Relations

Recap

Partial Order Relation

Website : https://www.goclasses.in/

Definition 5.1.1 A binary relation, $\leq$, on a set, X, is a *partial order* (or *partial ordering*) iff it is *reflexive*, *transitive* and *antisymmetric*, that is:

(1) (*Reflexivity*): $a \leq a$, for all $a \in X$;

(2) (*Transitivity*): If $a \leq b$ and $b \leq c$, then $a \leq c$, for all $a, b, c \in X$.

(3) (*Antisymmetry*): If $a \leq b$ and $b \leq a$, then $a = b$, for all $a, b \in X$.

A partial order is a *total order (ordering)* (or *linear order (ordering)*) iff for all $a, b \in X$, either $a \leq b$ or $b \leq a$.

When neither $a \leq b$ nor $b \leq a$, we say that *a and b are incomparable*.

If $\leq$ is a partial order on X, we say that the pair $\langle X, \leq \rangle$ is a *partially ordered set* or for short, a *poset*.

Maximal and Minimal Elements

Definition: Let *(A, R)* be a poset. Then *a* in *A* is a *minimal element* if there does not exist an element *b* in *A* such that *bRa*.

Similarly for a *maximal element*.

Note: there can be more than one minimal and maximal element in a poset.

Least and Greatest Elements

Definition: Let *(A, R)* be a poset. Then a in A is the *least element* if for every element b in A, aRb and b is the *greatest element* if for every element a in A, aRb.

Theorem: Least and greatest elements are unique.

Definition

Let $\langle A, \leq \rangle$ be a poset.

1. a is maximal if there does not exist $b \in A$ with $a \leq b$ and $a \neq b$.
2. a is minimal if there does not exist $b \in A$ with $b \leq a$ and $a \neq b$.
3. a is maximum if for every $b \in A$, we have $b \leq a$.
4. a is minimum if for every $b \in A$, we have $a \leq b$.

Let A = {a,b,c}. Consider the Set S = P(A) – { phi, {a,b,c} }
Make Hasses Diagram for (S, Subset).

S is obtained by deleting Ø and {a, b, c} from P({a,b,c}):

$(S, \subseteq)$

Poset ✓

NOT Toset

1. Ref ✓ $x \subseteq x$
2. Antisym ✓ $x \subseteq y$ and $y \subseteq x$ then $x = y$
3. Trans ✓ $x \subseteq y, y \subseteq z$ then $x \subseteq z$

$\{a\} \in S$, $\{b\} \in S$

not Comparable $\quad \{a\} \nsubseteq \{b\}$ $\quad \{b\} \nsubseteq \{a\}$

So NOT Toset

$\{a\}$, $\{a, b\}$ Comparable

$\{b, c\}$, $\{c, a\}$ NOT Comparable

$(S, \subseteq)$ — Poset $\longleftrightarrow$ Hasse Diagram

Partially ordered set

POR

$P(A)$ — $\{\phi, \{a,b,c\}\}$

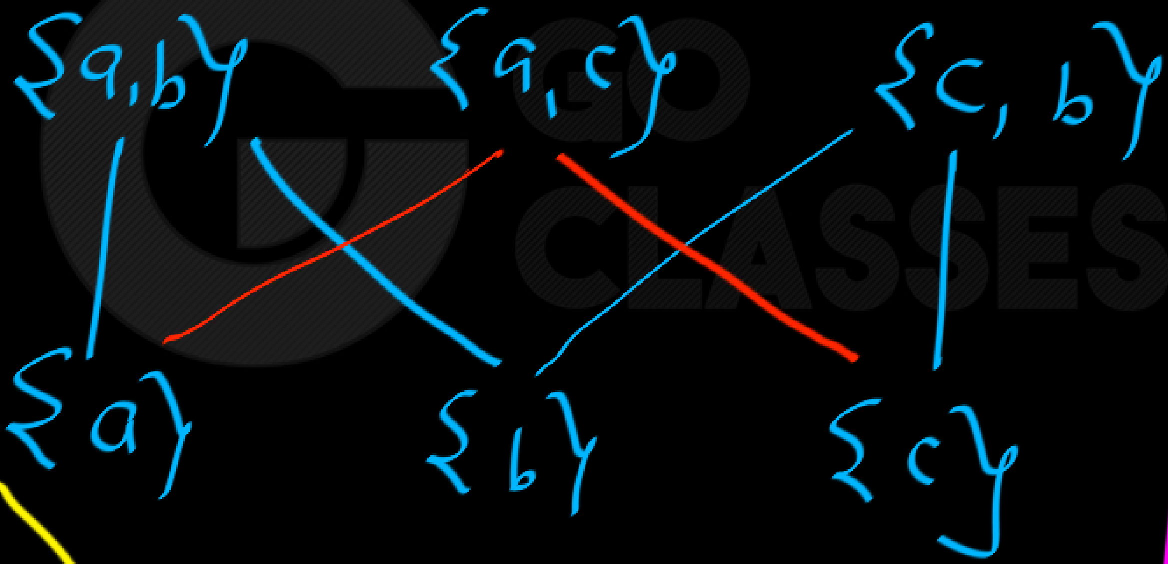
$\{a,b,c\}$

$|S| = 6$

Minimal : $\{a\}, \{b\}, \{c\}$

HD for $(S, \subseteq)$

Maximal : $\{a,b\}, \{a,c\}, \{c,d\}$



Greatest/ maximum : DNE

Least : DNE

$\{b\} \ R \ \{a,b\}$

$\{a,b\} \ \cancel{R} \ \{b\}$

minimal : no one else bows to me

m

$\forall a \quad a \not R\, m$

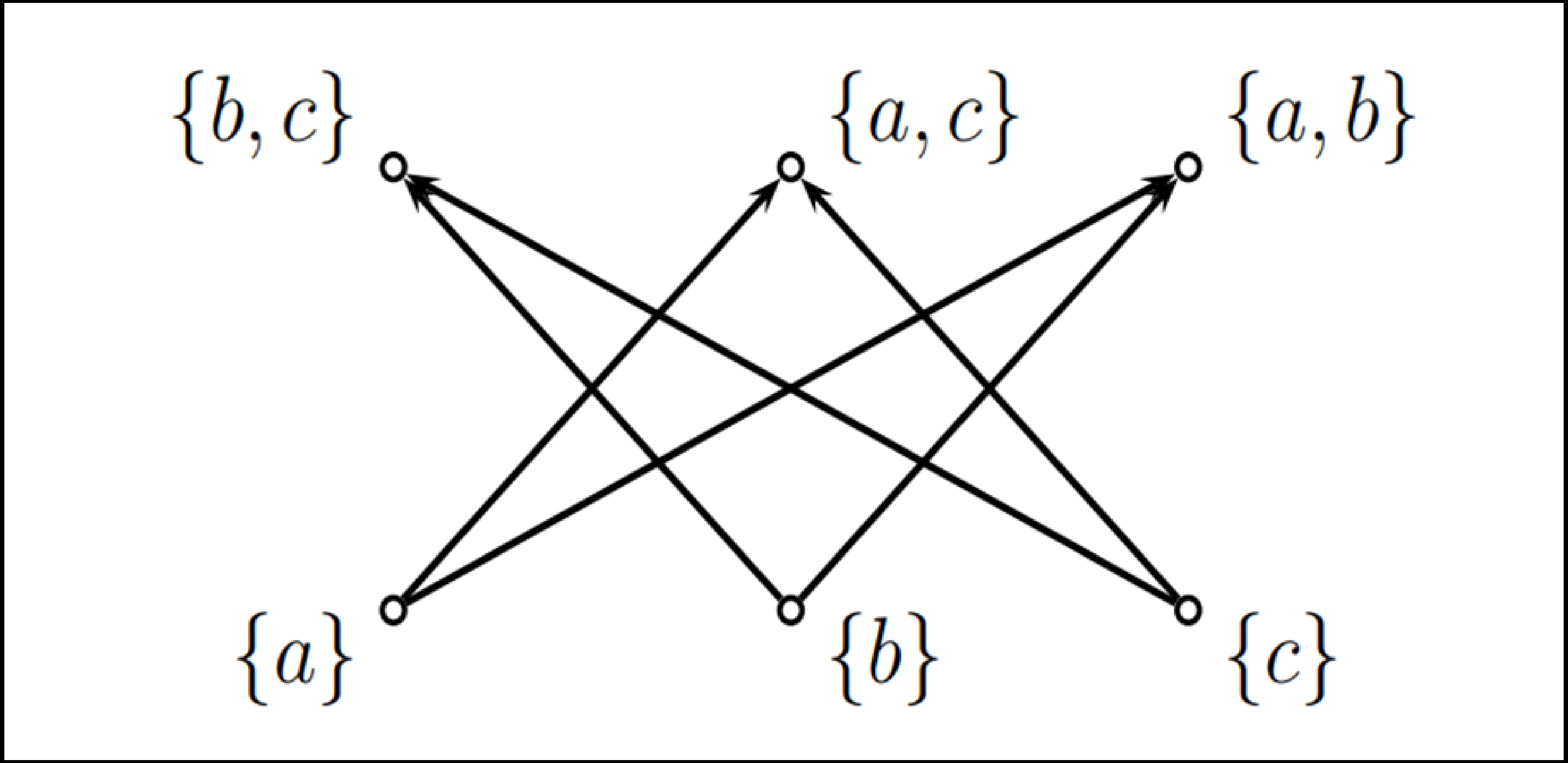
How many Elements are Related to $\{b\}$?

$= 1$

$\{b\} \, R \, \{b\}$

Make Hasses Diagram for (S, Subset).

S is obtained by deleting Ø and {a, b, c} from P({a,b,c}):

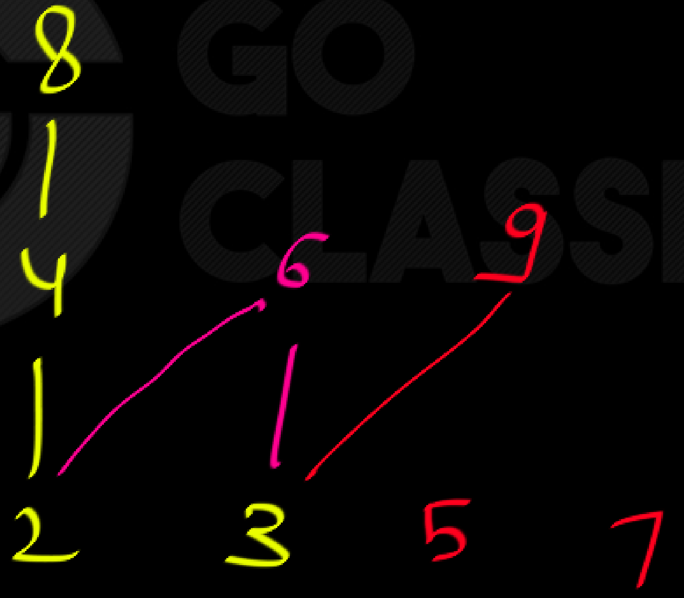


$(\{2, 3, 4, 5, 6, 7, 8, 9\}, |)$ — PoR

poset

#Edges in HD?

5 Edges



minimal: 2, 3, 5, 7

maximal: 8, 6, 9, 5, 7

Greatest: DNE

least: DNE

$(\{2, 3, 4, 5, 6, 7, 8, 9\}, \text{multiple of})$

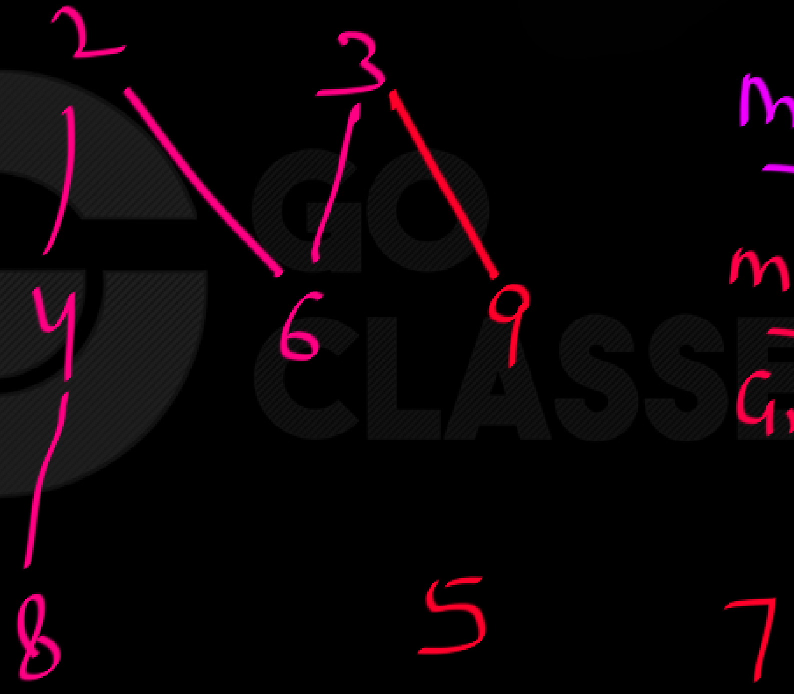
8R4R2

6R2

6R3

9R3

9 is multiple of 3.



minimal: 8, 5, 7, 6, 9

maximal: 2, 3, 5, 7

Greatest: DNE

least: DNE

Upper and Lower Bounds

Definition: Let S be a subset of A in the poset (A, R). If there exists an element a in A such that sRa for all s in S, then a is called an *upper bound*. Similarly for lower bounds.

Note: to be an upper bound you must be related to every element in the set. Similarly for lower bounds.

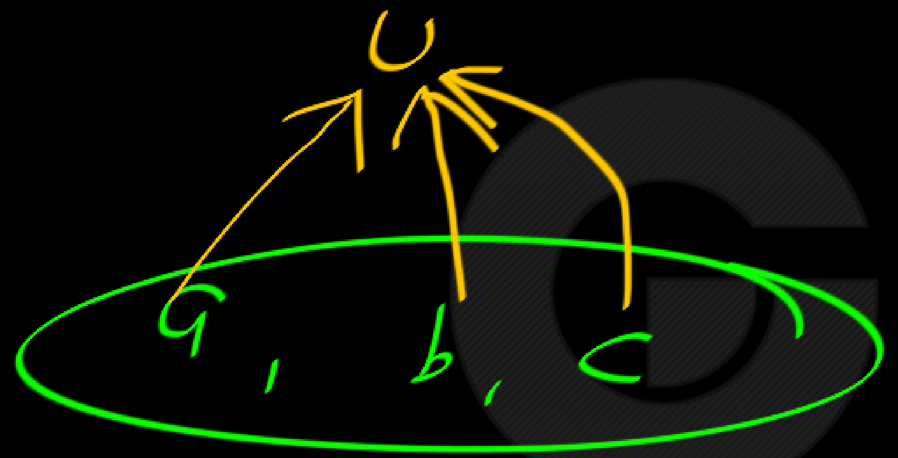
Poset (A, R) → POR on A

Subset $X \subseteq A$

Upper bound of X = $UB(X)$ = ?

lower bound of X = $LB(X)$ = ?

Upper bound of $\{a, b, c\}$:



$UB\{a,b,c\} = u$

iff

aRu, bRu,

cRu.

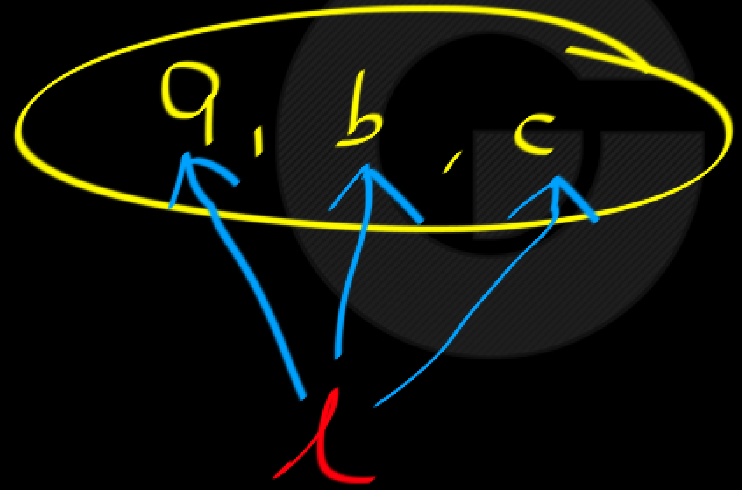
UB {majnu, uday}

movie = welcome

RDX

majnu, uday

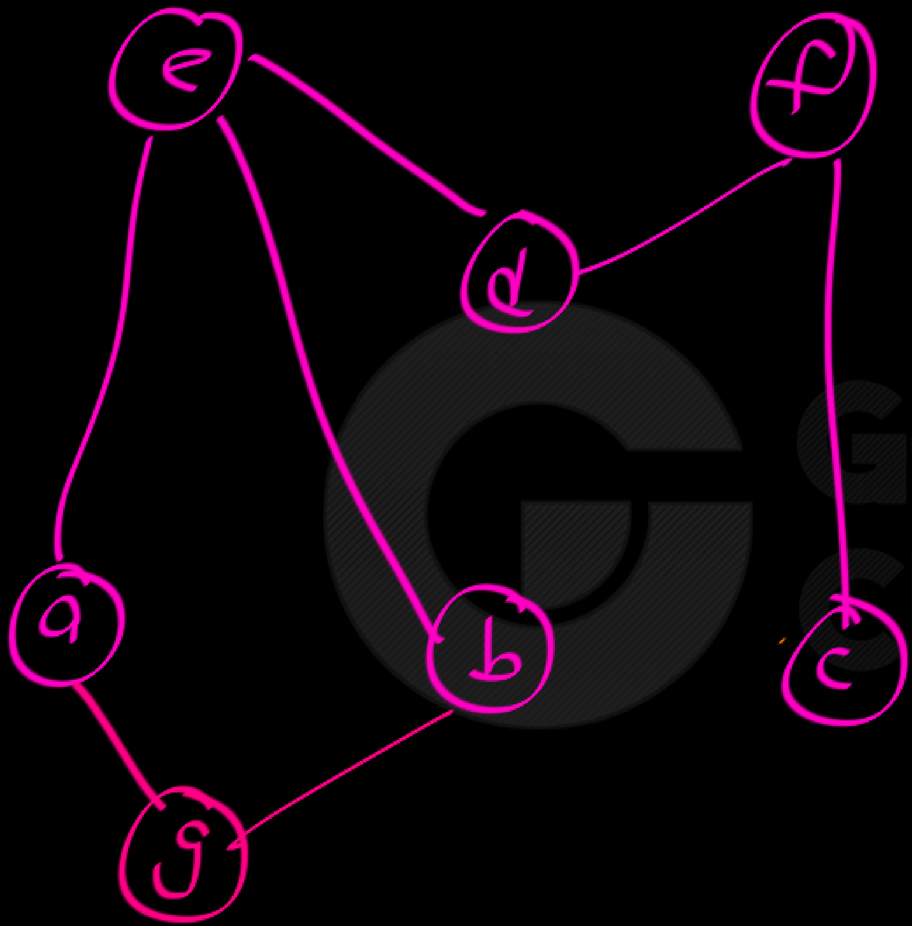
Lower bound of $\{a, b, c\}$:



l is LB $\{a,b,c\}$

iff

$l R a,\ l R b,\ l R c$



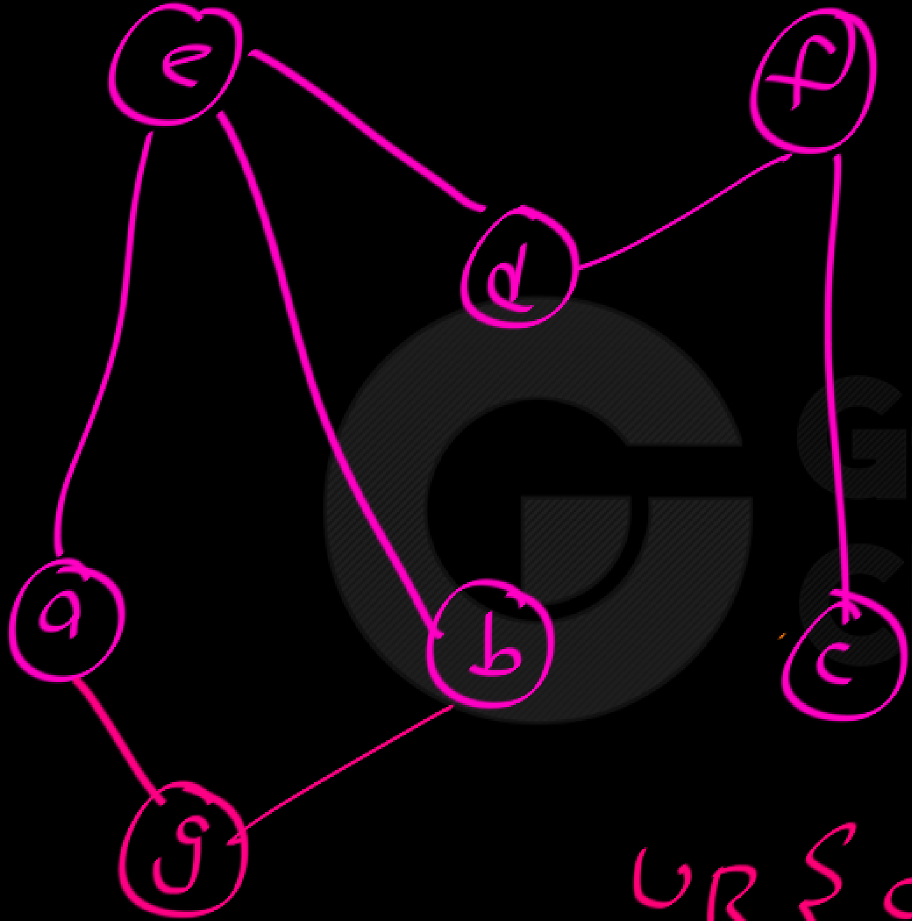
maximal: $\{e, f\}$

minimal: $\{g, d, c\}$

$UB\{a, g, b\} = \{e\}$

$UB\{a, g\} = \{a, e\}$

$LB\{a, g\} = g$



$UB\{e,f\} = \{d \quad \}$

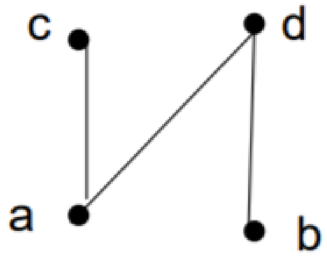
$UB\{e,f\} = \phi$

$UB\{a\} = \{a, e \quad \}$

$UB\{g\} = \{g, a, b, e\}$

$UB\{a,c\} = \phi$

Extremal Elements: Example 1



$LB\{c,b\} = \phi$

$LB\{c,d\} = a$

$UB\{a,b\} = d$

What are the minimal, maximal, minimum, maximum elements?

- Minimal: {a,b}
- Maximal: {c,d}


c, b
x
a, b
c

Definition 5.1.2 Let $\langle X, \leq \rangle$ be a poset and let $A \subseteq X$ be any subset of X. An element, $b \in X$, is a *lower bound of A* iff $b \leq a$ for all $a \in A$.

An element, $m \in X$, is an *upper bound of A* iff $a \leq m$ for all $a \in A$.

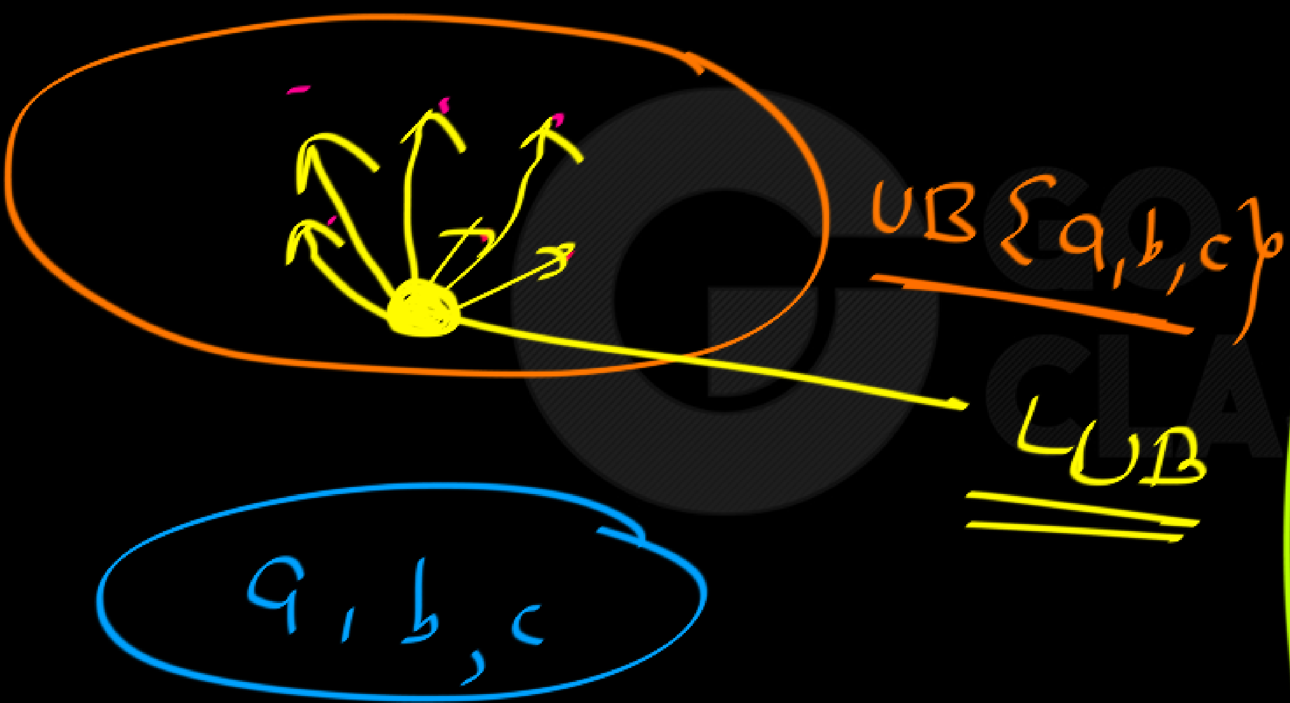
Least upper bound of a set X :

In UB(X), find least element.

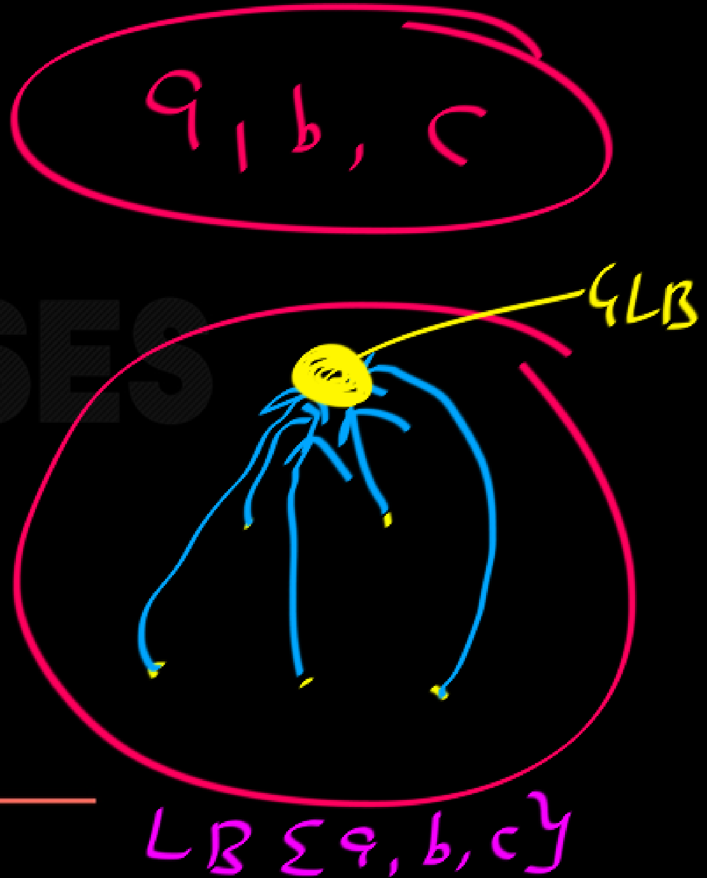
Greatest lower bound of a set X :

In LB(X), find Greatest element.

LUB{a, b, c}



GLB{a, b, c}



Least Upper and Greatest Lower Bounds

Definition: If a is an upper bound for S which is related to all other upper bounds then it is the *least upper bound*, denoted lub(S). Similarly for the *greatest lower bound*, glb(S).

Note:

If poset has :

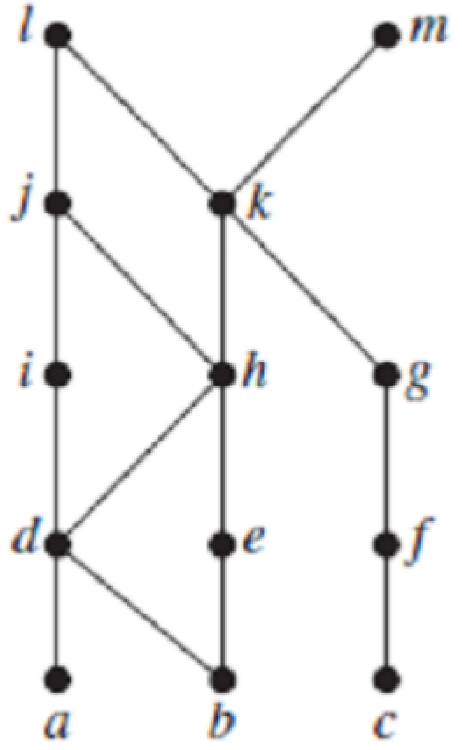
more than one maximal element ⟶ No Greatest

more than one minimal element ⟶ No least

1. For the Hasse diagram given below; find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets;
   (i) $\{d, k, f\}$

minimal — a, b, c

maximal — l, m



LB{d, k, f} = ϕ

UB{e, g} = {k, l, m}

LB{e, g} = ϕ

LB{l, m} = {k, g, f, c, h, e, b, a, d}

greatest — DNE

least — DNE

UB{d, k, f} = {k, l, m}

UB{l, m} = ϕ

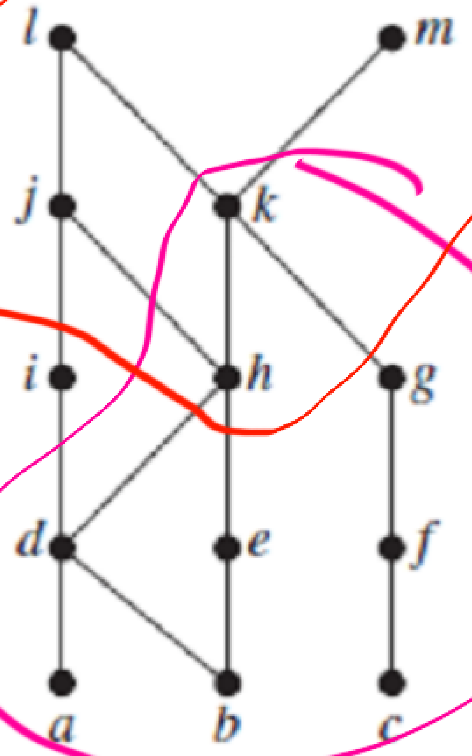
$$
\begin{tikzcd}
l & , & m \\
& k \arrow[ul] \arrow[ur] & \\
& x \arrow[u] &
\end{tikzcd}
$$

x in $LB\{l,m\}$

1. For the Hasse diagram given below; find maximal, minimal, greatest, least, LB, glb, UB, lub for the subsets;
 (i) $\{d, k, f\}$

GLB{l, m} = K

LUB{l, m} = DNE



UB{d, e}

LUB{d, e} = h

LB{l, m}

Greatest = k

UB{d, e} = {h, J, k, l, m}

Greatest element

least "

GLB

LUB

unique if exist

DNE

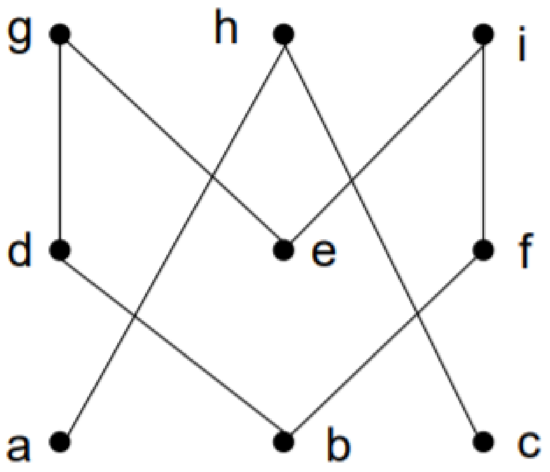
or

uniquely exist.

Extremal Elements: Example 2

Give lower/upper bounds & glb/lub of the sets:

{d,e,f}, {a,c} and {b,d}



{d,e,f}

- Lower bounds: ∅, thus no glb
- Upper bounds: ∅, thus no lub

{a,c}

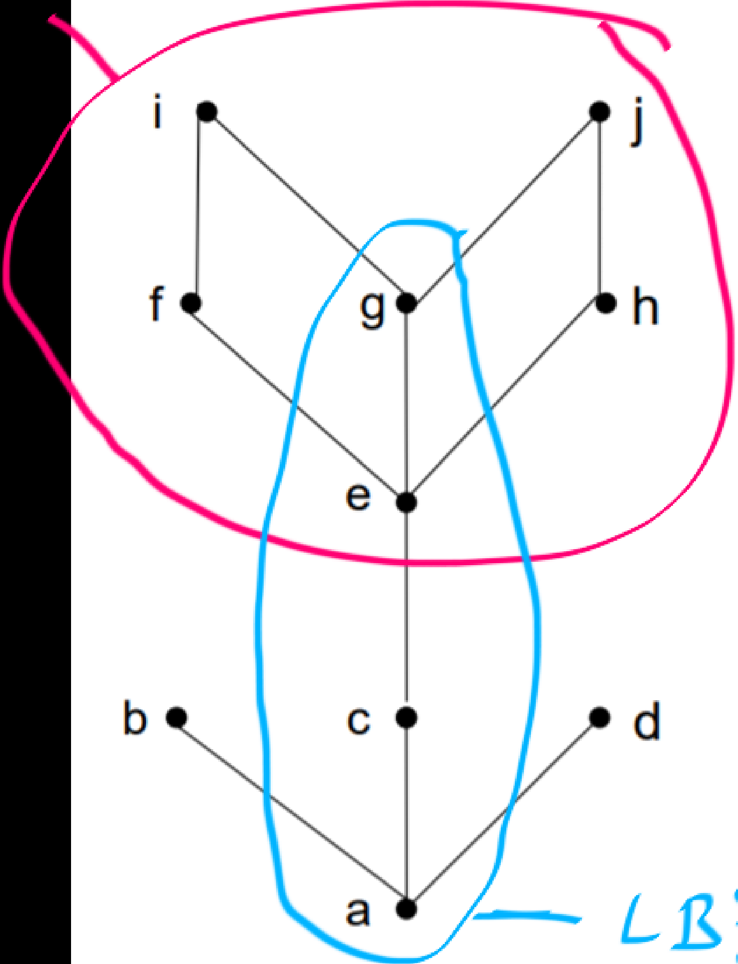
- Lower bounds: ∅, thus no glb
- Upper bounds: {h}, lub: h

{b,d}

- Lower bounds: {b}, glb: b
- Upper bounds: {d,g}, lub: d because $d \prec g$

Extremal Elements: Example 3

UB{c,e}



LB{i,j}

least

- Minimal/Maximal elements?
  - Minimal & Minimum element: a
  - Maximal elements: b,d,i,j

maximum —— DNE

- Bounds, glb, lub of {c,e}?
  - Lower bounds: {a,c}, thus glb is c
  - Upper bounds: {e,f,g,h,i,j}, thus lub is e
- Bounds, glb, lub of {b,i}?
  - Lower bounds: {a}, thus glb is [redacted] a
  - Upper bounds: ∅, thus lub DNE

a R all so a is least. ✓

noone else R a so a is minimal.

UB $\{c, e\} = \{e, f, i, g, j, h\}$

LUB $\{c, e\} = e$

LB $\{c, e\} = \{c, a\}$

GLB $\{c, e\} = c$

Q: If a is some $LB(X)$ then a definitely belongs to X? — wrong

$LB\{i,j\} = \{g, e, c, a\}$

$GLB(X) \in X$ — Incorrect

$GLB\{i,j\} = g$

An element, $b \in X$, is the *greatest lower bound of A* iff the set of lower bounds of A is nonempty and if b is the greatest element of this set.

An element, $m \in X$, is the *least upper bound of A* iff the set of upper bounds of A is nonempty and if m is the least element of this set.

Remarks:

1. If b is a lower bound of A (or m is an upper bound of A), then b (or m) may not belong to A.

4. The greatest lower bound (or the least upper bound) of A may not belong to A. We use the notation $\bigwedge A$ for the greatest lower bound of A and the notation $\bigvee A$ for the least upper bound of A.

In computer science, some people also use $\bigsqcup A$ instead of $\bigvee A$ and the symbol $\bigsqcup$ upside down instead of $\bigwedge$.

When $A = \{a, b\}$, we write $a \wedge b$ for $\bigwedge\{a, b\}$ and $a \vee b$ for $\bigvee\{a, b\}$.

The element $a \wedge b$ is called the *meet of a and b* and $a \vee b$ is the *join of a and b*. (Some computer scientists use $a \sqcap b$ for $a \wedge b$ and $a \sqcup b$ for $a \vee b$.)

Note:

mostly, we want to find GLB, LUB for pair of elements.

$$GLB\{a,b\} = a \wedge b = a \cap b$$
$$LUB\{a,b\} = a \vee b = a \cup b$$

Quickly finding GLB, LUB for two elements :

(1) $a \vee a = a$; $a \wedge a = a$

(2) If aRb then

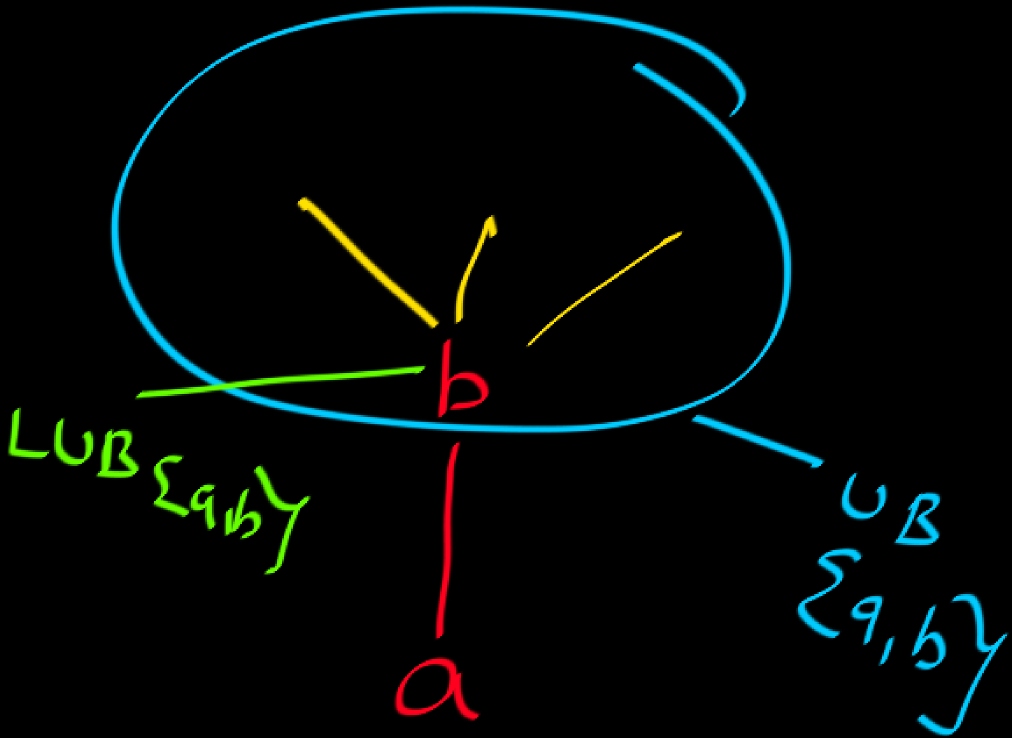
$a \vee b = b$; $a \wedge b = a$

aRb

$UB\{a,b\} = \{b, \text{----}\}$

b — $a \vee b$

a — $a \wedge b$



If $\underbrace{aRb}$ then

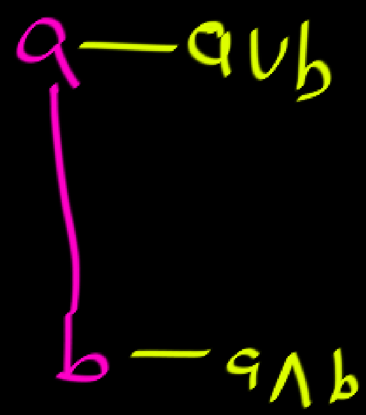
b — $a \vee b = LUB\{a,b\}$

a — $a \wedge b = GLB\{a,b\}$

Quickly finding GLB, LUB for two elements :

③ If a, b are Comparable then

Assume bRa



$a \vee b = a$; $a \wedge b = b$

Quickly finding GLB, LUB for two elements :

3 If a, b are Incomparable then

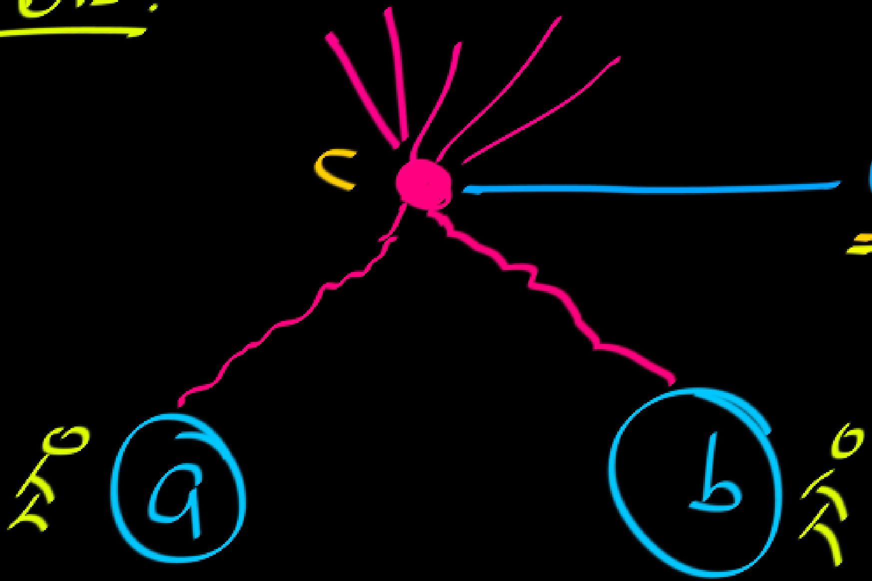
$a \vee b$ = "Unique" "first" Joining Point in upward Direction.

Quickly finding GLB, LUB for two elements :

3) If a, b are Incomparable then

$a \wedge b$ = "Unique" "first" meeting point in downward Direction.

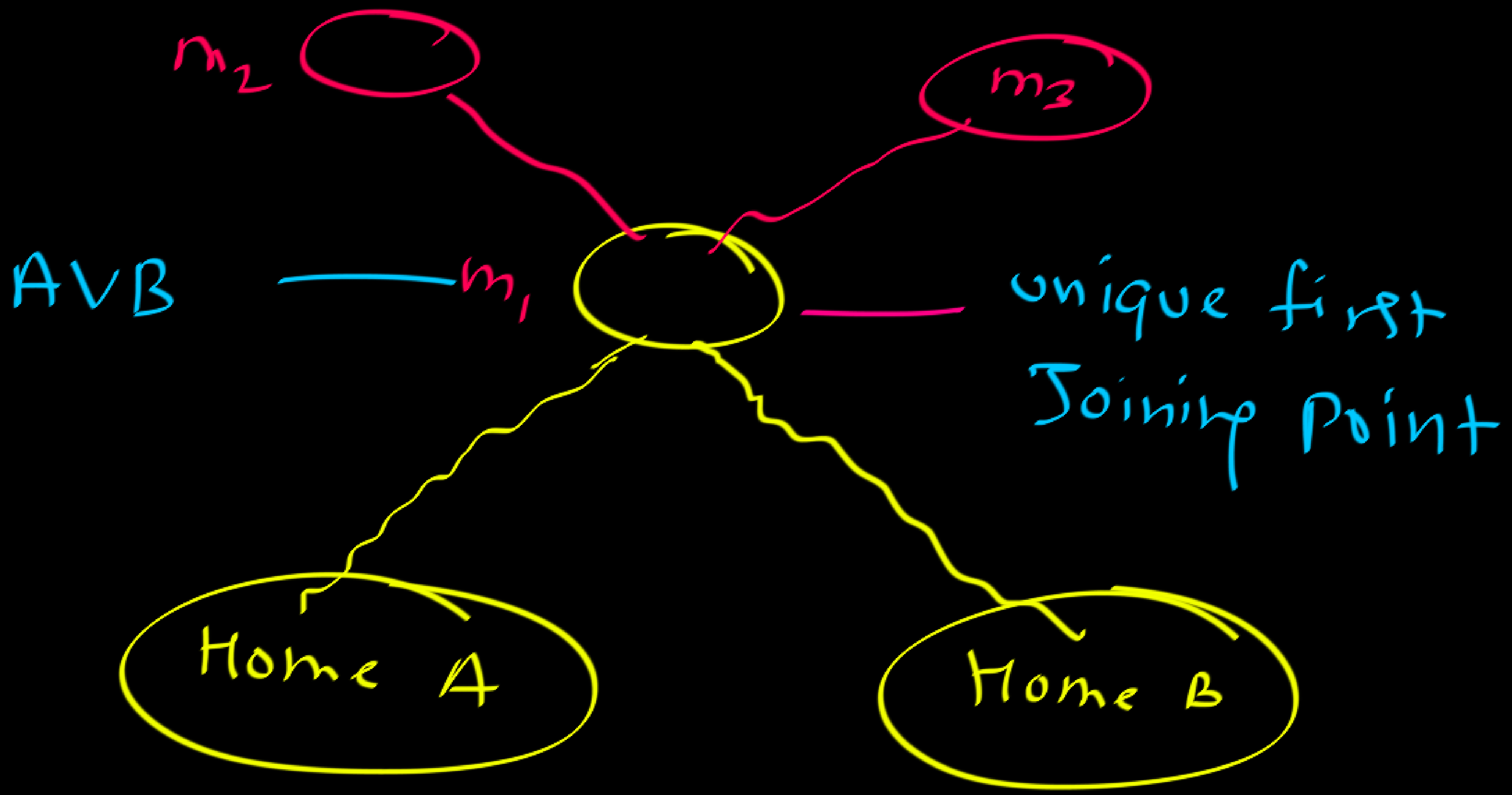
LUB:



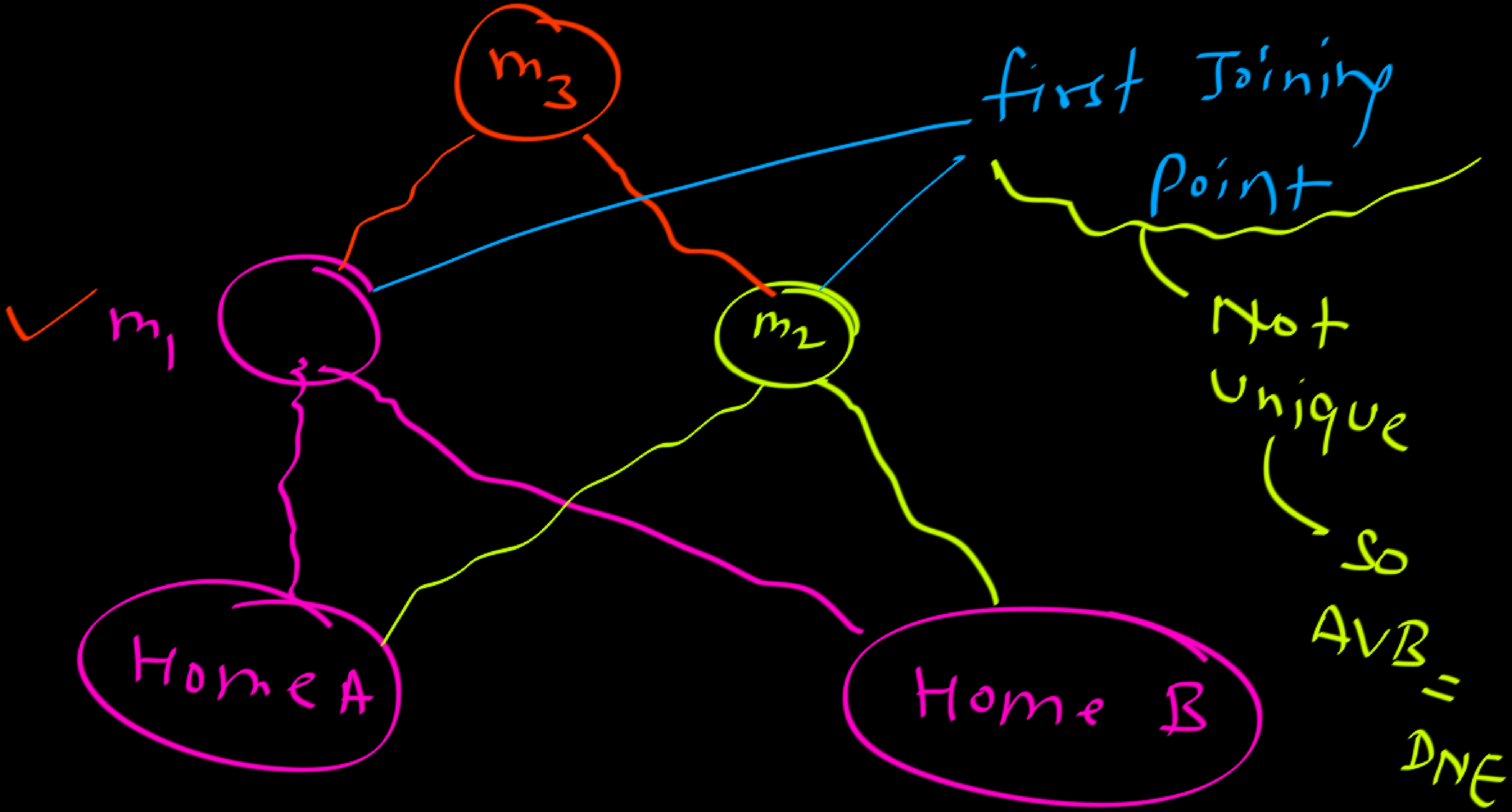
unique first Joining point then

$c = a \vee b$

LUB — unique if exist



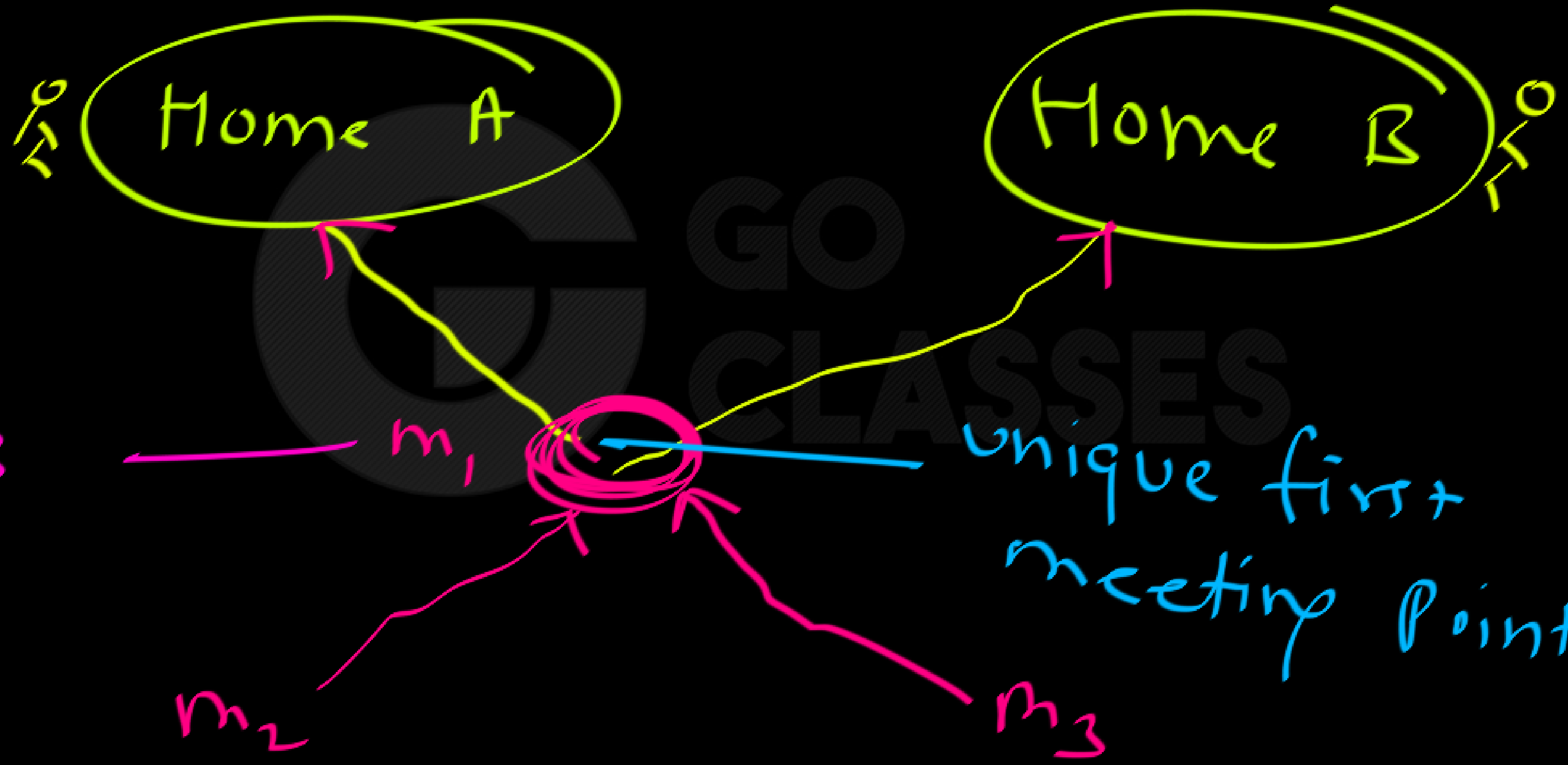
m_2
m_3
AVB
m_1
unique first
Joining Point
Home A
Home B



m_3
first Joining point
m_1
m_2
Not unique
so $A \vee B$ = DNE
Home A
Home B

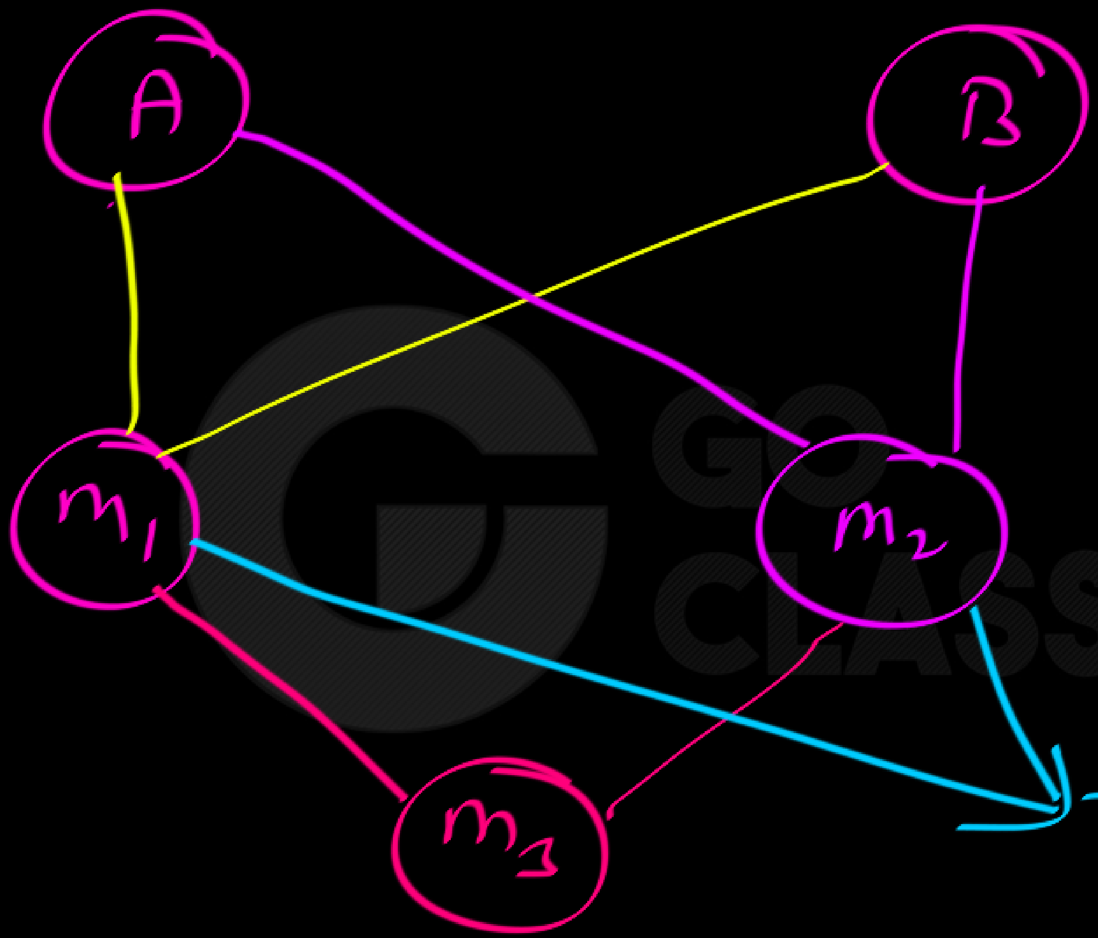
$A \wedge B$ — where A, B are not Comparable :

GLB



$A \wedge B$

unique first meeting point



$A \wedge B = DNE$

not unique

first meeting points

LUB$\{a,b\}$:

$$LUB\{a,b\} = a \vee b = a \cup b = \text{Join}\{a,b\}$$

GLB$\{a,b\}$:

$$GLB\{a,b\} = a \wedge b = a \cap b = \text{meet}\{a,b\}$$

Summary:

① $a \vee a = a, \quad a \wedge a = a$

② If aRb then $a \wedge b = a, \quad a \vee b = b$

b — $a \vee b$

a — $a \wedge b$

Summary:

(3) If a, b are not Comparable: