Set Theory

Recap

Poset, Lattice

Website : https://www.goclasses.in/

Partially Ordered Sets

- Partial Order

  A relation R on a set A is called a partial order if R is ***reflexive, antisymmetric and transitive***. The set A together with the partial order R is called a partially ordered set, or simply a poset, denoted by (A, R)

For instance,

1. Let A be a collection of subsets of a set S. The relation $\subseteq$ of set inclusion is a partial order on A, so (A, $\subseteq$) is a poset.

2. Let Z^+ be the set of positive integers. The usual relation $\leq$ is a partial order on Z^+, as is "$\geq$"

Partially Ordered Sets

- Comparable

If $(A, \leq)$ is a poset, elements a and b of A are comparable if

$$a \leq b \text{ or } b \leq a$$

In some poset, e.g. the relation of divisibility (a R b iff $a \mid b$), some pairs of elements are not comparable

$$2 \nmid 7 \text{ and } 7 \nmid 2$$

Note: if every pair of elements in a poset A is comparable, we say that A is **linear ordered** set, and the partial order is called a **linear order**. We also say that A is a **chain**.

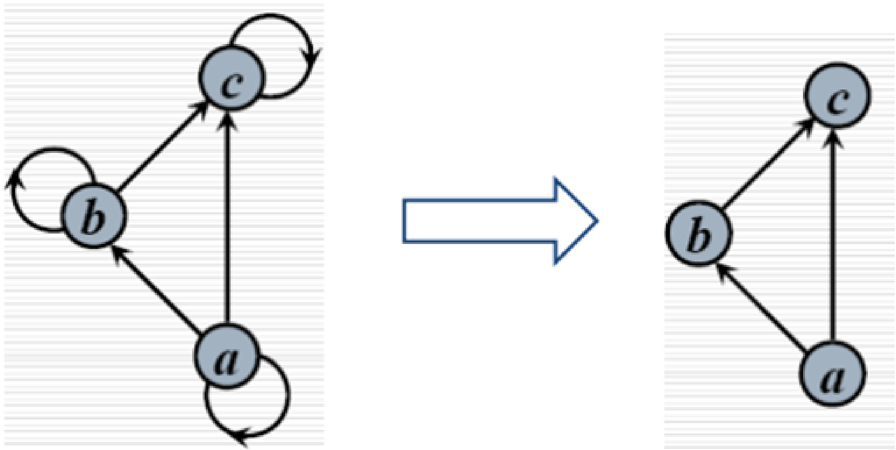
Partially Ordered Sets

- Hasse Diagrams

  Just a reduced version of the diagram of the partial order of the poset.

  a) Reflexive

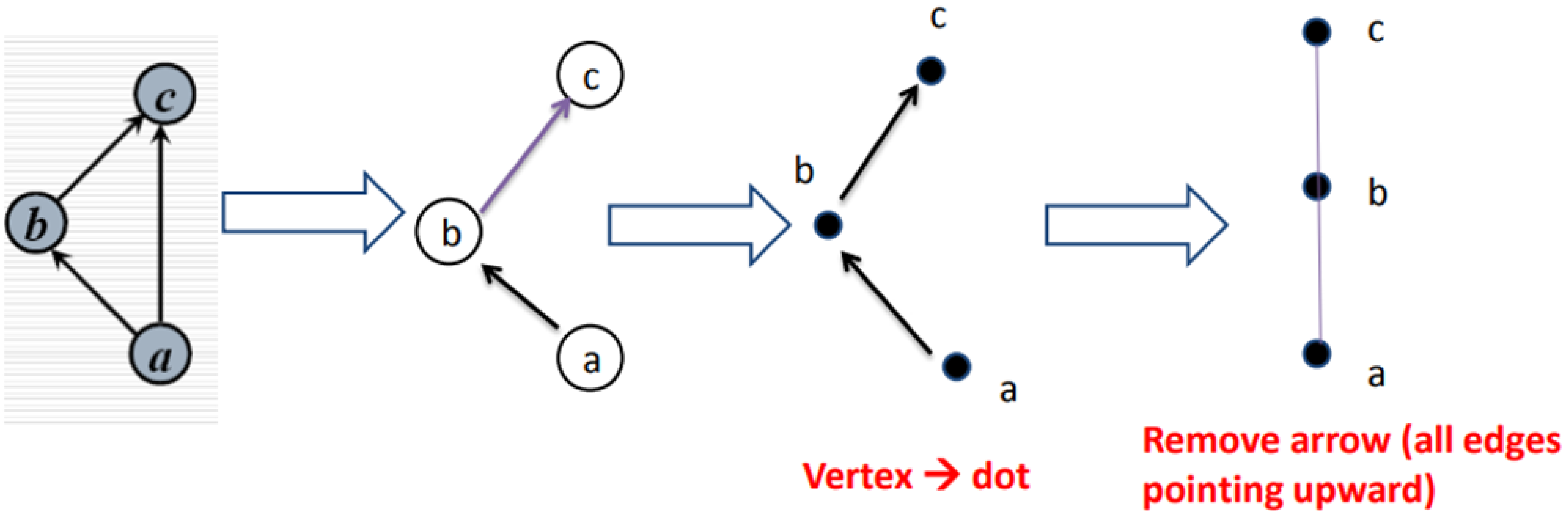
  Every vertex has a cycle of length 1 **(delete all cycles)**



Partially Ordered Sets

- Transitive

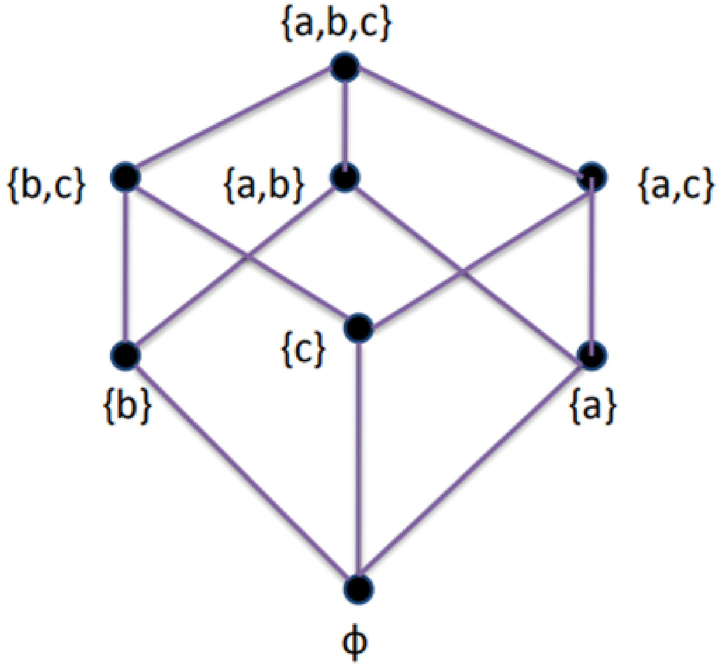
  $a \le b$, and $b \le c$, then $a \le c$ (delete the edge from a to c)



Partially Ordered Sets

- Example

  Let S={a,b,c} and A=P(S). Draw the Hasse diagram of the poset A with the partial order ⊆



Extremal Elements of Partially Ordered Sets

Consider a poset $(A, \leq)$

- Maximal Element

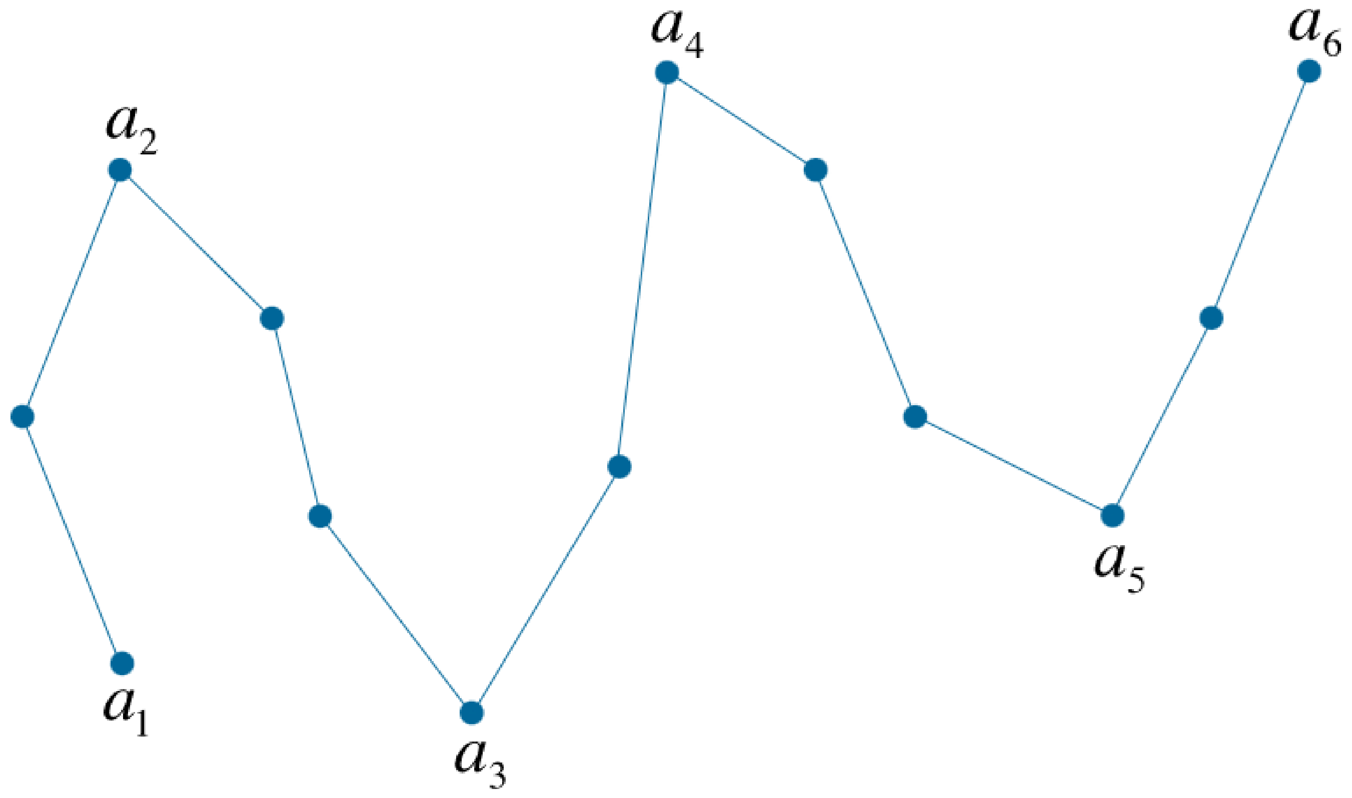
  An element a in A is called a maximal element of A if there is no element c in A such that $a<c$.

- Minimal Element

  An element b in A is called a minimal element of A if there is no element c in A such that $c<b$.

A partially ordered set may have one or many maximal or minimal elements.

$Maximal\ \ elements: \ a_2, a_4, a_6$

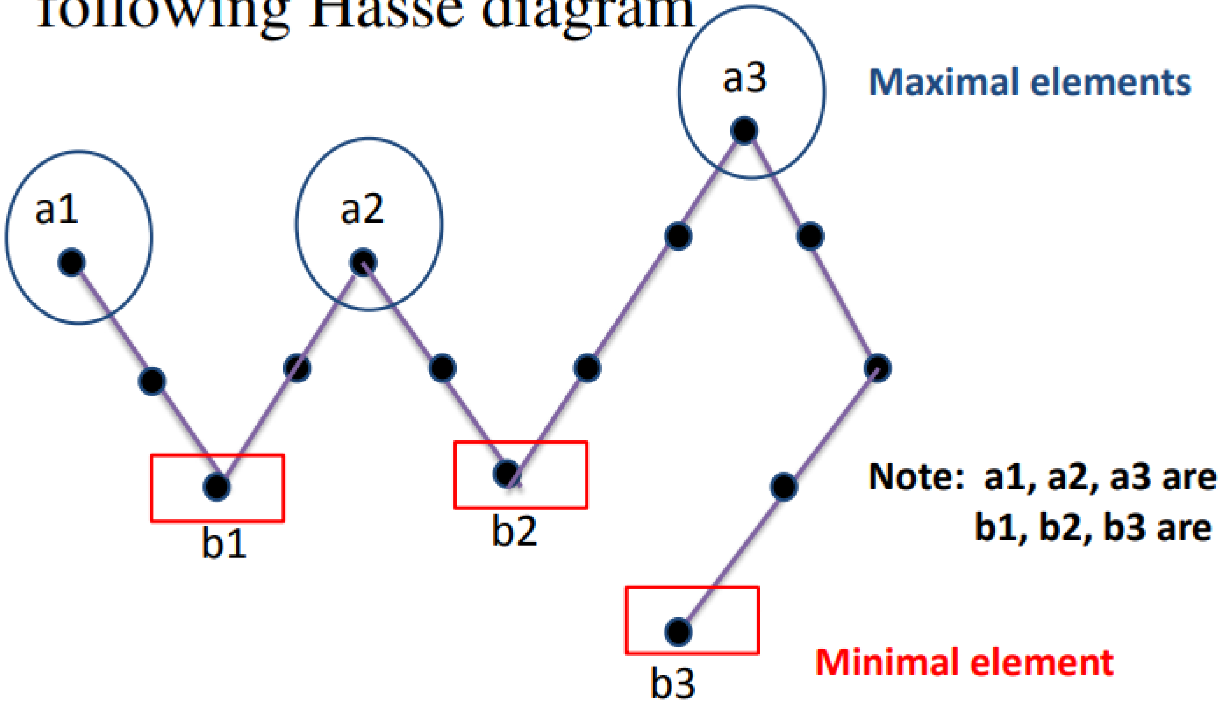


$Minimal\ \ elements: \ a_1, a_3, a_5$

Extremal Elements of Partially Ordered Sets

- Example 1

  Find the maximal and minimal elements in the following Hasse diagram



Note: a1, a2, a3 are incomparable
b1, b2, b3 are incomparable

Extremal Elements of Partially Ordered Sets

- Example 2

  Let A be the poset of nonnegative real number with the usual partial order $\leq$. Then 0 is a minimal element of A. There are no maximal elements of A

- Example 3

  The poset Z with the usual partial order $\leq$ has no maximal elements and has no minimal elements

Extremal Elements of Partially Ordered Sets

- Greatest element

  An element a in A is called a greatest element of A if

  $x \le a$ for all x in A.

- Least element

  An element a in A is called a least element of A if

  $a \le x$ for all x in A.

Extremal Elements of Partially Ordered Sets

- Example 5

  Let A be the poset of nonnegative real number with the usual partial order $\leq$. Then 0 is a least element of A. There are no greatest elements of A

- Example 7

  The poset Z with usual partial order has neither a least nor a greatest element.

Extremal Elements of Partially Ordered Sets

Consider a poset A and a subset B of A

- Upper bound

  An element a in A is called an upper bound of B if $b \leq a$ for all b in B

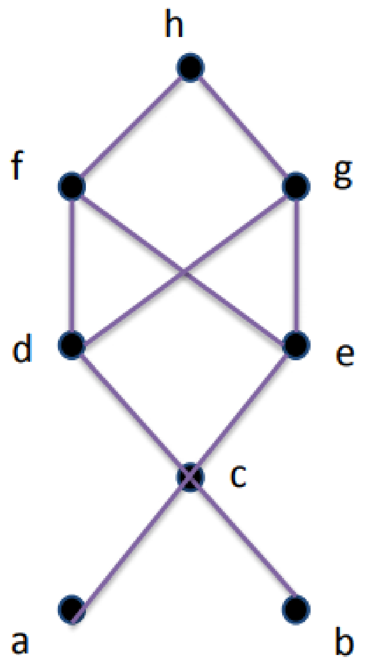
- Lower bound

  An element a in A is called a lower bound of B if $a \leq b$ for all b in B

Extremal Elements of Partially Ordered Sets

- Example 8

Find all upper and lower bounds of the following subset of A: (a) B_1={a, b}; B_2={c, d, e}



B_1 has no lower bounds; The upper bounds of B_1 are c, d, e, f, g and h

The lower bounds of B_2 are c, a and b
The upper bounds of B_2 are f, g and h

Extremal Elements of Partially Ordered Sets

Let A be a poset and B a subset of A,

- Least upper bound

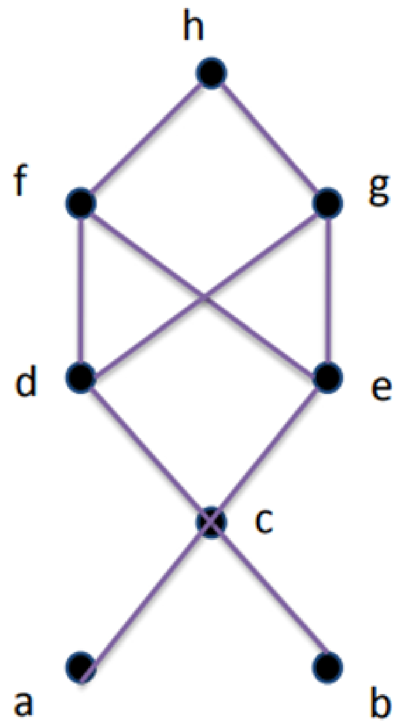
  An element a in A is called a least upper bound of B, denoted by (LUB(B)), if a is an upper bound of B and $a \leq a'$, whenever a' is an upper bound of B.

- Greatest lower bound

  An element a in A is called a greatest lower bound of B, denoted by (GLB(B)), if a is a lower bound of B and $a' \leq a$, whenever a' is a lower bound of B.

- Example 9

Find all least upper bounds and all greatest lower bounds of (a) $B_1=\{a, b\}$ (b) $B_2=\{c, d, e\}$



(a) Since B1 has no lower bounds, it has no greatest lower bounds; However,

LUB(B1)=c

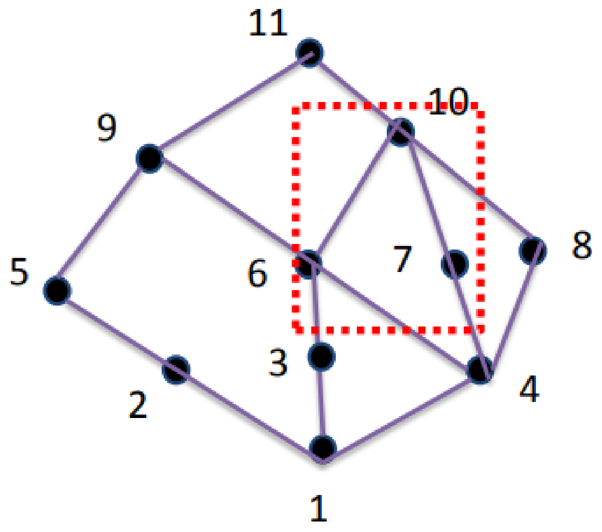
(b)Since the lower bounds of B2 are c, a and b, we find that GLB(B2)=c

The upper bounds of B2 are f, g and h. Since f and g are not comparable, we conclude that B2 has no least upper bound.

Extremal Elements of Partially Ordered Sets

- Example 10

  Let A={1,2,3,…,11} be the poset whose Hasse diagram is shown below. Find the LUB and GLB of B={6,7,10}, if they exist.



The upper bounds of B are 10, 11, and LUB(B) is 10 **(the first vertex that can be Reached from {6,7,10} by upward paths)**

The lower bounds of B are 1,4, and GLB(B) is 4 **(the first vertex that can be Reached from {6,7,10} by downward paths)**

Boolean Lattice:

$$\left\{ \begin{array}{c} \text{Bounded L} \\ \& \\ \text{Complemented L} \\ \& \\ \text{Distributive L} \end{array} \right\}$$

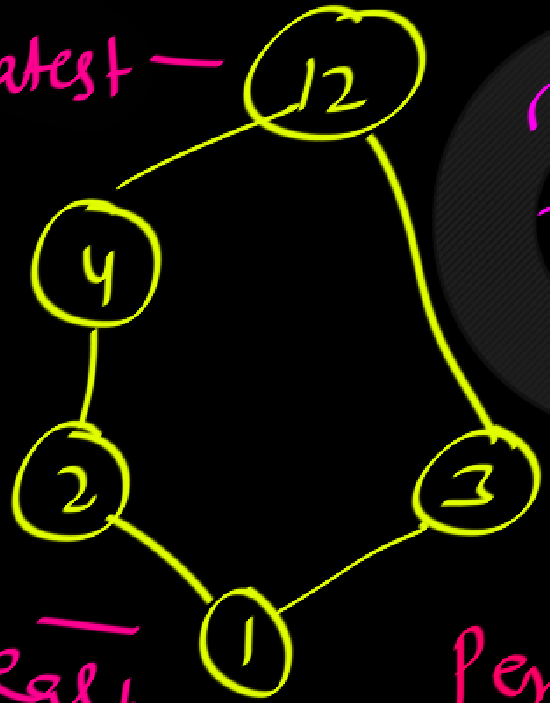
Complemented lattice $\longrightarrow$ bounded lattice.

"Complements" are only defined in Bounded lattices.

Boolean Lattice := Complementes Distributive lattice.

$$\left\{ \begin{array}{l} \text{Complementes L} \\ \& \\ \text{Distributive L} \end{array} \right\}$$

$(\{1, 2, 3, 4, 12\}, |)$ — lattice



— Not Distributive → So not boolean lattice.

— Complemented Lattice

2^{-1} = Comp of 2 = 3

$\bar{3}^{1} = 2, 4$; $\bar{4}^{1} = 3$

$\bar{1}^{1} = 12, \bar{12}^{1} = 1$

Pentagon lattice (N_5)

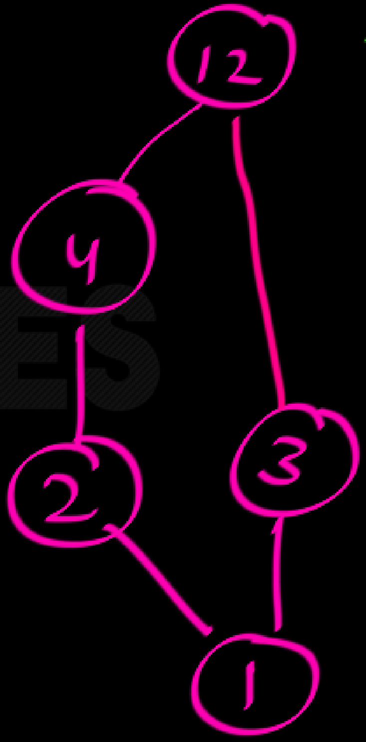
Partially Ordered Sets

- Example

Let A={1,2,3,4,12}. Consider the partial order of divisibility on A. Draw the corresponding Hasse diagram.

$1|2$, $1R2$

2,3 Incomparable

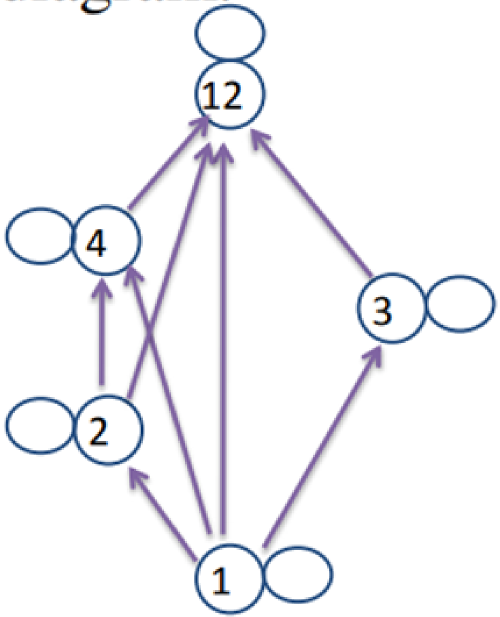


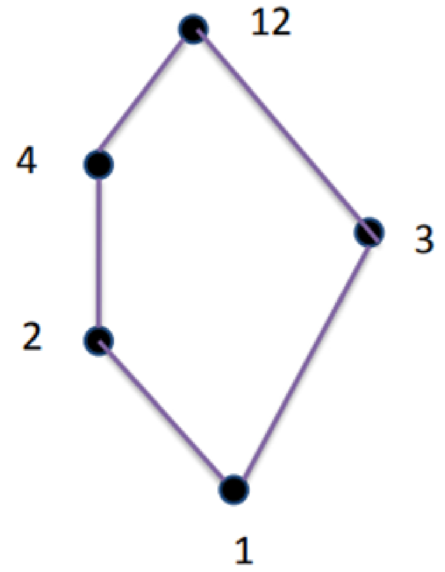
Hasse Diagram

Partially Ordered Sets

- Example

Let A={1,2,3,4,12}. Consider the partial order of divisibility on A. Draw the corresponding Hasse diagram.





Extremal Elements of Partially Ordered Sets

- Unit element

  The greatest element of a poset, if it exists, is denoted by I and is often called the unit element.

- Zero element

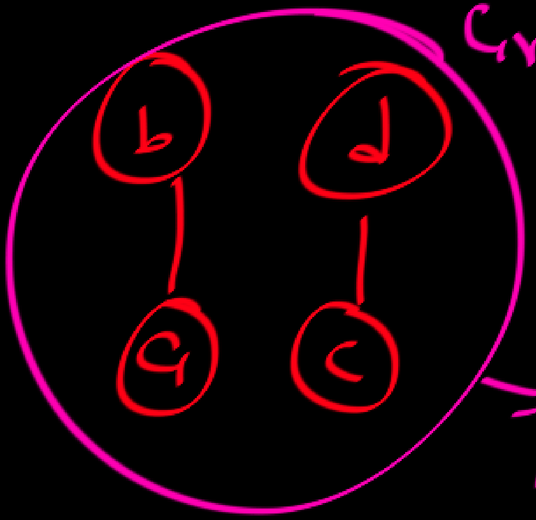
  The least element of a poset, if it exists, is denoted by 0 and is often called the zero element.

Q: does unit/zero element exist for a finite nonempty poset?

Q: finite (non-empty) poset Definitely has Greatest element ; least element?

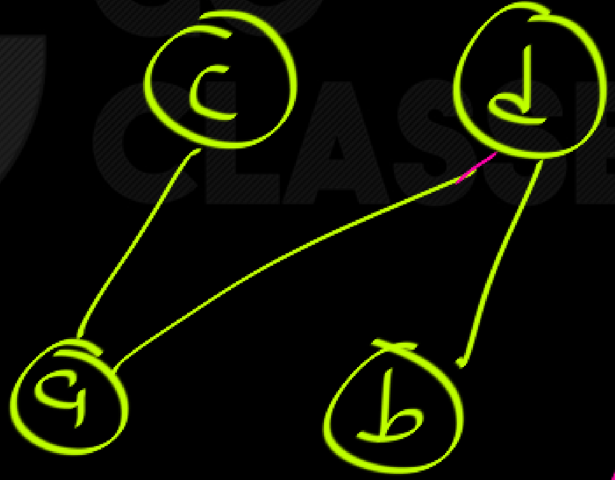
— <u>false</u>:

No Greatest



No least

;



<u>Poset</u>

Greatest = DNE
Least = DNE

Q: finite (non-empty) Lattice Definitely has Greatest element ; least element?

– True.

finite lattice is always bounded.

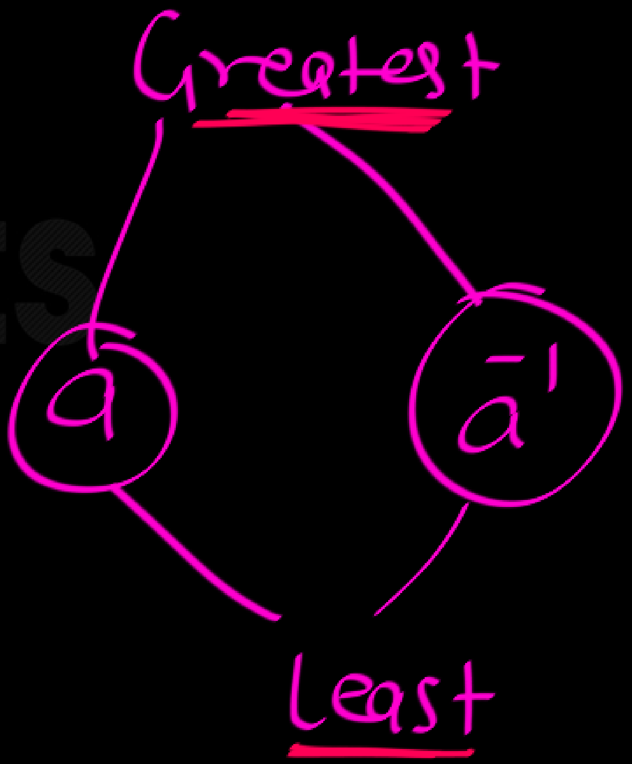
Complement: Complement of "a" is "b"

means

$a \vee b =$ Greatest

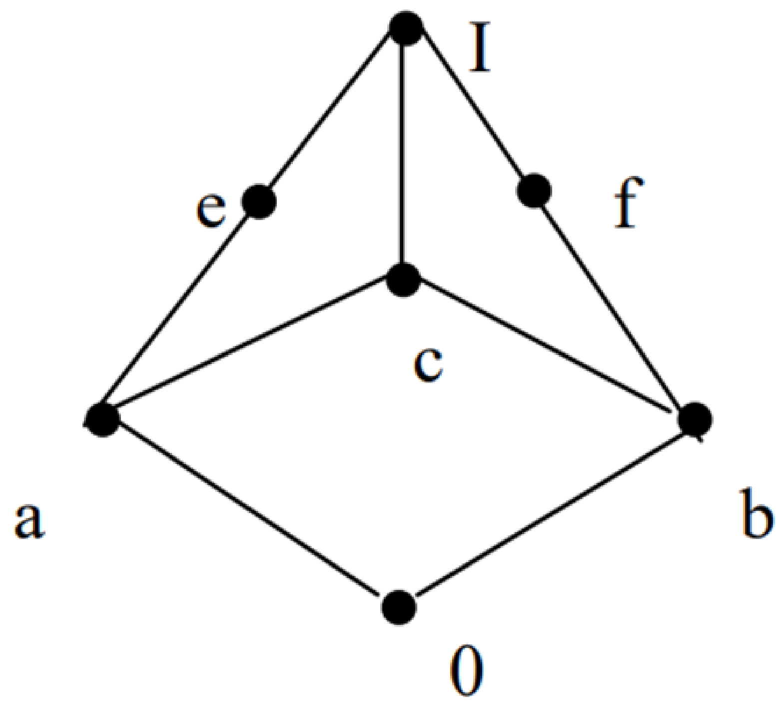
and

$a \wedge b =$ Least



Q : Do Complete Analysis of the following Hasse Diagram:

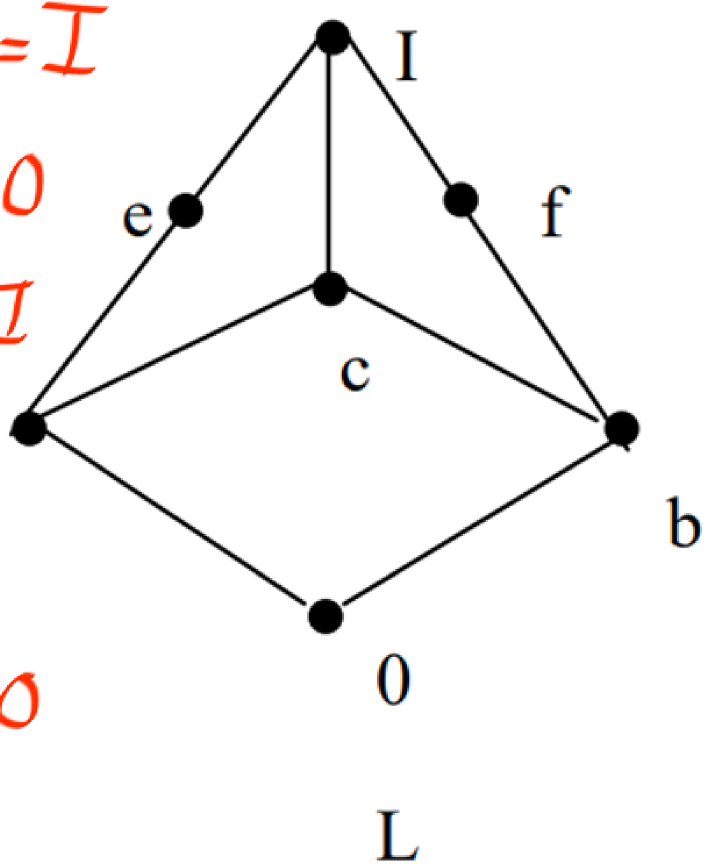
For example, consider the lattice shown in the diagram:



L

Find Extremal elements, GLB, LUB.

For example, consider the lattice shown in the diagram:



maximal = I

minimal = 0

Greatest = I

Least = 0

$e \wedge f = 0$

$e \wedge c = a$

$e \vee c = I$

$f \wedge c = b$

$f \vee c = I$

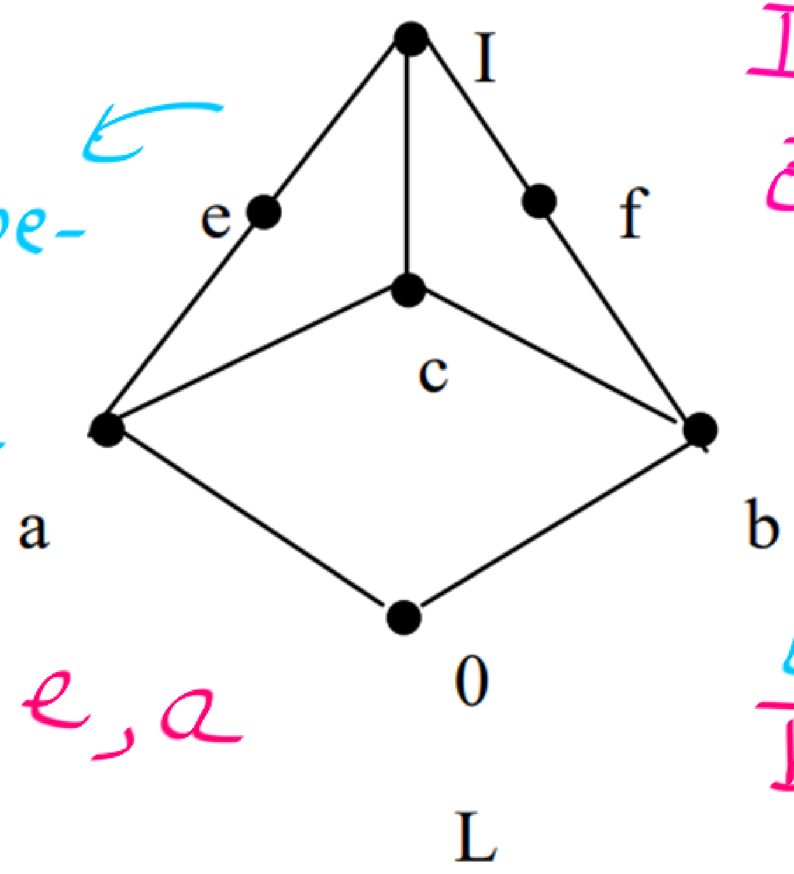
$UB\{f, c\} = \{I\}$

$LB\{f, c\} = \{b, 0\}$

$GLB\{f, c\} = b$

Find Complements of elements.

For example, consider the lattice shown in the diagram:



Not Compleme-
-ntes
Lattice

$\bar{f}' = e, a$

$\bar{I}' = 0$

$\bar{0}' = I$

$\bar{e}' = f, b$

$\bar{a}' = f$

$\boxed{\bar{c}' = DNE}$

$\bar{b}' = e$

$e \wedge c = a \neq 0$

so $\bar{e}' \neq c$

$\bar{a}' \neq b$

$a \vee b = c \neq I$

$\bar{a}' \neq c$

$a \vee c = c \neq I$

so Not Comp. lattice.

Since $\underbrace{e^{-1} = f, b}$

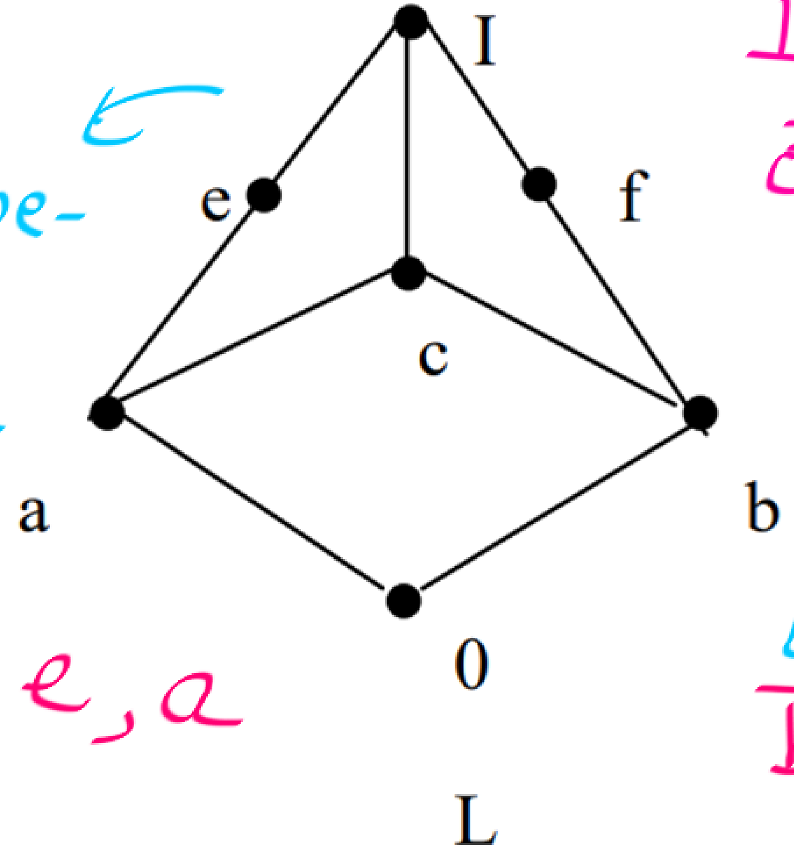
Some element has >1 Complement, So

not Distributive lattice.

Find Complements of elements.

For example, consider the lattice shown in the diagram:

Not Compleme- -nted Lattice



$\bar{f}' = e, a$

$\bar{I}' = 0$

$\bar{0}' = I$

$\bar{e}' = f, b$

$\bar{a}' = f$

$\boxed{\bar{c}' = DNE}$

$\bar{b}' = e$

$\bar{c}' = DNE$

$\bar{c}' \neq c$

$c \wedge c \neq 0$

$\bar{c}' \neq 0$

$c \vee 0 \neq I$

$\bar{c}' \neq e$

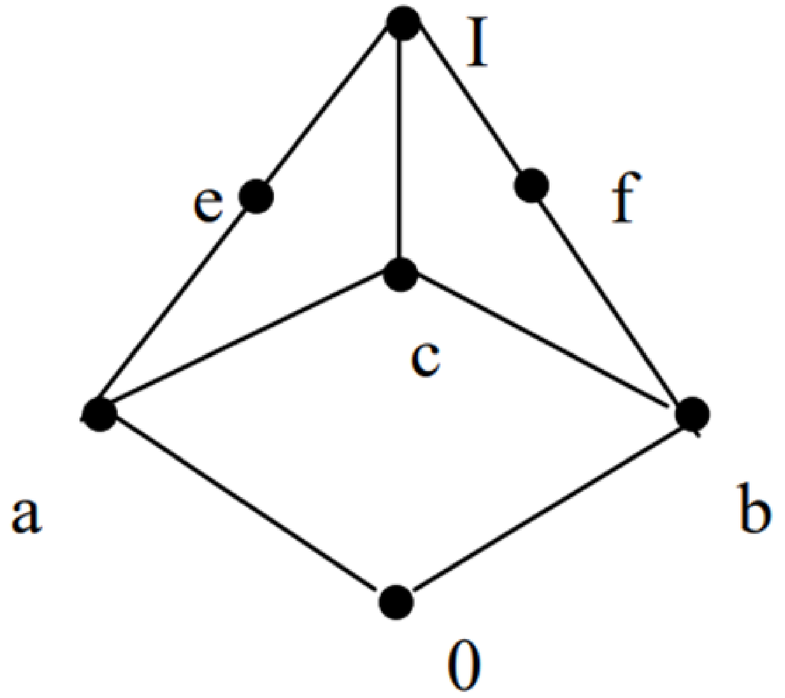
$c \wedge e \neq 0$

$\bar{c}' \neq b$

$c \vee b \neq I$

Find a "Sublattice" which is same as Kite Or Pentagon lattice.

For example, consider the lattice shown in the diagram:

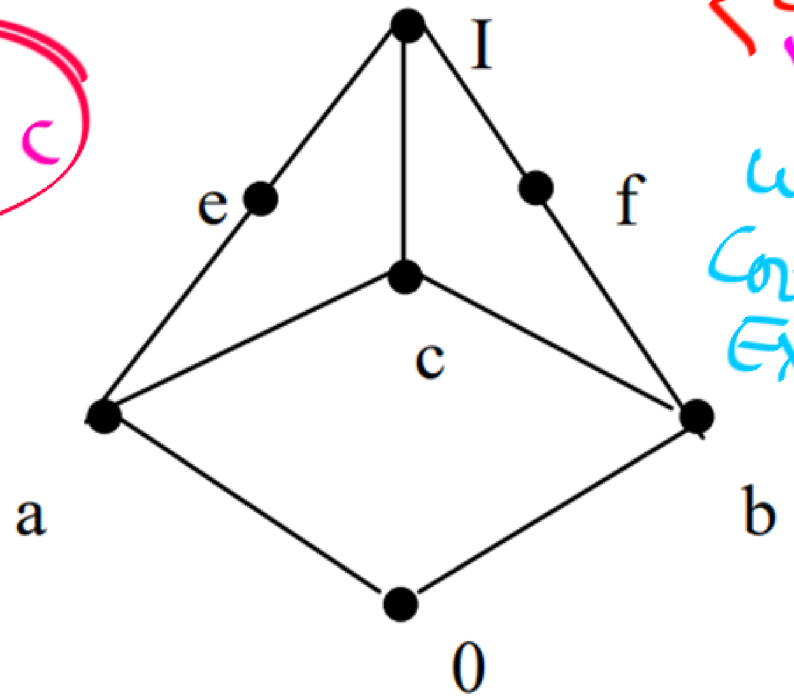


L

Find a "Sublattice" which is same as Kite Or Pentagon lattice.

For example, consider the lattice shown in the diagram:

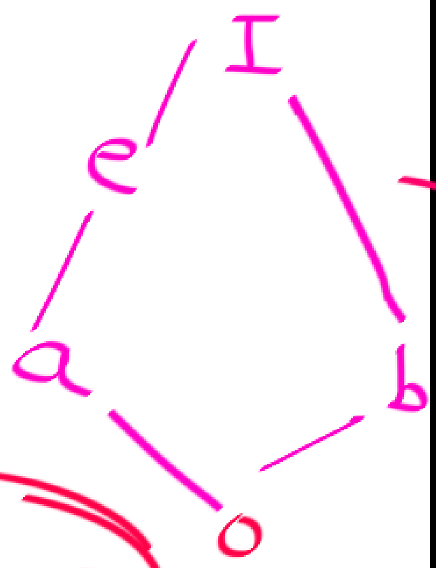
$a \vee b = c$



L

$\{a, e, I, b, 0\}$ — NOT Sub-lattice

wrong Counter Example



— Pentagon

$a \vee b = I$

Find a "Sublattice" which is same as Kite Or Pentagon lattice.

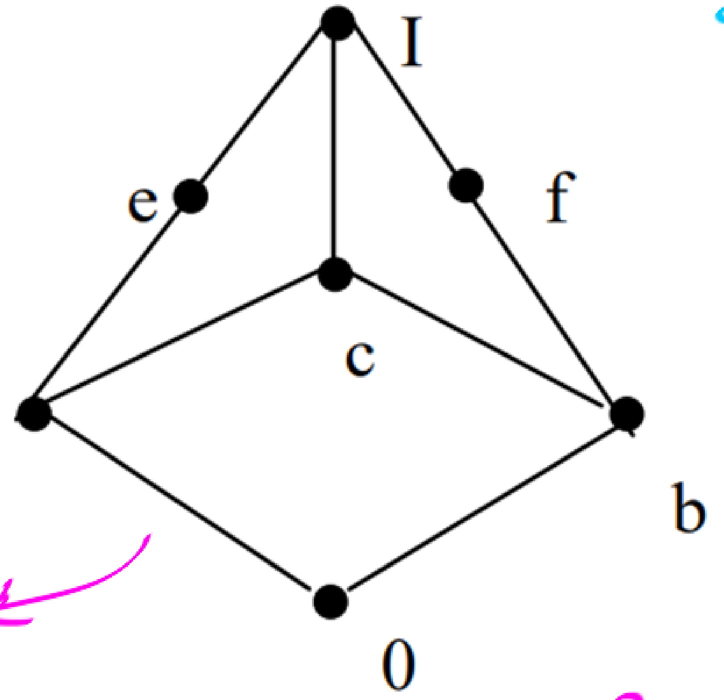
For example, consider the lattice shown in the diagram:

$a \vee f = I$

$a \wedge f = 0$

$e \vee f = I$

$e \wedge f = 0$



L

Not Distributive

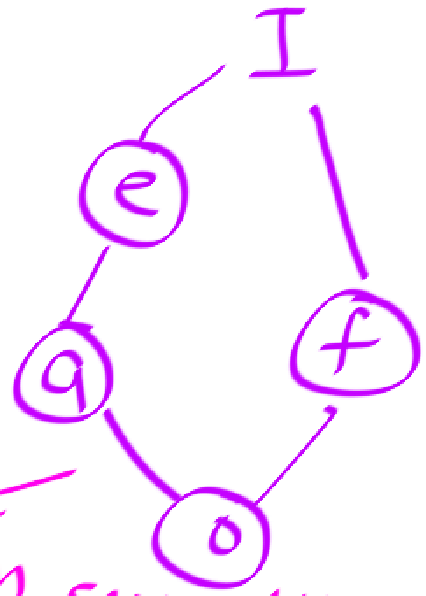
$\{a, e, I, f, 0\}$ Sub-lattice

$a \wedge f = 0$

$a \vee f = I$

$e \vee f = I$

$e \wedge f = 0$



Pentagon sublattice

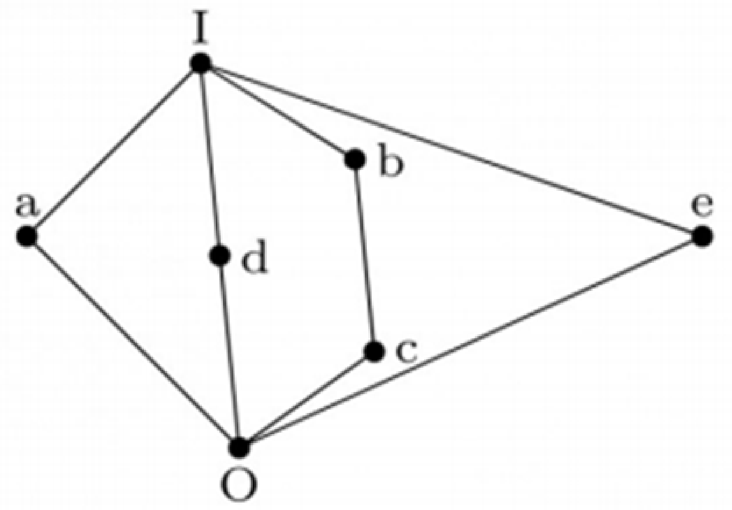
While Checking for Sub-lattice :

Just check LUB, GLB of Non-Comparable elements.

Is this Distributive lattice?

4.6.1 Lattice: GATE1988-1vii

The complement(s) of the element 'a' in the lattice shown in below figure is (are) ____



Is this Distributive lattice? — No

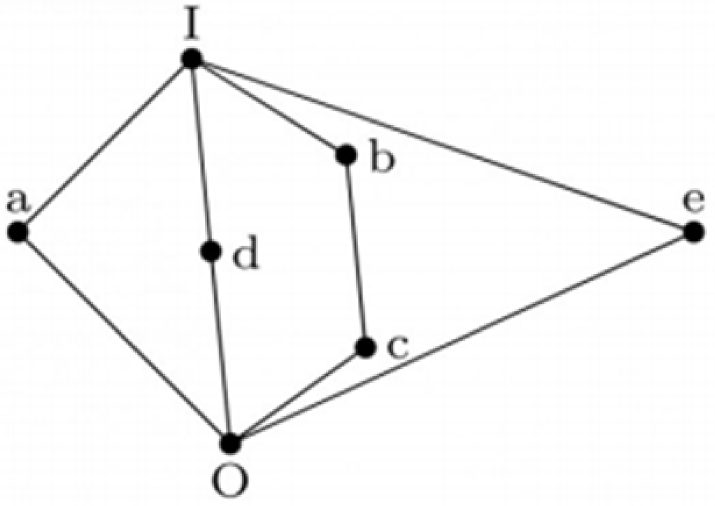
4.6.1 Lattice: GATE1988-1vii

The complement(s) of the element 'a' in the lattice shown in below figure is (are) _____

$\bar{a}' = d, b, c, e$

Some element has >1 Complement

So Not Distributive lattice.



Is this Distributive lattice? – No. Give a Sublattice which is kite or pentagon.

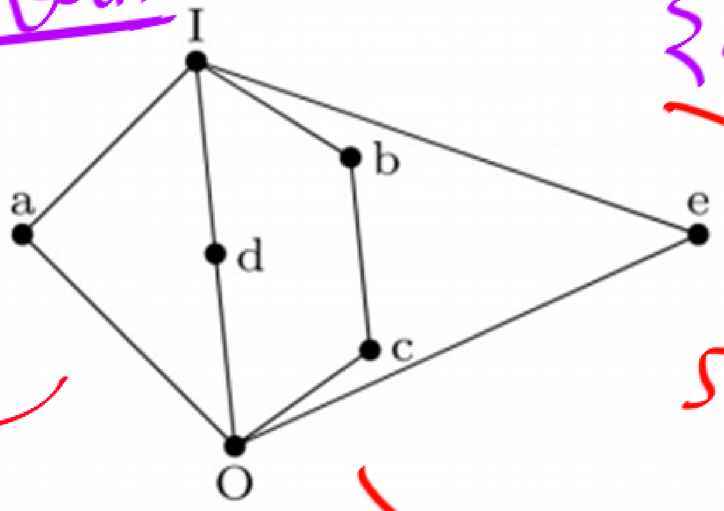
4.6.1 Lattice: GATE1988-1vii

The complement(s) of the element 'a' in the lattice shown in below figure is (are) ____

$\{a, d, e, I, 0\}$ — Sublattice

Sublattice kite (M_3)

So, Not Distributive



$\{a, b, c, I, 0\}$

Pentagon Sub-lattice

Not Dist.

Partial Order Relations

Next Sub-Topic:

Boolean Lattice/ Boolean Algebra

Website : https://www.goclasses.in/

least Greatest

Definition

A Boolean algebra is a lattice with 0 and 1 that is distributive and complemented.

Bounded

Example

Let A be a set. Then $\langle \text{pow}(A), \subseteq \rangle$ is a Boolean algebra.

$A = \{a, b\}$

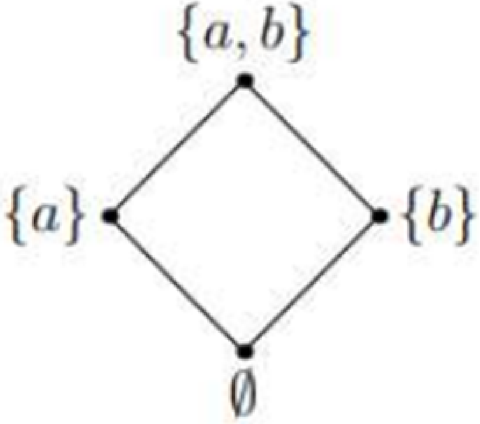
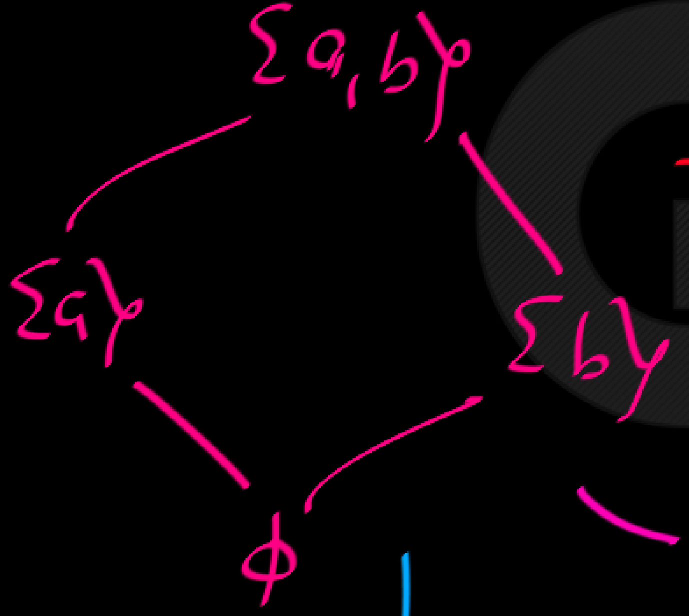


Figure: $A = \{a, b\}$.

$A = \{a, b\}$ then $(P(A), \subseteq)$ — Boolean Algebra? yes



— Bounded ✓ $1 = \{a,b\} =$ Greatest
$0 = \phi =$ least

— Distributive ✓ (< 5 elements)

Complemented ✓ $\phi^{-1} = \{a,b\}$; $\{a\}^{-1} = \{b\}$

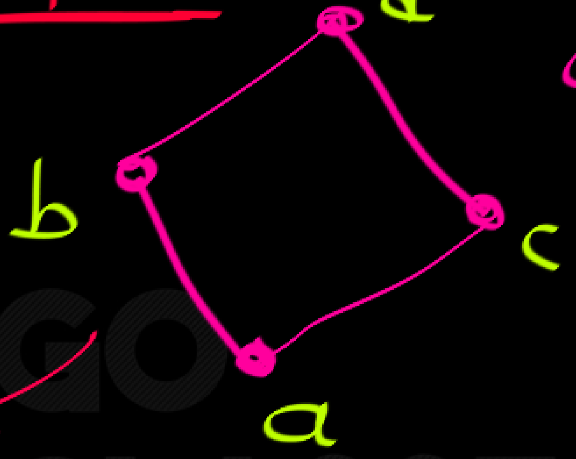
Boolean Lattices: Definition and Properties

Complemented distributive lattices are an important type of lattice. Rather than call them by this mouthful, they are given a special name: they are called *Boolean lattices*.

DEFINITION 7.2-7: *Boolean Lattices*

A lattice $\langle \mathcal{A}, \leq, ^{-}, 0, 1 \rangle$ *is a* ***Boolean lattice*** *iff it is a complemented distributive lattice.*

Note: Structure: Square:



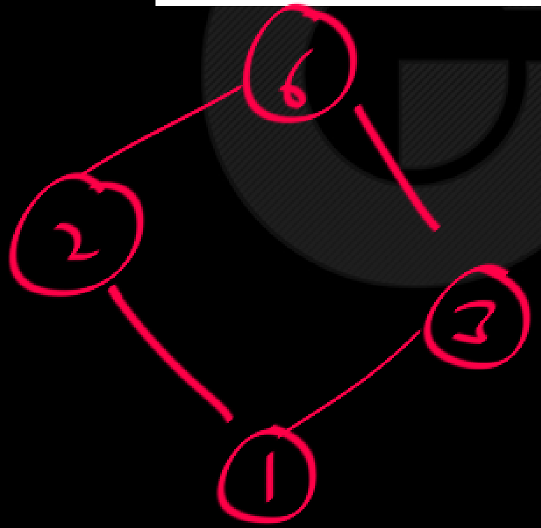
— bounded ✓
— Distri ✓
— Complem ✓

Square Structure is Boolean Algebra

$b' = c$
$c' = b$
$a' = d$
$d' = a$

Example

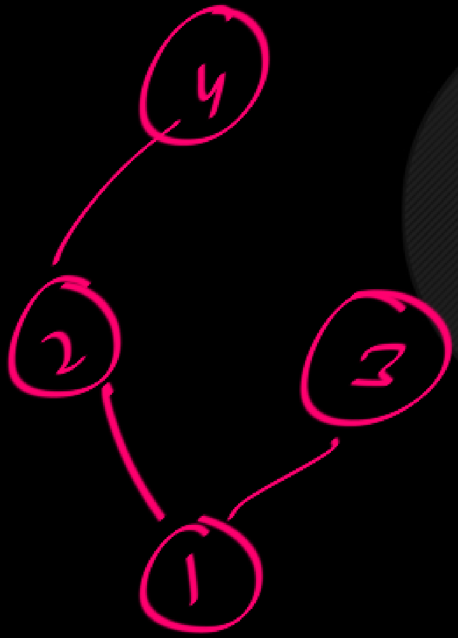
$\langle \{1, 2, 3, 6\}, | \rangle$ is a Boolean algebra.



Square Structure

Bool. Alg.

$(\{1, 2, 3, 4\}, |)$ — Not Bool. Alg.

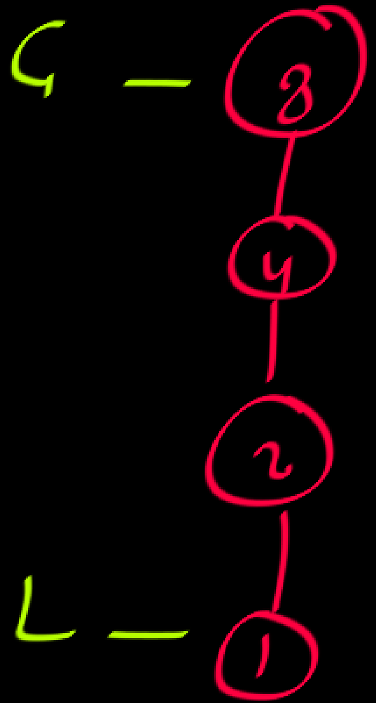


$3 \vee 4 = DNE$

$\underbrace{LUB(3,4)}$ $= DNE$

Not even a lattice

$(\{1, 2, 4, 8\}, 1)$ — Bool. Alg ✗



Bounded

Distributive (Chain is Dist.)

Not Comp.

$2^{-1} = \text{DNE}$; $1^{-1} = 8$; $8^{-1} = 1$

$4^{-1} = \text{DNE}$

Q: Single element Lattice :- Bool. Alg.

a

Dist ✓

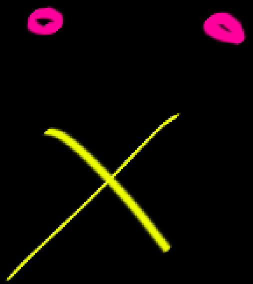
Comp ✓ $\bar{a}' = a$

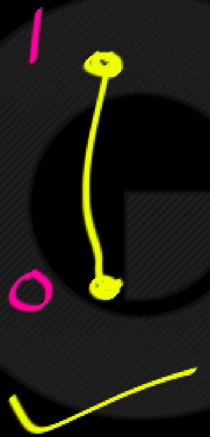
bounded ✓

$1 = a$

$0 = a$

Q: 2 Element Lattice Structures : = 1

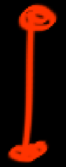




bounded ✓

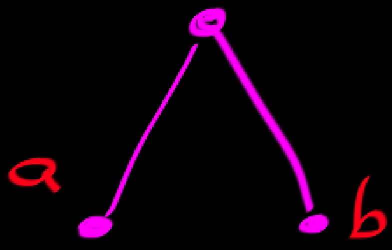
Compl ✓ $\quad 0^{-1} = 1$

Dist ✓ $\quad 1^{-1} = 0$

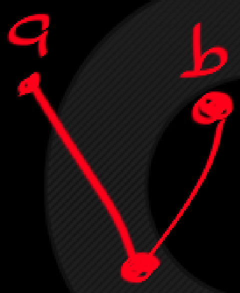


— bool. Alg.

Q: 3 element Lattice Structures : 1



$a \wedge b$ = DNE



$a \vee b$ = DNE

Bounded ✓

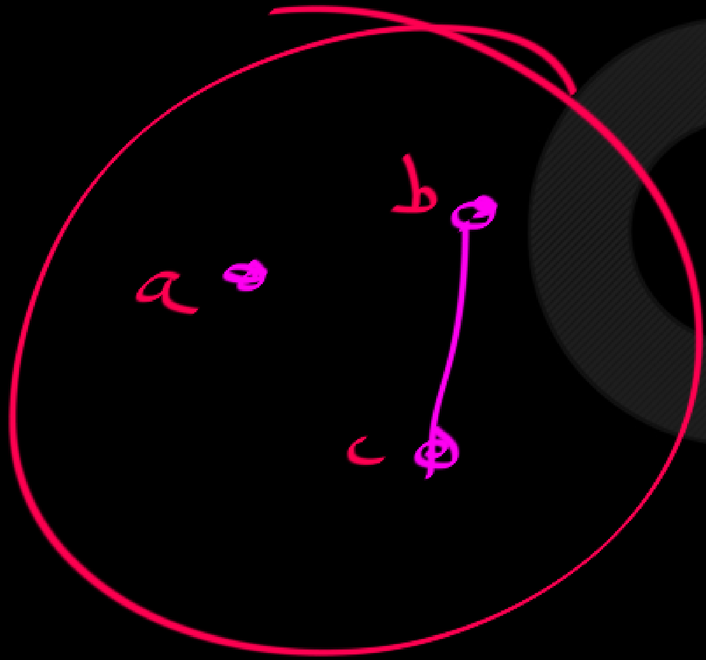
Dist ✓

Not Comp.

b^{-1} = DNE

NOT B.A.

Q: 3 element Lattice Structures:

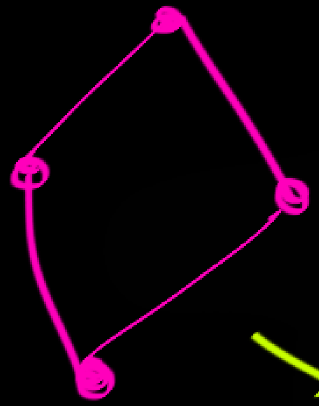


Not a Lattice

$a \vee b = DNE$

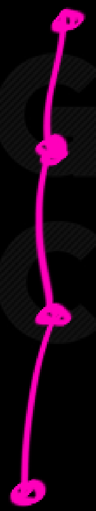
$a \wedge b = DNE$

Q: 4 element Lattice Structures : 2



Square

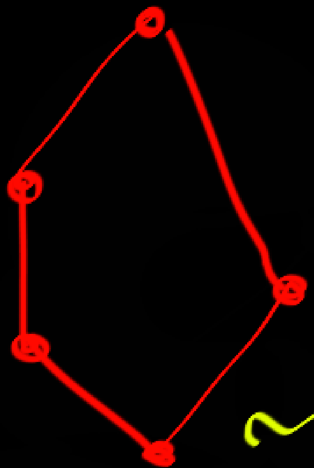
Bool Alg.

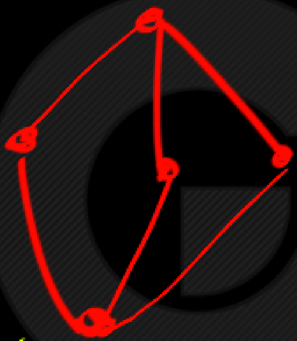


Not BA

Q: 5 element Lattice Structures: 5

None of them is BA.





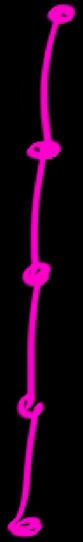
Not Dist

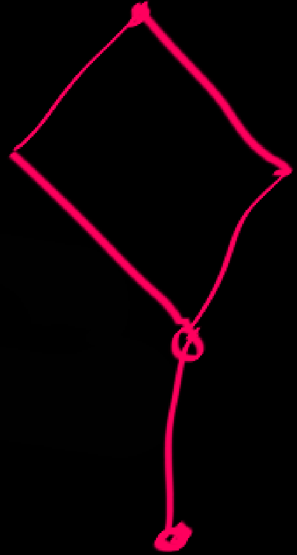
N_5

Pentagon

M_3

Kite



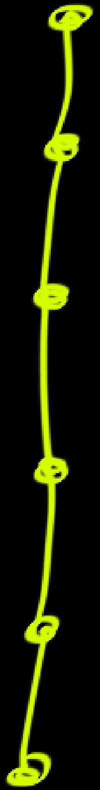


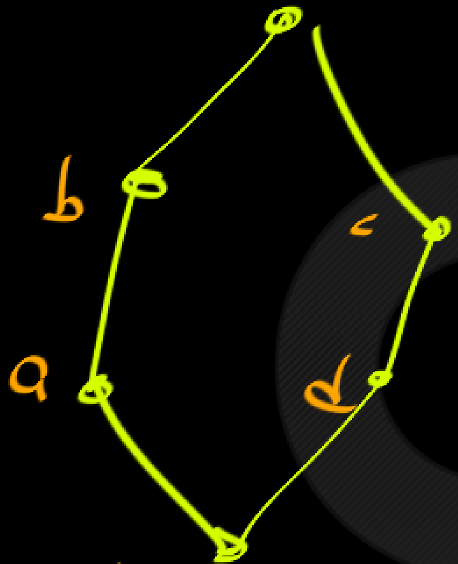
Chain

Not Comp

Q: 6 element Lattice Structures:

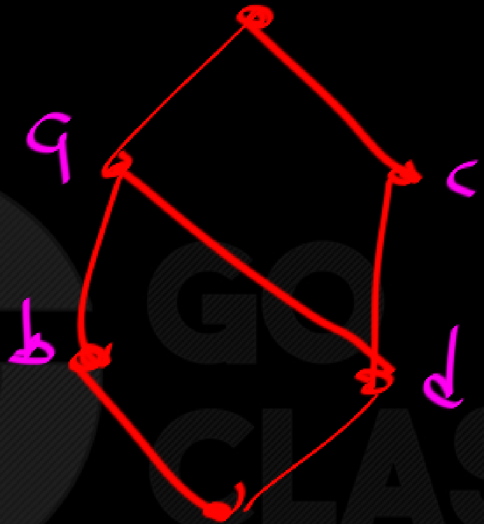
HW: Any lattice structure that you will make, will not be a BA.





Not Dist

$\boxed{\overline{a}' = d, c}$



Dist

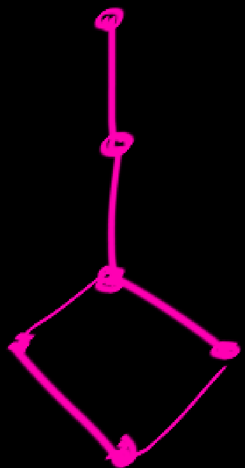
Not Comp

$\overline{c}' = b$

$\overline{a}' = \text{DNE}$

$\overline{b}' = c$

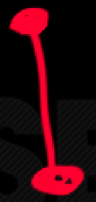
$\overline{d}' = \text{DNE}$



Not Comp.

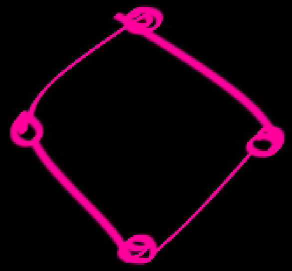
Boolean Algebras:

(1) 1 element BA : a • ✓

(2) 2 " " :  ✓

(3) 3 " " : None

Boolean Algebras:

4 element BA :  Square

5 " " : None

6 " " : None

7 " " : None

Note:

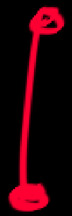
Every BA has the same structure as $(P(A), \subseteq)$ structure; for some A.

BA :

$(P(A), \subseteq)$

 $\equiv (P(A), \subseteq)$

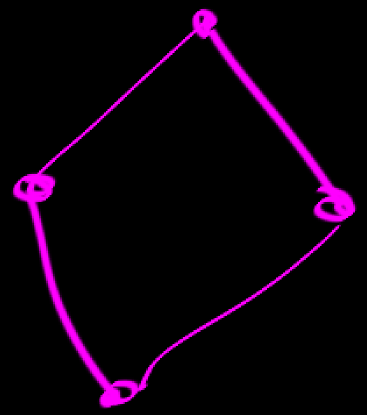
$A = \phi$

 $\equiv (P(A), \subseteq)$

$A = \{a\}$

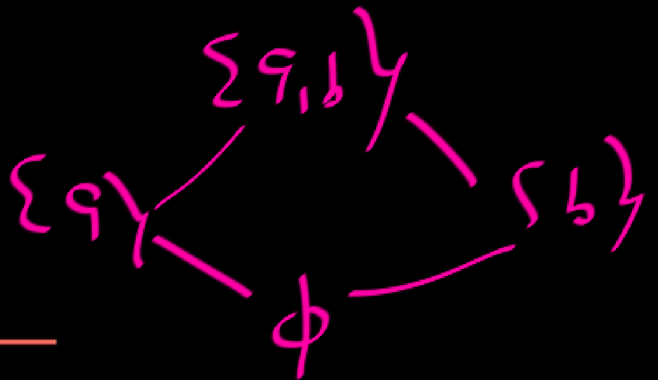
$|A| = 1$

BA :

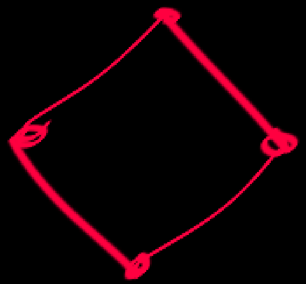


$\equiv \left(\underbrace{P(A)}_{\text{Base set}}, \subseteq\right)$ where $|A| = 2$

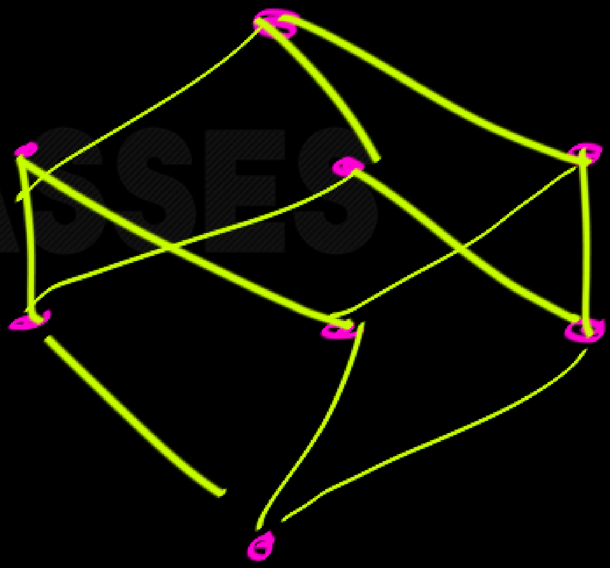
$A = \{a, b\}$; $(P(A), \subseteq)$



$|A| = 2 \; ; \; (P(A), \subseteq) \equiv$



$|A| = 3 \; ; \; (P(A), \subseteq) \equiv$



$(P\{a, b, c\}, \subseteq) \equiv$

$|A| = 4$; $(P(A), \subseteq)$ — BA

2^4 elements

$A = \phi$; $(P(A) = \{ \phi \}, \subseteq)$

Hasse Dig :

$\bullet\ \phi$

Note:

$$BA \equiv (P(S), \subseteq) \text{ for some } S.$$

Same structure

$\boxed{(P(N), \subseteq)}$ — BA ✓

Infinite BA.

$(P(S), \subseteq)$ — BA ✓

$N = \{1, 2, 3, ---\}$

$P(N) =$ infinite set

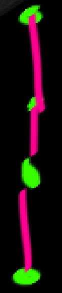
$(P(N), \subseteq)$ — $\underline{\underline{BA}}$.

Note:

Every BA has 2^n elements, $n \geq 0$

Q: Lattice with 2^n elements is BA?

No.

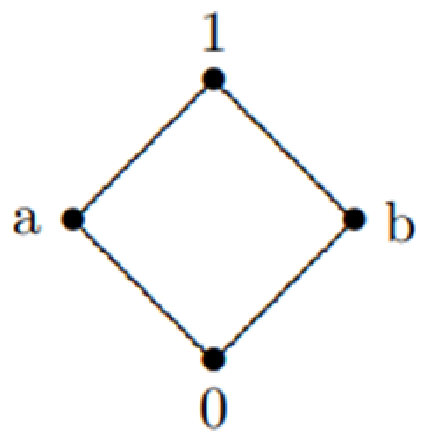


4 elements

Not BA.


1
0


1
0


1
a
b
0

Theorem (M. H. Stone, 1936)

Every finite Boolean algebra is isomorphic to the Boolean algebra $\langle \text{pow}(S), \subseteq \rangle$ of a finite set S.

Corollary

Every finite Boolean algebra has 2^n elements for some $n \in \mathbb{N}$.

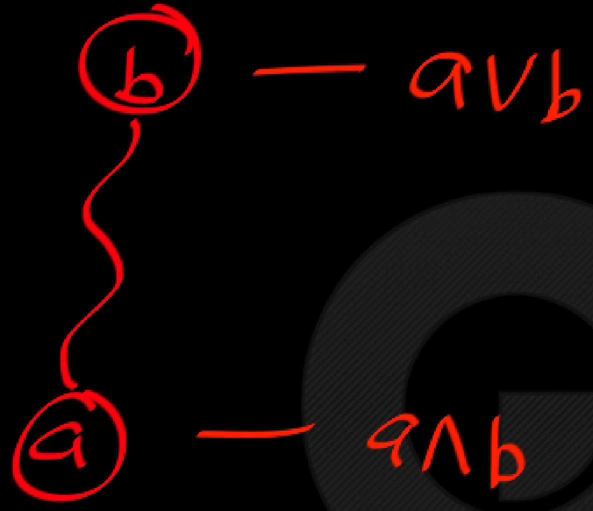
Partial Order Relations

Next Topic:

Practice Questions

Website : https://www.goclasses.in/

Q: Assume in a poset, aRb then a v b = b??



aRb then

$\left.\begin{array}{l} a \vee b = b \\ a \wedge b = a \end{array}\right\}$

Q: Assume in a poset, a v b = b then aRb??

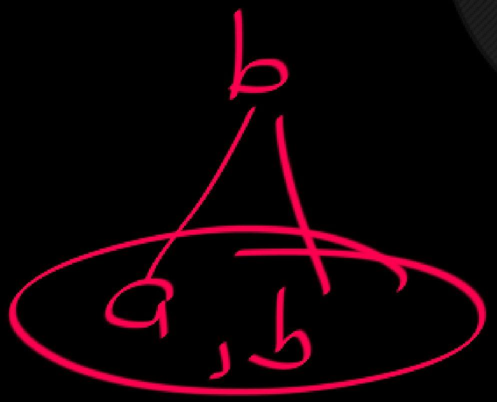
$a \vee b = b$ then aRb?

Q: Assume in a poset, a v b = b then aRb??

$a \vee b = b$ then aRb? — Yes ✓

Apply the Definition of <u>LUB</u>.

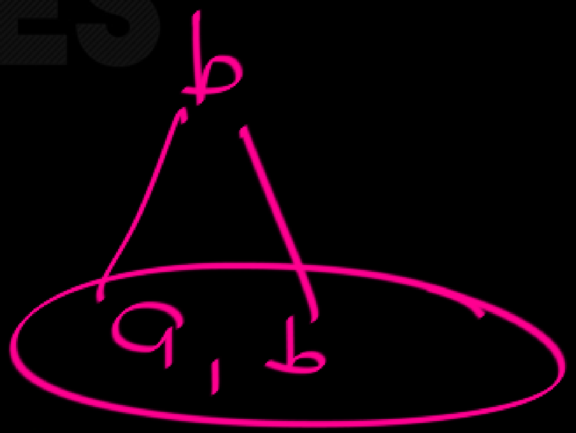
LUB is a UB.



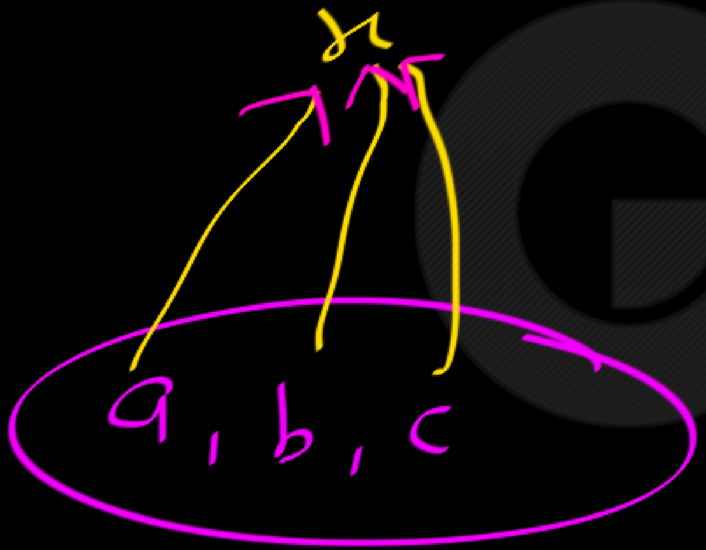
$LUB\{a, b\} = b$

means

aRb ✓



Upper bound:



LUB is also UB.

$LUB\{a,b,c\} = x$

means

$$\left.\begin{array}{l} aRx \\ bRx \\ cRx \end{array}\right\}$$

Lower bound:

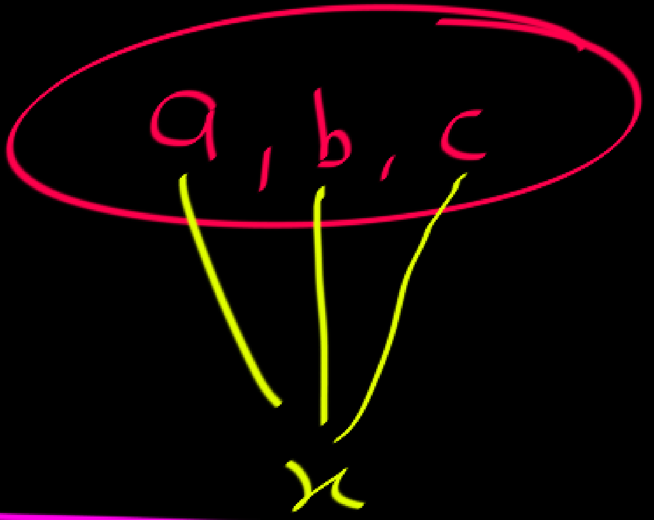
$LB\{a, b, c\} = x$

means $\cancel{aRx}$

xRa

xRb

xRc



GLB is also LB.

If $GLB\{a, b\} = b$

means: bRa

Note: In any poset :

$$aRb \text{ iff } a \vee b = b$$

$$aRb \text{ iff } a \wedge b = a$$

aRb then $a \wedge b = a$

$a \wedge b = a$ means $GLB\{a, b\} = a$

means

aRb

b — $a \vee b$

a — $a \wedge b$

Consistency Property in Lattices:

$a R b$ iff $a \vee b = b$

$a R b$ iff $a \wedge b = a$

Every Lattice Satisfies these:

Theorem 5.3.2. *Let* $[A, +, \cdot]$ *be a lattice and* $a, b, c \in A$.

i. $a + a = a$ *idempotency*

ii. $a + b = b + a$ *commutativity*

iii. $(a + b) + c = a + (b + c)$ *associativity*

iv. $a + (a \cdot b) = a$ *absorption*

v. $a + b = b \longleftrightarrow a \cdot b = a \longleftrightarrow a \leq b$ *consistency*

Match the Column:

Identity Property
Domination Property — Bounded Lattices

Demorgan Property — Boolean Lattice

Q: Why Boolean Lattice is Called Boolean Algebra?

In Digital logic : $\boxed{\text{2-element}}$ $\rightarrow$ 0, 1
Bool Alg.

Boolean Algebra: Any mathematical structure that satisfies All the properties

Commutative; Dist; Asso; Identity; Idempotent;
Complement; Demorgan

Propositional logic $\longrightarrow$ Bool Algebra

Set theory

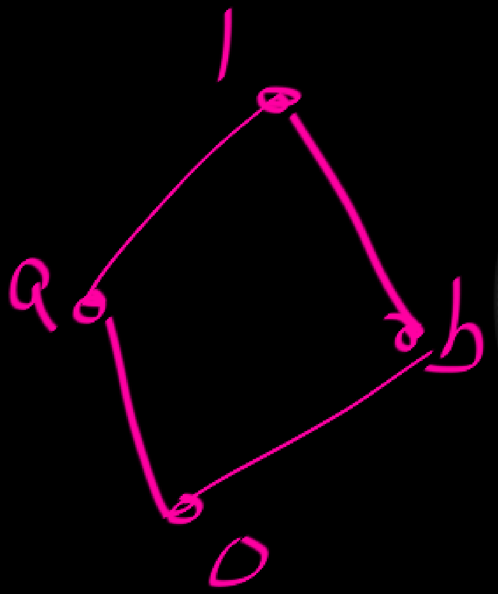
In Dig logic, 2-element BA

Boolean lattice

Boolean Lattice Satisfies:

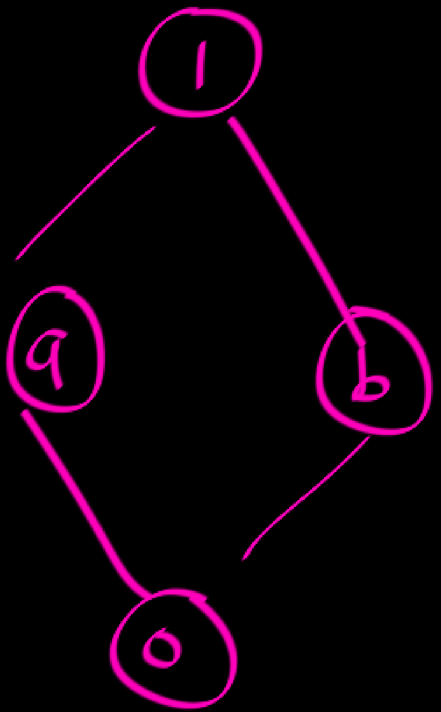
Commutative
Distributive = Distributive lattice
Identity = bounded lattice
Complement = Complemented lattice
Idempotent
Demorgan

Consider Boolean lattice:



Demorgan law:

$$\forall_{x,y} \begin{cases} \overline{(x \wedge y)} = \bar{x} \vee \bar{y} \\ \overline{(x \vee y)} = \bar{x} \wedge \bar{y} \end{cases}$$



$$\overline{(a \wedge b)} \overset{?}{=} \bar{a} \vee \bar{b}$$

$$\overline{0} \qquad b \vee a$$

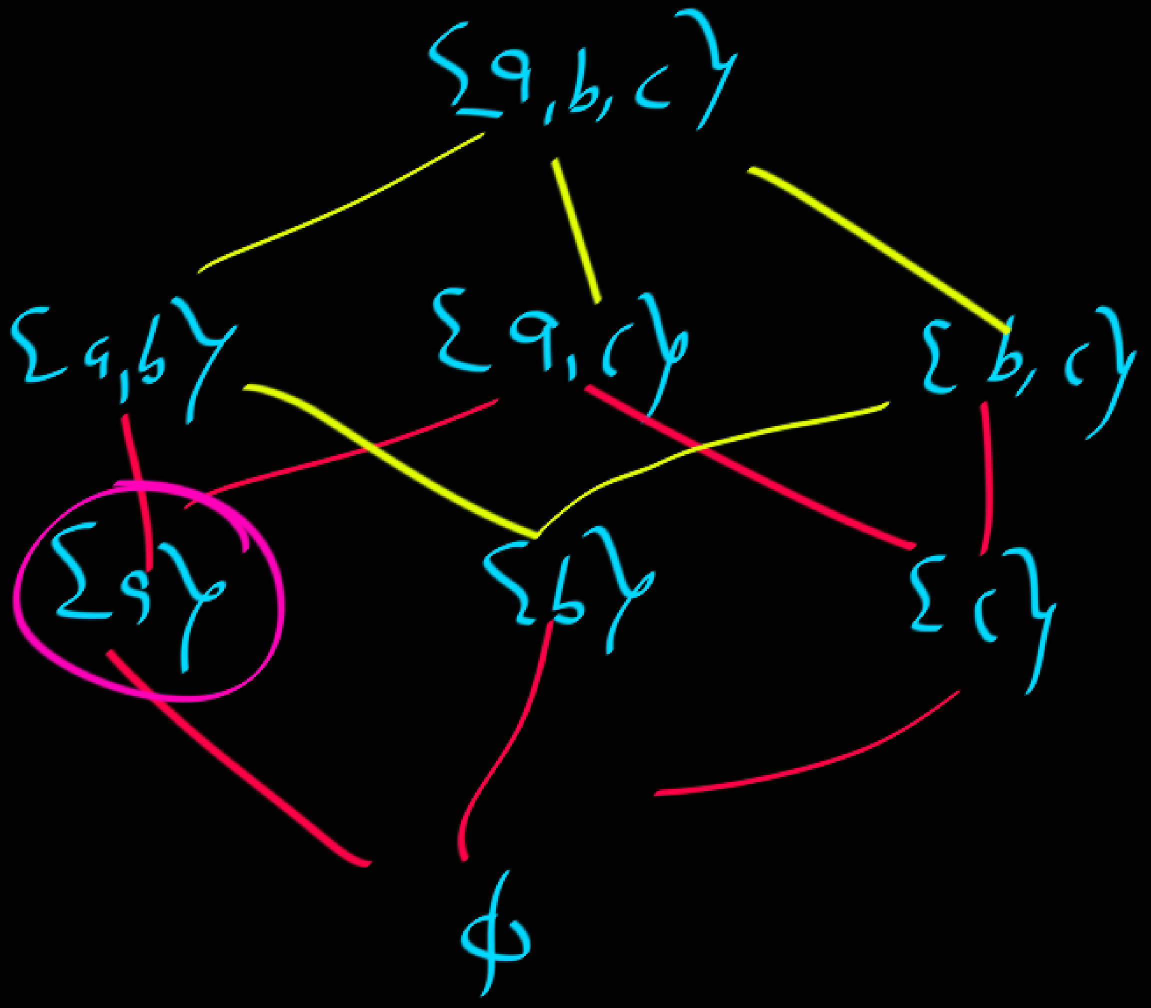
$$1 = 1$$

Boolean lattice

All Lattices Satisfy :

1. Commutative
2. Absorption
3. Idempotent
4. Associative
5. Consistency

$(\mathcal{P}\{a,b,c\}, \subseteq)$



$\overline{\{a\}} = \{b,c\}$

$\overline{\{b,c\}} = \{a\}$

Partial Order Relations

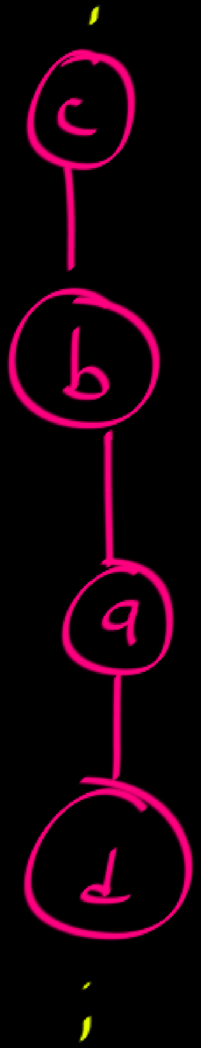
Next Topic:

Complete Analysis of Total Order Relations

Website : https://www.goclasses.in/

Total order relation:

1. Lattice ✓
2. Bounded Lattice — may or may not
3. Complemented iff ≤ 2 elements
4. Dist ✓
5. BA iff ≤ 2 elements.



Total order $\rightarrow$ every two elements Comparable.

$$\left.\begin{array}{l} a \vee b = b \\ a \vee c = c \\ c \wedge d = d \end{array}\right\}$$

Finite Total order:

1. Lattice ✓
2. Bounded Lattice ✓
3. Complemented iff ≤ 2 elements.
4. Distributive ✓
5. Boolean Algebra iff ≤ 2 elements.

Infinite Total order:

$([0,1], \leq)$ $(Z, \leq)$

1. Lattice ✓
2. Bounded Lattice — may or may not
3. Complemented — Never
4. Distributive ✓
5. Boolean Algebra — Never

Lattice: Every two non-comparable must have GLB, LUB.

Comparable elements Always have GLB, LUB.

aRb iff $a \vee b = b$, $a \wedge b = a$

$([0,1], \leq)$ — bounded Toset

$(Z, \leq)$ — Not bounded Toset.

$(N, \leq)$ — " " "

$(N, \geq)$ — " " "

In a Total order : Every two elements are comparable.

$$\boxed{a \vee b} = a \text{ or } b$$

for any a, b

$aRb \longrightarrow a \vee b = b$

or

$bRa \longrightarrow a \vee b = a$

In a Total order : Every two elements are comparable.

$$\boxed{a \wedge b} = a \text{ or } b$$

for any a, b

$a R b \longrightarrow a \wedge b = a$

or

$b R a \longrightarrow a \wedge b = b$

Problems 17-20: Totally Ordered Sets and Lattices

Let $\langle \mathcal{A}, \leq \rangle$ be a totally ordered set.

*17. Prove the following.

a. $\mathcal{A}$ is a lattice.

b. What is the meet and join of any two elements a and b in $\mathcal{A}$? Explain.

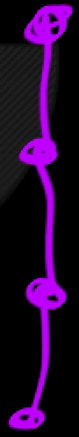
*18. Prove that if $\mathcal{A}$ has more than two elements, then it is not a complemented lattice, even if it has a minimum and a maximum.

totally ordered sets are distributive lattices.

Q: Not Complemented $\longrightarrow$ Not Distributive ?

false.

Counter Eg:



Partial Order Relations

Next Topic:

Complete Analysis of Subset Relations

Website : https://www.goclasses.in/

Let S be any non-empty set and let A be some collection of its subsets. i.e. A contains some of the subsets of S.

$A \subseteq P(S)$

Discuss the properties of the subset relation $\subseteq$ on elements of A.

Set S ; $A \subseteq P(S)$

$(A, \subseteq)$:

not $P(S)$

① Ref ✓
② Antisym ✓
③ Trans ✓

} Subset Relation is Always these

④ Symmetric — may or may not

$S = \{a, b, c\}$; $A = \{ \{a, b\} \} \subseteq P(S)$

$(A, \subseteq)$

→ Symmetric ✓

→ Ref ✓

→ Trans ✓

→ Antisym ✓

$S = \{a, b, c\}$; $A = \{\phi, \{a\}, \{a, b\}\}$

$(A, \subseteq)$

Symmetric ✗

Refl ✓

Trans ✓

Antisym ✓

$(\underbrace{\{x_1, x_2, \ldots x_n\}}_{\text{Base set}}, \subseteq)$

$x_1, x_2, \ldots x_n$: Set Set Set

$\subseteq$ → Subset Relation

Refl ✓ ; Antisym ✓ ; Trans ✓

Symmetric : Depend on BaseSet.

$(\{\{a\}, \{b\}\}, \subseteq)$ — Poset

Base set

Hasse Dig : $\{a\}\bullet \quad \bullet\{b\}$

Sym ✓
Ref ✓
Trans ✓
Antisym ✓

Analysis of $(P(S), \subseteq)$: $\rightarrow$ Boolean Algebra

① $S = \phi$; $P(S) = \{\phi\}$

$(P(S), \subseteq)$ — Poset ✓

Symm ✓

$(P(S), \subseteq)$

$|S| \geq 1$ then

$(P(S), \subseteq)$ — not symmetric

Boolean lattice

Ref ✓; Antisym ✓; Trans

$(P(S), \subseteq)$ — boolean lattice

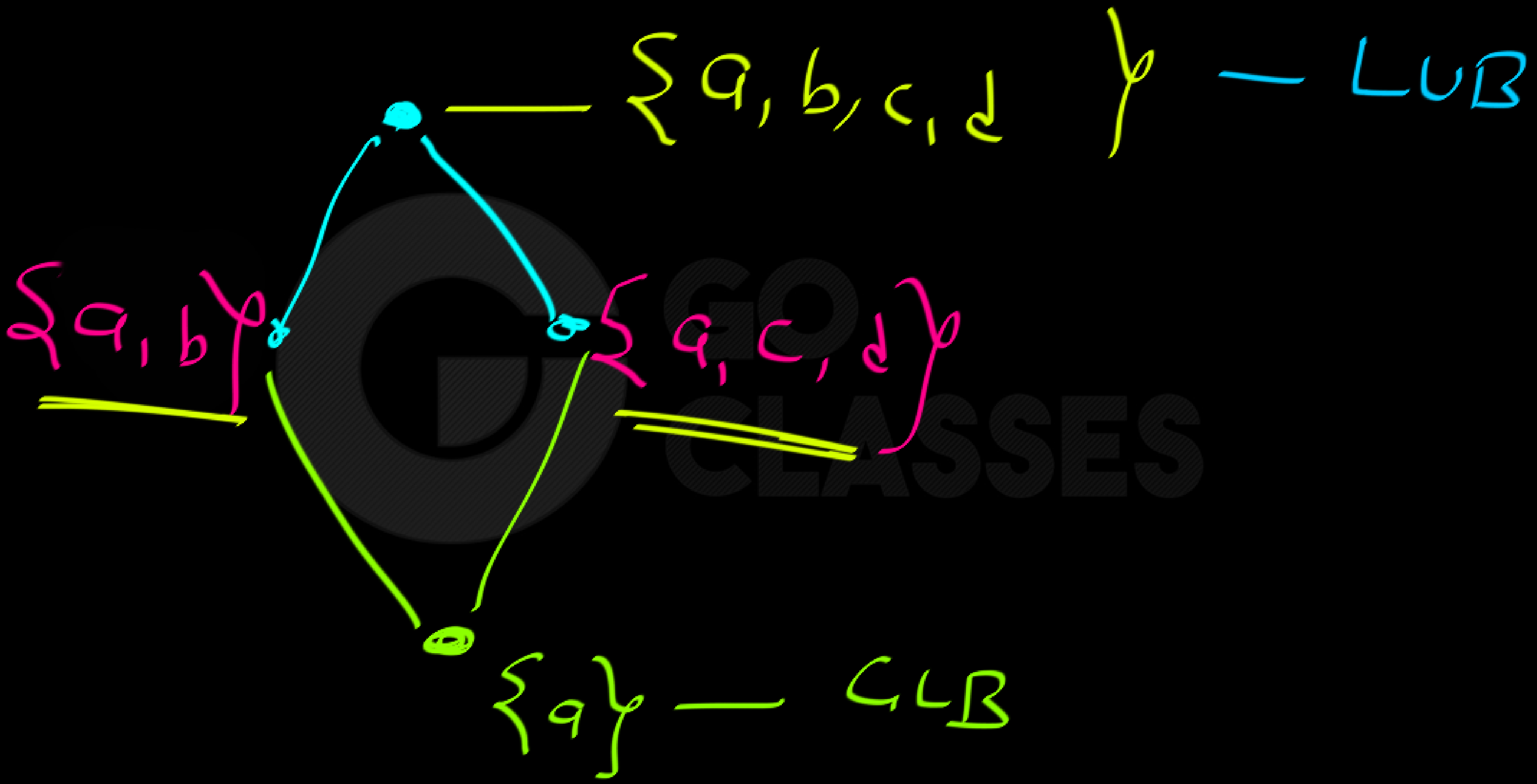
1. Lattice ✓
2. Distributive ✓
3. Complemented ✓
4. Bounded ✓

$(P(S), \subseteq)$:- Lattice

$$\left.\begin{array}{l} LUB = \text{Union} \\ GLB = \text{Intersection} \end{array}\right\}$$

Greatest = S ; $\forall x$, $x \subseteq S$

Least = ϕ


{a, b, c, d} — LUB
{a, b}
{a, c, d}
{a} — GLB

$(P(N), \subseteq)$ Base set $= \{X \mid X \subseteq N\}$

Greatest: N $\boxed{\forall x,\ x \subseteq N}$

Least: ϕ $\boxed{\forall x,\ \phi \subseteq x}$

$\{1,2,3\} \cup \{4,5,6\} = \{1,2,3,4,5,6\}$

Let S be any non-empty set and let A be the collection of all its subsets. i.e. A is the Powerset of S. So, $A = P(S)$.

Discuss the properties of the subset relation $\subseteq$ on elements of A.

$(P(S), \subseteq)$ — boolean Lattice

LUB = Union

GLB = Intersection

Greatest = S

Least = ϕ

$(P(S), \supseteq)$ — Boolean lattice

Greatest = ϕ

Least : S

Superset

LUB = Intersection

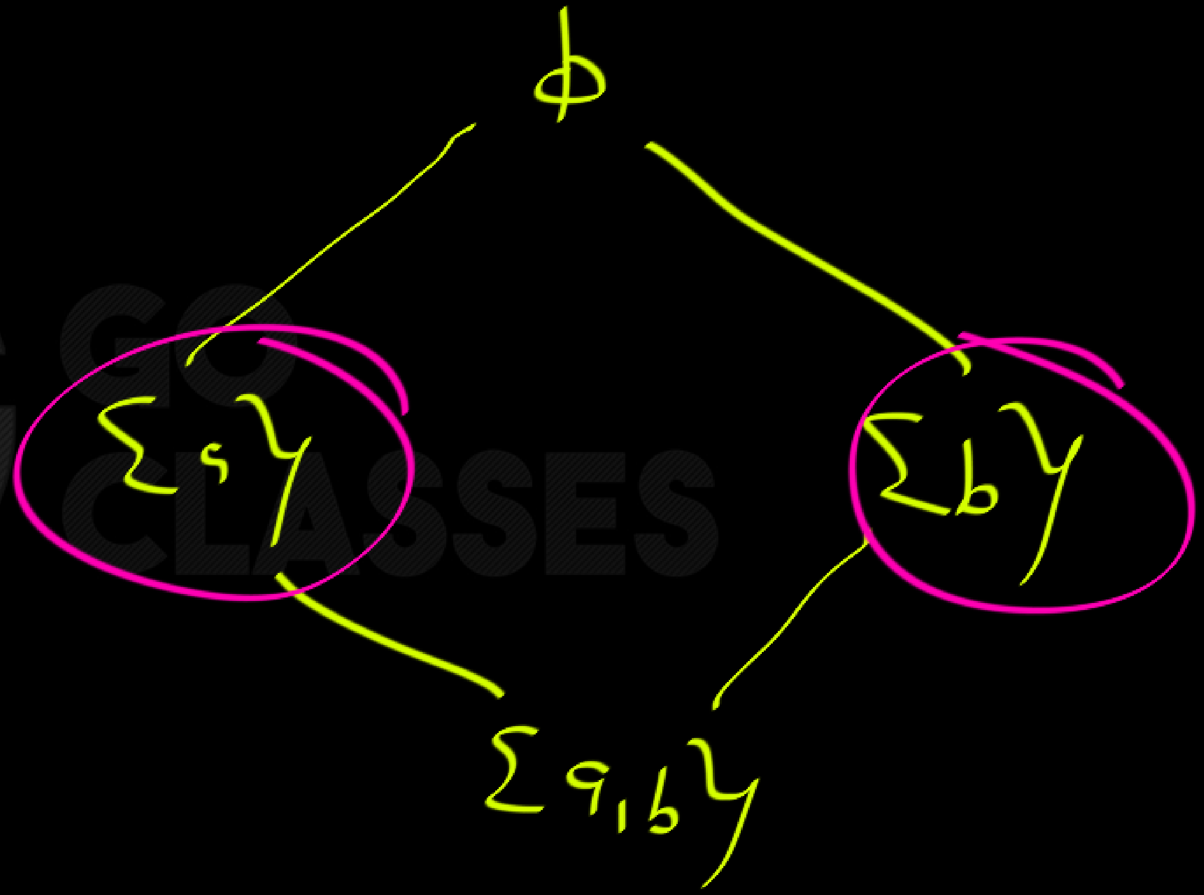
GLB = Union

$(P\{a,b\}, \supseteq)$

$\{a\} \vee \{b\} = \phi$

$\{a\} \vee \{b\} =$

$\{a\} \cap \{b\} = \phi$



Let S be any set and let $\mathcal{A}$ be some collection of its subsets. Discuss the properties of the subset relation $\subseteq$ on elements of $\mathcal{A}$.

Solution

The subset relation is reflexive and transitive, but it is generally not symmetric.
For if X is any element of $\mathcal{A}$, then $X \subseteq X$; and whenever $X \subseteq Y$ and $Y \subseteq Z$, $X \subseteq Z$.
Furthermore, $X \subseteq Y$ does not usually force $Y \subseteq X$. In particular, it fails if $\mathcal{A}$ is the full power set $\mathcal{P}(S)$. (Think: for what sort of collection $\mathcal{A}$ might symmetry hold?)
What is always true, however, is that if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
This is known as the *antisymmetric property* of $\subseteq$.