Group Theory

Next Topic

Properties of Groups

Unique Identity, Unique Inverse, Left(Right) Cancellation

Website : https://www.goclasses.in/

For Groups,

Proofs are VERY Important (as you will see while solving GATE PYQs) and VERY EASY(for GATE level).

Almost All the proofs need two things-

1. **Apply the definitions.**(Eg. Definition of Identity element, Inverse element, Associativity etc)
2. **Multiply both sides by Inverse** of some element to get desired result.

Proposition. *Let $(G, *)$ be a group. The identity element is unique.*

Proof. Assume $e, e' \in G$ both behave like the identity. Then $e = e * e' = e'$.

Proposition. *Let $(G, *)$ be a group. For $a \in G$ there is only one element which behaves like the inverse of a.*

Proof. Assume $a \in G$ has 2 inverses, $b, c \in G$. Then:

$$
\begin{aligned}
(a * b) &= e \\
c * (a * b) &= c * e \\
(c * a) * b &= c \quad \text{(associativity and identity)} \\
e * b &= c \\
b &= c
\end{aligned}
$$

Cancellation Law for Groups. *Let $a, b, c \in G$ a group. Then*

$$a * c = a * b \Rightarrow c = b \text{ and } c * a = b * a \Rightarrow c = b$$

Proof. Compose on left or right by $a^{-1} \in G$, then apply the associativity and inverses and identity axioms. □

Theorem 14.1
For any group G, the following properties hold:

(i) If $a, b, c, \in G$ and $ab = ac$ then $b = c$. (left cancellation law)
(ii) If $a, b, c, \in G$ and $ba = ca$ then $b = c$. (right cancellation law)
(iii) If $a \in G$ then $(a^{-1})^{-1} = a$. The inverse of the inverse of an element is the element itself.
(iv) If $a, b \in G$ then $(ab)^{-1} = b^{-1}a^{-1}$. That is the inverse of a product is the product of the inverses in reverse order.

In Group $\Rightarrow (ab)^{-1} = b^{-1}a^{-1}$

Q: Group $(G, *)$

$\boxed{ax = b}$ then what $x = ?$

Ans:

$ax = b$

$a^{-1}ax = a^{-1}b$

because of Group

$ex = a^{-1}b \Rightarrow \boxed{x = a^{-1}b}$

$x \neq ba^{-1}$

Q : Group $(G, *)$ $\quad ax = b$

We know $\boxed{x = a^{-1}b}$ is a Solution.

Can I say that $x = a^{-1}b$ is "Unique Solution" of $ax = b$?

Proof: Proof by Contradiction.

Assume two solutions of $\boxed{ax = b}$

y, z

$x = y$ is solution means

$ay = b$

$x = z$ is solution means

$az = b$

So, $\cancel{a}y = \cancel{a}z \Rightarrow \boxed{y = z}$

Note: In Group $(G, *)$

$ax = b$ has unique solution $\boxed{x = a^{-1}b}$

$xa = b$ has unique solution $\boxed{x = ba^{-1}}$

In Cayley Table of Group:

	y	z
a	b	b

$\boxed{ax=b} \Rightarrow$ has unique solution

Unique x will satisfy it.

Theorem 14.2

If G is a group and $a, b \in G$ then each of the equations $ax = b$ and $xa = b$ has a unique solution. In the first, the solution is $x = a^{-1}b$ whereas in the second $x = ba^{-1}$.

Theorem 14.2

If G is a group and $a, b \in G$ then each of the equations $ax = b$ and $xa = b$ has a unique solution. In the first, the solution is $x = a^{-1}b$ whereas in the second $x = ba^{-1}$.

Proof.

Consider first the equation $ax = b$. We want to isolate the x on the left side. Indeed, this can be done as follows.

$$\begin{aligned} x &= ex = (a^{-1}a)x \\ &= a^{-1}(ax) = a^{-1}b \end{aligned}$$

To prove uniqueness, suppose that y is another solution to the equation $ax = b$. Then, $ay = b = ax$. By the left cancellation property we have $x = y$. Finally, the proof is similar for the equation $xa = b$. ∎

Q: Group $(G, *)$ $a \in G$

$a^2 = a$ $\Rightarrow$ Does not mean Group is Idempotent because we are not saying $\forall a$.

then $a = ?$

$a^2 = a$

$aa = a$

$\cancel{a}a = \cancel{a}e$

$\boxed{a = e}$

$\underbrace{a = ae}$

$\underbrace{x = xe}$

Method 2

$\overset{x}{a}a = a$

$a = aa^{-1} = e$

Unique solution

Note: In Group, $\forall a$

$$\boxed{\begin{aligned} a &= ae \\ a &= ea \end{aligned}}$$

by Definition of Identity element

Lemma 1.2.1. *If $(G, *)$ is a group and $a \in G$, then $a * a = a$ implies $a = e$.*

Proof. Suppose $a \in G$ satisfies $a * a = a$ and let $b \in G$ be such that $b * a = e$. Then $b * (a * a) = b * a$ and thus

$$a = e * a = (b * a) * a = b * (a * a) = b * a = e$$

Idea 3:

$$a^2 = a$$

$$a\,a = a$$

working as e

$$xe = x$$

$$x \, \bigcirc = x$$

e

So $\boxed{a = e}$

Lemma 1.2.2. *In a group* $(G, *)$

(i) *if* $b * a = e$, *then* $a * b = e$ *and*

(ii) $a * e = a$ *for all* $a \in G$

True by Definition of Identity.

Group $(G, *)$

If $b * a = e$ then $a * b = ?$

$\Rightarrow ba = e$

$\Rightarrow b = a^{-1}$

$ab = ?$

$aa^{-1} = e$

$$\boxed{ba = e}$$

$$baa^{-1} = ea^{-1}$$

$$\boxed{b = a^{-1}}$$

$$b^{x}a = e$$

$$b = ea^{-1} = a^{-1}$$

Unique Solution

Idea 2 : Group $(G, *)$

Given $ab = e$

$xy = e$

$y = x^{-1}$

b working as a^{-1}

$\boxed{b = a^{-1}}$

$ba = ?$ $\quad ba = a^{-1}a = e$

Q: We know that In Group $(G, *)$

$$\left.\begin{array}{l} a\,a^{-1} = a^{-1}a \\ a\,e = e\,a \end{array}\right\}$$ Do these imply Abelian? $\Rightarrow$ No

these Commutative properties are in EVERY Group.

Theorem

- In a Group (G, *) the following properties hold good

1. Identity element is unique.
2. Inverse of an element is unique.
3. Cancellation laws hold good

   $a * b = a * c \Rightarrow b = c$ (left cancellation law)

   $a * c = b * c \Rightarrow a = b$ (Right cancellation law)

4. $(a * b)^{-1} = b^{-1} * a^{-1}$

- In a group, the identity element is its own inverse. $e^{-1} = e$

- ***Order of a group*** : The number of elements in a group is called order of the group.

- Finite group: If the order of a group G is finite, then G is called a finite group.

Important results :

1. If G is a monoid, then the identity element e is unique.
2. If G is a group then:

(i) If $c \in G$ and $cc = c$ then $c = e$.

(ii) For all $a, b \in G$, if $ab = ac$ then $b = c$, and if $ba = ca$ then $b = c$ (these properties of a group is called left cancellation and right cancellation, respectively).

(iii) For $a \in G$ the inverse of a, a^{-1}, is unique.

(iv) For all $a \in G$, we have $(a^{-1})^{-1} = a$.

(v) For all $a, b \in G$ we have $(ab)^{-1} = b^{-1}a^{-1}$.

(vi) For all $a, b \in G$, the equations $ax = b$ and $ya = b$ have unique solutions in G, namely $x = a^{-1}b$ and $y = ba^{-1}$, respectively.

Q.

In a Group G, let b,c are Different elements of G.

Is it POSSIBLE that a*b = a*c??

Q.

In a Group G, let b,c are Different elements of G.

Is it POSSIBLE that $a*b = a*c$?? —— No

If $ab = ac$ then $b = c$

So Contrapositive

If $b \neq c$ then $ab \neq ac$

Cayley table of Group.

	b	c
a	$h \neq$	g

If $b \neq c$ then $ab \neq ac$

NOTE:

In a Group G, If a*b = a*c then b=c.

By Left Cancellation Property.

Show that the following identities hold in any group. Explain your reasoning.

(i) $(a^{-1})^{-1} = a$. The fact that a^{-1} is the inverse of a is expressed as:

$$a^{-1} \cdot a = a \cdot a^{-1} = e.$$

But this also means a is the inverse of a^{-1}, i.e., $a = (a^{-1})^{-1}$.

(ii) $(a^{-1}ba)^3 = a^{-1}b^3a$. Prove this.

To prove: $(a^{-1}ba)^3 = a^{-1}b^3a$

$= a^{-1}b\underline{aa^{-1}}b\underline{aa^{-1}}ba$

$= a^{-1}bbba = \underline{\underline{a^{-1}b^3a}}$

Hence Proved

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(ii) $(a^{-1}ba)^3 = a^{-1}b^3a$.

$$(a^{-1}ba)^3 = a^{-1}ba \cdot a^{-1}ba \cdot a^{-1}ba = a^{-1}bebeba \cdot a^{-1} = a^{-1}b^3a.$$