First Order Logic

Next Chapter:

Practice

Bounded Variable, Free Variable
Scope of a Quantifier

- By Deepak Poonia (IISc Bangalore)

Instructor:

Deepak Poonia

IISc Bangalore

GATE CSE AIR 53; AIR 67;
AIR 107; AIR 206; AIR 256

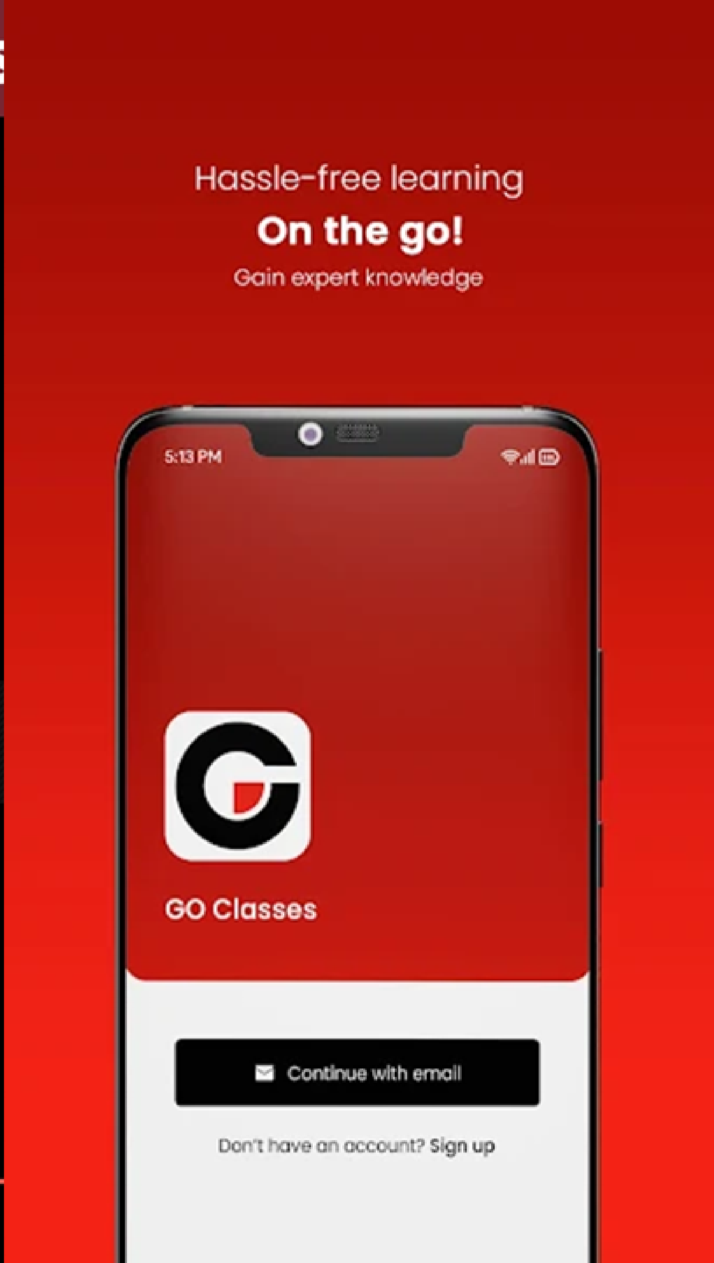
Discrete Mathematics Complete Course:

https://www.goclasses.in/courses/Discrete-Mathematics-Course

Download the GO Classes Android App:

https://play.google.com/store/apps/details?id=com.goclasses.courses

Search "GO Classes" on Play Store.



NOTE :

Complete Discrete Mathematics & Complete Engineering

Mathematics Courses, by GO Classes, are **FREE** for ALL learners.

Visit here to watch : https://www.goclasses.in/s/store/

SignUp/Login on Goclasses website for free and start learning.

We are on Telegram. **Contact us** for any help.

Link in the Description!!

Join GO Classes **Doubt Discussion** Telegram Group :

Username:

@GATECSE_Goclasses

We are on Telegram. **Contact us** for any help.

Join GO Classes Telegram Channel, Username: @GOCLASSES_CSE

Join GO Classes **Doubt Discussion** Telegram Group :

Username: @GATECSE_Goclasses

(Any doubt related to Goclasses Courses can also be asked here.)

Join GATEOverflow Doubt Discussion Telegram Group :

Username: @GateOverflow_CSE

First Order Logic

Next Topic:

Practice

Bounded Variable, Free Variable

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

Not a proposition

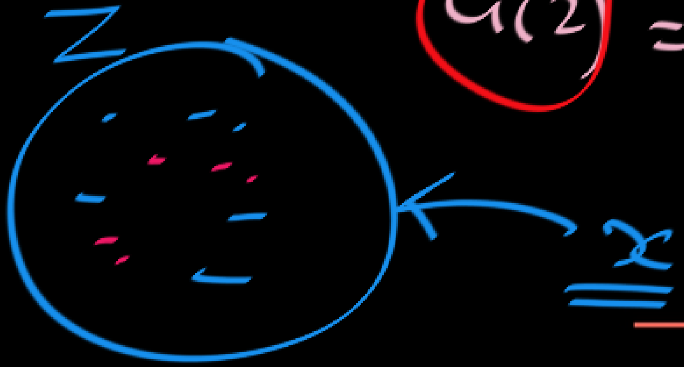
Same (free)

proposition

$G(x) = H(x) \land P(x)$: $\boxed{x \text{ is prime } \& \ x>5}$ G(x)

x is a free var.

$G(2) = \underset{}{\overset{F}{H(2)}} \land \overset{T}{P(2)} = F$



$G(7) : \overset{T}{H(7)} \land \overset{T}{P(7)} : T$

Mathematics:

$$y = x^2 + x + 2$$

free (same x)

$x = 2$

$$y = 2^2 + 2 + 2 \quad ; \quad y = (10)^2 + 10 + 2$$

$$\boxed{y = 3^2 + 4 + 2}$$ WRONG

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$\forall x \, [\, H(x) \wedge P(x) \,]$: No free Var.

Proposition

Scope

Same

Bounded Var.

Proposition

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$S : \forall x \, \underbrace{[\, H(x) \land P(x) \,]}_{\text{scope}}$$

No free var → proposition

for every integer x, x is prime & x > 5. : false

Counterex: 4, 2, 3, 6

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$\boxed{\forall x\, [\, H(x) \land P(x) \,]}$$; false

Counterex: $x = 2$

$x = 10$

$x = 8$

$x = 5$ ✓

$x = 4$ ✓

$x = -5$ ✓

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$\exists x \, [\, H(x) \wedge P(x) \,]$$

No free var

proposition

Scope

Bounded var

Assume: Domain: Set of Integers

P(x): x is Prime.   H(x): x > 5.

S : $\exists x\,[\,H(x) \land P(x)\,]$

x Z 7

Some integer x, x is prime & x>5 : True

witness: 7 ✓

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$\boxed{\exists x\, [\, H(x) \land P(x) \,]}$$: True

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$\exists x\, [\, H(x)] \land P(x)$$

Assume: Domain: Set of Integers

P(x): x is Prime.   H(x): x > 5.

$$\exists x\, [\, H(x)\,] \wedge P(x)$$

free var.

free var exists

Scope

Bounded

Nothing to do with each other

Not a proposition

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

free

$S(x)$: $\boxed{\exists x\, [\, H(x)]\wedge P(x)}$ = T ∧ P(x)

True = P(x)

$\boxed{S(x):\ x \text{ is prime}}$ → Predicate over var. x

Not a prop.

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

S : $\exists x\, [\, H(x)]\, \wedge\, \forall x\, [\, P(x)\,]$

Scope

Scope

No free var

proposition

Independent

Assume: Domain: Set of Integers

P(x): x is Prime.   H(x): x > 5.

prop.

S : $\underbrace{\exists x\, [\, H(x)]}_{\alpha} \wedge \underbrace{\forall x\, [\, P(x)\,]}_{\beta}$

S : $\alpha \wedge \beta$

$\alpha \to T$ $\quad \beta \to F$

$\boxed{S = F}$ ✓

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$\exists x\, [\, H(x)] \land \forall y\, [\, P(y)\,]$$

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

S : $\exists x\,[\,H(x)] \wedge \forall y\,[\,P(y)\,]$

No free var

Proposition

Assume: Domain: Set of Integers

P(x): x is Prime. H(x): x > 5.

$$S : \boxed{\underbrace{\exists x\,[\,H(x)\,]}_{\alpha}} \wedge \underbrace{\forall y\,[\,P(y)\,]}_{\beta}$$

$S : \alpha \wedge \beta$

$\alpha \to T$, $\beta \to F$

$\boxed{S = F}$

$$\boxed{\exists x\, P(x) \;\wedge\; \forall x\, P(x)}$$

$$\equiv \boxed{\exists x\, P(x) \wedge \forall y\, P(y)}$$

Same

What is the difference between these three formulae?

$G(x)$: $Hx \wedge Px$ → free var. x

same

S : $\forall x[Hx \wedge Px]$: proposition (No free var)

scope

$M(x)$: $\boxed{\forall x[Hx] \wedge Px}$ → Not a prop.

scope

free

predicate

An occurrence of x within the scope of $\forall x$ or $\exists x$ is <u>bound</u>, otherwise it is <u>free</u>.

$\forall x \, (P(x))$, $\exists x \, (P(x))$

Bounded (Quantified)

Bounded

Scope

$$\forall x \overbrace{(P(x))}^{} \wedge Q(x)$$

Bounded

free

Not in scope of any $\forall x$, $\exists x$

Bound variables: Occurrence of a variable is bound *iff* it is within the scope of a quantifier. **DESIRABLE**

Free variables: Occurrence of a variable is free *iff* it is NOT within the scope of a quantifier. **NOT DESIRABLE**

free var. exists $\longleftrightarrow$ Not a Proposition

Proposition $\longleftrightarrow$ No free var

Q: In the following predicate formula, which *variables* are *free* and which are *bound*?

$$\forall x[\exists y(R(x,y) \wedge T(z,x)) \rightarrow \forall z(N(z,x,y))]$$

free **bound**

free

Q: In the following predicate formula, which *variables* are *free* and which are *bound*?

Scope

free

$$\forall x[\exists y(R(x,y) \wedge T(z,x)) \rightarrow \forall z(N(z,x,y))]$$

free **bound**

free

Not in Scope of any ∀y or ∃y.

Q: In the following predicate formula, which *variables* are *free* and which are *bound*?

Scope

$$\forall x[\exists y(R(x,y) \wedge T(z,x)) \rightarrow \forall z(N(z,x,y))]$$

free **bound**

free

$$\exists x(xEy \ \& \ \forall z(zEy \rightarrow z=x))$$

find free & Bounded Var.

$$\exists x(xEy \ \& \ \forall z(zEy \rightarrow z=x))$$

Scope

free

Bounded

Bounded

$$\exists x(xEy \,\&\, \forall z(zEy \rightarrow z = x))$$

Bound occurrence

$$\exists x(xEy \,\&\, \exists y(\neg yEx))$$

find Bounded & free Var.

$$\exists x(xEy \;\&\; \exists y(\neg yEx))$$

Scope

Bounded

free

Bounded

$\exists x(xEy \ \& \ \exists y(\neg yEx))$

$$\exists x(xEy \ \& \ \exists y(\neg yEx))$$

y free

y bound

Quantifiers: Example

Consider this formula

$$\Phi = \forall y. ((P(y) \wedge Q(x)) \vee \forall x. (P(x) \vee Q(y) \wedge P(z)))$$

How many free variables does this formula contain? Clicker

1. One
2. Two
3. Three
4. Four
5. None

Quantifiers: Example

Consider this formula

Scope

Scope

$$\Phi = \forall y.((P(y) \wedge Q(x)) \vee \forall x.(P(x) \vee Q(y) \wedge P(z)))$$

Bounded

free

Bounded

Bounded

free

How many free variables does this formula contain? Clicker

1. One
2. Two
3. Three
4. Four
5. None

Q: For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness.
(Assume domain is set of real numbers)

(a) $\exists n(x > 2^n)$.

(b) $\exists x(x > 2^n)$.

Q: For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness. (Assume domain is set of real numbers)

(a) $\exists n(x > 2^n)$.

Bounded

free

(b) $\exists x(x > 2^n)$.

free

Bounded

Q: For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness. (Assume domain is set of real numbers)

(a) $\boxed{\exists n(x > 2^n).}$: $S(x)$

(b) $\boxed{\exists x(x > 2^n).}$: $G(n)$

free

Not propositions

Q: For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness. (Assume domain is set of real numbers)

(a) $\boxed{\exists n(x > 2^n)}$. : $S(x)$

$S(0)$: $\boxed{\exists n(0 > 2^n)}$

false

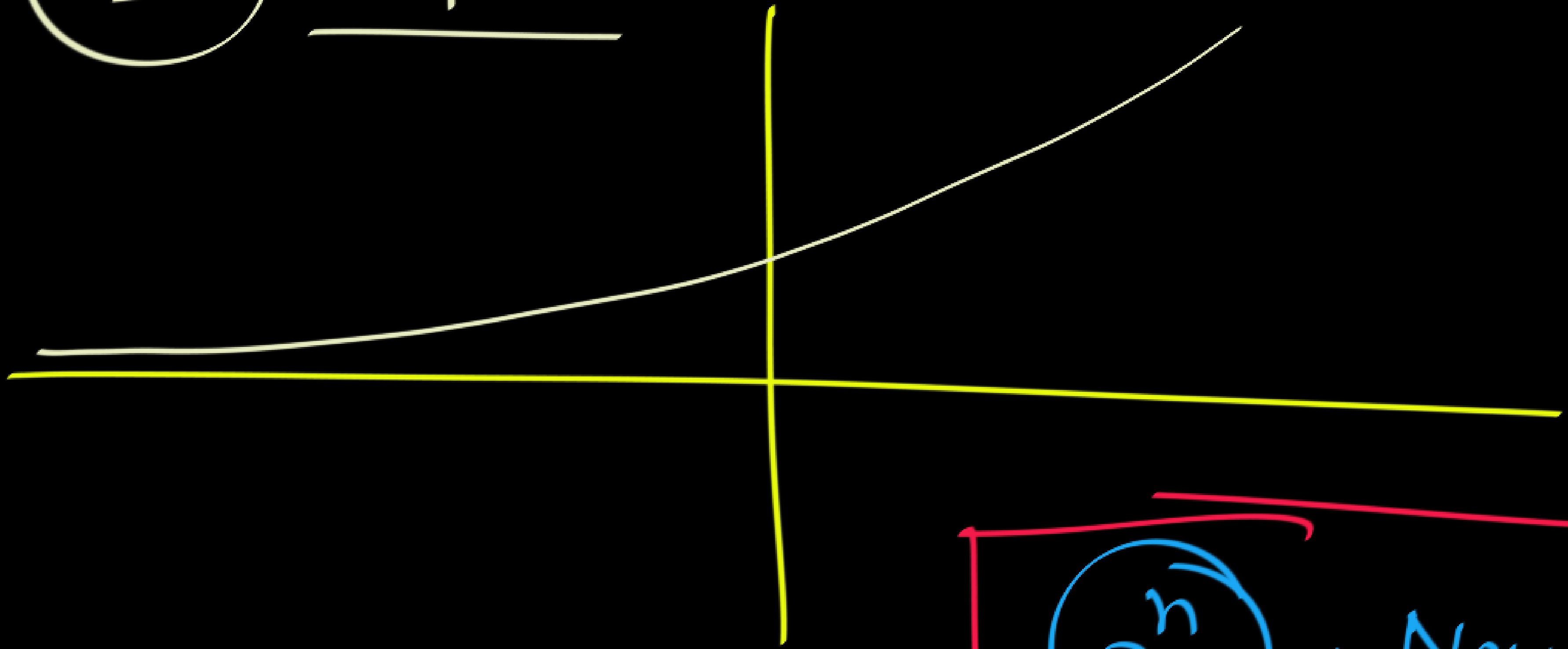
R

n

x

(b) $\exists x(x > 2^n)$.

2^n Graph



2^n : Never negative

Q: For each formula, determine which of its two variables is free. Determine whether the formula is true when the value of the free variable is 0. If it is true then find a witness.
(Assume domain is set of real numbers)

(a) $\exists n(x > 2^n)$.

(b) $\boxed{\exists x(x > 2^n).}$: $G(n)$

free

$G(0)$: $\exists x(x > 2^0)$

: $\boxed{\exists x(x > 1)}$

True

1. Scope of a Quantifier
2. Free Var
3. Bounded Var

Quantifiers

- We need quantifiers to formally express the meaning of the words "all" and "some".
- The two most important quantifiers are:
  - Universal quantifier, "For all". Symbol: $\forall$
  - Existential quantifier, "There exists". Symbol: $\exists$
- $\forall x\ P(x)$ asserts that $P(x)$ is true for **every** x in the domain.
- $\exists x\ P(x)$ asserts that $P(x)$ is true for **some** x in the domain.
- The quantifiers are said to bind the variable x in these expressions.
- Variables in the scope of some quantifier are called bound variables. All other variables in the expression are called free variables.
- A propositional function that does not contain any free variables is a proposition and has a truth value.

Free and bound variables

GO Classes

Two essentially different ways in which we use individual variables in first-order formulae:

1. Free variables: used to denote *unknown or unspecified objects*, as in $(\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0})$.

2. Bound variables: used *to quantify*, as in
   $\exists x((\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0}))$
   and $\forall x((\mathbf{5} < x) \vee (x^2 + x - \mathbf{2} = \mathbf{0}))$.

Note that the same variable can be *both free and bound in a formula*, e.g. x in the formula $x > \mathbf{0} \wedge \exists x(\mathbf{5} < x)$.

A formula with no bound variables is an open formula.
A formula with no free variables is a closed formula, or a sentence.