Set Theory

Next Chapter:

Special Types of Relations

(Reflexive, Symmetric, Transitive etc)

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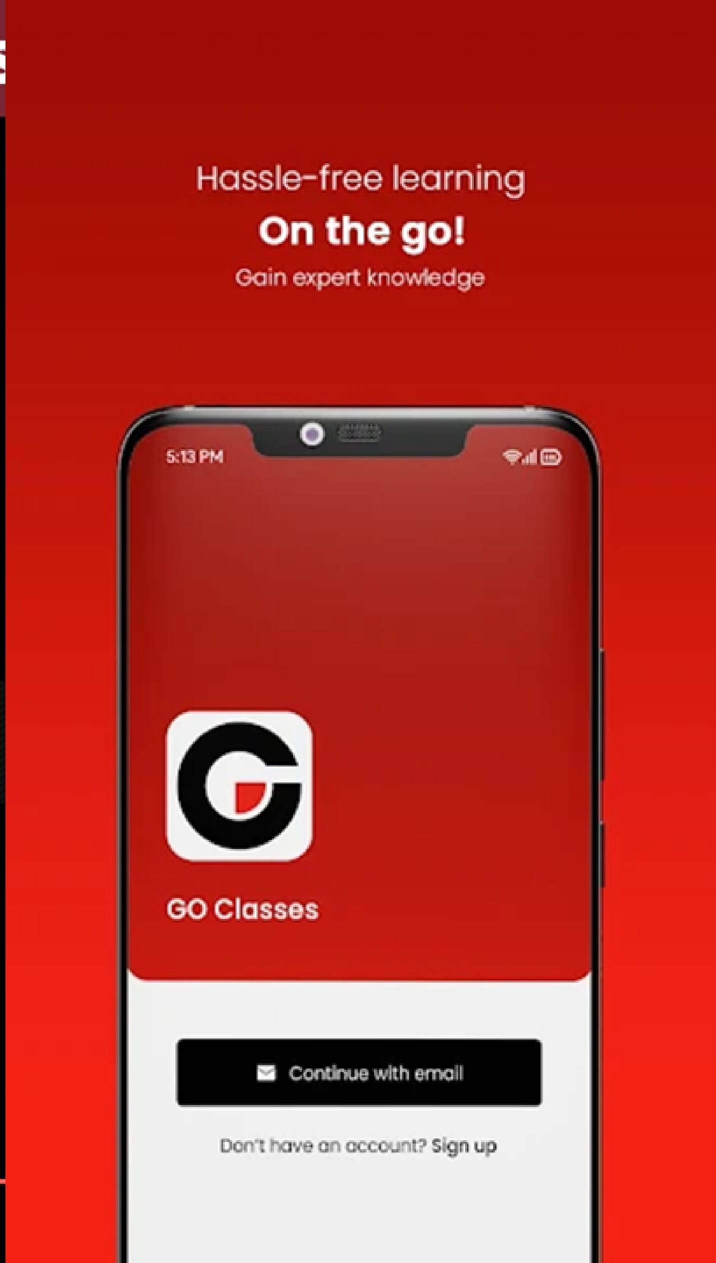

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Next Topic:

Practice (Relations)

1.

Relations, in mathematics, are one-way statements.

When we say "x is related to y", it doesn't mean "y is related to x".

For Example:

Define a relation R "son of" on Set of All People.
i.e. aRb iff a is son of b.

NOTE that If A is related to B, that means A is son of B… So, B is not related to A as B is not the son of A.

Relations in mathematics are one-way statements. When we say "x is related to y", doesn't mean "y is related to x".

For Example:

Define a relation T on Set of all integers as follows:

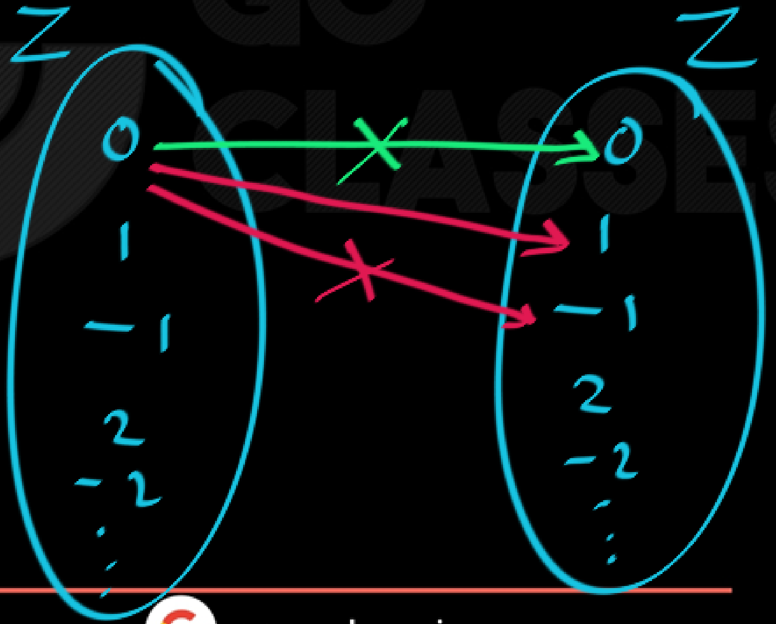
xTy iff $x < y$ Rule

$T: Z \to Z$; $T \subseteq Z \times Z$

0T0 : false

means $0 < 0$

false



1 ~~R~~ 0

0 R 1 ✓

0 ~~R~~ -1

0 ~~R~~ 0

$T : Z \to Z$

Rule: xTy iff $x < y$

x is Related to y under Relation T

$0 T 1$ ✓

$1 \cancel{T} 0$ ✓

$0 T 10$ ✓

$10 \cancel{T} 0$ ✓

"Relates" is one way.

$x R y$: x is Relates to y under relation R.

Note: Relation R on set A. → Base set

$a, b \in A$; Assume aRb

$R : A \to A$

① $\boxed{a, b \text{ are Related.}}$ Nonsense

② a is Related to b. ✓

2.

Binary Relation:

A relation between two sets (same or different)

R: A → B

or

R: A → A

Binary Relation:

$R : A \to B$

binary Relation

(A, B may be same OR Diff)

$\boxed{R \subseteq A \times B}$

binary Relation

Relation R on set A

$$\boxed{R : A \to A} \equiv R \subseteq A \times A$$

binary Relation

3.

Relation on a Set S:

A relation which relates elements of S to elements of S.

Relation R is on set A

$R: S \rightarrow S$

Here, S is called the Base Set.

$R_1 : N \rightarrow N$: R_1 is on set N.

$R : N \rightarrow N$

Base Set

$R_2 : N \rightarrow Z$:

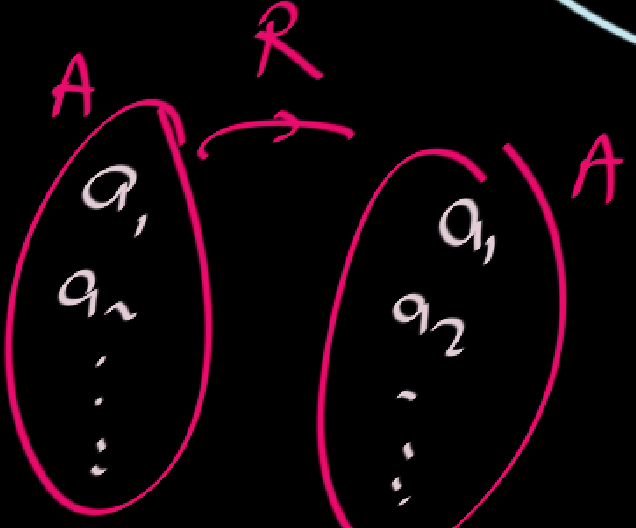
Don't use the word "Base set"

$$\boxed{R : A \longrightarrow A}$$

$\equiv$ R is on set A

Base set of Rel. R

$\equiv R \subseteq A \times A$



For sets A, B, any subset of $A \times B$ is called a (binary) relation from A to B.

Any subset of $A \times A$ is called a (binary) relation on A.

When “R” is a relation from Set A to Set B (Note that set A,B may be same as well) then R relates elements of A to elements of B, by some rule.

$R : A \rightarrow B$: $\boxed{R \subseteq A \times B}$

$R : A \rightarrow A$ $\boxed{R \subseteq A \times A}$

RELATION OR BINARY RELATION

Let A and B be two non-empty sets. A binary relation or simply a relation from A to B is a subset of $A \times B$. Given $x \in A$ and $y \in B$, we write $x \, R \, y$ if $(x, y) \in R$ and $x \not{R} \, y$ if $(x, y) \notin R$. If R is relation from A to A, then R is said to be relation on A.

A binary relation on a set S is a subset of $S \times S$.
A binary relation on a set S relates elements of S to elements of S. Here, S is called the Base Set, since, On S, we have defined the relation.
Such relations are very important in mathematics and in computer science.

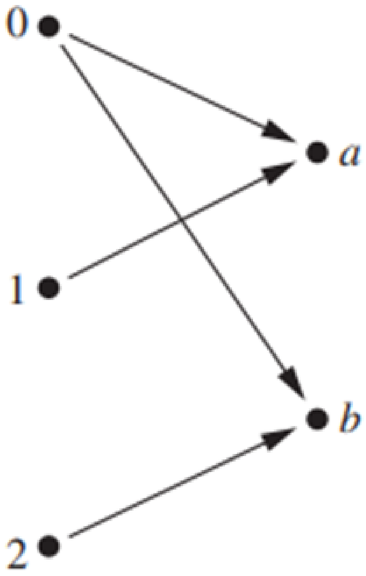
A **binary relation** R between two sets A and B (which may be the same) is a subset of the Cartesian product $A \times B$. If element a ∈ A is related by R to element b ∈ B, we denote this fact by writing **(a, b) ∈ R**, or alternately, by **a R b**. We say that **R is a relation on A and B**.

A **relation on a set A** is a subset of A × A.

Let A and B be sets. A *binary relation from* A *to* B is a subset of $A \times B$.

In other words, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B. We use the notation $a\,R\,b$ to denote that $(a, b) \in R$ and $a \not R\, b$ to denote that $(a, b) \notin R$. Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from A to B. This means, for instance, that $0\, R\, a$, but that $1 \not R\, b$. Relations can be represented graphically, as shown in Figure 1, using arrows to represent ordered pairs. Another way to represent this relation is to use a table, which is also done in Figure 1. We will discuss representations of relations in more detail in Section 9.3. ◀



R	a	b
0	×	×
1	×	
2		×

Relations on a Set

Relations from a set A to itself are of special interest.

A *relation on a set* A is a relation from A to A.

In other words, a relation on a set A is a subset of $A \times A$.

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) \mid a \text{ divides } b\}$?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}.$$

Example:

Let A = {1, 3, 5} be a set.

- Every relation on A is a subset of A × A.
- Every subset of A × A is a relation on set A.

$A \times A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$.

$A \times A = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5)\}$

Every subset of A x A is a Relation on set A

$A \times A \subseteq A \times A$

same

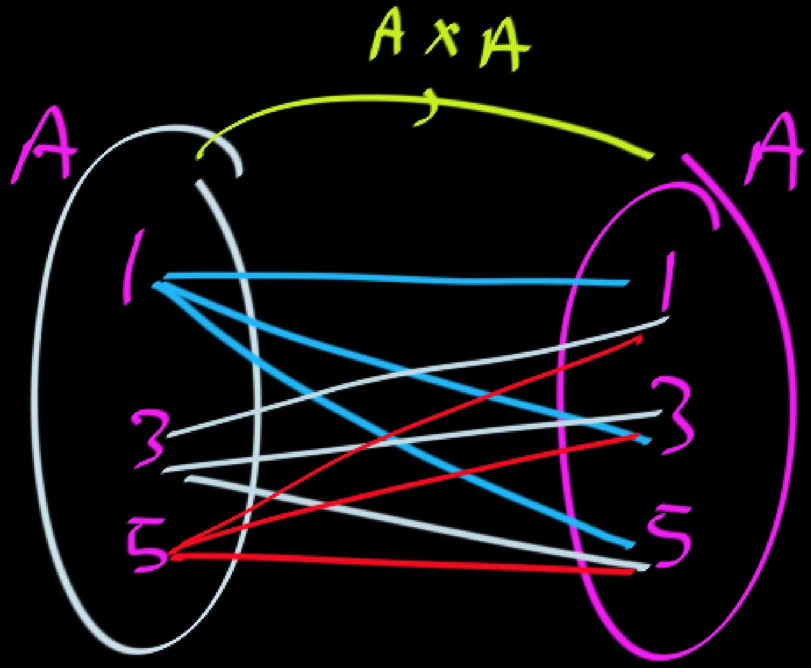
Also a Relation from A to A.

$A \times A$: a Relation on A.

Every Relation R on A is subset of $A \times A$

$A \times A$: Universal Relation on A
(full Relation on A)

$A : \{1, 3, 5\}$



$A \times A : A \to A$

a Relation on A

Relation

Set A : Base set :

Some Relations on Set A :

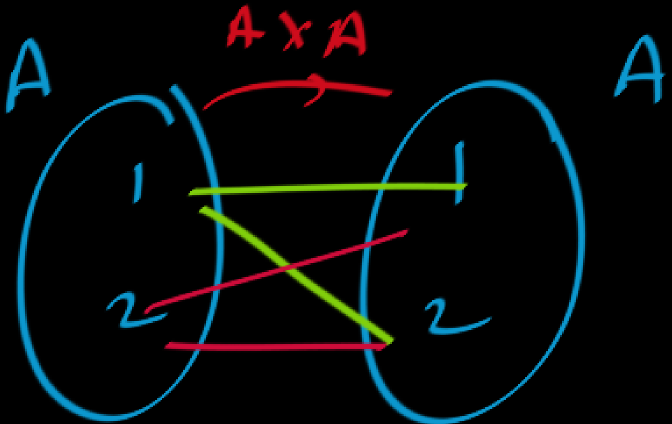
① A x A : full Relation (Universal Relation)

② ϕ : Empty Relation

$A : \{1, 2\}$

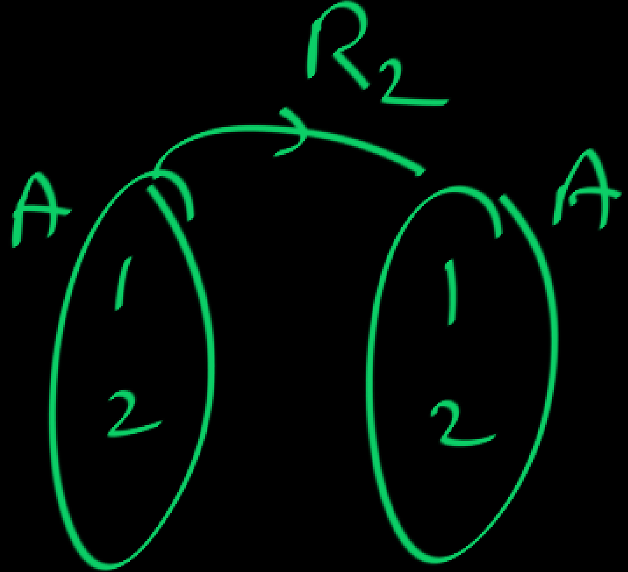
Full Relation on A

$A \times A$



Empty Relation on A

$R_2 = \phi = \{ \quad \}$



Example:
Let $A = \{1, 3, 5\}$ be a set. Then

$A \times A = \{(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)\}$.

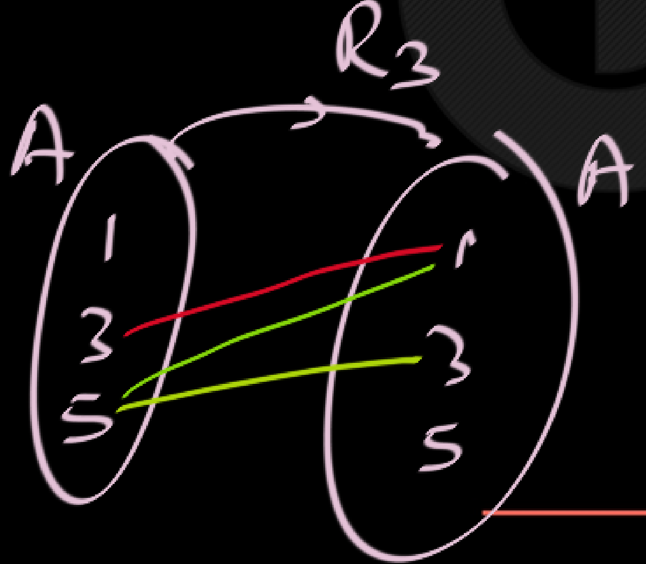
Any relation on A, is subset of $A \times A$.

For example, Let's define relation R_3 on A as: $x \, R_3 \, Y$ iff $x > y$

$A : \{1, 3, 5\}$

Relation R_3 on A : $\equiv$ $R_3 : A \to A$

$\equiv$ $R_3 \subseteq A \times A$



$x R_3 y$ iff $x > y$

$R_3 = \{(3,1), (5,1), (5,3)\}$

$\boxed{R_3 \subseteq A \times A}$

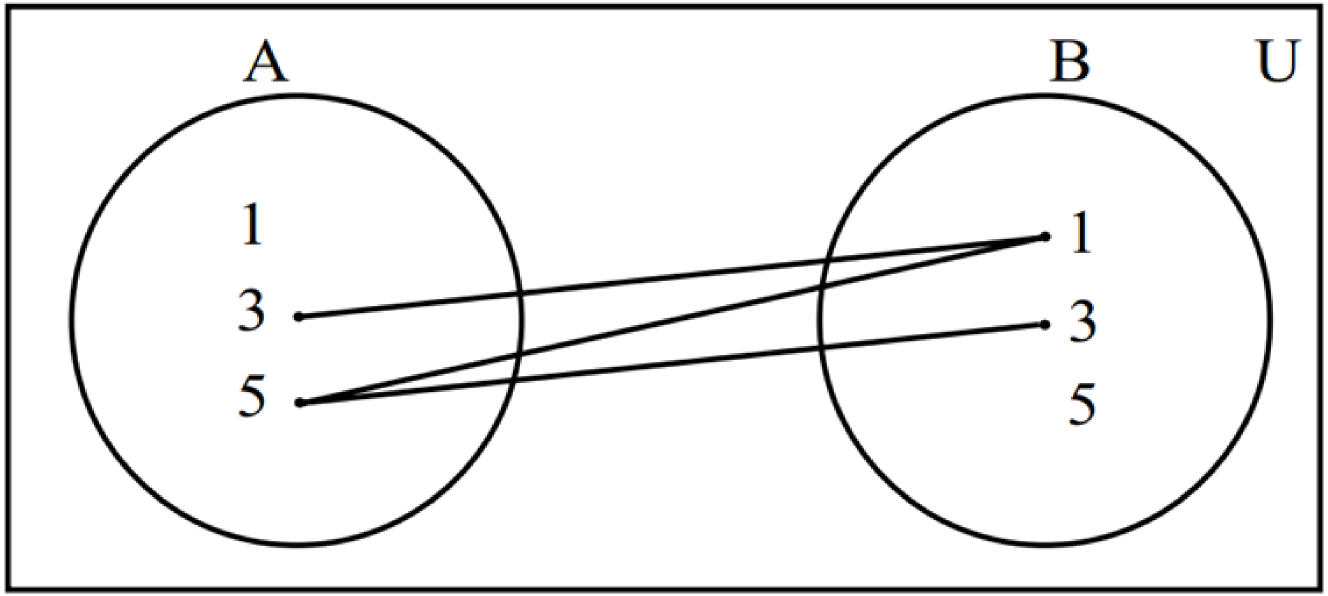
$|R_3| = 3$

Let A = {1, 3, 5} be a set. Then

A × A = {(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)}.

If $x > y$. The relation $x R_3 y$ means the ordered pairs in which the first element is greater than the second. These are (3, 1), (5, 1), and (5, 3).

$\therefore$ R_3 = {(3, 1), (5, 1), (5, 3)}, a subset of A × A and so is a relation of A.



Problem 1. The following table describes a binary relation. Find the set of ordered pairs that is this relation, as in the definition of a binary relation.

$\sim$	1	2	3	4	5	6
1	*					*
2		*				
3				*	*	
4			*		*	
5			*	*		
6	*					*

Problem 1. The following table describes a binary relation. Find the set of ordered pairs that is this relation, as in the definition of a binary relation.

matrix Representation of Relation ~

~	1	2	3	4	5	6
1	*					*
2		*				
3				*	*	
4			*		*	
5			*	*		
6	*					*

$1 \sim 1$

$1 \sim 6$ ($1 R 6$)

$3 \sim 5$

$4 \sim 5$

$6 \sim 6$

$6 \sim 1$

$2 \sim 2$

Relation

R

Problem 1. The following table describes a binary relation. Find the set of ordered pairs that is this relation, as in the definition of a binary relation.

Matrix Representation of Relation ~

~	1	2	3	4	5	6
1	*					*
2		*				
3				*	*	
4			*		*	
5			*	*		
6	*					*

$|\sim| = 11$

$4 \sim 3$

$4 \not\sim 2$; $2 \sim 2$

$\sim = \{(1,1), (1,6), (2,2), (3,5), (3,4), (4,3), (4,5), (5,3), (5,4), (6,1), (6,6)\}$

R Relation

Matrix Rep: Go Row Wise

R	a	b
a	×	✓
b	×	✓

means

aRb

bRb

$|R| = 2$

$R = \{(a,b), (b,b)\}$

Next:

Understanding Relations

19. Let R be a binary relation defined on the set $\{2, 3, 4, 5, 6\}$ by: $(a, b) \in R \Leftrightarrow a|b$.

    (a) List all the elements of R.

    (b) Represent the relation R using a digraph

19. Let R be a binary relation defined on the set $\{2, 3, 4, 5, 6\}$ by: $(a, b) \in R \Leftrightarrow a|b$.

A

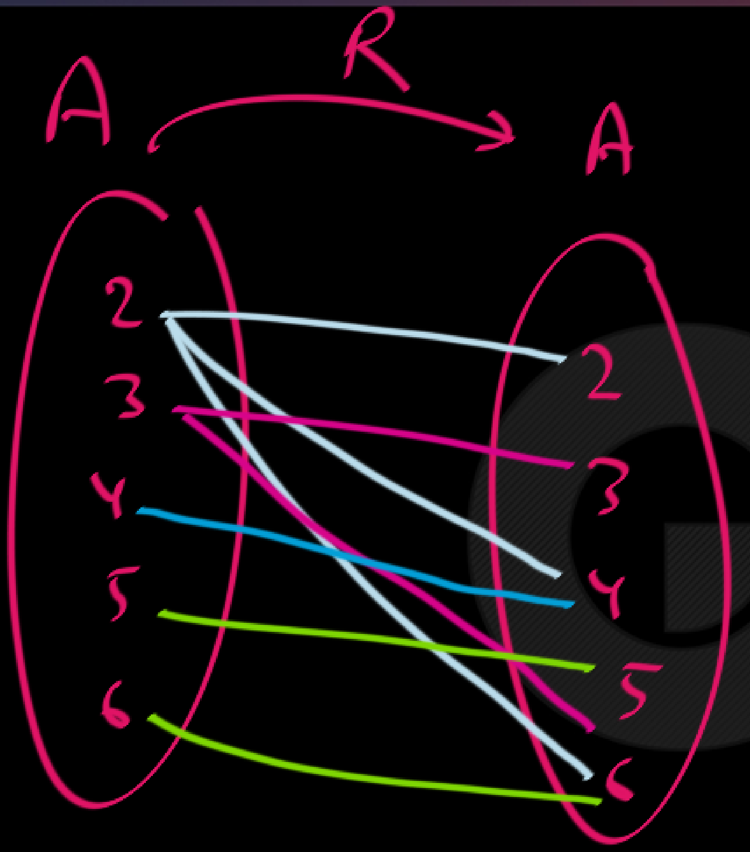
(a) List all the elements of R.

(b) Represent the relation R using a digraph

Graph Rep.

R on A: $\{2, 3, 4, 5, 6\}$

$R: A \to A \equiv \boxed{R \subseteq A \times A}$



$(a,b) \in R$ iff $a|b$.

same

$\boxed{(a,b) \in R} \equiv aRb$

$R = \{ (a,b), \ldots \}$

$A : \{2, 3, 4, 5, 6\}$

Relation R on A :

$\boxed{R \subseteq A \times A}$

xRy iff $x | y$

$|R| = 8$

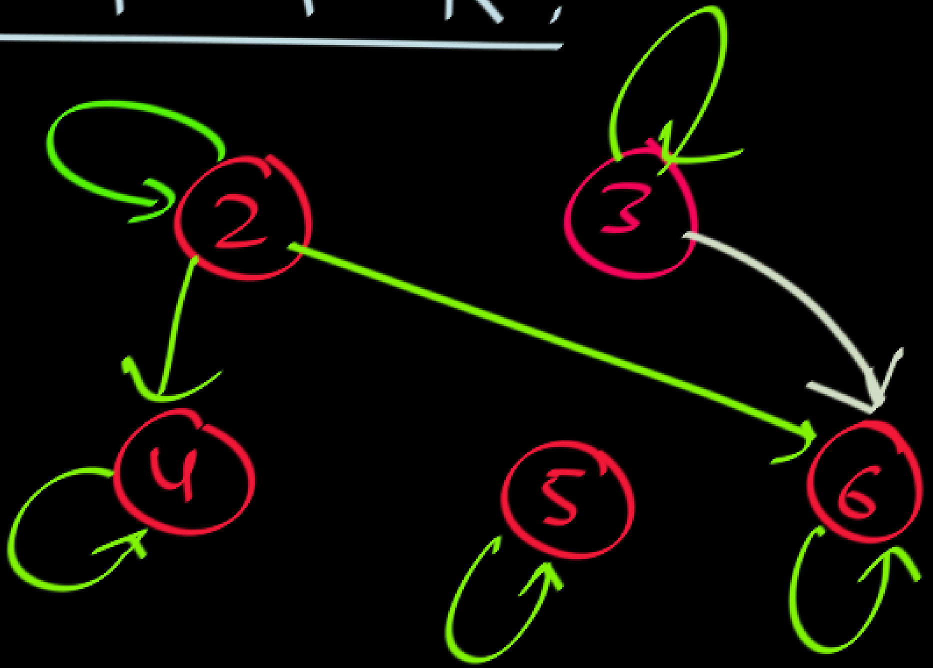
$R = \{(2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6)\}$

set

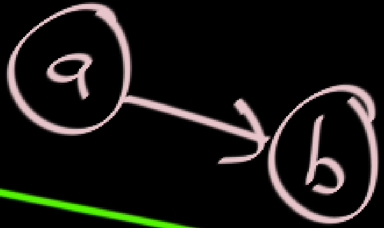
R on set A = {2, 3, 4, 5, 6}

A → Base set

Graph Rep of R:



If aRb then



R	2	3	4	5	6
2	✓		✓		✓
3		✓			✓
4			✓		
5				✓	
6					✓

20. Define a binary relation $\mathcal{R}$ on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ by:
$a\mathcal{R}b \Leftrightarrow ab$ is divisible by 8.

(a) Find the elements of the relation $\mathcal{R}$.

(b) Draw a directed graph representing the relation $\mathcal{R}$.

20. Define a binary relation $\mathcal{R}$ on the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ by:
$a\mathcal{R}b \Leftrightarrow ab$ is divisible by 8.

Base set

(a) Find the elements of the relation $\mathcal{R}$.

(b) Draw a directed graph representing the relation $\mathcal{R}$.

R on set A $\equiv$ $R: A \rightarrow A$ $\equiv$ $R \subseteq A \times A$

aRb iff $8 \mid ab$ $\rightarrow$ 8 Divides ab

$x \mid y$: x Divides y.

Divides

$a \mid b$ means: a Divides b

Divides

$A = \{1, 2, 3, 4, 5, 6, 7\}$

Relation R on $A \equiv \boxed{R : \underline{A} \longrightarrow \underline{A}}$

$\underline{x} R \underline{y}$ iff $8 \mid xy$

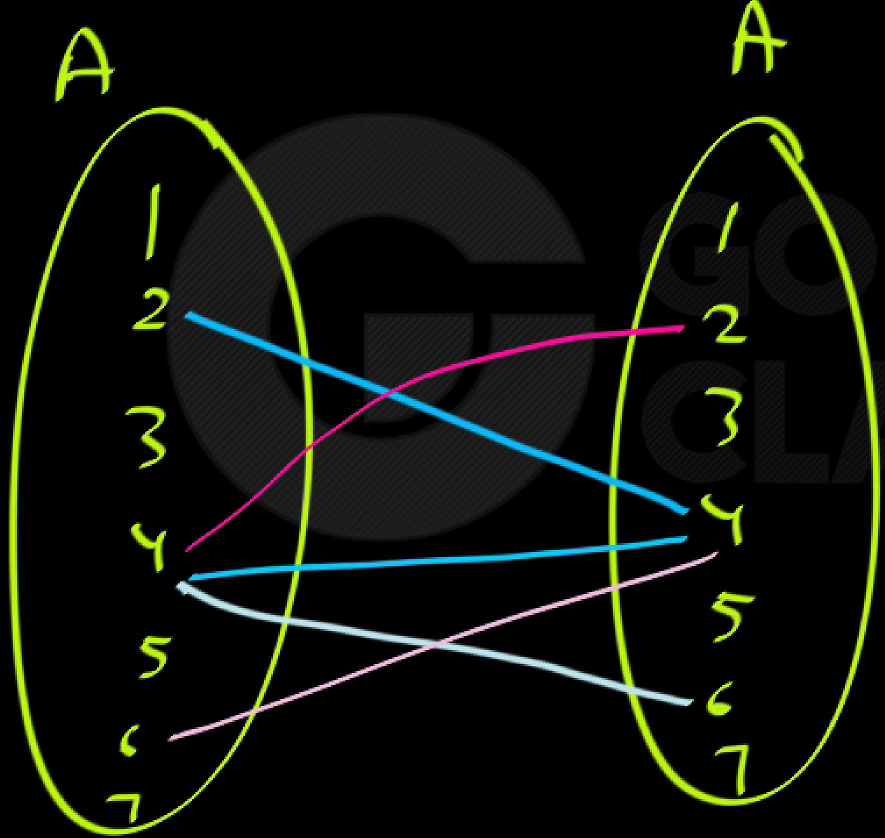
Base Set

$R : A \longrightarrow A$

$x R y$ iff $8 \mid xy$

$4 R 2$

$8 \mid 4 \times 2$



$1 \not R 1$ $\quad 8 \nmid 1 \cdot 1$

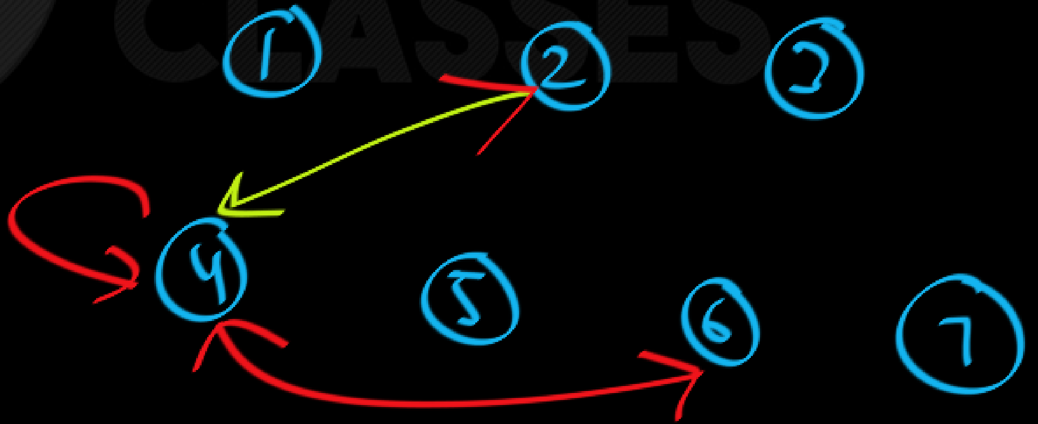
$1 \not R 2$ $\quad 8 \nmid 1 \cdot 2$

$2 R 4$, $8 \mid 2 \times 4$

$$R = \{(2,4), (4,2), (4,4), (4,6), (6,4)\}$$

$$|R| = 5$$

Graph Rep of R



$R = \{ (2,4), (4,2), (4,4), (4,6), (6,4) \}$

$|R| = 5$

Matrix Rep of R:

	1	2	3	4	5	6	7
1							
2				✓			
3							
4		✓		✓		✓	
5							
6				✓			
7							

Problem 5. Let A be the set of all ordered pairs of integers, that is, $A = Z \times Z$. Define a binary relation R on A as follows: for all $(a, b), (c, d) \in A$,

$$(a, b)R(c, d) \quad \Leftrightarrow \quad a \leq c \text{ and } b \leq d.$$

Problem 5. Let A be the set of all ordered pairs of integers, that is, $A = Z \times Z$. Define a binary relation R on A as follows: for all $(a, b), (c, d) \in A$,

$$(a, b)R(c, d) \Leftrightarrow a \le c \text{ and } b \le d.$$

$A : Z \times Z$ — Base set

$\boxed{R \text{ on } A} \equiv R : A \to A \equiv R \subseteq A \times A$

R on $Z \times Z \equiv R : Z \times Z \to Z \times Z \equiv R \subseteq (Z \times Z) \times (Z \times Z)$

Base set

R_1 on $Z \times Z$ → Base set

$Z \times Z$ → $Z \times Z$

$Z \times Z$: (2, 4), (−2, 0), ⋮

$Z \times Z$: (2, 4), (0, −2), ⋮

R_2 on Z ≡ → Base set

$R_2 : Z \to Z$

$Z \xrightarrow{R_2} Z$

Z: 0, 1, −1, 2, −2, ⋮

Z: 0, 1, −1, 2, −2, ⋮

$$Z \times Z = \{(a,b) \mid a \in Z, b \in Z\}$$

$$Z \times Z = \{(0,1), (0,0), (0,2), (-1,0), (0,-1), \ldots\}$$

R on $Z \times Z$ $\rightarrow$ Base set

Base set : $Z \times Z = \{(-2, 4), (4, -8), \ldots\}$

R on $Z \times Z$:

$$\underbrace{(a, b)}_{\in Z \times Z} \; R \; \underbrace{(c, d)}_{\in Z \times Z}$$

Base set : $Z \times Z$

$Z \times Z = \{ (x, y) \mid x \in Z, y \in Z \}$

R on $Z \times Z$:

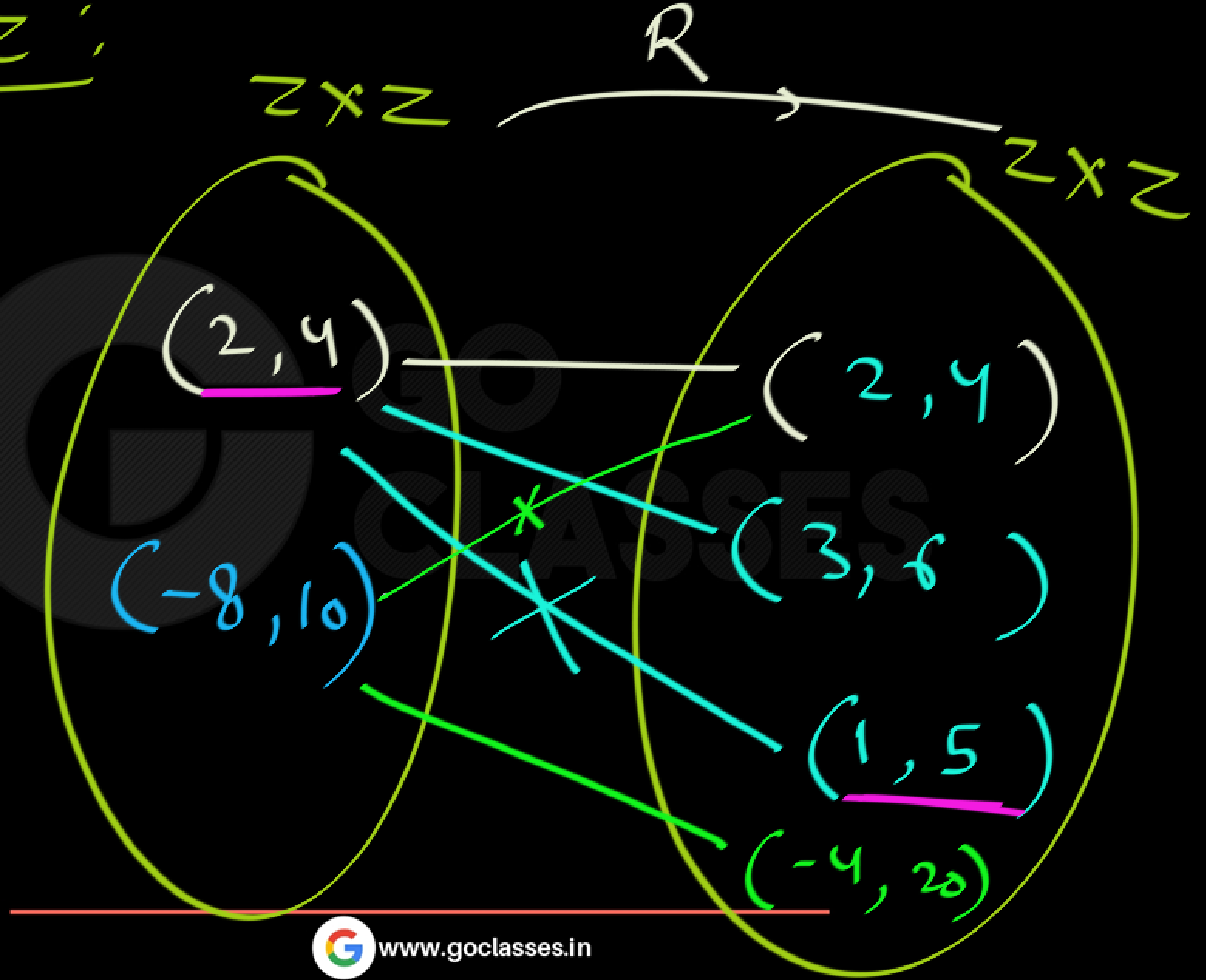
$(a, b) \, R \, (c, d)$ iff $a \leq c$ & $b \leq d$

R on $Z \times Z$:

$(a,b) \, R \, (c,d)$

iff

$a \leq c$ & $b \leq d$



R_1 on $Z \times Z$;

Base set

$R_1 : Z \times Z \to Z \times Z$

$R_1 \subseteq (Z \times Z) \times (Z \times Z)$

$(a,b) \, R \, (c,d)$

$(a,b) \in Z \times Z$

$(c,d) \in Z \times Z$

R_2 on Z

Base set

$R_2 : Z \to Z$

$R_2 \subseteq Z \times Z$

$a \, R \, b$

$a \in Z$

$b \in Z$

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

Base set

$$\cancel{R : N \to N}$$

$$R : N \times N \to N \times N$$

R_1 on $\underline{N}$ $\to$ Base set

R_2 on $\underline{N \times N}$ $\to$ Base set

$R_1 : \underline{N} \to \underline{N}$

$R_2 : \underline{N \times N} \to \underline{N \times N}$

$R_1 \subseteq N \times N$

$R_2 \subseteq (N \times N) \times (N \times N)$

$a R_1 b$

$a \in N$, $b \in N$

$(\underline{a,b}) R_2 (\underline{c,d})$

$$N \times N = \{ (a, b) \mid a \in N, b \in N \}$$

$$N \times N = \{ (1,1), (1,2), (1,3) - - - \\ (2,1), (2,2), (2,3) - - - - \\ - - - - - - - . \}$$

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

Base set

elements: (x, y)

$R : N \times N \rightarrow N \times N$

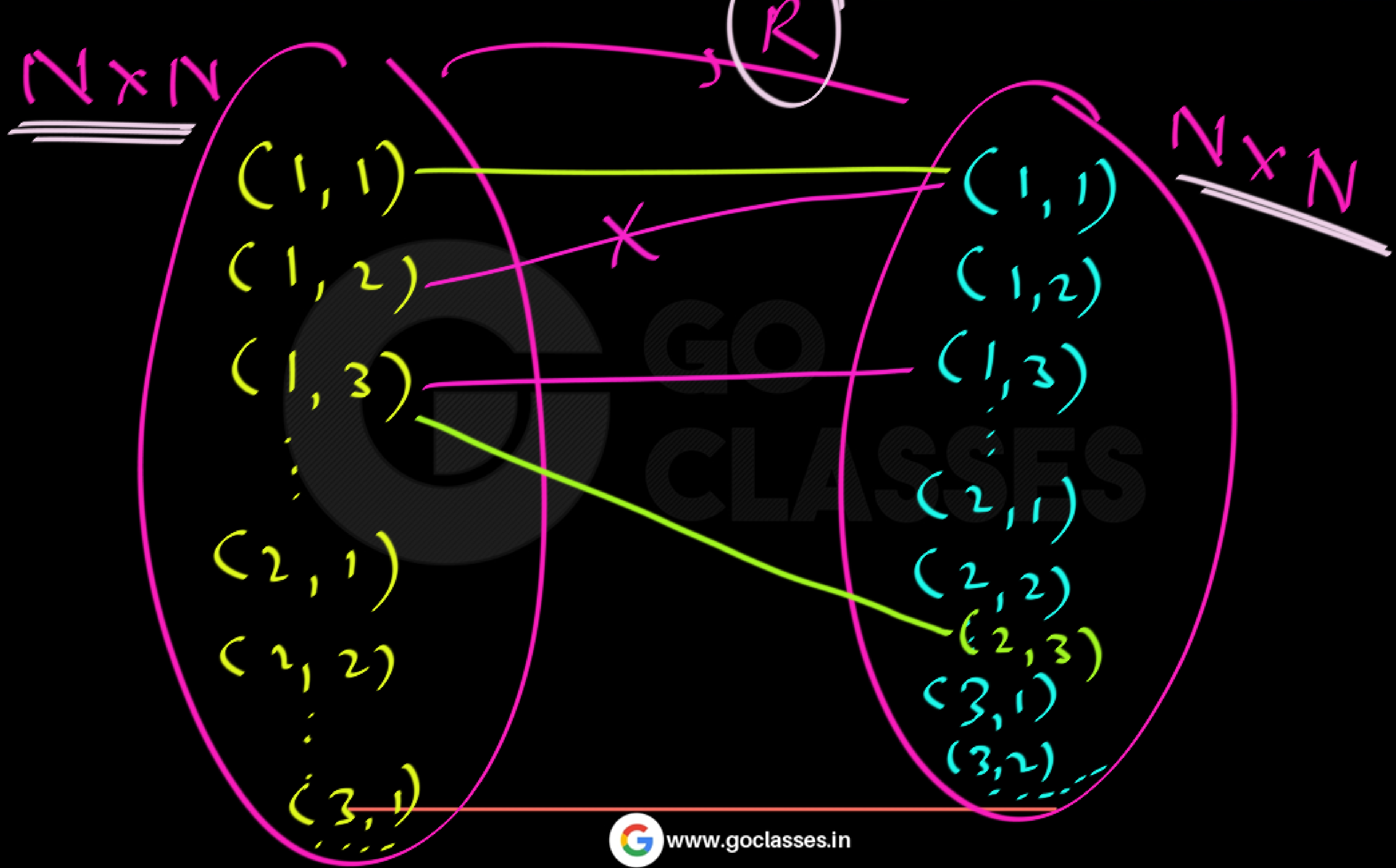
$(a, b)\ R\ (c, d)$ iff $a \mid c$ & $b \mid d$

$(4, 6)\ R\ (4, 6)$ ✓

$(4, 6)\ R\ (8, 36)$

$(4, 6)\ \cancel{R}\ (5, 7)$

$(8, 36)\ \cancel{R}\ (4, 6)$


N × N
R
(1, 1)
(1, 2)
(1, 3)
(2, 1)
(1, 2)
(3, 1)
N × N
(1, 1)
(1, 2)
(1, 3)
(2, 1)
(2, 2)
(2, 3)
(3, 1)
(3, 2)

Q1 (10 points)

$$R = \{(x, y) \in \mathbb{N}^2 : \exists n \in \mathbb{N}, x^n = y\}$$

[redacted]

Q1 (10 points)

$$R = \{(x, y) \in \mathbb{N}^2 : \exists n \in \mathbb{N}, x^n = y\}$$

$\in R$

Base set $\to N$

$R = \{ (x, y) \in N \times N \mid \exists_{n \in N}, x^n = y \}$

$(x, y) \in R \equiv x R y$

$x \to N$, $y \to N$

$R : N \to N$

$$(a, b) \in R \equiv aRb$$

Q1 (10 points)

$$R = \{(x, y) \in \mathbb{N}^2 : \exists n \in \mathbb{N}, x^n = y\}$$

$R : N \to N$

is a binary relation on the set of natural numbers $\mathbb{N}$.

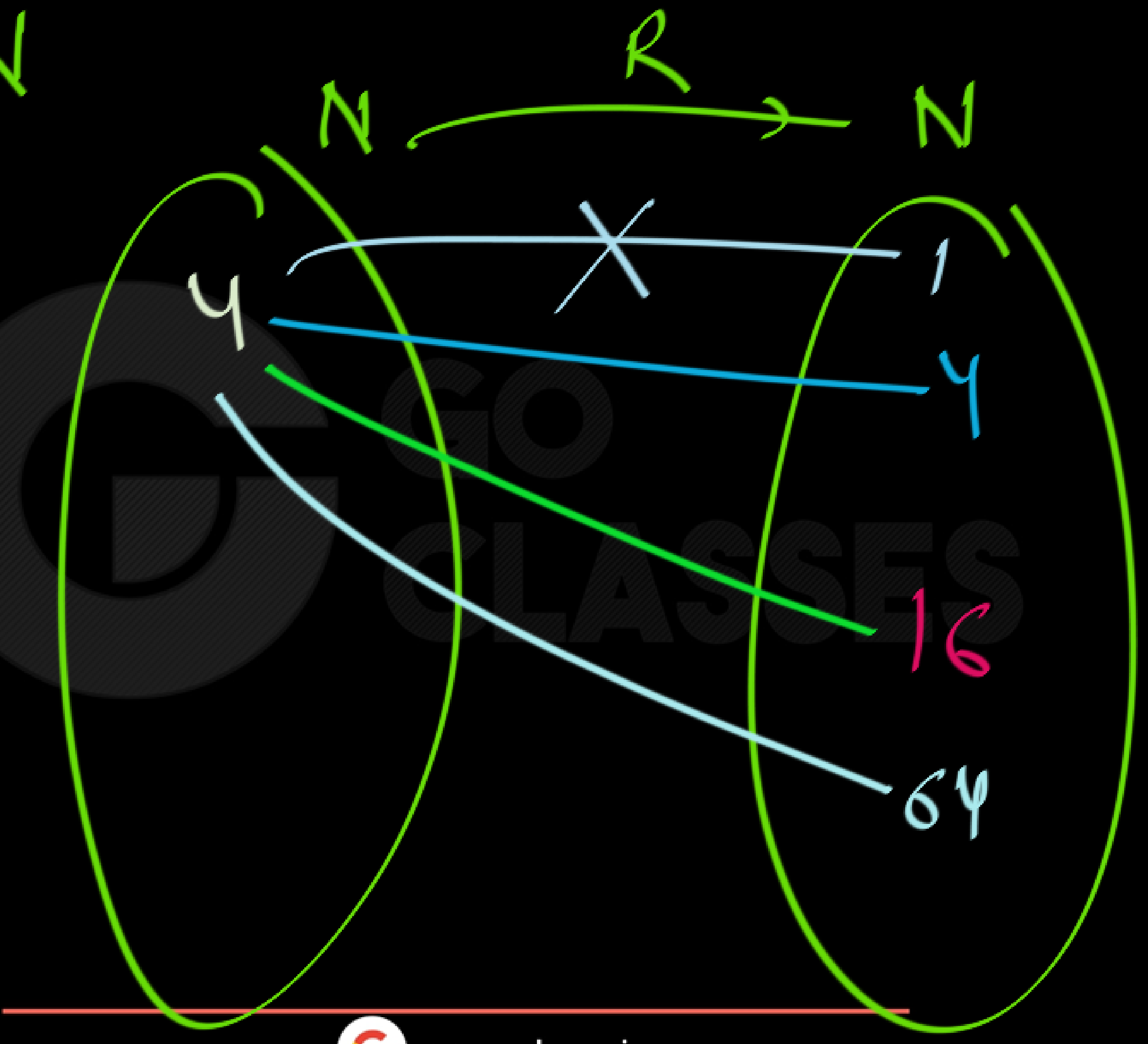
xRy iff $\exists_{n \in N}, x^n = y$

xRy iff $\boxed{\text{Some power of } x} = y$

natural

$R : N \to N$

$$\boxed{xRy \text{ iff } x^n = y}$$



$4R16$

$4^2 = 16$

$4^1 = 4$

xRy

$x^1 = y$

$4^2 = 16$

$4^3 = 64$

Q1 (10 points)

$$R = \{(x, y) \in \mathbb{N}^2 : \exists n \in \mathbb{N}, x^n = y\}$$

is a binary relation on the set of natural numbers $\mathbb{N}$.

$4 \not{R} 1$ $\quad 4^0 = 1$ $\quad$ not a natural no.

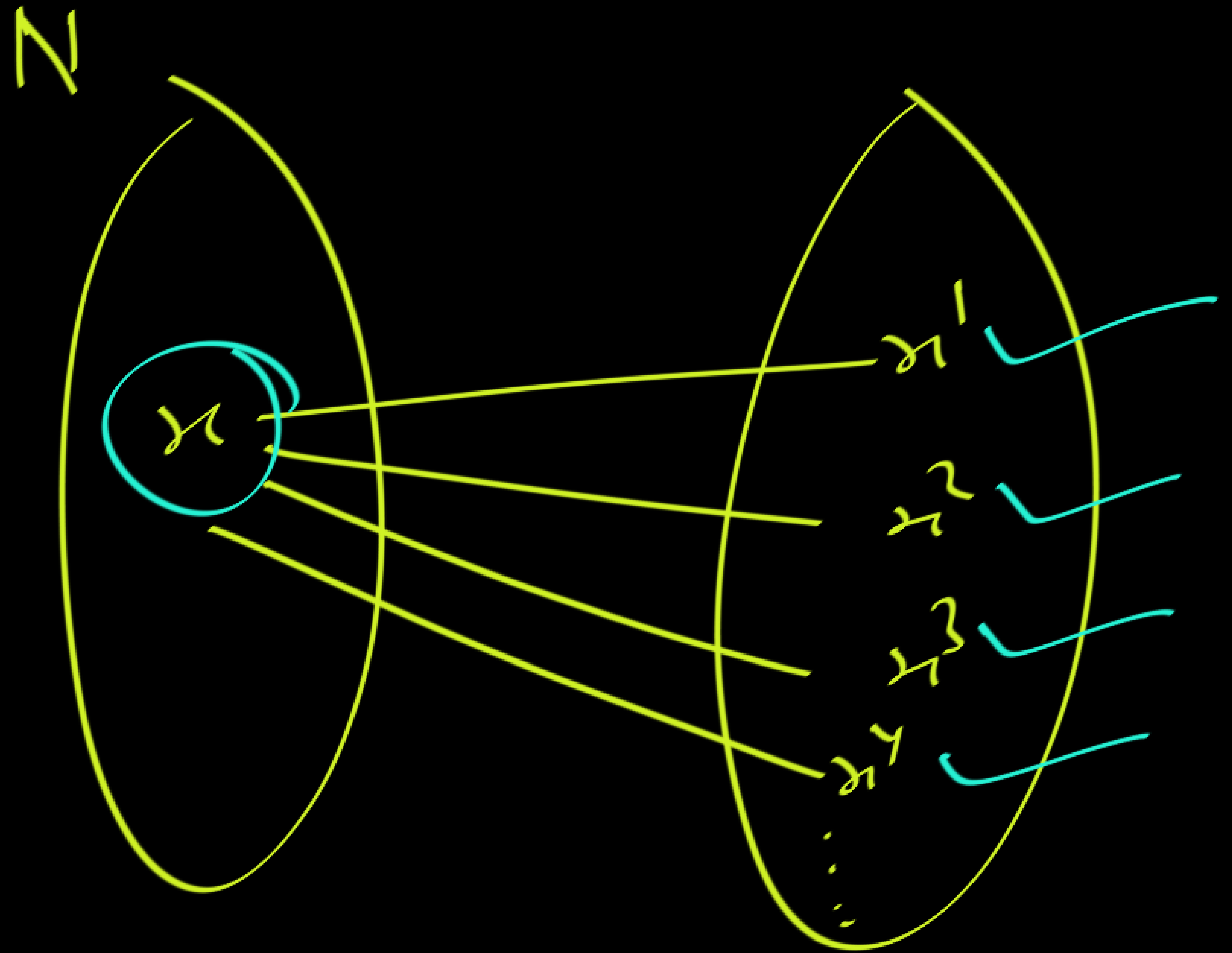
$\boxed{x R y \text{ iff } x^{n} = y, \; n \in \mathbb{N}}$

$4 R 64 \quad = 4^3 = 64$

$2 \;—\; 2^1, \; 2^2, \; 2^3, \; 2^4, \; 2^5, \; \vdots$

$3 \;—\; 3^1, \; 3^2, \; 3^3, \; \vdots$

$4 \;—\; 4^1, \; 4^2, \; 4^3, \; \vdots$


N
x
x'
x''
x'''
x''''

Let $R \subseteq \mathbb{N} \times \mathbb{N}$ *be a relation (a binary relation) on the set of natural numbers defined as follows:*

$$(x, y) \in R \Leftrightarrow x + y \geq 18.$$

Let $R \subseteq \mathbb{N} \times \mathbb{N}$ *be a relation (a binary relation) on the set of natural numbers defined as follows:*

$$\boxed{(x, y) \in R \Leftrightarrow x + y \geq 18.}$$

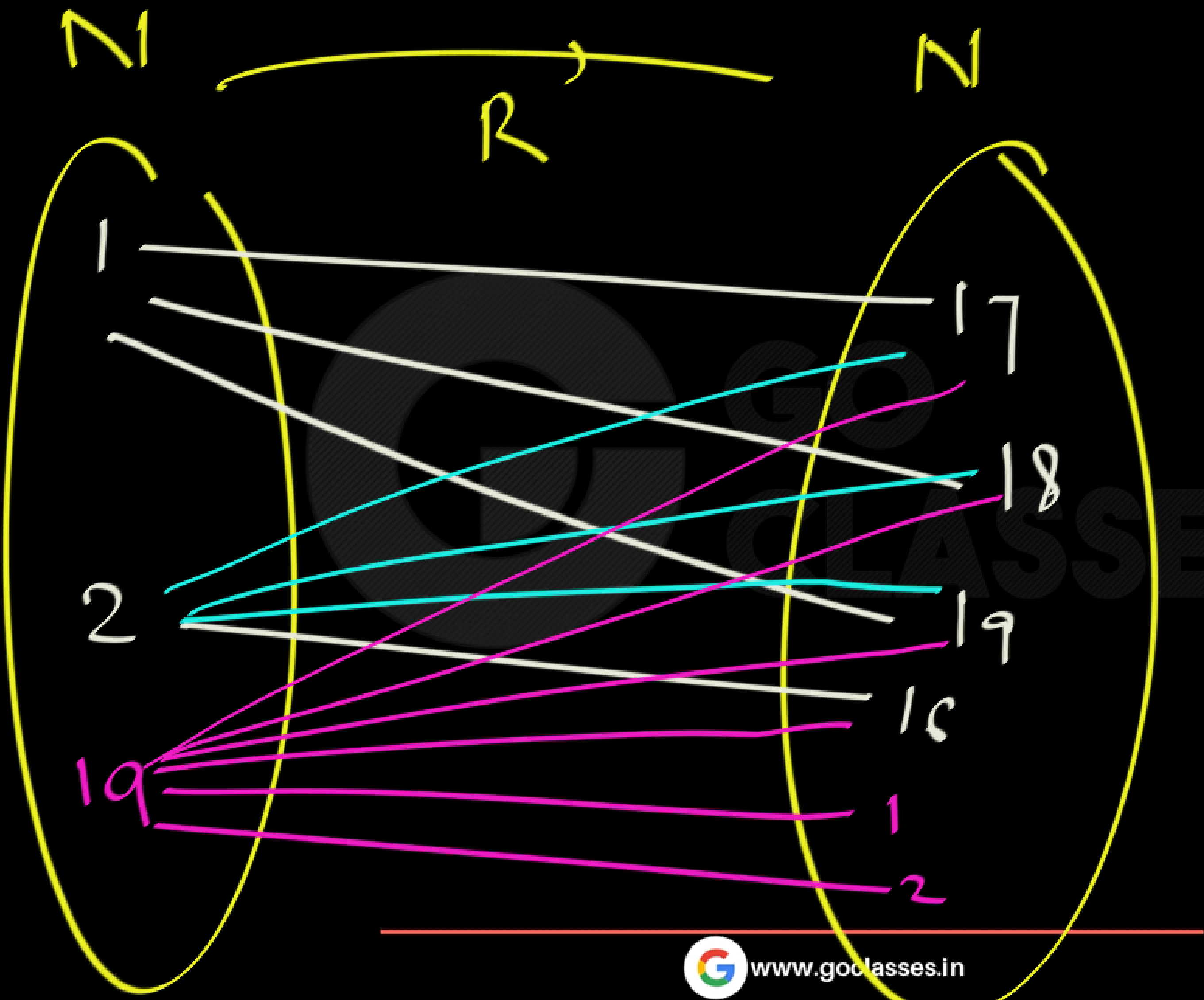
$R \subseteq N \times N \equiv R : N \rightarrow N$

$\equiv R$ on N

$x R y$ iff $\boxed{x + y \geq 18}$

$x \rightarrow N$

$y \rightarrow 17$



$1 \not R\ 10$

$1+10 \not> 18$

Question 2. Consider the binary relation $\preceq$ on $\mathbb{N}$ defined as follows:

$$m \preceq n \Leftrightarrow m \text{ divides } n$$

Question 2. Consider the binary relation $\preceq$ on $\mathbb{N}$ defined as follows:

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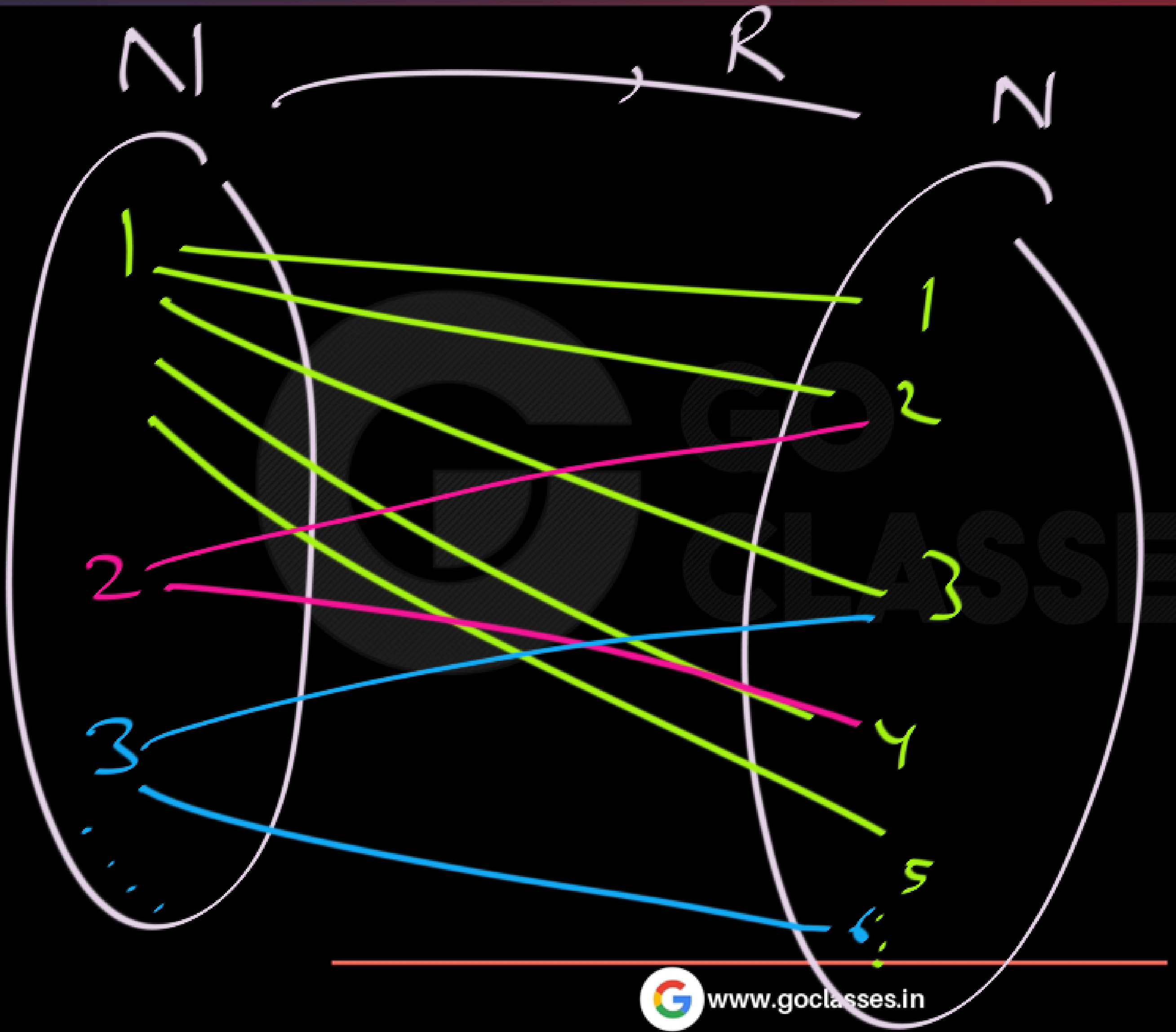
R

$R : N \to N \equiv R \subseteq N \times N$

$\boxed{m R n \text{ iff } m | n}$



xRy

iff

$x \mid y$

NOTE:

Before solving questions on Relations, First Understand the given relation.

Next Topic:

Special Types of Relations

(Reflexive, Symmetric, Transitive etc)

$R : A \rightarrow B$

$\boxed{R : A \rightarrow A}$

$\boxed{R \text{ on set } A}$

When a Relation is defined on a set A:

R : A → A

R : N → Z

R on a set A

Some Special Types of Relations:

1. Reflexive, Irreflexive

2. Symmetric, Anti-Symmetric, Asymmetric

3. Transitive

Some Special Types of Relations:

1. Reflexive, Irreflexive
2. Symmetric, Anti-Symmetric, Asymmetric
3. Transitive

Next Topic:

Reflexive Relation

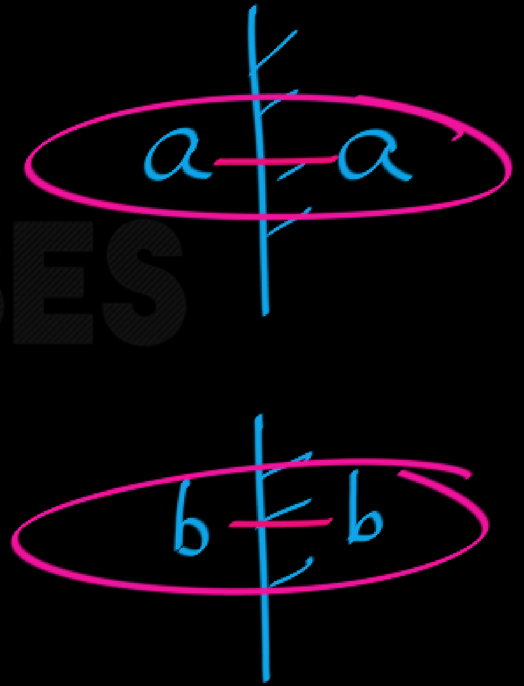
(Reflexive, Not Reflexive, Irreflexive)

Reflexive Relation: (IDEA):

Every element should have Reflexion.

means:

Every element is related to itself.



Set A

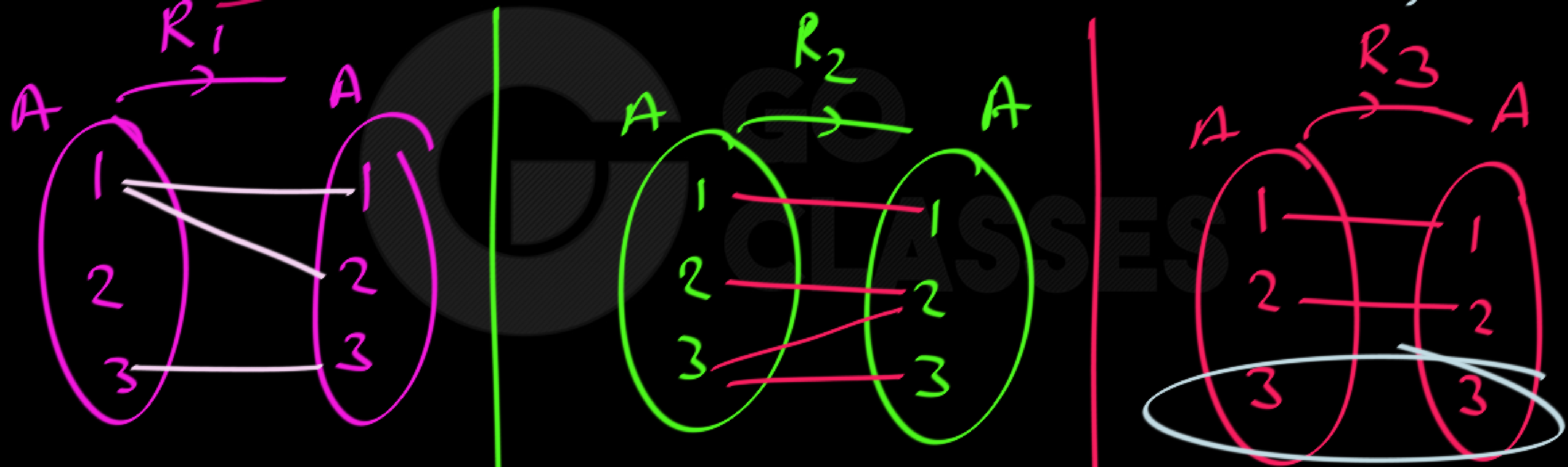
Base set

Relation $\boxed{R \text{ on set } A} \equiv R: A \to A$

Relation R is Reflexive iff every element $x \in A$ is relates to itself.

$A : \{1, 2, 3\}$

Not Reflexive



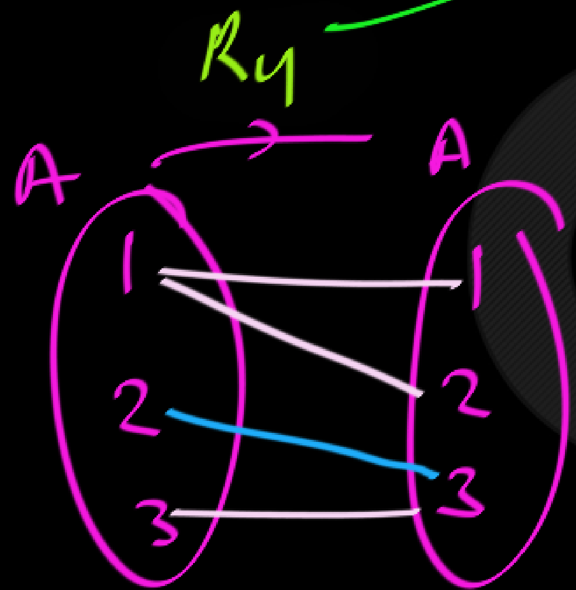
$2 \not{R_1} 2$

R_2 : Reflexive

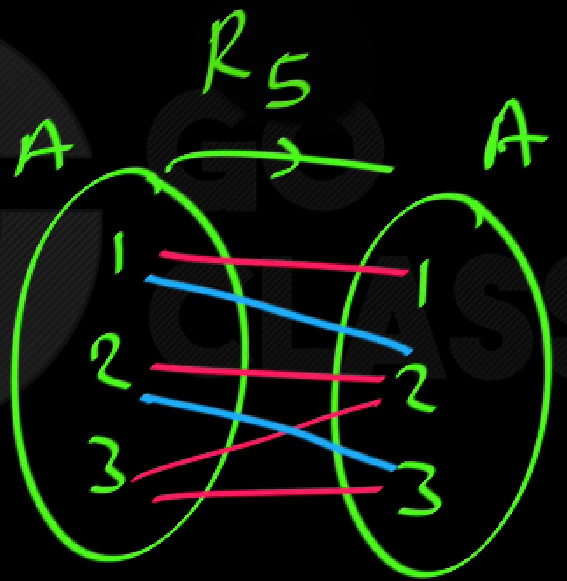
$3 \not{R_3} 3$

$A : \{1, 2, 3\}$

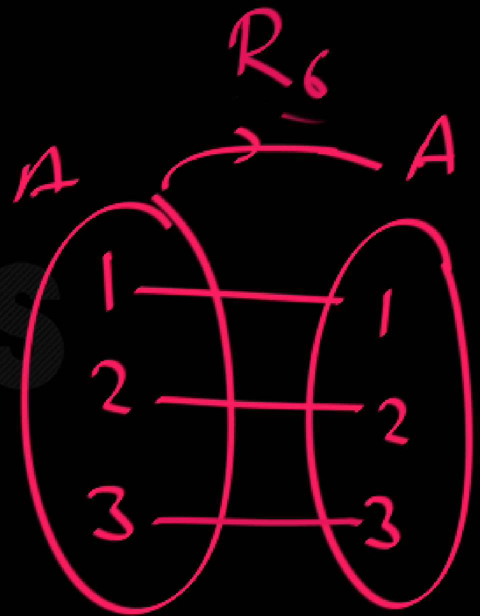
NoT Reflexive



$2 \not R_4 2$



R_5 : Reflexive



R_6 : Ref.

Relation R on set A → Base set

Relation R is Reflexive iff

$$\boxed{\forall_{x \in A} (xRx)}$$

Set $A = \{1, 2, 3\}$

Reflexive Relation R on A

$R = \{(1,1), (2,2), (3,3), \ldots\ldots\}$

Set $A = \{1, 2, 3\}$

Reflexive Relation R on A

$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$

Set $A = \{a_1, a_2, a_3, ----, a_n\}$

Reflexive Relation R on A:

$R = \{(a_1, a_1), (a_2, a_2), \ldots, (a_n, a_n), \ldots\ldots\}$

$a_i R a_i$

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

Consider the following relations on $\{1, 2, 3, 4\}$: Base set

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$R_2 = \{(1, 1), (1, 2), (2, 1)\}$,

$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\}$,

$3 \cancel{R_1} 3$

$2 \cancel{R_2} 2$

R_3: Ref

R_1: Not Ref $\leftarrow R_2$

Example 1:

Equality Relation on Z is Reflexive??

YES

$R : Z \to Z$; aRb iff $a = b$

$a \in Z$; $a = a$, So $\boxed{aRa}$

$2 \in Z$; $2 = 2$ So $2 R a$

Example 2:

Let Set A = {1,2,3}

Subset Relation on P(A) is Reflexive??

Example 2:

Let Set $A = \{1,2,3\}$

Subset Relation on $P(A)$ is Reflexive??

Base Set :

$\cancel{A}$

$P(A)$ ✓

Example 2:

Let Set A = {1,2,3}

Subset Relation on P(A) is Reflexive??

YES

$P(A) = \{ \phi, \{1\}, \{2\} - - - - . \{1,2,3\} \}$

R on P(A) ;

$\boxed{x R y \text{ iff } x \subseteq y}$

$R : P(A) \longrightarrow P(A)$

$P(A) = \{ x \mid x \subseteq A \}$ → Base set

set → Subset of A

$\boxed{m \, R \, n \text{ iff } m \subseteq n}$

$S \in P(A)$; $\boxed{S \subseteq S}$ $S R s$

Reflexivity

- Some relations always hold for any element and itself.
- Examples:
  - $x = x$ for any x.
  - $A \subseteq A$ for any set A.
  - $x \equiv_k x$ for any x.
  - $u \leftrightarrow u$ for any node u.
- Relations of this sort are called ***reflexive***.
- Formally speaking, a binary relation R over a set A is reflexive if the following is true:

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

Matrix Representation of Reflexive Relation:

Matrix Representation of Reflexive Relation:

$A = \{a, b, c, d\}$; Relation R on A

Reflexive R	a	b	c	d
a	1	–	–	–
b	–	1	–	–
c	–	–	1	
d	–	–	–	1

$1 \equiv$ ✓

$0 \equiv$ ✗

Matrix Representation of Reflexive Relation:

$A = \{a, b, c, d\}$; Relation R on A

Reflexive

$$
\begin{array}{c|cccc}
R & a & b & c & d \\
\hline
a & \boxed{1} & 0 & 1 & 0 \\
b & 0 & \boxed{1} & 1 & \\
c & & & \boxed{1} & \\
d & & & & \boxed{1}
\end{array}
$$

All 1's

1 ≡ ✓

0 ≡ ✗

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$

R_1	1	2	3	4
1	1	1	0	0
2	1	1	0	0
3	0	0	0	1
4	1	0	0	1

R_1: NOT Ref

Graph Representation of Reflexive Relation:

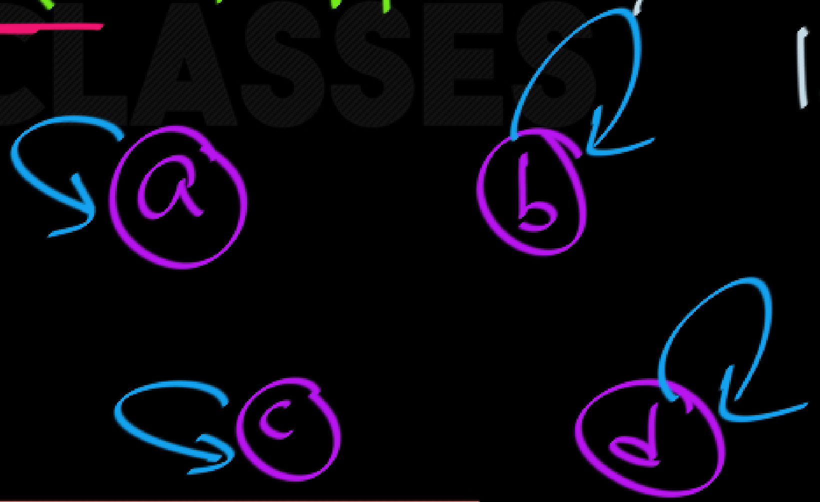
$A = \{a, b, c, d\}$ → Base set

Reflexive Relation R on A.

Graph Rep of R:

→ Every element Should have a self loop.

Self loop

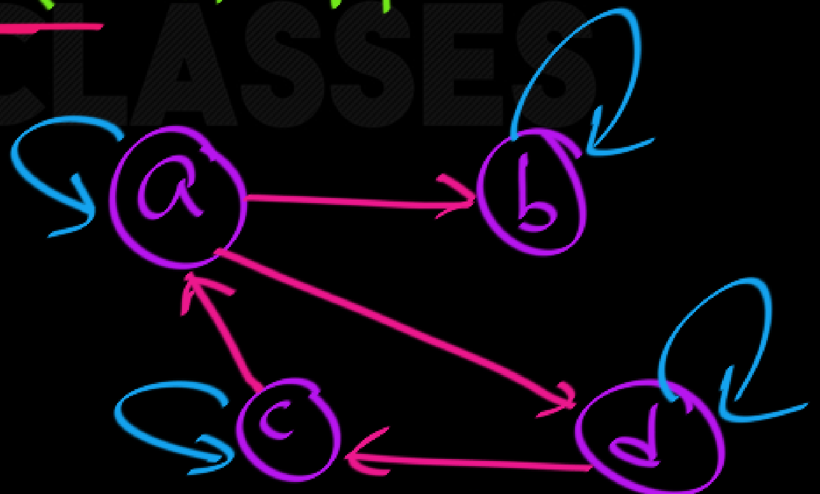


Graph Representation of Reflexive Relation:

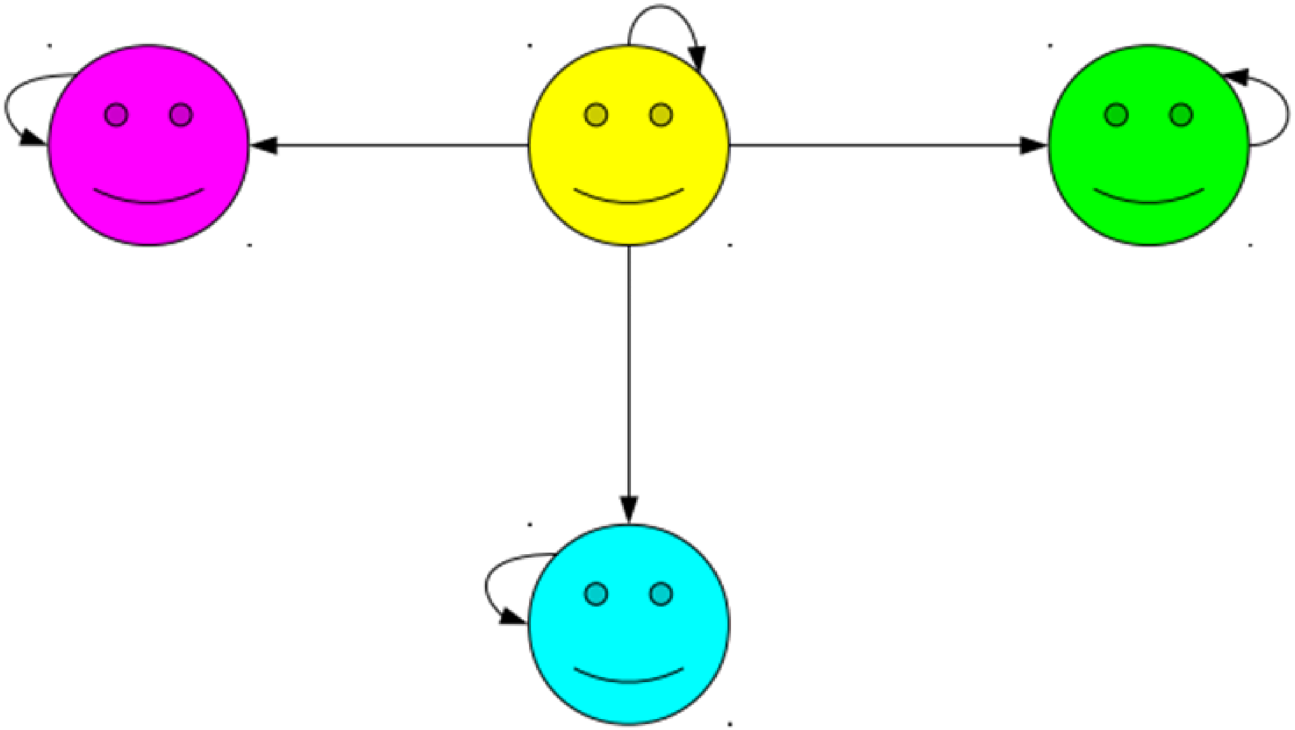
$A = \{a, b, c, d\}$ → Base set

Reflexive Relation R on A.

Graph Rep of R:

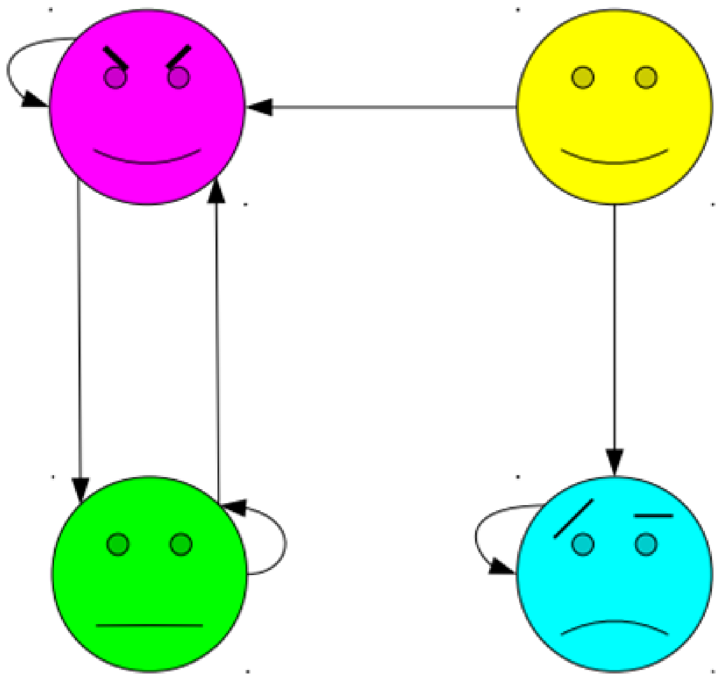


An Intuition for Reflexivity



$\forall a \in A.\ aRa$

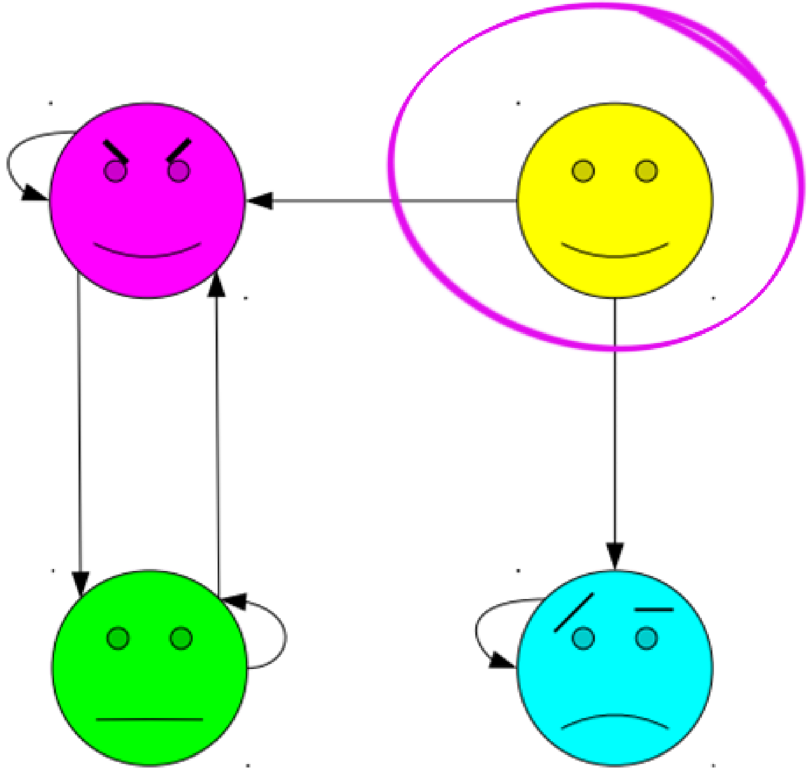
(*"Every element is related to itself."*)



Let R be the relation drawn to the left. Is R reflexive?

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

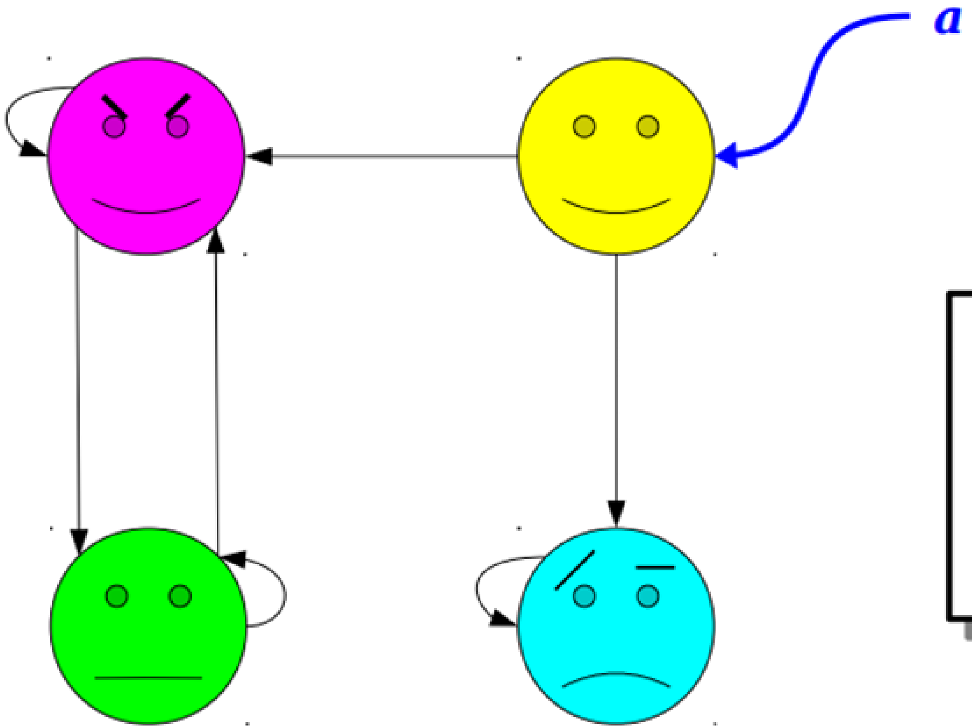


Let R be the relation drawn to the left. Is R reflexive?

No

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")



This means that R is not reflexive, since the first-order logic statement given below is not true.

$$\forall a \in A.\ aRa$$

("Every element is related to itself.")

"Not Reflexive" Relation:

Reflexive : $\forall_{x \in A} (xRx)$

Not Ref : $\exists_{x \in A} (x \cancel{R} x)$

“Irreflexive” Relation:

"Irreflexive" Relation:

Irreflexive Rel ;

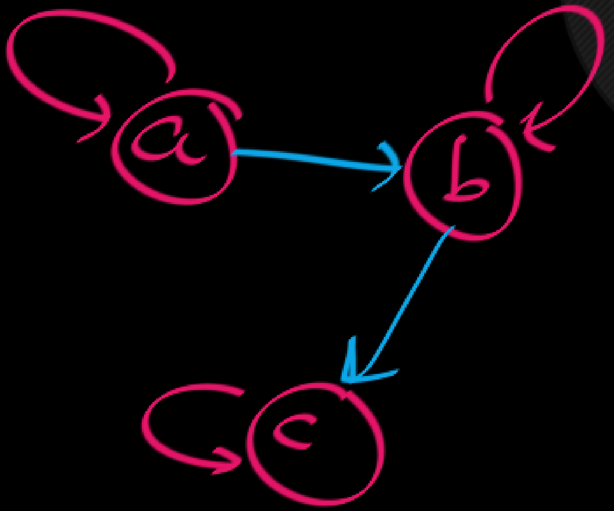
Extreme opposite of Reflexive Rel.

No element is Relates to itself.

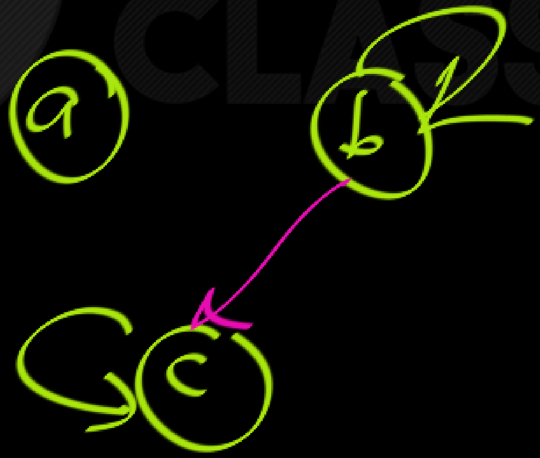
"Irreflexive" Relation:

Set A : $\{a, b, c\}$

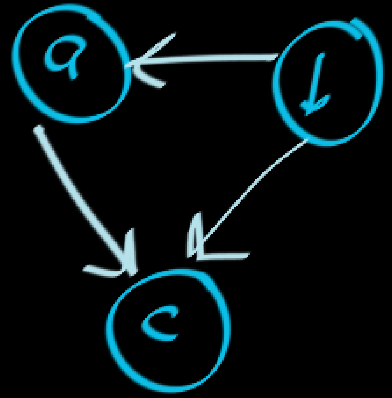
Reflexive



Not Reflexive



Irreflexive



"Irreflexive" Relation:

Base set : A

Reflexive: $\forall_{x \in A} (x R x)$

Not Reflexive: $\exists_{x \in A} (x \not R x)$

Irreflexive: $\forall_{x \in A} (x \not R x)$

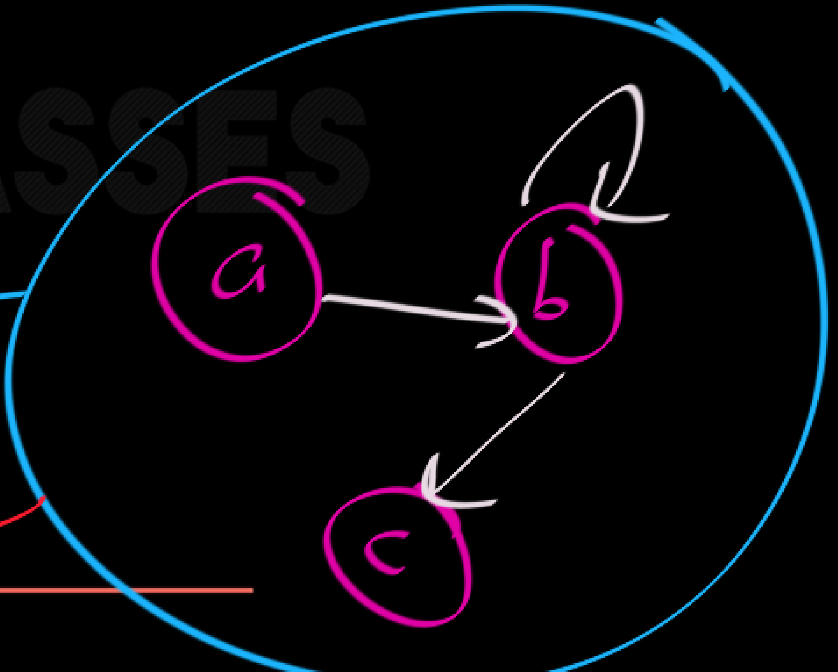
"Irreflexive" Relation:

Irreflexive $\longrightarrow$ NOT Ref

$A = \{1, 2, 3\}$

NOT Ref

NOT irref



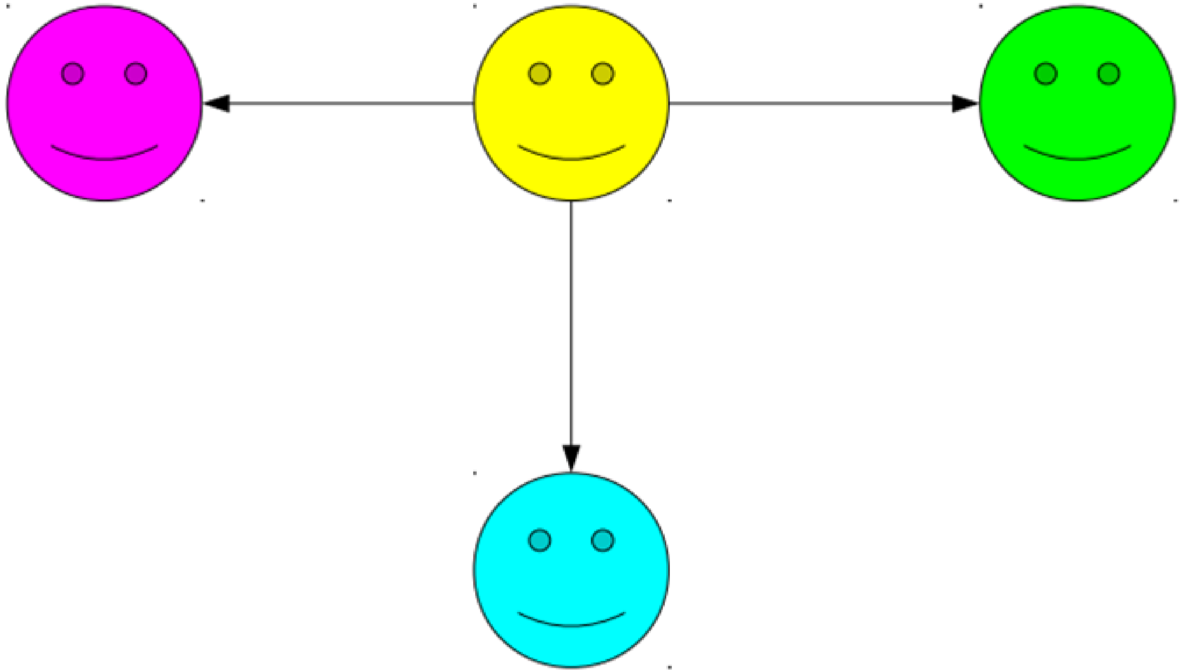
Irreflexivity

- Some relations *never* hold from any element to itself.
- As an example, $x \nless x$ for any x.
- Relations of this sort are called ***irreflexive***.
- Formally speaking, a binary relation R over a set A is ifreflexive if the following is true:

$$\forall a \in A.\ a \not R a$$

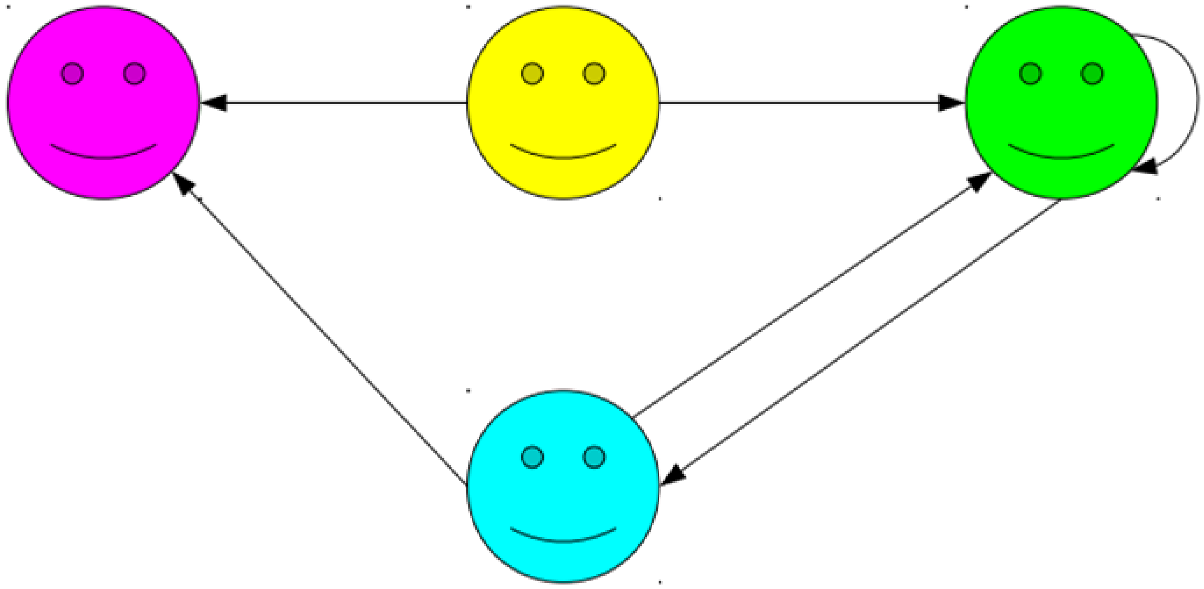
(*"No element is related to itself."*)

Irreflexivity Visualized



$$\forall a \in A.\ a \not R a$$

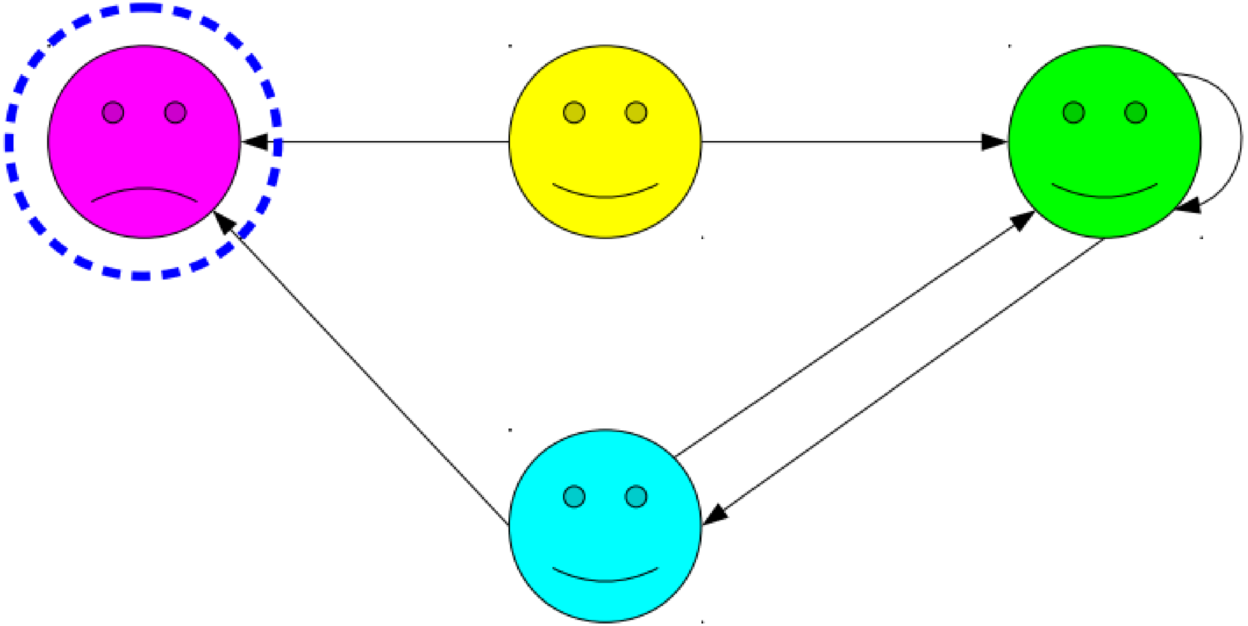
("No element is related to itself.")



Is this relation reflexive?

$$\forall a \in A.\ aRa$$

(*"Every element is related to itself."*)

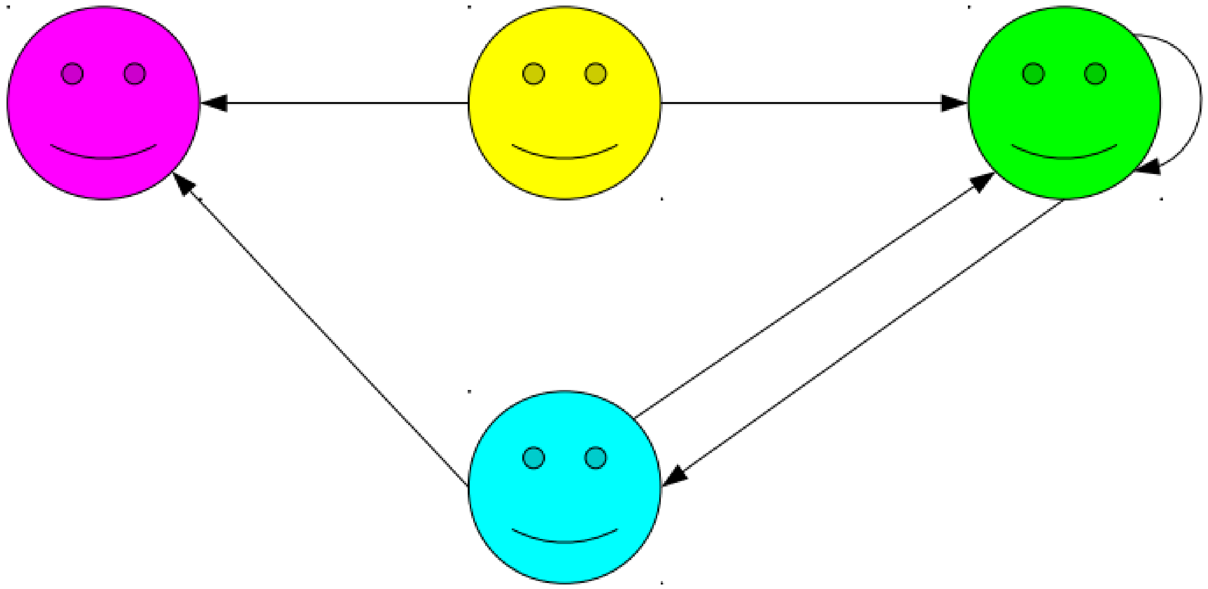


Is this relation reflexive?

Nope!

$\forall a \in A.\ aRa$

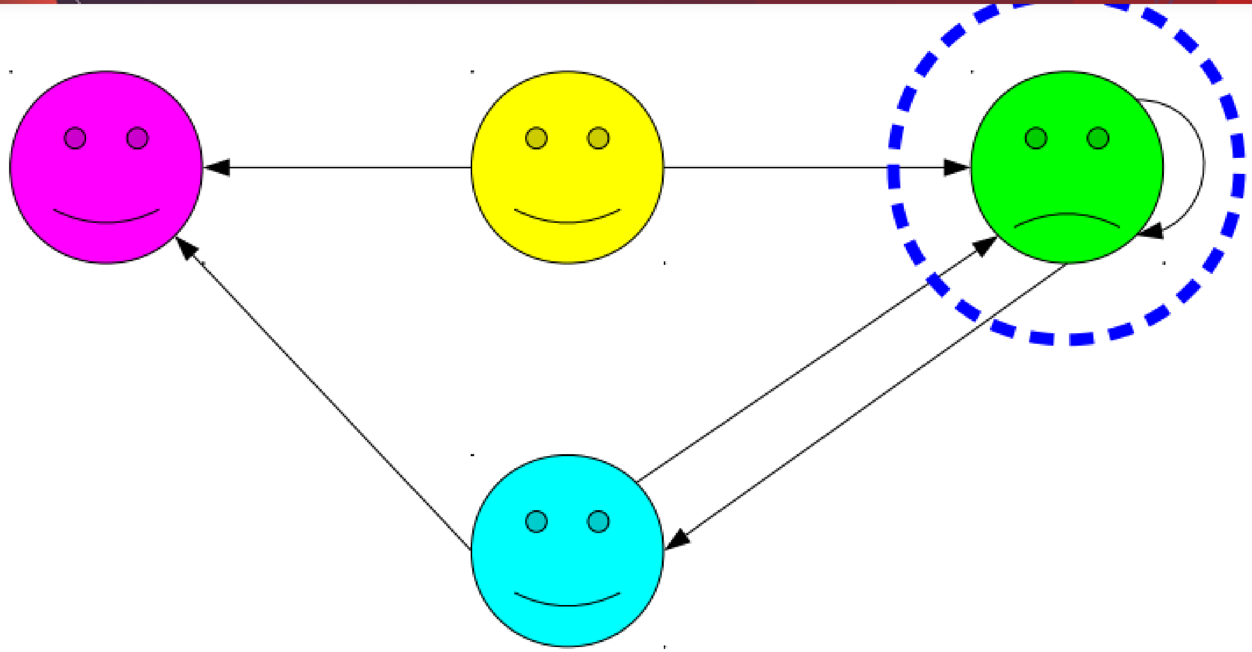
("Every element is related to itself.")



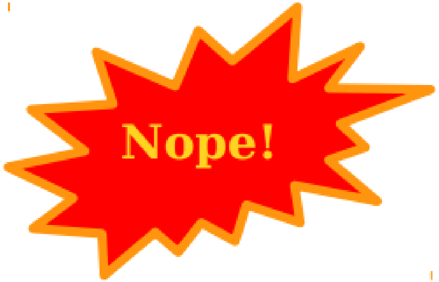
Is this relation irreflexive?

$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")



Is this relation irreflexive?



$$\forall a \in A.\ a \not R a$$

("No element is related to itself.")

Reflexivity and Irreflexivity

- Reflexivity and irreflexivity are ***not*** opposites!
- Here's the definition of reflexivity:

$$\forall a \in A.\ aRa$$

- What is the negation of the above statement?

$$\exists a \in A.\ a\not{R}a$$

- What is the definition of irreflexivity?

$$\forall a \in A.\ a\not{R}a$$

“Reflexive” Relation

Vs

“Not Reflexive” Relation

Vs

“Irreflexive” Relation

Base set $A = \{a, b, c, d\}$

Matrix Rep :

Reflexive

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & - & - & - \\ b & - & 1 & - & - \\ c & - & - & 1 & - \\ d & - & - & - & 1 \end{array}$$

NOT Ref

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & - & - & - \\ b & - & 0 & - & - \\ c & - & - & 1 & - \\ d & - & - & - & 0 \end{array}$$

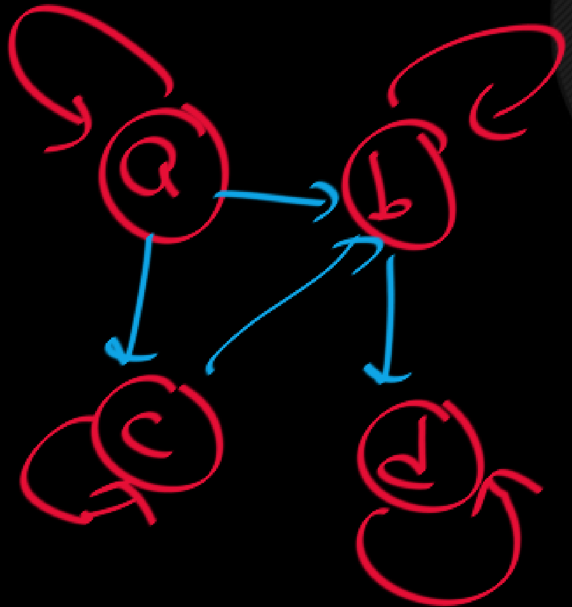
Irreflexive

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & & & \\ b & & 0 & & \\ c & & & 0 & \\ d & & & & 0 \end{array}$$

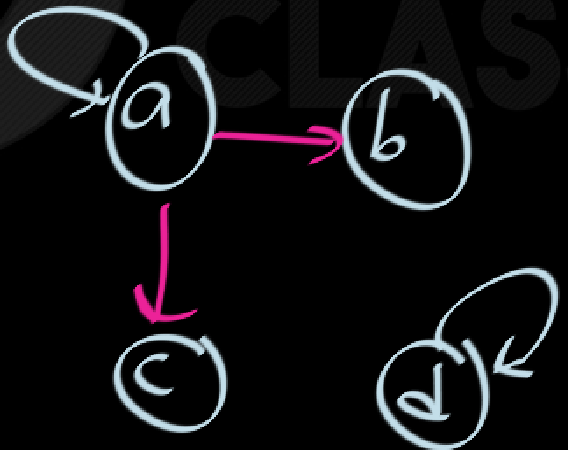
Base set $A = \{a, b, c, d\}$

Graph Rep:

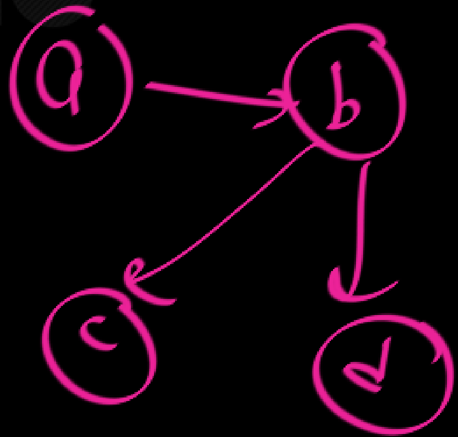
Reflexive



Not Ref



Irreflexive



Matrix Rep:

Reflexive: main Diagonal is All 1's.

Irreflexive: " " " All 0's.

NoT Reflexive: main Diagonal has at least one 0.

Graph Rep;

Reflexive: Every element has self loop.

Irreflexive: No " " " "

Not Reflexive: At least one element Doesn't have self loop.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Consider the following relations on $\{1, 2, 3, 4\}$: Base set

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Consider the following relations on {1, 2, 3, 4}: → Base set

$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\}$,

$3 \not R_1 3$

R_1: NOT Ref.

$4 R_1 4$

R_1: NOT Irref

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$

$2 \cancel{R_2} 2$ → R_2 : NOT Ref

1 R_2 1

R_2 : NOT Irref

Consider the following relations on {1, 2, 3, 4}:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$

Ref ✓

NOT Irref

Ref $\longrightarrow$ $\boxed{\text{Not Irref}}$

Irref $\longrightarrow$ Not Ref

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$
$$R_2 = \{(1,1), (1,2), (2,1)\},$$
$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$
$$R_4 = \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\},$$

$1 \cancel{R_4} 1$: R_4 : NOT Ref

Irref $\forall_{x \in A} (x \cancel{R} x)$

Consider the following relations on $\{1, 2, 3, 4\}$:

$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$,

$R_6 = \{(3, 4)\}$.

Which of these relations are reflexive?

R_5 : Ref

R_6 : Not Ref

R_6 : Irref

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

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Base set

$R : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$

$\cancel{R : \mathbb{N} \rightarrow \mathbb{N}}$

$$N \times N = \{ (a, b) \mid a \in N, b \in N \}$$

$$N \times N = \{ (1,1), (1,2), (1,3) \cdots$$

$$(2,1), (2,2), (2,3) \cdots \}$$

$R : N \times N \rightarrow N \times N$

$(a, b) \, R \, (c, d)$

$(a, b) \, R \, (c, d)$ iff $a \mid c$ AND $b \mid d$.

$(1,2)\ R\ (1,2)$

$(1,2)\ R\ (2,8)$

$(1,2)\ \cancel{R}\ (3,5)$

$(2,5)\ R\ (8,40)$

$(2,5)\ \cancel{R}\ (4,7)$

$(8,40)\ \cancel{R}\ (2,5)$

$R : N \times N \longrightarrow N \times N$

Base set

$(a,b) \, R \, (c,d)$ iff $a \mid c$ AND $b \mid d$.

Ref: Every element of Base set Should be Related to itself.

$$N \times N = \{ (a, b) \mid a \in N,\ b \in N \}$$

$$(a, b)\, R\, (a, b) : \text{Yes}$$

$(1,1)\ R\ (1,1)$ ✓

$(1,2)\ R\ (1,2)$ ✓

$(100,4)\ R\ (100,4)$ ✓

~~R : Ref~~ ✓

A relation R on a set A is called reflexive if for all $x \in A, (x, x) \in R$.

Example (7.4)

For $A = \{1, 2, 3, 4\}$, a relation $R \subseteq A \times A$ will be reflexive if and only if $R \supseteq \{(1, 1), (2, 2), (3, 3), (4, 4)\}$.

Consequently, $R_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not a reflexive relation on A, whereas $R_2 = \{(x, y) | x, y \in A, x \leq y\}$ is reflexive on A.

Next Topic:

Symmetric Relation

(Symmetric, Anti-Symmetric, Asymmetric)