The Fundamental Course

Chapter:

Proof Techniques

(Homework 1,2 Solutions)

- By Deepak Poonia(IISc Bangalore)

Instructor:

Deepak Poonia
IISc Bangalore
GATE CSE AIR 53; AIR 67; AIR 206; AIR 256;

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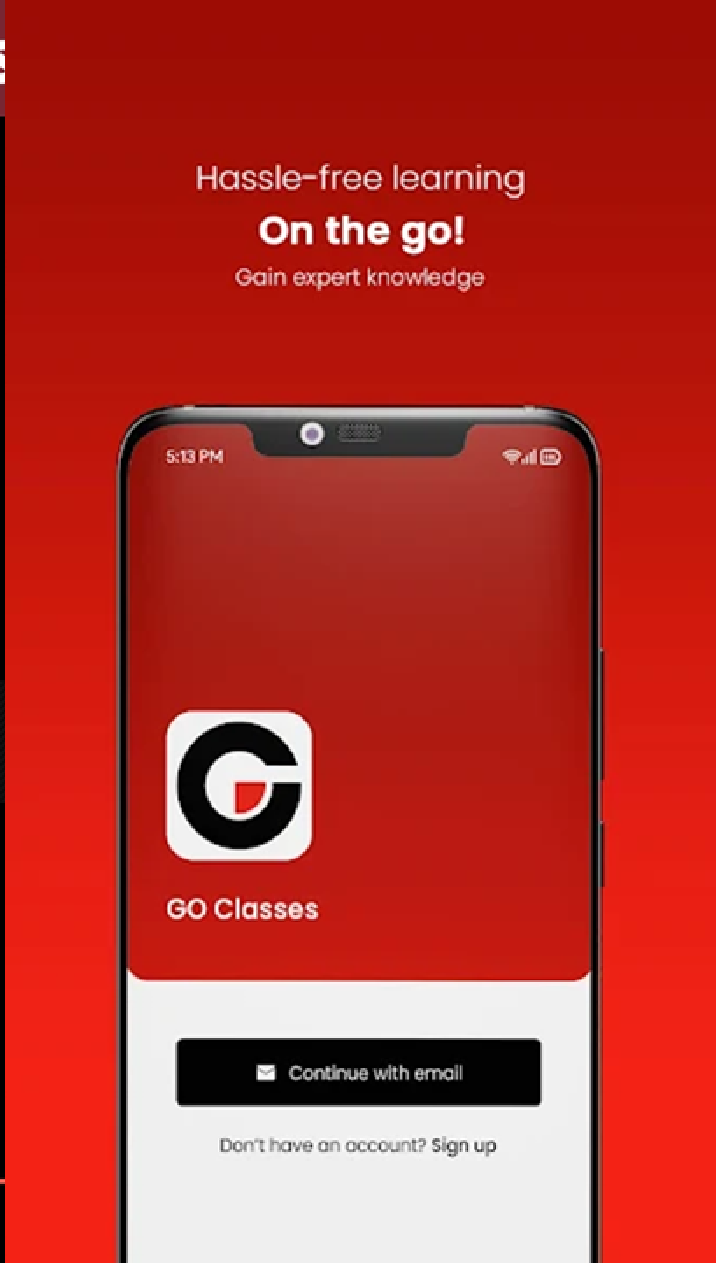

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HW 2 Q 7.

Prove the following proposition :

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Integer

Hint : Proof by Contraposition OR Proof by Contradiction.

n even $\longrightarrow$ n^2 even

n^2 even $\longrightarrow$ n even

n odd $\longleftrightarrow$ n^2 odd

n even $\longleftrightarrow$ n^2 even

Odd + odd = even ✓

If P then Q

Claim

A

How to Prove A is false:

Person

P True But Q false

$a, b, c \in \mathbb{Z} \rightarrow$ Integers

If $\underbrace{a^2 + b^2 = c^2}_{P}$ then $\underbrace{a \text{ or } b \text{ is even}}_{Q}$.

for purpose of Contra-diction

Assuming

Proof by Contradiction

$\boxed{a, b \text{ both odd}}$ $\neg Q$

$\downarrow$

a^2, b^2 both odd

$a = 2m+1$

$b = 2n+1$

$a^2 + b^2 = \text{Even} = c^2$

$c^2 = \text{even} \longrightarrow c = \text{even}$

$c = 2k$ int

$a^2 + b^2 = 4k^2$

$(2m+1)^2 + (2n+1)^2 = 4k^2$

$4m^2 + 4m \quad + 4n^2 + 4n + 2 = 4k^2$

even $\leftarrow \boxed{2m^2 + 2m + 2n^2 + 2n} + 1 = 2k^2$ even

even + 1 = even

$\boxed{\text{Odd = even}}$ Contradiction

So, a, b both can not be odd.

So, a or b must be even.

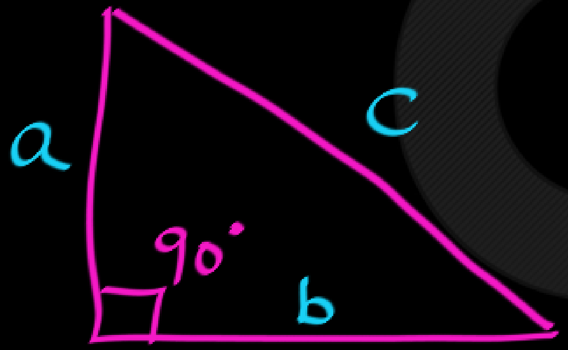
$a, b, c \in \mathbb{Z}$

If $\boxed{a^2 + b^2 = c^2}$ then a or b must be even.

Proved ✓

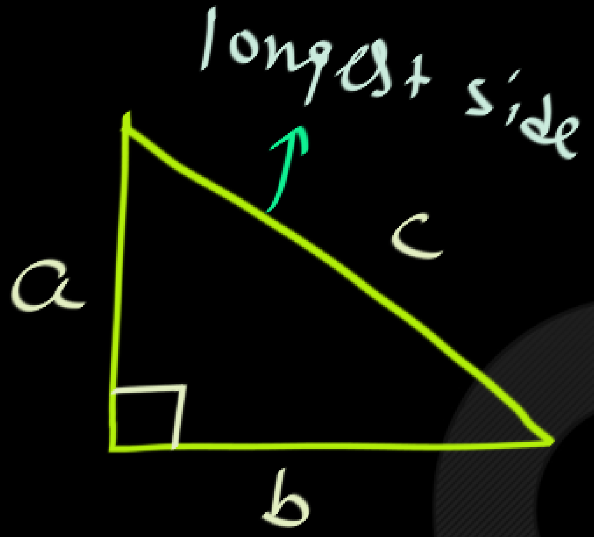
Applications ??

$$\boxed{a^2+b^2=c^2} \rightarrow \text{Pythogoras theorem}$$



$\rightarrow \boxed{a^2+b^2=c^2}$

Right Angle Triangle

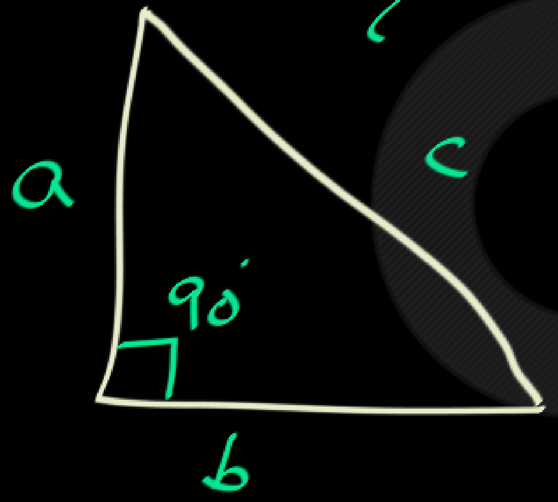


In a Right Angle Triangle,

If all sides are integers

then at least one of the Shorter sides (a, b) is Always even.

Rephrase:



If $a, b, c \in \mathbb{Z}$

then $\boxed{a, b \text{ both odds}}$ can Never happen.

HW 2 Q 7.

We have proved the following proposition :

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

Can you find One Application of this result???

Where have you seen this equation in your life??

Rephrase :

$$\boxed{(\text{odd})^2 + (\text{odd})^2 \neq (\text{int})^2}$$

Summation of two odd perfect squares is NEVER a perfect square.

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.

Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

This result can be re-phrased as following :

Cool application I: Sums of odd perfect squares. Can a sum of two perfect squares be another perfect square? Sure; for example, $3^2 + 4^2 = 5^2$, $5^2 + 12^2 = 13^2$, $6^2 + 8^2 = 10^2$, $7^2 + 24^2 = 25^2$. However, no matter how much you try, you won't find any examples in which the two perfect squares on the left are both odd. Your task is to prove this, i.e.:

Prove that a sum of two odd perfect squares is never a perfect square.

(An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

Something Beautiful:

1. Suppose $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

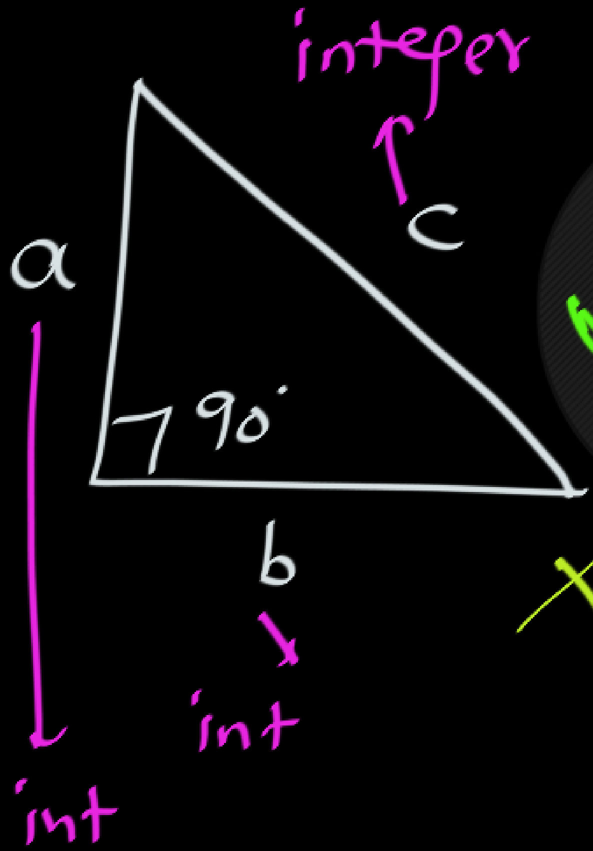
2. *Prove that a sum of two odd perfect squares is never a perfect square.*

3. (An interesting consequence of this result is that in any right triangle in which all sides have integer length at least one of the two shorter sides must be of even length.)

$\boxed{1 = 2 = 3}$ ✓

Possible Aptitude (GATE) Question:

which is True?

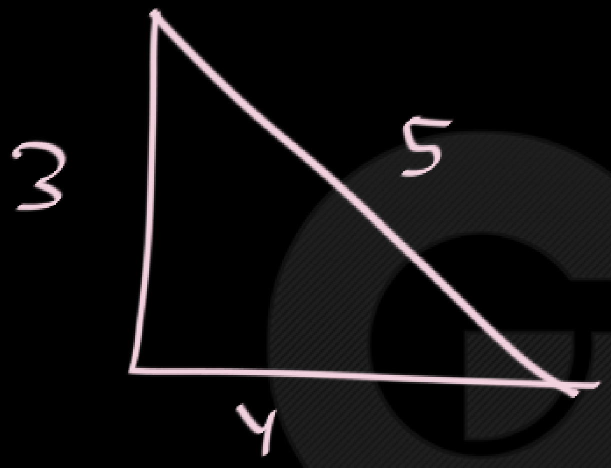


× (1) both a, b are odd → Never

(2) at least one of a, b is even → Always True

× (3) both a, b even → may or may not

(4)



$$3^2 + 4^2 = 5^2$$

Counter example of option 3.

Summary: $a, b, c \in \mathbb{Z}$

If $\boxed{a^2 + b^2 = c^2}$ then a or b even

Proof:

IDEA: Assume $\boxed{a^2 + b^2 = c^2}$ happens

BUT both a, b odd

leads to Contradiction

HW 2 Q 2.

Consider the following statement: for every prime number p, either $p = 2$ or p is odd. We can rephrase this: for every prime number p, if $p \neq 2$, then p is odd. Now try to prove it.

Hint: Prove using contrapositive.

Prime Numbers: → Usually only Associated with +ve integers.

2, 3, 5, 7, - - - - - - - - -

Divisible by Exactly two positive integers (1 & itself)

Is '1' prime? → No (Doesn't have two factors)

Prove: for every prime p;

either $p = 2$ OR $p = \text{odd}$

i.e. if $p \neq 2$ then $p = \text{odd}$.

Proof: prime $p \neq 2$ but $p = \text{even}$

for Contradiction:

factors of $p = 1, p, 2$

So p = NOT a prime → Contradiction

HW 2 Q 3.

If x and y are integers and $x^2 + y^2$ is even, prove that x+y is even.

Hint : Prove by Contraposition.

HW 2 Q 3.

If x and y are integers and $x^2 + y^2$ is even, prove that x+y is even.

Contraposition:

$\boxed{x + y = odd}$

? facts

$\boxed{x^2 + y^2 = odd}$

HW 2 Q 3.

If x and y are integers and $x^2 + y^2$ is even, prove that x+y is even.

Contraposition: $\boxed{x + y = \text{odd}}$

n odd $\downarrow$ n^2 odd

$(x+y)^2 = \text{odd} = \boxed{x^2 + y^2} + 2xy$

even

odd

So $\boxed{x^2 + y^2 = \text{odd}}$ Hence proved by contraposition

Note:

If you look at things too closely,

they start looking Blur.

HW 2 Q 4.

Prove the following proposition :

Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

HW 2 Q 4.

int $a \geq 2$

Prove the following proposition :

Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

int $a \geq 2$

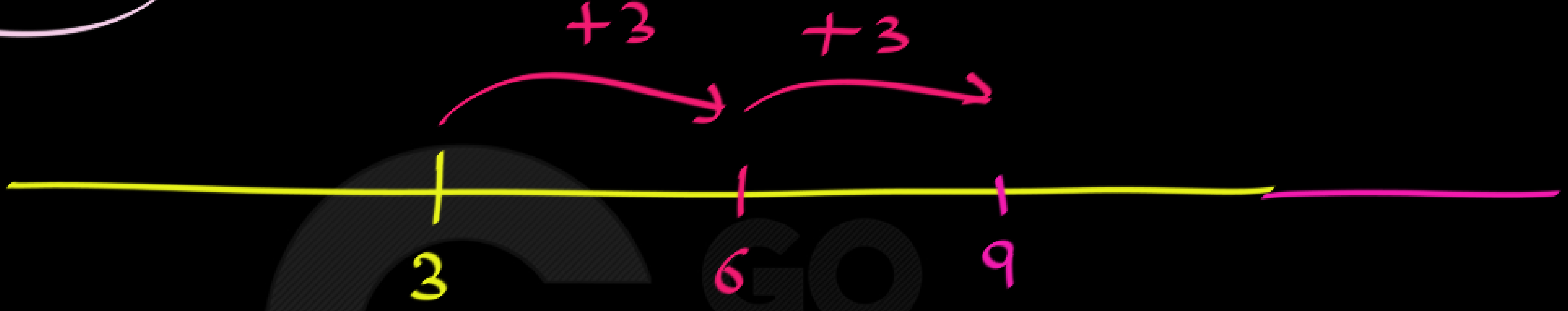
int b

$a \nmid b$ OR $a \nmid (b+1)$

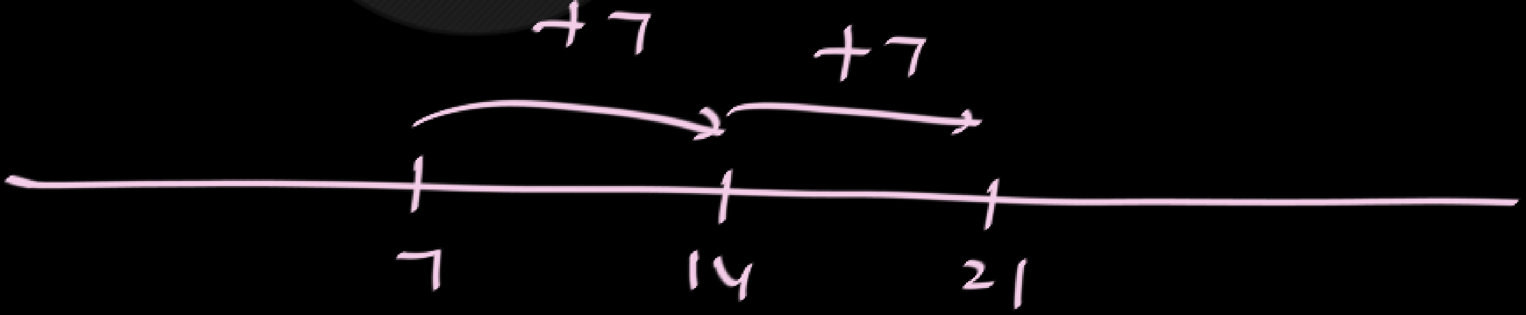
a Cannot Divide both

$b, (b+1)$

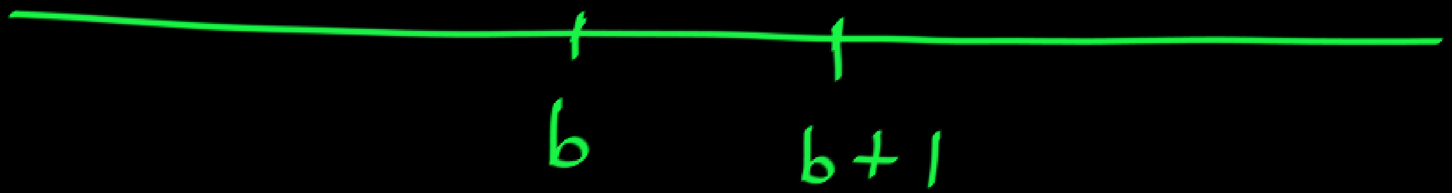
$a = 3$



$a = 7$



int $a \geq 2$



Can 'a' Divide both b, (b+1) ? No

So, $a \nmid b$ OR $a \nmid (b+1)$

Proof 2 ; int $a \geq 2$; int b

for Contradiction : $a \mid b$ and $a \mid (b+1)$

int $b = ak$ $b+1 = am$ int

$$\boxed{am - ak = 1}$$

+ve $a(m-k) = 1 \rightarrow$ +ve

must be +ve

$a(m-k) = 1$

+ve

Non-Zero

$a = \frac{1}{m-k}$

+ve integer

1, 2, 3, 4, ...

$a \leq 1$

Contradiction

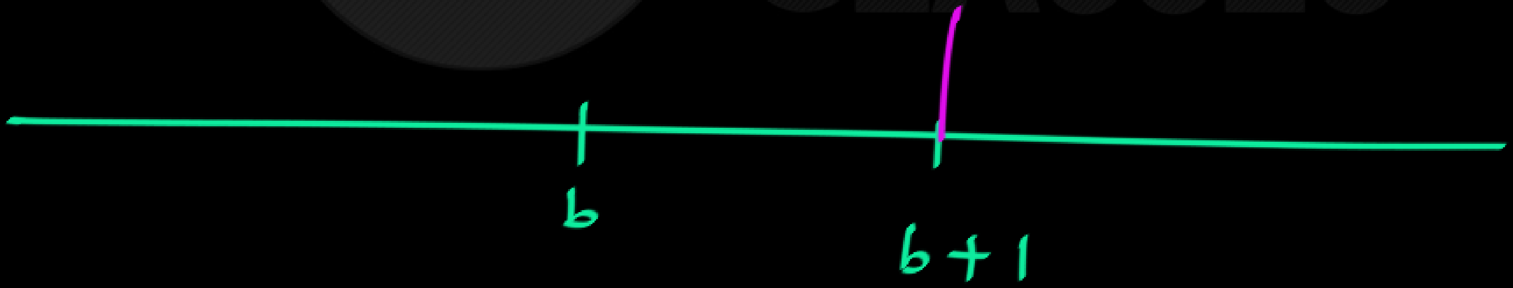
Given: $a \geq 2$

HW 2 Q 4.

Prove the following proposition :

Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

int $a \geq 2$



Proposition If $a, b \in \mathbb{Z}$ and $a \geq 2$, then $a \nmid b$ or $a \nmid (b+1)$.

Proof. Suppose for the sake of contradiction there exist $a, b \in \mathbb{Z}$ with $a \geq 2$, and for which it is not true that $a \nmid b$ or $a \nmid (b+1)$.
By DeMorgan's Law, we have $a \mid b$ and $a \mid (b+1)$.
The definition of divisibility says there are $c, d \in \mathbb{Z}$ with $b = ac$ and $b+1 = ad$.
Subtracting one equation from the other gives $ad - ac = 1$, or $a(d-c) = 1$.
Since a is positive, $d-c$ is also positive (otherwise $a(d-c)$ would be negative).
Then $d-c$ is a positive integer and $a(d-c) = 1$, so $a = 1/(d-c) < 2$.
Thus we have $a \geq 2$ and $a < 2$, a contradiction. ■

HW 2 Q 5. Prove the following propositions :

Prove that $\sqrt[3]{2}$ is irrational.

Prove that $\sqrt{6}$ is irrational.

Hint: Just like we proved that $\sqrt{2}$ is irrational in the lecture 2, these two can be proved in exactly similar way.

(1) $\sqrt[3]{2} = 2^{1/3}$

Prove: $\sqrt[3]{2}$ is Irrational.

for Contradiction; Assume $(2)^{1/3}$ is Rational.

$2^{1/3} = \frac{a}{b}$ int

in lowest form

$GCD(a,b) = 1$

$b \neq 0$

$$2^{1/3} = \frac{a}{b}$$

$$2 = \frac{a^3}{b^3} \Rightarrow a^3 = 2b^3 \rightarrow a^3 = \text{even}$$

$a = \text{even}$

$a = 2k$

$8k^3 = 2b^3$

$b^3 = 4k^3$ even

So $b^3 = \text{even}$ $\rightarrow$ $b = \text{even}$

$a = \text{even}$ $\rightarrow$ Contradiction

So

$\sqrt[3]{2}$ is Irrational.

Q: $\sqrt{6}$ is Irrational.

$\sqrt{6} = \frac{a}{b}$

$a^2 = 6b^2$

$a^2 =$ even

$a =$ even

$a = 2k$

$4k^2 = 6b^2$

$2k^2 = 3b^2$

even

$3b^2 =$ even

odd

even

$b =$ even

$a = \text{even}$
$b = \text{even}$

Contradiction ✓

Proposition: $\sqrt{6} \notin \mathbb{Q}$.

Proof: Suppose $\sqrt{6} \in \mathbb{Q}$. Then there exist integers $p, q \in \mathbb{Z}$, $q \neq 0$, such that $\frac{p}{q} = \sqrt{6}$ AND $\gcd(p, q) = 1$. (That is, the fraction is fully reduced.) Then:

$$\begin{aligned}
\sqrt{6} &= \frac{p}{q} \\
\left(\sqrt{6}\right)^2 &= \left(\frac{p}{q}\right)^2 \\
6 &= \frac{p^2}{q^2} \\
6q^2 &= p^2 \\
2(3q^2) &= p^2
\end{aligned}$$

Then p^2 is even. So, by a result from the book (Exercise 1, p110), p is even. That means there is some $a \in \mathbb{Z}$ such that $p = 2a$, so:

$$\begin{aligned}
2(3q^2) = (2a)^2 &= 4a^2 \\
3q^2 &= 2a^2
\end{aligned}$$

That is, $3q^2$ is even. Since the product of two odd integers is odd, and 3 is odd, we see that q^2 must be even, and so q itself is even. Then there exists some $b \in \mathbb{Z}$ such that $q = 2b$. But then 2 is a divisor of q as well as p, so $\gcd(p, q) \geq 2$, contradicting that $\gcd(p, q) = 1$. $\square$

Remark: we used proof by contradiction.

Note:

If P then Q

Proof by Contradiction

Assume P & NOT Q

HW 2 Q 8. Prove the following proposition :

Suppose a,b $\in$ R.

If a is rational and ab is irrational, then b is irrational.

Hint : Proof by Contradiction.

a = Rational
ab = Irrational
b = Rational

ab = Rational

Contradiction

HW 2 Q 9. Prove the following propositions :

There exist no integers a and b for which $21a + 30b = 1$.

There exist no integers a and b for which $18a + 6b = 1$.

Hint : Proof by Contradiction.

HW 2 Q 9. Prove the following propositions :

There exist no integers a and b for which $21a + 30b = 1$.

[redacted]

YOU

Proof by Contradiction.

Assume int a, b exists such that $21a + 30b = 1$

$3(7a + 10b) = 1$

int $\leftarrow$ $\boxed{7a + 10b} = \frac{1}{3}$ $\rightarrow$ NOT int

Contra-diction

HW 2 Q 9. Prove the following propositions :

you

[redacted]

There exist no integers a and b for which $18a + 6b = 1$.

Hint : Proof by Contradiction.

Assume : ~~int~~ int a, b

$18a + 6b = 1$

$3a + b = \frac{1}{6}$

int

Not int

Contradiction

HW 2 Q 10. Prove the following proposition :

If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

Hint : Proof by Contradiction.

HW 2 Q 10. Prove the following proposition :

Natural No.

If $b \in \mathbb{Z}$ and $b \nmid k$ for every $k \in \mathbb{N}$, then $b = 0$.

1, 2, 3, ...

Hint : Proof by Contradiction.

① If $b = 0$ then $b \nmid k$ for all $k \in N$

Does not Divide

$0 \mid 1$?

$1 = 0(k)$

int? → No

0×1

0×2 → $2 = 0(k)$

int? → No

② $b \neq 0$; $b \in Z$

then $b \mid b$; $b \mid -b$

$b = -5 \longrightarrow b \mid 5$

$b = 5 \longrightarrow b \mid 5$

$b \in \mathbb{Z}$; for all $k \in \mathbb{N}$

If $b \nmid k$ then $b = 0$

$b = -7 \longrightarrow b \mid 7$

$b = 8 \longrightarrow b \mid 8$

Prev. Q:

Every integer, except 0, Divides some Natural No.

$\equiv$ $\boxed{\text{If a int Does not Divide any Natural No then that int is 0.}}$

Given Prev. Q.

HW 2 Q 11. Prove the following proposition :

If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Hint : Proof by Contradiction.

HW 2 Q 11. Prove the following proposition :

If a and b are positive real numbers, then $a + b \geq 2\sqrt{ab}$.

Hint : Proof by Contradiction.

$\boxed{a+b < 2\sqrt{ab}}$

$(a+b)^2 < 4ab$

Important

why?

both sides positive

$a^2+b^2+2ab < 4ab$

$a^2+b^2-2ab < 0$

$$a^2 - 2ab + b^2 < 0$$

$$\boxed{(a-b)^2 < 0}$$

Contradiction!!

Never possible because a, b are real

Q: If $a < b$
then $a^2 < b^2$

True?

Q: If $a < b$ then $a^2 < b^2$ } True? No

$-4 < -3$

But $(-4)^2 \not< (-3)^2$

Note:

If a, b both positive.

then $\boxed{a < b \rightarrow a^2 < b^2}$

HW2 Q12. Prove the following proposition :

For every $n \in \mathbb{Z}$, $4 \nmid (n^2 + 2)$.

Hint : Proof by Contradiction.

Assume : $4 \mid n^2 + 2$

for you

$4 \mid n^2+2$

$n^2 + 2 = 4k$

int

even

even

even

$n = \text{even} = 2a$

$4a^2 + 2 = 4k$

$2a^2 + 1 = 2k$

even

even

odd = even

Contra-
-diction

HW 2 Q13.

both a,b are not odd

Prove the following proposition :

Suppose $a, b \in \mathbb{Z}$. If $4 \mid (a^2 + b^2)$, then a and b are not both odd.

$4 \mid a^2 + b^2$

Hint : Proof by Contradiction.

Assume both a, b odd

$a = 2m+1; \quad b = 2n+1$

$4 \mid a^2+b^2$ where $a = \underline{\underline{2m+1}}$; $b = \underline{\underline{2n+1}}$

$a^2+b^2 = 4k$

$4m^2+4n^2+4m+4n+2 = 4k$

$\underbrace{\boxed{2(m^2+n^2+m+n)}}_{\text{even}} \underline{\underline{+1}} = \underbrace{2k}_{\text{even}}$

odd = even

$\downarrow$

contradiction

HW1 Q 5.

A perfect number is a positive integer n such that the sum of the factors of n is equal to 2n (1 and n are considered factors of n). So 6 is a perfect number since $1 + 2 + 3 + 6 = 12 = 2 * 6$. Similarly, the divisors of 28 are 1, 2, 4, 7, and 14, 28 and $1 + 2 + 4 + 7 + 14+28 = 2^{*}28$.

perfect number

Prove that a prime number cannot be a perfect number.

Hint : What are the divisors of a prime number ?

6 → Perfect Number

factors of 6 = 1, 2, 3, 6

$1+2+3+6 = 2(6)$

Perfect Number: $n \longrightarrow$ Positive integer

$\boxed{2n = \text{Sum of All factors of } n}$

P : Prime No. → $P \geq 2$

factors: 1, P

$\boxed{1+P=2P}$

possible? No

$P+1 < P+P$

≥ 2

HW1 Q 6.

Prove that there does not exist an integer $n > 3$ such that n, $n+2$, $n+4$ are each prime.

i.e. For $n>3$, show that the integers n, $n+2$, and $n+4$ cannot all be prime (i.e. at least one of them must be Non-prime)

Hint : Every integer n can be written in one and only one of the following forms: $n=3k$ OR $n=3k+1$ OR $n=3k+2$, where k is some integer.

HW1 Q 6.

Prove that there does not exist an integer $n > 3$ such that $n, n+2, n+4$ are each prime.

i.e. For $n>3$, show that the integers $n, n+2$, and $n+4$ cannot all be prime (i.e. at least one of them must be Non-prime)

Hint : Every integer n can be written in one and only one of the following forms: $n=3k$ OR $n=3k+1$ OR $n=3k+2$, where k is some integer.

Every integer n:

$n = 2k$ OR $n = 2k+1$

Remainders when an int is Divided by 2:

0, 1

$0 \rightarrow n = 2k$

$1 \rightarrow n = 2k+1$

Every integer n :

$n = 3k$ OR $n = 3k+1$ OR $n = 3k+2$

Divide an int by 3 :

Possible Remainders : 0, 1, 2

$0 \rightarrow n = 3k$

$1 \rightarrow n = 3k+1$

$2 \rightarrow n = 3k+2$

Int $n > 3$

$n, \quad n+2, \quad n+4$

To prove:

At least one of them is Non-Prime

Case 1: If $n = 3k$ $\neq 1$

then n is Not Prime.

Case 1 Done

Int $n > 3$:

$n, \quad n+2, \quad n+4$

To prove:

At least one of them is Non-prime

Case 2: $n = 3k+1$

$n+2$ is Not prime

Case 2 Done

Int $n > 3$;

n, $n+2$, $n+4$

To prove:

At least one of them is Non-prime

Case 3: $n = 3k+2$

$n+4$ is NOT Prime

Case 3 Done

Int $n > 3$: $\rightarrow$ $\boxed{n, n+2, n+4}$ $\rightarrow$ At least one of them is Not prime.

$$\left\{\begin{array}{l} n = 3k \\ \text{OR} \\ n = 3k+1 \\ \text{OR} \\ n = 3k+2 \end{array}\right.$$

So; int $n > 3$

then $\boxed{n, n+2, n+4}$ $\rightarrow$ Can NOT be all primes.

HW1 Q 7.

Prove that if p, q are positive integers such that p|q and q|p, then p = q.

Important

$q = pa$ (a: +ve int)

$p = qb$ (b: +ve int)

$\Rightarrow p = pab \Rightarrow \boxed{ab = 1}$ (+ve int) $\Rightarrow a = b = 1$

$p = q$

Q:

$$a = ab \longrightarrow b = 1 \, ?$$

Q:

$a = ab \longrightarrow b = 1$?

No.

$0 = 0 * y \not\Rightarrow y = 1$

$\boxed{a = ab \longrightarrow b = 1}$ if $a \neq 0$

HW1 Q 8.

Prove that for any integer x, the integer $x(x+1)$ is even.

Case 1: $x = \text{even}$

$\boxed{x(x+1) = \text{even}}$

even

Case 2: $x = \text{odd}$

$x(x+1) = \text{even}$

even

Hope you liked the HWs.

From Tomorrow:

Discrete Mathematics

Main Course