Relations

Recap

Properties of Relation

Website : https://www.goclasses.in/

Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

Properties of Relations

Definitions

A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.

A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.

A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.

A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$

A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Properties of Binary Relations

A binary relation $R \subseteq A \times A$ is called

- Reflexive iff $\forall x\ (x, x) \in R$
- Symmetric iff $\forall x, y\ ((x, y) \in R \rightarrow (y, x) \in R)$
- Antisymmetric iff $\forall x, y\ ((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$
- Transitive iff $\forall x, y, z\ ((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$.

Examples:

- $\leq$ and $=$ are reflexive, but $<$ is not.
- $=$ is symmetric, but $\leq$ is not.
- $\leq$ is antisymmetric.
  Note: $=$ is also antisymmetric, i.e., $=$ is symmetric and antisymmetric.
  $<$ is also antisymmetric, since the precondition of the implication is always false.
  However, $R = \{(x, y) \mid x + y \leq 3\}$ is not antisymmetric, since $(1, 2), (2, 1) \in R$.
- All three, $=$, $\leq$ and $<$ are transitive.
  $R = \{(x, y) \mid y = 2x\}$ is not transitive.

Set Theory

Next Topic

“Partition” of a Set

Website : https://www.goclasses.in/

Apple

A

Partitial 1
(Bachelors)

2 parts

(mom)
Partition 2

A_1 A_2
A_4 A_3

4 parts

(child)
Partition 3

1 Part

Apple

A

$P : \{A_1, A_2, A_3, A_4\}$

(mom)

Partition

Partition $\{A_1 \quad A_2 \quad A_4 \quad A_3$

Part

Part

4 parts

Properties of Partition:

① $A_1 \cap A_2 = \phi$

② $A_1 \cup A_2 \cup A_3 \cup A_4 = A$

③ $A_i \neq \phi$

"Partition" of a set

Base set

$S = \{1, 2, 3, 4\}$

Part

$P : \{ \overset{S_1}{\{1\}}, \overset{S_2}{\{2, 3\}}, \overset{S_3}{\{4\}} \}$

Partition of S

$S_1 \cup S_2 \cup S_3 = S$ ✓ ; $S_1 \cap S_2 = \phi$

$S_1 \cap S_3 = \phi$

Disjoint Sets and Partitions

- Two sets are disjoint if their intersection is the empty set
- A partition is a collection of disjoint sets

A set S is **partitioned** into k nonempty subsets $A_1, A_2, \ldots, A_k$ if:

1. Every pair of subsets in disjoint: that is $A_i \cap A_j = \emptyset$ if $i \neq j$.
2. $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

Base set should be non-empty.

Base set S

Part

$P = \{ s_1, s_2, s_3, ---, s_k \}$

Partition of S

① $s_i \subseteq S$

② $s_1 \cup s_2 \cup s_3 \cup -- \cup s_k = S$

③ $s_i \neq \phi$ ④ $s_i \cap s_j = \phi \; ; \; i \neq j$

$S : \{1, 2\}$ # Partitions of $S = ? = 2$

1. $P_1 = \{ \{1,2\} \}$ → one Part

2. $P_2 = \{ \{1\}, \{2\} \}$ → 2 Parts

$S = \{ \text{VK}, \text{MSD} \}$

Partition:

① $P_1 = \{ \{\text{VK}\}, \{\text{MSD}\} \}$ — IPL Partition

② $P_2 = \{ \{\text{VK}, \text{MSD}\} \}$ — Team India Partition

Base set

Definition 1.5.8 (Partition of a set). *Let A be a non-empty set. Then a partition Π of A, into m-parts, is a collection of non-empty subsets $A_1, A_2, \ldots, A_m$, of A, such that*

1. $A_i \cap A_j = \emptyset$ *(empty set), for* $1 \leq i \neq j \leq m$ *and*
2. $\bigcup_{i=1}^{m} A_i = A$.

"Partition" of a set S : — Base set

a Collection (set) of Subsets of S which are Pairwise Disjoint, non-empty, and they cover ALL elements of Base set.

$S = \{1\}$ #partitions = ? = 1

$P = \underbrace{\{ \{1\} \}}_{\text{Partition}}$

$S = \{1, 2, 3\}$ #partitions = ?

① one part partitions: $\{ S \}$ — 1

② Two part partitions:

Short notation

1, 23 — $\{\{1\}, \{2, 3\}\}$ ✓

2, 13 — $\{\{2\}, \{1, 3\}\}$ ✓

3, 12 — $\{\{3\}, \{1, 2\}\}$ ✓

(3) 3 part partitions:

$$\{\{1\}, \{2\}, \{3\}\}$$ ✓

(4) 4 part partitions: none

Total partitions of S = 5 ✓

Set S ;

Every partition is a set.

Every part in a partition is also a set.

is a subset of S.

for "Convenience";

a part $\{a, b, c\}$ we will write abc.

ab means $\{a, b\}$

a means $\{a\}$

Partition $= \{\{a\}, \{b, c\}\}$

short: a, bc

Q: $S = \{a, b, c, d, e\}$

which is a partition of S?

1. $\{a, bc, de\}$ ✓
2. $\{a, bcd, de\}$ ✗ bcd, de not Disjoint.
3. $\{a, bc, d\}$ ✗ Doesnot cover all elements.

Q: $\underbrace{S = \{a, b, c, d, e\}}$ → base set

which is a partition of S?

① $\{a, bc, def\}$ ✗ f not in base sets,

② $\{a, bcde\}$ ✓

③ $\{a, bc, d, e\}$ ✓

Q: $S = \{a, b, c, d, e\}$

which is a partition of S?

1. $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}\}$ ✓
2. $\{\{a, b, c, d, e\}\}$ ✓
3. $\{\{a, b\}, \{c, d\}, \{e\}, \phi\}$ ✗

No part should be empty.

The idea of a partition is that you take a whole (the set X) and you divide it to parts.

Now if I cut off an apple into slices (and one core) I have several pairwise disjoint parts of the apple, but if I reassemble the parts I get a whole apple again.

Similarly we require this from a partition of a set. We want that the union of *all* the parts give us the entire set we partitioned.

The examples will help. Examples of partitions of $\{1, 2, 3\}$ are

$$\{1\}, \{2\}, \{3\}$$

$$\{1, 2\}, \{3\}$$

$$\{1\}, \{2, 3\}$$

$$\{1, 2, 3\}$$

$$\{2\}, \{1, 3\}$$

Example: Let $S = \{a,b,c,d,e\}$ and consider the following four partitions of S.

$$P_1 = \{\ \{a,b,c,d,e\}\ \}$$
$$P_2 = \{\ \{a,b\}, \{c,d,e\}\ \}$$
$$P_3 = \{\ \{a\}, \{b\}, \{c\}, \{d,e\}\ \}$$
$$P_4 = \{\ \{a\}, \{b\}, \{c\}, \{d\}, \{e\}\ \}$$

Q: $A = \{a, b, c, d\}$

_ , _ , _

Partitions with 3-parts? — 6

a, b, cd | ab, c, d | bc, a, d
a, c, bd | ac, b, d
$\boxed{b, c, ad}$ | $\boxed{ad, c, b}$

Same partition

Example 1.5.9. *1. The partitions of* $A = \{a, b, c, d\}$ *into*

(a) 3-*parts are* $a|\ b|\ cd$, $a|\ bc|\ d$, $ac|\ b|\ d$, $a|\ bd|\ c$, $ad|\ b|\ c$, $ab|\ c|\ d$, *where the expression* $a|bc|d$ *represents the partition* $A_1 = \{a\}$, $A_2 = \{b, c\}$ *and* $A_3 = \{d\}$.

(b) 2-*parts are*

$$a|\ bcd, \quad b|\ acd, \quad c|\ abd, \quad d|\ abc, \quad ab|\ cd, \quad ac|\ bd \quad \text{and} \quad ad|\ bc.$$

0

not +ve
not −ve

1

not Composite
not prime

$\mathbb{Z}$

① 2 part Partition: { even, odd } ✓

not a Partition of $\mathbb{Z}$ ⟵ { +ve, −ve } ✗

because 0 missing.

2. *Let* $A = \mathbb{Z}$ *and define*

   (a) $A_0 = \{2x : x \in \mathbb{Z}\}$ *and* $A_1 = \{2x + 1 : x \in \mathbb{Z}\}$. *Then* $\Pi = \{A_0, A_1\}$ *forms a partition of Zl into odd and even integers.*

   (b) $A_i = \{10n + i : n \in \mathbb{Z}\}$, *for* $i = 1, 2, \ldots, 10$. *Then* $\Pi = \{A_1, A_2, \ldots, A_{10}\}$ *forms a partition of* $\mathbb{Z}$.

$\mathbb{Z}$

$\mathbb{Z}$

10-part Partition : mod 10 Partition

$\{$ rem0, rem1, rem2, ---, rem9 $\}$

Note: let $n \in Z^+$

then Possible remainders when divides by n:

$\underbrace{0, 1, 2, ---, n-1}$

Divide by 3

Possible rem

0, 1, 2

{Prime, Composite} a Partition of $\mathbb{N}$?

No. 1 is missing.

Let $A = \{a, b, c, d\}$. *Then* $\Pi = \{\{a\}, \{b, d\}, \{c\}\}$ *is a partition of* A.

{ Prime,Composite, {1} } is a partition of N.

$A = \{a, b, c, d, e\}$

π is partition of A consisting 3 Parts. And a, b belong to same Part; c, d belong to same Part

How many such Partitions are possible?

$A = \{a, b, c, d, e\}$

$\pi = \{ \_\_, \_\_, \_\_ \}$

ab   cd   e

Answer : 1

Observation: a, c cannot be in same part.

$A = \{a, b, c, d, e, f\}$

π is partition of A Consisting 3 Parts. And a, b belong to same Part; c, d belong to Same Part

How many such Partitions are possible?

$A = \{a, b, c, d, e, f\}$

Π is partition of A consisting 3 Parts. And a, b belong to same Part; c, d belong to same Part

How many such Partitions are possible?

Answer = 6

$A = \{a, b, c, d, e, f\}$

$\pi =$ ___ , ___ , ___

ab   cd

Case 1: abcd in a part: — 1 such Partition

abcd , e, f

Case 2: ab, cd in diff parts.

ab | cd | ef —— 1

ab | cd | f —— 2 possible Partitions

ab | cd | e —— 2 possible Partitions.

Congruence Modulo m

- We say that two integers a and b are *congruent modulo m* if $a - b$ is a multiple of m ($m > 0$).
- Write $a \equiv b \pmod{m}$.
- For example, $2 \equiv 7 \pmod{5}$.

$$a \equiv b \pmod{n}$$

iff

$$\text{rem}(a,n) = \text{rem}(b,n)$$

Definitions

Two sets A and B are **disjoint** if they have no common elements; that is, if $A \cap B = \emptyset$.

A family of sets $A_1, A_2, \ldots, A_n$ is **pairwise disjoint** if for all $i, j \leq n$, if $i \neq j$, then A_i and A_j are disjoint.

A **partition** of a non-empty set S is a collection of pairwise disjoint non-empty subsets of S that have S as their union.

The last definition can also be written like this:

Let S be a non-empty set. A **partition** of S is a collection D of non-empty subsets of S such that

(i) if $P, Q \in D$ and $P \neq Q$, then $P \cap Q = \emptyset$; and

(ii) $\bigcup_{P \in D} P = S$.

The notation in condition (ii) means that the union of all the sets in the partition D is S. Given these definitions, we see that if R is an equivalence relation on S, the equivalence classes of R form a partition of S.