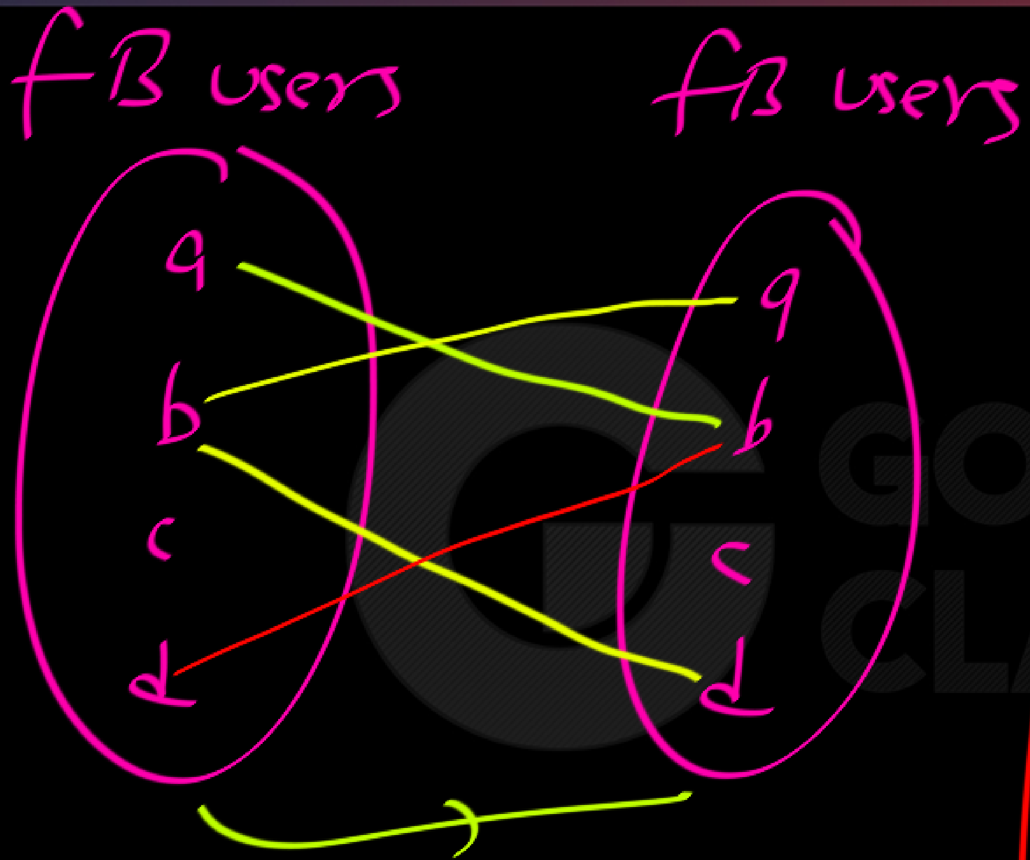
Relations

Next Topic

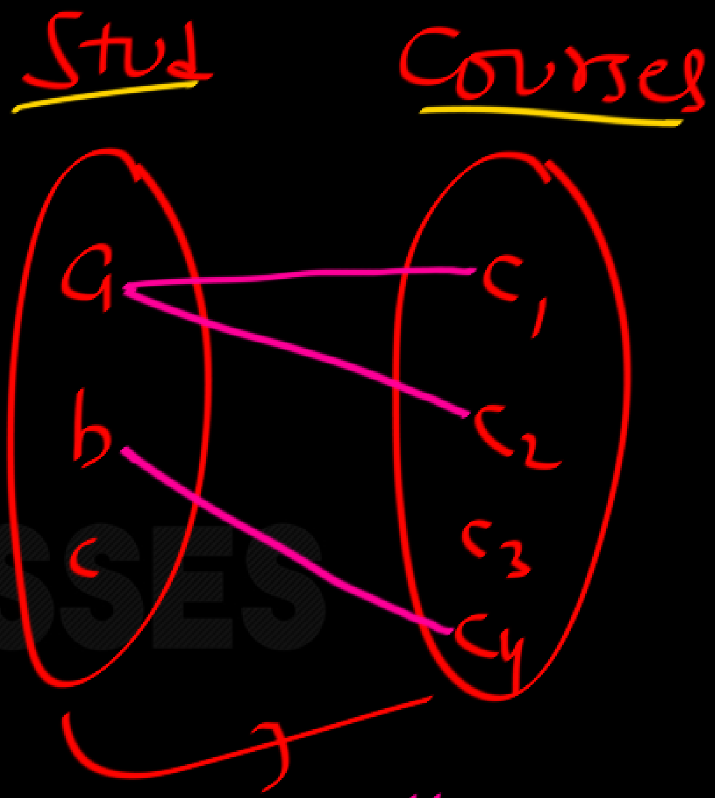
Relation

Definition, Terminology, Base Set

Website : https://www.goclasses.in/



FB friendship

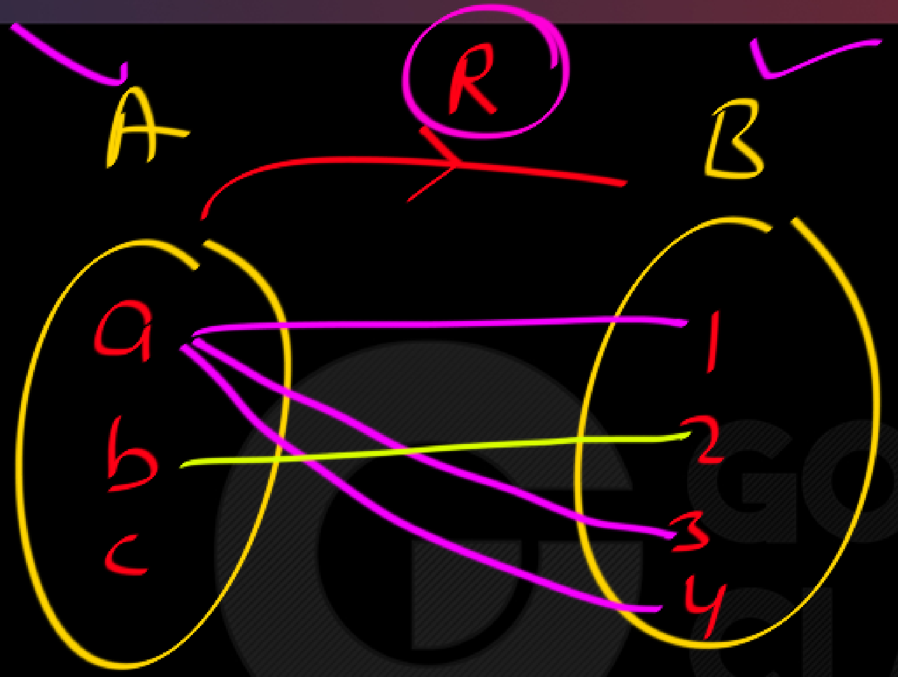


enroll

FB friendship : FB users → FB users

Enroll : Stud → Courses

In relation R;
a is Related to 1



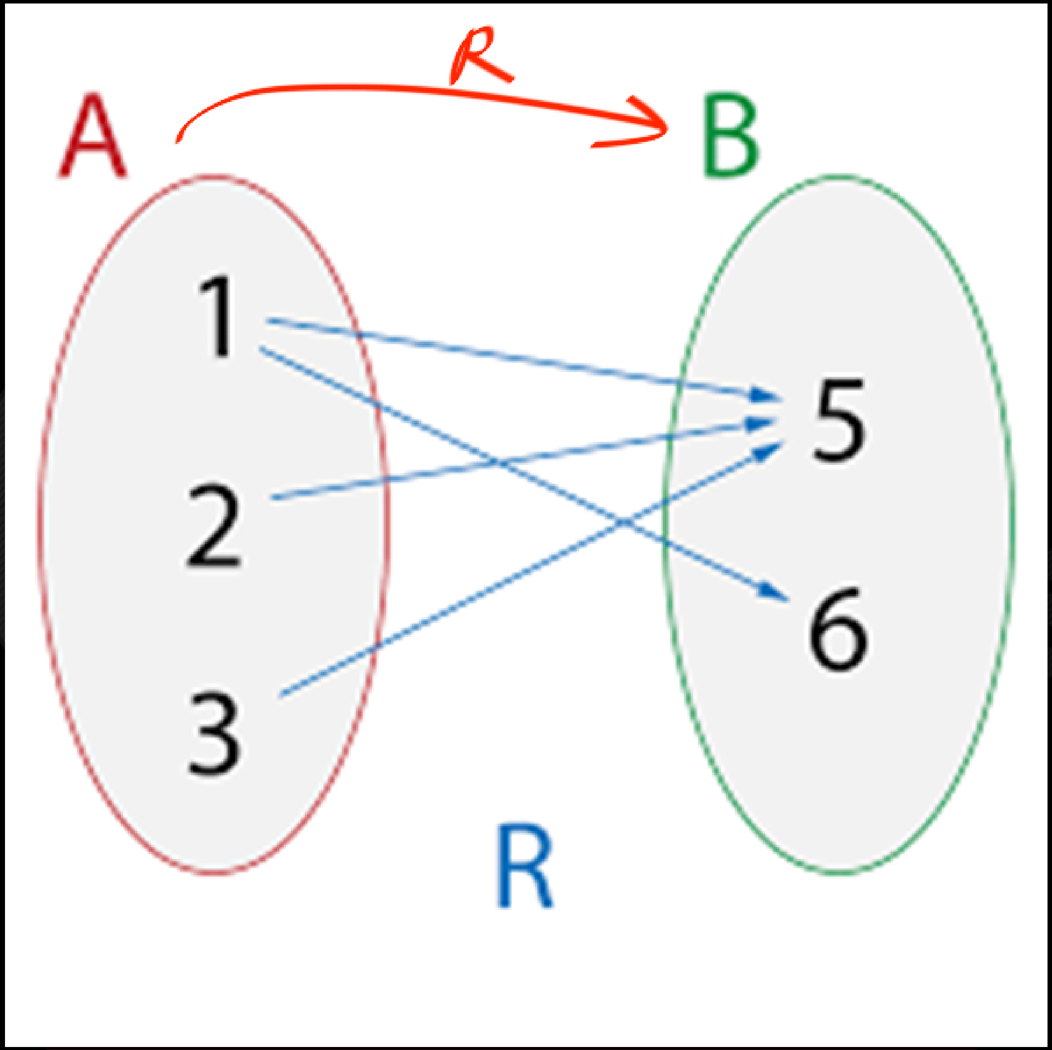
$\boxed{\text{a is Related to 3}}$

$aR3$

$(a,3) \in R$

$R: A \to B$

$R: A \rightarrow B$



1 R 5

1 R 6

2 R 5

$2 \cancel{R} 6$

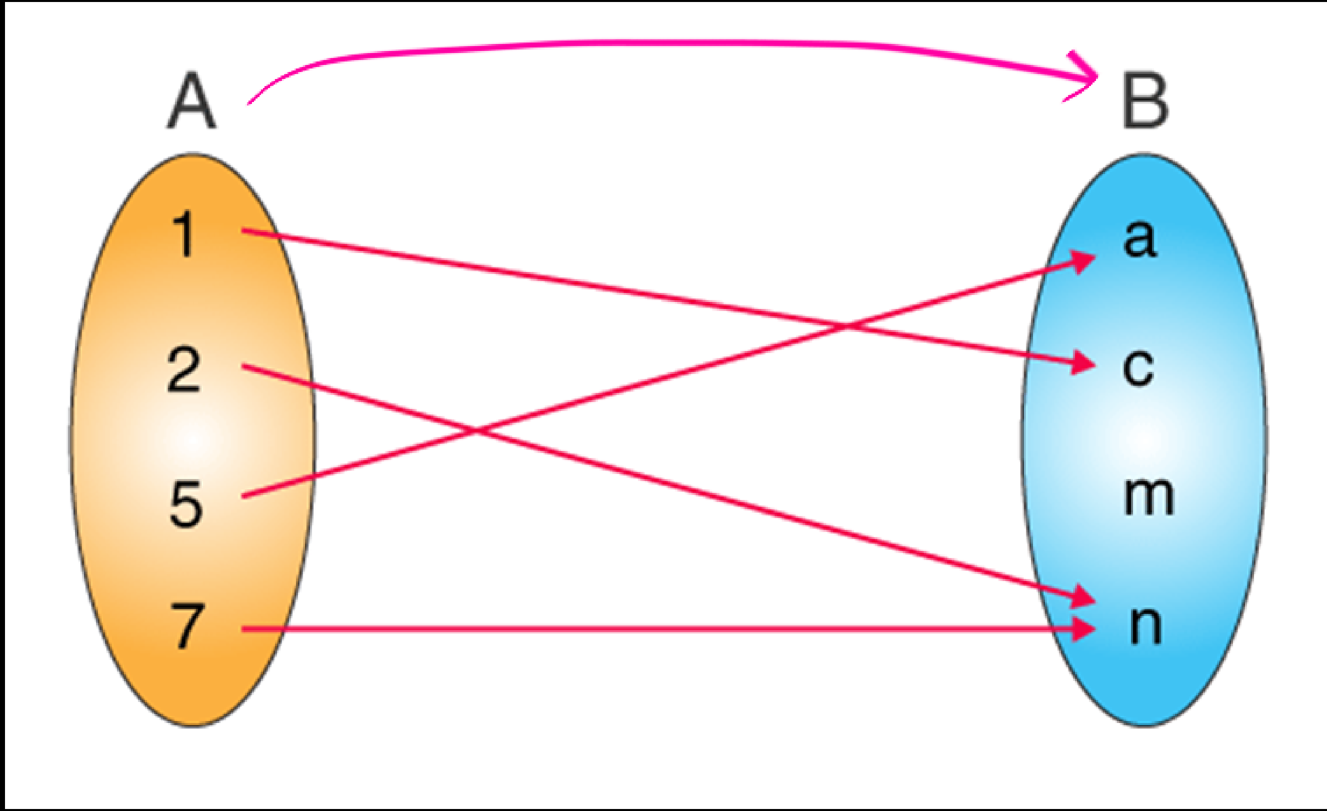
3 R 5

$3 \cancel{R} 6$

1Sc means

1 is Related to c under S.

S



1Sc

1 ~~S~~ a

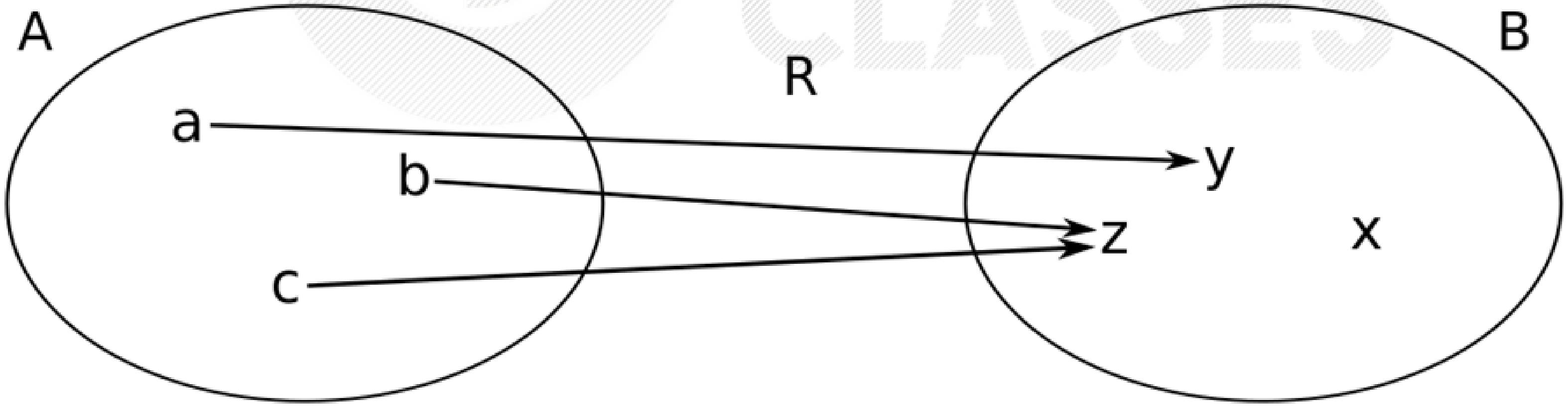
2S n

7Sn

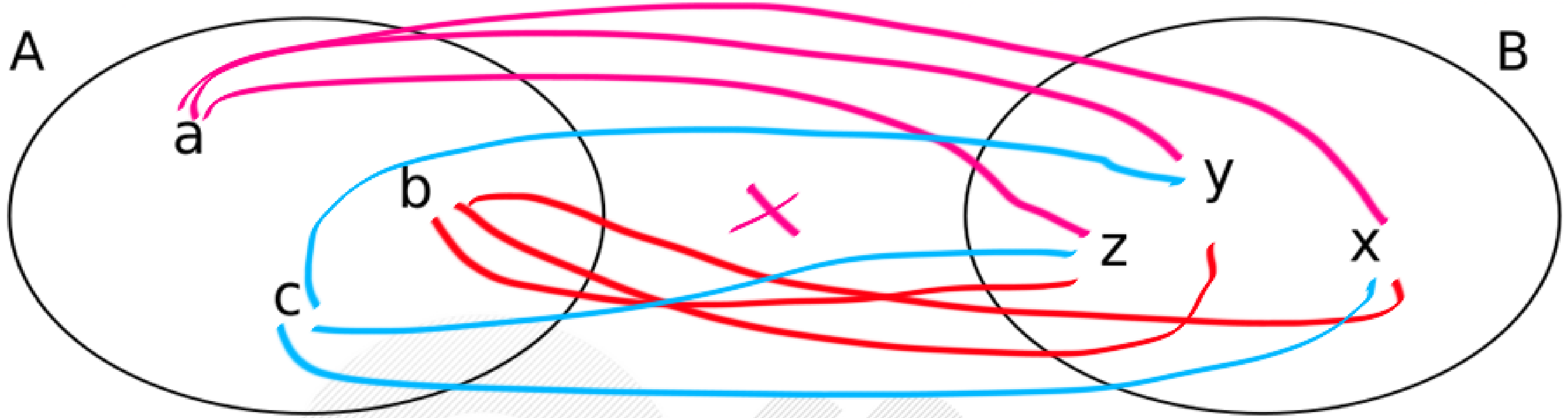
7 ~~S~~ m

$S : A \to B$

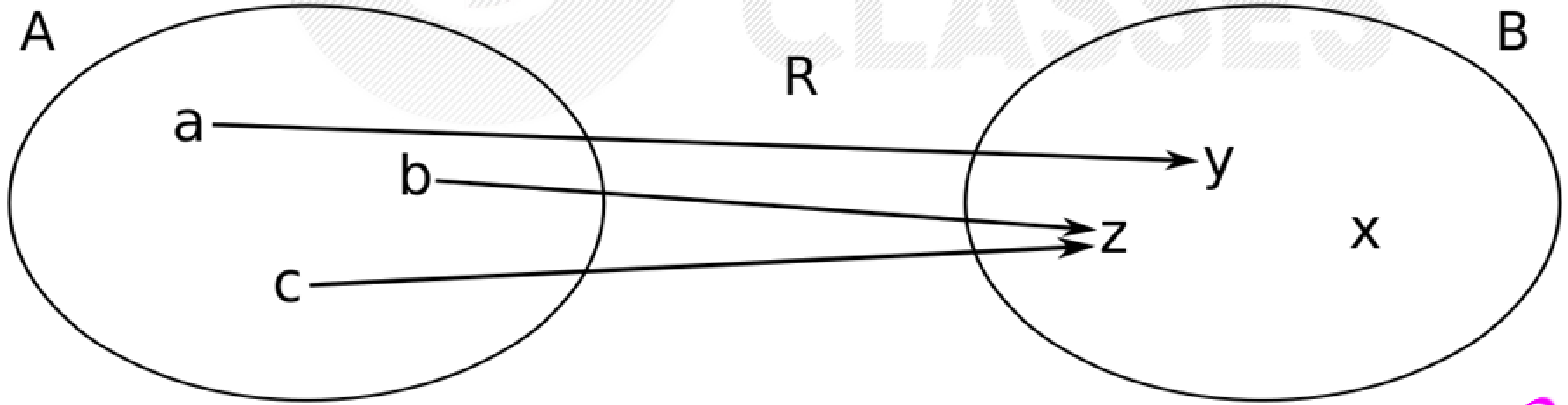
$A \times B = \{(a,x), (b,x), (c,x), (a,y), (b,y), (c,y), (a,z), (b,z), (c,z)\}$



$R = \{(a,y), (b,z), (c,z)\}$



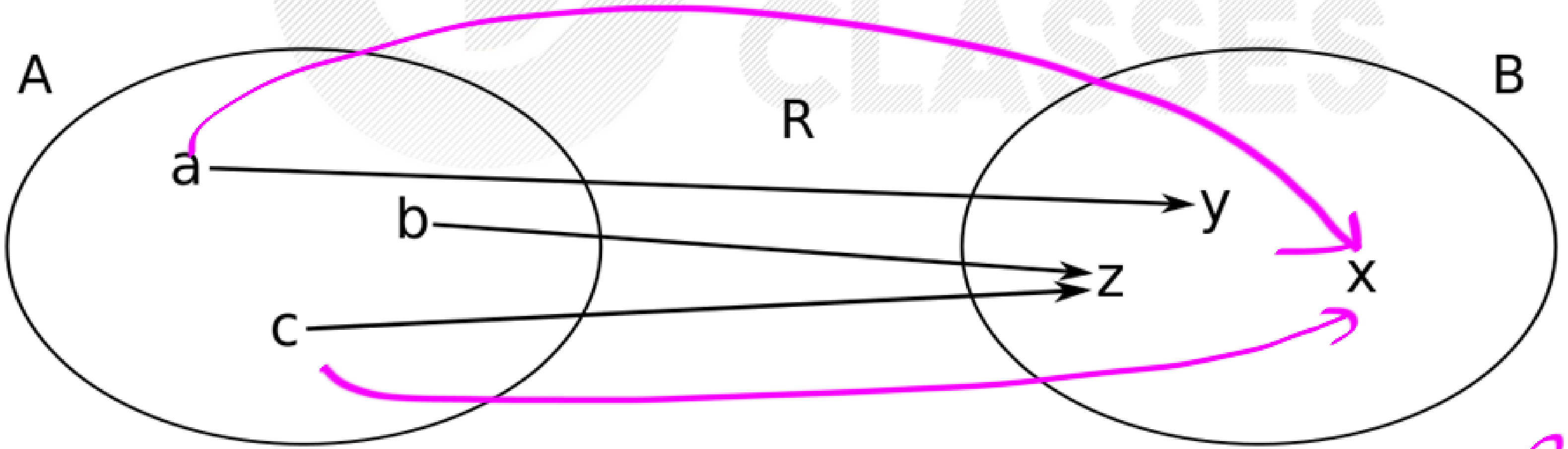
$A \times B = \{(a,x), (b,x), (c,x), (a,y), (b,y), (c,y), (a,z), (b,z), (c,z)\}$



$R = \{(a,y), (b,z), (c,z)\}$

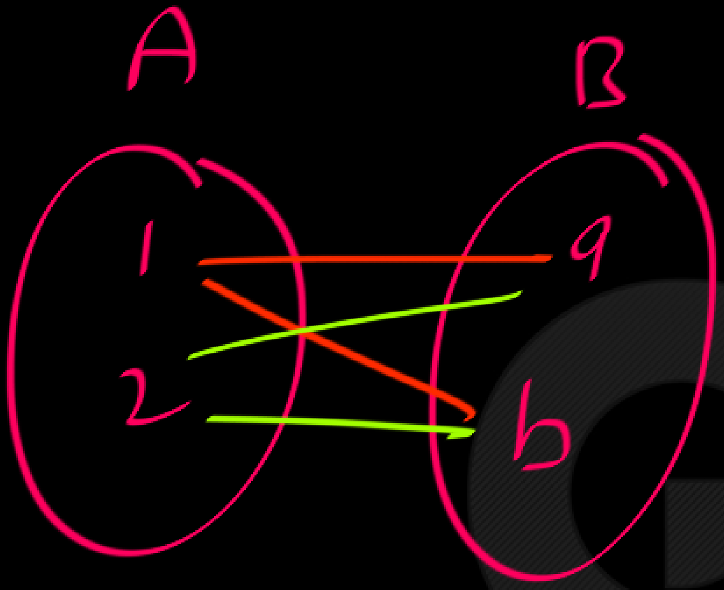
$R \subseteq A \times B$

A x B = {(a,x), (b,x), (c,x), (a,y), (b,y), (c,y), (a,z), (b,z), (c,z)}



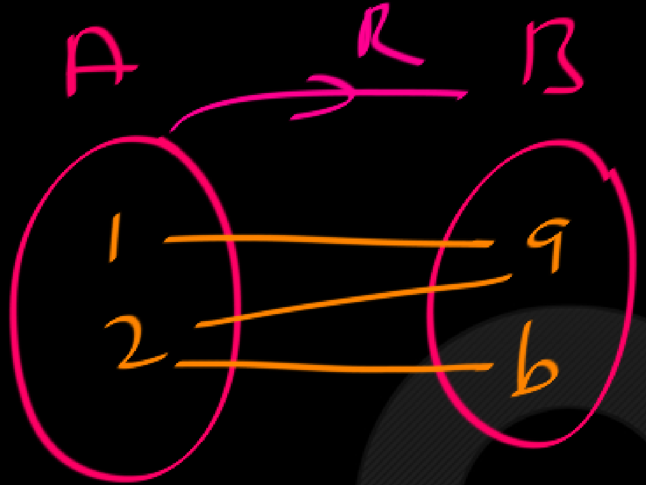
$R \subseteq A \times B$

R = {(a,y), (b,z), (c,z)}



A × B is a Relation from A to B.

$$\underbrace{A \times B}_{} = \{(1,a),(1,b),(2,a),(2,b)\}$$



$R : A \longrightarrow B$

$R \subseteq A \times B$

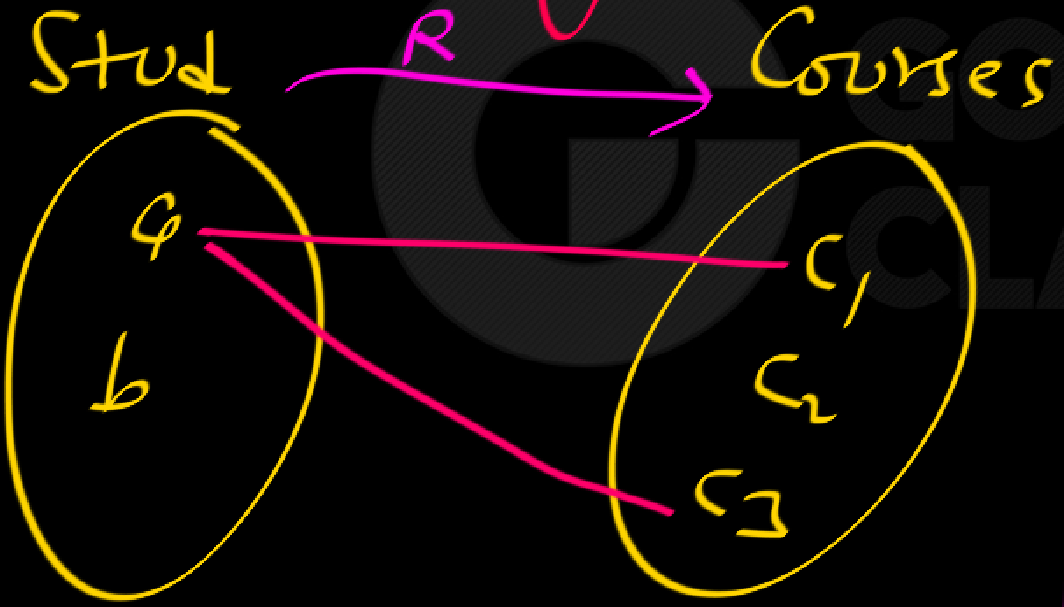
$R = \{ (1, a), (2, a), (2, b) \}$

Set A ; set B

R : A $\rightarrow$ B is subset of A×B.

Every subset of A×B is a Relation from A to B.

1. Relation from A to B are one-way.



xRy

x: Stud, y: course

xRy iff x takes y.

aRc_1 ✓

~~c_1Ra~~

$A = \{1, 2, 3, 5\}$ $B = \{2, 3, 4\}$

$R : A \to B$

$\boxed{x R y \text{ iff } x \leq y}$

x — A, y — B

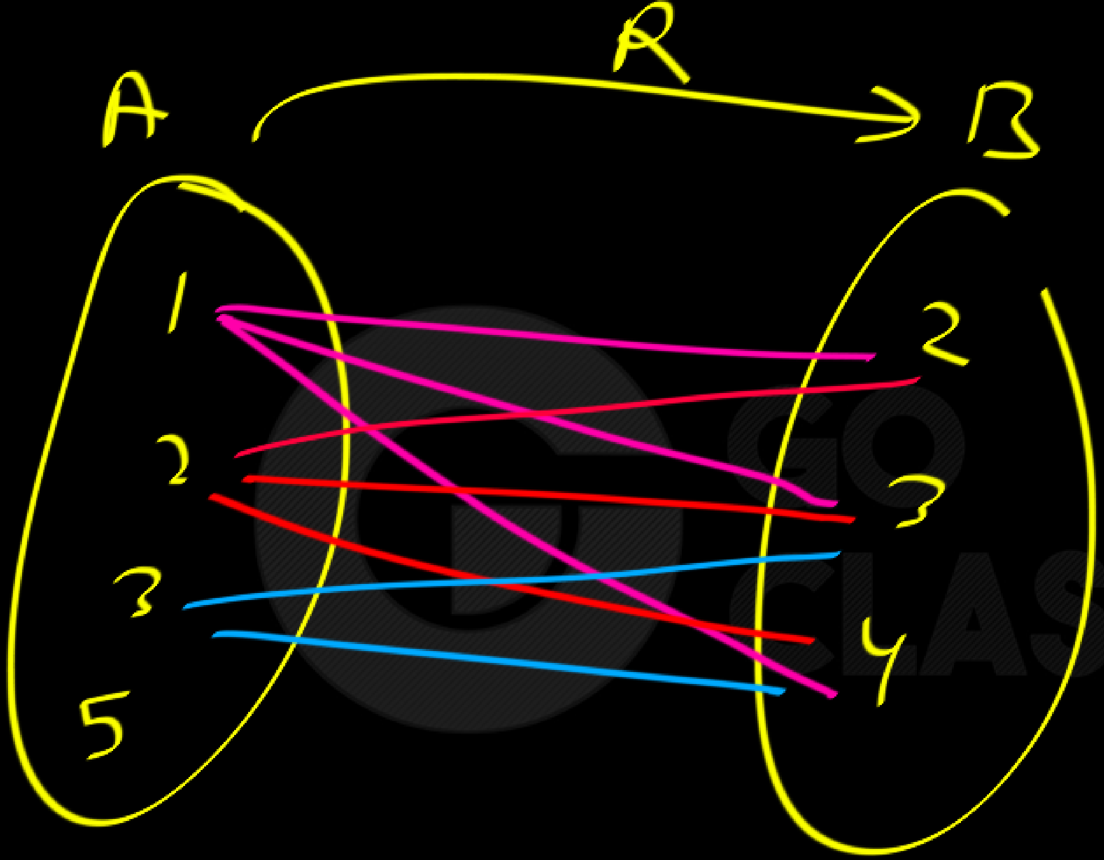
$2 R 3$ ✓

$3 R 4$ ✓

$3 R 2$ ✗

$4 R 5$ ✗

Not in A



xRy

A B

iff

$x \le y$

$1R2$

$5 \not R 4$

$\cancel{4R5}$

Note:

$\boxed{\text{a is Related to b}}$ a → R

Doesnot mean that
b is Relates to a.

a is Related to b under Relation S.

aSb ✓ ; (a,b) ∈ S ✓

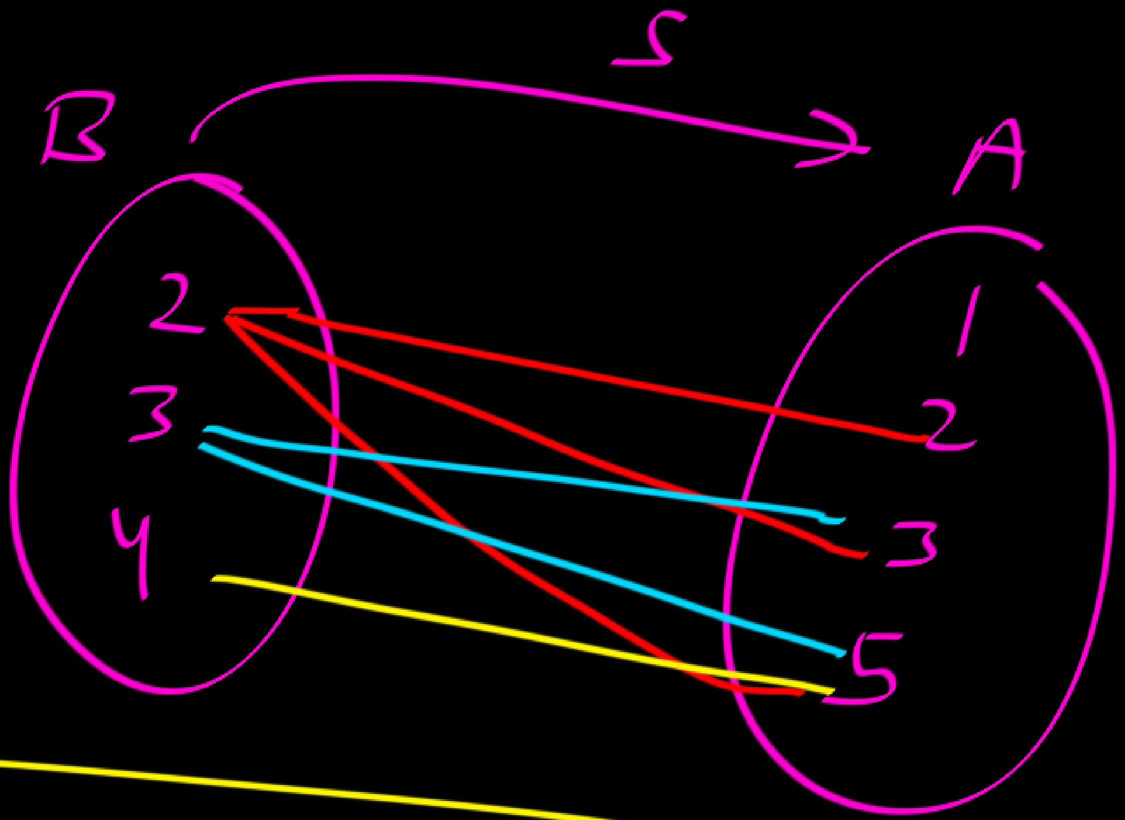
$A = \{1, 2, 3, 5\}$ $B = \{2, 3, 4\}$

$S : B \to A$ $S \subseteq B \times A$

$x R y$ iff $x \leq y$

$|S| = 7 = 6$

$S = \{(2,2), (2,3), (2,5), (3,3), (3,5), (4,5)\}$



$x \, S \, y$

iff

$x \leq y$

$(2, 4) \in S$??

not in A

$$\boxed{S \subseteq B \times A}$$

A **binary relation** R between two sets A and B (which may be the same) is a subset of the Cartesian product $A \times B$. If element a ∈ A is related by R to element b ∈ B, we denote this fact by writing **(a, b) ∈ R**, or alternately, by **a R b**. We say that **R is a relation on A and B**.

A **relation on a set A** is a subset of A × A.

Relations

Definition

Given sets $A_1, \ldots, A_n$, a subset $R \subseteq A_1 \times \cdots \times A_n$ is an n-ary relation.

Example: Database R contains tuples (Street name, House number, currently inhabited flag), i.e., $R \subseteq \mathit{Strings} \times \mathbb{N} \times \mathbb{B}$. So R is a 3-ary relation.

Definition

Given sets A and B, $R \subseteq A \times B$ is a binary relation from A to B.

The property $(x, y) \in R$ is also written as xRy.
Example: $R \subseteq \mathbb{R} \times \mathbb{Z}$ where $(x, y) \in R$ iff $y = \lfloor x \rfloor$ (rounding down).

Definition

$R \subseteq A \times A$ is called a relation on A.

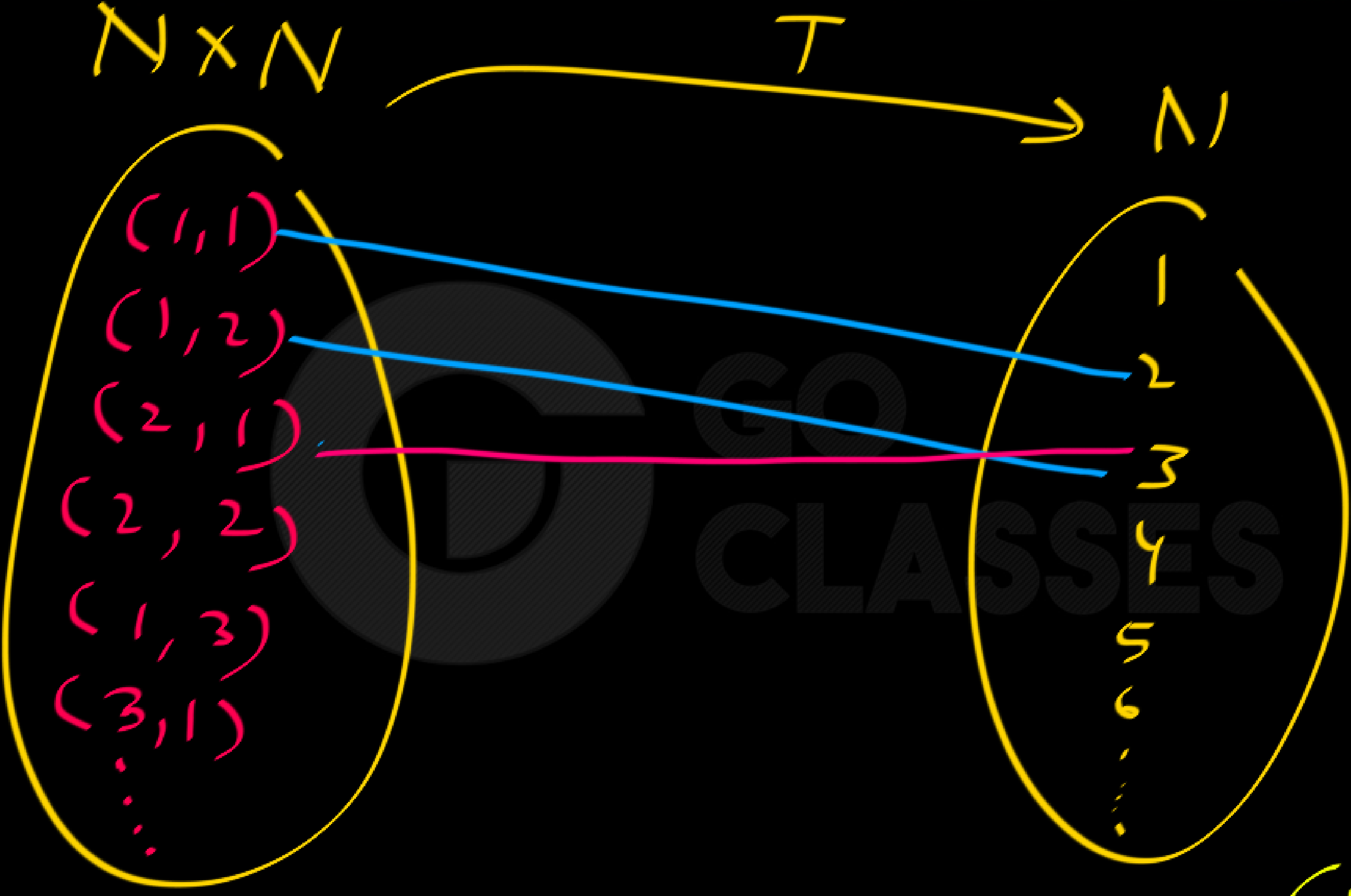
Example: $\leq \subseteq \mathbb{Z} \times \mathbb{Z}$ is the 'less or equal' relation on the integers.

Q: $T : N \times N \rightarrow N$

$(a, b)\, T\, y$ iff $a + b = y$

$T \subseteq (N \times N) \times N$ ✓

$T \subseteq N \times N \times N$ ✗



$(1,1)\,T\,2$

$(1,2)\,T\,3$

$(2,1)\,T\,3$

$(2,2)\,T\,4$

$(100,2)\,T\,102$

8 is Related to (4,4)

(4,4) " " " 8

$R : A \to B$

means $R \subseteq A \times B$

$S : B \to A$ means $S \subseteq B \times A$

$T : A \times B \to B \times A$ means $T \subseteq (A \times B) \times (B \times A)$

Q: Set A ; Set B

#Relations from A to B?

$R : A \rightarrow B$

$R \subseteq A \times B$

Every Relation from A to B is subset of $A \times B$.

Every subset of $A \times B$ is a Relation from A to B.

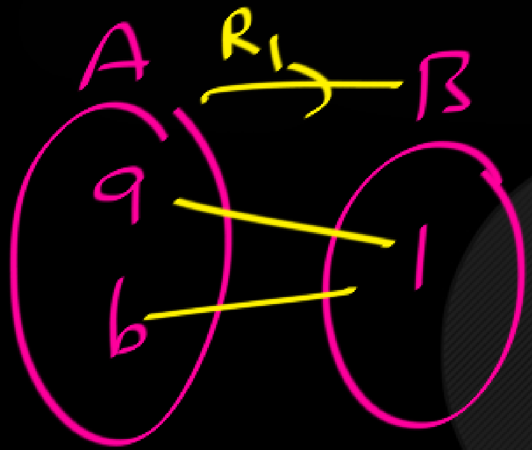
$$|A \times B| = |A| \cdot |B|$$

$$\#\text{subset of } A \times B = 2^{|A||B|}$$

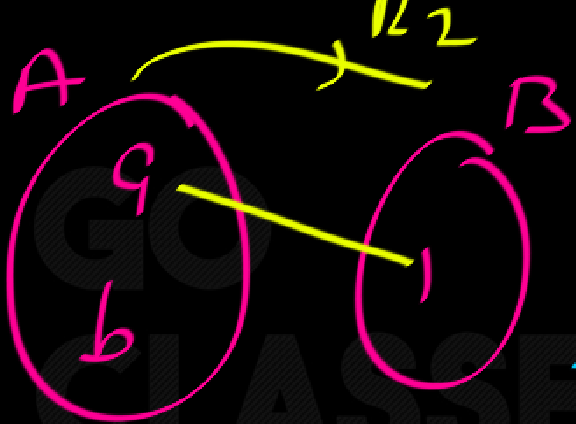
is same as

no. of Relations from A to B.

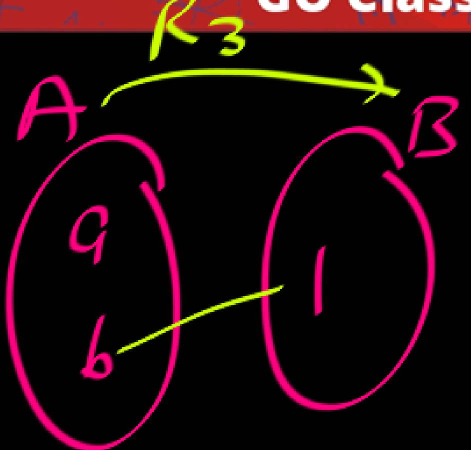
$A = \{a, b\}$; $B = \{1\}$



$R_1 = \{(a,1), (b,1)\}$



$R_2 = \{(a,1)\}$



$R_3 = \{(b,1)\}$

$R_4 = \phi$

$|A| = 2 \quad ; \quad |B| = 1$

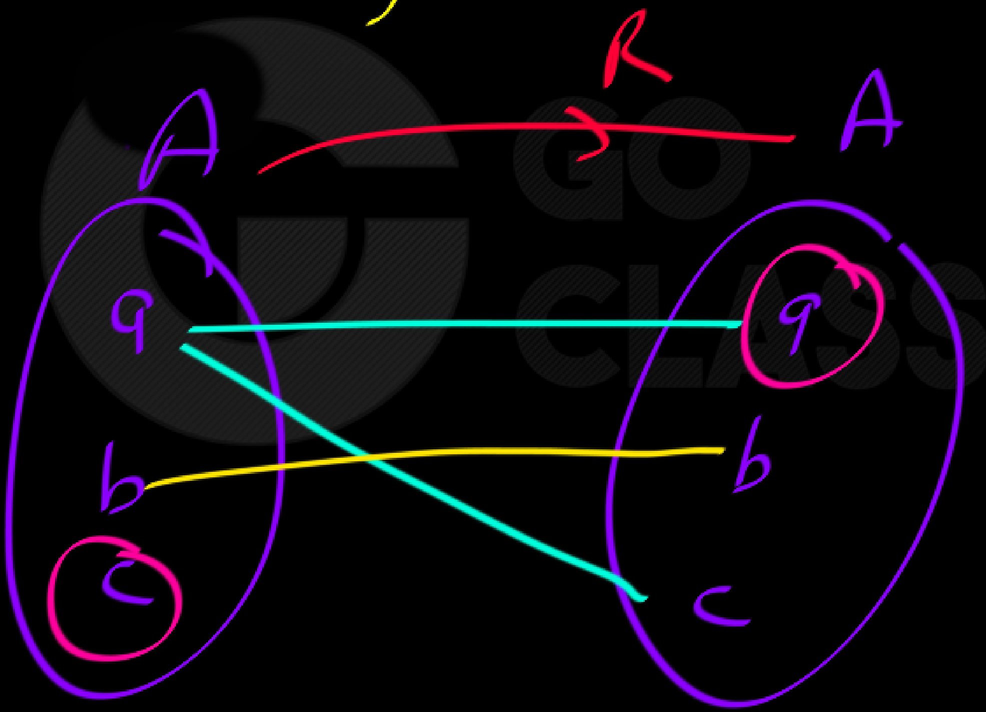
$|A \times B| = 2 \cdot 1 = 2$

$\#$ subsets of $A \times B = 2^2 = 4$

Let A and B be sets. A *binary relation from A to B* is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

$R : A \to A$ } very Imp. in CS, maths



$R : A \to A$

$a R a$

$a R c$

$b R b$

$a \not R b$

$c \not R a$

a is Related to c ✓

c " " " a ✗

$\boxed{\text{a, c are Related}}$ — Ignore writing this.

$\boxed{\text{a, c are " to each other.}}$ ✗

means aRc and cRa

$R : A \to A$ means $R \subseteq A \times A$

Base set

$R : N \times N \to N \times N$ means

$R \subseteq (N \times N) \times (N \times N)$

$R \subseteq (N \times N)$ means $R : N \to N$

$\boxed{R : A \longrightarrow A}$ we call it

R is on set A.

R is on set N×N

✗ $R \subseteq N \times N$ ✓ $R \subseteq (N \times N) \times (N \times N)$

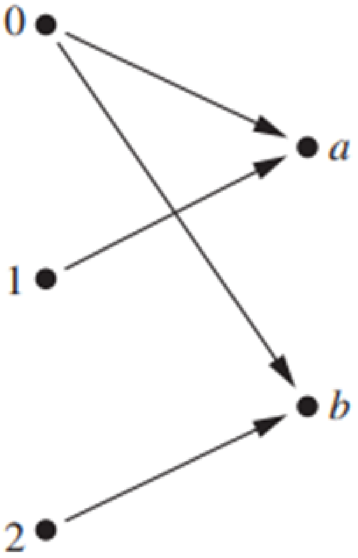
$\boxed{R \subseteq N \times N}$ means R is on set N

means

$\boxed{R ; N \rightarrow N}$

Base set

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0 \, R \, a$, but that $1 \not R \, b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1. We will discuss representations of relations in more detail in Section 9.3.



R	a	b
0	×	×
1	×	
2		×

Relations on a Set

Relations from a set A to itself are of special interest.

A *relation on a set* A is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Consider these relations on the set of integers:

$R : \mathbb{Z} \to \mathbb{Z}$; $R \subseteq \mathbb{Z} \times \mathbb{Z}$

Base set

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

Consider these relations on the set of integers:

$R : Z \to Z$; $R \subseteq Z \times Z$

Base set

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

R_1, R_3, R_4

R_1, R_3, R_4, R_6

R_2, R_5, R_6

Which of these relations contain each of the pairs (1, 1), (1, 2), (2, 1), (1, −1), and (2, 2)?

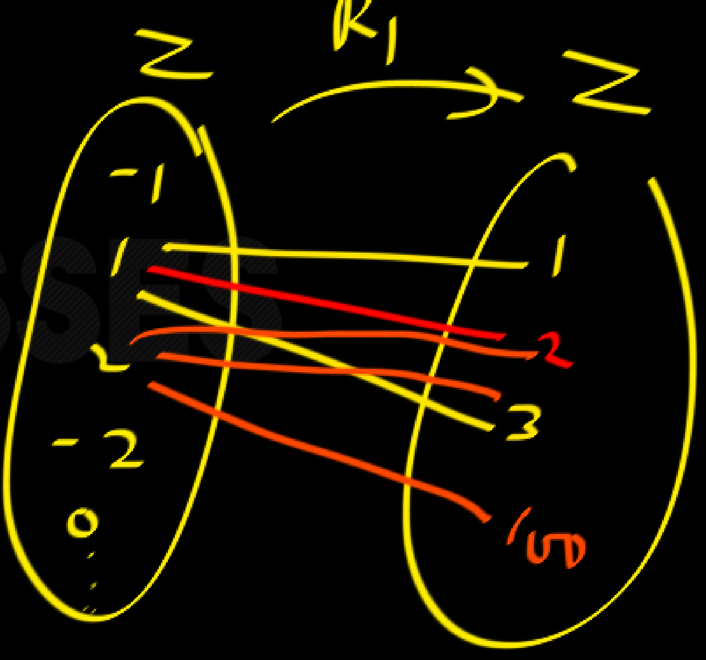
R_1, R_6

R_2, R_3, R_6

$$R_1 = \{(\underbrace{a}_{Z}, \underbrace{b}_{Z}) \mid a \le b\}$$

$$R_1 : Z \to Z$$

Base set
Z



$$R_1 = \{(0,0), (0,1), (-1,-1), (-1,100) \cdots\}$$

Q: $R : \underline{N \times N} \rightarrow \underline{N \times N}$

$\underline{(a, b)}\ R\ (c, d)$ iff

$a = c$ OR $b = d$

Base set $\rightarrow$ $\underline{\underline{N \times N}}$

$\boxed{R \text{ is on } N \times N}$ ✓

$(1,1)\, R\, (1,1)$ ✓

$\boxed{(0,1)}\, R\, (0,2)$ ✗

not in our Base set $\boxed{N \times N}$

$\underbrace{(-1,1)}\, R\, (-1,1)$ ✗

not in Base set

$(2, 3) \cancel{R} (3, 2)$

$(a, b) \in N \times N$

$(c, d) \in N \times N$

$(2, 3) \, R \, (2, 4)$

$R = \{ ((a, b), (c, d)) \mid a = c, d = b \}$

$$R = \{ ((1,1),(1,2)),\ ((1,2),(2,2)),\ ((100,1),(100,4)) \ldots \}$$

How many relations are there on a set with n elements?

Solution: A relation on a set A is a subset of $A \times A$. Because $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of $A \times A$. Thus, there are 2^{n^2} relations on a set with n elements. For example, there are $2^{3^2} = 2^9 = 512$ relations on the set $\{a, b, c\}$. ◀

Representations of Relations :

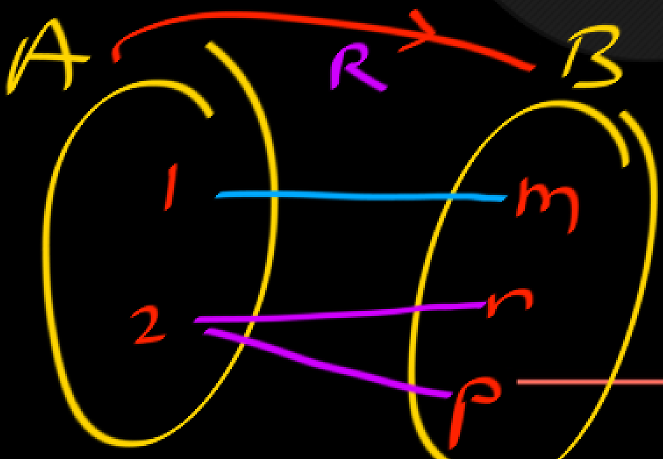
$R : A \rightarrow B$

$R \subseteq A \times A$

$A = \{1, 2\}$

$B = \{m, n, p\}$

$1 R m, \; 2 R n, \; 2 R p$



Arrow Diagram Rep

Set Rep.

$R = \{(1,m), (2,n), (2,p)\}$

Relations on a set A: very special

$$\boxed{R : A \rightarrow A}$$

Additional Rep.
- matrix Rep
- Graph Rep

$A = \{1, 2, 3\}$ → Base set

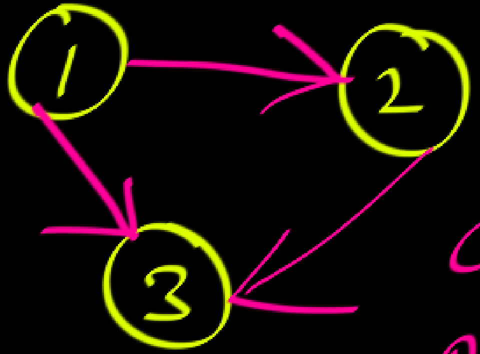
R on A

$R : A \to A$

$R \subseteq A \times A$

$x R y$ iff $x < y$

$R = \{(1,2), (1,3), (2,3)\}$ Set Rep



Graph Rep

	1	2	3
1	×	✓	✓
2	×	×	✓
3	×	×	×

$A = \{1, 2, 3, 4\}$ → Base set

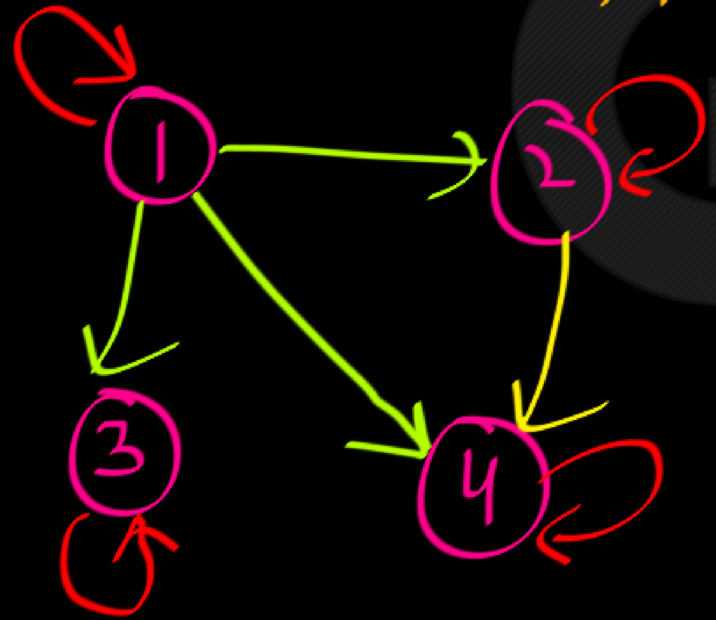
R on A

$R : A \to A$

$R \subseteq A \times A$

$x R y$ iff $x \mid y$.

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), \ldots$



	1	2	3	4
1	✓	✓	✓	✓
2	✗	✓	✗	✓
3	✗	✗	✓	✗
4	✗	✗	✗	✓

Representing Binary Relations

- Let A be a finite set
- The binary relation on A can be convenient represented in two different ways :

- Method 1: Matrix Form

Base set

A = { 1, 2, 3, 4 }

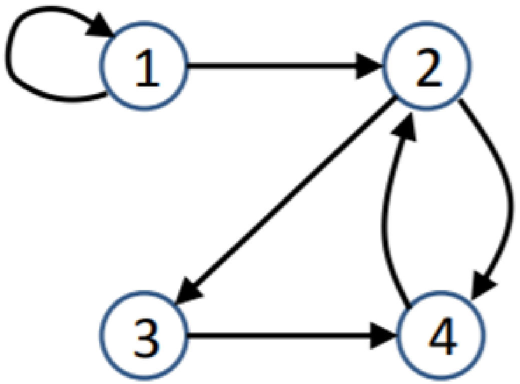
R = { (1,1), (1,2), (2,3),
(2,4), (3,4), (4,2) }

	1	2	3	4
1	✓	✓		
2			✓	✓
3				✓
4		✓		

Representing Binary Relations

- Method 2: Directed Graph
- $A = \{ 1, 2, 3, 4 \}$

  $R = \{ (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) \}$

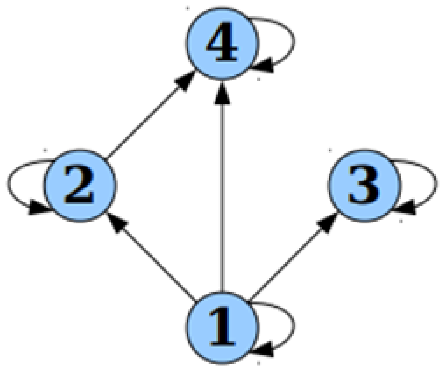


Graph Rep: If aRb then

$$a \longrightarrow b$$

Visualizing Relations

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y if and only if xRy.
- Example: the relation $a \mid b$ (meaning "a divides b") over the set $\{1, 2, 3, 4\}$ looks like this:



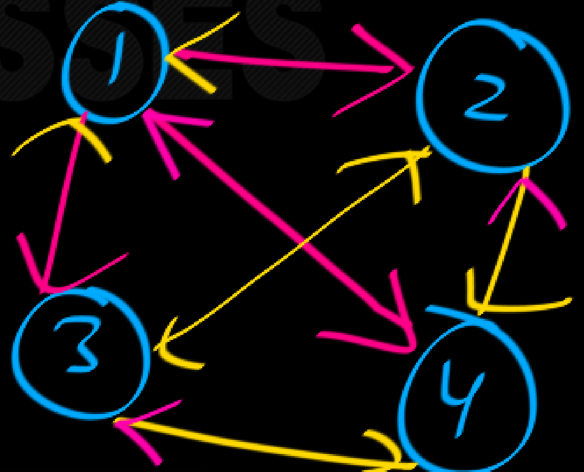
$A = \{1, 2, 3, 4\}$ → Base set

$\boxed{R \text{ on } A}$

xRy iff $x \neq y$.

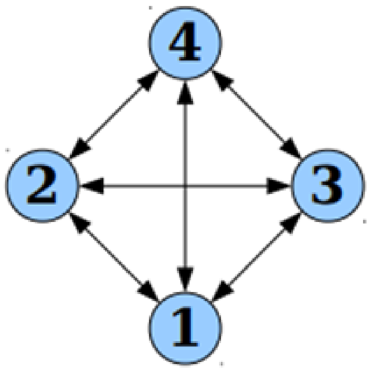
$1 \not R 1$ $\quad$ $2 R 1$

$1 R 2$ $\quad$ $2 \not R 2$



Visualizing Relations

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y if and only if xRy.
- Example: the relation $a \neq b$ over $\{1, 2, 3, 4\}$ looks like this:



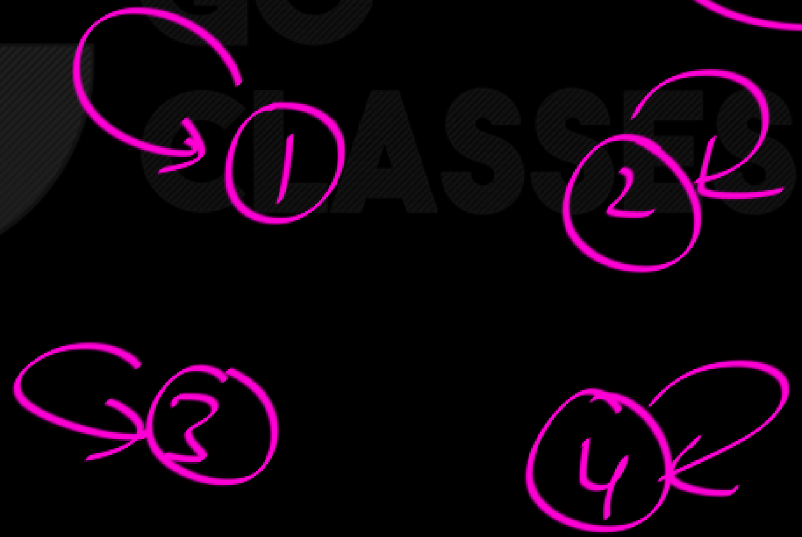
$A = \{1, 2, 3, 4\}$

$\boxed{R \text{ on } A}$

$R : A \to A$

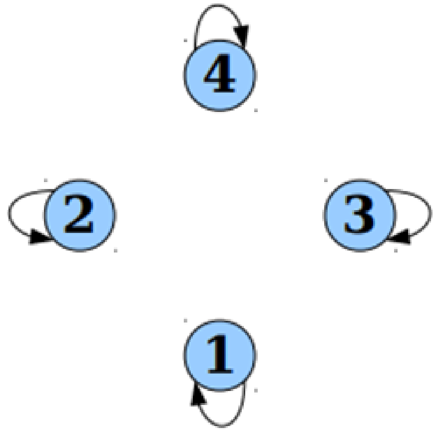
$R \subseteq A \times A$

$x R y$ iff $x = y$



Visualizing Relations

- We can visualize a binary relation R over a set A as a graph:
  - The nodes are the elements of A.
  - There is an edge from x to y if and only if xRy.
- Example: the relation $a = b$ over $\{1, 2, 3, 4\}$ looks like this:



Q: $R = \{((a,b),(c,d)) \mid c,a \in Z;\ d,b \in N;\ a+c = b+d\}$

① Base set: $Z \times N$ ✓

$Z \times N \leftarrow (a,b)\ R\ (c,d) \rightarrow \in Z \times N$

$a \to Z$, $b \to N$, $c \to Z$, $d \to N$

$R : Z \times N \to Z \times N$

R is on $\underbrace{Z \times N}$.

not in $\boxed{Z \times N}$

2 $\boxed{(-2, 1) \, R \, \boxed{(-3, -6)}}$ ~~X~~

Nonsense

not even in Base set.

$(a,b)\, R\, (c,d)$ iff $a+c=b+d$

3

$(-2, 1)\, R\, (3, x)$

$\boxed{-2+3 = 1+x}$

$x=0$

not in N.

no such x (Hint: $x \in N$)

4

$(-2, 2)\, R\, (5, x)$

$x = 1$

$-2+5 = 2+x$

$x=1$

1. $\boxed{\text{Related}}$ word is one way.
2. Take Care of Base Set.

$\boxed{R \text{ on set } A}$ → Base set for R

means

$R : A \to A$

means

$R \subseteq A \times A$

R on $\underbrace{N \times Z}$ → Base set

$R : N \times Z \to N \times Z$

$R \subseteq (N \times Z) \times (N \times Z)$