Propositional Logic

Next Chapter:

Propositional Formula (Revisited)

Truth Table

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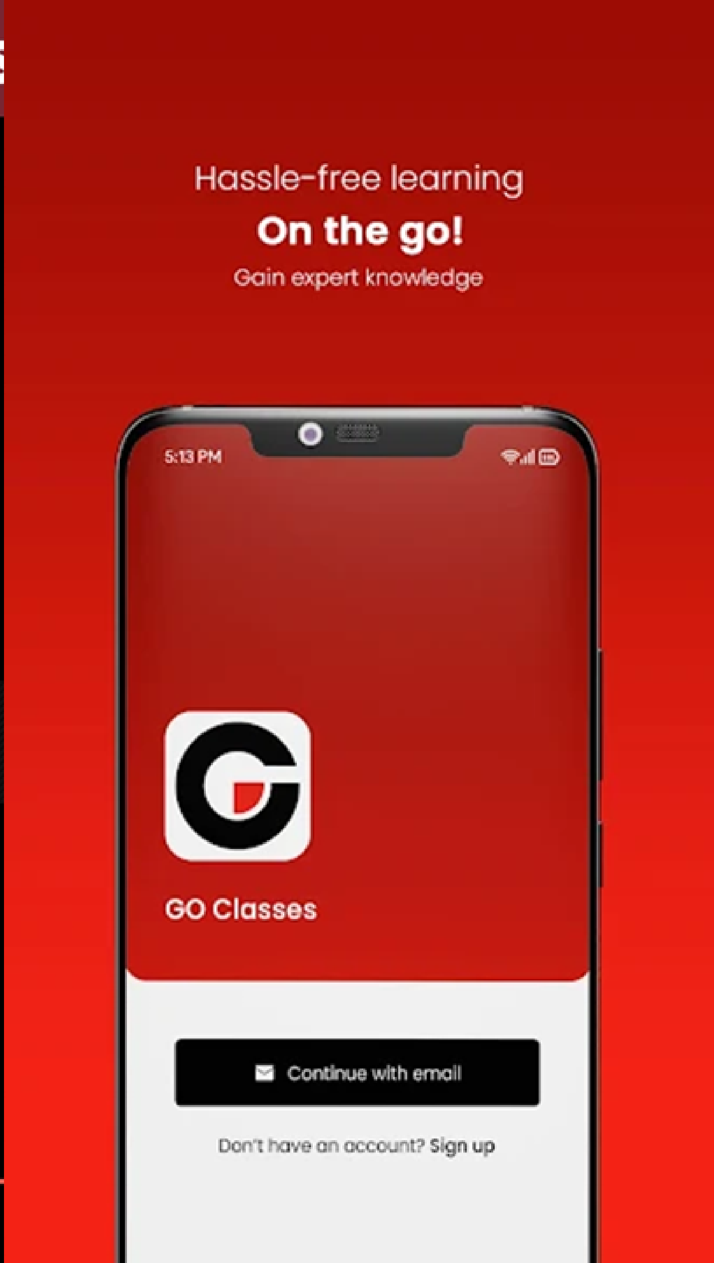

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Recap So Far

- A **propositional variable** is a variable that is either true or false.
- The **logical connectives** are
  - Negation: $\neg p$
  - Conjunction: $p \wedge q$
  - Disjunction: $p \vee q$
  - Implication: $p \rightarrow q$
  - Biconditional: $p \leftrightarrow q$

Some Popular Boolean Operators

Formal Name	**Nickname**	**Arity**	**Symbol**
Negation operator	NOT	Unary	$\neg$
Conjunction operator	AND	Binary	$\wedge$
Disjunction operator	OR	Binary	$\vee$
Exclusive-OR operator	XOR	Binary	$\oplus$
Implication operator	IMPLIES	Binary	$\rightarrow$
Biconditional operator	IFF	Binary	$\leftrightarrow$

Example	Name	Meaning
$\neg p, \bar{p}$	Negation	Not p
$p \vee q$	Inclusive Or	Either p or q or both
$p \wedge q$	And	Both p and q
$p \oplus q$	Exclusive Or	Either p or q, but not both
$p \Rightarrow q$	Implies	If p, then q
$p \Leftrightarrow q$	Equivalence	p if and only if q

Logical Connectives

- **Logical NOT: $\neg p$**
  - Read "**not** p"
  - $\neg p$ is true if and only if p is false.
  - Also called **logical negation**.
- **Logical AND: $p \wedge q$**
  - Read "p **and** q."
  - $p \wedge q$ is true if both p and q are true.
  - Also called **logical conjunction**.
- **Logical OR: $p \vee q$**
  - Read "p **or** q."
  - $p \vee q$ is true if at least one of p or q are true (inclusive OR)
  - Also called **logical disjunction**.

1. A **proposition** is a statement that is either true (T) or false (F), but not both.

   Standard notation for a proposition: p, q, r, ...

2. New propositions, called **compound proposition**, are formed from existing propositions using logical operations

   Four basic compound propositions:

   (a) **Negation** of p: $\neg p =$ "It is not the case that p"

   (b) **Conjunction** of p and q: $p \wedge q =$ "p and q"

   (c) **Disjunction** of p and q: $p \vee q =$ "p or q" ("inclusive or")

   (d) **Exclusive disconjunction** of p and q: $p \oplus q =$ "this is true either p or q but not both are ture"

Truth tables display the relationships between the true values of propositions:

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

3. (a) The **Implication** $p \to q$ is the proposition "if p then q"

   p is called *hypothesis*, and q is called the *conclusion*.

   Some other common ways of expressing this implications are: "p implies q", "p only if q", "p sufficient for q", and "q necessary for p"

(b) **Biconditional** proposition $p \leftrightarrow q$ is the compound proposition "p if and only if q",

(c) Truth tables of $p \to q$ and $p \leftrightarrow q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

You should now know the truth tables for $\wedge$, $\vee$, $\sim$, $\Rightarrow$ and $\Leftrightarrow$.

They should be understood as well as memorized.

You must understand the symbols thoroughly.

Next Topic:

Propositional Formula

(Compound Proposition)

Revisited...

Next Topic:

Priority

Precedence Order

of Logical Connectives

In Number theory :

$2 + 5 \times 3$ → 17 ✓

$2 + 5 \times 3$ → $(2+5) \times 3$ ✗

$2 + 5 \times 3$ → $2 + (5 \times 3)$ ✓

$$\boxed{(\ ) > \times > +}$$ → Precedence order

In propositional logic:

$(\;) > \neg > \wedge > \vee > \rightarrow > \leftrightarrow$

Eg: $\neg p \vee q$

$\neg(p \vee q)$

$(\neg p) \vee q$ ✓

In propositional logic:

$p \vee q \wedge r$

$(p \vee q) \wedge r$ ✗

$p \vee (q \wedge r)$ ✓

$\wedge > \vee$

In Propositional logic:

( )

$\neg$ — Highest

$\wedge$

$\vee$

$\rightarrow$

$\leftrightarrow$ — lowest

What about $\oplus$, $\uparrow$, $\downarrow$?

precedence

NOT Defined

Must be (will be) given.

Give proper Parentheses

$p \oplus q \uparrow r \rightarrow$ Poor Expression

$p \oplus (q \uparrow r)$ ✓

$(p \oplus q) \uparrow r$ ✓

must be Given

Precedence of Logical Operators

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$.

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Finally, it is an accepted rule that the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$. Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$. We will use parentheses when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator. Table 8 displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

Precedence of Logical Operators

TABLE 8
Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Next Topic:

Propositional Formula

$||$

(Compound Proposition)

cs.ecu.edu/karl/2400/fall20/Notes/logic1.pdf

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1 Propositional Logic

1.1 Syntax of propositional logic

Rosen and Zybooks use term *compound proposition* for an expression written using logic. I will use the more common term *propositional formula* instead.

Propositional formula ≡ Compound Proposition

term more used

Prop. formula ≡ Prop. Expression

≡ well formed formula ≡ Compons prop.

Proposition

Atomic Proposition (prop. Variable)

P, Q, R, S, - - - - - -

prop. Variable

Compound Proposition (prop. formula)

P, Q, R, S - - - - -

T, F, $P \wedge (P \to Q)$, - - -

- - - - - - - -

Prop. formula $\equiv$ Compound Proposition

Prop. Variable $\equiv$ Atomic Proposition

Prop. formula (Compound prop)

① True ≡ T ≡ 1 ✓
False ≡ F ≡ 0 ✓

② Every prop. variable : P, Q, R, ...

③ $P \vee \bar{P}$, $P \rightarrow Q$, $P \wedge (P \rightarrow R)$, $\neg P$, ...

$\alpha : (P \rightarrow Q)$; $\beta : P \wedge R$

prop. variable

NOT a prop variable

prop. formula (Compound prop)

New prop. formulas :

$\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$;

$\alpha \oplus \beta$; $\alpha \leftrightarrow \beta$ - - - -

Definition 1.1. A *propositional formula* is defined as follows.

1. Symbols **T** and **F** are propositional formulas.
2. A *propositional variable* is a propositional formula. We will use p, q, r and s, possibly with subscripts, as propositional variables and X for talking about an arbitrary variable.
3. If A and B are propositional formulas then so are
   (a) $A \vee B$,
   (b) $A \wedge B$,
   (c) $\neg A$,
   (d) (A).
   (e) $A \rightarrow B$
   (f) $A \leftrightarrow B$
   (g) $A \oplus B$

For example, each of the following is a propositional formula.

- p
- $p \vee q$
- $p \wedge \neg q$
- $p \wedge q \wedge r$

- $q \vee p \wedge r$
- $(r \wedge \mathbf{T}) \vee \neg q$

Operator $\vee$ is read "or", $\wedge$ is read "and", and $\neg$ is read "not".

Rules of *precedence* and *associativity* determine how you break a propositional formula into subformulas. Higher precedence operators are done first. The following lists operators by precedence, from highest to lowest.

Precedence	
parentheses	high
$\neg$	
$\wedge$	
$\vee$	low

For example, $p \lor q \land r$ is understood to have the same structure as $p \lor (q \land r)$ since $\land$ has higher precedence than $\lor$.

Atomic propositions and logical connectives

An *atomic proposition* is a statement or assertion that must be true or false.

Examples of atomic propositions are: "5 is a prime" and "program P terminates".

Propositional formulas are constructed from atomic propositions by using *logical connectives*.

Connectives	
0	false
1	true
$\neg$	not
$\wedge$	and
$\vee$	or
$\rightarrow$	conditional (implies)
$\Leftrightarrow$	biconditional (equivalent)

A typical propositional formula is $A \wedge (B \vee C) \rightarrow B$.

The *truth value* of a propositional formula can be calculated from the truth values of the atomic propositions it contains.

α : $p \vee (p \to q)$ → prop. formula

α → prop. formula

p, q → prop. Variable

$p = T,\ q = F \longrightarrow \alpha = T$

$p = F,\ q = T \longrightarrow \alpha = T$

1.2 Meaning of propositional logic

The meaning of a propositional formula can only be defined when the values of all of its variables are given. Each variable can be true or false.

Prop. Variables : a, b → atomic proposition

~~Atomic Prop.~~

Truth Table:

a	b	- - - -
F	F	
F	T	
T	F	
T	T	

a — F or T

b — F or T

Prop. formula (Compound prop):

α, β

Truth Table:

α	β	
F	F	
F	T	
T	F	
T	T	

Prop. formula (Compound prop);

α, β

NOT prop. variables

$$\begin{cases} \alpha = p \wedge q \\ \beta = p \vee \bar{p} \end{cases}$$

Prop. variable

Prop. formula (Compound prop):

α, β

$\alpha : p \wedge q$

$\beta : p \vee \bar{p}$

prop. variable

p	q	α	β
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Q: 3 prop. variables.

\# Rows in Truth Table?

P, Q, R

$2^3 = 8$

8 Truth values Combinations

Q: 3 prop. formulas α, β, γ

Rows in Truth Table ?

Q: 3 prop. formulas α, β, γ

Rows in Truth Table?

α	β	γ
F	F	F
F	F	T
F	T	F
⋮		
T	T	T

$\rightarrow 8$

Q: 3 prop. formulas α, β, γ

Rows in Truth Table?

$(p \vee q) \wedge (R \to s)$

$p \vee \bar{p}$

Q: 3 prop. formulas α, β, γ

Rows in Truth Table? $\Rightarrow$ 2

$\alpha = p$ ✓

$\beta = \neg p$ ✓

$\gamma = p \vee \bar{p}$ ✓

p	α	β	γ
T	T	F	T
F	F	T	T

3 prop. formula: α, β, γ

Compound prop.

$\alpha : p \wedge q$

$\beta : p \rightarrow q$

$\gamma : p$

p	q	α	β	γ
F	F	F	T	F
F	T	F	T	F
T	F	F	F	T
T	T	T	T	T

Prop. formula Vs Prop. Variable

(Compound prop) (Atomic prop)

$\alpha = \boxed{p \wedge (p \rightarrow (q \vee r))}$

Compound prop.

Truth Table:

prop. variables

P	Q	R	α	β	---

prop. formulas

8 Rows

α, β : prop. formula ;

- - - - - - -	α	β
	⋮	⋮

α, β : prop. formula ;

α	β
F	F
F	T
T	F
T	T

Next Topic:

Truth Table

Question 1 :

Let p represent a true statement, while q and r represent false statements. Find the truth value of the compound statement.

(i) : $\sim[(\sim p \wedge \sim q) \vee \sim q] \longrightarrow$ F

(ii) : $\sim(p \wedge q) \wedge (r \vee \sim q) \longrightarrow$ T

(iii): $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim p) \longrightarrow$ T

$P = \text{True}$; $q, r = F$

Question 1 :

Let p represent a true statement, while q and r represent false statements. Find the truth value of the compound statement.

(i) : $\sim[(\sim p \wedge \sim q) \vee \overset{T}{\sim q}]$ $\rightarrow \neg(T) = F$

(ii) : $\sim(p \wedge q) \wedge (r \vee \sim q)$ $\rightarrow \neg(\overset{F}{p \wedge q}) \wedge (\overset{F}{r} \vee \overset{T}{\bar{q}})$

$T \wedge (T)$

$= T$

(iii): $\sim(\sim p \wedge \sim q) \vee (\sim r \vee \sim p)$

True

$\neg(F \wedge T) \vee (T \vee F)$

$= T \vee T = T$

$T \vee \text{anything} = T$

Question 2 :

Suppose the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false.

Find the truth values of P,Q,R and S.

Question 2 :

Suppose the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false.

Find the truth values of P,Q,R and S.

T T F F

$\underbrace{[(P \wedge Q) \vee R]}_{\alpha} \Rightarrow \underbrace{(R \vee S)}_{\beta}$

$\alpha \rightarrow \beta \equiv$ false

$\alpha = T$

& $\beta = F$

$$(P \wedge Q) \vee \overset{F}{R} \equiv \text{True}$$

and

$R \vee S = \text{false}$

$R = \text{false}$ &

$S = \text{false}$

$(P \wedge Q) \vee F \equiv \text{True}$

$P \wedge Q = \text{True}$

$P = T$ & $Q = T$

$$\boxed{\alpha \vee f = \alpha}$$

Constant (fixed)

$$\alpha \vee F = \boxed{} \rightarrow \alpha$$

$$T \rightarrow \boxed{T}$$

$$F \rightarrow \boxed{F}$$

Truth Table :

Truth table tells us about the Truth Value of a Compound Proposition(Propositional Formula) for each combination of truth values of atomic propositions.

Eg: $\alpha = p \wedge (p \rightarrow q)$

Atomic prop

p	q	α
F	F	?
F	T	
T	F	
T	T	

Truth Tables of Compound Propositions

p	q	$p \land (p \to q)$
F	F	F
F	T	F
T	F	F
T	T	T

Compound prop

(prop formula)

when $p = T \Rightarrow p \land (\overset{T}{p} \to q)$

$\underbrace{T \land q}_{q}$

q

$$\boxed{T \rightarrow Y \equiv Y}$$

More Elaborate Truth Tables

We can't evaluate this until we have a value for $p \to q$.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

We can't evaluate this until we have a value for $p \to q$.

p	q	$p \land (p \to q)$
F	F	F ∧ T = F
F	T	F ∧ T = F
T	F	T ∧ F = F
T	T	T ∧ T = T

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

More Elaborate Truth Tables

so let's start by evaluating this.

p	q	$p \wedge (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

Now we can go evaluate this.

p	q	$p \land (p \to q)$
F	F	T
F	T	T
T	F	F
T	T	T

More Elaborate Truth Tables

This gives the final truth value for the expression.

p	q	$p \wedge (p \rightarrow q)$	
F	F	F	T
F	T	F	T
T	F	F	F
T	T	T	T

$$(P \vee Q) \wedge \sim (P \wedge Q),$$

which literally means:

P or Q is true, and it is not the case that both P and Q are true.

This statement will be true or false depending on the truth values of P and Q. In fact we can make a truth table for the entire statement. Begin as usual by listing the possible true/false combinations of P and Q on four lines. The statement $(P \vee Q) \wedge \sim (P \wedge Q)$ contains the individual statements $(P \vee Q)$ and $(P \wedge Q)$, so we next tally their truth values in the third and fourth columns. The fifth column lists values for $\sim (P \wedge Q)$, and these are just the opposites

of the corresponding entries in the fourth column. Finally, combining the third and fifth columns with $\wedge$, we get the values for $(P \vee Q) \wedge \sim (P \wedge Q)$ in the sixth column.

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	**F**
T	F	T	F	T	**T**
F	T	T	F	T	**T**
F	F	F	F	T	**F**

This truth table tells us that $(P \vee Q) \wedge \sim(P \wedge Q)$ is true precisely when one but not both of P and Q are true, so it has the meaning we intended. (Notice that the middle three columns of our truth table are just "helper columns" and are not necessary parts of the table. In writing truth tables, you may choose to omit such columns if you are confident about your work.)

$\alpha : \boxed{(P \vee Q) \wedge \neg(P \wedge Q)}$

Compound statement (prop. formula)

P	Q	α
F	F	T
F	T	T
T	F	T
T	T	F

3 prop. variables: P, Q, R

	P	Q	R	Truth Value
P=F	F	F	F	
	F	F	T	
	F	T	F	
	F	T	T	
P=T	T	F	F	
	T	F	T	
	T	T	F	
	T	T	T	

8 Combinations

3 prop. variables: P, Q, R

	P	Q	R	$P \Leftrightarrow (Q \vee R)$
P = F	F	F	F	T
	F	F	T	F
	F	T	F	F
	F	T	T	F
P = T	T	F	F	F
	T	F	T	T
	T	T	F	T
	T	T	T	T

Making a truth table for $P \Leftrightarrow (Q \vee R)$ entails a line for each T/F combination for the three statements P, Q and R. The eight possible combinations are tallied in the first three columns of the following table.

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	T	**T**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	T	**F**
F	F	T	T	**F**
F	F	F	F	**T**

Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q). \quad \alpha$$

p	q	α
F	F	F
F	T	T
T	F	F
T	T	T

NOTE:

Creating "Truth Table" is Inefficient, time-consuming method.

We will RARELY use Truth Tables to solve questions.

We will see some Analytical Ways to solve questions quickly & improve our Analytical Skills as well.

Next Topic:

Tautology, Contradiction, Contingency in Propositional Logic