Partial Order Relations

Next Topic:

Types of Lattices

Bounded, Complemented, Distributive

Website : https://www.goclasses.in/

Every lattice will satisfy :

Proposition

Any lattice has the following properties:

1. *Commutativity*: $a \cap b = b \cap a$ and $a \cup b = b \cup a$.
2. *Associativity*: $(a \cap b) \cap c = a \cap (b \cap c)$ and $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law*: $a \cap a = a$ and $a \cup a = a$.
4. *Absorption law*: $(a \cup b) \cap a = a$ and $(a \cap b) \cup a = a$.

Proposition 5.2.2 *If X is a lattice, then the following identities hold for all $a, b, c \in X$:*

$L1$	$a \vee b = b \vee a,$	$a \wedge b = b \wedge a$
$L2$	$(a \vee b) \vee c = a \vee (b \vee c),$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$L3$	$a \vee a = a,$	$a \wedge a = a$
$L4$	$(a \vee b) \wedge a = a,$	$(a \wedge b) \vee a = a.$

Properties (L1) correspond to commutativity, properties (L2) to associativity, properties (L3) to idempotence and properties (L4) to absorption. Further-

Let L be a lattice. Define the **meet** ($\wedge$) and **join** ($\vee$) operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y, and z of a lattice L.

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ **(commutative laws)**

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ **(associative laws)**

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ **(absorption laws)**

d) $x \wedge x = x$ and $x \vee x = x$ **(idempotent laws)**

Set Identities

A, B and C are sets, and we consider them to be subsets of a universal set U. Remember that $\emptyset$ is the empty set, and that A^c means"the complement" of A.

1. Commutative Laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

2. Associative Laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

3. Distributive Laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

4. Identity Laws:

$$A \cup \emptyset = A$$
$$A \cap U = A$$

5. Complement Laws:

$$A \cup A^c = U$$
$$A \cap A^c = \emptyset$$

6. Double Complement Laws:

$$(A^c)^c = A$$

7. Idempotent Laws:

$$A \cup A = A$$
$$A \cap A = A$$

8. Universal Bound Laws:

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$

9. De Morgan's Laws:

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

10. Absorption Laws:

$$A \cup (A \cap B) = A$$
$$A \cap (A \cup B) = A$$

11. Complements of U and $\emptyset$:

$$U^c = \emptyset$$
$$\emptyset^c = U$$

12. Set Difference Law:

$$A - B = A \cap B^c$$

The table below shows the most important logical equivalences of propositional logic. These equivalences are also often referred to as *logical axioms* and *logical identities*. For the sake of readability, 1 represents `true`, 0 represents `false`, and $=$ represents $\leftrightarrow$. Notice that each equivalence has a corresponding **dual** equivalence that is obtained by replacing $\vee$ with $\wedge$ (or $\wedge$ with $\vee$) and 1 with 0 (or 0 with 1). For example, the dual of $p \vee 1 = 1$ is $p \wedge 0 = 0$. Notice also that these equivalences match the set identities from the Sets lecture, with $\vee$ corresponding to $\cup$, $\wedge$ to $\cap$, 1 to $\mathcal{U}$, and 0 to $\emptyset$. In the parlance of mathematics we say that, together with their respective operations, the set $\mathcal{L}$ of all logical propositions and the set $\mathcal{S}$ of sets are both examples of a mathematical structure called a **Boolean algebra**, and the table below lists the axioms (expressed in logical form) of such an algebra.

Equivalence	**Name**
$p \wedge 1 = p$ $p \vee 0 = p$	Identity
$p \vee 1 = 1$ $p \wedge 0 = 0$	Domination
$p \vee p = p$ $p \wedge p = p$	Idempotency
$\neg(\neg p) = p$	Double negation
$p \vee q = q \vee p$ $p \wedge q = q \wedge p$	Commutativity
$(p \vee q) \vee r = p \vee (q \vee r)$ $(p \wedge q) \wedge r = p \wedge (q \wedge r)$	Associativity
$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$	Distributivity
$\neg(p \vee q) = \neg p \wedge \neg q$ $\neg(p \wedge q) = \neg p \vee \neg q$	De Morgan

Logic Equivalences

GO Classes

Equivalence	Name
$p \wedge T \equiv p, \quad p \vee F \equiv p$	Identity laws
$p \vee T \equiv T, \quad p \wedge F \equiv F$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q$ $\neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv T, \quad p \wedge \neg p \equiv F$	Negation laws

Identity Property:

fixes $\forall a$

we say "e" is Identity element for operation # iff

$$\boxed{\begin{aligned} a \# e &= a \\ e \# a &= a \end{aligned}} \quad \forall a$$

Addition: $\longrightarrow$ $e = 0$

$\forall a, \quad a + 0 = a = 0 + a$

Multiplication $\longrightarrow$ $e = 1$

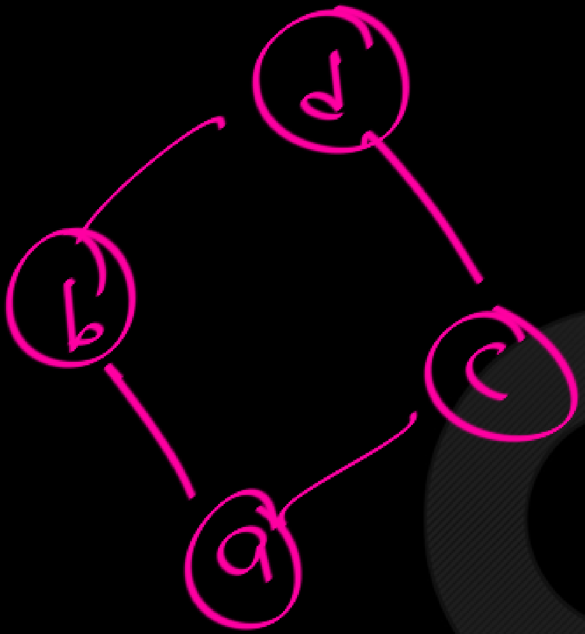
$\forall a, \quad a \times 1 = a = 1 \times a$

Subtraction: $\longrightarrow$ $e = DNE$

$a - 0 = a \neq 0 - a$

Division: $\longrightarrow$ e = DNE

$$a \div 1 = a \neq 1 \div a$$



Identity element for GLB := d ??

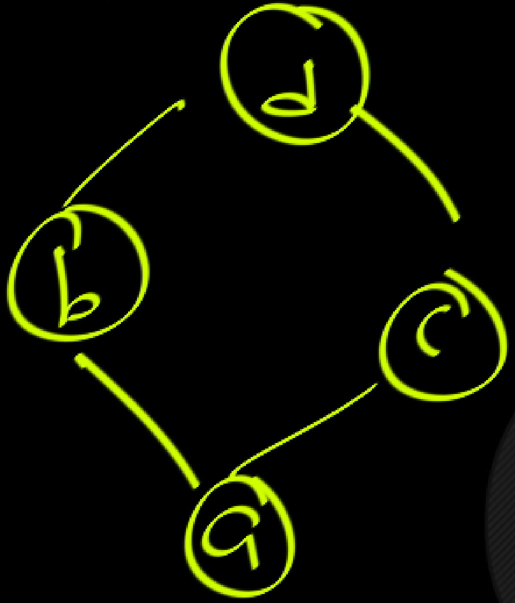
$$\boxed{x \wedge e = x} \quad \forall x$$

$$e \wedge x = x$$

fixes for all elements

Lattice ✓

Commutative ✓



$d \wedge d = d$

$b \wedge d = b$

$c \wedge d = c$

$a \wedge d = a$

d Is for GLB

Lattice: $(N, \leq)$

Id. for GLB = ?

Id. for LUB = ?

Lattice: $(N, \leq) \Rightarrow a \vee b = \max(a, b)$

Id. for GLB = ? — DNE

Id. for LUB = ? — 1

$1 \vee b = b$

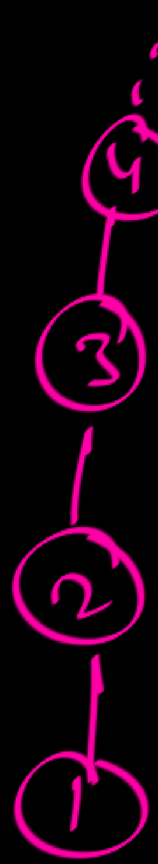
$1 \vee 1 = 1$
$1 \vee 2 = 2$

$1 \vee 3 = 3$

$\forall x, \; 1 \vee x = x$

→ Id for $\vee$ operation

$(\mathbb{N}, \leq)$ — Toset ✓



$2 \wedge 10 = 2$

$2 \vee 10 = 10$

$2 \vee 100 = 100$

$2 \vee 1 = 2$

$a \vee b = \max(a, b)$

$a \wedge b = \min(a, b)$

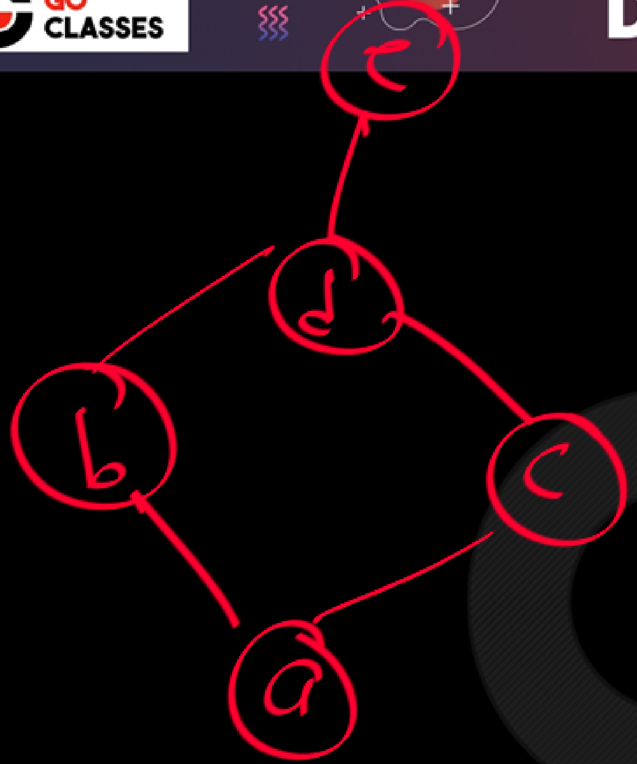
$(N, \leq)$

$a \wedge e = a \;, \forall a$

$\min(a, e) = a \;, \forall a$

There is no such "e".

Why? because if there is such "e" then $e \wedge (e+1) = e \neq e+1$



Is for $\vee = a$

Is for $\wedge = e$

$$\left.\begin{array}{l}\forall x,\ x \vee a = x \\ \forall x,\ x \wedge e = x\end{array}\right\}$$

$e \wedge e = e$

$e \wedge d = d$

$e \wedge c = c$

$(\mathbb{Z}, \leq)$

Id for $\vee$ —— DNE

Id for $\wedge$ —— DNE

Q: In a lattice, what is Identity element for GLB?

Ans: Greatest element

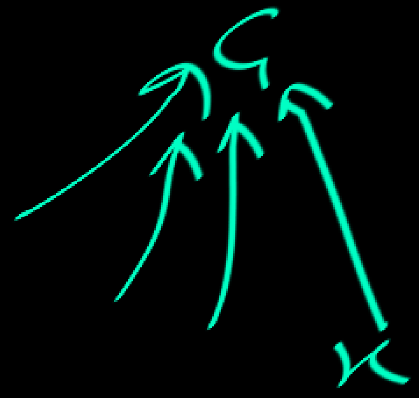
Q: In a lattice, what is Identity element for LUB ?

— Least element

If G is Greatest in a lattice

then

$\forall x$, $x \wedge G = x$

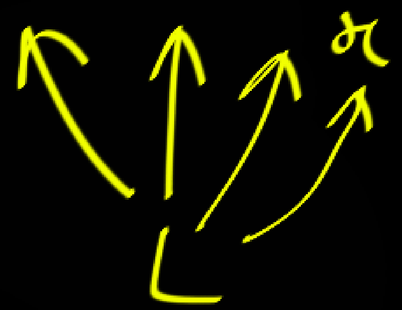


So, G is Identity element for GLB.

If L is least in a lattice

then

$\forall x$, $x \vee L = x$



So, L is Identity element for LUB

Conclusion: In any lattice:

Identity element for GLB = Greatest

" " " LUB = Least

Q: In lattice, Greatest Definitely exists? — No.

Eg: $(Z, \leq)$ — lattice

no Greatest

Bounded lattice:

- Lattice with Greatest and least element.

- Lattice with Identity element for both GLB, LUB.

Greatest G

$\forall x,\quad x \wedge G = x$

Least L

$\forall x,\quad x \vee L = x$

Logic

for smart

Trick: —— for Dumb

Greatest = Is element for GLB

Least = Is element for LUB

Partial Order Relations

Next Sub-Topic:

Bounded Lattice

(Identity element, Domination Law for Join, Meet)

Website : https://www.goclasses.in/

A lattice L is **bounded** if it has both an **upper bound**, denoted by 1, such that $x \preccurlyeq 1$ for all $x \in L$ and a **lower bound**, denoted by 0, such that $0 \preccurlyeq x$ for all $x \in L$.

41. Show that if L is a bounded lattice with upper bound 1 and lower bound 0 then these properties hold for all elements $x \in L$.

a) $x \vee 1 = 1$ **b)** $x \wedge 1 = x$

c) $x \vee 0 = x$ **d)** $x \wedge 0 = 0$

42. Show that every finite lattice is bounded.

Bounded :

$(\{1, 2, 3, 6\}, |)$ ✓

$([1, 2], \leq)$ ✓

$(N, \leq)$ ✗ — no Greatest

$(Z, \leq)$ ✗ — no Greatest
no least

Domination law:

$\mathcal{U}$ - universal set

In Set Theory:

$$S \cup \mathcal{U} = \mathcal{U}$$
$$S \cap \phi = \phi$$

In Prop. logic:

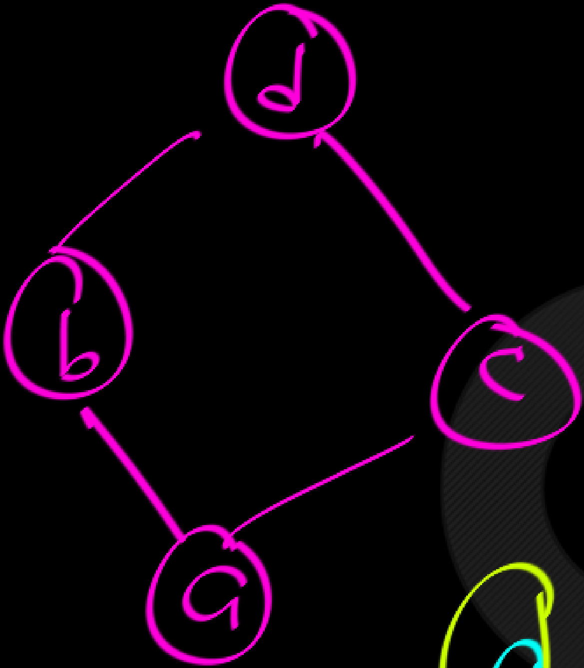
$$P \vee T = T$$
$$P \wedge F = F$$

Universal set (M) in "Dominator" for union operation.

$\forall$ sets S, $S \cup M = M$

Empty set ϕ is "Dominator" for intersection operation.

$\forall$ sets S, $S \cap \phi = \phi$



Dominator for GLB =

$d \wedge b = b \neq d$

not Dominator for GLB.

$a \wedge d = a$

$a \wedge a = a$
$a \wedge b = a$
$a \wedge c = a$

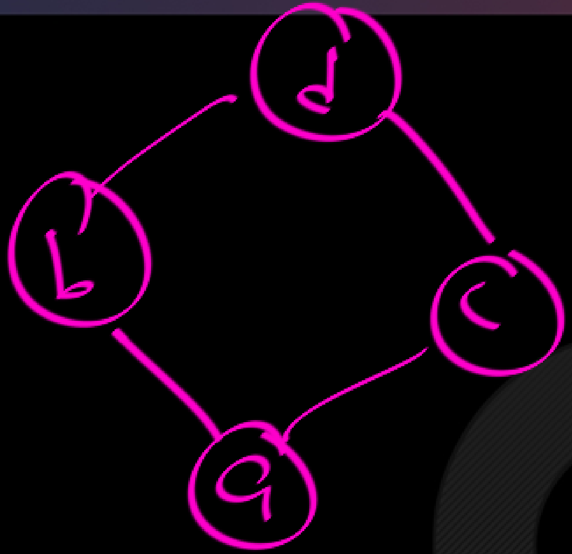
a is the Dominator for GLB.

Dominator (D) for GLB : fixes $\forall x$

$$\forall x, \quad \boxed{x \wedge D = D}$$

Dominator (D) for LUB :

$$\forall x, \quad x \vee D = D$$



Dominator for LUB

$= d$

$d \vee d = d$

$b \vee d = d$

$c \vee d = d$

$a \vee d = d$

$(N, \leq)$ — Dominator for GLB := 1

$1 \wedge 1 = 1$

$1 \wedge 2 = 1$

$1 \wedge 3 = 1$

$\vdots$

$1 \wedge x = 1, \forall x$

Dominator for GLB

$(N, \leq)$ — Dominator for LUB

$= \underline{DNE}$

no element D

$$\boxed{x \vee D = D, \ \forall x}$$

why?

$(D+1) \vee D = D+1 \neq D$

Q: In a lattice,

Dominator of GLB =

" " LUB =

Q: In a lattice,

Dominator of GLB = Least

" " LUB = Greatest

Least L :

Dominator for GLB

$$x \wedge L = L, \quad \forall x$$

Greatest G :

Dominator for LUB

$$x \vee G = G, \quad \forall x$$

Bounded lattice:

Def 1: Lattice with 1, 0

Greatest

least

Def 2: Lattice with Ids for GLB, LUB

Def 3: " " Dominator for GLB, LUB.

In a Lattice:

Greatest $\equiv$ Is element for GLB
$\equiv$ Dominator for LUB

In a Lattice:

Least ≡ Id element for LUB

≡ Dominator for GLB

DEFINITION 7.2 - 4: ***Bounded Lattices***

A lattice $\langle \mathcal{A}, \leq \rangle$ is **bounded** *iff it has a minimum element and a maximum element. These are denoted by* 0 *and* 1 *respectively.*

The extreme elements of bounded lattices interact with other elements of the lattice in the obvious ways, captured by the next proposition.

PROPOSITION 7.2 - 3: ***Extreme Elements in a Bounded Lattice***

Suppose $\langle \mathcal{A}, \leq \rangle$ is a bounded lattice having minimum 0 *and maximum* 1*, and let x be any element in $\mathcal{A}$. Then*

a) $0 \vee x = x = x \vee 0; \qquad 1 \wedge x = x = x \wedge 1$

b) $0 \wedge x = 0 = x \wedge 0; \qquad 1 \vee x = 1 = x \vee 1$

Partial Order Relations

Next Sub-Topic:

Complemented Lattice

Website : https://www.goclasses.in/

DEFINITION 7.2 - 4: ***Bounded Lattices***

A lattice $\langle \mathcal{A}, \leq \rangle$ *is* ***bounded*** *iff it has a minimum element and a maximum element. These are denoted by* 0 *and* 1 *respectively.*

DEFINITION 7.2 - 5: ***Complements in a Bounded Lattice***

Suppose $\langle \mathcal{A}, \leq \rangle$ *is a bounded lattice with minimum* 0 *and maximum* 1. *A* ***complement of an element x*** *is an element* z *such that* $x \wedge z = 0$ *and* $x \vee z = 1$.

DEFINITION 7.2 - 6: ***Complemented Lattices***

A bounded lattice $\langle \mathcal{A}, \leq \rangle$ *is* ***complemented*** *iff every element has a complement.*

at least one Complement

Complement of an element "a":

"b" is Complement of a iff

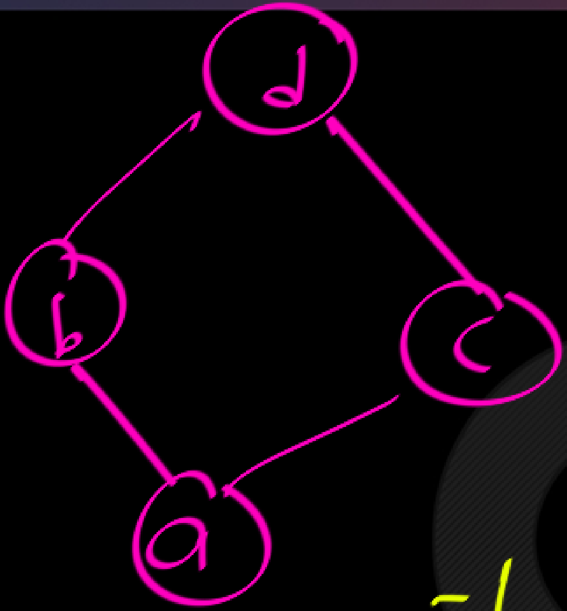
$$\begin{cases} a \vee b = 1 & \text{Greatest} \\ a \wedge b = 0 & \text{least} \end{cases}$$

In Bounded Lattice

If lattice is not bounded then Complement Can not be Defined.

In a bounded lattice:

a $\quad b = a'$ iff $\boxed{\begin{array}{l} a \vee b = G \\ a \wedge b = L \end{array}}$



Bounded lattice L

Complement of b ; = c

$$a^{-1} = d \quad \boxed{\begin{matrix} a \vee d = d \\ a \wedge d = a \end{matrix}}$$

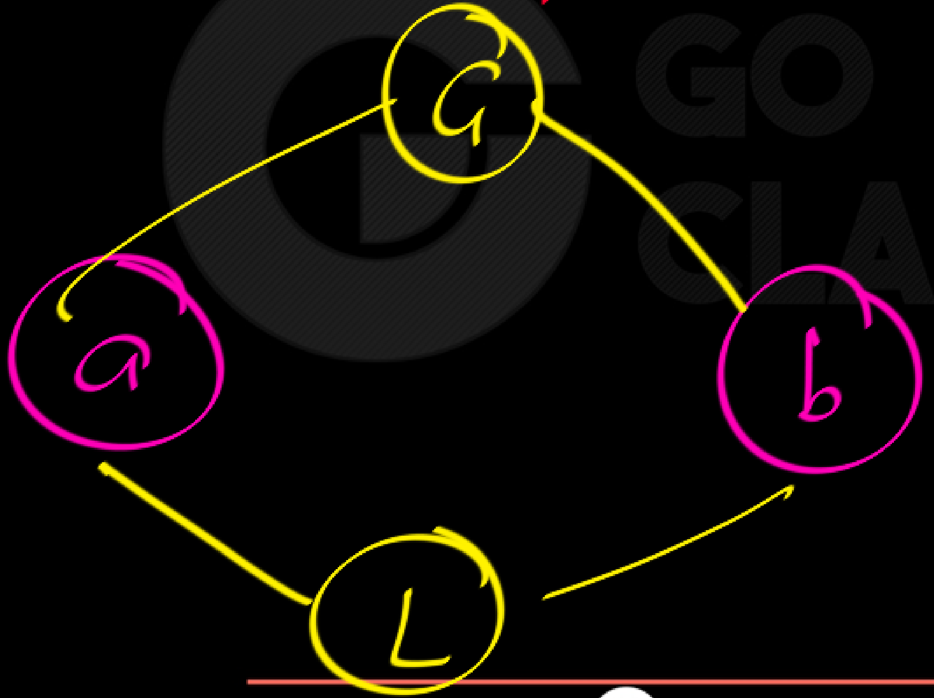
$b \vee c = d = G$

$b \wedge c = a = L$

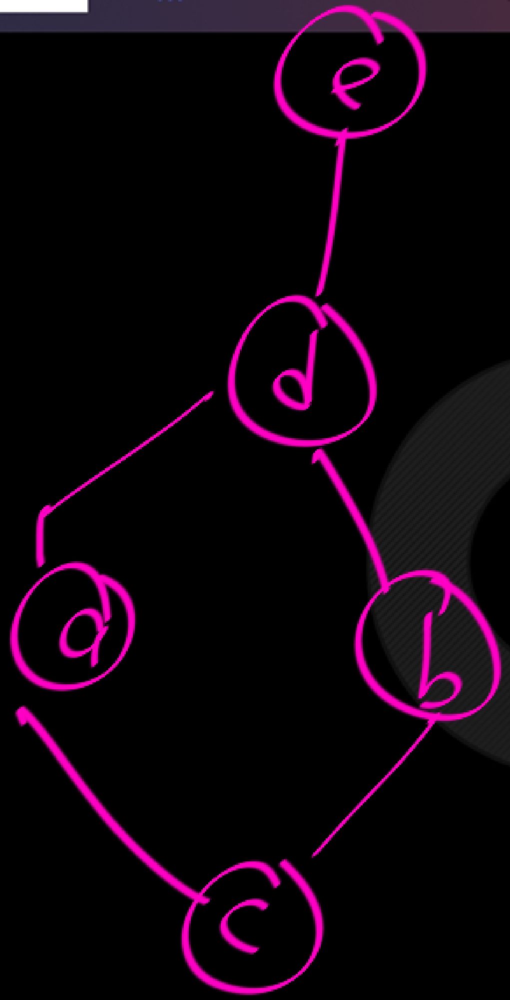
$b^{-1} = c$; c^{-1} = Complement of c = b

Idea: In a bounded lattice :

Complement of a = $\bar{a}'$



$\underbrace{b = \bar{a}'}$



a^{-1} = Complement of a

$a \vee b = d \neq e$

$a \wedge b = c$

$b \neq a^{-1}$

$a^{-1} = DNE$

$c^{-1} = e$

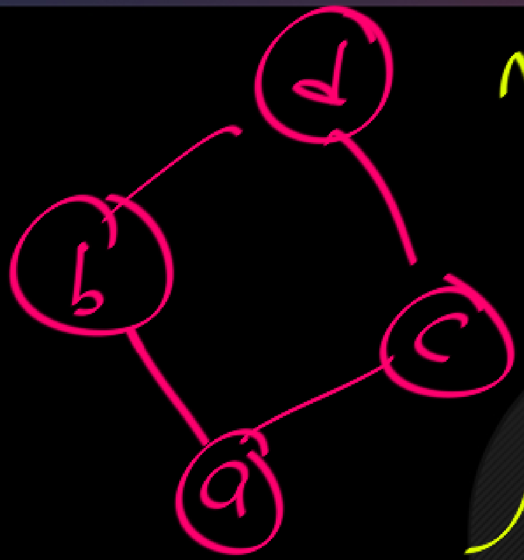
$e^{-1} = c$

$d^{-1} = DNE$

$$(\text{least})^{-1} = \text{Greatest}$$

$$(\text{Greatest})^{-1} = \text{least}$$

$$\boxed{a^{-1} = b \iff b^{-1} = a}$$



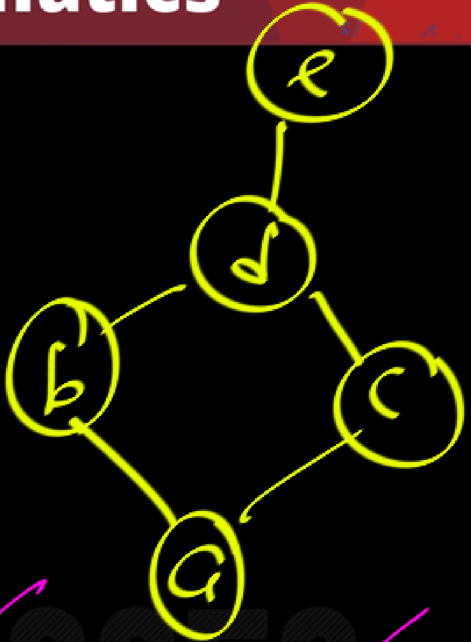
Complemented Lattice

$a' = d$

$d' = a$

$b' = c$

$c' = a$



$a' = e$

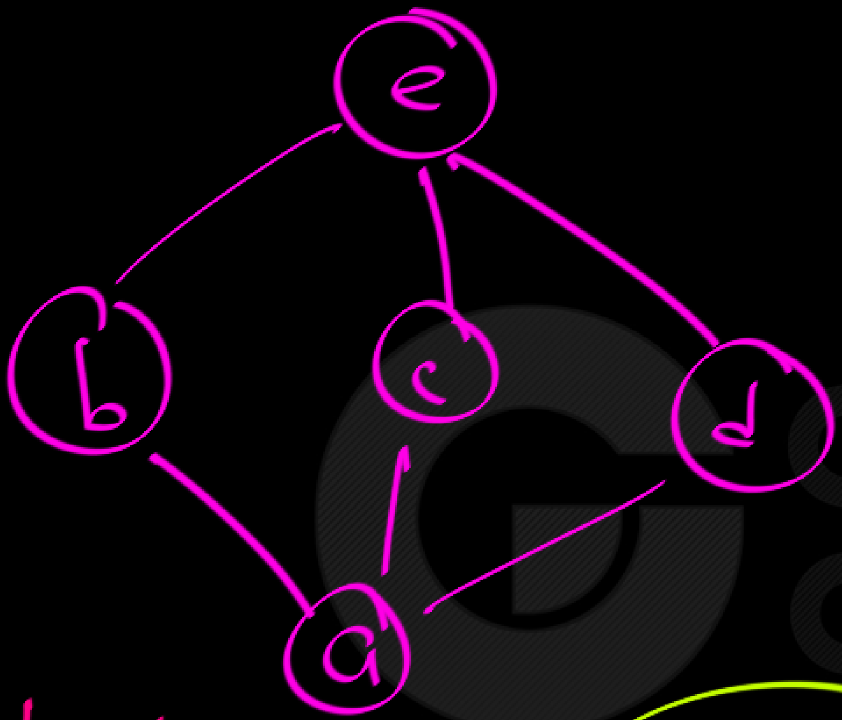
$e' = a$

$b' = \text{DNE}$

$c' = $ "

$d' = $ "

Not Complemented Lattice



$b' = c, d$

Complemented lattice

$b \vee c = e$
$b \wedge c = a$

so $b' = c$

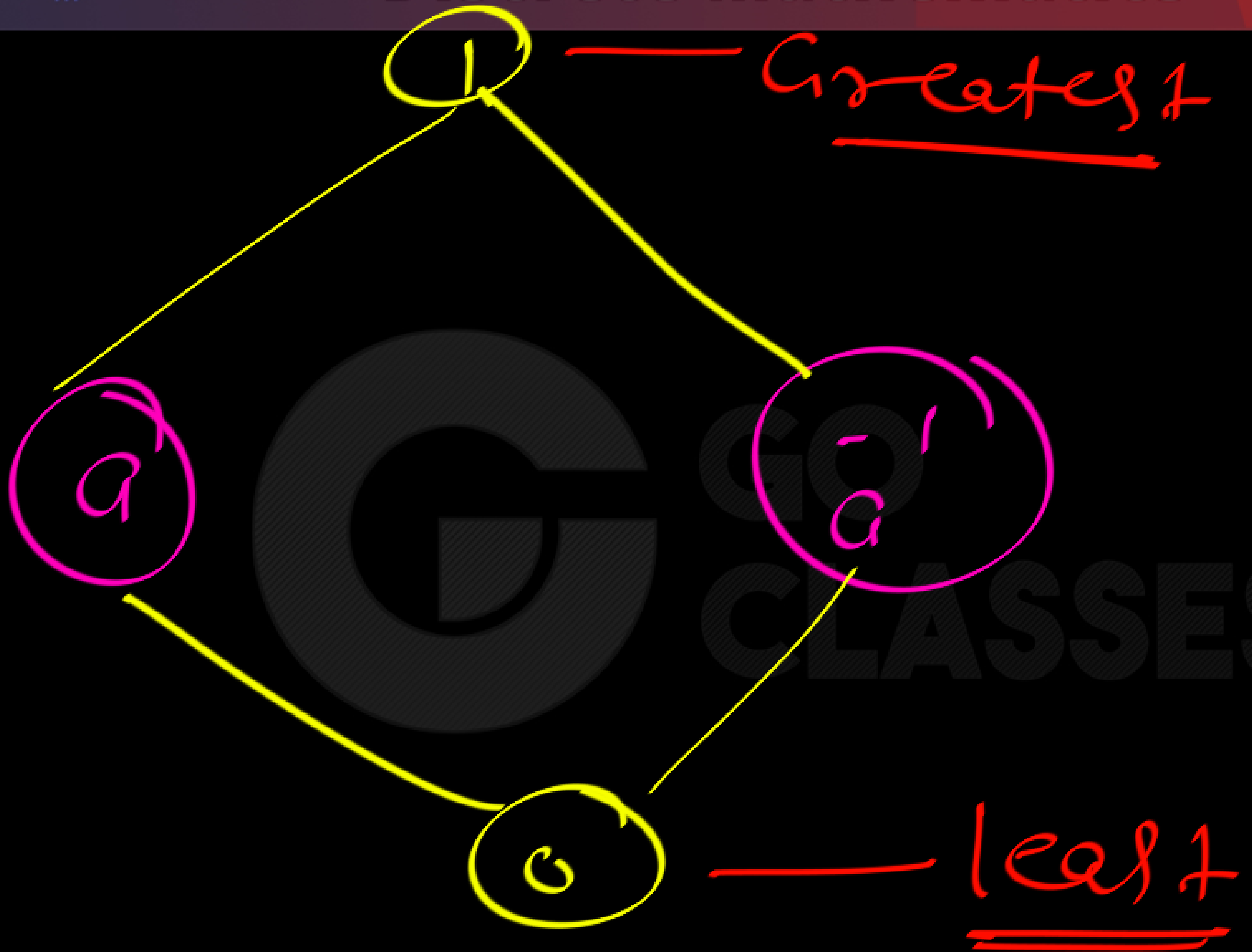
$c' = b, d$

$d' = c, b$

$a' = e, \; e' = a$

$b \vee d = e$
$b \wedge d = a$

so $b' = d$


1
Greatest
a
$\bar{a}'$
0
least

Complemented lattices (cont'd)

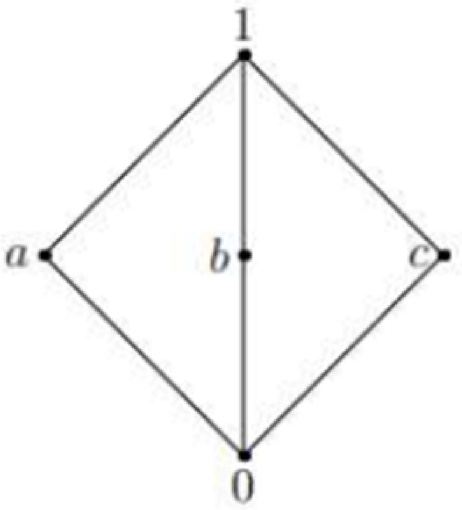


Figure: A complemented lattice where some complements are not unique

$\bar{1}' = 0$

$\bar{c}' = a, b$

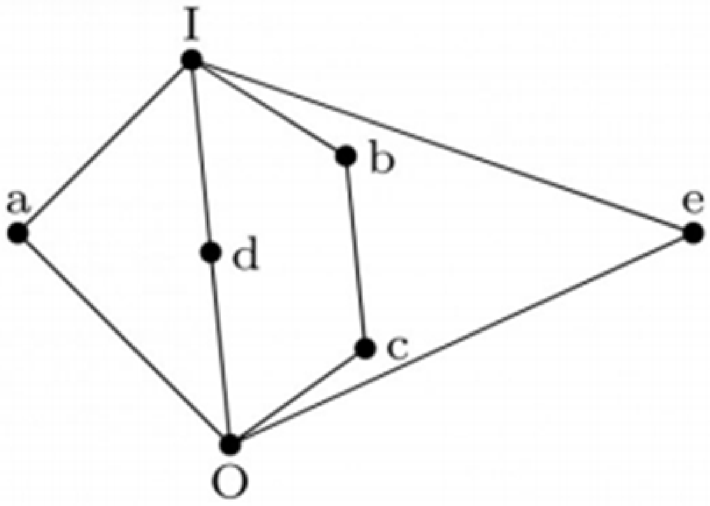
$\bar{a}' = b, c$

$\bar{b}' = a, c$

$\bar{0}' = 1$

4.6.1 Lattice: GATE1988-1vii

The complement(s) of the element 'a' in the lattice shown in below figure is (are) ____



4.6.1 Lattice: GATE1988-1vii

The complement(s) of the element 'a' in the lattice shown in below figure is (are) _____

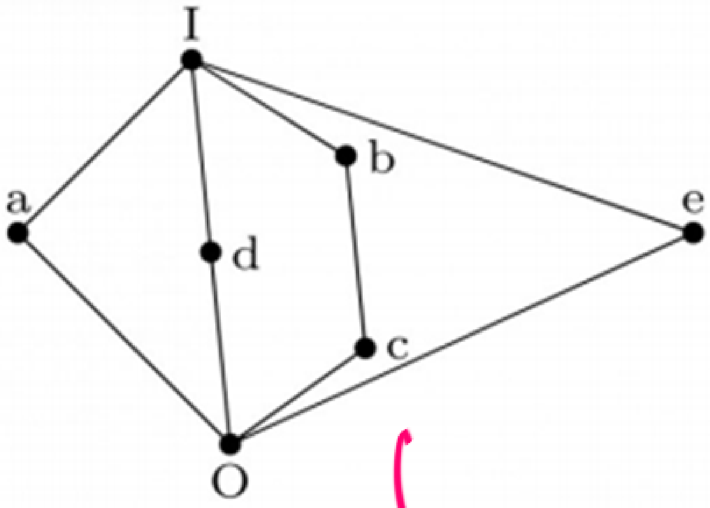
$\bar{0}' = 1$

$\bar{1}' = 0$

$\bar{b}' = a, d, e$

$\bar{e}' = a, d, b, c$

$\bar{a}' = d, b, c, e$



$\boxed{a \wedge b = 0 \quad a \vee b = 1}$

so

$\bar{a}' = b$

$\bar{d}' = a, b, c, e$

$\bar{c}' = a, d, e$

Complemented lattice

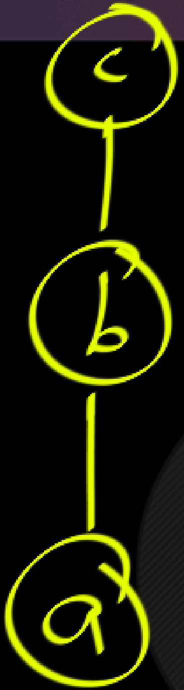
$\bar{0} = 1$?

$$\boxed{\begin{array}{l} 0 \wedge 1 = 0 = \text{least} \\ 0 \vee 1 = 1 = \text{Greatest} \end{array}}$$

So, $\bar{0} = 1$

$\bar{1} = 0$

Eg:



Not Complemented Lattice.

$b^{-1} = ?$

$b^{-1} = DNE$

$b^{-1} \neq b$ why?

$b \wedge b \neq$ least

$b^{-1} \neq c$ why?

$b \wedge c \neq$ least

$b^{-1} \neq a$ why?

$b \vee a \neq$ greatest

Q: If aRb, and $a \neq 0$, $b \neq 1$

then is it possible $a^{-1} = b$? —— No

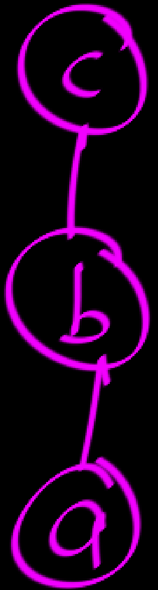
$$\begin{array}{c} b \\ | \\ a \end{array}$$

$a \wedge b = a \neq 0$

$a \vee b = b \neq 1$

So $a^{-1} \neq b$

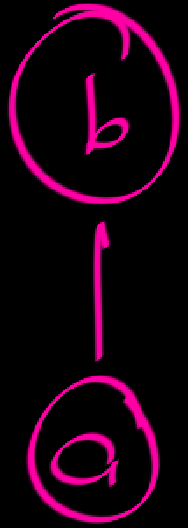
Q: Can a Toset be Complemented lattice? — iff ≤ 2 elements.



$\overline{b}' = \text{DNE}$

$\overline{b}' = \text{DNE}$

$\overline{c}' = \text{DNE}$



$\left.\begin{array}{l} \bar{a}' = b \\ \bar{b}' = a \end{array}\right\}$ Complemented

Single element Lattice :

(a) } Bounded or not?

Author 1	Author 2
$0 \neq 1$ in bounded Lattice	0, 1 Can be same. in bounded lattice.

least, Greatest must be Diff

We will follow this.