Set Theory

Next Chapter:

Special Types of Relations

(Anti-Symmetric, Transitive etc)

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GATE CSE AIR 53; AIR 67;
AIR 107; AIR 206; AIR 256

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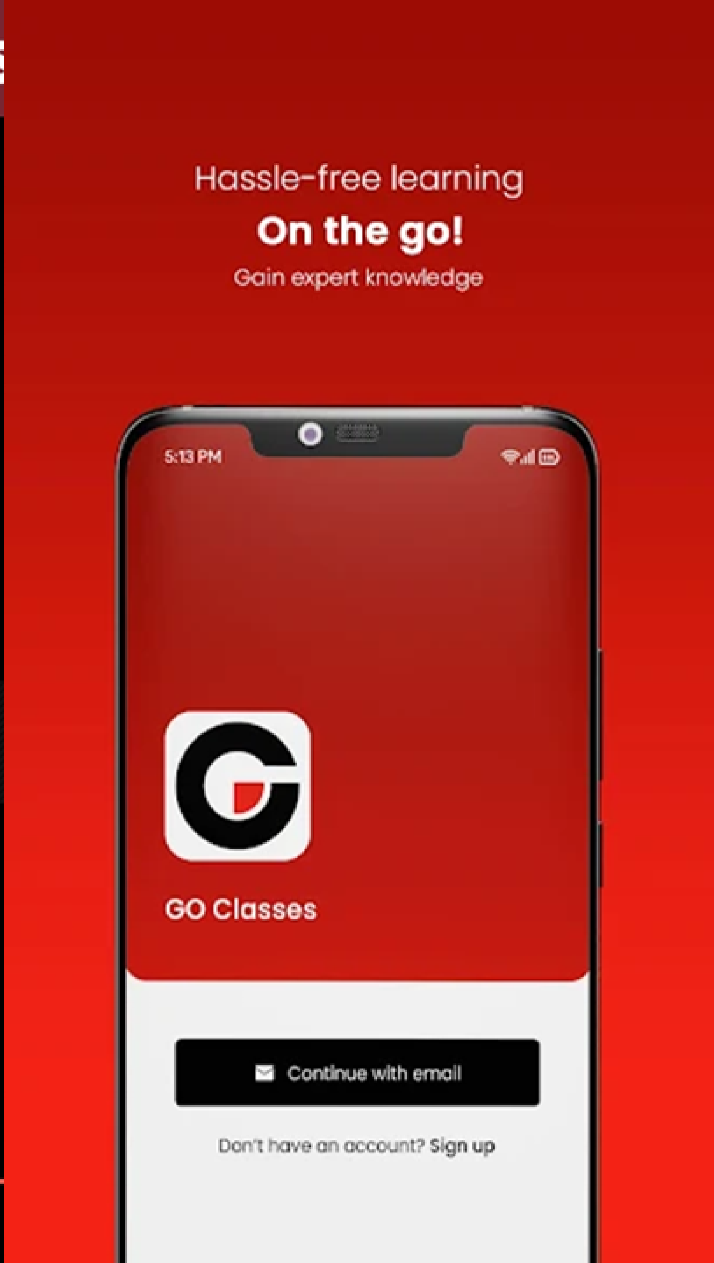

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"Anti-Symmetric" Relation:

"Anti Symmetric" Relation:

$\boxed{R : A \rightarrow A}$

$A = \{a, b, c, \ldots\}$

R is Anti-Sym :

$a \neq b$

$a, b \in A$

$\boxed{aRb \ \& \ bRa}$

Not happen.

"Anti Symmetric" Relation:

$R : A \longrightarrow A$

R is Antisym:

$\forall_{a,b \in A}$ (If $a \neq b$ and aRb then $b \not R a$)

"Anti Symmetric" Relation:

Definition !

$R : A \to A$

R is Antisym:

$$\forall_{a,b \in A} \left(\boxed{(a \neq b) \wedge aRb} \to b \not R a \right)$$

"Anti Symmetric" Relation:

Definition 2

$R : A \to A$

R is AntiSym :

$$\forall_{a,b \in A} \left([aRb \wedge bRa] \longrightarrow a = b \right)$$

Anti-Sym Relation:

Relation R on set A.

R is Antisym:

Definition 1:

for all a, b : if $a \neq b$ ~~&~~ aRb then $b \not R a$

Definition 2:

for all a, b : if aRb ~~&~~ bRa then $a = b$.

Anti-Sym Relation:

Relation R on set A.

R is Antisym:

Definition 1:

$$\forall_{a,b \in A} \left(\left[(a \neq b) \wedge aRb \right] \longrightarrow b \not R a \right)$$

Definition 2:

$$\forall_{a,b \in A} \left(\left[aRb \wedge bRa \right] \longrightarrow a = b \right)$$

Anti-Sym Relation:

Relation R on set A.

R is Antisym :

Definition 1:

$$\forall_{a,b \in A} \left(\left[(a \neq b) \wedge aRb \right] \longrightarrow b \not R a \right)$$

Definition 2:

$$\forall_{a,b \in A} \left(\left[aRb \wedge bRa \right] \longrightarrow a = b \right)$$

Assume, Base Set A ; R on A.

For any x, y ∈ A (Assume x,y are different elements):

R is Antisym

- $x \cancel{R} y$ & $y \cancel{R} x$ Allowed
- $x R y$ & $y \cancel{R} x$ "
- $x \cancel{R} y$ & $y R x$ "
- $\boxed{x R y \text{ \& } y R x}$ NOT Allowed
- $x R x$ ✓ Allowed
- $y \cancel{R} y$ ✓ Allowed

Assume, Base Set A ;

For any x, y ∈ A (Assume x,y are different elements):

If $x \neq y$

- $x \cancel{R} y$ & $y \cancel{R} x$
- $x R y$ & $y \cancel{R} x$
- $x \cancel{R} y$ & $y R x$
- $\boxed{x R y \text{ \& } y R x}$ NOT Allowed
- $x R x$
- $y \cancel{R} y$

In Anti-Sym Relation

“Asymmetric” Relation:

(Counter-Symmetric Relation)

"Asymmetric" Relation:

Anti-sym & Irreflexive

No element is related to each other.

$R : A \to A$

"Asymmetric" Relation:

$$\forall_{a,b \in A} \left(aRb \longrightarrow b \not{R} a \right)$$

"Asymmetric" Relation:

$R : A \to A$

① $\forall_{a,b \in A} \left(aRb \longrightarrow b \not R a \right)$ ✓

② Asym = Antisym & Irref.

"Asymmetric" Relation:

$R : A \to A$

$$\forall_{a,b \in A} \left(aRb \longrightarrow b \cancel{R} a \right)$$

If $a = b$

$$\overset{F}{aRa} \longrightarrow \overset{T}{a \cancel{R} a} = \text{True}$$

aRa

"Asymmetric" Relation:

$R : A \to A$

$$\forall_{a,b \in A} \left(aRb \longrightarrow b \cancel{R} a \right)$$

If $a = b$

True False

$aRa \longrightarrow a \cancel{R} a$ = false

aRa

Assume, Base Set A ; R : A → A

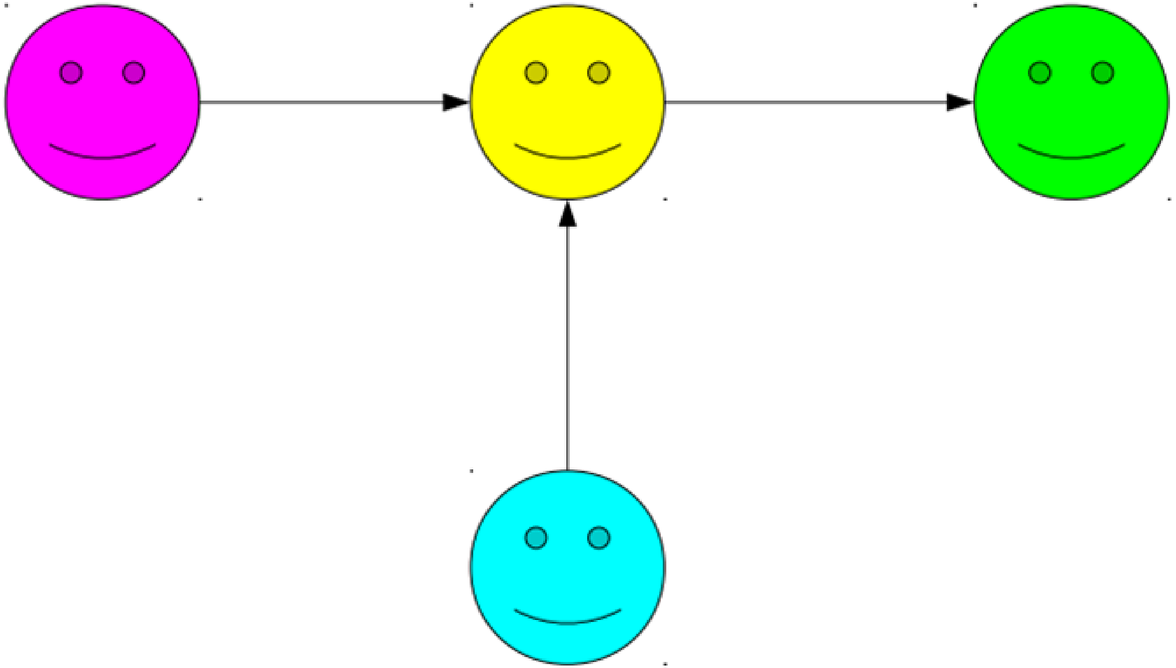
For any x, y ∈ A (Assume x,y are different elements):

If R Asym ⇒

- x ~~R~~ y & y ~~R~~ x — Allowed
- x R y & y ~~R~~ x — "
- x ~~R~~ y & y R x — "
- $\boxed{\text{x R y \& y R x}}$ Not Allowed
  - x R x — Not Allowed
  - y ~~R~~ y — Allowed

Asym = Antisym & Irref

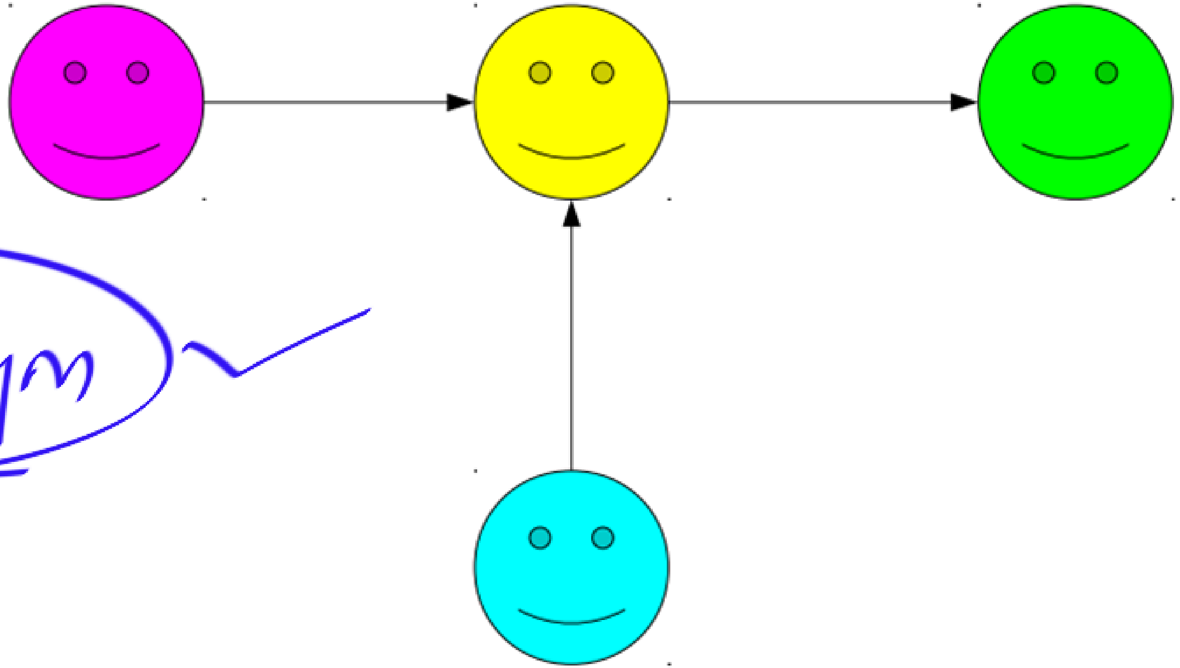
Asymmetry Visualized



$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b \not R a)$$

("If a relates to b, then b does not relate to a.")

Asymmetry Visualized



Antisym

No self loops

Irref

$$\forall a \in A.\ \forall b \in A.\ (aRb \rightarrow b\not{R}a)$$

("If a relates to b, then b does not relate to a.")

"Symmetric" Relation

Vs

"Anti Symmetric" Relation

Vs

"Asymmetric" Relation

Assume, Base Set A ; $\boxed{R : A \to A}$

For any x, y ∈ A (Assume x,y are different elements):

- $x \cancel{R} y$ & $y \cancel{R} x$ — Allowed
- $x R y$ & $y \cancel{R} x$ — NOT Allowed
- $x \cancel{R} y$ & $y R x$ — NOT Allowed
- $x R y$ & $y R x$ — Allowed
- $x R x$ — Allowed
- $y \cancel{R} y$ — Allowed

R is Symmetric

Assume, Base Set A ; $\boxed{R : A \rightarrow A}$

For any x, y ∈ A (Assume x,y are different elements):

- $x \cancel{R} y$ & $y \cancel{R} x$ Allowed
- $x R y$ & $y \cancel{R} x$ Allowed
- $x \cancel{R} y$ & $y R x$ Allowed
- $\boxed{x R y \ \& \ y R x}$ NOT Allowed
- $x R x$ — Allowed
- $y \cancel{R} y$ — Allowed

R Anti-sym

Assume, Base Set A ; $\boxed{R : A \to A}$

For any x, y ∈ A (Assume x,y are different elements):

- $x \cancel{R} y$ & $y \cancel{R} x$ Allowed
- $x R y$ & $y \cancel{R} x$ ''
- $x \cancel{R} y$ & $y R x$ ''
- $\boxed{x R y \;\&\; y R x}$ NOT Allowed
- $x R x$ — NOT Allowed
- $y \cancel{R} y$ — Allowed

R

~~Asym.~~ = Antisym & Irref

Graph Representation:

Sym Vs Anti-Sym

Vs Asym.

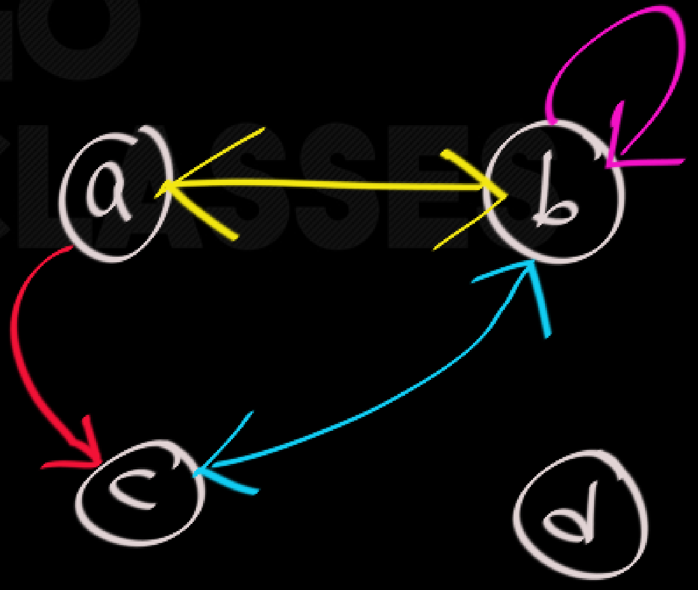
Graph Representation:

$\boxed{R : A \rightarrow A}$

R : Symmetric

$A = \{a, b, c, d\}$

Graph Rep. of R :



Violation of Symmetric Relation

Graph Representation:

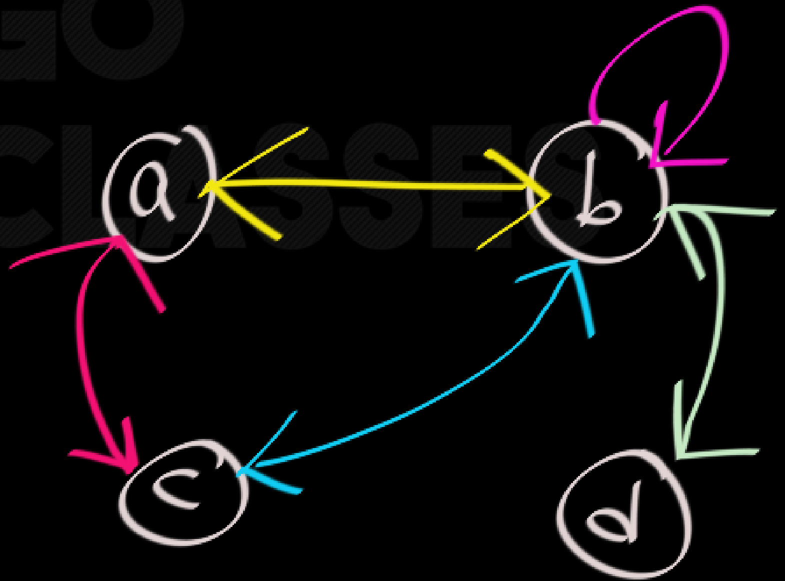
$\boxed{R : A \rightarrow A}$

R : Symmetric

A = {a, b, c, d}

Graph Rep. of R :

No Unidirectional Edges.



Graph Representation:

$R: A \to A$

$A = \{a, b, c, d\}$

R : Anti-Sym

if $a \neq b$

Graph Rep. of R :

$a \leftrightarrow b$

Violation of Anti-Sym Relation

Graph Representation:

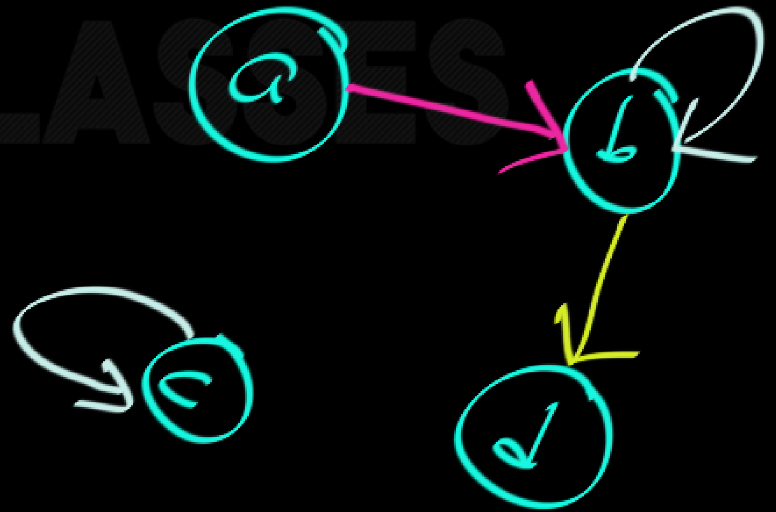
$R: A \to A$

R: Anti-Sym

$A = \{a, b, c, d\}$

Graph Rep. of R:

No bi-directional Edges b/w different elements.



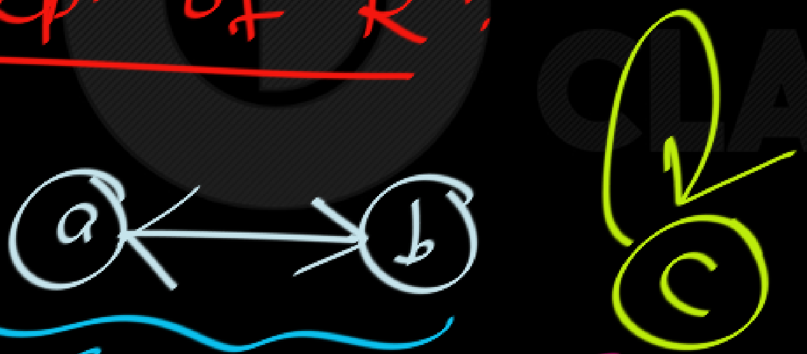
Graph Representation:

$R : A \to A$

$A = \{a, b, c, d\}$

R : Asym = Anti-sym & Irref

Graph Rep. of R:



Violation of Asym Relation

Graph Representation:

$R: A \rightarrow A$

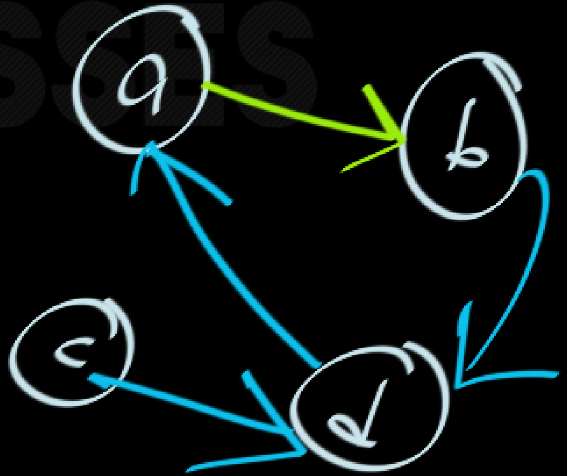
$A = \{a, b, c, d\}$

$\boxed{R: \text{Asym}}$

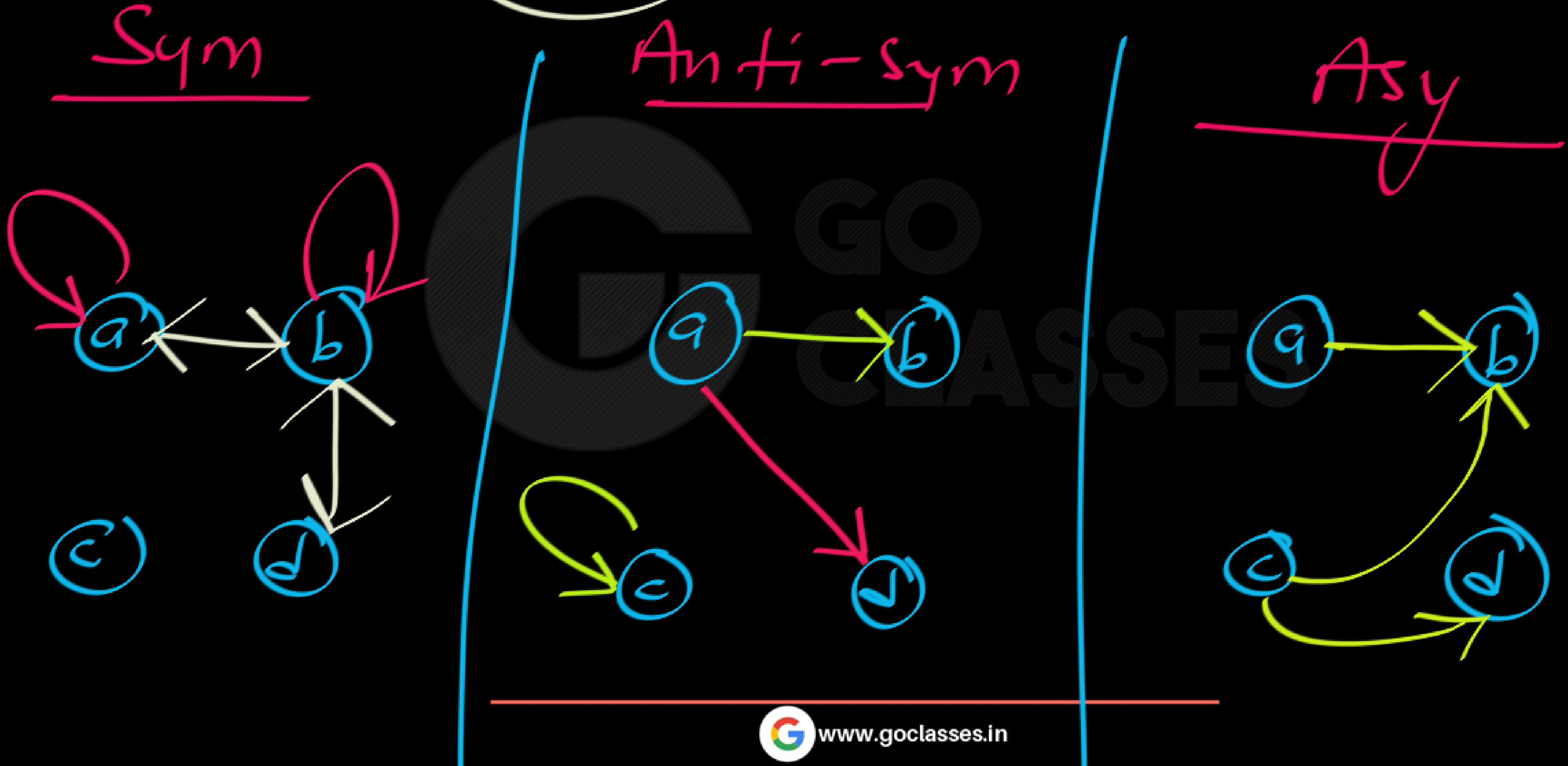
Graph Rep of R:

① No self loop.

② No bidirectional Edges b/w different elements.



Graph Representation:



Matrix Representation:

Sym Vs Antisym

Vs Asym

Matrix Representation:

$R : A \to A$

$\boxed{R : \text{Sym} \checkmark}$

Sym R	i	j
i	✓	$\boxed{}$
j	$\boxed{}$	✗

R

(i,j)	(j,i)
✓	✓
✗	✗

Matrix Representation:

$R : A \to A$

R: Sym

Sym R	i	j
i		
j		

matrix M

$\boxed{M = M^T}$

Symmetric matrix

Matrix Representation:

$R : A \to A$

R: Anti-Sym

Anti-sym R	i	j
i	✗	
j		✓

R: Anti-sym

(i, j)	(j, i)
✓	✗
✗	✓
✗	✗

Matrix Representation:

$R : A \to A$

$R :$ Asym = Anti-Sym + Irref

Anti-sym R

R	i	j
i	✗	
j		✗

R : Anti-sym

(i,j)	(j,i)
✓	✗
✗	✓
✗	✗

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which Relation is Symmetric, Anti-sym, Asymmetric ?

Consider the following relations on $\{1, 2, 3, 4\}$: → Base set

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$,

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which Relation is Symmetric, Anti-sym, Asymmetric ?

Consider the following relations on {1, 2, 3, 4}: Base set

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

R_1 : NOT Sym

$3 R_1 4$ but $4 \cancel{R_1} 3$

R_1 . NOT Antisym

$1 \neq 2$

$\boxed{1 R_1 2 \; \& \; 2 R_1 1}$

Violation of Antisymmetric

NOT Asym

R_1:

$$\boxed{\begin{array}{l}\text{Not Sym} \\ \text{NOT Antisym} \\ \text{NOT Asym}\end{array}}$$

$$\boxed{\text{Asym} = \text{Antisym} \ \& \ \text{Irref}}$$

$$\boxed{\text{Not Anti-Sym}} \longrightarrow \boxed{\text{Not Asym}}$$

$$\text{Asym} \longrightarrow \text{Antisym}$$

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

Violation of anti-sym Relation

R_2 : Sym ✓

Not Anti-Sym

Not Asym

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$
$$R_2 = \{(1,1), (1,2), (2,1)\},$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$

NOT Irref.

Violation of Asym Rel.

R_3 : Sym ✓

R_3 : NOT Asym

R_3 : NOT Antisym

$1 R_3 2$

$2 R_3 1$

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$

$$R_2 = \{(1,1), (1,2), (2,1)\},$$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$

$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$

R_4: Sym ✗ $\quad 2 R_4 1$ But $1 \not R_4 2$

R_4: Antisym ✓

R_4: Asym ✓ = Antisym ✓ + Irref ✓

R : Antisym

$$\forall_{a,b \in A} \left(\left[a \neq b \land aRb \right] \longrightarrow b \not R a \right)$$

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_5 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$,

[redacted]

Violation of Asym

R_5: Anti-sym ✓

NOT Irref

R_5: NOT Asym

R_5: Sym ✗

$1 R_5 2$
BUT
$2 \cancel{R_5} 1$

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$. — Irref ✓

NOT Sym : $\boxed{3 R_6 4 \text{ BUT } 4 \not R_6 3}$

Anti-sym : ✓

Asym ✓

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

[redacted]

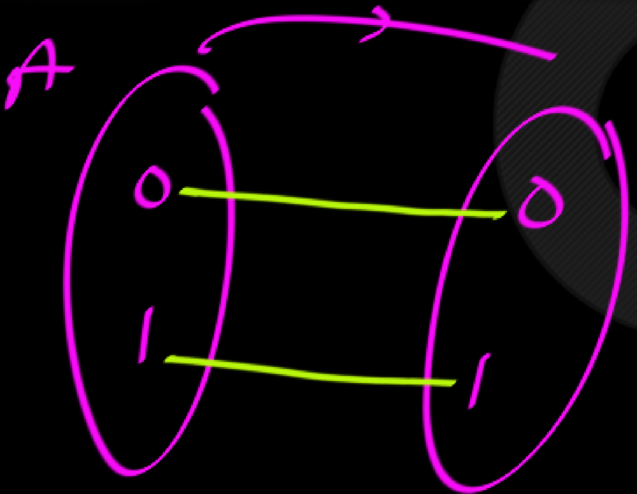
Base set

0.2. R on $A \equiv R : A \to A$

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

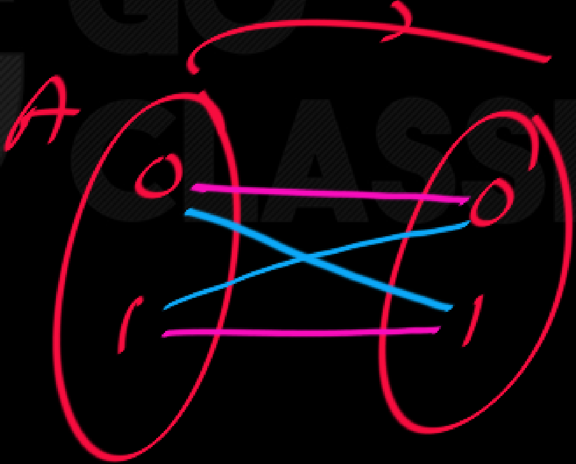
Base set A

R_1



Ref & Sym

R_2



Ref & Sym

φ

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

mat. Rep:

R	0	1
0	✓	
1		✓

(0,1)

(1,0)

(0,1)	(1,0)
✓	✓
✗	✗

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

mat. Rep:

R	0	1
0	✓	✓
1	✓	✓

Ref & Sym

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

mat. Rep:

$$\begin{array}{c|cc} R_2 & 0 & 1 \\ \hline 0 & \checkmark & \times \\ 1 & \times & \checkmark \end{array}$$

R_2: Ref & Sym

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

Graph Rep:

0.2.

Problem 2. Find all the relations on $\{0,1\}$ that are reflexive and symmetric.

Graph Rep: R_1

Ref

Sym

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

Graph Rep:

R_2

Refl ✓

Sym ✓

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

3 methods

2 Rel.

$R_1 : \{(0,0), (1,1)\}$

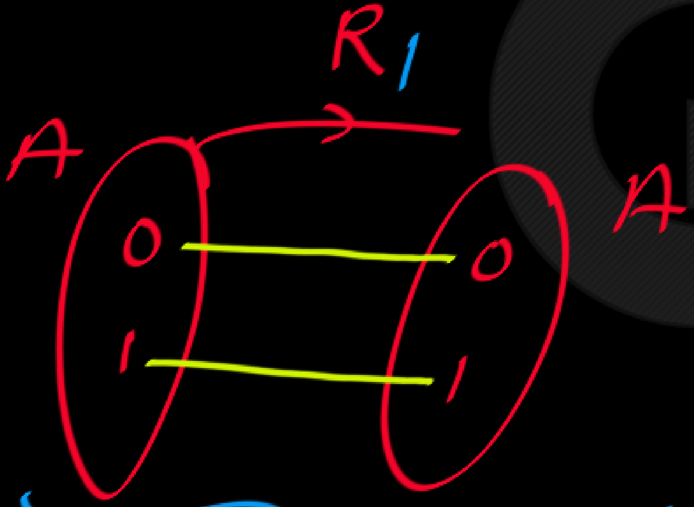
$R_2 = \{(0,0), (1,1), (0,1), (1,0)\}$

0.2. $R : A \to A$

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.



Refl ✓; Anti-sym ✓

Violation of Anti-sym:

$a \neq b$

$a \leftrightarrow b$

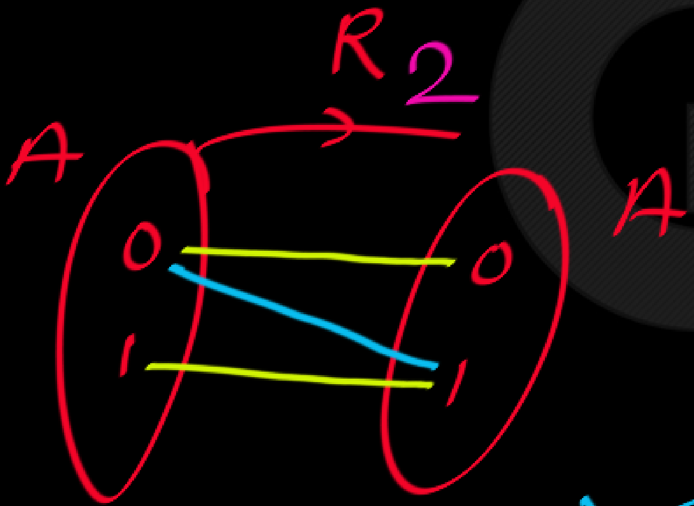
0.2. $R : A \rightarrow A$

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

A

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.



Ref ✓ Antisym ✓

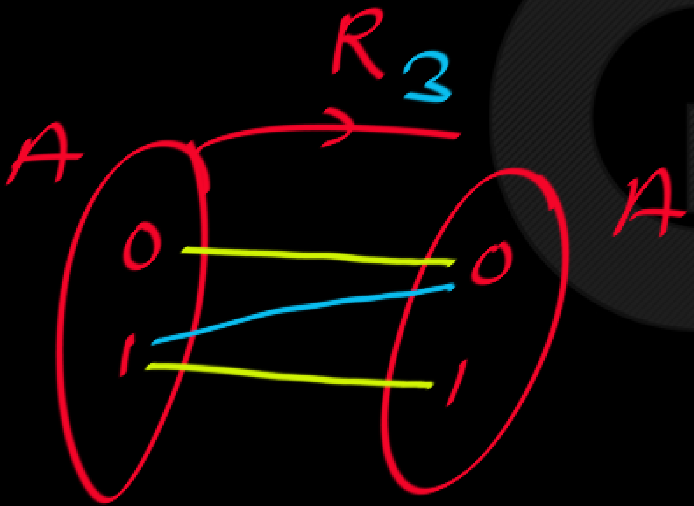
0.2. $R : A \rightarrow A$

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

$\rightarrow A$

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.



Ref & Antisym

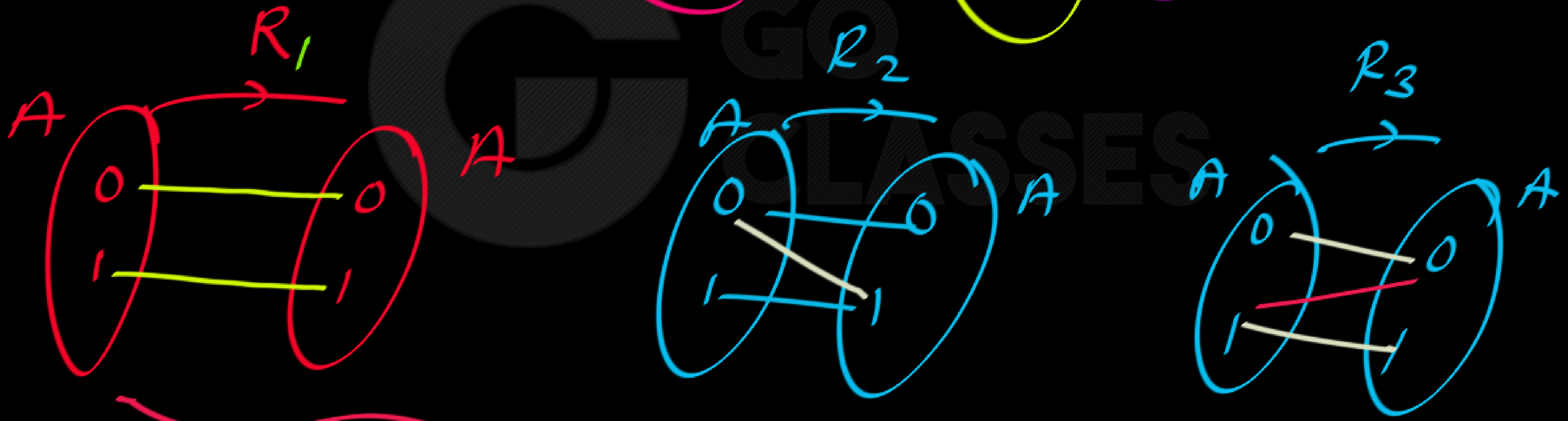
0.2. $R : A \rightarrow A$

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

A



Ref & AntiSym

0.2.

Problem 2. Find all the relations on $\{0,1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0,1\}$ that are reflexive and antisymmetric.

mat. Rep:

	0	1
0	✓	$\boxed{}$ (0,1)
1	$\boxed{}$ (1,0)	✓

(0,1)	(1,0)
✗	✗
✓	✗
✗	✓

3 cases

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

mat. Rep:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \checkmark & \boxed{\times} \\ 1 & \boxed{\times} & \checkmark \end{array} \quad R_1$$

(0, 1)

(1, 0)

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

mat. Rep:

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \checkmark & \boxed{\checkmark} \\ 1 & \boxed{\times} & \checkmark \end{array} \quad R_2$$

$(0,1)$

$(1,0)$

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

mat. Rep:

$(0,1)$

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \checkmark & \boxed{\times} \\ 1 & \boxed{\checkmark} & \checkmark \end{array}$$

R_3

$(1,0)$

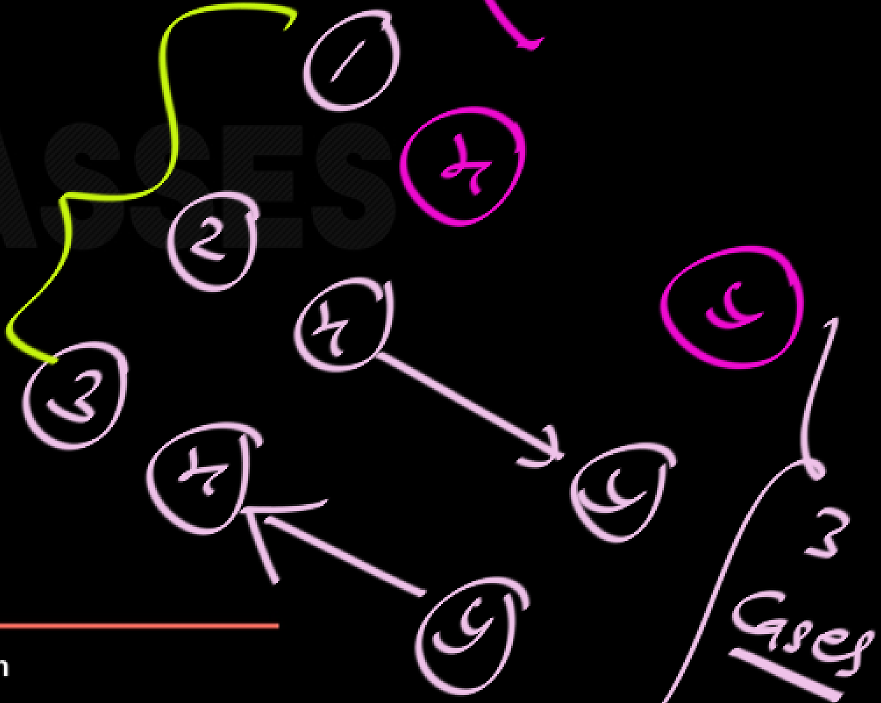
0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

Graph Rep:



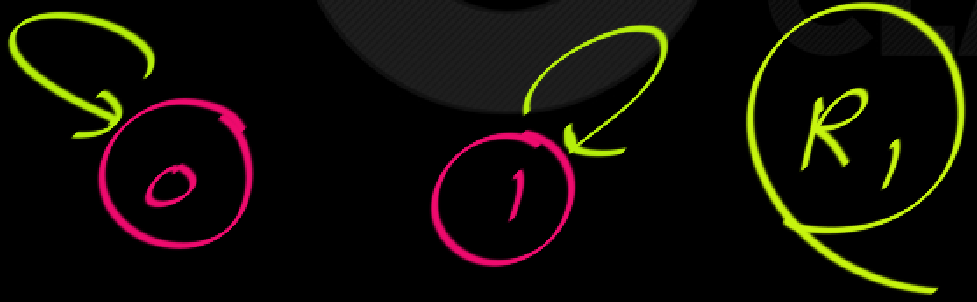
0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

Graph Rep :



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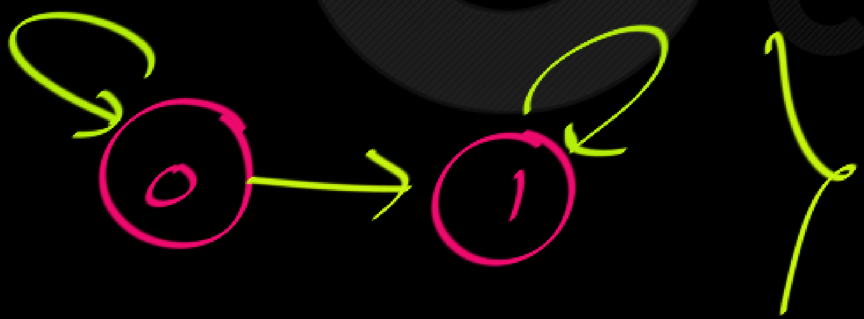


R_2

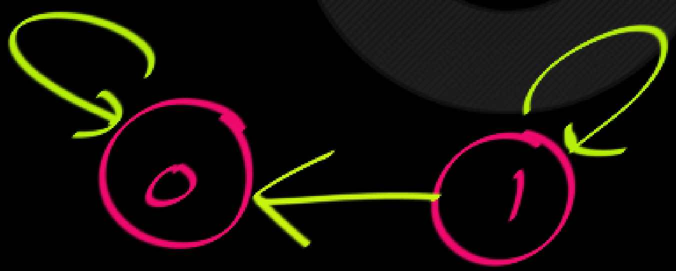
0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

Graph Rep :



R_3

0.2.

Problem 2. Find all the relations on $\{0, 1\}$ that are reflexive and symmetric.

0.3.

Problem 3. Find all the relations on $\{0, 1\}$ that are reflexive and antisymmetric.

$R_1 = \{ (0,0), (1,1) \}$ ✓

$R_2 = \{ (0,0), (1,1), (0,1) \}$ ✓

$R_3 = \{ (0,0), (1,1), (1,0) \}$ ✓

3 Rel

Relation R on set A is called symmetric if
$(x, y) \in R \Rightarrow (y, x) \in R$, for all $x, y \in A$.

Example (7.6)

With $A = \{1, 2, 3\}$, we have:

a) $R_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ a symmetric, but not reflexive, relation on A;

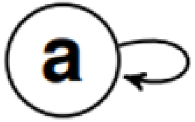
b) $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$ a reflexive, but not symmetric, relation on A;

c) $R_3 = \{(1, 1), (2, 2), (3, 3)\}$ and
$R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, two relations on A that are both reflexive and symmetric; and

d) $R_5 = \{(1, 1), (2, 2), (3, 3)\}$, a relation on A that is neither reflexive nor symmetric.

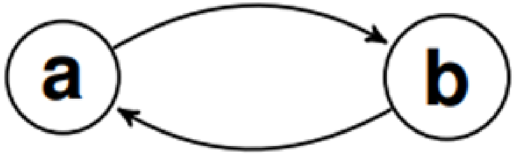
Properties

Reflexive: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.



Every vertex has a self-loop.

Symmetric: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.



If there is an edge from one vertex to another, there is an edge in the opposite direction.

Antisymmetric: A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

There is at most one edge between distinct vertices.

Next Topic:

Transitive Relation

Transitive Relation :

If $\underline{aRb}$ & $\underline{bRc}$

then aRc

Transitive Relation :

$$
\begin{tikzcd}
a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c
\end{tikzcd}
$$

Transitive Edge

Transitive Relation :

friendship :

NOT Transitive

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

Transitive Relation:

fatherhood:

NOT Transitive

$a \longrightarrow b \longrightarrow c$

Transitive Relation:

Sibling:

Transitive

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

Transitive Relation:

Rel. R on set A

R is Transitive iff

$$\forall_{a,b,c \in A} \left([aRb \wedge bRc] \longrightarrow aRc \right)$$

Transitive Relation:

$\boxed{R : A \to A}$

$\forall \underbrace{a, b, c}_{\text{Same, Different}} \in A \left(\underbrace{\boxed{a \to b \wedge b \to c}}_{aRc} \right)$

Transitive Relation:

$\forall a, b, c \in A$

$$
\begin{tikzcd}
a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c
\end{tikzcd}
$$

→ Transitive Edge

Transitive Relation:

Equality

$$a \xrightarrow{a=b} b \xrightarrow{b=c} c$$

$$a = b = c \longrightarrow a = c$$

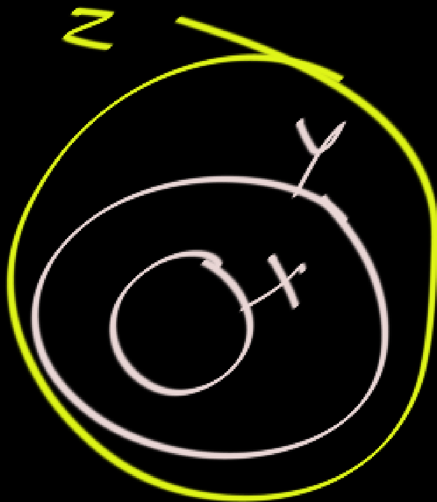
Transitive Rel.

Transitive Relation:

Subset Relation: ✓

$$X \xrightarrow{X \subseteq Y} Y \xrightarrow{Y \subseteq Z} Z$$

$$X \subseteq Z$$



Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

Consider the following relations on $\{1, 2, 3, 4\}$: Base set

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

R_1: $\boxed{3 \rightarrow 4 \rightarrow 1}$ BUT $3 \cancel{R_1} 1$

NOT Trans.

Consider the following relations on {1, 2, 3, 4}:

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$

$R_2 = \{(1, 1), (1, 2), (2, 1)\},$

$\because \boxed{2 \rightarrow 1 \rightarrow 2}$ But $2 \cancel{R_2} 2$

Not Trans.

$\forall\, a, b, c \in A$ → Same / Different

If

$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend right] & b \arrow[r] & c \end{tikzcd}$$

then

Trans. Edge

Example 1:

Equality Relation on Z is Transitive??

YES

Base set: Z ; R on Z ; R: Z → Z

$\boxed{aRb \text{ iff } a=b}$

$$\begin{tikzcd} a \arrow[r, "a=b"] \arrow[rr, bend right] & b \arrow[r, "b=c"] & c \end{tikzcd}$$

If

$$\underline{\underline{a=b}}\,\underline{\underline{=c}} \longrightarrow \underline{\underline{a=c}} \longrightarrow aRc$$

Example 2:

Let Set A = {1,2,3}

Subset Relation on P(A) is Transitive??

set

P(A) : Base set = $\{ X \mid X \subseteq A \}$

$= \{ \phi, \{1\}, \{2,3\}, \{2\}, \ldots \{1,2,3\} \}$

Example 2:

Let Set A = {1,2,3}

Subset Relation on P(A) is Transitive??

$R : P(A) \rightarrow P(A)$

$\boxed{x R y \text{ iff } x \subseteq y}$

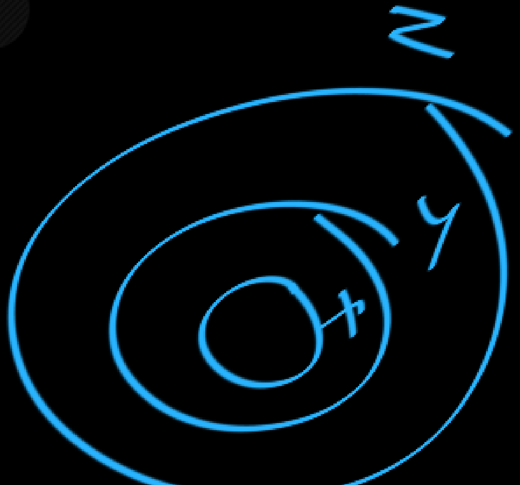
every element is a subset of A

YES

$$\begin{tikzcd} X \arrow[r, "x \subseteq y"] \arrow[rr, bend right] & Y \arrow[r, "y \subseteq z"] & Z \end{tikzcd}$$

If

$x \subseteq y \subseteq z \longrightarrow x \subseteq z \longrightarrow xRz$



Transitivity

- Many relations can be chained together.
- Examples:
  - If $x = y$ and $y = z$, then $x = z$.
  - If $u \leftrightarrow v$ and $v \leftrightarrow w$, then $u \leftrightarrow w$.
  - If $x \equiv_k y$ and $y \equiv_k z$, then $x \equiv_k z$.
- These relations are called ***transitive***.
- A binary relation R over a set A is called *transitive* if

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

("Whenever a is related to b and b is related to c, we know a is related to c.)

"Not Transitive" Relation:

Violation of Transitivity :

$\exists\, a, b, c \in A$

Same / diff.

$\boxed{a \rightarrow b \rightarrow c \quad \text{But } a \not{R}\, c}$

"Not Transitive" Relation:

Violation of Transitivity :

$$\exists_{a,b,c \in A} \left(aRb \wedge bRc \wedge a \not{R} c \right)$$

Same / diff.

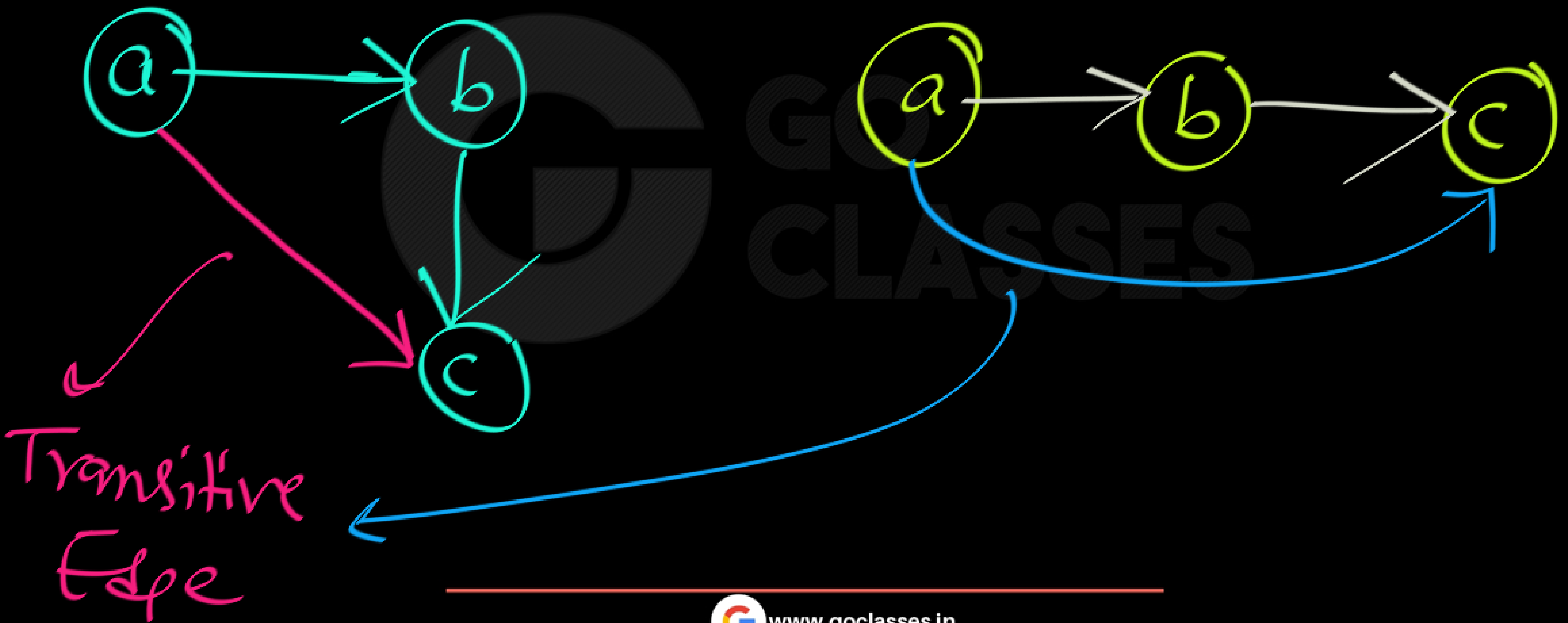
$$\begin{tikzcd} a \arrow[r] \arrow[rr, bend left, "\times"] & b \arrow[r] & c \end{tikzcd}$$

Graph Representation of Transitive Relation:

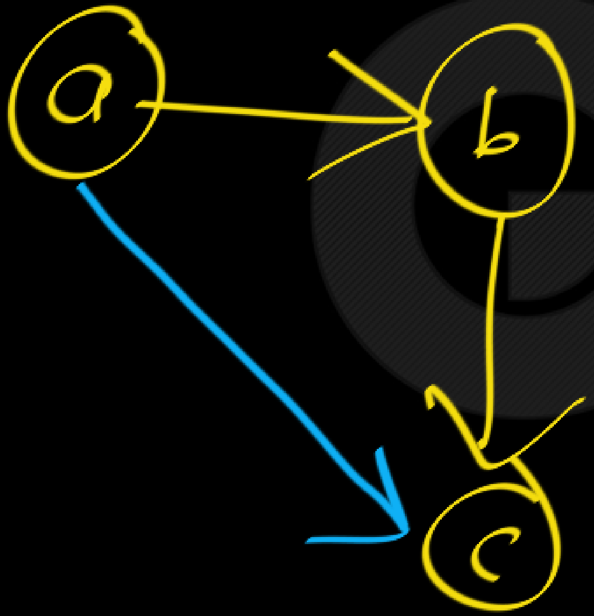
Very Useful in Q.

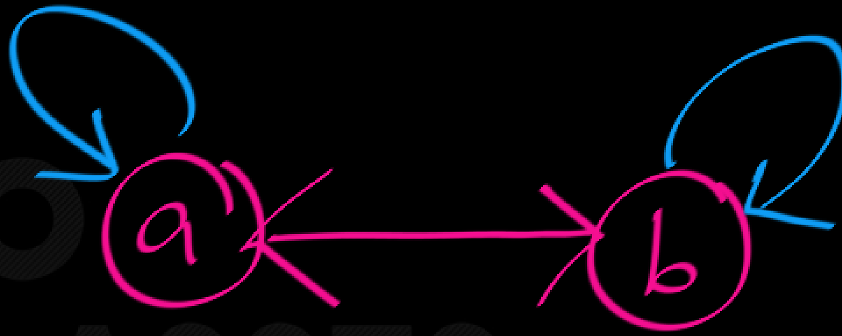
Relation Transitivity

Graph Representation of Transitive Relation:

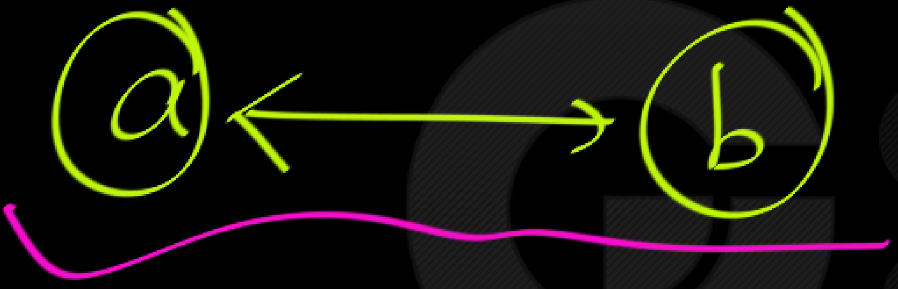


Graph Representation of Transitive Relation:





Graph Representation of Transitive Relation:



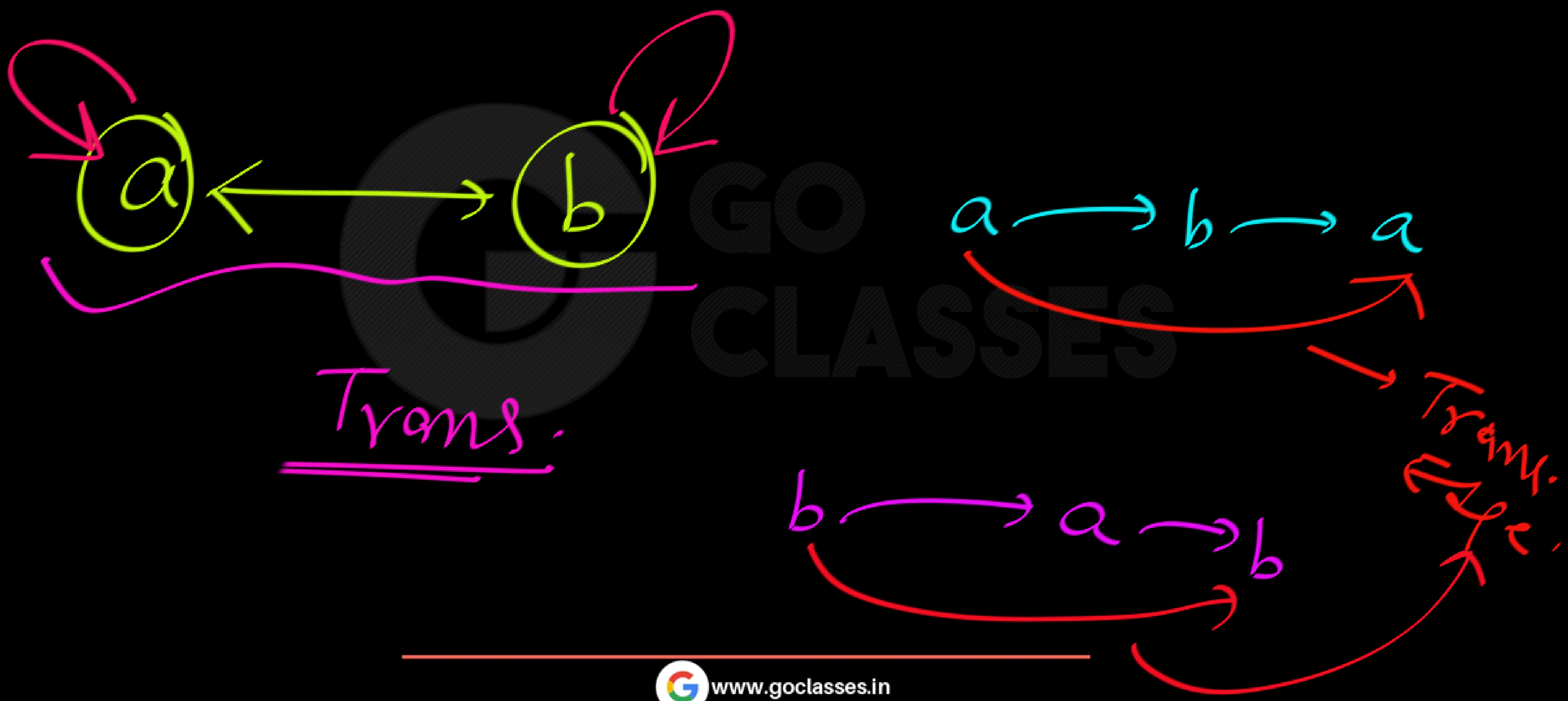
Not Trans.

$a \to b \to a$

$b \to a \to b$

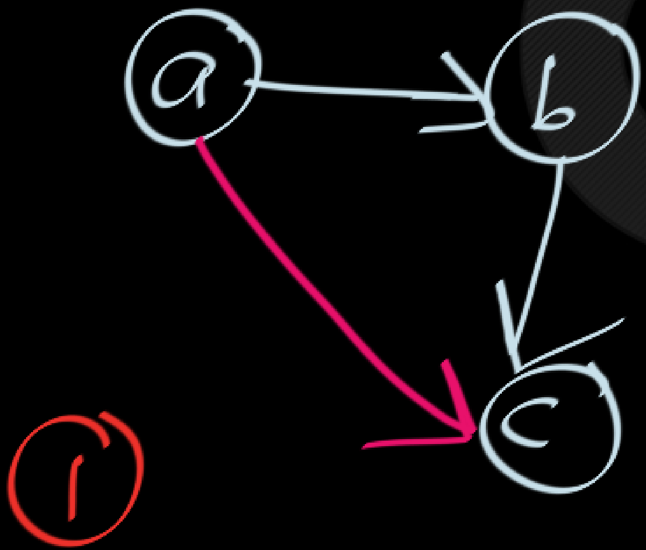
Trans. Edge.

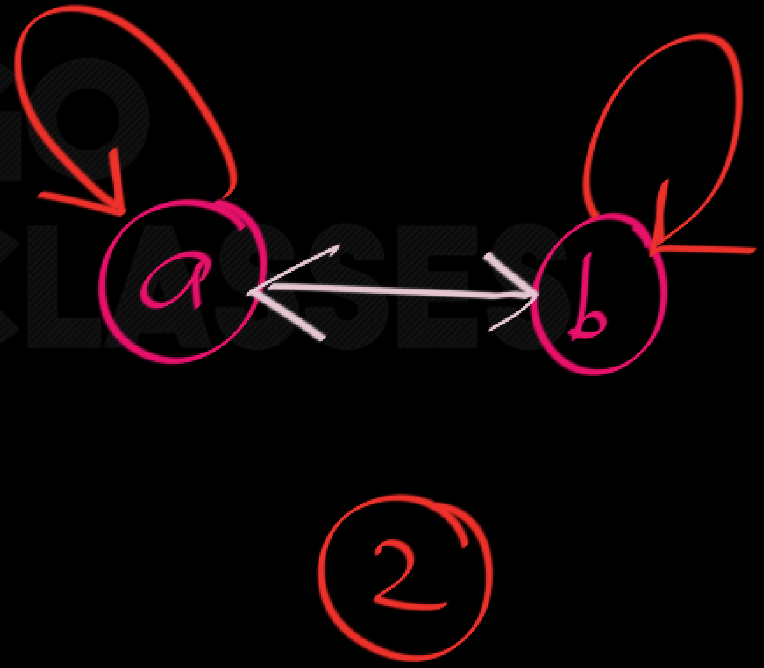
Graph Representation of Transitive Relation:



Graph Representation of Transitive Relation:

Final Conclusion:

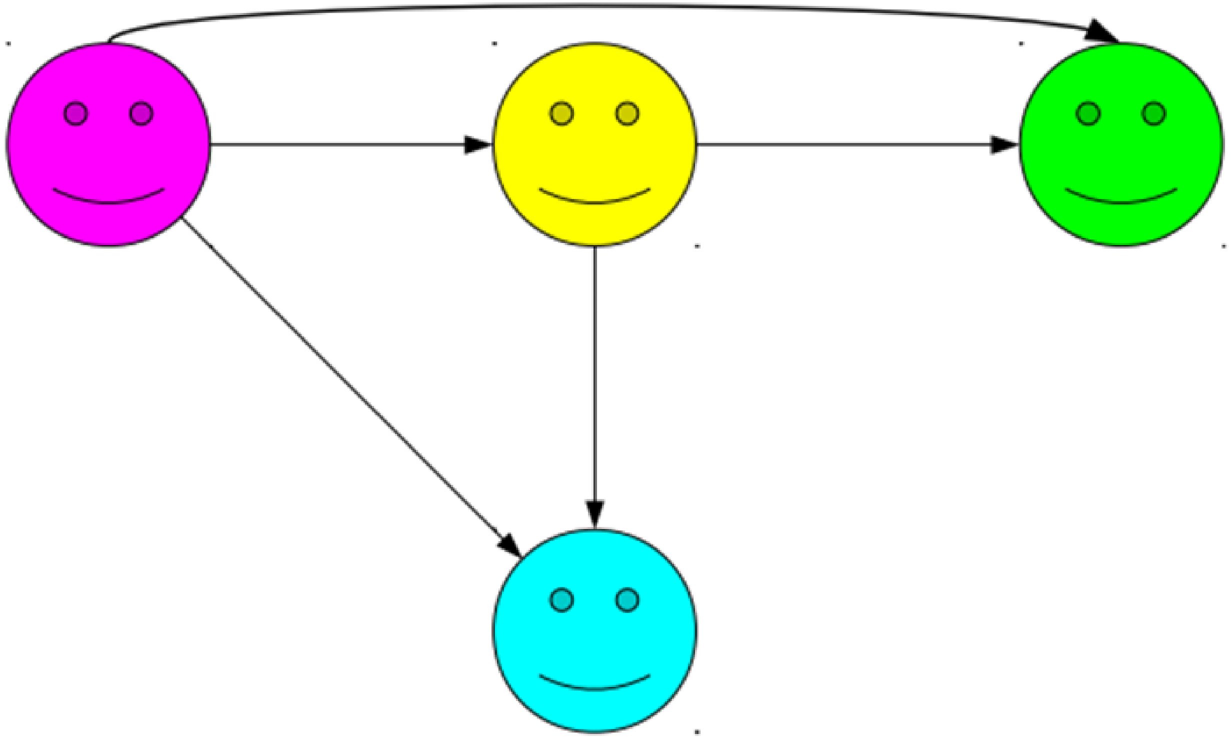




Violation of Transitive Relation:

final Conclusion:

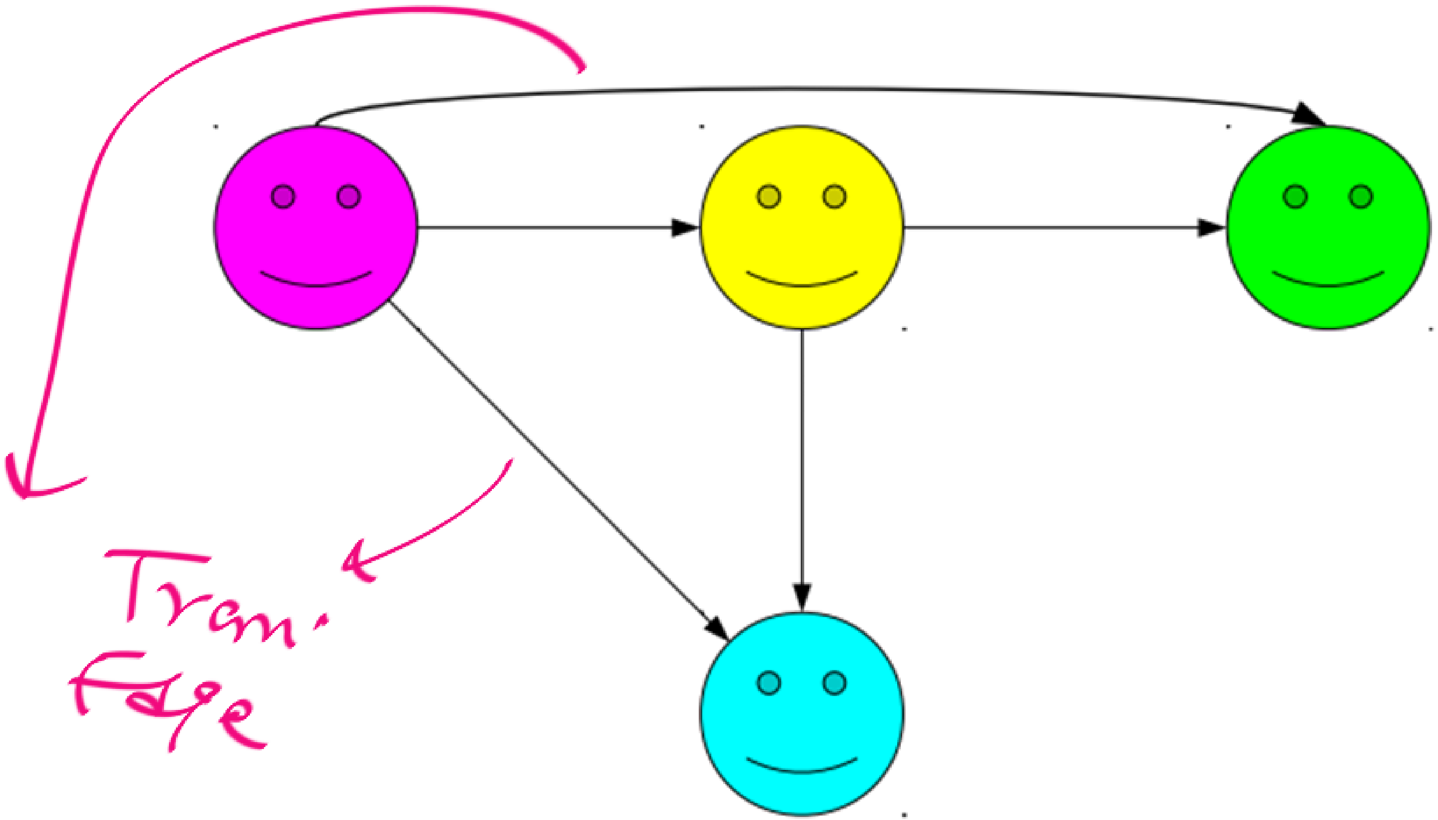
An Intuition for Transitivity



$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$

("Whenever a is related to b and b is related to c, we know a is related to c.)

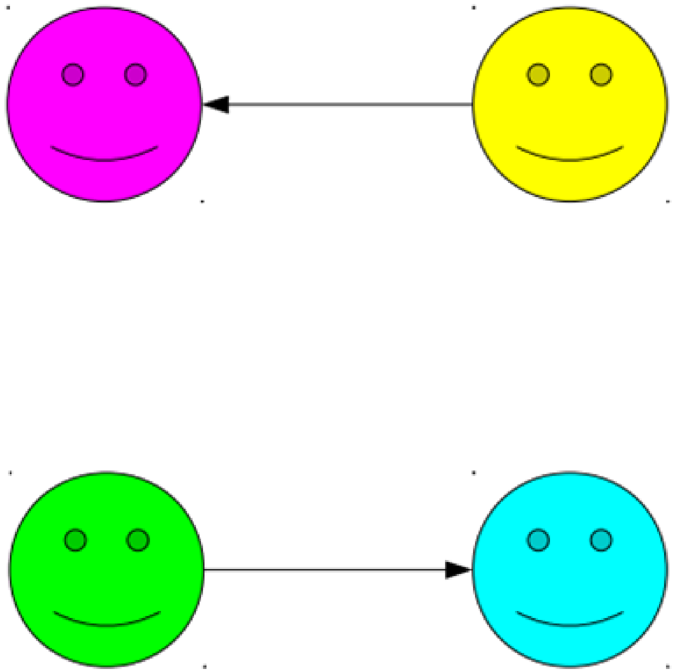
An Intuition for Transitivity



$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$

("Whenever a is related to b and b is related to c, we know a is related to c.)

Is This Relation Transitive?



$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$

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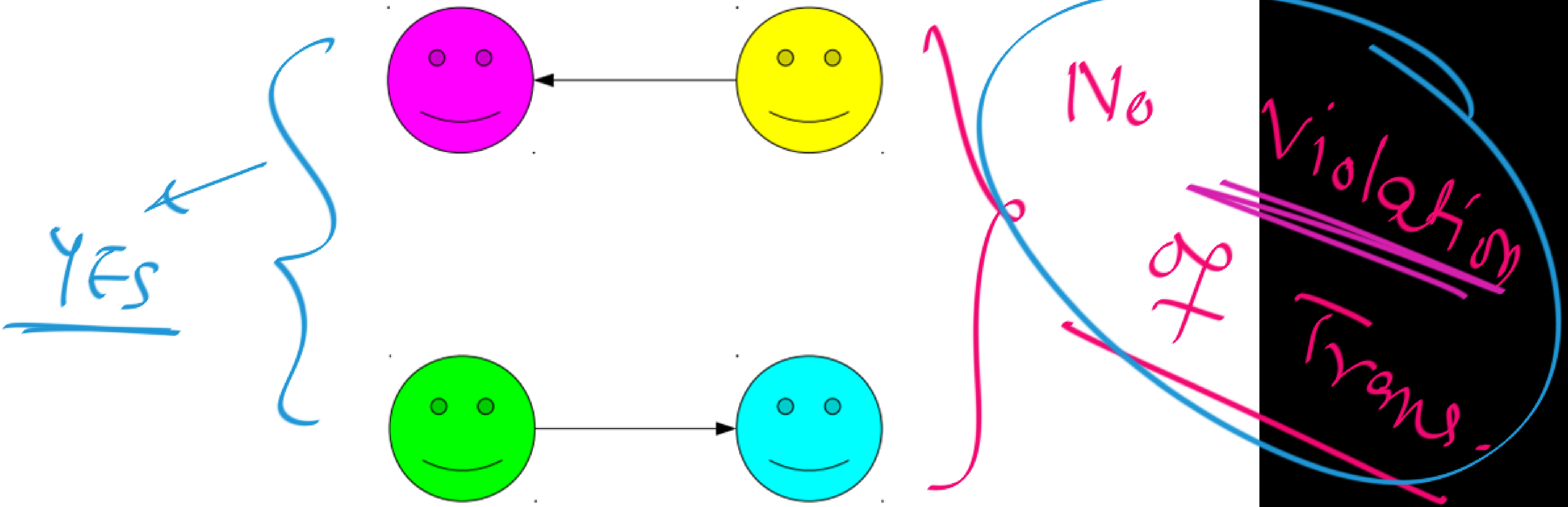
Is This Relation Transitive?



$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \to aRc)$

("Whenever a is related to b and b is related to c, we know a is related to c.)

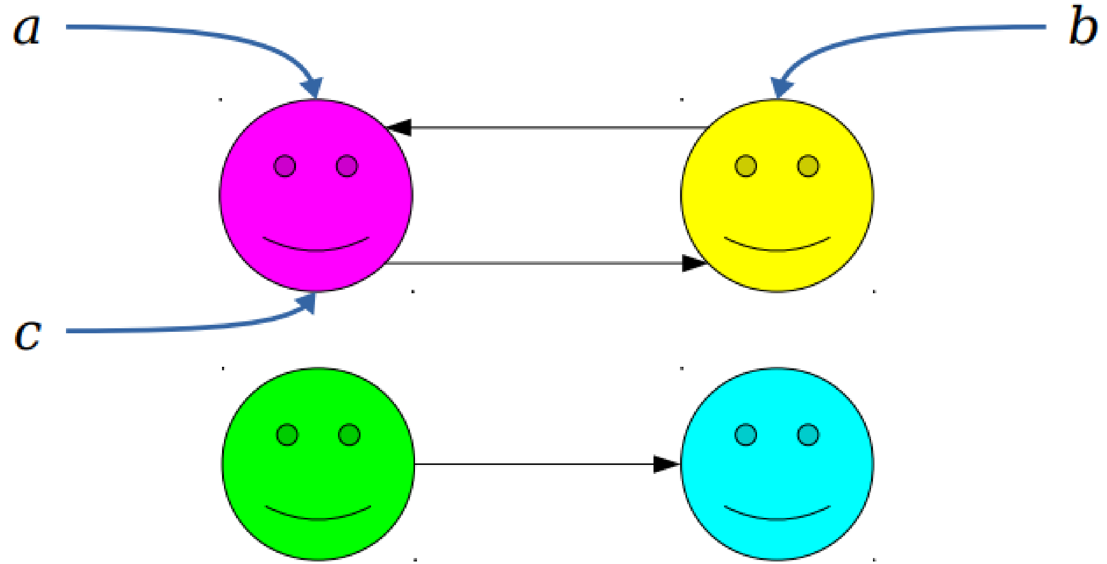
Violation of Trans;

(1) $a \longrightarrow b \longrightarrow c$

$a \not\longrightarrow c$

(2) $a \longleftrightarrow b$ But $a \not R a$ or $b \not R b$

Is This Relation Transitive?



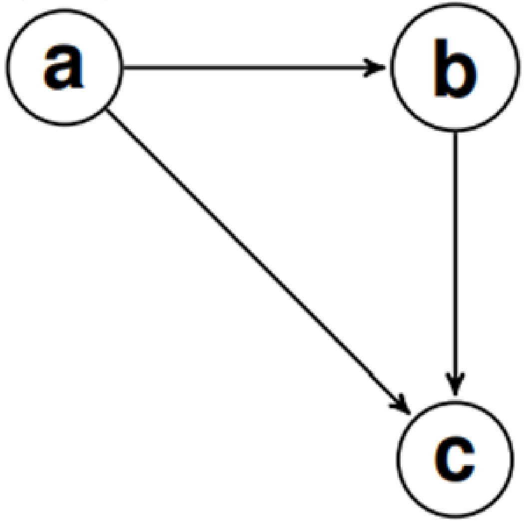
$\forall a \in A.\ \forall b \in A.\ \forall c \in A.\ (aRb \land bRc \to aRc)$

("Whenever a is related to b and b is related to c, we know a is related to c.)

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Transitive: A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.



If there is a path from one vertex to another, there is an edge from the vertex to another.

Properties of Relations

Definitions

A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.

A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.

A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.

A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$

A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

Properties of Binary Relations

A binary relation $R \subseteq A \times A$ is called

- Reflexive iff $\forall x\ (x, x) \in R$
- Symmetric iff $\forall x, y\ ((x, y) \in R \rightarrow (y, x) \in R)$
- Antisymmetric iff $\forall x, y\ ((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$
- Transitive iff $\forall x, y, z\ ((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$.

Examples:

- $\leq$ and $=$ are reflexive, but $<$ is not.
- $=$ is symmetric, but $\leq$ is not.
- $\leq$ is antisymmetric.
  Note: $=$ is also antisymmetric, i.e., $=$ is symmetric and antisymmetric.
  $<$ is also antisymmetric, since the precondition of the implication is always false.
  However, $R = \{(x, y) \mid x + y \leq 3\}$ is not antisymmetric, since $(1, 2), (2, 1) \in R$.
- All three, $=$, $\leq$ and $<$ are transitive.
  $R = \{(x, y) \mid y = 2x\}$ is not transitive.