1	0	
⋮	⋮	⋮	⋮	
1	1	1	1	

16 Rows

$a <^{0}_{1}$ or

$c <^{0}_{1}$ or

$d <^{0}_{1}$ or

Boolean Algebra :

Next Topic :

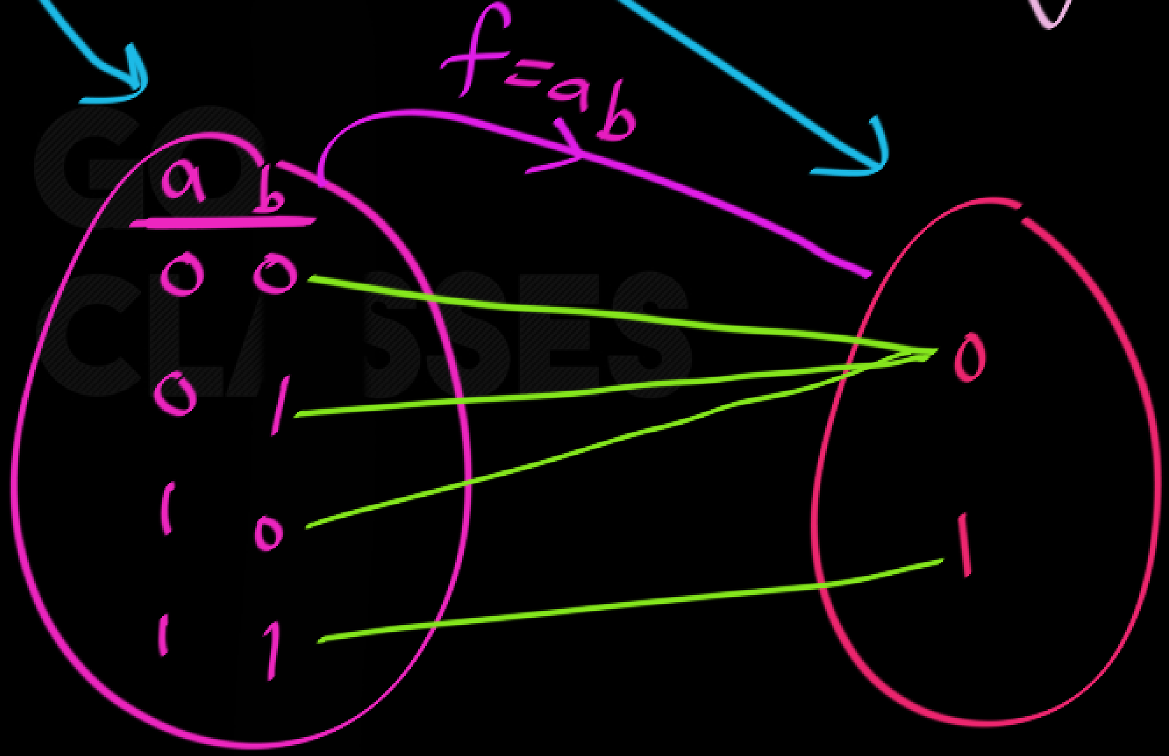
Boolean Expression == Boolean Function

Why?

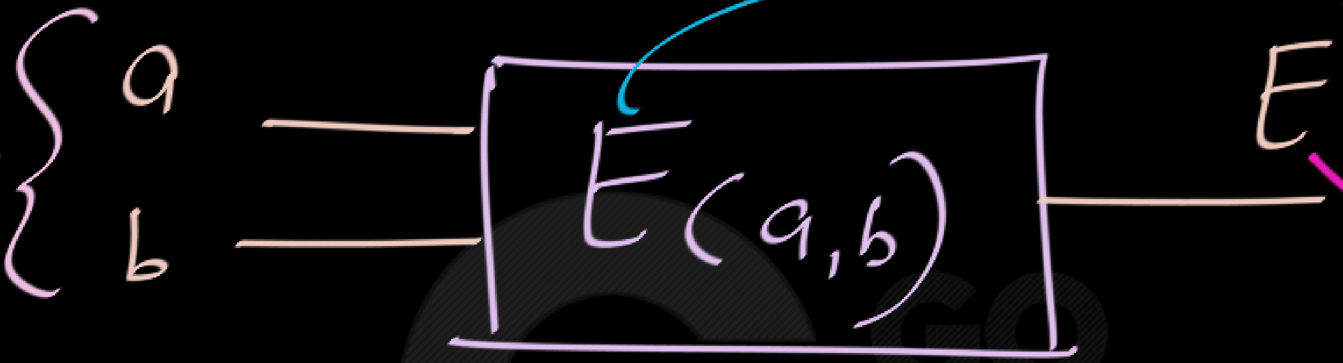
function : Domain, Co-Domain, mapping

Boolean Exp $= ab = f$

a	b	$f = ab$
0	0	0
0	1	0
1	0	0
1	1	1



mapping of Boolean function.



Input Combinations

outputs possible

0	0
0	1
1	0
1	1

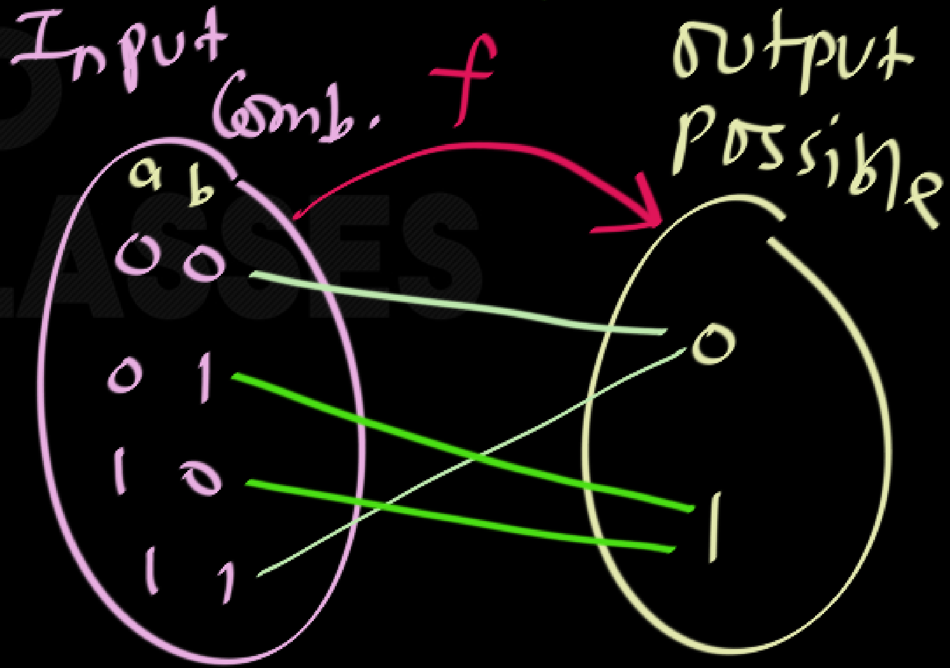
Domain of Boolean function

0
1

CoDomain of Bool fun.

NOTE: Why Boolean Exp are called Boolean functions?

a	b	$E(a,b) = a \oplus b = f(a,b)$
0	0	0
0	1	1
1	0	1
1	1	0



Boolean expressions are formed by application of the basic operations to one or more variables or constants. The simplest expressions consist of a single constant or variable, such as 0, X, or Y'. More complicated expressions are formed by combining two or more other expressions using AND or OR, or by complementing another expression. Examples of expressions are

$$AB' + C \tag{2-1}$$

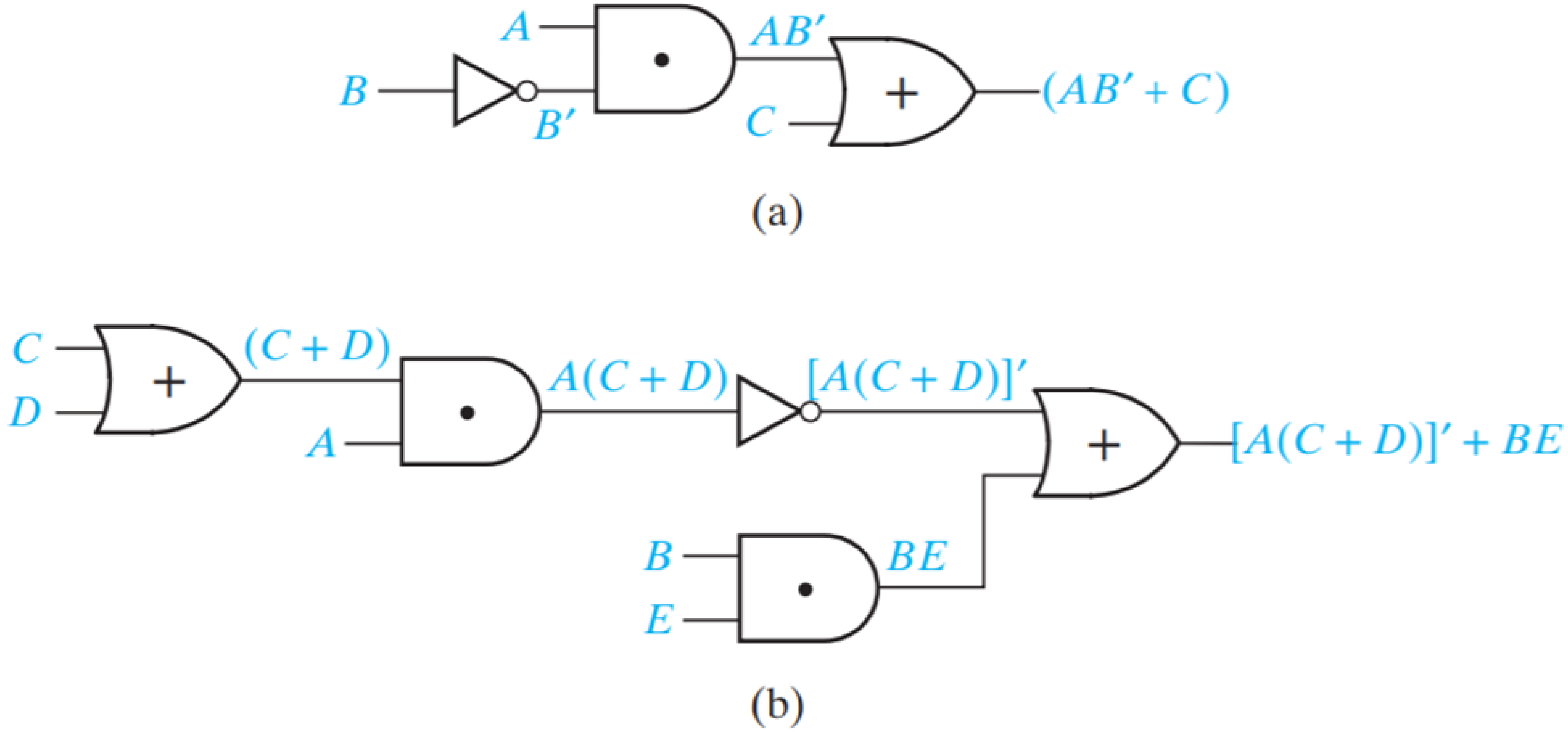
$$[A(C + D)]' + BE \tag{2-2}$$

Parentheses are added as needed to specify the order in which the operations are performed. When parentheses are omitted, complementation is performed first followed by AND and then OR. Thus in Expression (2-1), B' is formed first, then AB', and finally $AB' + C$.

Each expression corresponds directly to a circuit of logic gates.

$$AB' + C \tag{2-1}$$
$$[A(C + D)]' + BE \tag{2-2}$$

Each expression corresponds directly to a circuit of logic gates. Figure 2-1 gives the circuits for Expressions (2-1) and (2-2).



An expression is evaluated by substituting a value of 0 or 1 for each variable. If $A = B = C = 1$ and $D = E = 0$, the value of Expression (2-2) is

$$[A(C + D)]' + BE = [1(1 + 0)]' + 1 \cdot 0 = [1(1)]' + 0 = 0 + 0 = 0$$

Each appearance of a variable or its complement in an expression will be referred to as a *literal*. Thus, the following expression, which has three variables, has 10 literals:

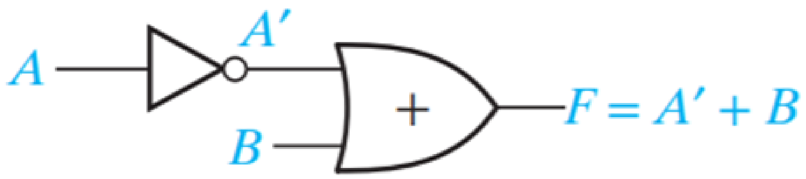
$$ab'c + a'b + a'bc' + b'c'$$

$$f_{(A,B)} = A' + B$$

Circuit Diagram:

Truth Table

A truth table (also called a table of combinations) specifies the values of a Boolean expression for every possible combination of values of the variables in the expression.



(a)

A	B	A'	$F = A' + B$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	0	1

(b)

Boolean Logic 2

- Precedence rules just like decimal system
- Implied precedence: NOT > AND > OR
- Use parentheses as necessary

$$AB + C = (AB) + C$$

$$(\overline{A} + B)C = ((\overline{A}) + B)C$$

Boolean Logic: Example

D	X	A	$L = \overline{DX} + A$
0	0	0	
0	0	1	
0	1	0	
0	1	1	
1	0	0	
1	0	1	
1	1	0	
1	1	1	

Boolean Logic: Example

D	X	A	$\overline{X}$	$\overline{DX}$	$L=\overline{DX}+A$
0	0	0	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	0	1

Boolean Logic: Example 2

X	Y	$XY + \overline{X}\overline{Y}$
0	0	
0	1	
1	0	
1	1	

Boolean Logic: Example 2

X	Y	$\overline{X}$	$\overline{Y}$	XY	$\overline{X}\,\overline{Y}$	$XY + \overline{X}\,\overline{Y}$
0	0	1	1	0	1	1
0	1	1	0	0	0	0
1	0	0	1	0	0	0
1	1	0	0	1	0	1

$= x \odot y$

If an expression has n variables, and each variable can have the value 0 or 1, the number of different combinations of values of the variables is

$$\underbrace{2 \times 2 \times 2 \times \ldots}_{n \text{ times}} = 2^n$$

Therefore, a truth table for an n-variable expression will have 2^n rows.

Boolean Algebric Function

- A Boolean function can be expressed algebraically with binary variables, the logic operation symbols, parentheses and equal sign.

- For a given combination of values of the variables, the Boolean function can be either 1 or 0.

- Consider for example, the Boolean Function:

$F1 = x + y'z$

The Function F1 is equal to 1 if x is 1 or if both y' and z are equal to 1; F1 is equal to 0 otherwise.

- The relationship between a function and its binary variables can be represented in a truth table. To represent a function in a truth table we need a list of the $\mathbf{2}^n$ combinations of the n binary variables.

- A Boolean function can be transformed from an algebraic expression into a logic diagram composed of different Gates

Boolean Algebra :

Next Topic :

Complement of a Boolean Function

a	b	$f(a,b)$	$\overline{f}$
0	0	1	0
0	1	0	1
1	0	1	0
1	1	0	1

$f(a,b) = \overline{b}$; $\overline{f} = b$

a	b	$f(a,b)$	$\overline{f}$
0	0	1	0
0	1	1	0
1	0	1	0
1	1	1	0

$f(a,b)=1$; $\overline{f}(a,b)=0$

Q: Is it possible that

$$f = \overline{f} \, ?$$

Q: Is it possible that

$f = \overline{f}$? = NO

f	$\overline{f}$
0	1
1	0
1	0
0	1

$f = 1 \leftrightarrow \overline{f} = 0$

So $\boxed{f \neq \overline{f}}$ Always

Boolean Algebra :

Next Topic :

Laws in Boolean Algebra

Laws of Boolean Algebra

Propositional logic

Set Theory

Group Theory

Boolean Lattices

Same

Similar

Isomorphic

① Domination Law / Annulment law :

$A + 1 = 1$
$A \cdot 0 = 0$

$B + 1 = 1$
$C + 1 = 1$
$BCA + 1 = 1$

$\alpha + 1 = 1$
$\alpha \cdot 0 = 0$

$A + 1 = ? = 1$

0 → 1
1 → 1

$A \cdot 0 = ? = 0$

0 → 0
1 → 0

$\boxed{} + 1 = 1$

$\boxed{} \cdot 0 = 0$

$A \oplus 1 = ? = \overline{A}$

$0 \rightarrow 1$

$1 \rightarrow 0$

$A \odot 1 = ? = A$

$0 \rightarrow 0$

$1 \rightarrow 1$

$A \odot 0 = ? = \overline{A}$

$0 \rightarrow 1$

$1 \rightarrow 0$

$\overline{A} \oplus 0 = ? = \overline{A}$

$0 \rightarrow 0$

$1 \rightarrow 1$

$$\boxed{A \oplus 0 = A}$$

$$\overline{A} \oplus 0 = \overline{A}$$

$$\overline{A} \oplus 1 = A$$

$$\left.\begin{array}{l} \alpha \oplus 0 = \alpha \\ \\ \alpha \oplus 1 = \alpha' \end{array}\right\}$$

2 Identity law:

$$\left\{ \begin{array}{l} A + 0 = A \\ A \cdot 1 = A \end{array} \right\}$$

$A + 0 = ? = A$

$0 \rightarrow 0$

$1 \rightarrow 1$

$A \cdot 1 = ? = A$

$0 \longrightarrow 0$

$1 \longrightarrow 1$

3 Idempotent Law : (law of Repetition)

$A + A = A$

$A \cdot A = A$

$A + A + A + A = A$

proof:

$A + A = ? = A$

0	0 →	0
1	1 →	1

$$A \oplus A = 0$$

$$A \odot A = 1$$

$$A \uparrow A = \overline{AA} = \overline{A}$$

$$A \downarrow A = \overline{A + A} = \overline{A}$$

(4) Complement Law :

$$A + \overline{A} = 1$$

$$A\overline{A} = 0$$

$$A \oplus \overline{A} = 1$$

$$A \odot \overline{A} = 0$$

$$A \uparrow \overline{A} = \overline{A\overline{A}} = \overline{0} = 1$$

$$A \downarrow \overline{A} = \overline{A + \overline{A}} = \overline{1} = 0$$

Boolean Algebra: Identities and Theorems

OR	AND	NOT	
$X+0 = X$	$X1 = X$		(identity)
$X+1 = 1$	$X0 = 0$		(null)
$X+X = X$	$XX = X$		(idempotent)
$X+\overline{X} = 1$	$X\overline{X} = 0$		(complementarity)
		$\overline{\overline{X}} = X$	(involution)
$X+Y = Y+X$	$XY = YX$		(commutativity)
$X+(Y+Z) = (X+Y)+Z$	$X(YZ) = (XY)Z$		(associativity)
$X(Y+Z) = XY + XZ$	$X+YZ = (X+Y)(X+Z)$		(distributive)
$\overline{X+Y} = \overline{X}\,\overline{Y}$	$\overline{XY} = \overline{X} + \overline{Y}$		(DeMorgan's theorem)

5 Double Complementation ;

$$\overline{\overline{A}} = A$$

$$\overline{(\overline{\alpha})} = \alpha$$

6 Commutative law;

$$A + B = B + A$$

$$AB = BA$$

$$A \oplus B = B \oplus A$$

$$A \odot B = B \odot A$$

$$A \uparrow B = B \uparrow A$$

$$A \downarrow B = B \downarrow A$$

Operations with 0 and 1:

$$X + 0 = X$$

$$X + 1 = 1$$

$$X \cdot 1 = X$$

$$X \cdot 0 = 0$$

Commutative, Associative, Distributive, and DeMorgan's Laws

Associative Law:

$$(ab)c = a(bc)$$

$$(a+b)+c = a+(b+c)$$

Prove: $$\underbrace{(a+b)+c}_{\text{LHS}} = \underbrace{a+(b+c)}_{\text{RHS}}$$

by Case method:

Case 1: $a=0$	**Case 2:** $a=1$
$\left.\begin{array}{l} \text{LHS} = b+c \\ \text{RHS} = b+c \end{array}\right\}$ Same	$\left.\begin{array}{l} \text{LHS} = 1 \\ \text{RHS} = 1 \end{array}\right\}$ Same

So, In Every Case, $(a+b)+c = a+(b+c)$

NOTE: "By Case" method :

	a	b	c	$f(a,b,c)$
$a=0$	0	0	0	
	0	0	1	
	0	1	0	
	0	1	1	
$a=1$	1	0	0	
	1	0	1	
	1	1	0	
	1	1	1	

for any variable x

$x=0$ or $x=1$ } 2 cases possible.

Associative Law: $a \oplus b \oplus c$

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

Proof by Case:

Case 1: $a = 0$

$LHS = b \oplus c$

$RHS = b \oplus c$

Same

Case 2: $a = 1$

$LHS = \bar{b} \oplus c$

$RHS = \overline{b \oplus c}$

Same

$$\underbrace{\bar{b} \oplus c}_{\text{LHS}} \stackrel{?}{=} \underbrace{\overline{b \oplus c}}_{\text{RHS}}$$

by Case:

$b = 0$

$$\left.\begin{aligned} \text{LHS} &= \bar{c} \\ \text{RHS} &= \bar{c} \end{aligned}\right\} \text{Same}$$

$b = 1$

$$\left.\begin{aligned} \text{LHS} &= c \\ \text{RHS} &= c \end{aligned}\right\} \text{Same}$$

Associative Law:

Associative opⁿ	Non-Asso opⁿ
AND	$\uparrow$ (NAND)
OR	$\downarrow$ (NOR)
$\oplus$	
$\odot$ = HW	

NAND: $(A \uparrow B) \uparrow C \neq A \uparrow (B \uparrow C)$

by Case:

$A = 0$

LHS $= \overline{C}$

RHS $= 1$

NOT Same

$A = 1$

No need to check.

$0 \uparrow \alpha = \overline{0 \cdot \alpha} = \overline{0} = 1$

$0 \uparrow \square = 1$

$1 \uparrow \alpha = \overline{1 \cdot \alpha} = \overline{\alpha}$

$1 \uparrow \square = \overline{\square}$

$0 \downarrow a = \overline{a}$

$0 \downarrow a = \overline{0 + a} = \overline{a}$

$1 \downarrow a = 0$

$1 \downarrow a = \overline{1 + a} = \overline{1}$
$= 0$

X	Y	Z	XY	YZ	(XY)Z	X(YZ)
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	1	0	0
1	0	0	0	0	0	0
1	0	1	0	0	0	0
1	1	0	1	0	0	0
1	1	1	1	1	1	1

De-Morgan Law: Very Important

$$\overline{(a+b)} = \bar{a}\,\bar{b}$$

$$\overline{ab} = \bar{a} + \bar{b}$$

Proof: $\overline{(a+b)} = \bar{a}\,\bar{b}$

"Prove by Case":

$a=0$	$a=1$
LHS $= \bar{b}$	LHS $= 0$
RHS $= \bar{b}$	RHS $= 0$
Same	Same

X	Y	X'	Y'	$X+Y$	$(X+Y)'$	$X'Y'$	XY	$(XY)'$	$X'+Y'$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

=

=

Operations with 0 and 1:

1. $X + 0 = X$ 1D. $X \cdot 1 = X$
2. $X + 1 = 1$ 2D. $X \cdot 0 = 0$

Idempotent laws:

3. $X + X = X$ 3D. $X \cdot X = X$

Involution law:

4. $(X')' = X$

Laws of complementarity:

5. $X + X' = 1$ 5D. $X \cdot X' = 0$

Commutative laws:

6. $X + Y = Y + X$ 6D. $XY = YX$

Associative laws:

7. $(X + Y) + Z = X + (Y + Z)$
$= X + Y + Z$
7D. $(XY)Z = X(YZ) = XYZ$

Distributive laws:

8. $X(Y + Z) = XY + XZ$

8D. $X + YZ = (X + Y)(X + Z)$

DeMorgan's laws:

9. $(X + Y)' = X'Y'$

9D. $(XY)' = X' + Y'$

Distributive Law:

$$a \cdot (b + c) = ab + ac$$

$$a + (b\,c) = (a+b)(a+c)$$

Prove by Case. → HW

$$a + (bc) = (a+b)(a+c)$$

Proof by Case :

$a = 0$	$a = 1$
$\left.\begin{array}{l} LHS = bc \\ RHS = bc \end{array}\right\}$ Same	$\left.\begin{array}{l} LHS = 1 \\ RHS = 1 \end{array}\right\}$ Same

Q: AND, OR are Complement of each other?

Q: AND, OR are Complement of each other? = NO

$f = ab \Rightarrow \overline{f} = a+b$ NO

$f = ab \Rightarrow \overline{f} = \overline{a}+\overline{b}$

$$\overline{\alpha + \beta} = \overline{\alpha}\ \overline{\beta}$$

$$\overline{\alpha\beta} = \overline{\alpha} + \overline{\beta}$$

Laws in Boolean Algebra

- **Commutative Law**

A.B = B.A

A+B = B+A

- **Associative Law**

(A.B).C = A.(B.C)

(A+B) + C = A+ (B+C)

- **Distributive Law**

A.(B+C)=A.B+A.C

A+(B.C)=(A+B).(A+C)

- **Absorption**

A+ (A.B)=A

A.(A+B)=A

A+AB = A
A+A'B =A+B
(A+B)(A+C) = A+BC

- **AND Law**

A.0 = 0

A.1 =A

A.A = A

A.A' =0

- **OR law**

A+0 = A

A+1=1

A+A=A

A+A' = 1

- **Inversion Law(Involution)**

A'' = A

- **DeMorgan's Theorm**

(x.y)' = x' + y'

(x+y)' = x' . y'

Idempotent Law

Complement Law

Prove DeMorgan's Theorem

$$\boxed{\overline{XY} = \overline{X} + \overline{Y}}$$

X	Y	$\overline{XY}$
0	0	1
0	1	1
1	0	1
1	1	0

X	Y	$\overline{X} + \overline{Y}$
0	0	1
0	1	1
1	0	1
1	1	0

Boolean Algebra :

Next Topic :

Complementing Boolean Expressions Using De-Morgan Law

NOTE:

a $\bar{a}$

If Boolean expression only contains Literals, AND, OR, NOT operations (no other operation) then we can use De-morgan law to find expression for Complement of a Function.

$$\overline{a+b} = \bar{a}\,\bar{b}$$

$$\overline{\alpha+\beta} = \bar{\alpha}\,\bar{\beta}$$

$$\overline{(\bar{a}+b)} = a\,\bar{b}$$

$$\overline{a\cdot b} = \bar{a}+\bar{b}$$

$$\overline{\bar{a}\cdot b} = a+\bar{b}$$

$$\bar{a} + b = E(\bar{a}, b, +)$$

$$\overline{E} = \overline{\bar{a} + b} = a \cdot \bar{b} = a\bar{b}$$

$$E(a,\ b,\ \bar{a},\ \bar{b},\ \cdot,\ +,\ 0,\ 1)$$

$$\overline{E}(\bar{a},\ \bar{b},\ a,\ b,\ +,\ \cdot,\ 1,\ 0)$$

$E($ literal l, $+$, $\cdot$, 0, $1)$

$\downarrow$

$\overline{E}(\ \bar{l}$, $\cdot$, $+$, 1, $0)$

DeMorgan's Theorem

- Procedure for complementing expressions
- Replace...
  - AND with OR, OR with AND
  - 1 with 0, 0 with 1
  - X with $\overline{X}$, $\overline{X}$ with X

$$\overline{XY} = \overline{X} + \overline{Y}$$

$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

$$\overline{xy} = \overline{x} + \overline{y}$$

$$\overline{x+y} = \overline{x} \cdot \overline{y}$$

$$\overline{(a+\overline{b})\cdot\overline{c}} = (\overline{a}\cdot b) + c = \overline{a}\,b + c$$

$$\overline{(a+\overline{b})\cdot(\overline{c}\cdot\overline{b})} = (\overline{a}\cdot b) + (c+b)$$

NOTE:

$$xy = \overline{x} + \overline{y}$$

mistake

$$\overline{\overline{ab} + c} = \overline{\overline{ab}} \cdot \overline{c} = (ab)\overline{c}$$

$$\overline{(\overline{a+b})\overline{c}} = (a+b) + c$$

$$\overline{ab} = \overline{a} + \overline{b}$$

$$\overline{\underbrace{(\overline{a+b})}_{x}\;\underbrace{\overline{c}}_{y}} = \overline{xy} = \overline{x} + \overline{y}$$

$$= \overline{\overline{a+b}} + \overline{\overline{c}}$$

$$= a+b+c$$

$$\overline{(\overline{A})} = A$$

DeMorgan's Practice

$$F = \overline{\overline{A\overline{B}C} + \overline{ACD} + B\overline{C}}$$

DeMorgan's Practice

$$F = \overline{A\overline{B}C + \overline{ACD} + B\overline{C}}$$

$A\overline{B}C$

$$= (A\overline{B}C) \cdot (ACD) \cdot (\overline{B} + C)$$

DeMorgan's Practice

$$F = \overline{\overline{A\overline{B}}C + \overline{ACD} + B\overline{C}}$$

$$= (A\overline{B}C)(ACD)(\overline{B\overline{C}})$$

$$= (A\overline{B}CD)(\overline{B}+C)$$

$$= A\overline{B}CD + A\overline{B}CD$$

$$= A\overline{B}CD$$

$$(A\bar{B}C)(ACD)(\bar{B}+C)$$

$$= (AC\bar{B}D)(\bar{B}+C)$$

$$= AC\bar{B}D\bar{B} + AC\bar{B}DC$$

$$= AC\bar{B}D + AC\bar{B}D = AC\bar{B}D$$

$$AA = A$$

$$CC = C$$

$$A(B+C) = AB + AC$$

Boolean Algebra: Example 2

Find the complement of F.

$$F = A\overline{B} + \overline{A}B$$

$$\overline{F} =$$

$$f = \overline{A}B + A\overline{B}$$

$$\overline{f} = \overline{\overline{A}B + A\overline{B}}$$

$$= (A + \overline{B}) \cdot (\overline{A} + B)$$

$$f = A \oplus B$$

$$\overline{f} = \underline{\underline{A \odot B}}$$

Boolean Algebra: Example 2

Find the complement of F.

$$F = A\overline{B} + \overline{A}B$$

$$\overline{F} = \overline{A\overline{B} + \overline{A}B}$$

$$(\overline{A\overline{B}})(\overline{\overline{A}B}) \qquad \text{(by DeMorgan's)}$$

$$(\overline{A} + \overline{\overline{B}})(\overline{\overline{A}} + \overline{B}) \qquad \text{(by DeMorgan's)}$$

$$(\overline{A} + B)(A + \overline{B}) \qquad \text{(by involution)}$$

Boolean algebra practice 3

Find the complement of F.

$$F = (\overline{V}W + X)Y + \overline{Z}$$

Boolean algebra practice 3

Find the complement of F.

$$F = (\overline{V}W + X)Y + \overline{Z}$$

$$\overline{F} = \left(\left((V+\overline{W})\cdot\overline{X}\right) + \overline{Y}\right)\cdot\left(Z\right)$$

Boolean algebra practice 3

Find the complement of F.

$$F = (\overline{V}W + X)Y + \overline{Z}$$

$$\overline{F} = \overline{(\overline{V}W + X)Y + \overline{Z}}$$

$$\overline{((\overline{V}W + X)Y)}\,\overline{\overline{Z}} \qquad \text{(by DeMorgan's)}$$

$$(\overline{(\overline{V}W + X)} + \overline{Y})Z \qquad \text{(by DeMorgan's \& involution)}$$

$$(\overline{\overline{V}W}\,\overline{X} + \overline{Y})Z \qquad \text{(by DeMorgan's)}$$

$$((\overline{\overline{V}} + \overline{W})\overline{X} + \overline{Y})Z \qquad \text{(by DeMorgan's)}$$

$$((V + \overline{W})\overline{X} + \overline{Y})Z \qquad \text{(by null)}$$

Q:

$$\overline{a + bc}$$

$$= \cancel{\bar{a} \cdot \bar{b} + \bar{c}}$$

Q:

$$\overline{a+bc} = \overline{a+(bc)}$$

$$= (\bar{a} \cdot \bar{b}) + \bar{c}$$

✓ $\bar{a} \cdot (\bar{b} + \bar{c})$

Operator Presedence

- The operator Precedence for evaluating Boolean expression is:
  - 1. Parentheses
  - 2. NOT
  - 3. AND
  - 4. OR

Some More "Useful" Properties :

(1) Absorption Law :

$$a + (ab) = a$$

$$a \cdot (a+b) = a$$

$$a \cdot 1 + ab = a(1 + b) = a \cdot 1$$
$$= a$$

$$a \cdot (a + b) = aa + ab = a + ab$$
$$= a$$

$$\boxed{\alpha + \alpha\beta = \alpha}$$

$$\alpha \cdot (\alpha + \beta) = \alpha$$

(2) $A + \overline{A}B = A + B$ } prove by case

$\alpha + \overline{\alpha}\beta = \alpha + \beta$

$\overline{A} + AB = \overline{A} + B$

$$\overline{A} + \overline{A}B = \overline{A} \quad \Big\} \text{ Absorption law}$$

$$\overline{A} + AB = \overline{A} + B \quad \checkmark$$

(3) Consensus Property;

$$ab + \bar{a}c + bc = ab + \bar{a}c$$

$$ab + \bar{a}c + bc = ab + \bar{a}c$$

Proof by Case:

Case 1: $a = 0$

$LHS = c$
$RHS = c$
Same

Case 2 $a = 1$

$LHS = b$
$RHS = b$
Same

Removal Redundancy:

$$a + a = a$$

$$a + aa = a$$

$$a + ab = a$$

$$ab + \bar{a}c + bc = ab + \bar{a}c$$

$$a + \overline{a}c = a + c$$

$$\overline{a} + \overline{a}c = \overline{a}$$

$$\overline{a} + ac = \overline{a} + c$$

Boolean Algebra :

Next Topic :

Dual of a Boolean Functions

Laws and Theorems of Boolean Algebra

1a.	$X \bullet 0 = 0$	1b.	$X + 1 = 1$	Annulment Law
2a.	$X \bullet 1 = X$	2b.	$X + 0 = X$	Identity Law
3a.	$X \bullet X = X$	3b.	$X + X = X$	Idempotent Law
4a.	$X \bullet \overline{X} = 0$	4b.	$X + \overline{X} = 1$	Complement Law
5.	$\overline{\overline{X}} = X$			Double Negation Law
6a.	$X \bullet Y = Y \bullet X$	6b.	$X + Y = Y + X$	Commutative Law
7a.	$X\ (Y\ Z) = (X\ Y)\ Z = (X\ Z)\ Y = X\ Y\ Z$			Associative Law
7b.	$X + (Y + Z) = (X + Y) + Z = (X + Z) + Y = X + Y + Z$			Associative Law
8a.	$X \bullet (Y + Z) = X\ Y + X\ Z$	8b.	$X + Y\ Z = (X + Y) \bullet (X + Z)$	Distributive Law
9a.	$\overline{X \bullet Y} = \overline{X} + \overline{Y}$	9b.	$\overline{X + Y} = \overline{X} \bullet \overline{Y}$	de Morgan's Theorem
10a.	$X \bullet (X + Y) = X$	10b.	$X + X\ Y = X$	Absorption Law
11a.	$(X + Y) \bullet (X + \overline{Y}) = X$	11b.	$X\ Y + X\ \overline{Y} = X$	Redundancy Law
12a.	$(X + \overline{Y}) \bullet Y = XY$	12b.	$X\ \overline{Y} + Y = X + Y$	Redundancy Law

The Boolean algebra laws are given in pairs to show the algebra satisfies a duality.

Identity law:

$$\left.\begin{array}{l} a+0=a \\ a\cdot 1=a \end{array}\right\} \text{Dual of each other}$$

Distributive law:

$$\left.\begin{cases} a \cdot (b+c) = ab + ac \\ a + bc = (a+b)(a+c) \end{cases}\right\} \text{In Pairs}$$

Dual of each other

Dual of a function:

$f_{(a,b)} = ab$

Dual

$f^d = ?$

?

NOTE:

If Boolean expression only contains Literals, AND, OR, NOT operations (no other operation) then we can use find expression for Dual of a Function Easily.

$$f(a,b) = a \cdot b$$

$$f^d = (a + b)$$

$$(ab)^d = a+b$$

$$f(a,b) = a \cdot b$$

$$\overline{f} = \overline{a} + \overline{b}$$

$$f(a,b) = a + b$$

$$f^d = a \cdot b$$

$$f(a,b) = a + b$$

$$\overline{f}(a,b) = \overline{a} \cdot \overline{b}$$

$$f(a,\ b,\ \bar{a},\ \bar{b},\ +,\ \cdot,\ 1,\ 0)$$

$$f^d(a,\ b,\ \bar{a},\ \bar{b},\ \cdot,\ +,\ 0,\ 1)$$

$$\overline{f}(\bar{a}, \bar{b}, a, b, \cdot, +, 1, 0)$$

$$f(a, b, \bar{a}, \bar{b}, +, \cdot, 0, 1)$$

$$f^{d}(a, b, \bar{a}, \bar{b}, \cdot, +, 1, 0)$$

The Boolean algebra laws are given in pairs to show the algebra satisfies a duality.

Given a Boolean algebra expression the dual of the expression is obtained by interchanging the constants 0 and 1 and interchanging the operations of AND and OR. Variables and complements are left unchanged.

$$(XYZ\ldots)^D = X + Y + Z + \cdots \qquad (X + Y + Z + \cdots)^D = XYZ\ldots$$

$$(XYZ\ldots)^D = X + Y + Z + \cdots \qquad (X + Y + Z + \cdots)^D = XYZ\ldots$$

Q: $f(a,b,c) = (a+\bar{b})(\bar{a}c) + (b\bar{a})$

$f^d = ?$

$\overline{f} = ?$

Q: $f(a,b,c) = (a+\bar{b})(\bar{a}c) + (b.\bar{a})$

$f^d = ((a\bar{b}) + (\bar{a}+c))(b+\bar{a})$

$\bar{f} = ((\bar{a}b) + (a+\bar{c}))(\bar{b}+a)$

$$f(a, \bar{a}, +, \cdot, 0, 1)$$

$$\overline{f}(\bar{a}, a, \cdot, +, 1, 0)$$

$$f^d(a, \bar{a}, \cdot, +, 1, 0)$$

$$f = \overline{a} + b\overline{c}$$

$$f^c = \overline{f} = a(\overline{b} + c)$$

$$f^d = \overline{a}(b + \overline{c})$$

f	$\overline{f}$	f_d
0	1	1
1	0	0
a	$\overline{a}$	a
b	$\overline{b}$	b

f	$\overline{f}$	f_d
$a+b$	$\overline{a}\,\overline{b}$	$a\,b$
$a\,b$	$\overline{a}+\overline{b}$	$a+b$
$a+0$	$\overline{a}.1$	$a.1$

Basically to find Dual of a function, Don't Disturb the Literals.

Eg: $F = a + \bar{b}$

$\bar{a}b$ $\quad\quad$ $a\bar{b}$

a	b	$F = a + \bar{b}$	$\bar{F}$	F_d
0	0	1	0	0
0	1	0	1	0
1	0	1	0	1
1	1	1	0	0

$f \neq f_d$

$f \neq \bar{f}$

Q: for all functions,

$f = f_d$?

Q: for all functions,

$f = f^d$? No

$f = ab$; $f^d = a + b$

Different

Q: Is it possible?

$f = f_d$?

Q: Is it possible?

$f = f^d$? Yes.

$f = a$

$f^d = a$

Same

$f = a + b$

$f^d = ab$

Different

Q: Is it possible?

$$f = \overline{f}\ ?$$

Q: Is it possible?

$$f = \overline{f}\ ?$$

NO

Q: Is it possible for some f

$$f^d = \overline{f} \, ?$$

Q: Is it possible for some f

$f^d = \overline{f}$? Yes.

$f = a \oplus b$

$f^d = ?$; $\overline{f} = ?$

$f = 0$ — $f^d = 1$, $\overline{f} = 1$

$$f = a \oplus b = \underbrace{a\bar{b} + \bar{a}b}$$

$$f^d = (a+\bar{b})(\bar{a}+b) = a\bar{a} + ab + \bar{b}\bar{a} + \bar{b}b = ab + \bar{a}\bar{b}$$

$$\bar{f} = a \odot b = ab + \bar{a}\bar{b}$$

for $f = a \oplus b$ $\quad \underbrace{f^d = \bar{f}}$

Q: for All f

$f^d = \overline{f}$?

NO

$f = a + b$

$f^d = ab$

$\overline{f} = \overline{a}\,\overline{b}$

Different

$$\text{function } f \xrightarrow{\text{new function}} f^d$$

$$\text{function } f \xrightarrow{\text{new function}} \overline{f}$$

f is NEVER equal to $\overline{f}$

f may or may not be same as f_d.

F	$\overline{F}$
0	1
1	0
0	1
1	0

$f = a$

a	$f = a$	$f_d = a$
0	0	0
1	1	1

F is NEVER same as $\overline{F}$

$f = f_d$

Boolean Algebra :

Next Topic :

Simplification of Boolean Functions

Simplification Theorems

The following theorems are useful in simplifying Boolean expressions:

Uniting:	$XY + XY' = X$	(2-15)	$(X + Y)(X + Y') = X$	(2-15D)
Absorption:	$X + XY = X$	(2-16)	$X(X + Y) = X$	(2-16D)

$$ab + \bar{a}b = (\underbrace{a+\bar{a}}_{1})b = b$$

$$a + ab = a$$

$$\underbrace{a + ab = a}_{\text{equation 1}}$$

$$\underbrace{a(a+b) = a}_{\text{Dual of Eq 1}}$$

Equation:

$$E = F \quad \text{--- eq 1}$$

Dual of equation:

$$E^d = F^d \quad \text{--- eq 2}$$

If $E = F$

then $E^c = F^c$

Elimination: $X + X'Y = X + Y$ (2-17) $X(X' + Y) = XY$ (2-17D)

Consensus: $XY + X'Z + YZ = XY + X'Z$ (2-18)

$(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$ (2-18D)

In switching algebra, each of the above theorems can be proved by using a truth table. In a general Boolean algebra, they must be proved algebraically starting with the basic theorems.

Proof of (2-15): $XY + XY' = X(Y + Y') = X(1) = X$

Proof of (2-16): $X + XY = X \cdot 1 + XY = X(1 + Y) = X \cdot 1 = X$

Proof of (2-17): $X + X'Y = (X + X')(X + Y) = 1(X + Y) = X + Y$

Proof of (2-18): $XY + X'Z + YZ = XY + X'Z + (1)YZ =$
$XY + X'Z + (X + X')YZ = XY + XYZ + X'Z + X'YZ =$
$XY + X'Z$ (using absorption twice)

After proving one theorem in a pair of theorems, the other theorem follows by the duality property of Boolean algebra.

$$\boxed{a(b+c) = ab + ac}$$

Dual

$$\boxed{a + bc = (a+b)(a+c)}$$

Simplify

$$F = \overline{X}YZ + \overline{X}\,\overline{Y}Z + XZ$$

Simplify

$$F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$$

$$f = \overline{x}yz + \overline{x}y\overline{z} + xz$$

$$(\overline{x}y + x)z + \overline{x}y\overline{z}$$

$$(x+y)z + \overline{x}y\overline{z} = xz + yz + \overline{x}y\overline{z}$$

$$xz + \cancel{yz} + \bar{x}y$$

$$= \underline{xz + \bar{x}y}$$

$$F = \overline{X}YZ + \overline{X}Y\overline{Z} + XZ$$

$\overline{X}Y(Z + \overline{Z}) + XZ$ (by reverse distribution)

$\overline{X}Y1 + XZ$ (by complementarity)

$\overline{X}Y + XZ$ (by identity)