First Order Logic

Next Topic:

Nested Quantifiers

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Nested quantifiers

Two quantifiers are nested if one is within the scope of the other.

Example:

$$\forall x\ \underbrace{\exists y\ \overbrace{(x + y = 0)}^{P(x,y)}}_{Q(x)}$$

$\forall x\ Q(x)$

$Q(x)$ is $\exists y\ P(x,y)$

$P(x,y)$ is $(x + y = 0)$

$$= \forall x\, \underbrace{\exists y\, \underbrace{(x+y=0)}_{\text{scope of } \exists y}}_{\text{scope of } \forall x}$$

$\forall x\, P(x)$

$P(x) : \exists y\, (x+y=0)$

$$\underbrace{\exists_y\ (x+y=0)} = P(x)$$

free variable $= x$

Bounded variable $= y$

"Propositional function" $\equiv$ Predicate

$$\forall x \boxed{\forall y \, (xy = yx)} \quad Q(x)$$

$\forall x$

$\forall y$

$$\forall x \, Q(x)$$

$$Q(x) = \boxed{\forall y \, (xy = yx)}$$

Nested quantifiers mean that one quantifier is within the scope of another, such as

$$\forall x \exists y (x + y = 0).$$

Note that everything within the scope of a quantifier can be thought of as a propositional function. For example,

$$\forall x \exists y (x + y = 0)$$

is the same thing as $\forall x Q(x)$, where $Q(x)$ is $\exists y P(x, y)$, where $P(x, y)$ is $x + y = 0$.

We have defined the existential and universal quantifiers and showed how they can be used to represent mathematical statements.

We also saw how they can be used to translate English sentences into logical expressions.

Every natural number is a real number.

Domain: Real number

$$\forall_x \left(N(x) \rightarrow R(x) \right)$$

Some even number is prime.

$$\exists_x \left(even(x) \wedge prime(x) \right)$$

No even number greater than 2 is prime.

$$\forall_x \left((\text{Even}(x) \wedge x > 2) \longrightarrow \neg \text{Prime}(x) \right)$$

$$\equiv \neg \exists_x \left(\text{even}(x) \wedge x > 2 \wedge \text{Prime}(x) \right)$$

Domain: Set of all natural numbers.

For every natural number n, n+0 = n.

$\forall_n (n+0=n)$ ✓

$\forall_n (Nat(n) \rightarrow (n+0=n))$ ✓

For every natural number n, n*1 = n.

$\forall_n (n*1=n)$ ✓

$\forall_n (Nat(n) \rightarrow (n*1=n))$ ✓

Domain: Set of all Integers

For every natural number n, n+0 = n.

$\forall_n (n+0=n)$ ✗

$\forall_n (Nat(n) \rightarrow (n+0=n))$ ✓

For every natural number n, n*1 = n.

$\forall_n (n*1=n)$ ✗

$\forall_n (Nat(n) \rightarrow (n*1=n))$ ✓

Why do we need Nested Quantifiers?

$$\forall x\ \exists y\ P(x,y)$$

$$\exists y\ \underbrace{\forall x\ P(x,y)}_{\text{scope of } \exists y}$$

Why do we need Nested Quantifiers?

- Many interesting statements in first-order logic require a combination of quantifiers.

Eg: For every natural number, there exists a greater natural number.

$\forall \exists$

Eg: Some natural number is less than or equal to every natural number.

$\exists \forall$

Domain: Set of all natural numbers

Eg: For every natural number n, 2n is even.

$\forall_n (\text{Even}(2n))$

Eg: For every two natural numbers a,b; a+b = b+a.

$\forall_a \forall_b$

Eg: For every three natural numbers a,b,c; (a+b)+c = a+(b+c).

$\forall_a \forall_b \forall_c$

Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
- Example: "Every person loves someone else"

$\forall \exists$

- Alert:
  - The quantifiers must be read from left to right

$$\forall x \, \exists y \; P(x, y)$$

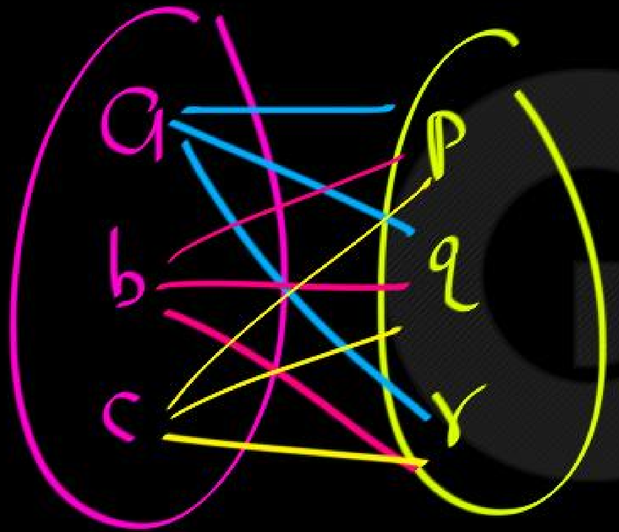
for all x, there is a y, P(x,y) is True.

1. $\forall x \forall y \, P(x,y)$
2. $\forall x \exists y \, P(x,y)$
3. $\exists x \forall y \, P(x,y)$
4. $\exists x \exists y \, P(x,y)$

1. $\forall x \forall y \, P(x,y)$

$x \rightarrow \{a, b, c\}$

$y \rightarrow \{p, q, r\}$



$P(x,y)$

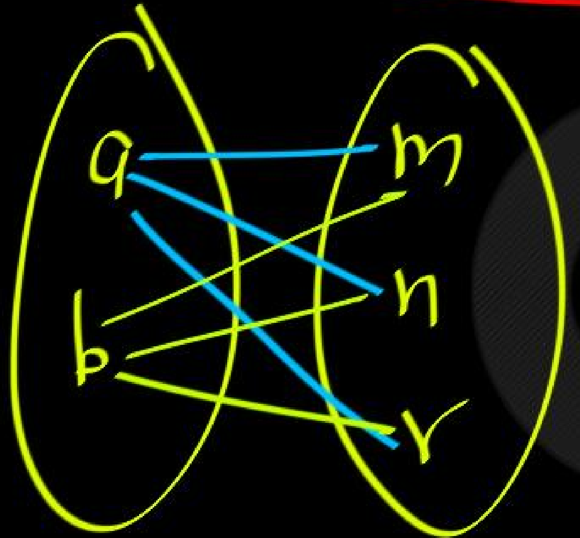
$P(a,p)$ ✓, $P(a,q)$ ✓, $P(a,r)$ ✓

$P(b,p)$ ✓, $P(b,q)$ ✓, $P(b,r)$ ✓

$P(c,p)$ ✓ $P(c,r)$ ✓

$P(c,q)$ ✓

$\boxed{\forall_x \forall_y \, T(x,y)}$



$T(x,y)$

Every student has taken every course.

$x \longrightarrow$ set of students $\{a, b\}$

$y \longrightarrow$ set of courses $\{m, n, r\}$

$T(x,y)$: x has taken course y

$x, y \rightarrow N$

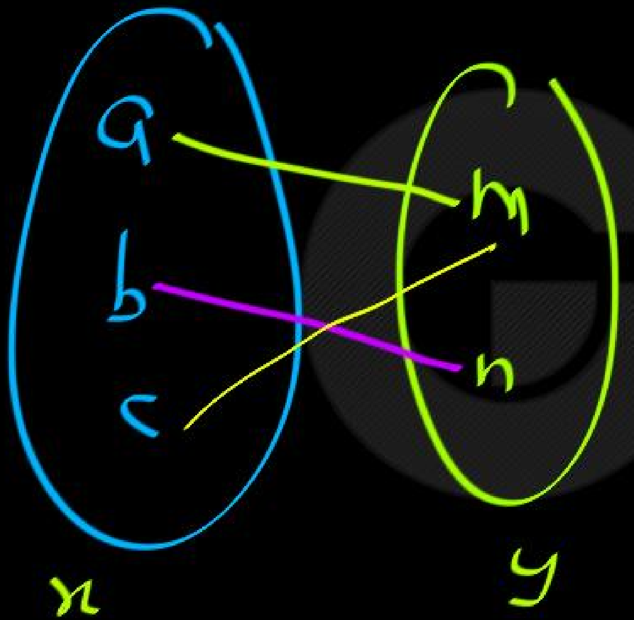
$\boxed{\forall_x \forall_y (x+y = y+x)}$: True

a b

$(a+b = b+a)$

for every pair (a,b) of natural numbers,
$a+b = b+a$

② $\forall x \, \exists y \, P(x,y)$



$P(x,y)$

$P(a,m) = T$

$P(b,n) = T$

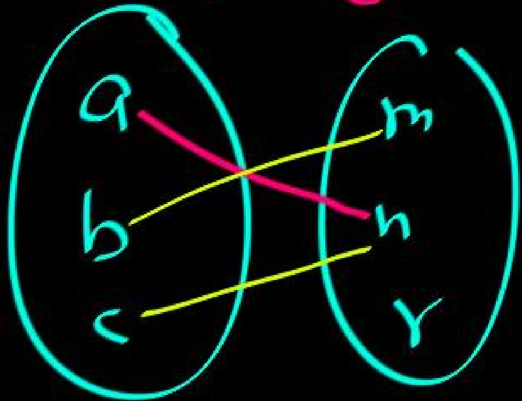
$P(c,m) = T$

$x \longrightarrow$ set of Students (a, b, c)

$y \longrightarrow$ set of Courses (m, n, r)

$T(x,y)$: Student x has taken Course y

$\forall_x \exists_y T(x,y)$: Every student has taken at least one Course.

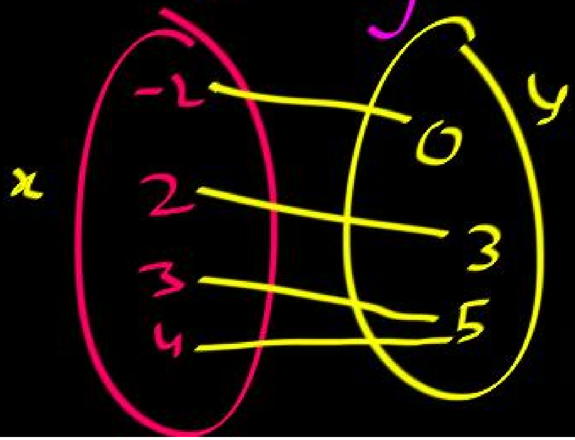


$T(x,y)$

eg: $P(x,y) : x < y$ Domain: $\mathbb{Z}$

$\forall x \exists y\, P(x,y) =$ True

$\forall x \exists y\, (x<y)$: for every integer x, there is some integer y such that $x < y$.



(for every integer, there is a greater integer)

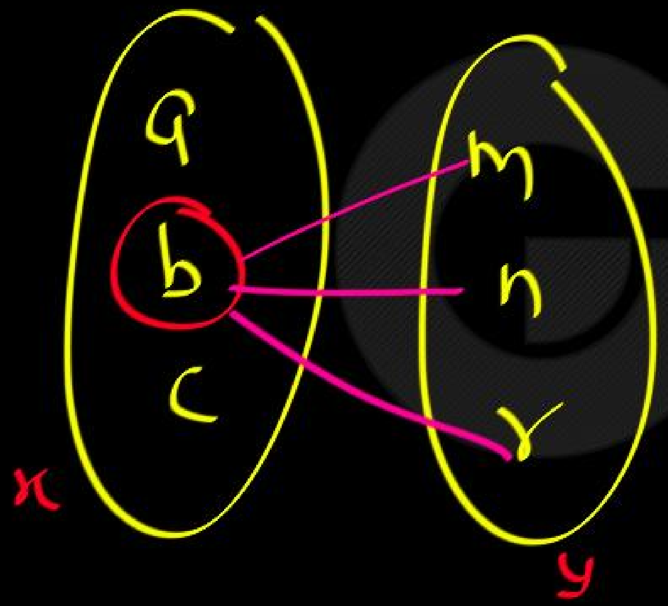
Eg: $P(x,y)$: $x < y$ Domain: $\mathbb{N}$

$\forall x \exists y \, P(x,y)$: for every natural number there is a greater natural number.

$x < y$

True

(3) $\exists_x \forall_y P(x, y)$



$P(x, y)$

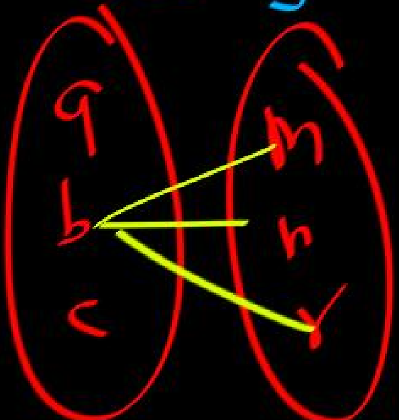
$$\begin{cases} P(b, m) = T \\ P(b, n) = T \\ P(b, r) = T \end{cases}$$

$x \longrightarrow$ set of Students (a, b, c)

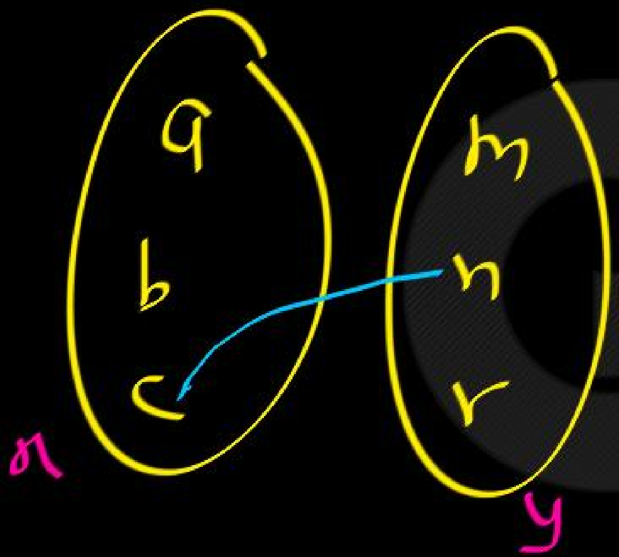
$y \longrightarrow$ set of Courses (m, n, r)

$T(x, y)$: Student x has taken Course y

$\exists_x \forall_y \, T(x, y)$: Some Student has taken all the Courses.



(4) $\exists x \, \exists y \, P(x,y)$



$P(x,y)$

$P(c,n) = T$

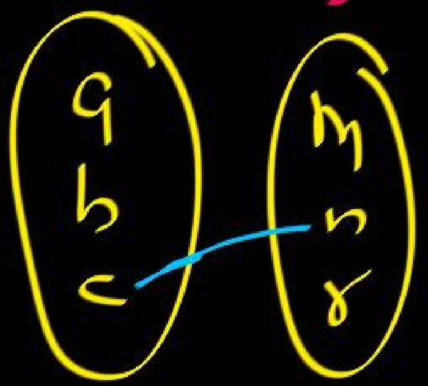
(c,n)

$x \longrightarrow$ set of Students (a, b, c)

$y \longrightarrow$ set of Courses (m, n, r)

$T(x, y)$: Student x has taken Course y

$\exists_x \exists_y T(x, y)$: Some Student has taken some Course.



$T(x, y)$

Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

find the Truth Value of :

1. $\forall x \forall y\, P(x,y)$
2. $\forall x \exists y\, P(x,y)$
3. $\exists x \forall y\, P(x,y)$
4. $\exists x \exists y\, P(x,y)$

Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

find the Truth Value of :

1. $\forall x \forall y \, P(x,y)$ — F
2. $\forall x \exists y \, P(x,y)$ — T
3. $\exists x \forall y \, P(x,y)$ — F
4. $\exists x \exists y \, P(x,y)$ — T

Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

① $\forall x \forall y\, P(x, y)$: false

$\forall x \forall y\, (x + y = 17)$

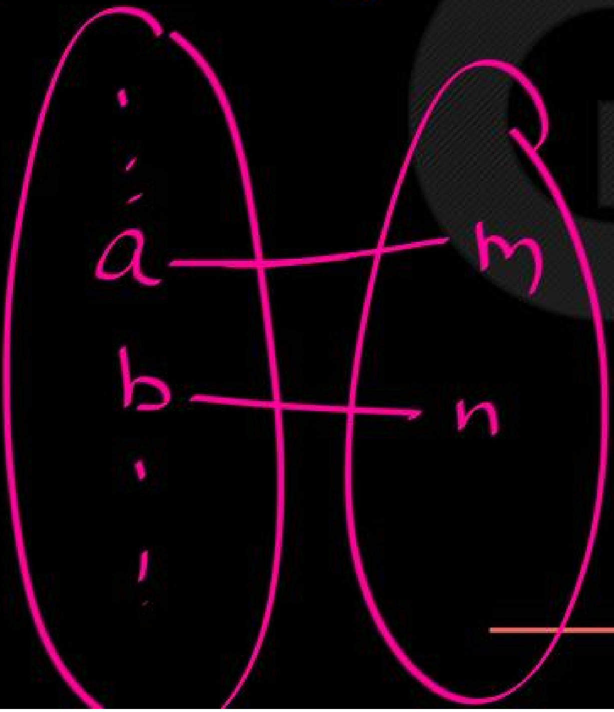
$x = 16$

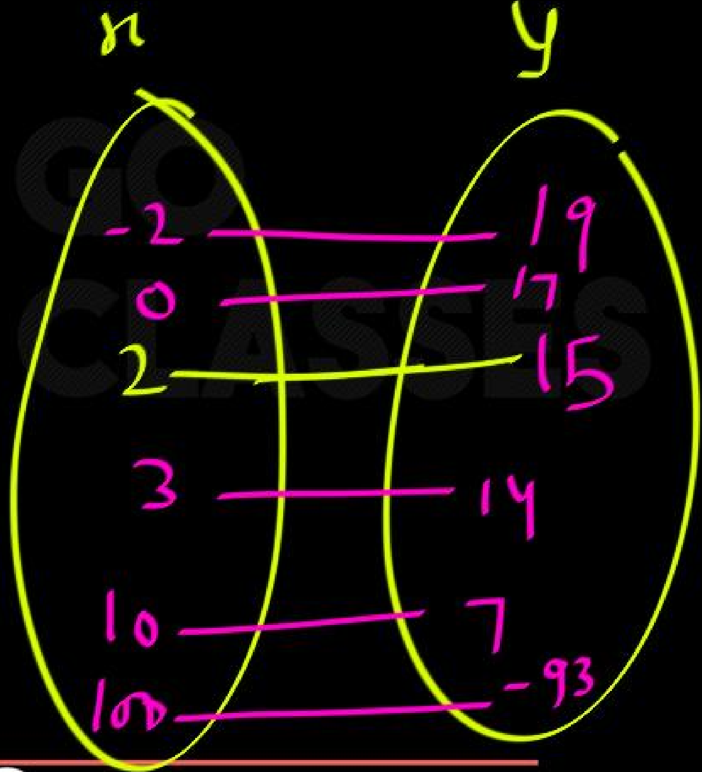
$y = 1$

$P(\underset{x}{\uparrow}, \underset{y}{\uparrow}) = \text{True}$

Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

2) $\forall x \exists y \; P(x, y)$ → True

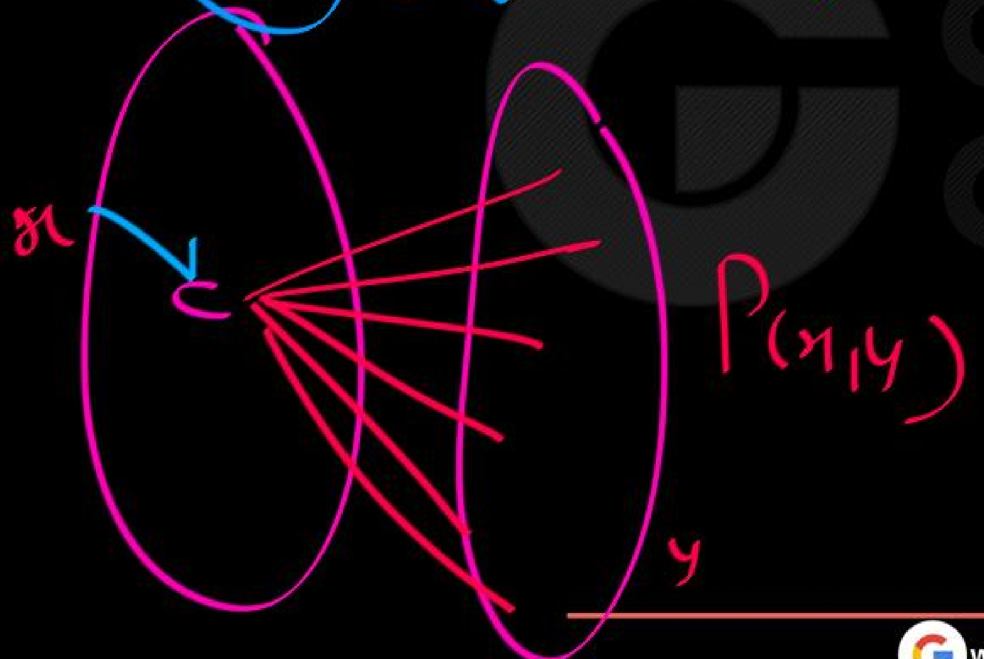


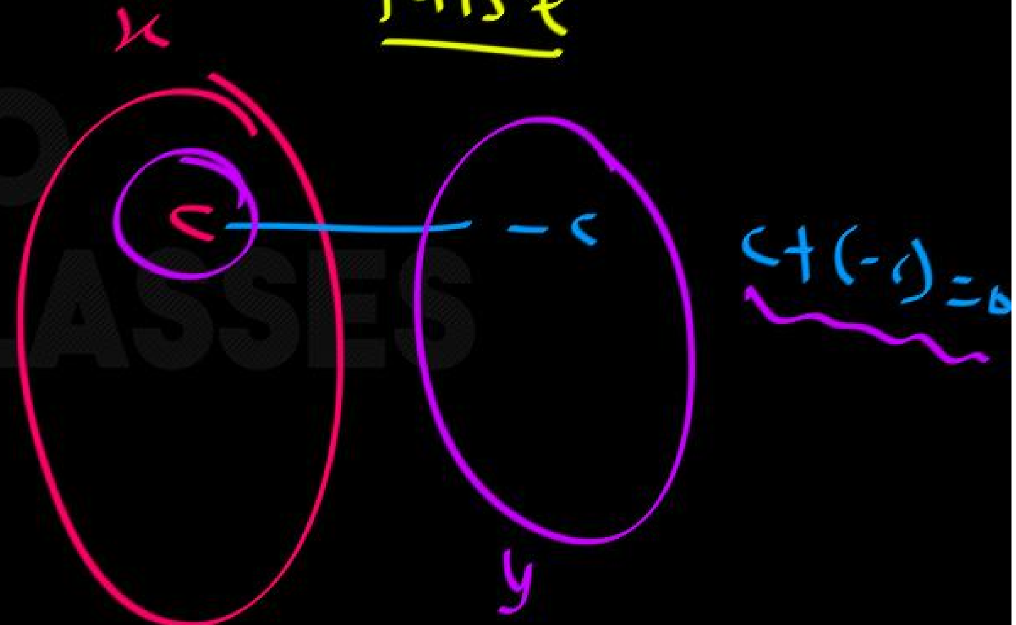


$x + y = 17$

Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

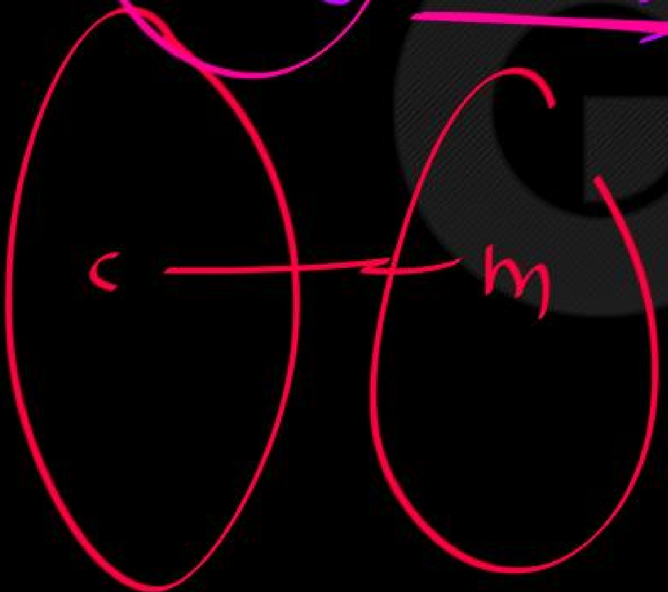
(3) $\exists x \forall y \, P(x,y)$ false

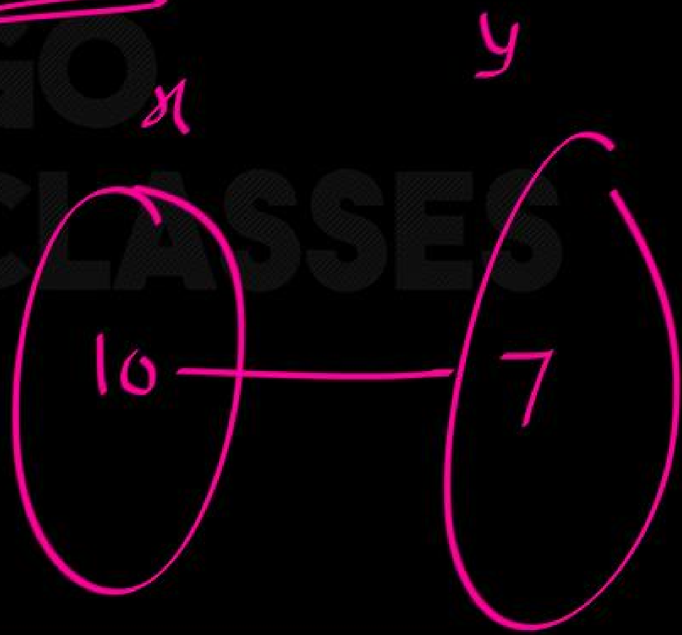




Let $P(x, y)$ denote $x + y = 17$, and let U be the set of integers.

(4) $\exists x \exists y \; P(x, y) = \text{True}$





$x + y = 17$

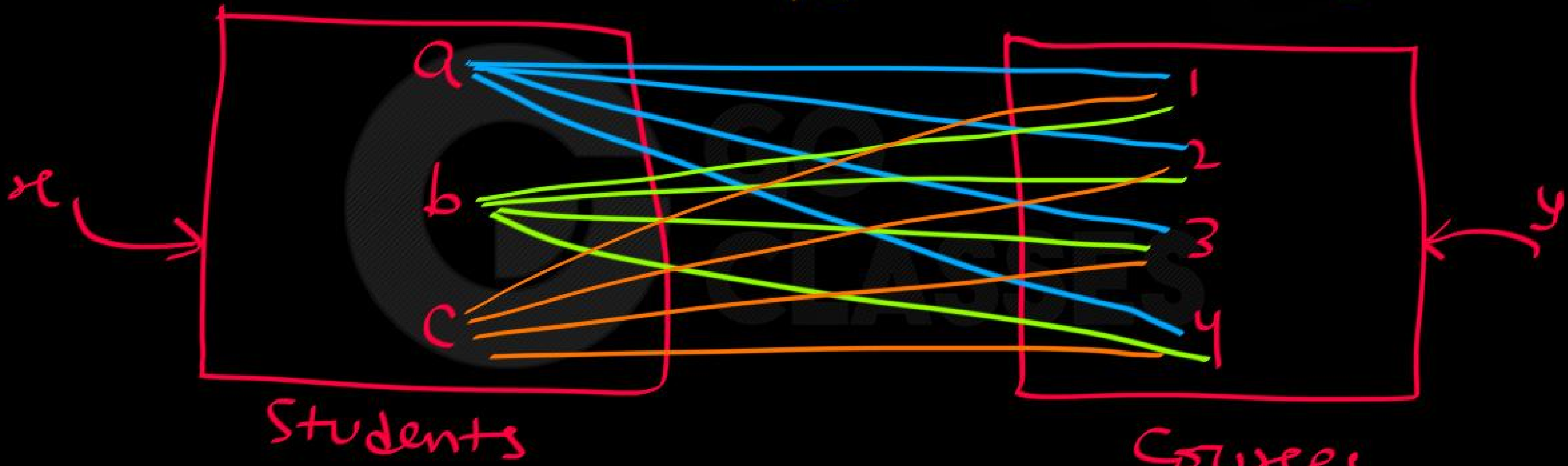
Domain of x : set of all students $\{a, b, c\}$

Domain of y : set of all Courses $\{1, 2, 3, 4\}$

$T(x, y)$: Student x has taken Course y.

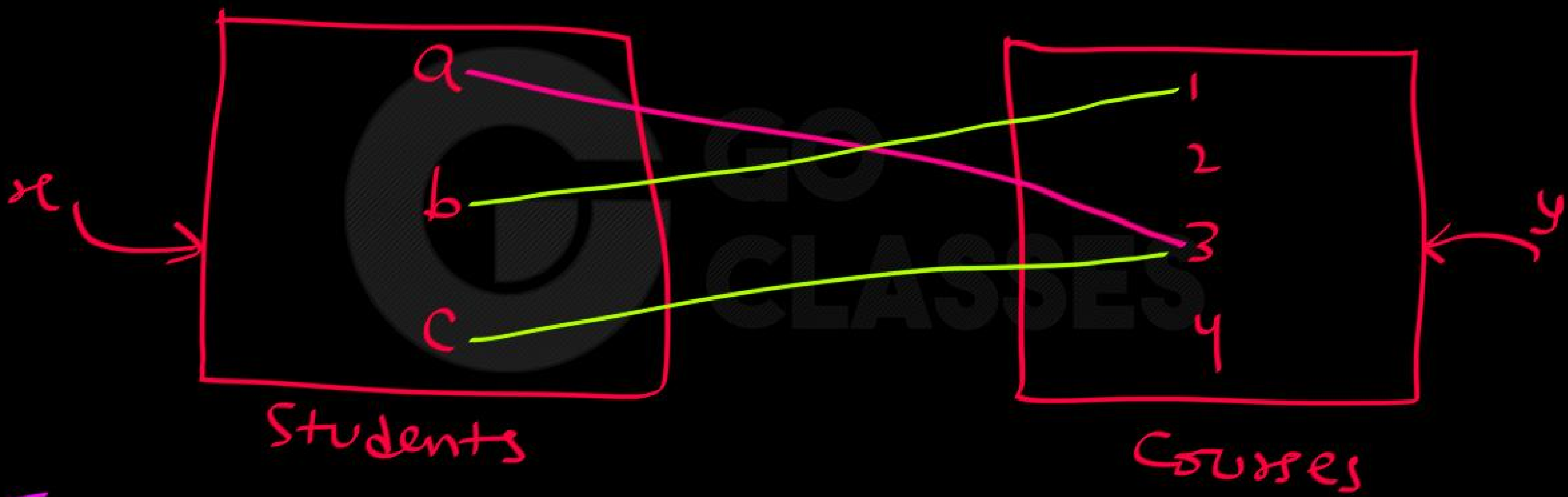
$\forall x \forall y\, T(x, y)$ means $T(\ ,\ ) = \text{True}$

$\forall x$ $\forall y$



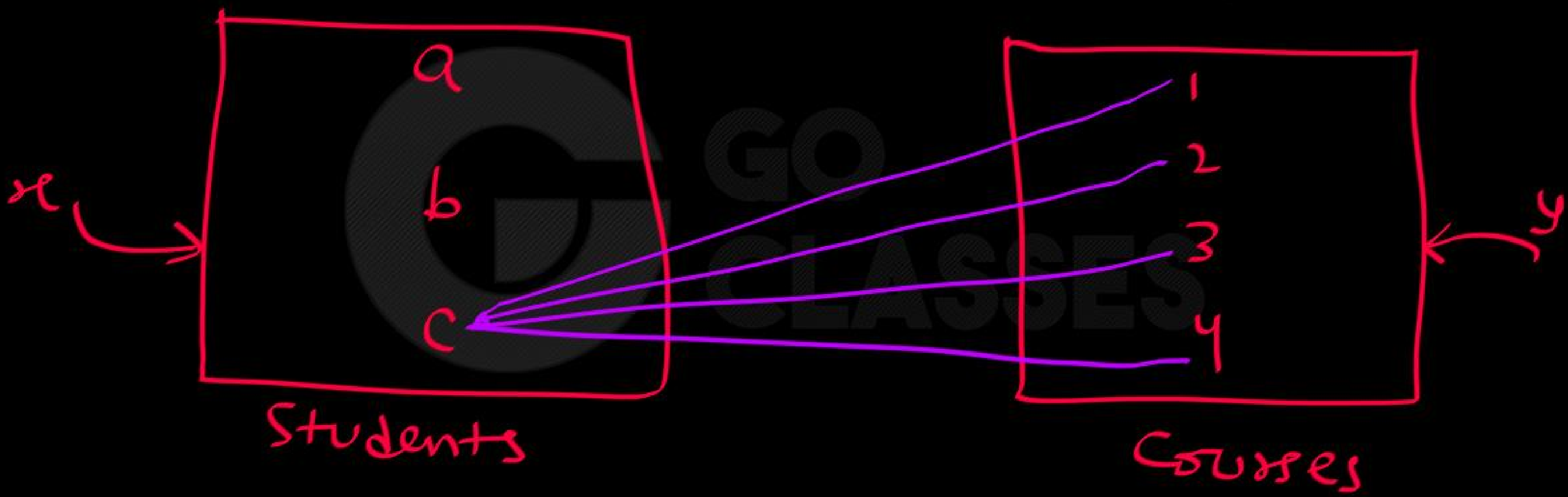
Every student has taken every course.

$\forall x\, \exists y\, T(x, y)$: for everyone, there is someone



Every Student has taken some course.

$\exists_x \forall_y T(x,y)$: There is someone, who is for everyone



Some student has taken all courses.

$\underbrace{\exists x}\ \underbrace{\exists y}\ T(x, y)$

Some student has taken some Course.

$$\forall y\, \exists x\, T(x,y)$$

$$\exists y\, \forall x\, T(x,y)$$

$$\exists y\, \exists x\, T(x,y)$$

$$\forall y\, \forall x\, T(x,y)$$

!!

$$\forall x\, \exists y\, T(x,y)$$

$$\exists x\, \forall y\, T(x,y)$$

$$\exists x\, \exists y\, T(x,y)$$

$$\forall x\, \forall y\, T(x,y)$$

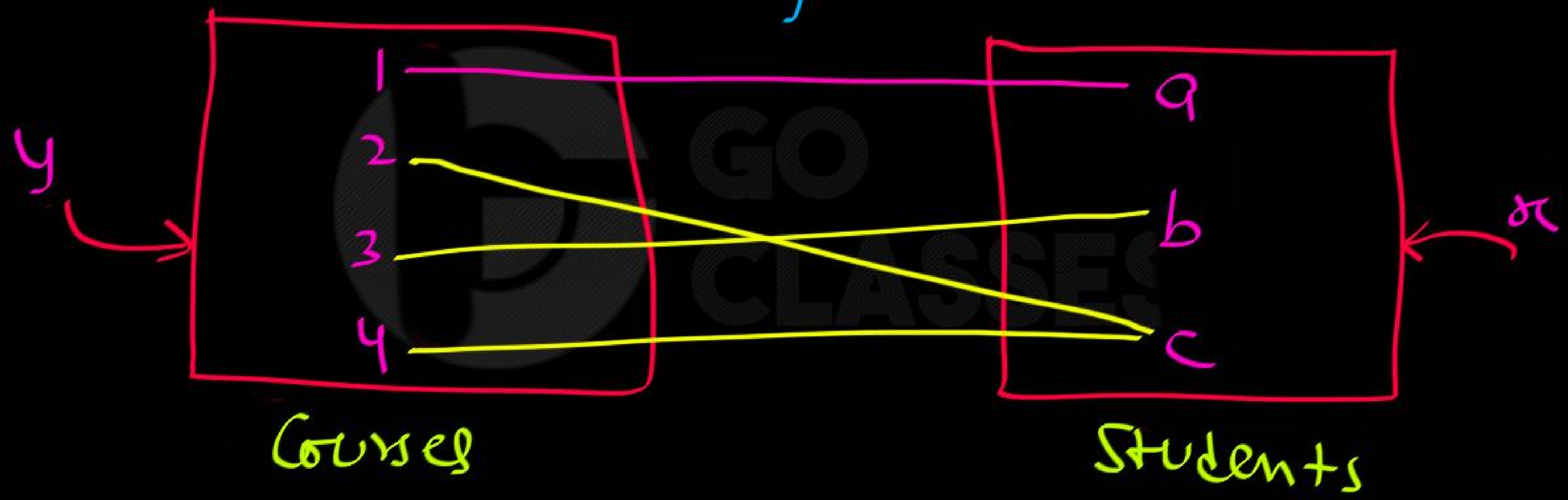
✓

Quantifiers are read from left to right.

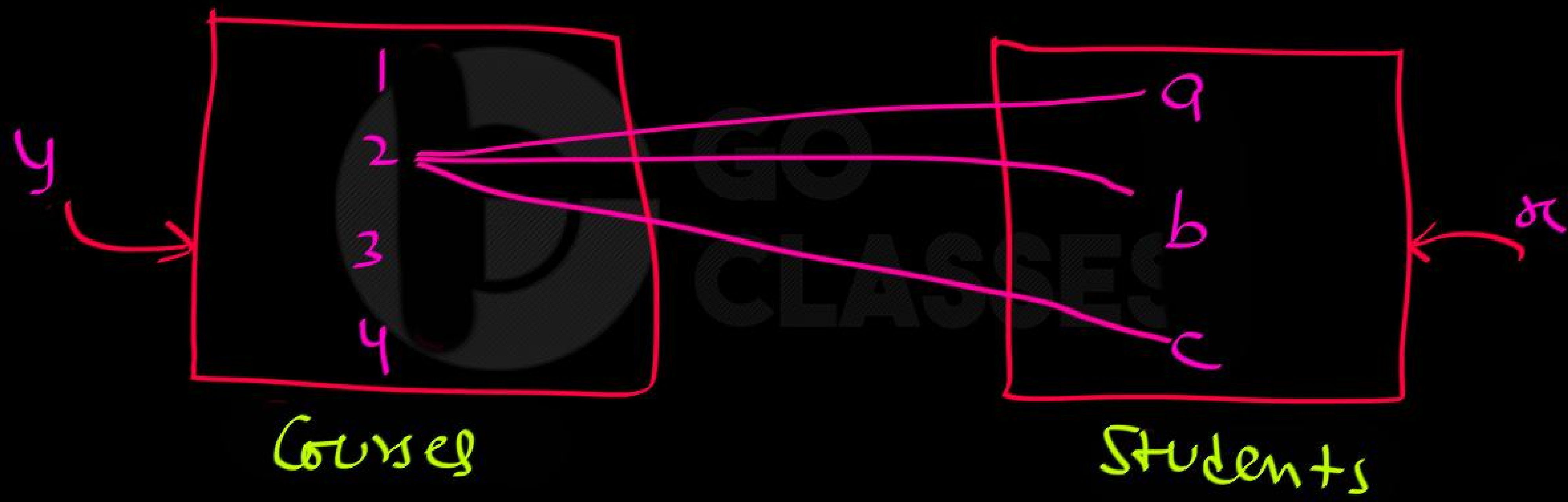
$$\overrightarrow{\forall_x \, \exists_y \, P(x,y)}$$

$$\overrightarrow{\forall_y \, \exists_x \, P(x,y)}$$

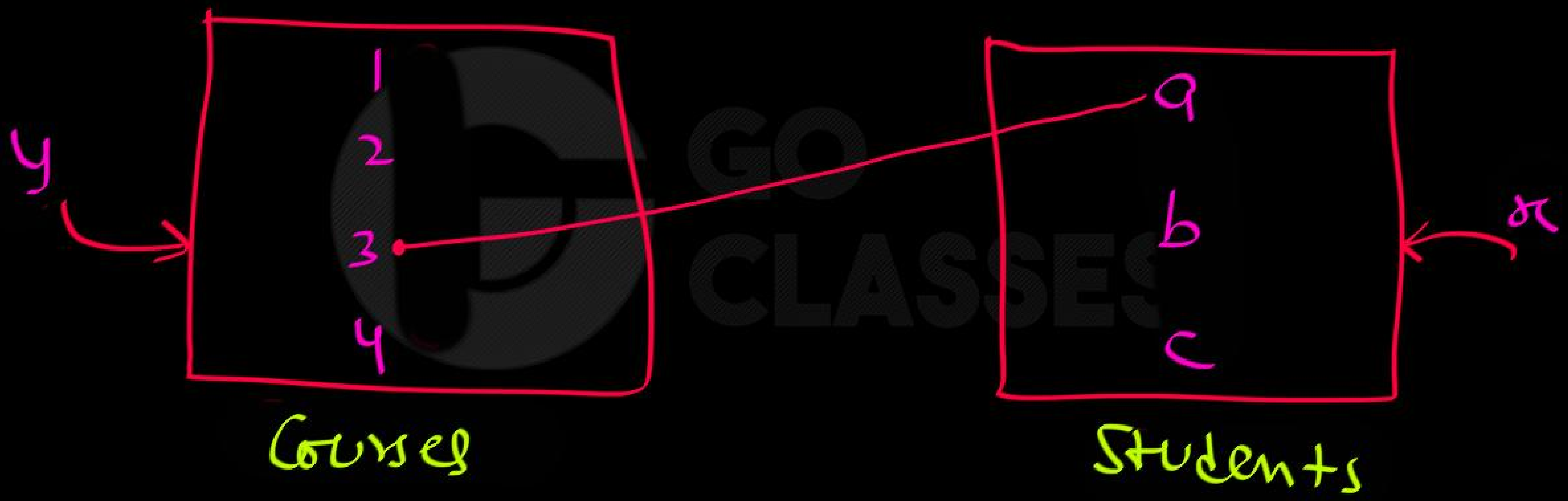
$\forall_y \exists_x T(x,y)$: Every Course is taken by some Student.



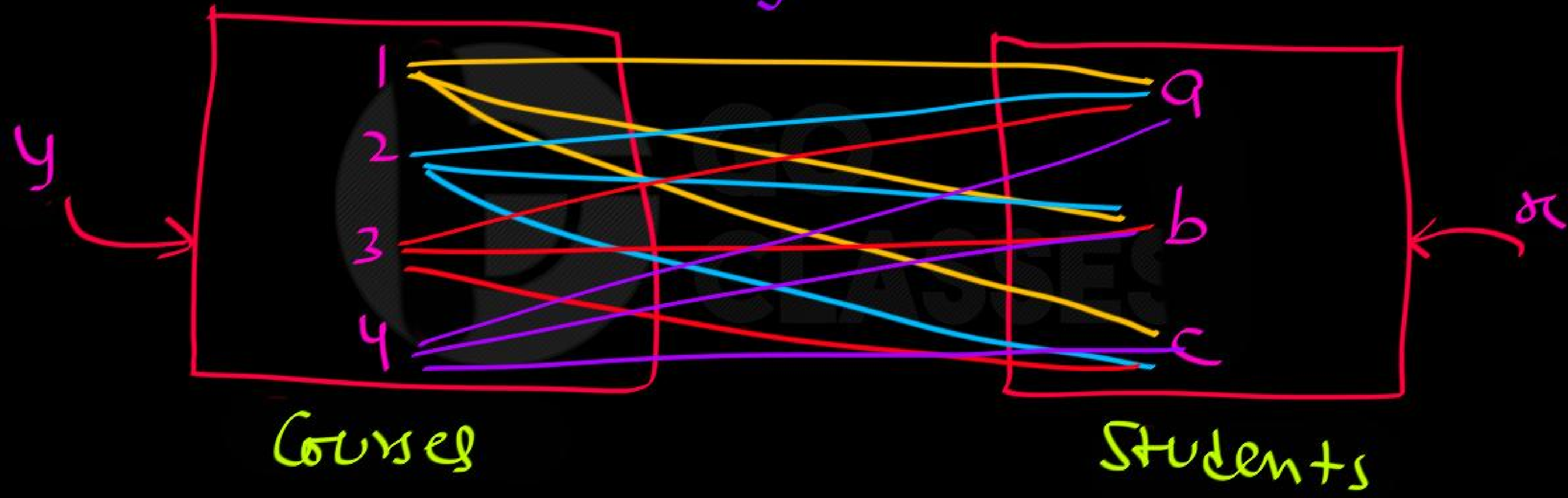
$\exists y \forall x\, T(x, y)$: Some Course is Taken by all the students.



$\exists y \exists x \, T(x, y)$; Some Course is taken by some Student.



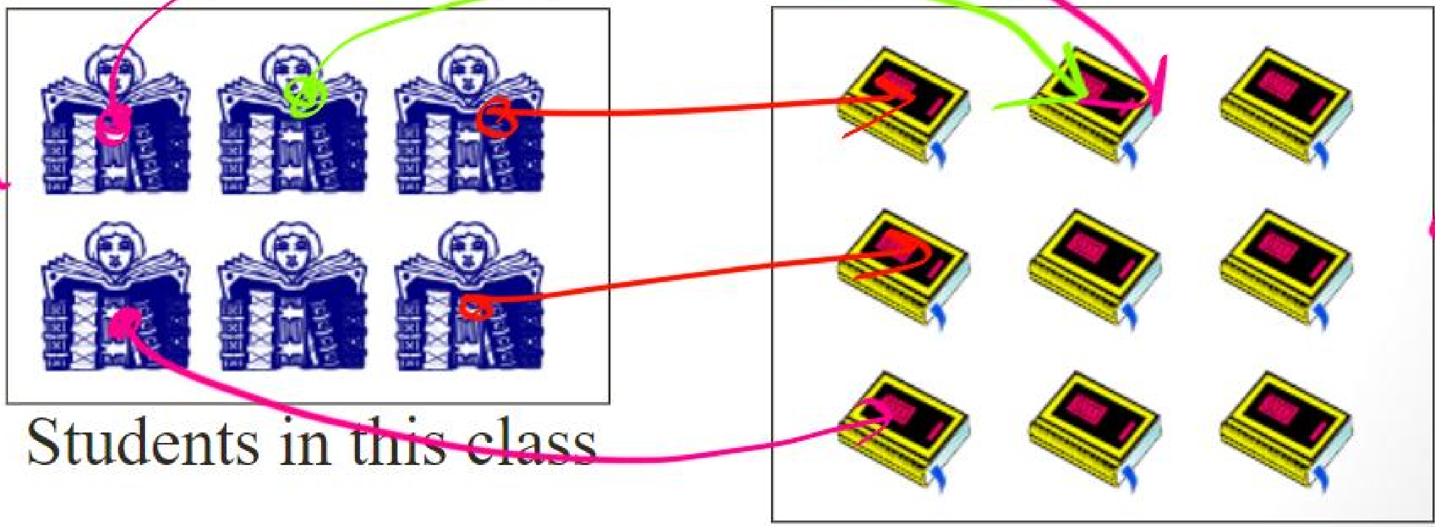
$\forall_y \forall_x \, T(x,y)$; Every Course is Taken by every student.



$$\forall x \exists y P(x, y)$$

For every student in this class, there is a CS class that student has taken.

x



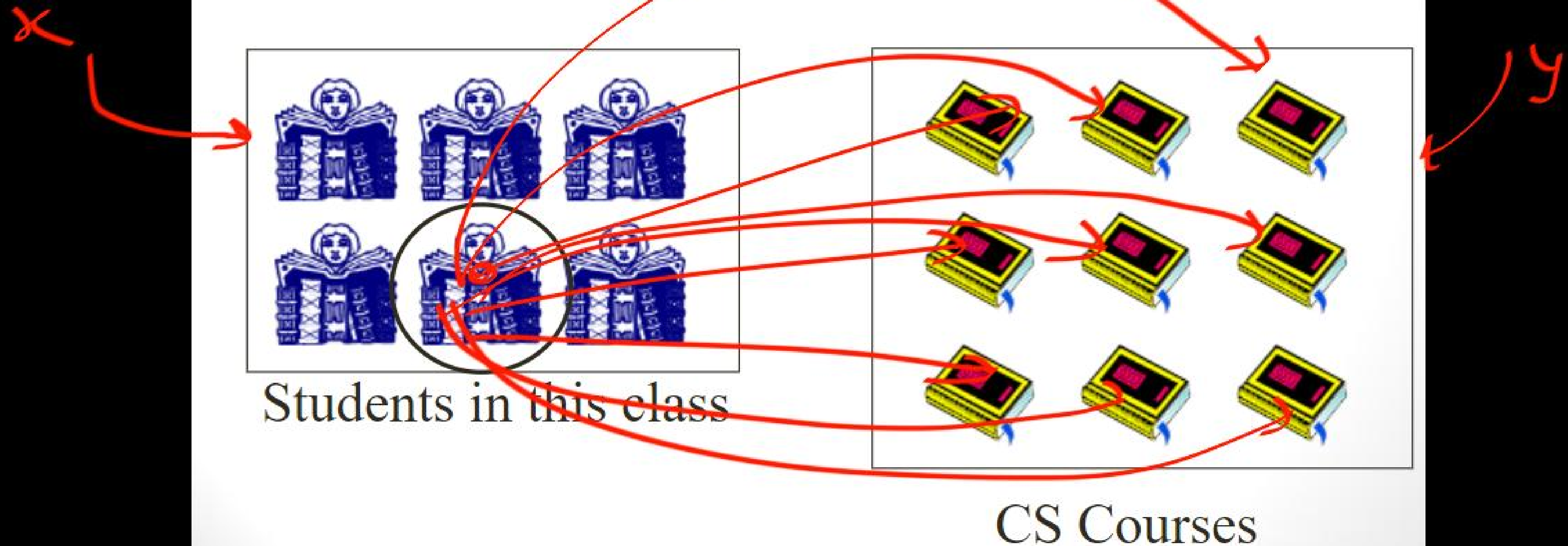
Students in this class

CS Courses

y

$$\exists x \forall y P(x, y)$$

There is a student in this class who has taken every CS class.



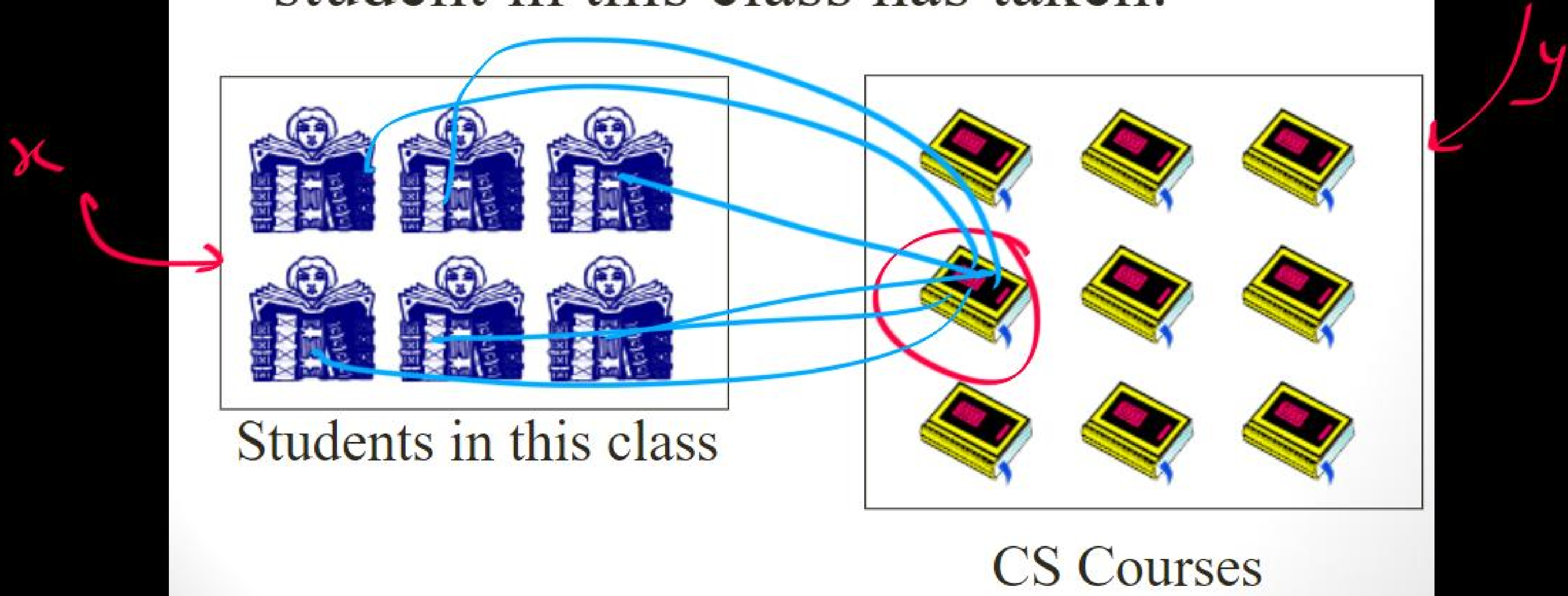
GO Classes

GO CLASSES

GO Classes

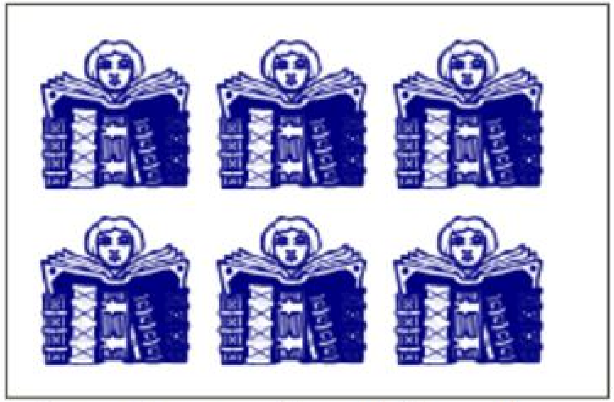
$$\exists y \forall x P(x, y)$$

There is a CS course that every student in this class has taken.

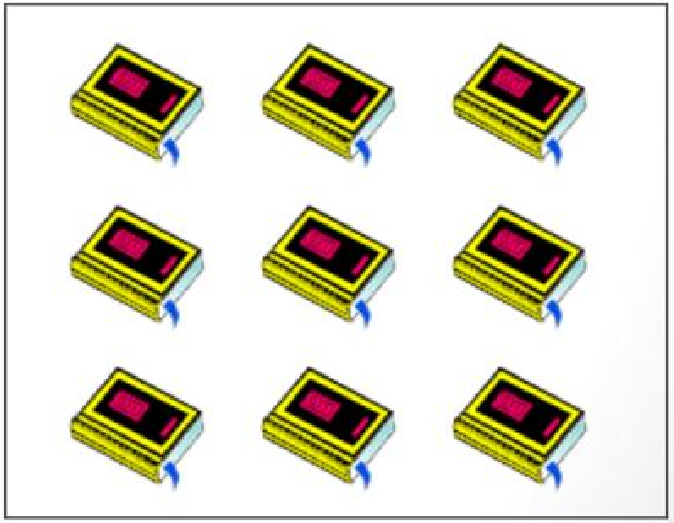


$$\forall y \exists x P(x, y)$$

For every CS course, there is a student in this class who has taken the course.

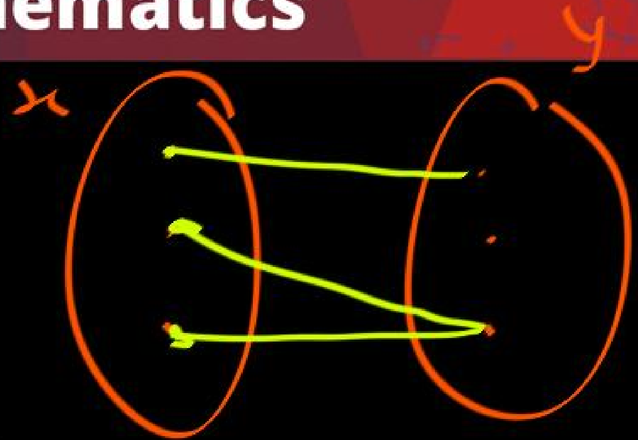


Students in this class

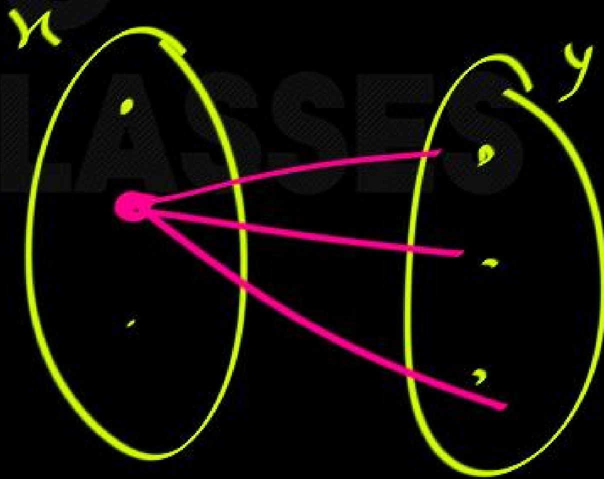


CS Courses

$\forall x \; \exists y \; P(x,y)$



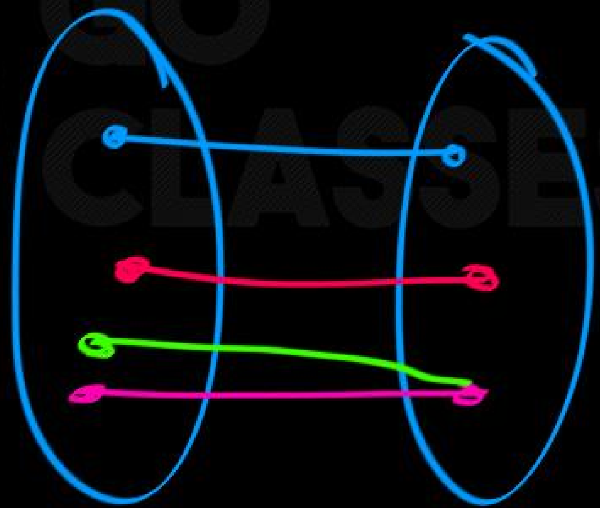
$\exists x \; \forall y \; P(x,y)$



"Every person loves someone"

$\forall_x \exists_y L(x,y)$

For every person p…
… there is a person y…
… they(p) love



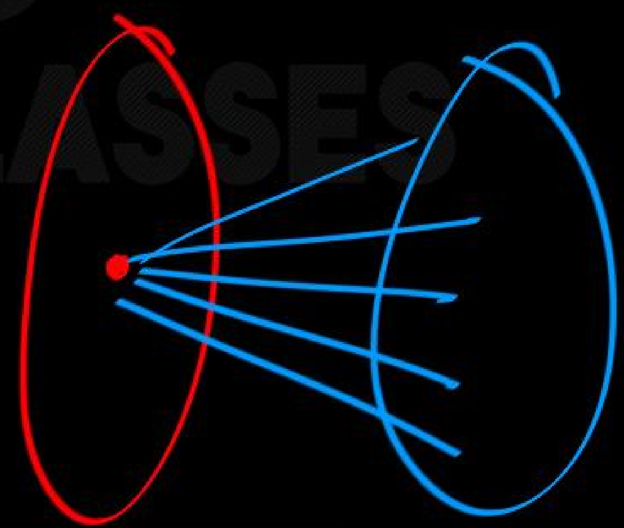
"Someone loves everyone"

$\exists_x \forall_y L(x, y)$

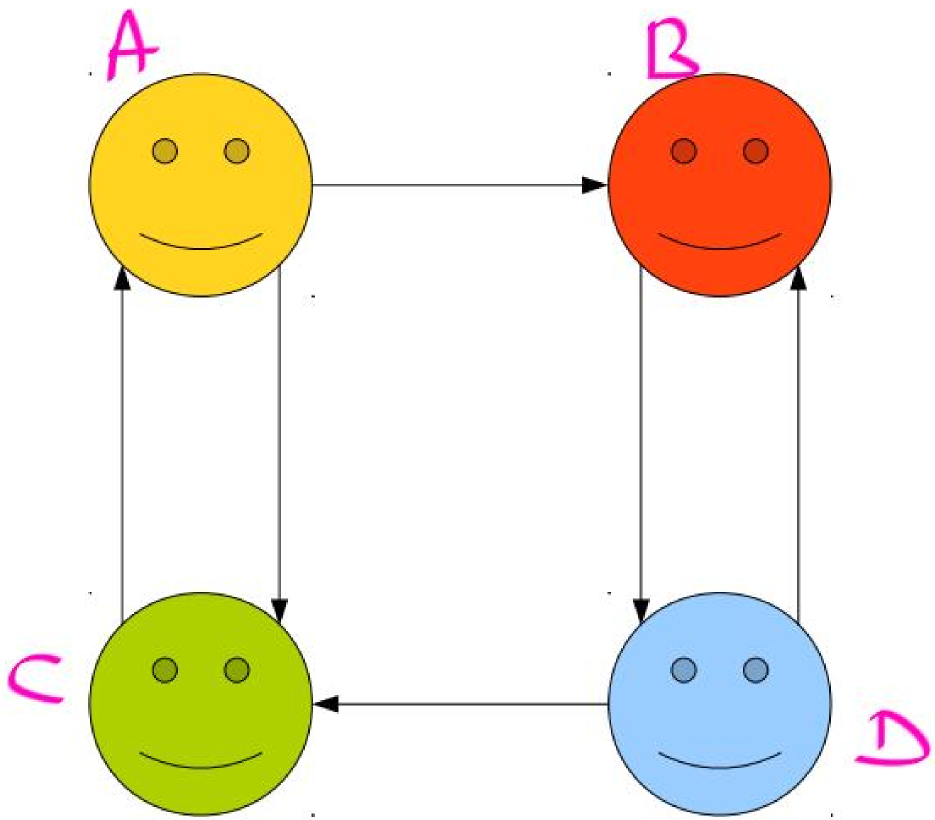
There is a person p…

… for every person person y…

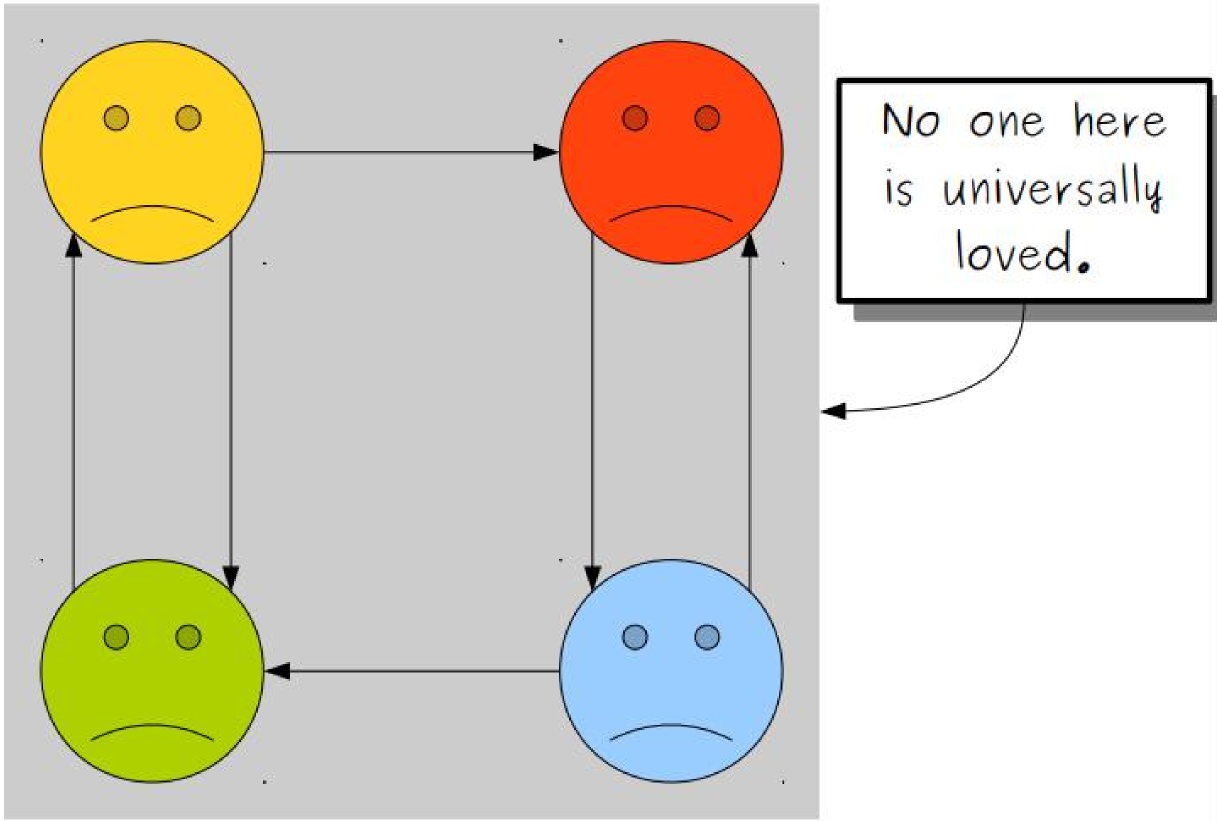
… p loves y



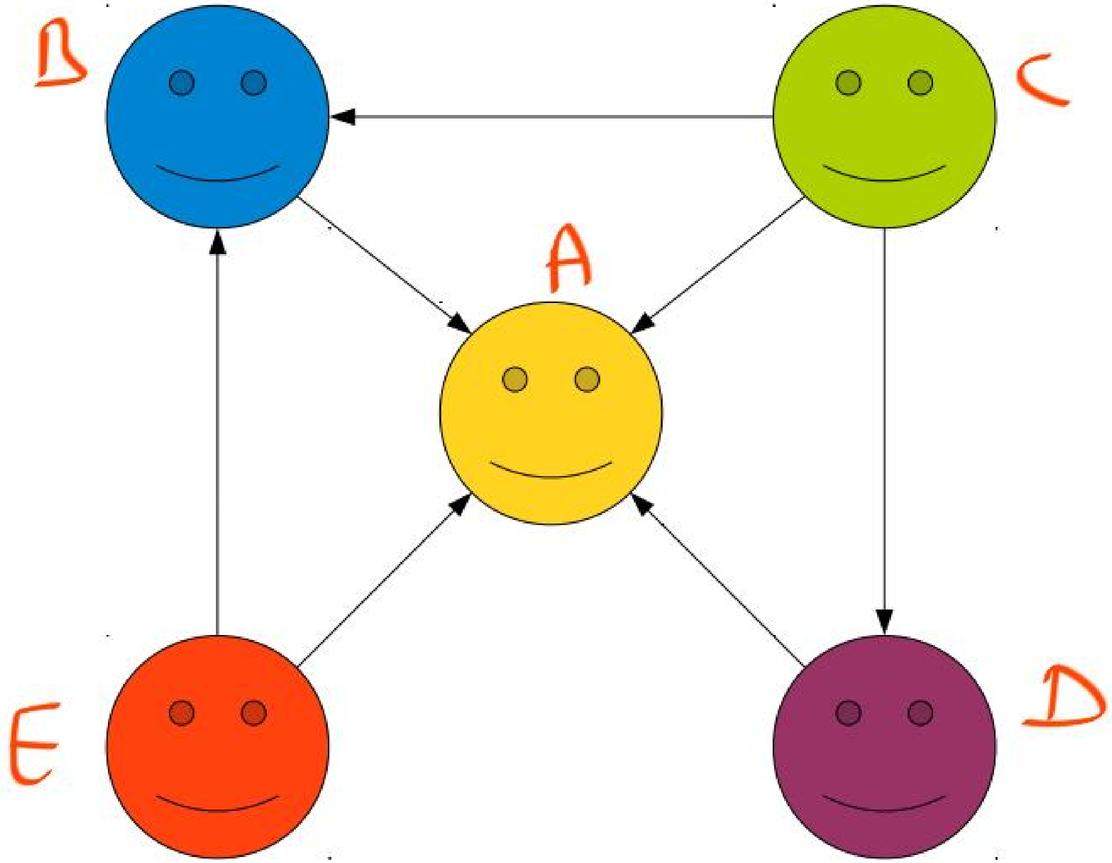
Every Person Loves Someone Else



Every Person Loves Someone Else

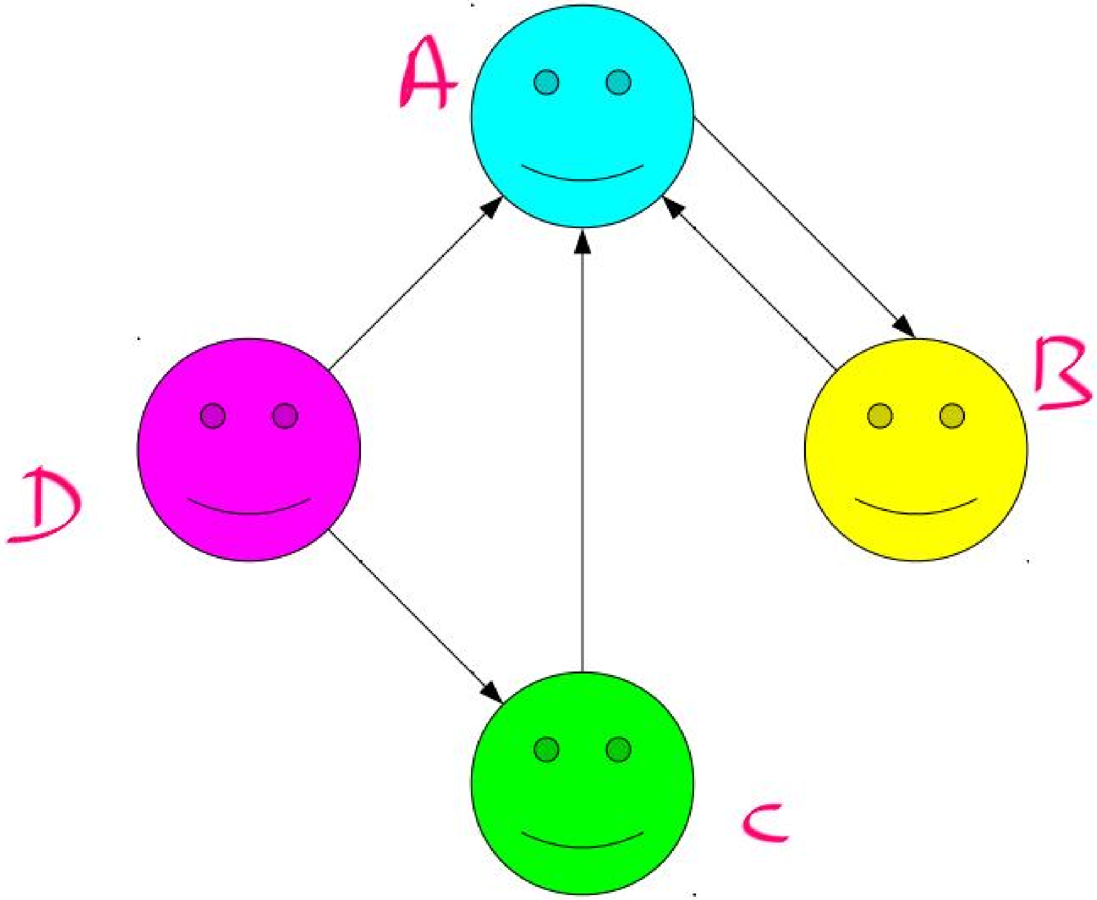


There is Someone Everyone Else Loves



$\exists_x \forall_y$

Every Person Loves Someone Else ***and*** There is Someone Everyone Else Loves

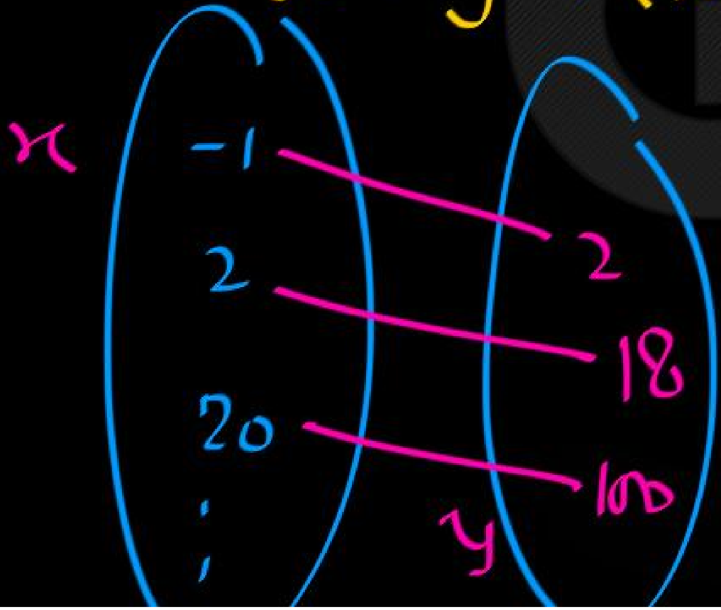


Q: Domain: set of integers.

$$P(x,y) : x < y$$

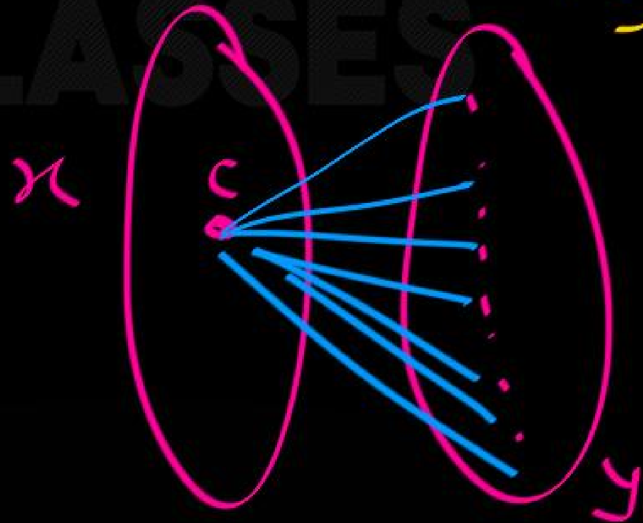
True

(1) $\forall x \exists y \; P(x,y)$



$x<y$

(2) $\exists x \forall y \; P(x,y)$: false



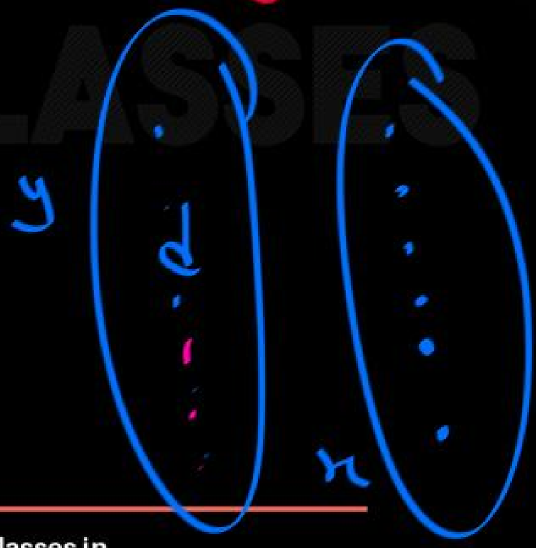
$\underbrace{\infty, -\infty}$ not numbers

Q: Domain: set of integers.

$P(x,y)$: $x < y$

① $\forall_x \exists_y P(x,y)$ → True

② $\exists_y \forall_x (x<y)$ → false



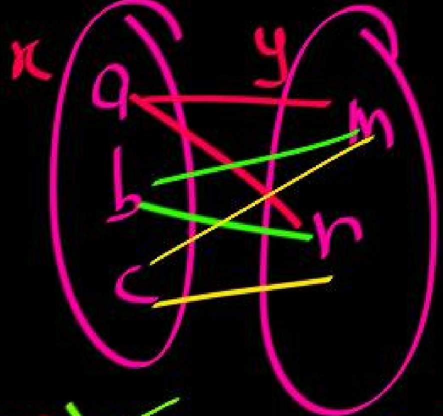
$x < d$

$$\left\{ \begin{array}{l} \forall_x \exists_y \alpha \not\equiv \exists_x \forall_y \alpha \\ \forall_x \exists_y \alpha \not\equiv \exists_y \forall_x \alpha \end{array} \right\}$$

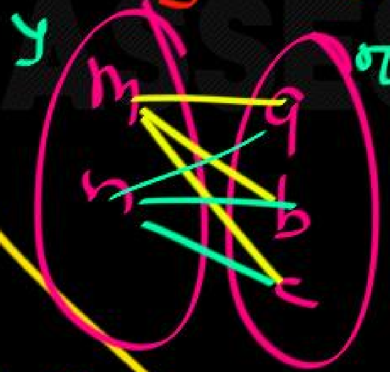
Eg: $x \rightarrow \{a, b, c\}$

$y \rightarrow \{m, n\}$

$$\forall x \, \forall y \, P(x,y) \equiv \forall y \, \forall x \, P(x,y)$$



$P(x,y)$



$P(x,y)$

$P(a,m)$, $P(a,n)$, $P(b,m)$, $P(b,n)$, $P(c,m)$, $P(n,c)$

$P(c,n)$
$P(c,m)$
$P(b,n)$
$P(b,m)$
(m,a) $P(a,m)$
$P(a,n)$

$$\begin{cases} \forall x \, \forall y \, P(x,y) \equiv \forall y \, \forall x \, P(x,y) \checkmark \\ \exists x \, \exists y \, P(x,y) \equiv \exists y \, \exists x \, P(x,y) \checkmark \end{cases}$$

$$\exists x \exists y\, P(x,y) \equiv \exists y \exists x\, P(x,y)$$

a b c

m n

$P(b,m)$

True

y

m n

x

a b c

$P(x,y)$

$P(b, n)$

True

Quantifier Ordering

- The statement

$$\forall x.\ \exists y.\ P(x, y)$$

means "for any choice of x, there's some choice of y where $P(x, y)$ is true."

- The choice of y can be different every time and can depend on x.

Quantifier Ordering

- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some x where for any choice of y, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of y, this places a lot of constraints on what x can be.

Order matters when mixing existential and universal quantifiers!

Nested quantifiers (example)

Translate the following statement into English.

$$\forall x\, \forall y\, (x + y = y + x)$$

Domain: real numbers

Nested quantifiers (example)

Translate the following statement into English.

$$\forall x \, \forall y \, (x + y = y + x)$$ — True

Domain: real numbers

Solution:

For all real numbers x and y, $x + y = y + x$.

Nested quantifiers (example)

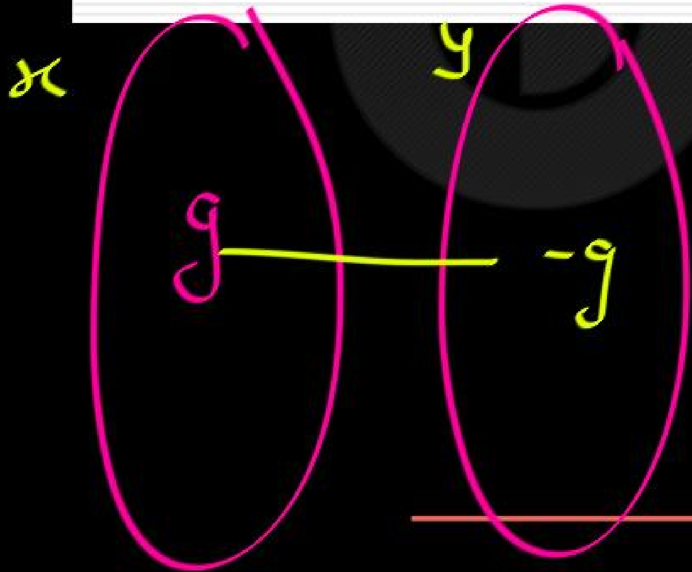
Translate the following statement into English.

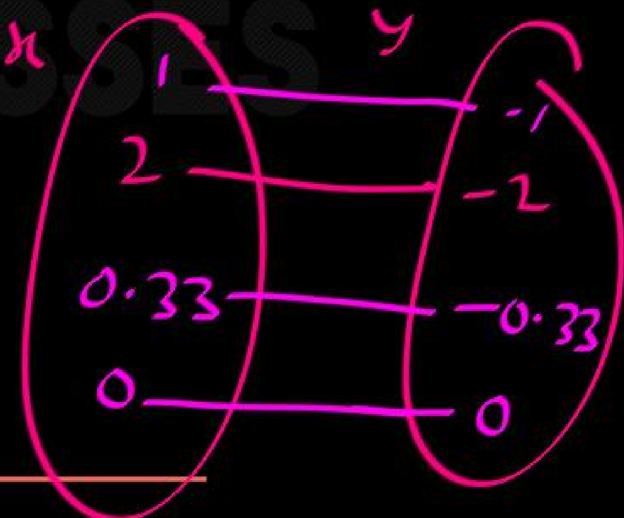
$\forall x \, \exists y \, (x = -y)$ — True

Domain: real numbers

$2 = -(-2)$

$x = -y$





Nested quantifiers (example)

Translate the following statement into English.

$\forall x \, \exists y \, (x = -y)$

Domain: [redacted] natural numbers

Nested quantifiers (example)

Translate the following statement into English.

$\forall x \, \exists y \, (x = -y)$ —— false

Domain: [redacted] natural numbers

x: 1, 2, 3, 4, ...

y: 1, 2, 3, ...

Nested quantifiers (example)

Translate the following statement into English.

$\forall x \, \exists y \, (x = - y)$

Domain: real numbers

Solution:

For every real number x, there is a real number y such that $x = - y$.

Nested quantifiers (example)

Translate the following statement into English.

$$\forall x\ \forall y\ ((x > 0) \wedge (y < 0) \rightarrow (xy < 0))$$

Domain: real numbers

for every real numbers x, y;
If $x > 0$, $y < 0$ then $xy < 0$.

Nested quantifiers (example)

Translate the following statement into English.

$$\forall x\, \forall y\, ((x > 0) \land (y < 0) \rightarrow (xy < 0))$$

Domain: real numbers

Solution:

For every real numbers x and y, if x is positive and y is negative then xy is negative.

The product of a positive real number and a negative real number is always a negative real number.

The order of quantifiers (example)

Assume $P(x,y)$ is $(xy = yx)$.

Translate the following statement into English.

$\forall x\ \forall y\ P(x,y)$ domain: real numbers

Solution:

For all real numbers x, for all real numbers y, $xy = yx$.

For every pair of real numbers x, y, $xy = yx$.

The order of quantifiers

The order of nested **universal** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

The order of quantifiers (example)

Assume $P(x,y)$ is $(xy = 6)$.

Translate the following statement into English.

$\exists x \, \exists y \, P(x,y)$ domain: integers

Solution:

There is an integer x for which there is an integer y that $xy = 6$.

There is a pair of integers x, y for which $xy = 6$.

The order of quantifiers

The order of nested **existential** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

The order of quantifiers (example)

Assume P(x,y) is (x + y = 10).

$\forall x \, \exists y \, P(x,y)$ domain: real numbers

For all real numbers x there is a real number y such that x + y = 10.

True (y = 10 - x)

$\exists y \, \forall x \, P(x,y)$ domain: real numbers

There is a real number y such that for all real numbers x, x + y = 10.

False

So, $\forall x \, \exists y \, P(x,y)$ and $\exists y \, \forall x \, P(x,y)$ are not logically equivalent.

Nested quantifiers

Example: for P(x,y) that is "x > y" and domain being the real numbers:

$\forall_x \forall_y P(x, y)$ "each real is greater than all reals"(false)

$\forall_x \exists_y P(x, y)$ "for each real x there exists real y so that x is greater than y" (true)

$\exists_x \forall_y P(x, y)$ "there is x that is greater than any real" (false)

$\exists_x \exists_y P(x, y)$ "there exist some real x and y such that x is greater than y" (true)

The order of nested quantifiers matters!

If the quantifiers are of the same kind they relative order (if they are neighbours) does not matter, however the order of different kinds may matter:

Examples:
$\forall_x \forall_y P(x, y)$
is logically equivalent to:
$\forall_y \forall_x P(x, y)$
and both can be read as "each real is greater than all reals" (false)

However:

$\forall_x \exists_y P(x, y)$ "for each real x there exists real y so that x is greater than y" (true)
$\exists_y \forall_x P(x, y)$ "there is y that is greater than any real" (false)

The Order of Quantifiers

- Let $P(x, y)$ be that statement "$x + y = 0$", what is the truth values of $\exists y \forall x P(x, y)$ and $\forall x \exists y P(x, y)$?
- For $\exists y \forall x P(x, y)$
  - **False**. There is no real number such that $x + y = 0$ for all real numbers x.
- For $\forall x \exists y P(x, y)$
  - **True**. Given a real number x, there is a real number y such that $x + y = 0$ $(y = -x)$
- The order of quantifiers will make different results.

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y \, P(x,y)$

4. $\exists x \, \exists \, y \, P(x,y)$

Questions on Order of Quantifiers

Example 1: Let U be the real numbers,

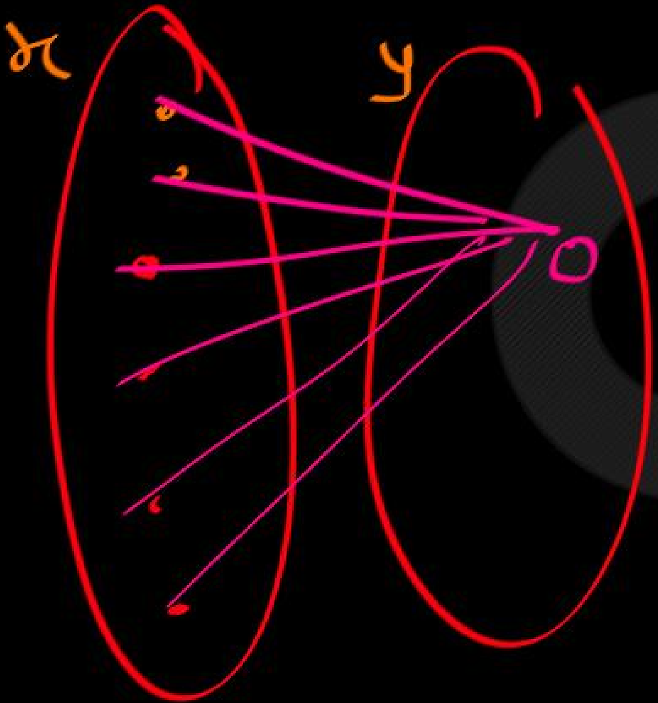
Define $P(x,y) : x \cdot y = 0$

What is the truth value of the following:

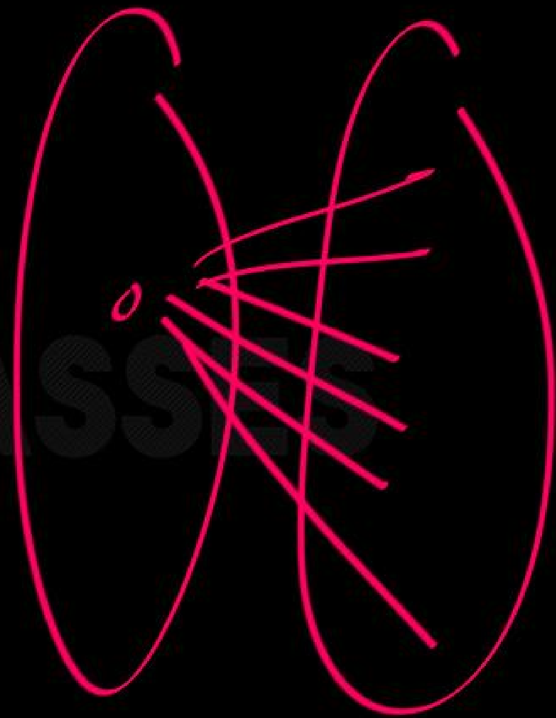
1. $\forall x \forall y P(x,y)$ false : Counter Eg : P(4, 2) = false
2. $\forall x \exists y P(x,y)$ True
3. $\exists x \forall y\, P(x,y)$ True
4. $\exists x \exists y\, P(x,y)$ True witness : P(0, 2) = True

$\forall x \exists y \, (xy = 0)$

$\exists x \forall y \, (xy = 0)$



$xy = 0$



Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$

2. $\forall x \exists y P(x,y)$

3. $\exists x \forall y\, P(x,y)$

4. $\exists x \exists y\, P(x,y)$

Questions on Order of Quantifiers

Example 2: Let U be the real numbers,

Define $P(x,y) : x / y = 1$

What is the truth value of the following:

1. $\forall x \forall y P(x,y)$; False $x=2$; $y=4$ $P(2,4) = \text{false}$
2. $\forall x \exists y P(x,y)$ false   for 0, there is no one.
3. $\exists x \forall y\, P(x,y)$ false
4. $\exists x \exists y\, P(x,y)$ True $P(2,2) = \text{True}$

$\forall x \exists y\,(x/y = 1)$

$\exists x \forall y\,(x/y = 1)$

$c/y = 1$

Quantification of two variable

- **$\forall x \, \forall y \, P(x,y)$**
  - When true?

    P(x,y) is true for every pair x,y.
  - When false?

    There is a pair x, y for which P(x,y) is false.

- **$\forall x \, \exists y \, P(x,y)$**
  - When true?

    For every x there is a y for which P(x,y) is true.
  - When false?

    There is an x such that P(x,y) is false for every y.

Quantification of two variable

- $\exists x \, \forall y \, P(x,y)$
  - When true?
    There is an x for which P(x,y) is true for every y.
  - When false?
    For every x there is a y for which P(x,y) is false.

- $\exists x \, \exists y \, P(x,y)$
  - When true?
    There is a pair x, y for which P(x,y) is true.
  - When false?
    P(x,y) is false for every pair x, y.

Nested Quantifiers

Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
Example: $\forall x \exists y P(x, y)$

Quantifier order matters!
$\forall x \exists y P(x, y) \neq \exists y \forall x P(x, y)$

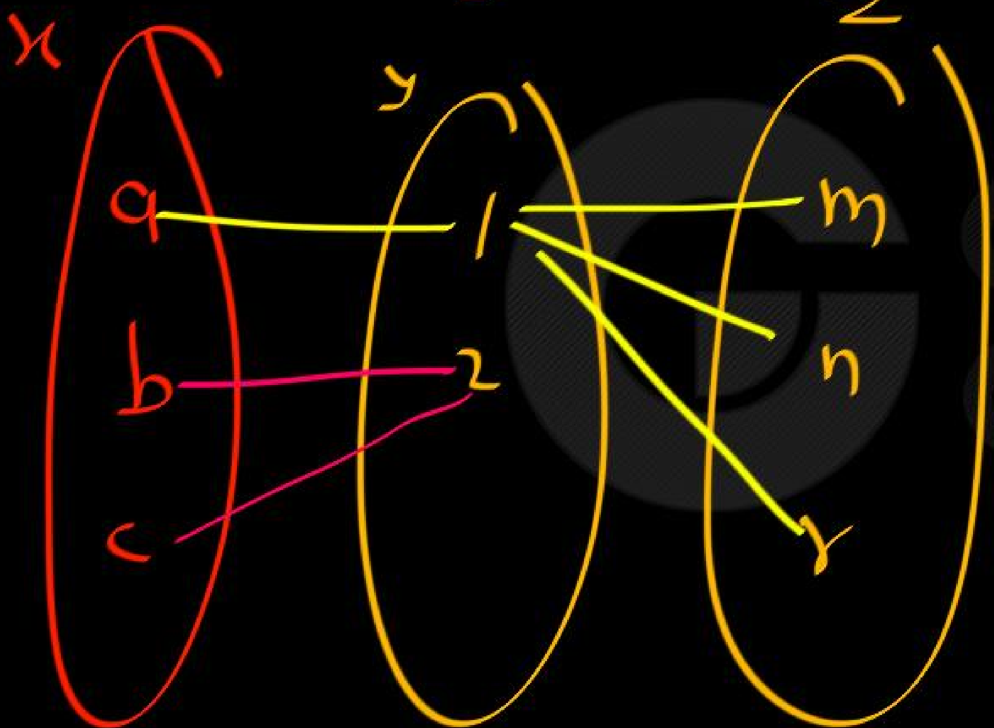
The order of quantifiers

The **order** of nested existential and universal quantifiers in a statement is important.

Nested Quantifiers for more than 2 Quantifiers :

$$\forall x \, \exists y \, \forall z \; P(x, y, z)$$

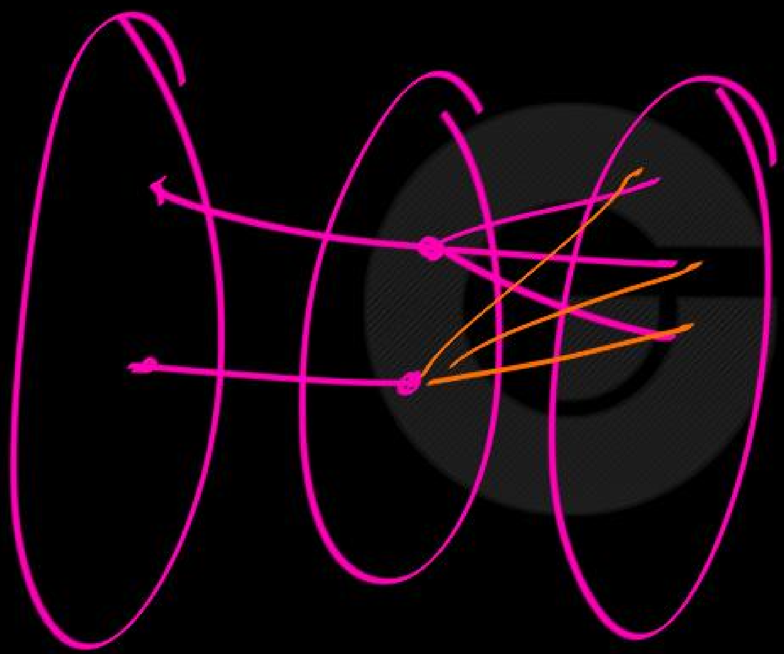
$\forall_x \exists_y \forall_z \; P(x,y,z)$



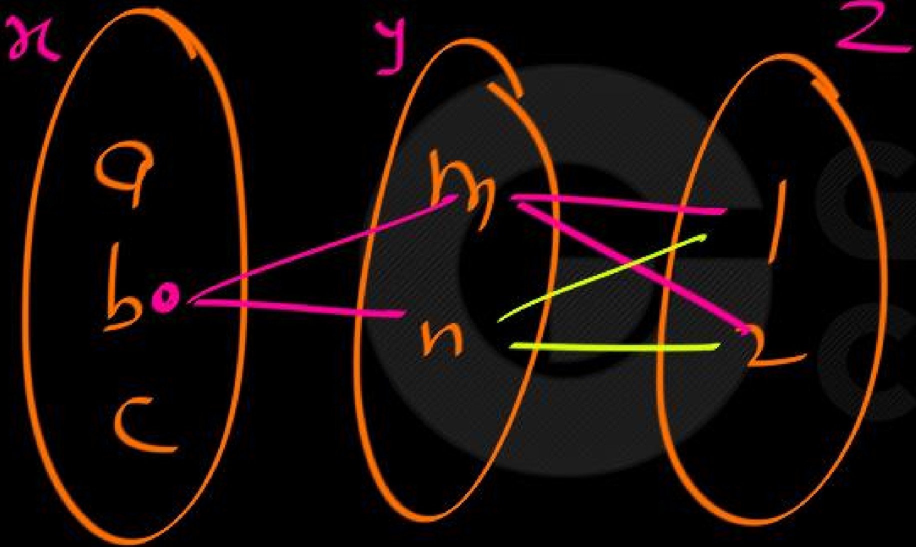
$$\left.\begin{array}{l} P(a,1,m) \\ P(a,1,n) \\ P(a,1,r) \end{array}\right\} \text{True}$$

$$\left.\begin{array}{l} P(b,2,m) \\ P(b,2,n) \\ P(b,2,r) \end{array}\right\} \text{True}$$

$\forall \quad \exists \quad \forall$



$$\exists_x \forall_y \forall_z P(x, y, z)$$



The order of quantifiers

Assume P(x,y,z) is (x + y = z).

∀x ∀y ∃z P(x,y,z)   domain: real numbers

For all real numbers x and y there is a real number z such that x + y = z.

True

∃z ∀x ∀y P(x,y,z)   domain: real numbers

There is a real number z such that for all real numbers x and y x + y = z.

False

So, ∀x ∀y ∃z P(x,y,z) and ∃z ∀x ∀y P(x,y,z) are not logically equivalent.

$P(x, y, z) = \boxed{x + y = z}$

Domain : real

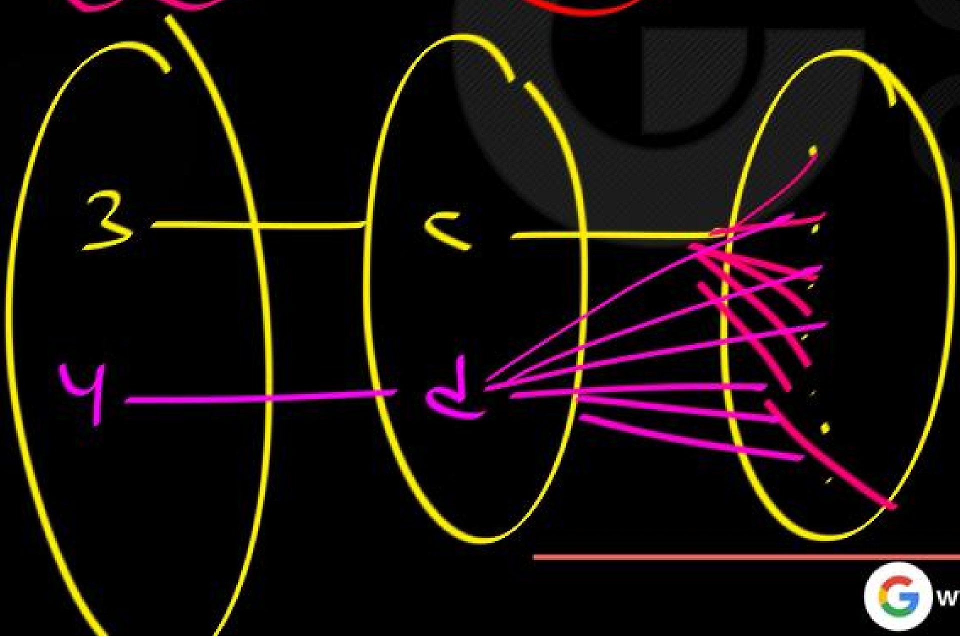
$\forall x \forall y \forall z \; P(x, y, z)$: false

$P(2, 2, 10) =$ false

$P(x,y,z) = \boxed{x+y=z}$

Domain : real

$\forall_x \exists_y \forall_z P(x,y,z)$: false



$x+y=z$

$x \quad y \quad \forall z$

$\downarrow \quad \downarrow \quad \downarrow$

$3 — 5 \quad z$

$3+5=10$

$P(x, y, z) = \boxed{x + y = z}$

Domain : Real

$\underbrace{\forall x \, \forall y}\, \exists z \; P(x, y, z)$: True

z

(2, 4, 6)

(3, 100, 103)

(−3, 3, 0)

$P(\;\;,\;\;,\; \exists z)$: True

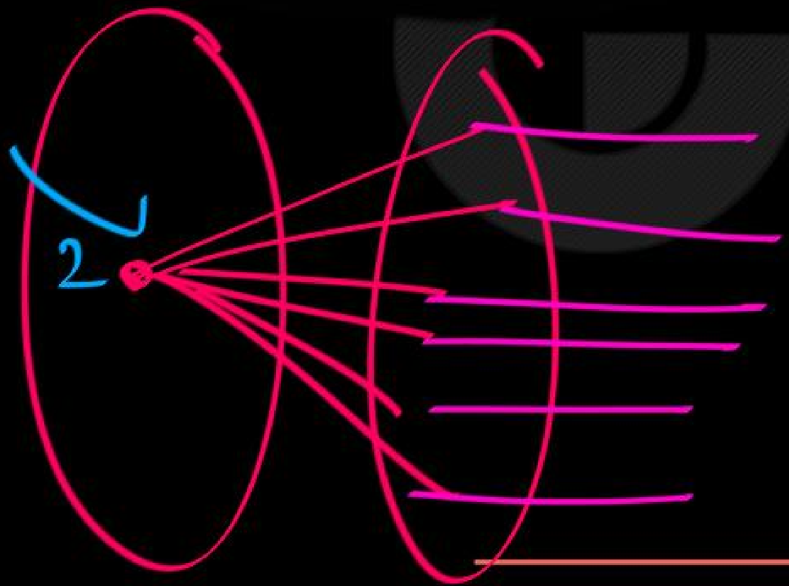
$\forall x$

$\forall y$

$P(x, y, z) = \boxed{x + y = z}$

Domain : real

$\exists_x \forall_y \exists_z P(x, y, z)$: True



2, 5, 7

2, 6, 8

2, -2, 0

2, 10, 12

1) Determine the truth value of each expression below if the domain is the set of all real numbers.

a) $\exists x \forall y\ (xy = 0)$ (If true, give an example.)

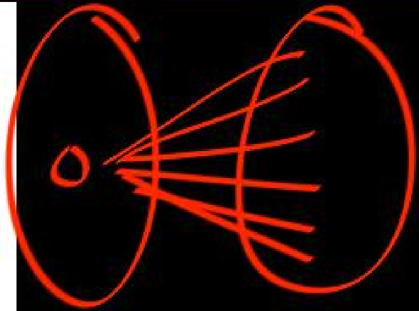
b) $\forall x \forall y \exists z\ (z = (x - y)/3)$ (If false, give a counterexample.)

c) $\forall x \forall y\ (xy = yx)$ (If false, give a counterexample.)

d) $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$ (If true, give an example.)

1) Determine the truth value of each expression below if the domain is the set of all real numbers.

True — a) $\exists x \forall y\ (xy = 0)$ (If true, give an example.)



True — b) $\forall x \forall y \exists z\ (z = (x - y)/3)$ (If false, give a counterexample.)

$2,\ 5,\ \frac{-3}{3}$

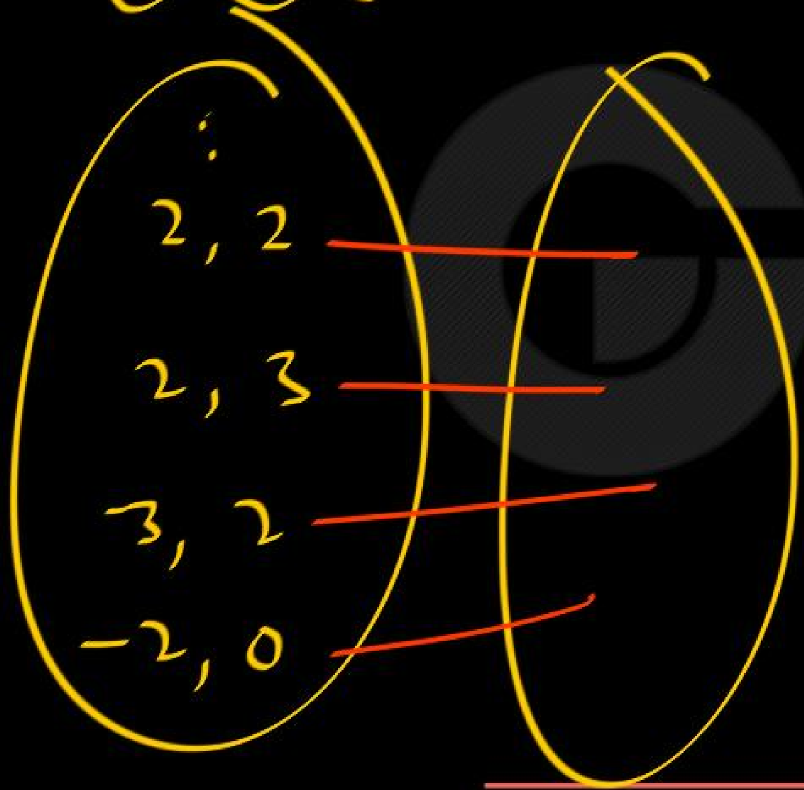
True — c) $\forall x \forall y\ (xy = yx)$ (If false, give a counterexample.)

True — d) $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$ (If true, give an example.)

$3^2 + 4^2 = 5^2$

$(3, 4, 5)$

$\underbrace{\forall x \forall y} \exists z$



Nested Quantifiers

- Nested quantifiers are often necessary to express the meaning of sentences in English as well as important concepts in computer science and mathematics.

  Example: "Every real number has an additive inverse" is
  $\forall x \exists y (x + y = 0)$
  where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:
  $\forall x \exists y (x + y = 0)$ can be viewed as $\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ is $(x + y = 0)$

Nested Quantifiers

- The nested quantifier means one quantifier is within the **scope** of another.
  - Example: $\forall x \exists y(x + y = 2)$ with two quantifiers.
  - $\forall x$ is applied to $\exists y(x + y = 2)$
  - $\exists y$ is applied to $(x + y = 2)$
- The part of the logical expression where a quantifier is applied is called the scope of that quantifier
- $\forall x \exists y(x + y = 2)$ says that for every real number x, there is a real number y such that $x + y = 2$

THINKING OF QUANTIFICATION AS LOOPS In working with quantifications of more than one variable, it is sometimes helpful to think in terms of nested loops. (Of course, if there are infinitely many elements in the domain of some variable, we cannot actually loop through all values. Nevertheless, this way of thinking is helpful in understanding nested quantifiers.) For example, to see whether $\forall x \forall y P(x, y)$ is true, we loop through the values for x, and for each x we loop through the values for y. If we find that $P(x, y)$ is true for all values for x and y, we have determined that $\forall x \forall y P(x, y)$ is true. If we ever hit a value x for which we hit a value y for which $P(x, y)$ is false, we have shown that $\forall x \forall y P(x, y)$ is false.

Similarly, to determine whether $\forall x \exists y P(x, y)$ is true, we loop through the values for x. For each x we loop through the values for y until we find a y for which $P(x, y)$ is true. If for every x we hit such a y, then $\forall x \exists y P(x, y)$ is true; if for some x we never hit such a y, then $\forall x \exists y P(x, y)$ is false.

To see whether $\exists x \forall y P(x, y)$ is true, we loop through the values for x until we find an x for which $P(x, y)$ is always true when we loop through all values for y. Once we find such an x, we know that $\exists x \forall y P(x, y)$ is true. If we never hit such an x, then we know that $\exists x \forall y P(x, y)$ is false.

Finally, to see whether $\exists x \exists y P(x, y)$ is true, we loop through the values for x, where for each x we loop through the values for y until we hit an x for which we hit a y for which $P(x, y)$ is true. The statement $\exists x \exists y P(x, y)$ is false only if we never hit an x for which we hit a y such that $P(x, y)$ is true.