First Order Logic

Next Chapter:

English-FOL Translation

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GATE CSE AIR 53; AIR 67;
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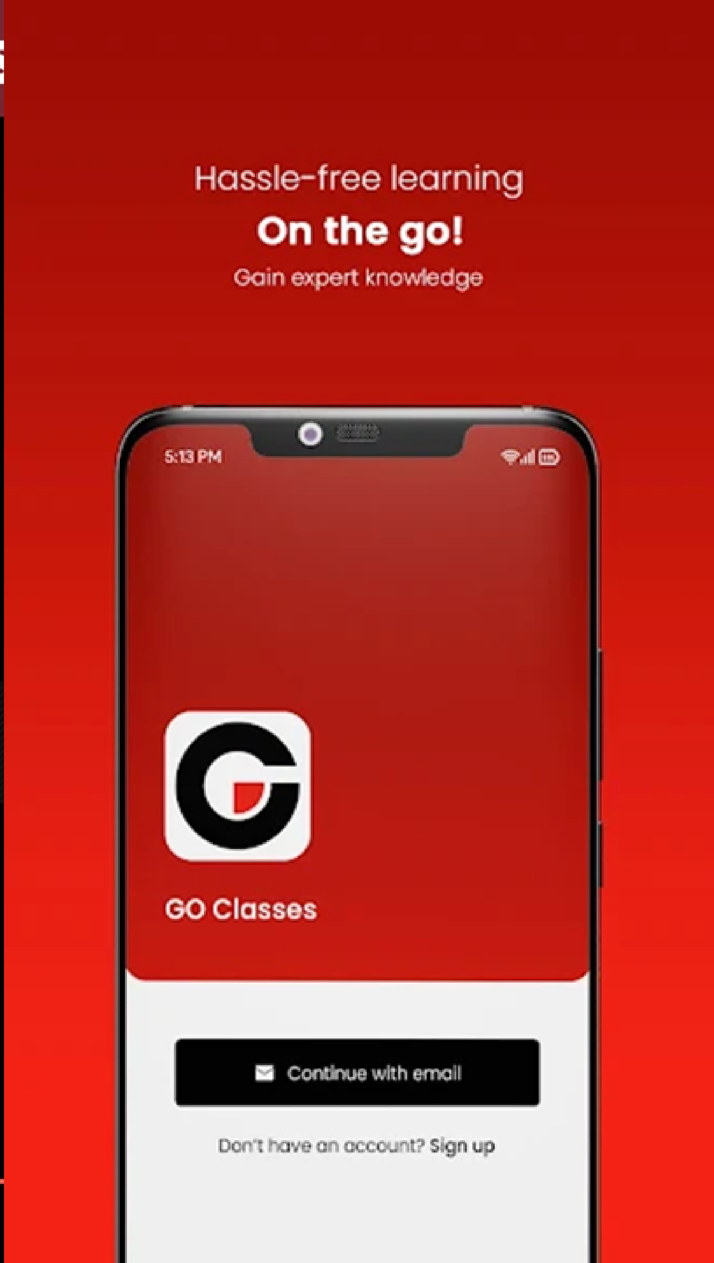

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First Order Logic

Next Topic:

Quantifier Tricky Notes

1.

Quantifiers, when Domain is empty??

NOTE:

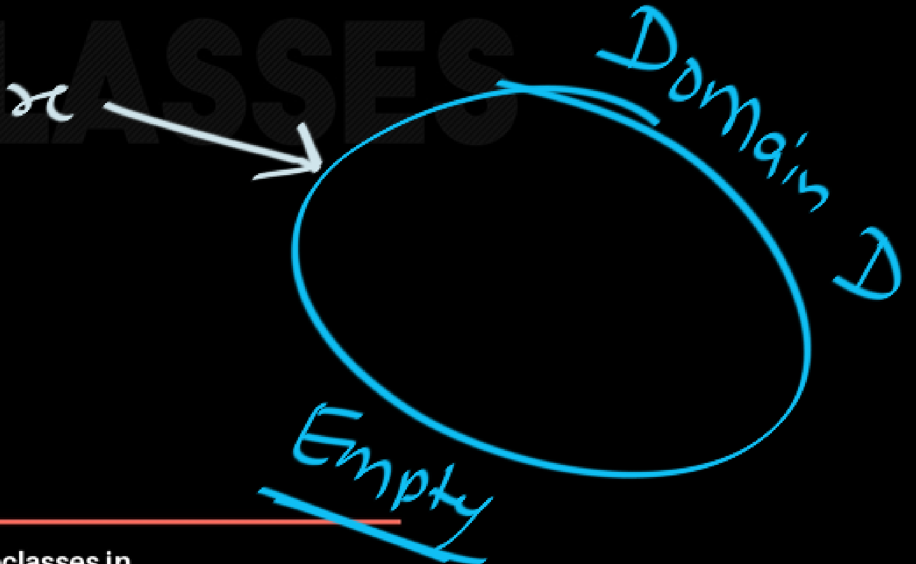
In FOL, Domain is ALWAYS Non-empty, unless it is explicitly given as Empty.

If the domain is explicitly given as Empty, then

How do Quantifiers behave??

$\forall_x P(x) = ? = T/F?$

$\exists_x P(x) = ? = T/F?$



Domain: Empty

$$\boxed{\forall_x P(x) = \text{True}}$$

$\equiv$ for every element x in the Domain, $P(x)$ is True.

Domain → Empty

No counterexample for $\forall x P(x)$

Useful Intuition:

Universally-quantified statements are true if there's no counterexample.

$\forall_x P(x) = \text{false}$ iff there is at least one Counterexample.

Domain: Empty

$$\boxed{\forall_x \left(q(x) \right)} = \text{True}$$

Reason: No Counterexample.

Useful Intuition:

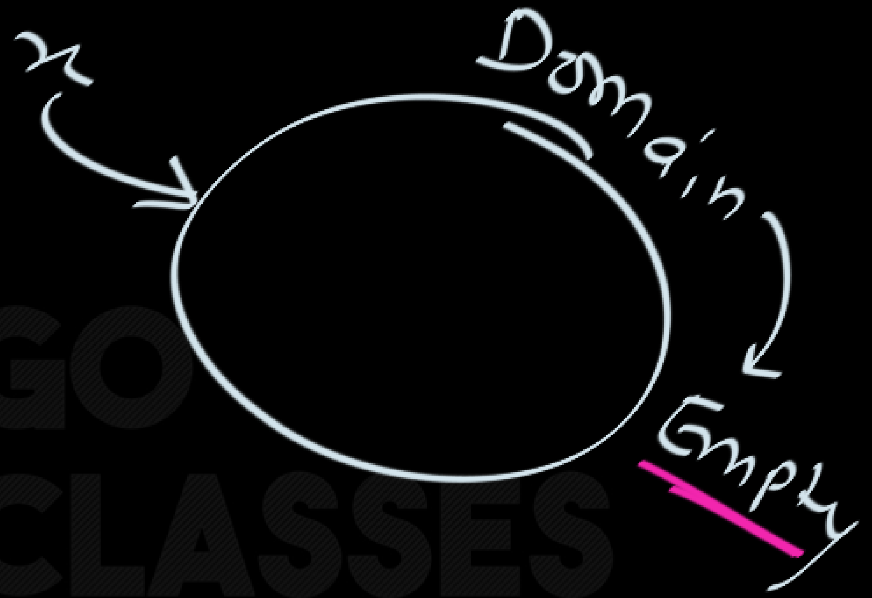
Existentially-quantified statements are False if there's no witness.

$\exists x\, P(x) = \text{True}$ iff there is at least one element in the Domain for which P is True.

Domain: Empty

$\exists x\, P(x) = \text{false}$

Existential statement



Quantifiers, when Domain is empty??

$\rightarrow \forall x P(x) = \text{True}$

$\rightarrow \exists x P(x) = \text{false}$

Fun with Edge Cases

Domain

Empty

$\boxed{\forall x.\ Smiling(x)}$ = T/F ?

Fun with Edge Cases

$$\forall x.\ Smiling(x)$$

Fun with Edge Cases

Universally-quantified statements are said to be ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Fun with Edge Cases

$$\exists x.\ Smiling(x)$$

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since nothing exists, period!

$\cancel{\exists x.\ Smiling(x)}$

2.
Quantifiers, when No free variable??

Domain: D

P : $\boxed{2+2=4}$ → No free variable

→ proposition

$\forall_x P \equiv \forall_x (2+2=4)$

$= T/F$??

D

a, b, c, d

Domain: D

P : $\boxed{2+2=4}$ → No free variable

→ proposition

→ True

$\forall_x P \equiv \forall_x (2+2=4)$

$= \text{True}$

D

a, b, c, d,

Domain: D

P : $\boxed{2+2=5}$

No free variable

proposition

false

$\forall_x P = ? = T/F?$

D

a, b, c, d

Domain: D

P : $\boxed{2+2=5}$ → No free variable

→ proposition

→ false

$\forall_x P = \forall_x (2+2=5)$

$=$ false.

D

a, b, c, d,

$P : \boxed{2+2=6}$ — False

$Q : 2+2=4$ — True

Propositions

$\forall_x P \equiv P = \text{false}$

$\forall_x P \equiv P \equiv \text{false}$

No free x

Domain

$a_1, a_2, \ldots$

$p : \boxed{2+2=6}$ False

$\varphi : 2+2=4$ True

Propositions

$\forall_x\, p \equiv p = \text{false}$

$\forall_x\, \varphi \equiv \varphi \equiv \text{True}$

No free x

Domain

$a_1, a_2, \ldots$

$P: \quad 2+2=6$ — False

$Q: \quad 2+2=4$ — True

Propositions

$\exists_x P = P = \text{false}$

$\exists_x P = \text{false}$

$$\left.\begin{array}{l} P : \; 2+2=6 \quad \text{False} \\ \varphi : \; \boxed{2+2=4} \quad \text{True} \end{array}\right\} \underline{\text{Propositions}}$$

$$\exists_x P = P = \text{false}$$

$$\exists_x \varphi = \text{True} \checkmark$$

$$\exists_x \varphi \equiv \varphi = \text{True} \checkmark$$

Domain

$a_1, a_2, \ldots$

Domain: D

P : $\boxed{2+2=4}$ → No free variable → proposition

$\forall_x P = P = \text{True}$

$\exists_x P = P = \text{True}$

D

a, b, c, d

$P(x): x > 2$; Domain $= \mathbb{N}$

$\exists x \, (2+2=4)$: True

$\exists x \, (2+2=5)$: false

$\forall x \, (2+2=5)$: false

Domain

1, 2, 3, ...

P with No free x

OR

P with No variable x

$$\boxed{\begin{aligned} \forall_x P &= P \\ \exists_x P &= P \end{aligned}}$$

3.

Predicate with No variable, is a proposition.

Predicate:

Variable

$P(x_1, x_2, ---, x_n)$: n-ary predicate

$P(a_1, a_2, ---, a_n)$ = Proposition

$a_1, a_2, --- , a_n ---$

Domain

Predicate :

Variable

$P(x_1, x_2, ---, x_n)$: n-ary predicate

$P(x)$: unary predicate

$P(x,y)$: binary "

P : proposition

P : $\boxed{2+2=5}$

proposition

Predicate with 0 variable.

1. If Empty Domain:

$$\forall x P(x) = \text{True} \quad ; \quad \exists x P(x) = \text{false}$$

2. If P has No free variable x

$$\forall x P = P \quad ; \quad \exists x P = P$$

3. Predicate with 0 variable, is called Proposition.

First Order Logic

Next Topic:

English – FOL Translation

Example 1:

Let, Domain: Set of All Animals.
Let, Predicate, cute(x): x is cute.

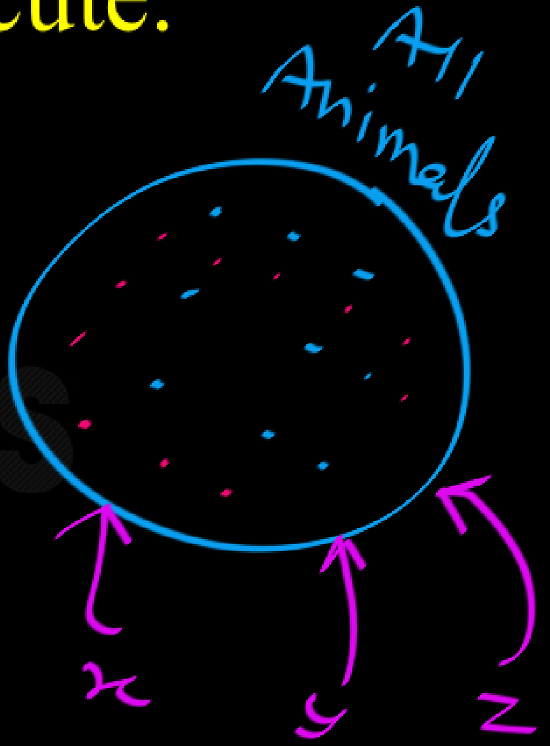
Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

1. Every animal is cute.

$$\boxed{\forall_x \; c(x)}$$

for all x in the Domain, x is cute.



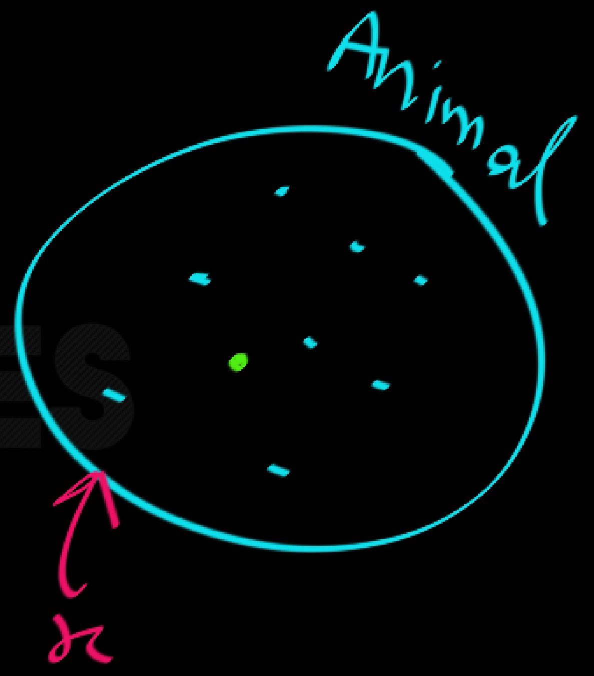
Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

2. Some animal is cute.

$$\boxed{\exists_x \, c(x)}$$

for some element x in the Domain, x is cute.

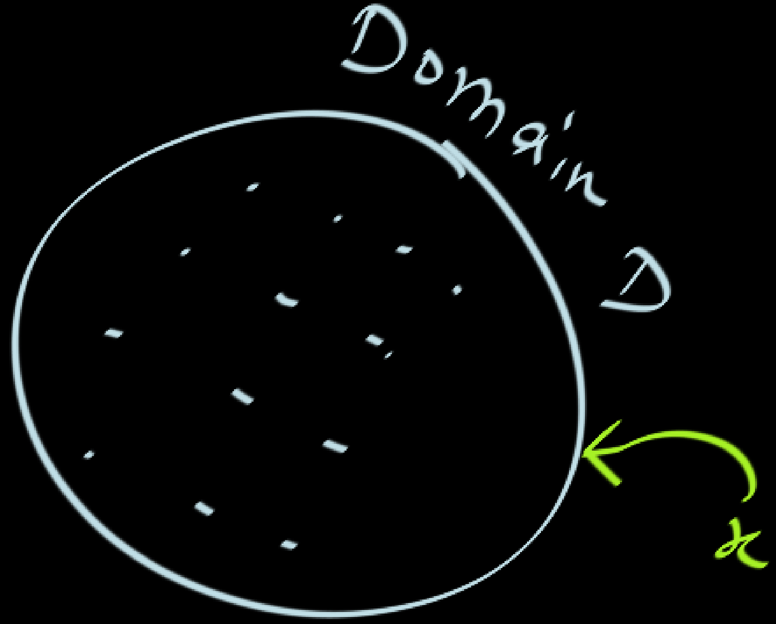


for all $x \in D$, $P(x)$

$= \forall_x P(x)$

for some $x \in D$, $P(x)$

$= \exists_x P(x)$



Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

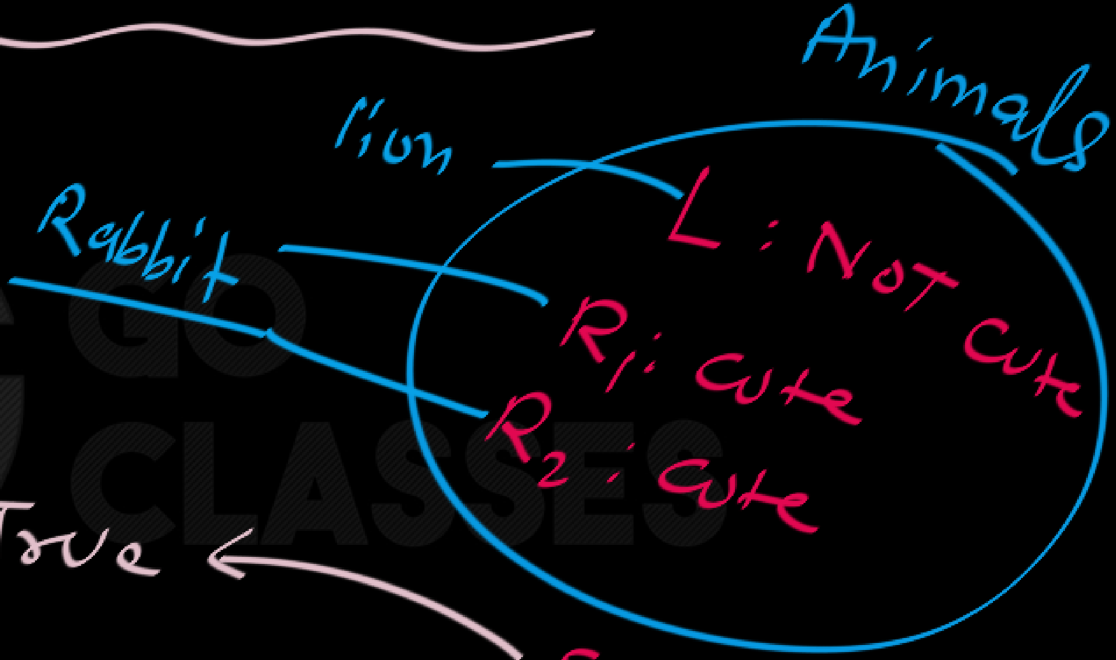
3. Every rabbit is cute.

$\boxed{\forall_x \; c(x)}$ ✗

Every animal is cute.

S : Every rabbit is cute.

E : $\boxed{\forall_x \, c(x)}$



$\Leftarrow$ S : True

E : false

Scenario

E does not Express S

S : Every rabbit is cute.

$\equiv$ for every animal x,

if x is a Rabbit then

x is cute.

$$\equiv \forall_x \left(\text{Rabbit}(x) \longrightarrow \text{cute}(x) \right)$$

S : Every rabbit is cute.

$\equiv$ for every animal x,
if x is a Rabbit then
x is cute.

$$\equiv \forall_x \left(\text{Rabbit}(x) \longrightarrow \text{Cute}(x) \right)$$ ✓

S : Every rabbit is cute.

E : $\boxed{\forall_x \left(\text{Rabbit}(x) \land \text{Cute}(x) \right)}$

for every animal x, x is Rabbit & cute.

$\equiv$ every animal is Rabbit & cute.

S : Every rabbit is cute.

$E : \forall x \, (\text{Rabbit}(x) \wedge \text{Cute}(x))$

Animal

L, T

R_1 : cute

R_2 : cute

Scenario

S : True

E : False

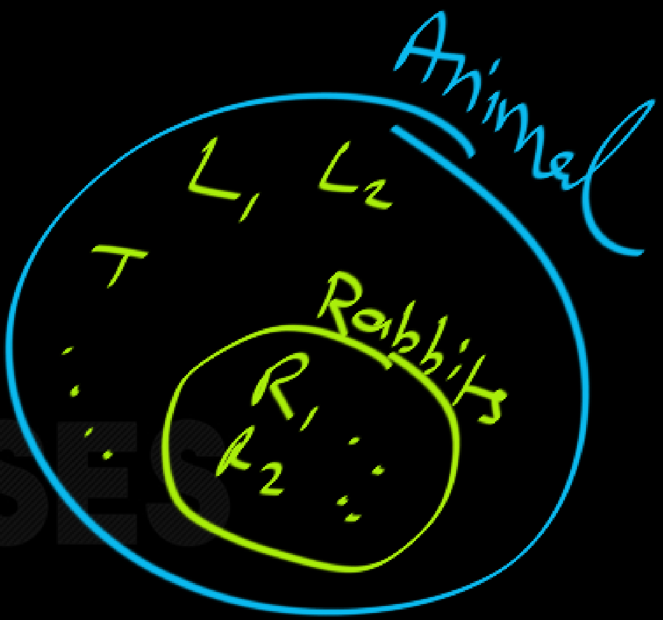
E Does not Express S.

S : Every rabbit is cute.

① $\forall x \; c(x)$ ✗

② $\forall x \left(R(x) \land C(x) \right)$ ✗

③ $\forall x \left(R(x) \rightarrow C(x) \right)$ ✓



Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

4. Some rabbit is cute.

$\boxed{\exists_x\, c(x)}$

Some animal is cute.

S : Some rabbit is cute.

≡ for some animal x, if x is Rabbit then x is cute. ✗

$$\equiv \boxed{\exists_x \left(R(x) \longrightarrow C(x) \right)}$$ ✗

E :

S : Some rabbit is cute.

E : $\exists x \, (R(x) \rightarrow C(x))$

Animals

L, T

R_1 : Not cute

R_2 : Not cute

Scenario

E Does not Express S.

S : false

E : True

Some rabbit is cute.

$\equiv$ there is some animal x,
if x is rabbit, x is cute.

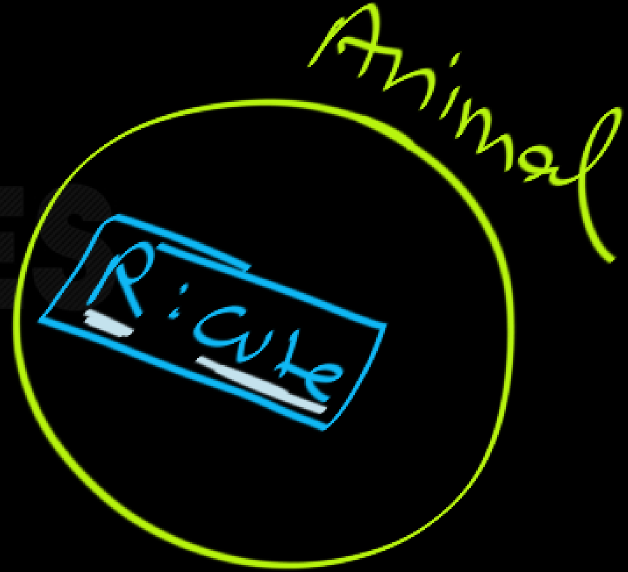
Some rabbit is cute.

$$\exists x \left(R(x) \rightarrow C(x) \right)$$

Some rabbit is cute.

≡ At least one Rabbit is cute.

✓ $\exists x \left(R(x) \wedge C(x) \right)$



Some Rabbit is cute.

1. $\exists x \; C(x)$ ✗

2. $\exists x \left(R(x) \rightarrow C(x) \right)$ ✗

3. $\boxed{\exists x \left(R(x) \wedge C(x) \right)}$ ✓

Using the predicates

- $Happy(x)$, which states that x is happy, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a sentence in first-order logic that says

every happy person wears a hat.

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- $Happy(x)$, which states that x is happy, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a sentence in first-order logic that says

every happy person wears a hat.

$\forall x\ (\text{happy}(x) \longrightarrow \text{wearingHat}(x))$ ✓

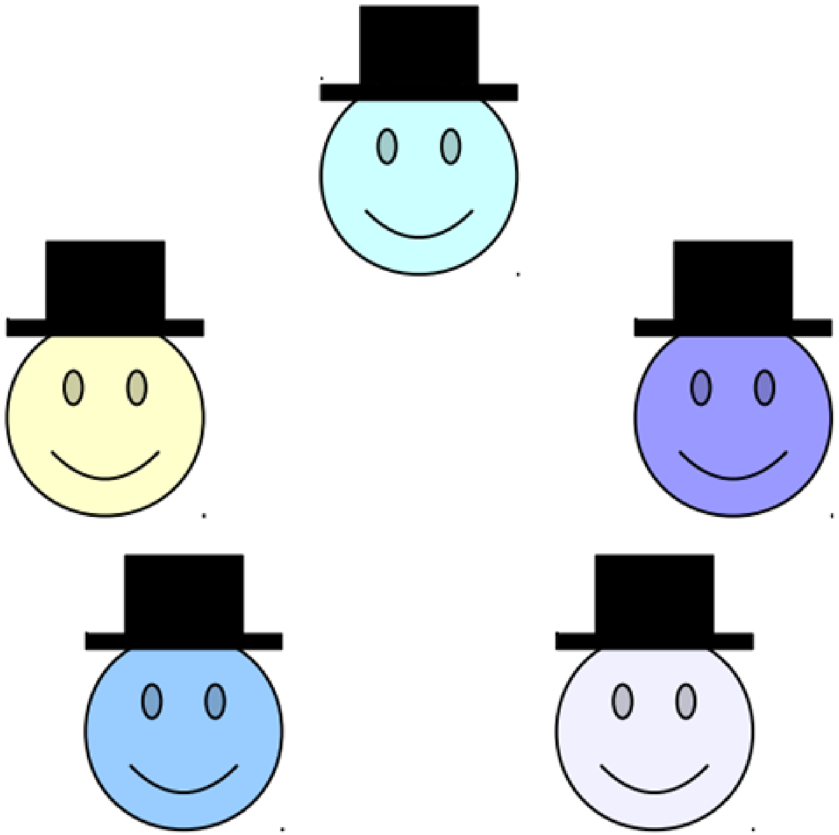
Using the predicates

- *Happy*(x), which states that x is happy, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

every happy person wears a hat.

$\forall x\ (\text{happy}(y) \wedge \text{wearHat}(x))$ ✗



"Every happy person wears a hat."	***True***
$\forall x.\ (Happy(x) \land WearingHat(x))$	***True***
$\forall x.\ (Happy(x) \to WearingHat(x))$	***True***

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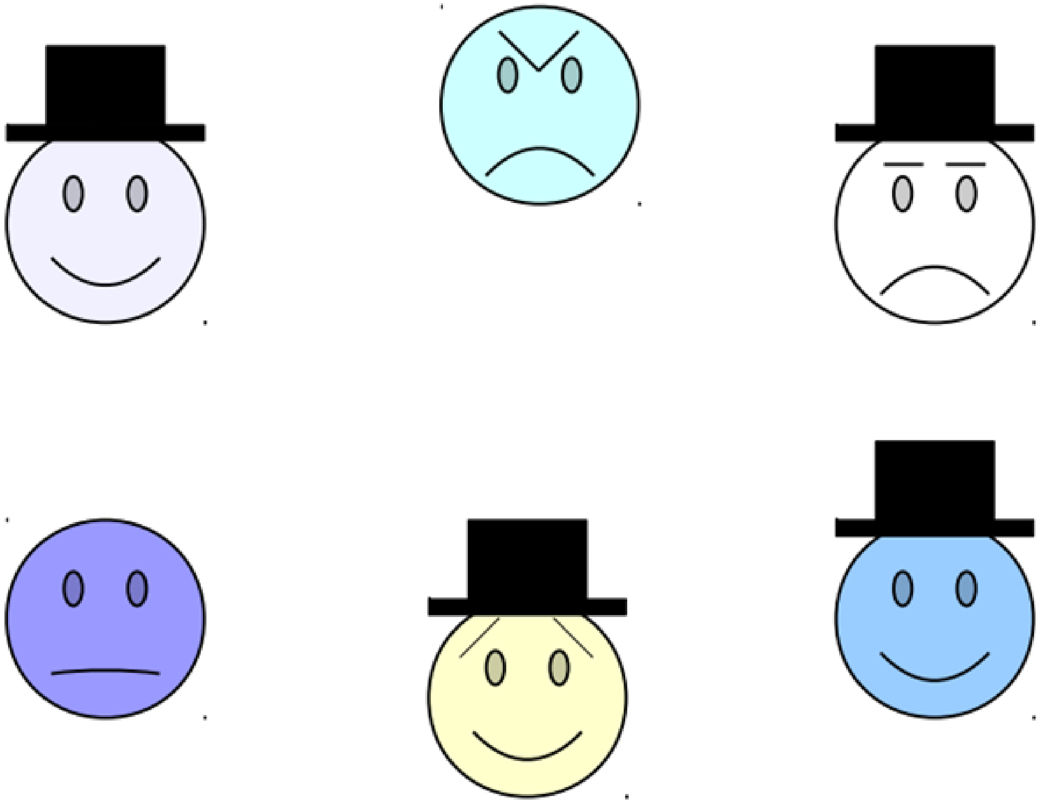
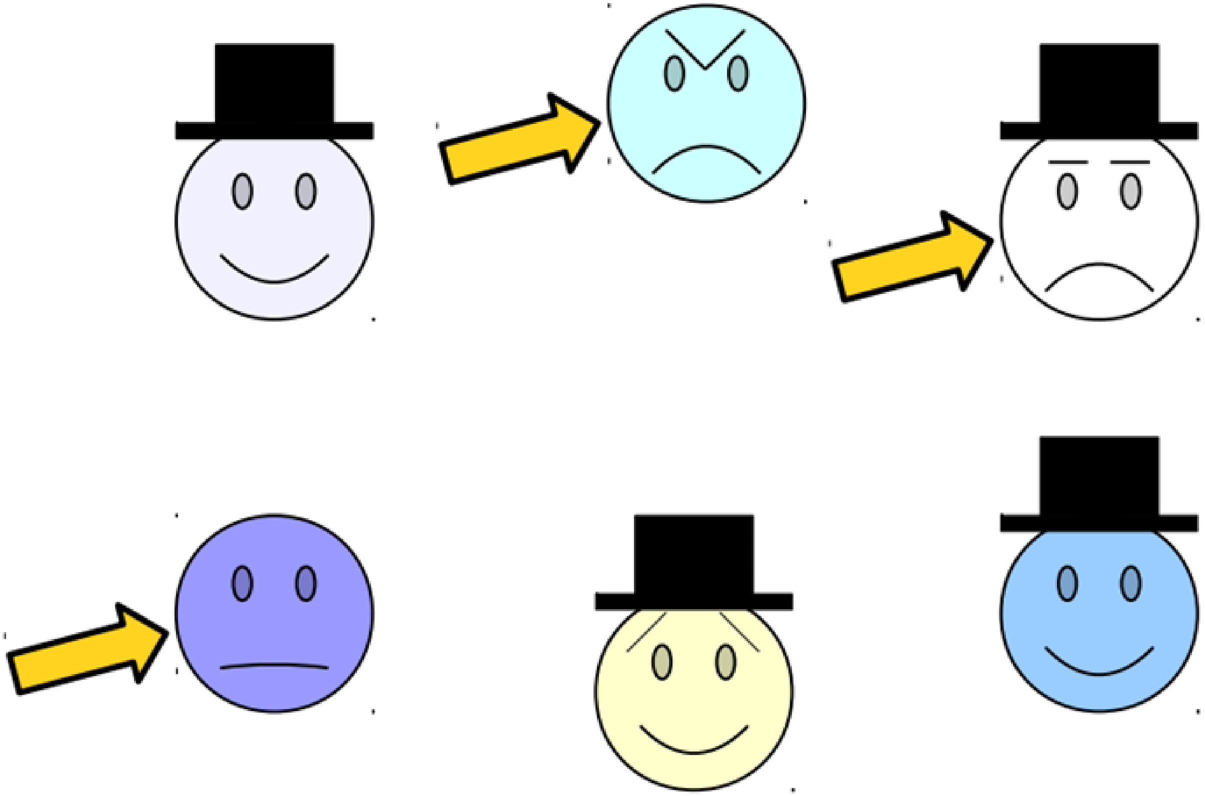


"Every happy person wears a hat." ***True***

$$\forall x.\ (Happy(x) \land WearingHat(x))$$

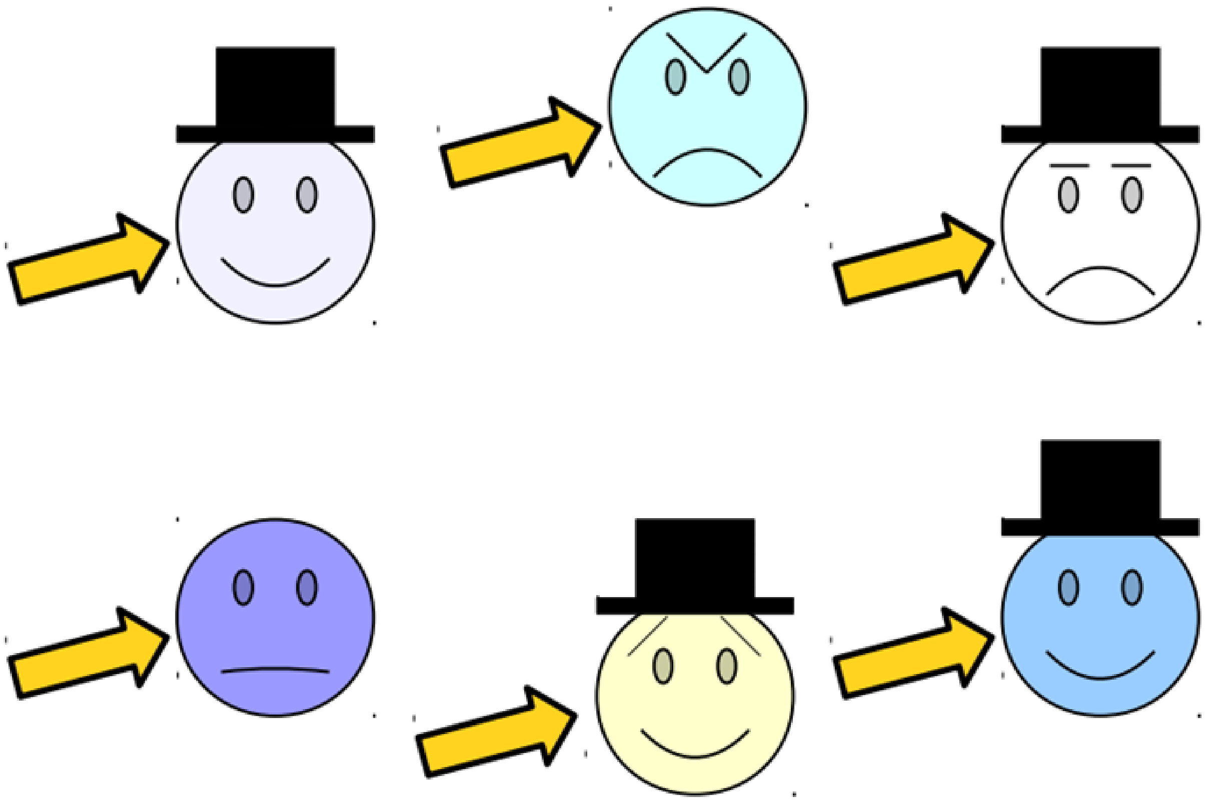
$$\forall x.\ (Happy(x) \to WearingHat(x))$$



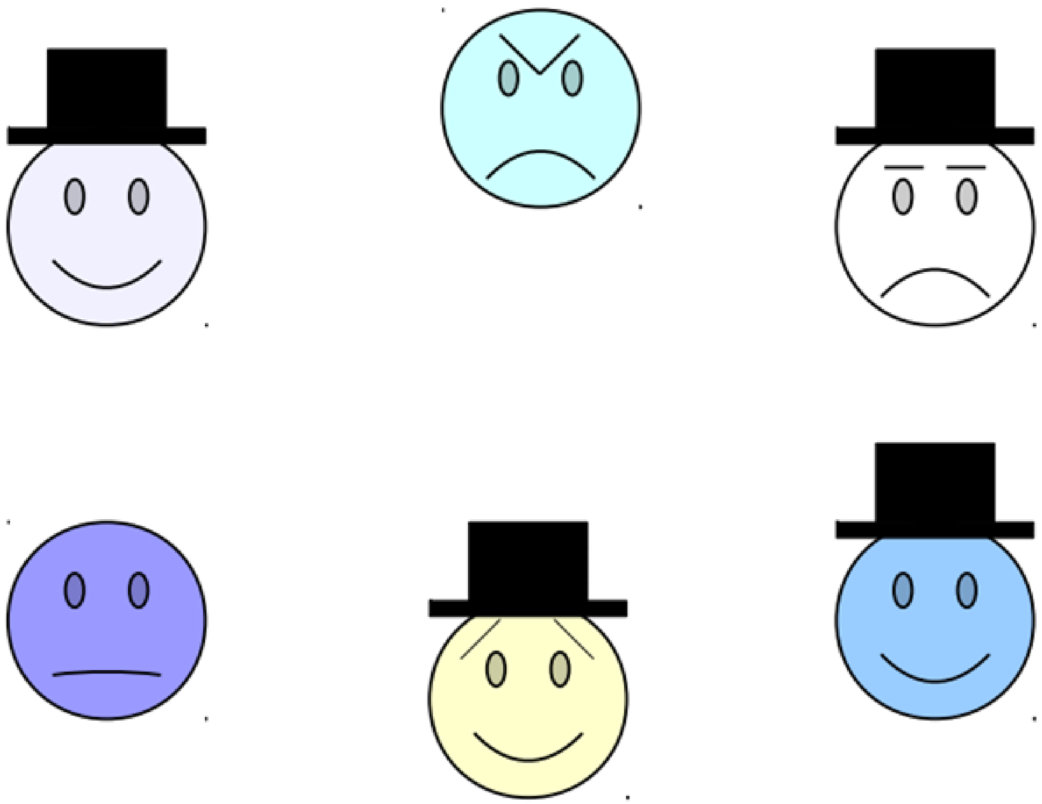
"Every happy person wears a hat." ***True***

$\forall x. \, (Happy(x) \land WearingHat(x))$ ***False***

$\forall x. \, (Happy(x) \rightarrow WearingHat(x))$



“Every happy person wears a hat.”	***True***
$\forall x. (Happy(x) \land WearingHat(x))$	***False***
$\forall x. (Happy(x) \to WearingHat(x))$	***True***



"Every happy person wears a hat."	***True***
$\cancel{\forall x. (Happy(x) \land WearingHat(x))}$	***False***
$\forall x. (Happy(x) \to WearingHat(x))$	***True***

“All P's are Q's”

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property P but not have property Q.

All A's are B's

for every element x, if $A(x)$ then $B(x)$

$$\equiv \forall_x \left(A(x) \rightarrow B(x) \right) \checkmark$$

Using the predicates

- $Happy(x)$, which states that x is happy, and
- $WearingHat(x)$, which states that x is wearing a hat,

write a sentence in first-order logic that says

***some happy person wears a hat*.**

Using the predicates

- *Happy*(x), which states that x is happy, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

some happy person wears a hat.

$$\equiv \quad \exists x\,(\text{happy}(x) \wedge \text{WearHat}(x))$$

Using the predicates

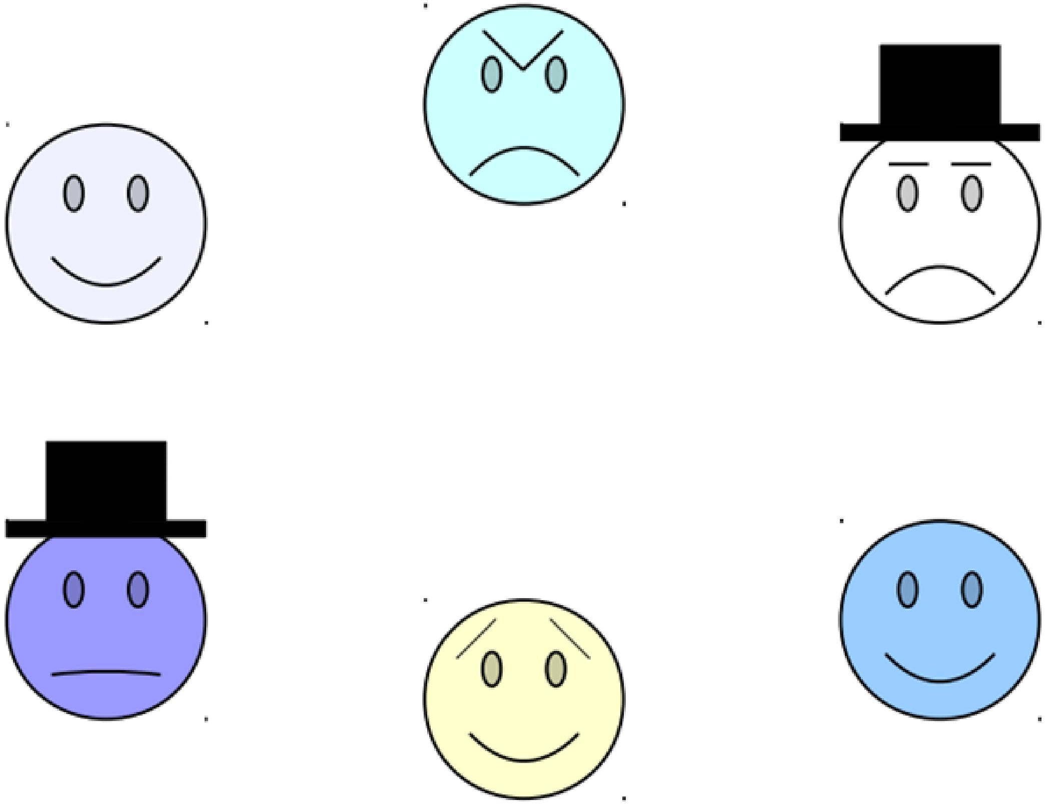
- *Happy*(x), which states that x is happy, and
- *WearingHat*(x), which states that x is wearing a hat,

write a sentence in first-order logic that says

false **some happy person wears a hat.**

scenario

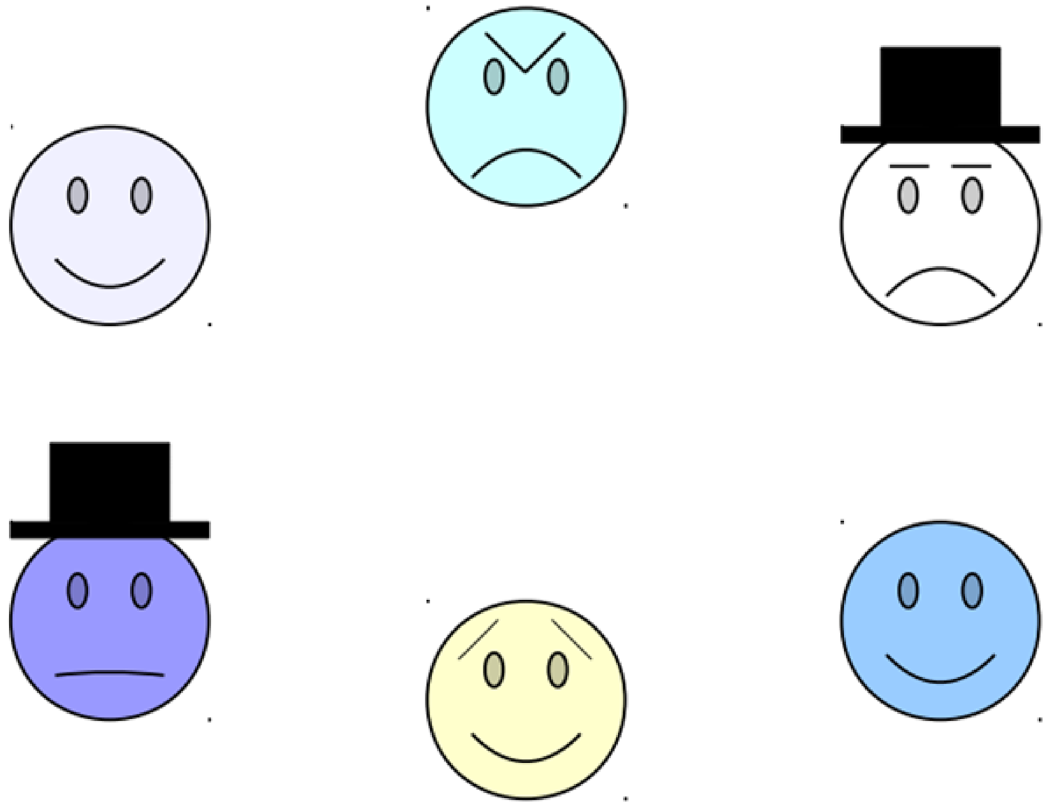
True $\exists x \left(H(x) \rightarrow WH(x) \right)$ F



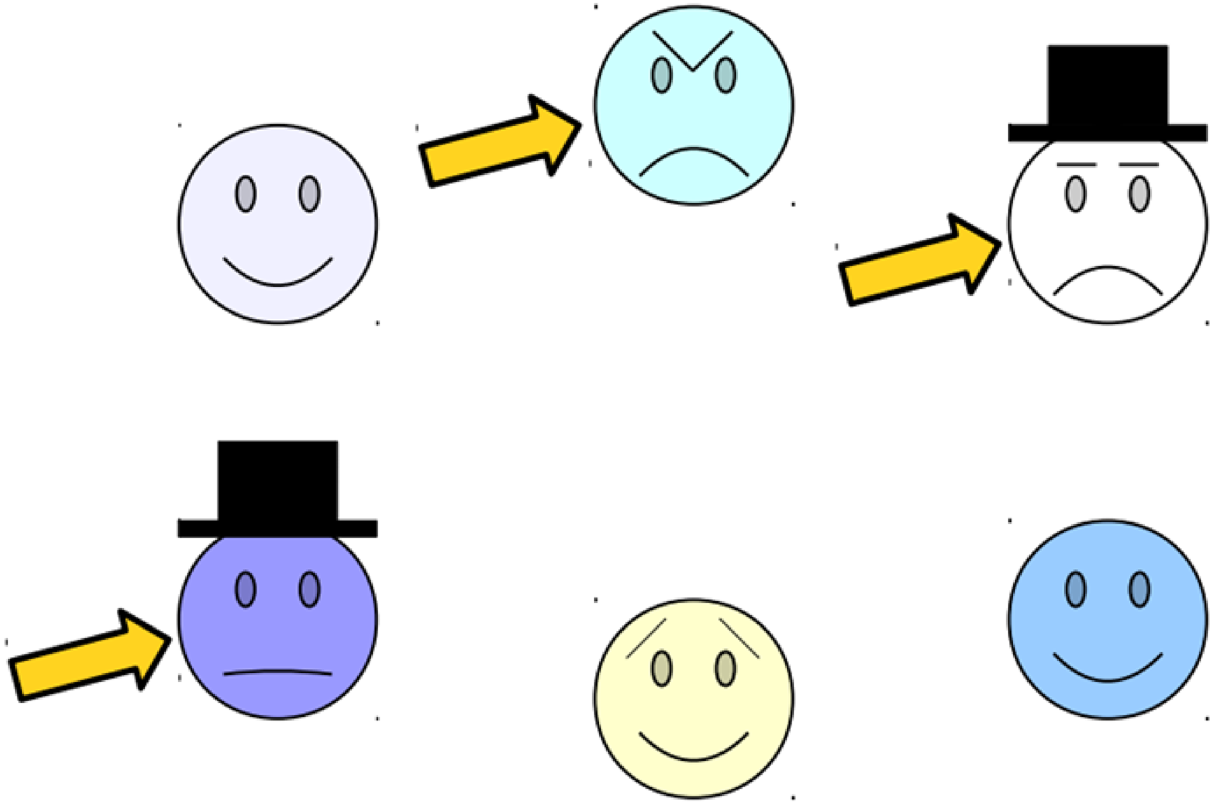
"Some happy person wears a hat."

$$\exists x.\ (Happy(x) \land WearingHat(x))$$

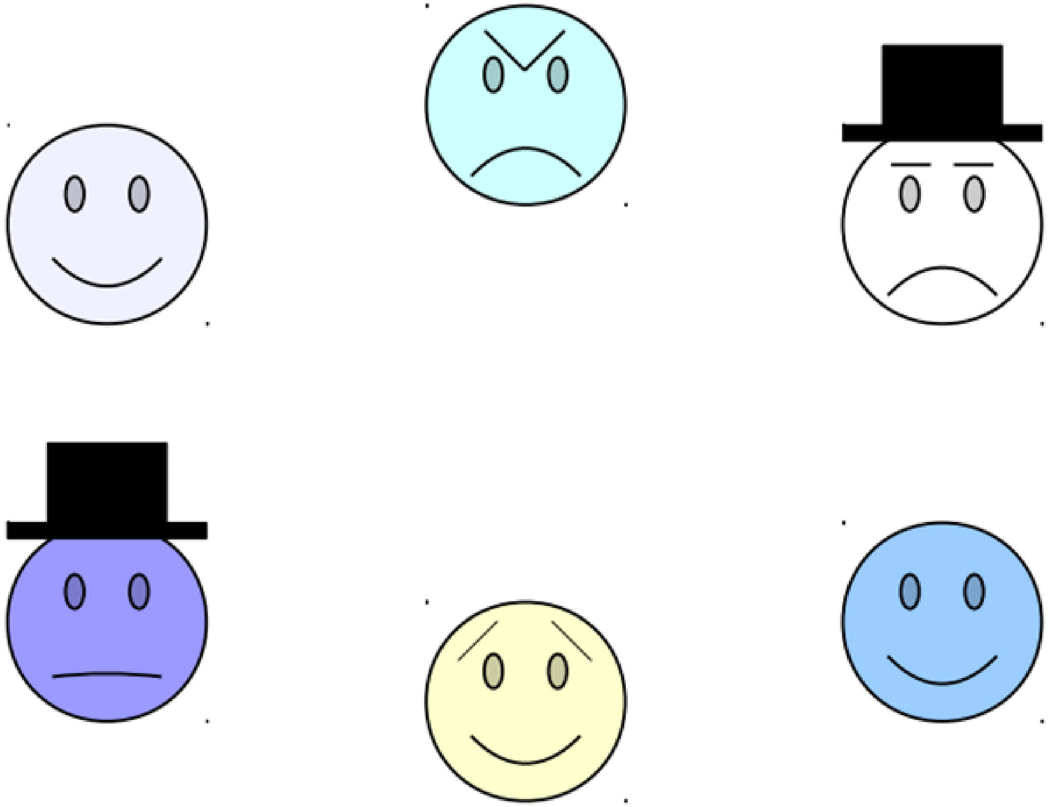
$$\exists x.\ (Happy(x) \to WearingHat(x))$$



"Some happy person wears a hat."	***False***
$\exists x.\ (Happy(x) \land WearingHat(x))$	***False***
$\exists x.\ (Happy(x) \rightarrow WearingHat(x))$	



"Some happy person wears a hat."	***False***
$\exists x.\ (Happy(x) \land WearingHat(x))$	***False***
$\exists x.\ (Happy(x) \to WearingHat(x))$	***True***

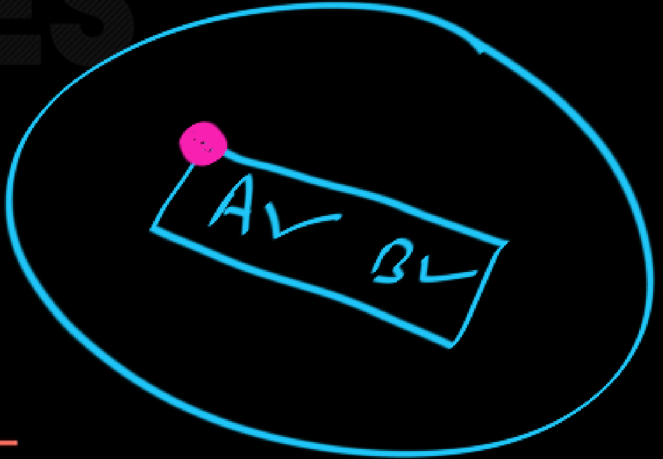


"Some happy person wears a hat."	***False***
$\exists x. \, (Happy(x) \land WearingHat(x))$	***False***
$\cancel{\exists x. \, (Happy(x) \to WearingHat(x))}$	***True***

Some A's are B's

$\equiv$ At least one A must be B.

$\equiv \exists x \left(A(x) \wedge B(x) \right)$



“Some P is a Q”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x. \, (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

Every A is B.

$$\forall x \left(A(x) \rightarrow B(x) \right)$$

Some A is B.

$$\exists x \left(A(x) \land B(x) \right)$$

Every A is B.

$\forall x \, (A(x) \rightarrow B(x))$

Some A is B.

$\exists x \, (A(x) \wedge B(x))$

Good Pairings

- The $\forall$ quantifier *usually* is paired with $\rightarrow$.

$$\forall x.\ (P(x) \rightarrow Q(x))$$

- The $\exists$ quantifier *usually* is paired with $\wedge$.

$$\exists x.\ (P(x) \wedge Q(x))$$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$

b) $\forall x(R(x) \land H(x))$

c) $\exists x(R(x) \rightarrow H(x))$

d) $\exists x(R(x) \land H(x))$

8. Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the domain consists of all animals.

a) $\forall x(R(x) \rightarrow H(x))$

b) $\forall x(R(x) \land H(x))$

c) $\exists x(R(x) \rightarrow H(x))$

d) $\exists x(R(x) \land H(x))$

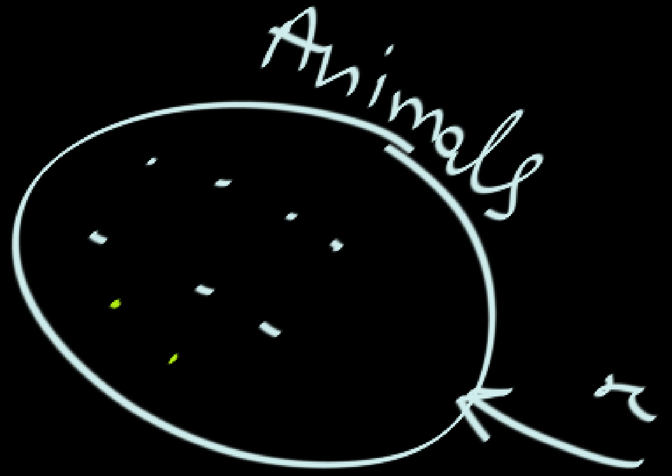
Every animal is a Rabbit & Hops.

Every Rabbit hops.

Some animal is "NOT Rabbit" OR Hops.

Some Rabbit Hops.

(a) $\forall_x \left(R(x) \rightarrow H(x) \right)$

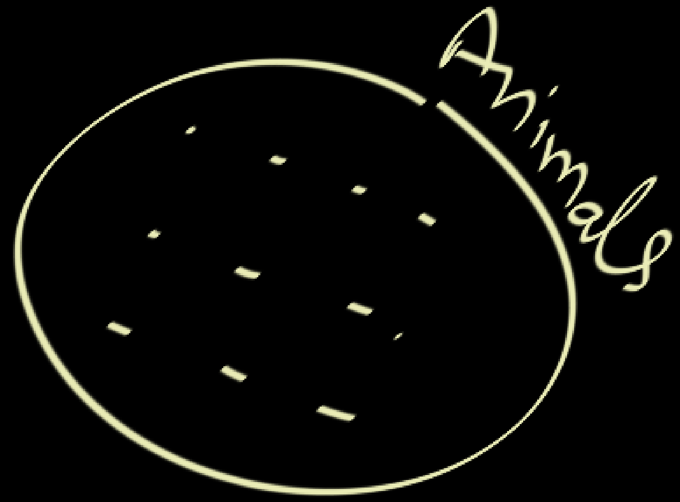


$\equiv$ for every animal x, if $R(x)$ then x hops.

$\equiv$ Every Rabbits Hops $\equiv \forall_x \left(R(x) \rightarrow H(x) \right)$

(b) $\forall x \, (R(x) \wedge H(x))$

$\equiv$ for every animal x, x is Rabbit & x hops.



$\equiv$ Every animal is a Rabbit & hops.

Every Rabbit Hops.

$$\equiv \quad \forall x \left(R(x) \rightarrow H(x) \right) \checkmark$$

$$\forall x \left(R(x) \wedge H(x) \right) \times$$

(c) $\exists x \left(R(x) \rightarrow H(x) \right)$

Some Rabbit Hops.

$\exists x \left(R(x) \wedge H(x) \right)$

$\exists x \left(R(x) \rightarrow H(x) \right)$

(C) $G : \exists x \left(R(x) \rightarrow H(x) \right)$ $\not\equiv$

S: $\boxed{\text{Some Rabbit Hops}}$

Scenario

L : T
R_1 : NOT Hop
R_2 : NOT Hop

Animals

S : false
G : True

G is NOT Correct Exp for S.

(c) $\exists x \left(R(x) \rightarrow H(x) \right)$

$\equiv \exists x \left(\overline{R(x)} \vee H(x) \right)$

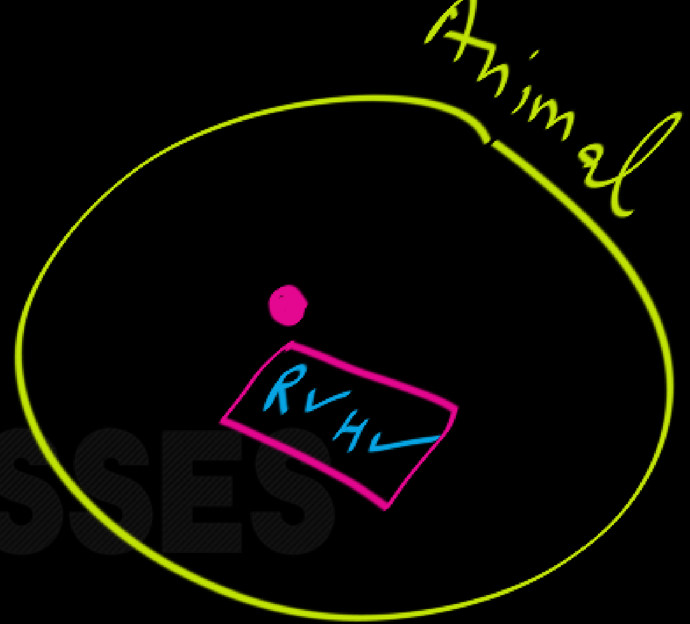
$\boxed{\alpha \rightarrow \beta \equiv \bar{\alpha} \vee \beta}$

$\equiv$ Some animal is "Not Rabbit" OR hops.

(D) $\exists x \left(R(x) \wedge H(x) \right)$

$\equiv$ Some animal is Rabbit & Hops

$\equiv$ Some Rabbit Hops. ✓



First Order Logic

Next Topic:

English – FOL Translation

Variations

Example 1:

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

1. Every animal is cute.

$$\boxed{\forall_x \; \text{cute}(x)}$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

2. Some animal is cute.

$\boxed{\exists_x \ cute(x)}$

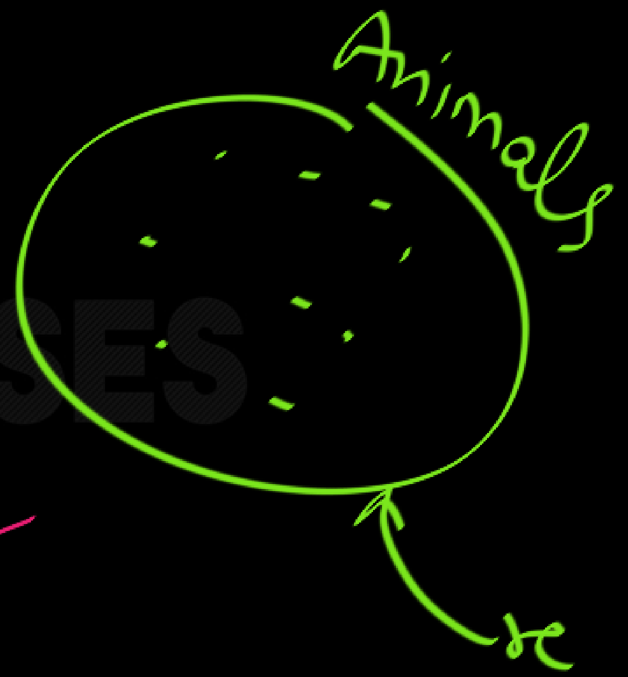
Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

3. Every rabbit is cute.

$\forall x \; cute(x)$ ✗

$\forall x \left(R(x) \rightarrow cute(x) \right)$ ✓

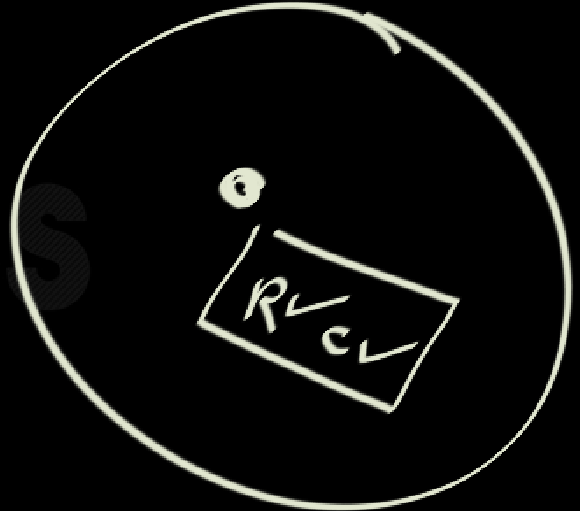


Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

4. Some rabbit is cute.

$$\exists x\big(R(x) \wedge \text{cute}(x)\big) \checkmark$$



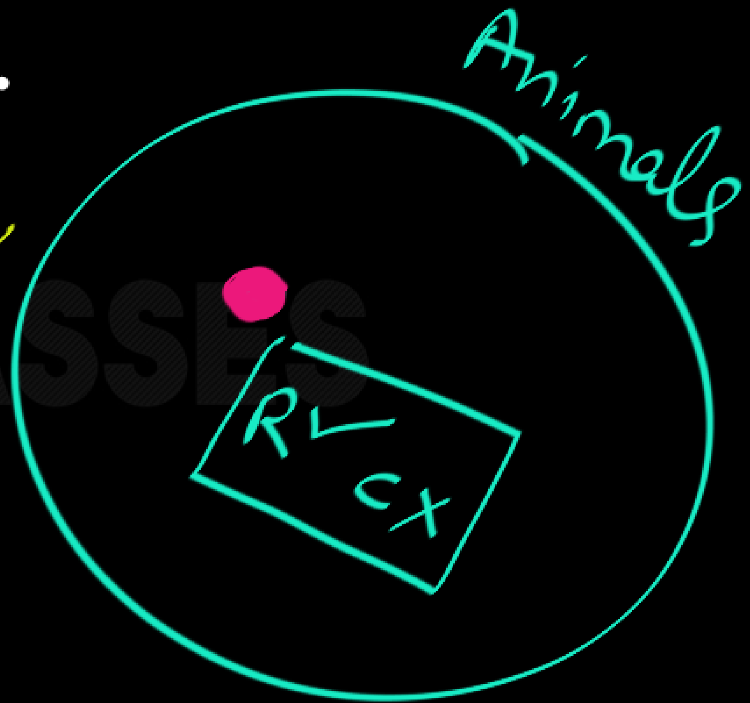
Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

5. Some rabbit is not cute.

$\exists x\ (R(x) \land \neg Cute(x))$

But



Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

6. All rabbits are non-cute.

$$\forall x \left(R(x) \longrightarrow \neg \text{cute}(x) \right)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

7. It is not the case that some rabbit is cute.

$$\boxed{\neg \exists_x \left(R(x) \wedge cute(x) \right)}$$

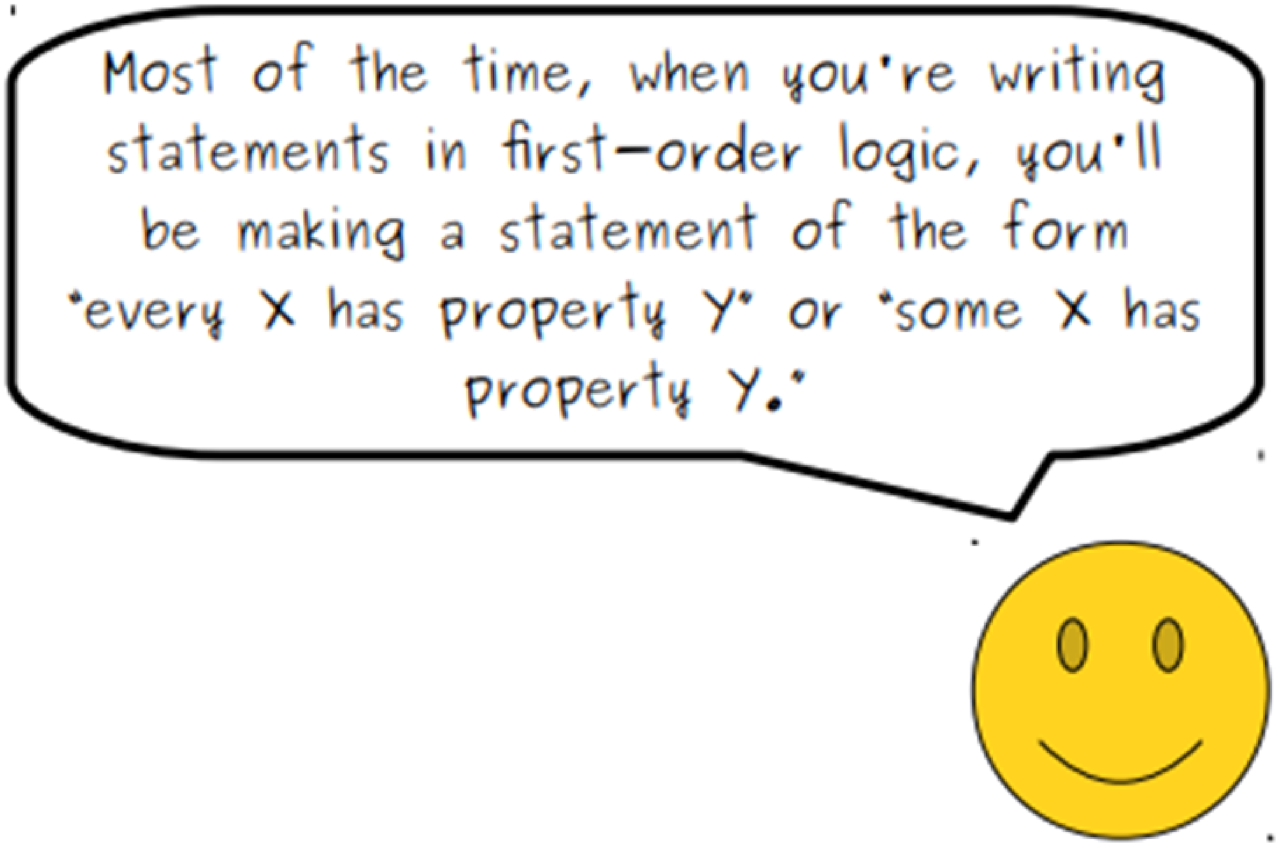
Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

8. It is not the case that all rabbits are cute.

$$\boxed{\neg \forall x \, (R(x) \rightarrow cute(x))}$$

Translating English to FOL



Translating English to FOL

Most of the time, when you're writing statements in first-order logic, you'll be making a statement of the form "every X has property Y" or "some X has property Y."

$$\forall_x \left(X_{(x)} \rightarrow Y_{(x)} \right)$$

$$\exists_x \left(X_{(x)} \wedge Y_{(x)} \right)$$

No p's are Q's.

$$\equiv \forall x \left(P(x) \rightarrow \neg Q(x) \right) ✓$$

$$\equiv \neg \exists x \left(P(x) \land Q(x) \right) ✓$$

No Rabbit is Cute.

$\equiv$ All Rabbits are Non-cute.

$$\equiv \forall x \left(R(x) \rightarrow \neg \, \text{cute}(x) \right)$$

No Rabbit is Cute.

$\equiv$ It is not the case that Some Rabbit is cute.

$$\neg \exists x \left(R(x) \wedge Cute(x) \right)$$

No Rabbit is Cute.

$\equiv$ It is not the case that some Rabbit is cute. $\neg \exists x (R(x) \wedge cute(x))$ ✓

$\equiv$ All Rabbits are Non-cute

$\forall x (R(x) \longrightarrow \neg cute(x))$ ✓

No Rabbit is Cute.

$\equiv \forall x \left(R(x) \longrightarrow \neg cute(x) \right)$ ✓

$\equiv \neg \exists x \left(R(x) \wedge cute(x) \right)$ ✓

"All Ps are Qs."

"Some Ps are Qs."

"No Ps are Qs."

"Some Ps aren't Qs."

"All *P*s are *Q*s."

$\forall x\,(P(x) \rightarrow Q(x))$

"Some *P*s are *Q*s."

$\exists x\,(P(x) \land Q(x))$

"No *P*s are *Q*s."

$\forall x\,(P(x) \rightarrow \neg Q(x))$

"Some *P*s aren't *Q*s."

$\exists x\,(P(x) \land \neg Q(x))$

"All A's are B's"

translates as

$$\forall x. \, (A(x) \rightarrow B(x))$$

"Some *A* is a *B*"

translates as

$$\exists x.\ (A(x) \land B(x))$$

"All *P*s are *Q*s."

$\forall x. (P(x) \to Q(x))$

"Some *P*s are *Q*s."

$\exists x. (P(x) \land Q(x))$

"No *P*s are *Q*s."

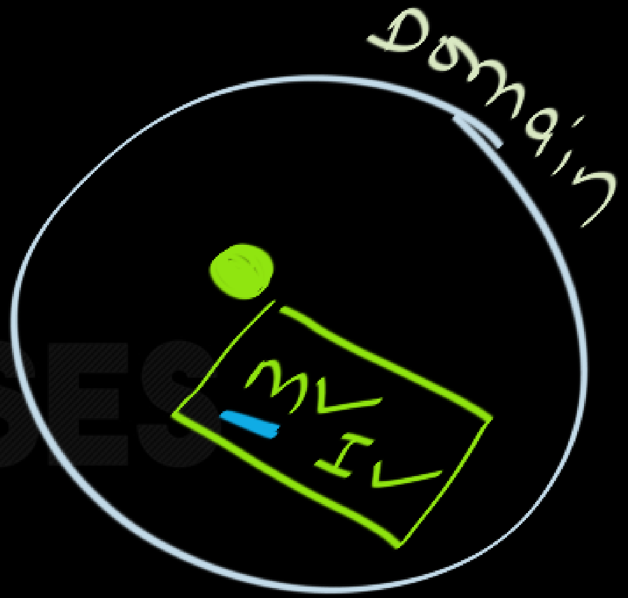
$\forall x. (P(x) \to \neg Q(x))$

"Some *P*s aren't *Q*s."

$\exists x. (P(x) \land \neg Q(x))$

Some muggle is intelligent.

$$\exists_x \big(M(x) \wedge I(x) \big)$$



Domain : Set of muggles

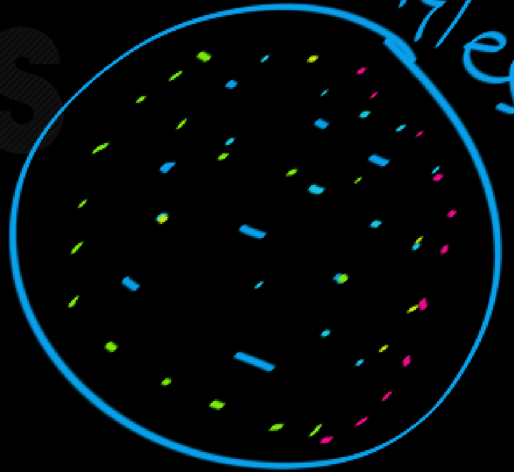
for every element

m is True.

Some muggle is intelligent.

$\exists_x (m(x) \wedge I(x))$ ✓

muggles



$\exists_x I(x)$ ✓

Some muggle is intelligent.

Domain : Set of muggles

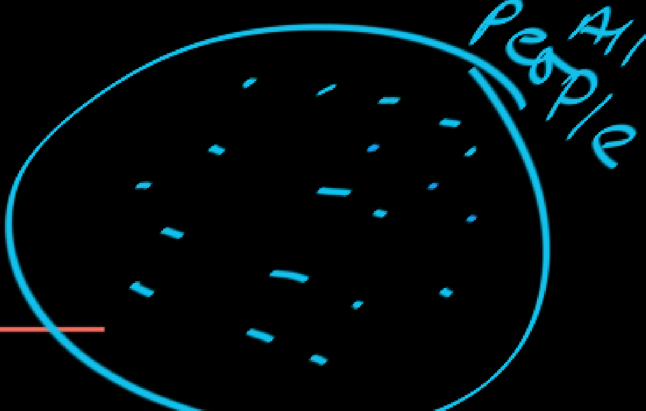
$\boxed{\exists_x I(x)}$

$\exists_x \left(M(x) \wedge I(x) \right)$

Domain : Unrestricted(by default, if domain is not given in the question)

$\cancel{\exists_x I(x)}$

$\exists_x \left(M(x) \wedge I(x) \right)$



Some muggle is intelligent.

$$\boxed{\exists m.}\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

"All *As* are *Bs*"

$$\forall x.\ (A(x) \to B(x))$$

"Some *As* are *Bs*"

$$\exists x.\ (A(x) \land B(x))$$

"No *As* are *Bs*"

$$\forall x.\ (A(x) \to \neg B(x))$$

"Some *As* aren't *Bs*"

$$\exists x.\ (A(x) \land \neg B(x))$$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

First Order Logic

Next Topic:

English – FOL Translation

MORE Variations

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

Only males are in army.

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

$\boxed{\text{Only males are in army.}}$

$male(x) \longrightarrow Army(x)$ ✗

$\boxed{Army(x) \longrightarrow male(x)}$ ✓

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

Only males are in army.

$\boxed{\neg male(x) \longrightarrow \neg Army(x)}$ ✓

$\equiv \boxed{Army(x) \longrightarrow male(x)}$ ✓

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

Only males are in army.

$\equiv$ for every x, if x is in Army then x is a male.

$$\equiv \forall x \left(Army(x) \rightarrow Male(x) \right)$$

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

Only males are in army.

$\equiv \forall x\,(\text{Army}(x) \rightarrow \text{male}(x))$

$\equiv \forall x\,(\neg\,\text{male}(x) \rightarrow \neg\,\text{Army}(x))$

Same

Only A's are B's.

$\equiv \quad \neg A(x) \longrightarrow \neg B(x)$

$\equiv \quad B(x) \longrightarrow A(x)$

same

Only A's are B's.

$$\equiv \forall x \left(B(x) \rightarrow A(x) \right) \checkmark$$

$$\equiv \forall x \left(\neg A(x) \rightarrow \neg B(x) \right) \checkmark$$

Only males are in Army.

$\equiv \forall x \left(Army(x) \rightarrow male(x) \right)$

$\equiv$ Every army person is a male.

Same

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

Only males are in army.

$$\forall x\,(\text{Army}(x) \rightarrow \text{male}(x))$$

All males are in army.

$$\forall x\,(\text{male}(x) \rightarrow \text{Army}(x))$$

Domain: Set of all people

Male(x): x is a male. ; Army(x): x is in army.

All & Only males are in army.

$$\forall x\,(\text{male}(x) \leftarrow\rightarrow \text{Army}(x))$$

$$\equiv \forall x\,(\text{male}(x) \longleftrightarrow \text{Army}(x))$$

All & Only A's are B's.

A → B

B → A

A ↔ B

All & Only A's are B's.

$\equiv \forall x \left(A(x) \longleftrightarrow B(x) \right)$

Only A's are B's.

$$\equiv \forall x \left(B(x) \rightarrow A(x) \right)$$

All & only A's are B's.

$$\equiv \forall x \left(A(x) \longleftrightarrow B(x) \right)$$

Example 1:

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

1. Every animal is cute.

$\forall_x \text{ cute}(x)$ ✓

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

2. Some animal is cute.

$\exists_x \text{ cute}(x)$ ✓

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

3. Every rabbit is cute.

$\forall x \left(R(x) \rightarrow cute(x) \right)$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

4. Some rabbit is cute.

$$\exists x \left(R(x) \wedge cute(x) \right)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

5. Some rabbit is not cute.

$$\exists x \left(R(x) \wedge \neg cute(x) \right)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

6. All rabbits are non-cute.

$$\forall x\,\big(R(x) \longrightarrow \neg\, cute(x)\big)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

7. It is not the case that some rabbit is cute.

$$\neg \exists_x \left(R(x) \wedge cute(x) \right)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

8. It is not the case that all rabbits are non-cute.

$$\neg \forall x \left(R(x) \longrightarrow \neg cute(x) \right)$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

9. Only rabbits are cute.

$$\forall x(\neg R(x) \rightarrow \neg cute(x))$$

$$\equiv \forall x(cute(x) \rightarrow R(x))$$

Let, Domain: Set of All Animals.

Let, Predicate, cute(x): x is cute.

10. All & Only rabbits are cute.

$$\forall x \left(R(x) \leftrightarrow cute(x) \right)$$

First Order Logic

Next Topic:

English – FOL Translation

MORE Practice

Available Predicates:

Orange(x)
Cat(x)
Fluffy(x)

Imagine that we have these predicates available to us to use...

Every orange cat is fluffy.

$$\forall x \left(\text{Orangecat}(x) \rightarrow \text{fluffy}(x) \right)$$

$$\equiv \forall x \left(\left(\text{orange}(x) \wedge \text{cat}(x) \right) \rightarrow \text{fluffy}(x) \right)$$

No A's are B's.

$\equiv \forall x \, (A(x) \rightarrow \neg B(x))$

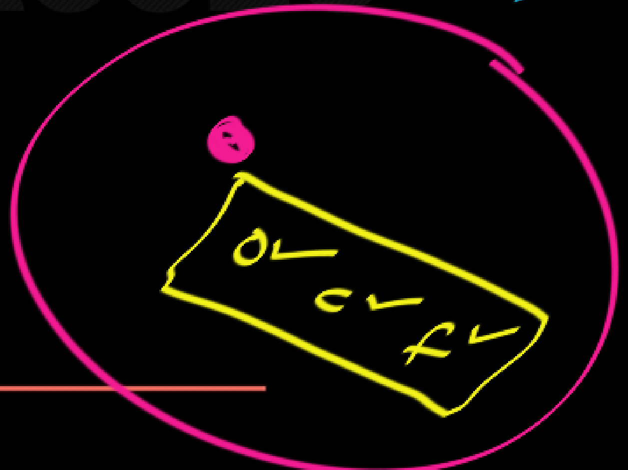
$A(x) \longrightarrow \neg B(x)$

No orange cat is fluffy

$$\forall x \left((orange(x) \wedge cat(x)) \longrightarrow \neg fluffy(x) \right)$$

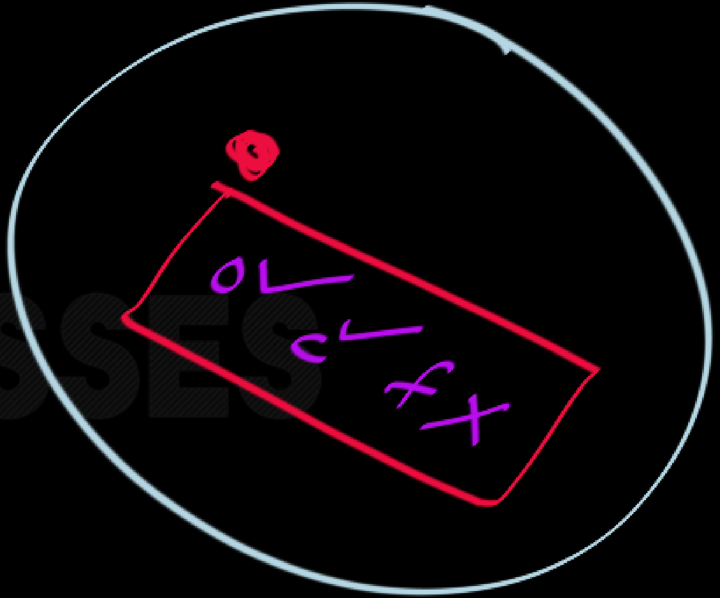
Some orange cats are fluffy

$$\exists x \left(orange(x) \land cat(x) \land fluffy(x) \right) \checkmark$$



Some orange cats are not fluffy

$$\exists x\,(\text{orange}(x) \land \text{Cat}(x) \land \neg\,\text{fluffy}(x))$$



Not all Cats are fluffy.

$\equiv$ Some Cat is not fluffy.

$\equiv \exists x \left(Cat(x) \wedge \neg fluffy(x) \right)$

Not all A's are B's.

$\equiv$ Some A is not B

$\equiv \exists x \left(A(x) \land \neg B(x) \right)$

Not all orange cats are fluffy

$\equiv$ Some orange cat is not fluffy.

$\equiv \exists x \big(orange(x) \wedge Cat(x) \wedge \neg fluffy(x) \big)$

OVCVFX

Only orange cats are fluffy

$$\forall x \Big(\text{fluffy}(x) \longrightarrow \big(\text{orange}(x) \land \text{Cat}(x)\big) \Big)$$

All and only orange cats are fluffy

$$\forall x\left(\left[orange(x) \wedge cat(x) \right] \longleftrightarrow fluffy(x) \right)$$

Express the statement “Every student in this class has studied calculus” using predicates and quantifiers.

Express the statement "Every student in this class has studied calculus" using predicates and quantifiers.

$$\forall x \left(\text{Student}(x) \longrightarrow \text{Calculus}(x) \right)$$

Express the statement “Every student in this class has studied calculus” using predicates and quantifiers.

Solution: First, we rewrite the statement so that we can clearly identify the appropriate quantifiers to use. Doing so, we obtain:

> “For every student in this class, that student has studied calculus.”

Next, we introduce a variable x so that our statement becomes

> “For every student x in this class, x has studied calculus.”

Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution: The statement "Some student in this class has visited Mexico" means that

> "There is a student in this class with the property that the student has visited Mexico."

We can introduce a variable x, so that our statement becomes

> "There is a student x in this class having the property that x has visited Mexico."

Express the statements "Some student in this class has visited Mexico" and "Every student in this class has visited either Canada or Mexico" using predicates and quantifiers.

Solution: The statement "Some student in this class has visited Mexico" means that

> "There is a student in this class with the property that the student has visited Mexico."

We can introduce a variable x, so that our statement becomes

> "There is a student x in this class having the property that x has visited Mexico."

$\exists x \left(\text{student}(x) \wedge \text{mexico}(x) \right)$

Note:

If Domain is Not given, then Assume that Domain Contains everything.

Translating from English to Logic

Example 1: Translate the following sentence into predicate logic: "Every student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, define a propositional function J(x) denoting "x has taken a course in Java" and translate as $\forall x\, J(x)$.

Solution 2: But if U is all people, also define a propositional function S(x) denoting "x is a student in this class" and translate as $\forall x\, (S(x) \rightarrow J(x))$.

$\forall x\, (S(x) \wedge J(x))$ is not correct. What does it mean?

Translating from English to Logic

Example 2: Translate the following sentence into predicate logic: "Some student in this class has taken a course in Java."

Solution:

First decide on the domain U.

Solution 1: If U is all students in this class, translate as

$$\exists x\, J(x)$$

Solution 2: But if U is all people, then translate as

$$\exists x\, (S(x) \land J(x))$$

$\exists x\, (S(x) \rightarrow J(x))$ is not correct. What does it mean?

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

2. "Every student in this class has visited Canada or Mexico."

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

$\exists x \, (S(x) \wedge M(x))$

2. "Every student in this class has visited Canada or Mexico."

$\forall x \, (S(x) \longrightarrow (C(x) \vee M(x)))$

Translation from English to Logic

Examples:

1. "Some student in this class has visited Mexico."

   Solution: Let $M(x)$ denote "x has visited Mexico" and $S(x)$ denote "x is a student in this class," and U be all people.

   $$\exists x\ (S(x) \land M(x))$$

2. "Every student in this class has visited Canada or Mexico."

   Solution: Add $C(x)$ denoting "x has visited Canada."

   $$\forall x\ (S(x) \rightarrow (M(x) \lor C(x)))$$

All P's are Q's.

"Whenever $P(x)$, then $Q(x)$"

translates as

$$\forall x.\ (P(x) \rightarrow Q(x))$$

Some p is q :

"There is some $P(x)$ where $Q(x)$"

translates as

$$\exists x.\ (P(x) \land Q(x))$$

The Takeaway Point

- Be careful when translating statements into first-order logic!
- $\forall$ is usually paired with $\rightarrow$.
- $\exists$ is usually paired with $\land$.

All A's are B's

$$\equiv \forall x \left(A(x) \longrightarrow B(x) \right)$$

"For any natural number n,
n is even if and only if n^2 is even"

"For any natural number n,
$\boxed{n \text{ is even if and only if } n^2 \text{ is even"}}$

$$\forall_n \left(\text{Natural}(n) \rightarrow \left(\text{Even}(n) \leftrightarrow \text{Even}(n^2) \right) \right)$$

"For any natural number n,
n is even if and only if n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier***
and says "for any choice of n,
the following is true."

- $S(x)$: x is a student
- $I(x)$: x is intelligent
- $M(x)$: x likes music

- Write wffs than express the following statements:
  - All students are intelligent.
  - Some intelligent students like music.
  - Everyone who likes music is a stupid student.
  - Only intelligent students like music.

For anything, if it is a student, then it is intelligent $\Leftrightarrow$ $(\forall x)[S(x) \to I(x)]$

There is something that is intelligent and it is a student and it likes music $\Leftrightarrow$ $(\exists x)[I(x) \land S(x) \land M(x)]$

For anything, if that thing likes music, then it is a student and it is not intelligent $\Leftrightarrow$

$(\forall x)(M(x) \to S(x) \land [I(x)]')$

For any thing, if it likes music, then it is a student and it is intelligent $\Leftrightarrow$

$(\forall x)(M(x) \to S(x) \land I(x))$

First Order Logic

Next Topic:

English – FOL Translation

One Final Variation… I promise

Quantifiers with Restricted Domains

What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Quantifiers with Restricted Domains

What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

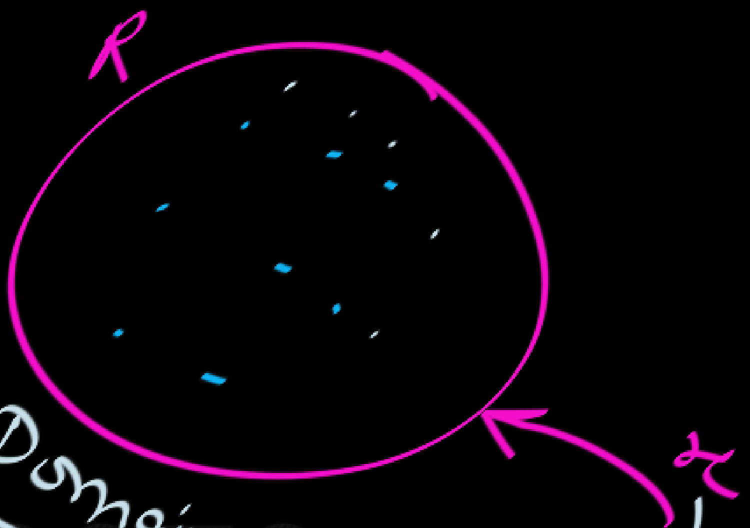
Domain: R

(a) $\forall_{x<0} \left(x^2 > 0 \right)$

for all $x<0$, $x^2>0$.

$\equiv$ for all x; if $x<0$ then $x^2>0$

$\equiv \forall_x \left(x<0 \longrightarrow x^2>0 \right)$



Domain

Real

$$\forall_{x<0}(x^2>0) \equiv \forall_x(x<0 \rightarrow x^2>0)$$

for all real x; if $x<0$ then $x^2>0$

True

Domain: $\mathbb{R}$

$$\forall_{P(x)} \left(Q(x) \right) \equiv \forall_x \left(P(x) \rightarrow Q(x) \right)$$

(b) $\forall_{\underbrace{y \neq 0}_{P(x)}} \left(\overbrace{y^3 \neq 0}^{Q(x)} \right)$

Domain:

for all $y \neq 0$, $y^3 \neq 0$.

$\equiv$ for all y, if $y \neq 0$ then $y^3 \neq 0$

$$\equiv \forall_y \left(y \neq 0 \longrightarrow y^3 \neq 0 \right)$$

$$\forall_{y \neq 0} \left(y^3 \neq 0 \right)$$

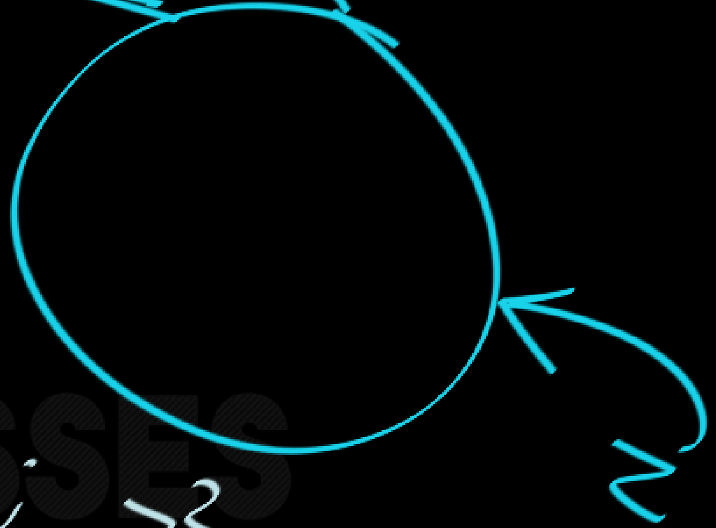
$\equiv$ for all real y, if $y \neq 0$ then $y^3 \neq 0$

True

$$\forall_{P(x)} \left(Q(x) \right) \equiv \forall x \left(P(x) \rightarrow Q(x) \right)$$

(c) $\exists_{z>0}\,(z^2 = 2)$

Domain: $\mathbb{R}$



$\equiv$ there is some real $z>0$; $z^2=2$.

$\equiv$ there is some real z; $z>0$ & $z^2=2$

$\equiv \exists_z\,(z>0 \wedge z^2=2)$

$\exists_{z>0}(z^2=2)$: True

witness: $\sqrt{2}$ ✓

$\equiv \exists_z (z>0 \wedge z^2=2)$

$$\exists_{P(x)} \left(Q(x) \right) \equiv \exists x \left(P(x) \wedge Q(x) \right)$$

$$\forall_{P(x)} \left(Q(x) \right) \equiv \forall x \left(P(x) \rightarrow Q(x) \right)$$

What do the statements $\forall x < 0\,(x^2 > 0)$, $\forall y \neq 0\,(y^3 \neq 0)$, and $\exists z > 0\,(z^2 = 2)$ mean, where the domain in each case consists of the real numbers?

Solution: The statement $\forall x < 0\,(x^2 > 0)$ states that for every real number x with $x < 0$, $x^2 > 0$. That is, it states "The square of a negative real number is positive." This statement is the same as $\forall x(x < 0 \rightarrow x^2 > 0)$.

The statement $\forall y \neq 0\,(y^3 \neq 0)$ states that for every real number y with $y \neq 0$, we have $y^3 \neq 0$. That is, it states "The cube of every nonzero real number is nonzero." Note that this statement is equivalent to $\forall y(y \neq 0 \rightarrow y^3 \neq 0)$.

Finally, the statement $\exists z > 0\,(z^2 = 2)$ states that there exists a real number z with $z > 0$ such that $z^2 = 2$. That is, it states "There is a positive square root of 2." This statement is equivalent to $\exists z(z > 0 \wedge z^2 = 2)$. ◀

Note that the restriction of a universal quantification is the same as the universal quantification of a conditional statement. For instance, $\forall x < 0\,(x^2 > 0)$ is another way of expressing $\forall x(x < 0 \rightarrow x^2 > 0)$. On the other hand, the restriction of an existential quantification is the same as the existential quantification of a conjunction. For instance, $\exists z > 0\,(z^2 = 2)$ is another way of expressing $\exists z(z > 0 \wedge z^2 = 2)$.

First Order Logic

Next Topic:

English – FOL Translation

Heart wants MORE Practice