



# Translating English to FOL

English statement :  $P$ : All crows are black

$\overbrace{P}$   
English  
Statement

$\overbrace{\psi}$   
FOL  
Expression

$$P \equiv \psi$$

$$\left. \begin{array}{l} P \text{ True} \rightarrow \psi \text{ True} \\ P \text{ false} \rightarrow \psi \text{ false} \end{array} \right\}$$

$P$

English Statement

$\psi$

FOL Expression

$$\begin{array}{l} \text{whenever } \underline{\underline{P \text{ True}}} \rightarrow \underline{\underline{\psi \text{ True}}} \\ \text{whenever } \underline{\underline{P \text{ false}}} \rightarrow \underline{\underline{\psi \text{ false}}} \end{array}$$

Most of the time, when you're writing statements in first-order logic, you'll be making a statement of the form "every X has property Y" or "some X has property Y."





## Four fundamental types of statements :

"All Ps are Qs."

"Some Ps are Qs."

"No Ps are Qs."

"Some Ps aren't Qs."

**"All A's are B's"**

"All crows are black"

All Natural numbers are integers,

All Indians are Asians,

① When Domain is Not specified , then  
Domain Contains Everything.

All crows are black

$\forall$  for every element in the Domain, If it is crow  
then it is black

$\forall x \text{ (crow}(x) \rightarrow \text{black}(x))$



P : All crows are black.

Q :  $\forall x (crown(x) \wedge \text{Black}(x))$

Everything in the Domain  
is crow and black.

P: All cows are black.

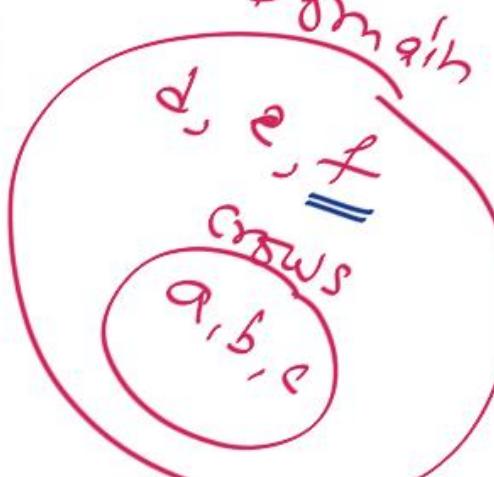
0:   $(\text{row}(c)) \wedge \text{Black}(x)$

Everything in the Domain  
is cow and black.

P:T

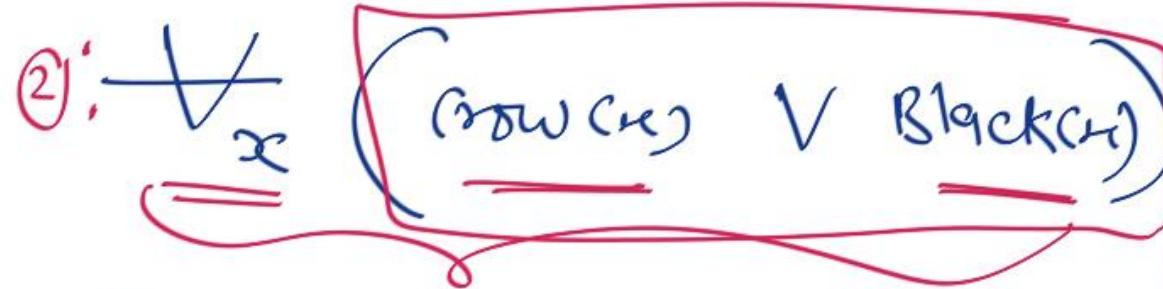
0: false

Scenario:

Domain:  
  
 $\text{row}(c) = \{a, b, c\}$

$\text{row}(x) = \{d, e, f\}$

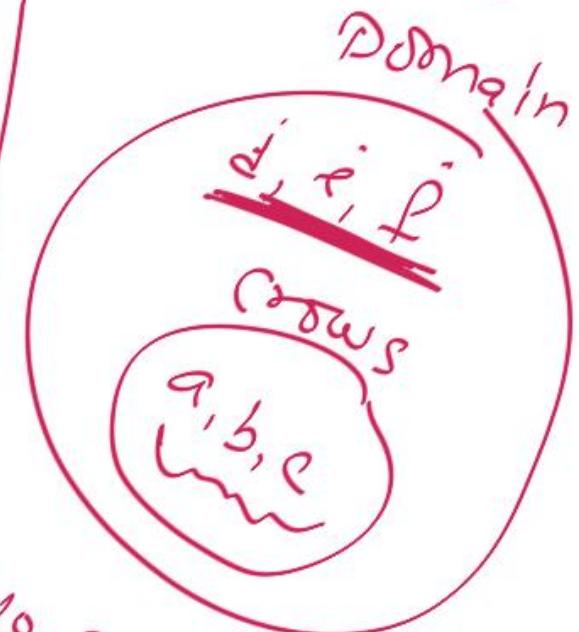
P: All crows are Black.



P: F

T: T

$\neg$  Scenario



P: All crows are black

③:  $\forall_x (\text{crow}(x) \leftrightarrow \text{Black}(x))$

Everything in the Domain is Black iff it is crow.

P: True

3: False

Scenario

Domain



$\neg \Rightarrow$  Not crow  
 $A_1, A_2 \in \text{Crows}$   
 $B_1 \in \text{Black}$

P: All crows are black,

Everything in the Domain,  
If it is crow then  
it is black.

Q:  ~~$\forall x (\text{Crow}(x) \rightarrow \text{Black}(x))$~~

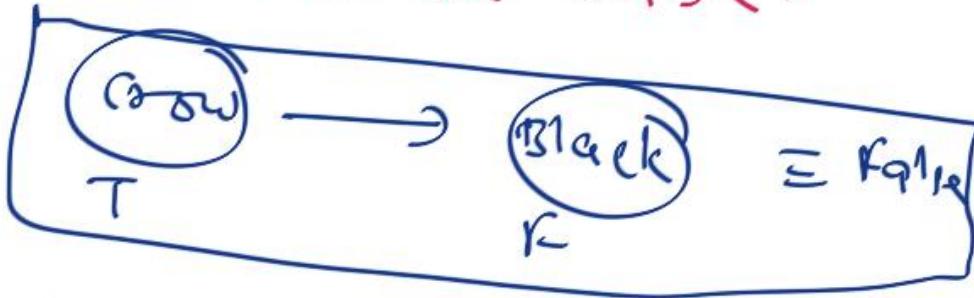
$P$ : All Crows are Black

$\Psi$ :

$\forall x \left( \text{crow}(x) \rightarrow \text{Black} \right)$

When  $P$  is True

$\Psi$  cannot be False.

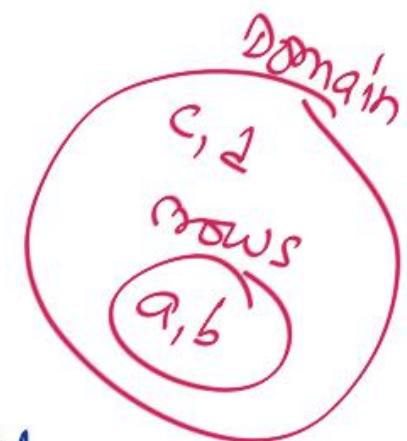


then  $\Psi$  is True

When  $P$  is False.

there is some crow which  
is Not Black,

$\Psi$  will False.



All crows are black

$\equiv \forall_{x} (\text{Crow}(x) \rightarrow \text{Black}(x))$

Why??

Imp:

Template in FOL

"All A's are B's"

translates as

$$\forall x (A(x) \rightarrow B(x))$$

$$\forall x. (A(x) \rightarrow B(x))$$

**“Some A is a B”**



Some crows are black.

At least one crow is black.  $\equiv$

$$\exists \underline{x} \left( \text{crow}(x) \wedge \text{black}(x) \right)$$

C

~~These exist an element in the Domain such that it is a crow and it is black,~~

p: Some crows are black.

①:  $\exists x \text{ crow}(x) \rightarrow \text{Black}(x)$

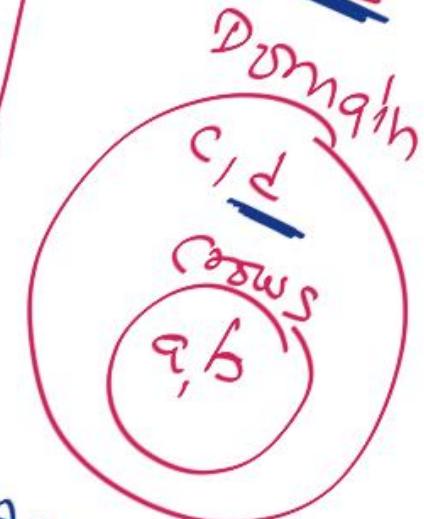
$\equiv \exists x (\text{crow}(x) \vee \text{Black}(x))$

$\equiv$  there is some element in Domain which is Black or not crow.

P: false

Q: True

Scenario



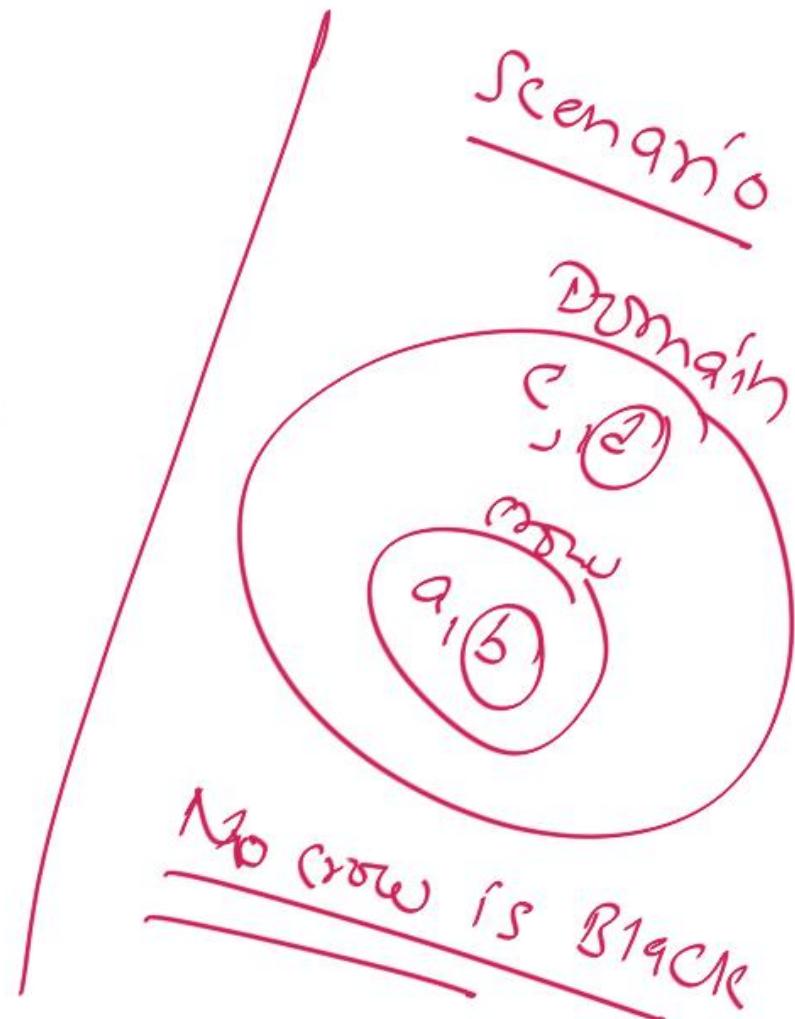
Assume No  
cow is  
Black.

②:  $\exists x$  (cow( $x$ )  $\vee$  Black( $x$ ))

P: Some cows are Black

P: False

②: True



$\neg P$ : Some crows are black ✓

$\neg \psi$ :  $\exists x$   $(\text{crow}(x) \wedge \text{black}(x))$

When  $P = \text{True}$  ✓

$\psi = \text{True}$  ✓

When  $P = \text{False}$

No crow is black

$\psi$  cannot be true  
 $\psi$  False

✓ Some Crows are Black

Crow(x)  $\wedge$  Black(x)

$\exists x$  (Crow(x)  $\wedge$  Black(x))

$P \rightarrow Q \equiv \neg P \vee Q$

Imp- Template in fOL

"Some A is a B"

translates as

$$\exists x (A(x) \wedge B(x))$$

**$\exists x. (A(x) \wedge B(x))$**



✓  
"All As are Bs"

✓  
 $\forall x. (A(x) \rightarrow B(x))$

"No As are Bs"

$\forall x. (A(x) \rightarrow \neg B(x))$

✓  
"Some As are Bs"

✓  
 $\exists x. (A(x) \wedge B(x))$

"Some As aren't Bs"

$\exists x. (A(x) \wedge \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

Some A's are Not B's

$\exists x (A(x) \wedge \neg B(x))$

Some

crows are Not Black ,

$$\exists_x (\text{crow}(x) \wedge \neg \text{Black}(x))$$

↙ No A's are B's

{  $\cancel{\forall}_x (A(x) \rightarrow \neg B(x))$  ✓ }  
{  $\neg \exists_x (A(x) \wedge B(x))$  ✓ }

No crows are black,

There does not exist an element which is crow and black.

$\neg \exists x (\text{crow}(x) \wedge \text{black}(x))$

for every element,  
If it is crow then  
It is Not Black.

$\forall x (\text{crow}(x) \rightarrow \neg \text{black}(x))$

$\exists_x$  : There exists  $x$

$\forall \exists_x$  : There Does not exist  $x$





"All *P*s are *Q*s."

$$\forall x. (P(x) \rightarrow Q(x))$$

"No *P*s are *Q*s."

$$\forall x. (P(x) \rightarrow \neg Q(x))$$

"Some *P*s are *Q*s."

$$\exists x. (P(x) \wedge Q(x))$$

"Some *P*s aren't *Q*s."

$$\exists x. (P(x) \wedge \neg Q(x))$$

All A's are B's :  $\forall_x (A(x) \rightarrow B(x))$

Some A's are B's :  $\exists_x (A(x) \wedge B(x))$

No A's are B's :  $\forall_x (A(x) \rightarrow \neg B(x))$

$$\equiv \neg \exists_x (A(x) \wedge B(x))$$

Some A's are Not B's  $\equiv \exists_x (A(x) \wedge \neg B(x))$

*Available Predicates:*

Orange(x) ✓  
Cat(x) ✓  
Fluffy(x) ✓

Imagine that we have these predicates available to us  
to use...



~~Every orange cat is fluffy.~~

$\forall x$  (Orange cat  $\rightarrow$  fluffy)

$\forall x$  ((Orange( $x$ )  $\wedge$  Cat( $x$ ))  $\rightarrow$  fluffy( $x$ ))

No orange cat is fluffy

$\forall x$  (orange cat  $\rightarrow$   $\neg$  fluffy)

$\forall x$  ((orange( $x$ )  $\wedge$  cat( $x$ )  $\rightarrow$   $\neg$  fluffy( $x$ ))  
 $\neg \exists x$  (orange( $x$ )  $\wedge$  cat( $x$ )  $\wedge$  fluffy( $x$ ))



Some orange cats are fluffy

$\exists x$  ( orange(x)  $\wedge$  cat(x)  $\wedge$  fluffy(x) )



Some orange cats are not fluffy

$$\exists x \left( \text{orange}(x) \wedge \text{Cat}(x) \wedge \neg \text{Fluffy}(x) \right)$$

Not all crows are Black. }  
≡ Some crows are Not Black }

≡  $\exists x (Cow(x) \wedge \neg Black(x))$

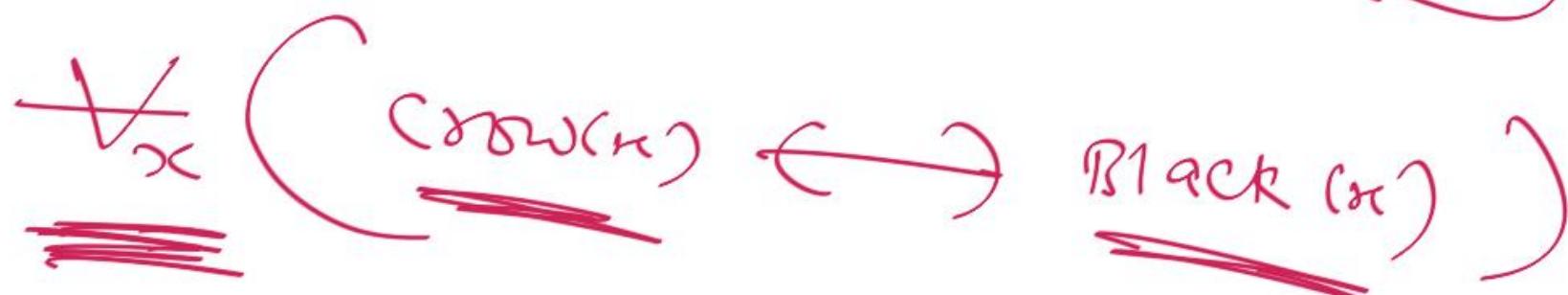
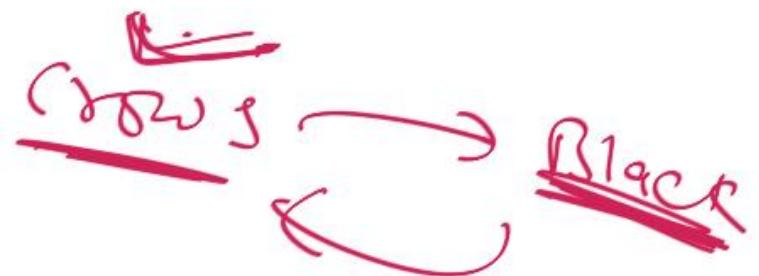
✓

Not all orange cats are fluffy

Some orange cats are Not fluffy.

$\exists x (Orange(x) \wedge Cat(x) \wedge \neg Fluffy(x))$

All and only crows are black.



~~All and only A's are B's~~

~~$\forall x (A(x) \leftrightarrow B(x))$~~



only cows are black.

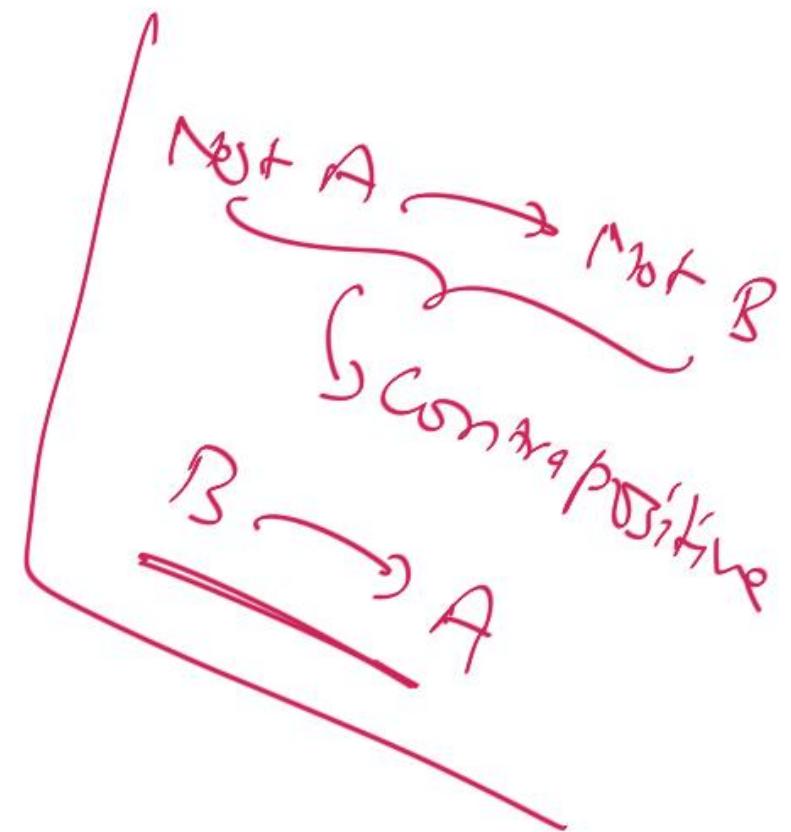
$\equiv$  If it is Not cow then it Not Black  
 $\equiv$  If it is Black then it is cow

$\equiv \forall x (Black(x) \rightarrow Cow(x)) \}$   
 $\equiv \forall x (Cow(x) \rightarrow Black(x))$

only A's are B's

$$\forall x (B(x) \rightarrow A(x))$$

$$\equiv \forall x (\neg A(x) \rightarrow \neg B(x))$$



$\exists p'$

All and only orange cats are fluffy

$\forall x$  (Orange Cat ( $x$ )  $\leftrightarrow$  fluffy ( $x$ ))

$\equiv \forall x$  ((Orange ( $x$ )  $\wedge$  cat( $x$ )  $\leftrightarrow$  fluffy( $x$ ))

① All A's are B's :  $\forall_x (A(x) \rightarrow B(x))$

② Some A's are B's :  $\exists_x (A(x) \wedge B(x))$

③ No A's are B's :

$$\forall_x (A(x) \rightarrow \neg B(x)) \equiv \neg \exists_x (A(x) \wedge B(x))$$

④

Some A's are Not B's :

$$\exists x (A(x) \wedge \neg B(x))$$

⑤

Not All A's are B's :

Some A's are not B's

$$\equiv \exists x (A(x) \wedge \neg B(x)) \leftarrow$$

~~Not~~ "all A's are B's":

$$\neg \forall_x (A(x) \rightarrow B(x)) \Leftarrow$$

$$\equiv \exists_x (A(x) \wedge \neg B(x))$$

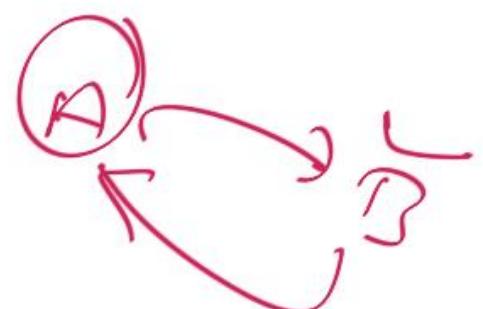
⑥

Only A's are B's

$$\forall x (\neg A(x) \rightarrow \neg B(x)) \\ \equiv \forall x (B(x) \rightarrow A(x))$$

⑦ All and only A's are B's

~~$\forall x (A(x) \leftrightarrow B(x))$~~



English → f<sub>o</sub>L



{ "All Ps are Qs." }  
 $\forall x. (P(x) \rightarrow Q(x))$





Some  $P$ s are  $Q$ s."

$\exists x. (P(x) \wedge Q(x))$

$$\exists x (P(n) \wedge Q(n))$$





"No Ps are Qs."

$$\forall x. (P(x) \rightarrow \neg Q(x))$$

$\equiv$

$$\neg \exists x (P(x) \wedge Q(x))$$

$$\overbrace{\forall x (P(x) \rightarrow \neg Q(x))}^{\text{III}}$$

$$\neg \exists x (P(x) \wedge Q(x))$$

Note:

There exists  $x$  :  $\exists x$

There Does not exist  $x$  :  $\neg \exists x$

"Some Ps aren't Qs."

$$\exists x. (P(x) \wedge \neg Q(x)) \checkmark$$

It is Not the case that

All p's are q's

$$\neg \forall x (P_{(n)} \rightarrow Q_{(n)})$$

$$\equiv \neg \forall x (P_{(n)} \rightarrow Q_{(n)}) \checkmark$$



All  $p^1$ 's are  $q$ 's ↗

Some  $p^1$ 's are  $q$ 's ↗

Only  $p^1$ 's are  $q$ 's ↗

$\forall x (q_{(x)} \rightarrow p_{(x)})$

- Only  $A'$ 's are  $B'$ 's  $\equiv$  All  $B'$ 's are  $A'$
- $\equiv$  If Not  $A$  then Not  $B$
- $\equiv$  All  $B$ 's are  $A$ 's
- $\equiv$   $\forall x (B_{(x)} \rightarrow A_{(x)})$   $\equiv$   $\forall x (\neg A_{(x)} \rightarrow \neg B_{(x)})$

only

Vaccinated people are in the flight.

- ≡ If you are in flight then you are Vaccinated,
- ≡ If you are Not Vaccinated then you are Not in flight.
- ≡  $\forall x$  (In-flight(x)  $\rightarrow$  Vaccinated(x))

All and only A's are B's

$\equiv A \text{ iff } B$

$\equiv \forall_x (A_{(x)} \leftrightarrow B_{(x)})$



All A's are B's :  $\frac{A_{(n)} \rightarrow B_{(n)}}{B_{(n)} \rightarrow A_{(n)}}$

Not all A's are B's  $\equiv$  It is

$$\equiv \exists x (A(x) \wedge \neg B(x))$$

Not the  
case that

$$\equiv \neg \forall x (A(x) \rightarrow B(x))$$

All A's  
are B's

$$\exists x (A(x) \wedge \neg B(x))$$



All students are smart.

$$\forall x \rightarrow (\text{student}(x) \rightarrow \text{smart}(x))$$



There exists a student.

---

$$\exists_x (\text{Student}(x))$$

---



There exists a smart student.

$$\exists \underline{x} (\text{Student}(\underline{x}) \wedge \underline{\text{smart}}(\underline{x}))$$



Every student loves someone.



✓ Love(x,y) : x loves y

x loves someone

$$\forall_x (\text{Student}(x) \rightarrow \exists_y (\text{Love}(x,y))) \quad //$$



✓ Every student loves some student.

$$\forall x (\text{student}(x) \rightarrow \boxed{x \text{ loves some student}})$$

$$\forall x (\text{student}(x) \rightarrow \exists y (\text{student}(y) \wedge \text{loves}(x, y)))$$

~~Every student loves some other student.~~

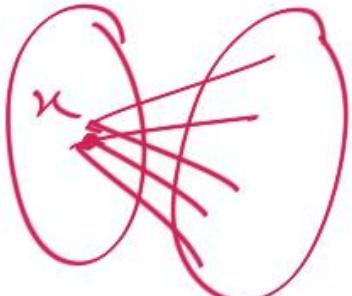
$\forall_x (\text{stu}(x) \rightarrow x \text{ loves some other student})$

$\forall_x (\text{stu}(x) \rightarrow \exists_y (\text{student}(y) \wedge (x \neq y) \wedge \text{loves}(x,y)))$



$\exists x$  There is a student who is loved by every other student.

$$\exists x \left( \underline{\text{student}(x)} \right) \wedge \forall y \left[ \underline{\text{stu}(y) \wedge (x \neq y)} \right] \downarrow \underline{\text{loves}(y, x)}$$



$$\exists x = \underline{\underline{\text{stud}(x)}} \wedge$$

GO  
CLASSES



Bill is a student.

Student(x) : x is a  
student.

Predicate.

Student(Bill)

Bill takes either Analysis or Geometry (but not both).

Take(x,y) : x takes y

Given predicate

Take(Bill, Analysis)



+ Take(Bill, Geometry)



Bill takes Analysis or Geometry (or both).

$$\underline{\text{Take(Bill, Analysis)}} \quad \checkmark \quad \underline{\text{Take(Bill, Geometry)}}$$

= =



Bill takes Analysis and Geometry.

Take(Bill, Analysis)

Take(Bill, Geo)



Bill does not take Analysis

$\neg \text{Take}(\text{Bill}, \text{Analysis})$

No student loves Bill.

$\exists x$  : There Does  
not exist

$\exists x$  ( student ( $x$ )  $\wedge$  loves ( $x$ , Bill) )

$\forall x$  ( student ( $x$ )  $\rightarrow$  loves ( $x$ , Bill) )

Bill has no sister.

Given Predicate

Sisterof(x,y) =  
x is sister of y.

$\exists \exists_x (\text{Sisterof}(x, \text{Bill}))$

$\forall_x (\neg \text{Sisterof}(x, \text{Bill}))$

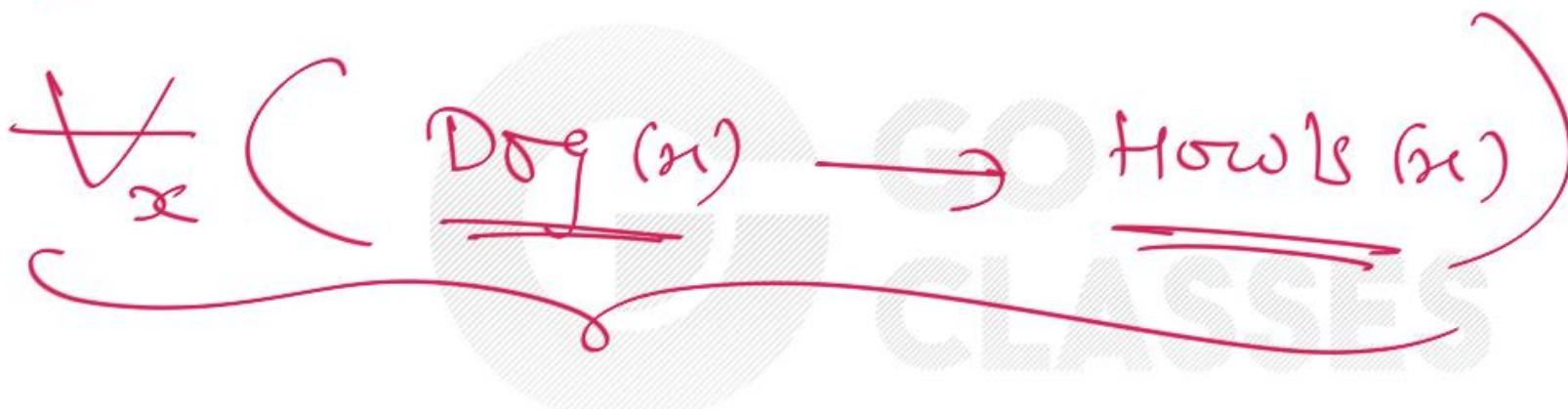


Bill has at least one sister.

$$\exists_x (\text{Sisterof}(x, \text{Bill}))$$



All dogs howl at night.



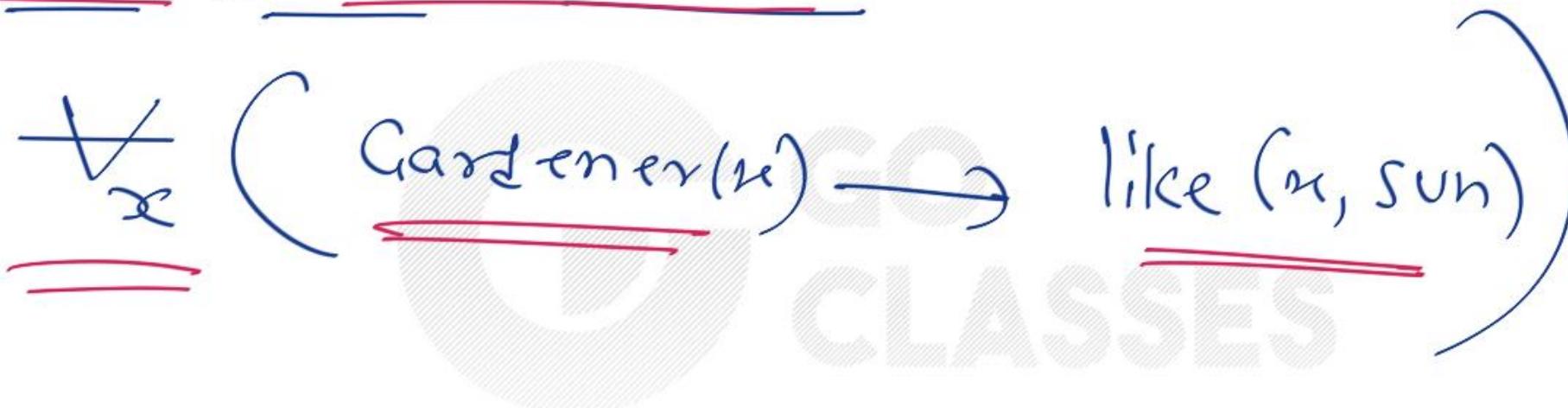
All purple mushrooms are poisonous.

$$\forall x (\underline{\text{Purple}}(\underline{x}) \rightarrow \underline{\text{Poison}}(\underline{x}))$$

$$\forall x ((\underline{\text{Purple}}(x) \wedge \underline{\text{mush}}(x)) \rightarrow \underline{\text{Poison}}(x))$$



Every gardener likes the sun.



No gardener likes the sun.

$$\neg \exists x \left( \underline{\text{Gardener}(x)} \wedge \underline{\text{like}(x, \text{sun})} \right)$$
$$\forall x \left( \underline{\text{Gardener}(x)} \rightarrow \underline{\neg \text{like}(x, \text{sun})} \right)$$

Not all gardeners like the sun.

$$\exists \underline{x} \left( \underline{\text{Gardener}(x)} \wedge \neg \underline{\text{like}(u, \text{sun})} \right)$$

$$\neg \forall \underline{x} \left( \underline{\text{Gardener}(x)} \rightarrow \underline{\text{like}(u, \text{sun})} \right) \checkmark$$



Every student walks or talks.

$$\forall x \left( \text{STUD}(x) \rightarrow (\underline{\text{walk}(x)} \vee \underline{\text{talk}(x)}) \right)$$



✓ Some student walks or talks.

$$\exists x \left( \text{stud}(x) \wedge (\text{walk}(x) \vee \text{talk}(x)) \right)$$

No student walks or talks.

$$\neg \exists x (\text{student}(x) \wedge (\text{walk}(x) \vee \text{talk}(x)))$$
$$\forall x (\text{student}(x) \rightarrow \neg (\text{walk}(x) \vee \text{talk}(x)))$$



Only students walk or talk.



Only A's are B's  $\equiv$  All B's are A's



All and Only students walk or talk.

$$\forall x \left( \underline{\text{student}(x)} \leftrightarrow (\text{walk}(x) \vee \text{talk}(x)) \right)$$



## Numerical quantification :

Now we will learn how to use FOL to express numerical quantifiers as the following:

at least two,

at most one,

exactly one,

at least three,

at most two,

exactly two, etc.





"Every cube is large."

$$\forall x \left( \text{Cube}(x) \rightarrow \text{Large}(x) \right)$$



No cube is large.

$$\left\{ \begin{array}{l} \cancel{\exists x} (\text{cube}(x) \wedge \text{large}(x)) \\ \forall x (\text{cube}(x) \rightarrow \cancel{\exists} \text{large}(x)) \end{array} \right\}$$



"

Nothing is large.

"

{

$$\neg \exists_x (\underline{\text{large}}(x))$$

$$\forall_x (\neg \underline{\text{large}}(x))$$



$\exists_x$  : there exists ;  $\neg \exists_x$  : There Does not exist.



Some Cube is large.

---

---

$$\exists x (\text{Cube}(x) \wedge \text{large}(x))$$

All large cubes are nice. ✓

$$\forall x \left( \underline{\text{large}(x) \wedge \text{cube}(x)} \rightarrow \underline{\text{nice}(x)} \right) \checkmark$$



At least one:

There is at least one cube.

$\exists_x$  Cube( $x$ )

''

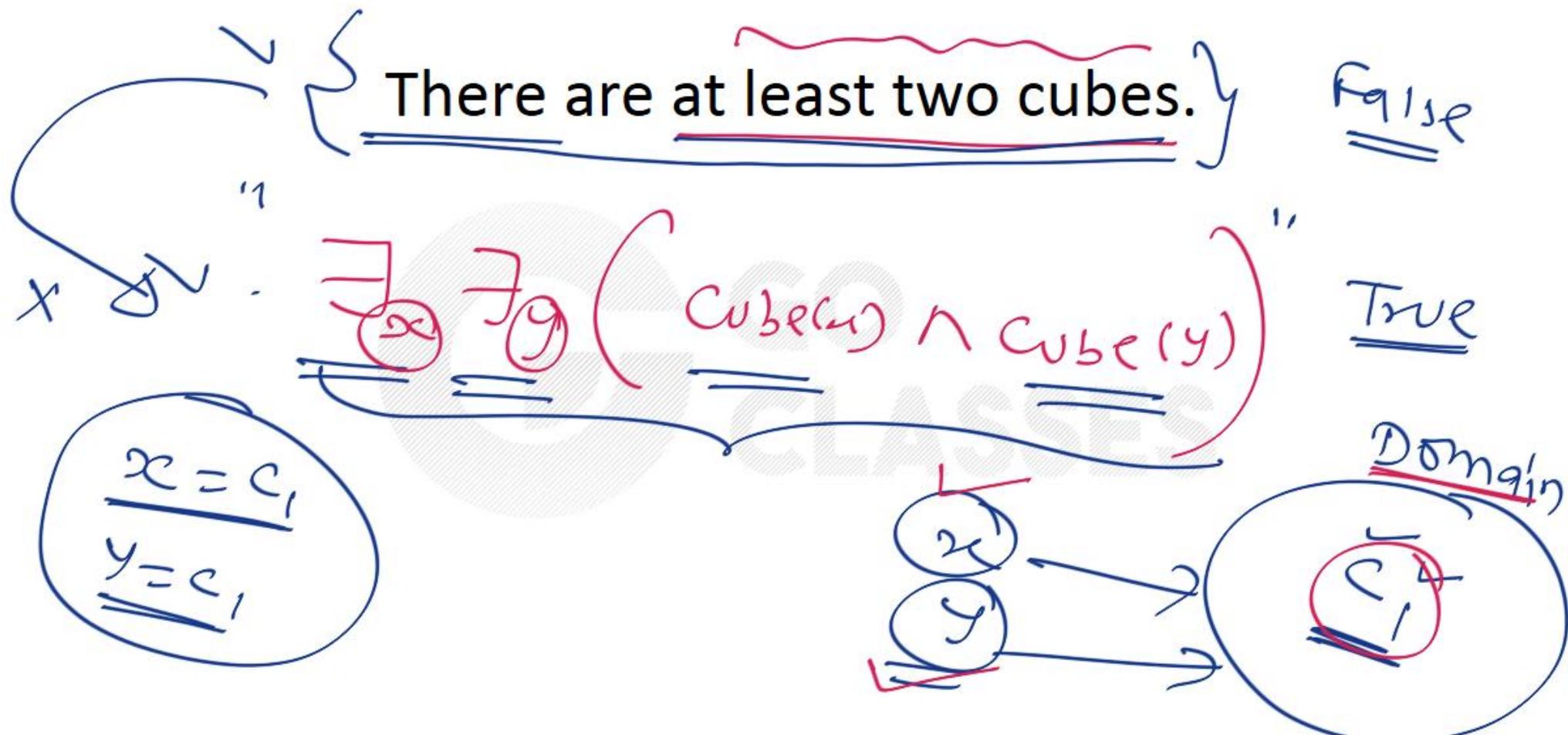
There are at least two cubes.

''



$$\exists_x \exists_y (\text{cube}(x) \wedge \text{cube}(y) \wedge (x \neq y))$$

$$\exists_x \exists_y (\text{cube}(x) \wedge \text{cube}(y) \wedge (x \neq y))$$





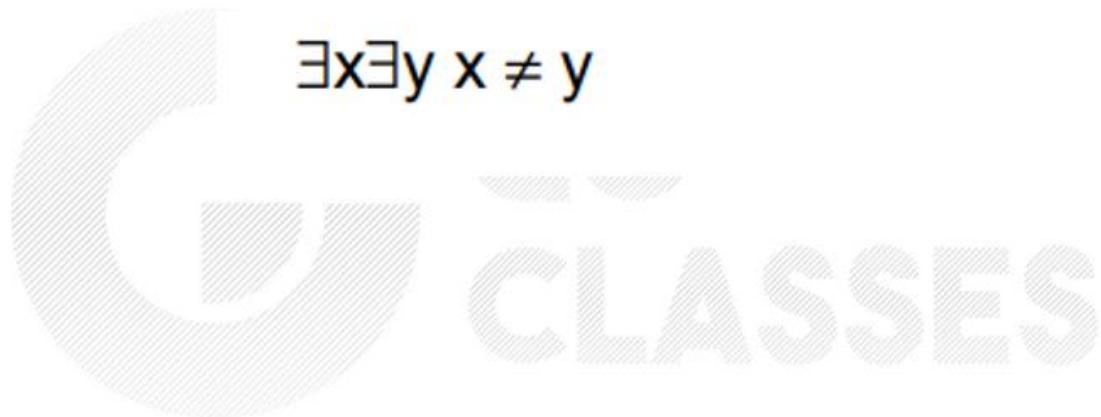
"These are at least 2 elements in the Domain."

$$\exists x \exists y ((x \neq y))$$



If we want to say simply that there are at least two things

$$\exists x \exists y \ x \neq y$$





There are at least three cubes.



$$\exists_x \exists_y \exists_z \left( \underbrace{\text{cube}(x) \wedge \text{cube}(y) \wedge \text{cube}(z)}_{\text{Three distinct cubes}} \wedge \underbrace{(x \neq y) \wedge (y \neq z)}_{\text{All pairs are distinct}} \wedge \underbrace{(x \neq z)}_{\text{All three are distinct}} \right)$$



we can say simply that there are at least three things

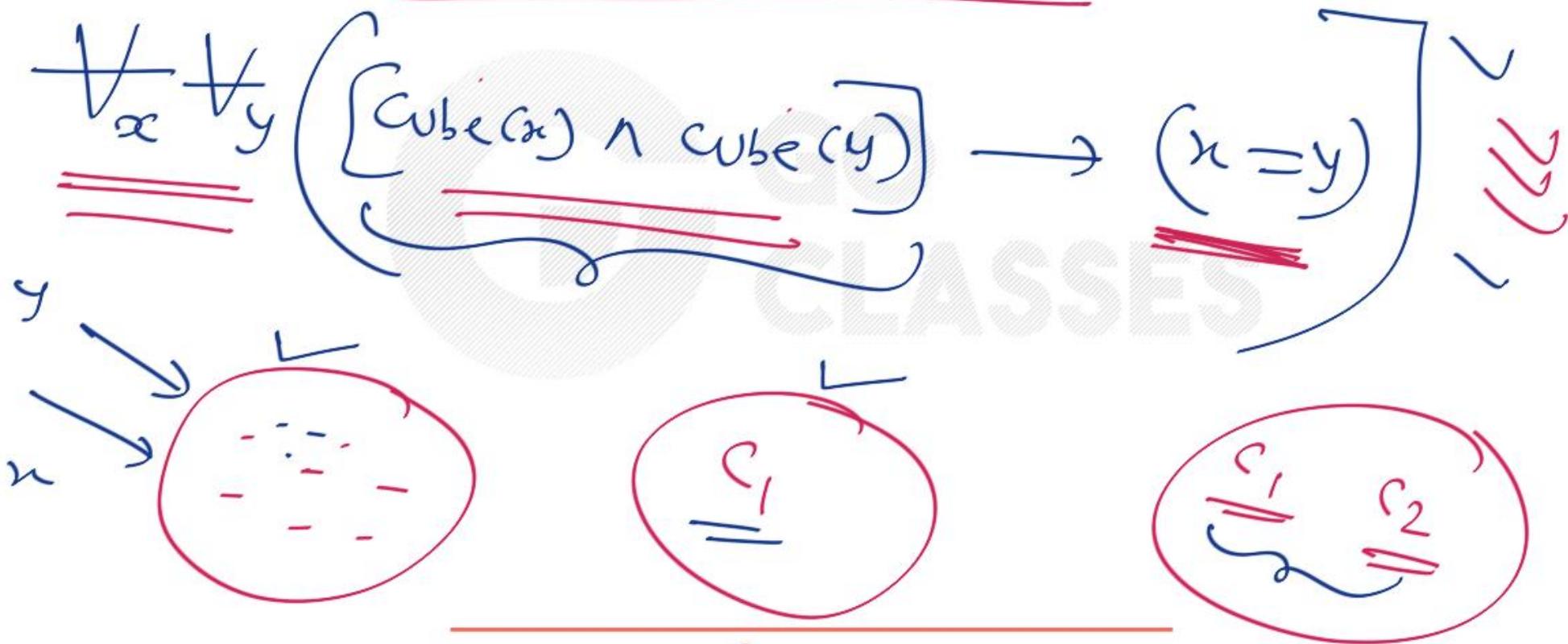
11

$$\exists \underline{x} \exists \underline{y} \exists \underline{z} (\underline{x} \neq \underline{y} \wedge \underline{y} \neq \underline{z} \wedge \underline{x} \neq \underline{z}).$$

CLASSES



There is at most one cube.





There is at most one cube.

No cube

OR

"Exactly one cube"

$\neg \exists x \text{ Cube}(x)$

∨

Exactly one cube

✓ There is at most one element in the Domain.

~~$\forall x \forall y$~~

$x = y$

✓

No element

one element

a

a, b



11

If we want to say simply that there is at most one thing

$$\forall x \forall y \ x = y$$

CLASSES

These is at most one bird

A diagram illustrating a function mapping from two inputs,  $x$  and  $y$ , to an output. The inputs  $x$  and  $y$  are shown on the left, with  $x$  having three parallel lines below it and  $y$  having two parallel lines below it. An arrow points from each input into a large circle representing the function. Inside the circle, the expression  $\text{bird}(x) \wedge \text{bird}(y)$  is written above a horizontal line, which is then followed by a wavy brace underneath. To the right of the circle, an arrow points to the output  $x = y$ , which also has three parallel lines below it.



There is exactly one cube.

~~At least one cube~~



~~at most one cube~~

~~$\exists x \text{ Cube}(x)$~~

$\lambda x \lambda y ((\underline{\text{Cube}(x) \wedge \text{Cube}(y)}) \wedge \underline{(x = y)})$

There is exactly one cube.

$$\exists_x \left( \text{cube}(x) \wedge \forall_y \left( \text{cube}(y) \rightarrow x = y \right) \right)$$

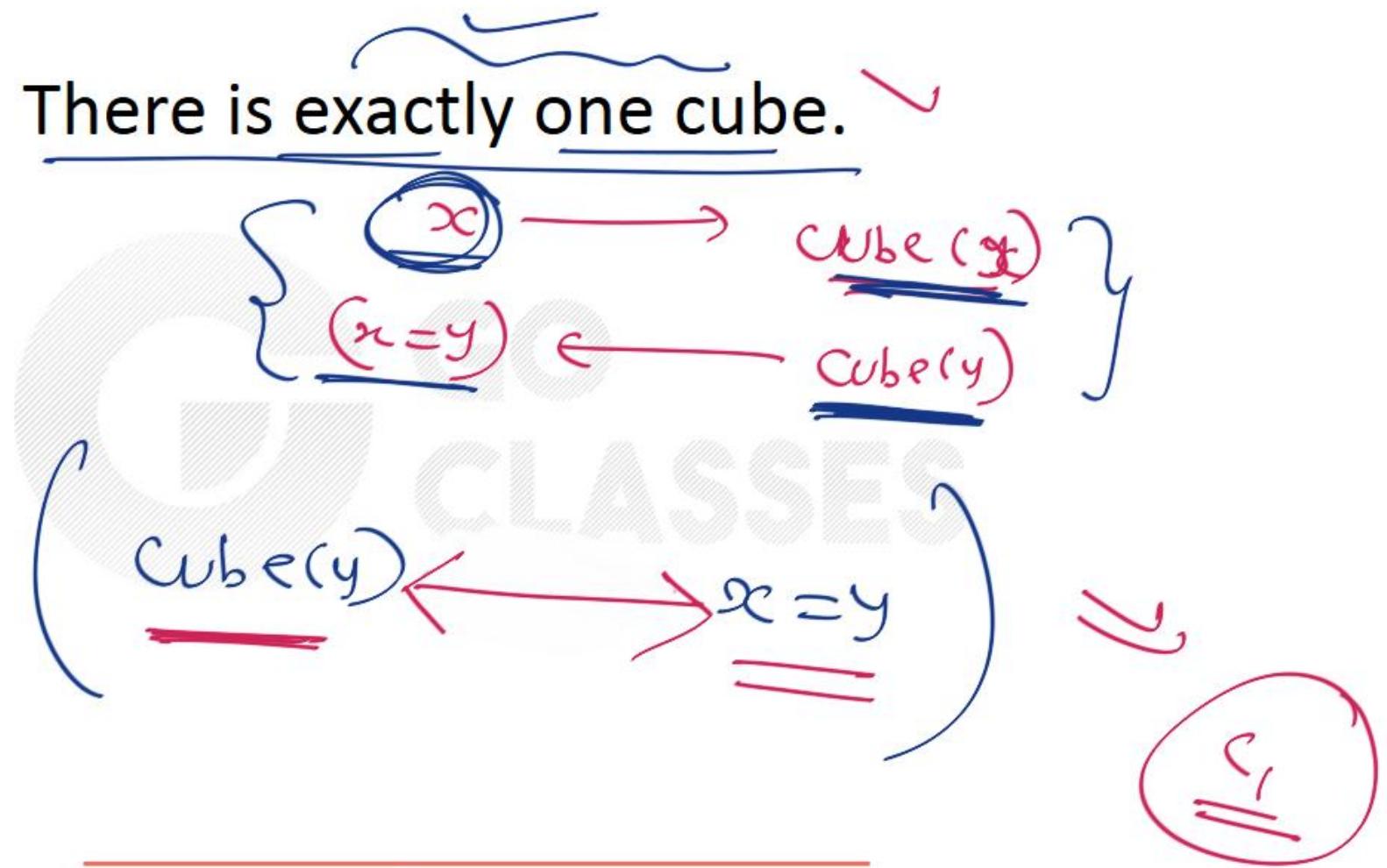
Diagram illustrating the proof:

- The top part shows the expression  $\exists_x \text{cube}(x)$  with a checkmark and a red bracket under it, followed by  $\wedge \forall_y \text{cube}(y) \rightarrow x = y$ .
- The bottom part shows the expression  $\exists_x \forall_y \left( \text{cube}(x) \wedge \left( \text{cube}(y) \rightarrow x = y \right) \right)$ .
- Both parts have red strikethroughs through the entire expression and the quantifiers  $\exists_x$  and  $\forall_y$ .
- A large blue bracket groups the entire top expression.
- A large blue bracket groups the entire bottom expression.



There is exactly one cube. ✓

$$\exists x \forall y \neg (x = y) \rightarrow c_1$$



There is Exactly one bird.

$$\exists x \forall y (\text{bird}(x) \wedge \text{bird}(y) \rightarrow x = y)$$

Diagram illustrating the logical expression:

- The expression  $\exists x$  is connected by a brace to the term  $x$  above it.
- The expression  $\forall y$  is connected by a brace to the term  $y$  above it.
- The predicate  $\text{bird}(x)$  is connected by a brace to the term  $\text{bird}(x)$  above it.
- The predicate  $\text{bird}(y)$  is connected by a brace to the term  $\text{bird}(y)$  above it.
- The equality  $x = y$  is connected by a brace to the term  $x = y$  above it.
- A large brace at the bottom groups the entire quantified statement  $\exists x \forall y$  with the predicate  $(\text{bird}(x) \wedge \text{bird}(y))$ .
- A small brace at the bottom groups the two terms  $\text{bird}(x)$  and  $x = y$ .

There is exactly one cube.

$$\exists x \text{ Cube}(x) \wedge \forall x \forall y ((\text{Cube}(x) \wedge \text{Cube}(y)) \rightarrow x = y)$$

$$\exists x (\text{Cube}(x) \wedge \forall y (\text{Cube}(y) \rightarrow y = x))$$

$$\exists x \forall y (\text{Cube}(y) \leftrightarrow y = x)$$



There is at most one cube.

$$\forall x \forall y ((\text{cube}(x) \wedge \text{cube}(y)) \rightarrow x = y)$$

The diagram illustrates a logical proof. On the left, there are two sets of three parallel horizontal lines, each labeled with a 'V' above it. A large curved bracket groups these two sets. To the right of the bracket is a logical expression:  $(\text{cube}(x) \wedge \text{cube}(y)) \rightarrow x = y$ . Another large curved bracket groups this expression and the result of the implication. Below the entire diagram is a large, faint watermark-like text "GO CLASSES".



there are at most two cubes.

$$\cancel{\forall x} \cancel{\forall y} \cancel{\forall z}$$

---

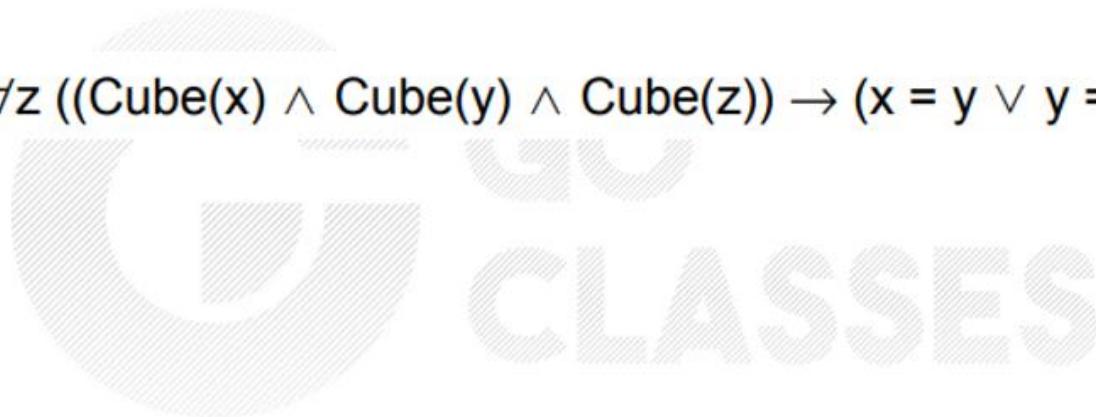
---

$$\left( \text{cube}(x) \wedge \text{cube}(y) \wedge \text{cube}(z) \right) \rightarrow \left( \begin{array}{l} (x=y) \vee (y=z) \vee (x=z) \\ \hline \end{array} \right)$$



there are at most two cubes.

$$\forall x \forall y \forall z ((\text{Cube}(x) \wedge \text{Cube}(y) \wedge \text{Cube}(z)) \rightarrow (x = y \vee y = z \vee x = z)).$$





Exactly one cube!

$$\exists x \left( \underline{\text{cube}(x)} \wedge \cancel{\forall y} (\text{cube}(y) \rightarrow y=x) \right)$$



There are exactly two cubes.

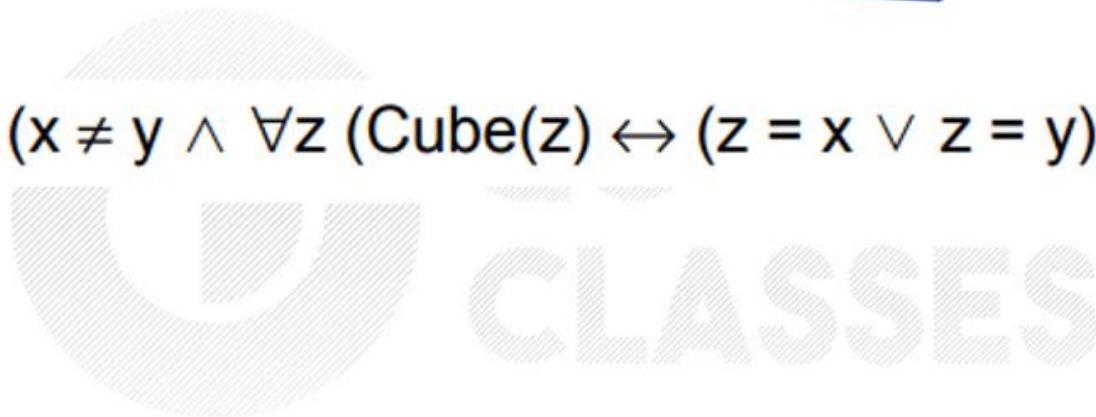
$$\exists \underline{x} \underline{y} \left( (\underline{x} \neq \underline{y}) \wedge \underline{\text{cube}(x)} \wedge \underline{\text{cube}(y)} \wedge \right. \\ \left. \forall \underline{z} (\underline{\text{cube}(z)} \rightarrow (\underline{x} = \underline{z}) \vee (\underline{z} = \underline{y})) \right)$$



There are exactly two cubes.

---

$$\exists x \exists y (x \neq y \wedge \forall z (\text{Cube}(z) \leftrightarrow (z = x \vee z = y))).$$





“For every number there is a larger number.”  
Assume domain is set of numbers.

$$\nexists x \exists y (y > x)$$

GO  
CLASSES



There is a number that is larger than every other number.  
Assume domain is set of numbers.

$$\exists \underline{x} \forall \underline{y} \left( (\underline{x} \neq \underline{y}) \rightarrow (\underline{x} > \underline{y}) \right)$$



✓ ✓

Every positive number is a square.  
Assume domain is set of numbers.

$$\left. \begin{array}{l} x = y^2 \\ 4 = 2^2 \\ 9 = 3^2 \end{array} \right\}$$

$$\forall x \left( \underline{(x > 0)} \rightarrow \exists z \left( \underline{z * z = x} \right) \right)$$



✓ If one number is less than another, then there is a number properly between the two.

Assume domain is set of numbers.

$$\forall_{\underline{x}} \forall_{\underline{y}} (\underline{(x < y)}) \rightarrow \exists_{\underline{z}} (\underline{(x < z)} \wedge \underline{(z < y)})$$



There are at least two numbers which are prime.  
Assume domain is set of numbers.

$$\exists \underline{x} \exists \underline{y} \left( \underline{(x \neq y)} \wedge \underline{\text{prime}(x)} \wedge \underline{\text{prime}(y)} \right)$$



There are at least three numbers which are prime.  
Assume domain is set of numbers.

$$\exists_{\underline{x}} \exists_{\underline{y}} \exists_{\underline{z}} \left( \underbrace{\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Prime}(z)}_{\text{Three distinct primes}} \wedge \right. \\ \left. \wedge (x \neq y) \wedge (y \neq z) \wedge (x \neq z) \right)$$

✓ There are infinitely many numbers which are prime.  
Assume domain is set of numbers.

$$\forall \underline{x} \exists \underline{y} \left( \underline{(y > x)} \wedge \underline{(\text{Prime}(y))} \right)$$



There is at most one number which is prime.  
Assume domain is set of numbers.

$$\forall x \forall y ((\text{prime}(x) \wedge \text{prime}(y)) \rightarrow (x = y))$$



There is exactly one number which is prime.

Assume domain is set of numbers.

$$\exists x \left( \underline{\text{Prime}}(x) \wedge \forall y \left( \underline{\text{Prime}}(y) \rightarrow y=x \right) \right)$$

a divides b

Divides  $a \mid b$  :

Assume domain is set of integers.

$$\underbrace{a \cdot x = b}$$

$$2 \mid 4 ; \underline{2(2)=4}$$

$$\exists x (a \cdot x = b)$$



✓ ✓

x is an even number

Assume domain is set of integers.

$$\text{Circled } x = 2(y)$$

✓

$$\exists y (x = 2y)$$


$$\begin{aligned}4 &= 2(2) \\6 &= 2(3) \\10 &= 2(5) \\9 &\neq 2(\quad)\end{aligned}$$

x is prime

$y|x \rightarrow y=1 \text{ or } y=x$

Assume domain is set of integers.

$$\underline{(x > 1)} \wedge \left( \forall y \left( \underline{y|x} \rightarrow \underline{(y=1) \vee (y=x)} \right) \right)$$



x is prime

Assume domain is set of integers.

$$\underline{\underline{(1 < x)}} \wedge \forall y \left( (y|x) \rightarrow ((y \approx 1) \vee (y \approx x)) \right)$$



"The relation "divides" is transitive.  
Assume domain is set of integers.

$a|b, b|c \text{ then } a|c$

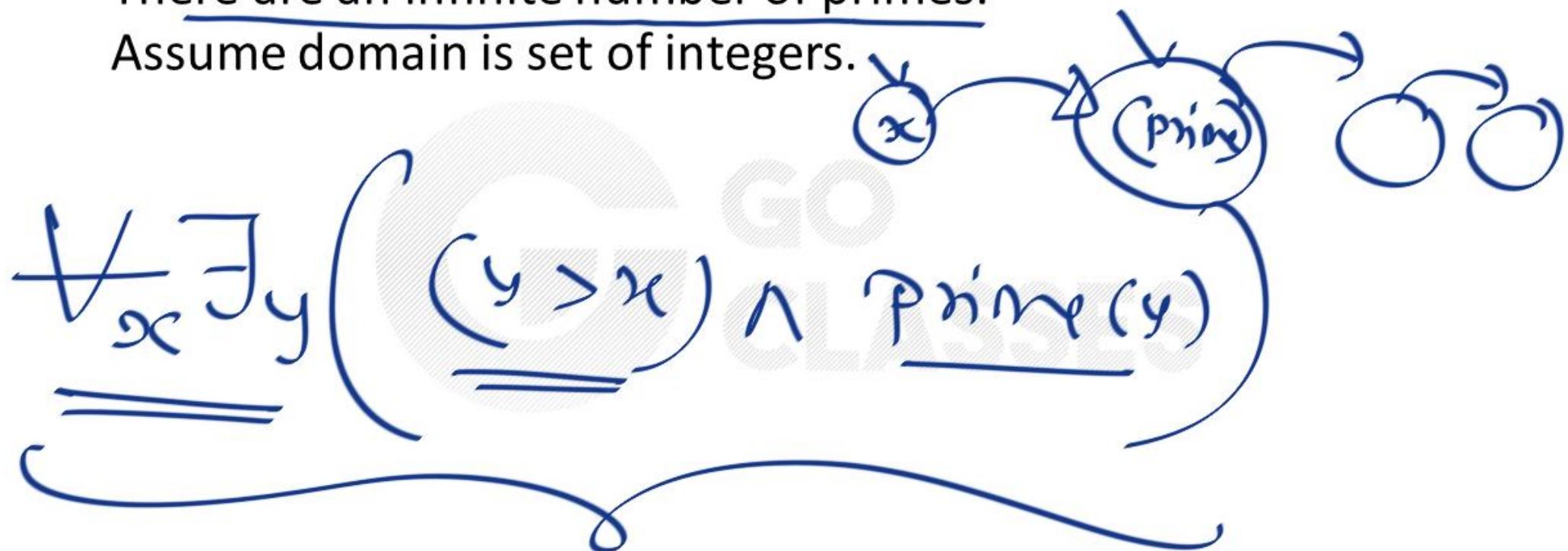
$$\forall a \forall b \forall c \left( \left[ \begin{array}{l} \underline{(a|b)} \\ \underline{(b|c)} \end{array} \right] \rightarrow \underline{\underline{(a|c)}} \right)$$

Diagram illustrating the logical structure of the proof:

- Variables  $a, b, c$  are introduced with underlines.
- A large bracket groups the condition  $(a|b) \wedge (b|c)$ .
- An arrow points from this bracket to the conclusion  $\underline{\underline{(a|c)}}$ .
- Curved arrows point from the variables  $a, b, c$  to their respective positions within the condition and conclusion.

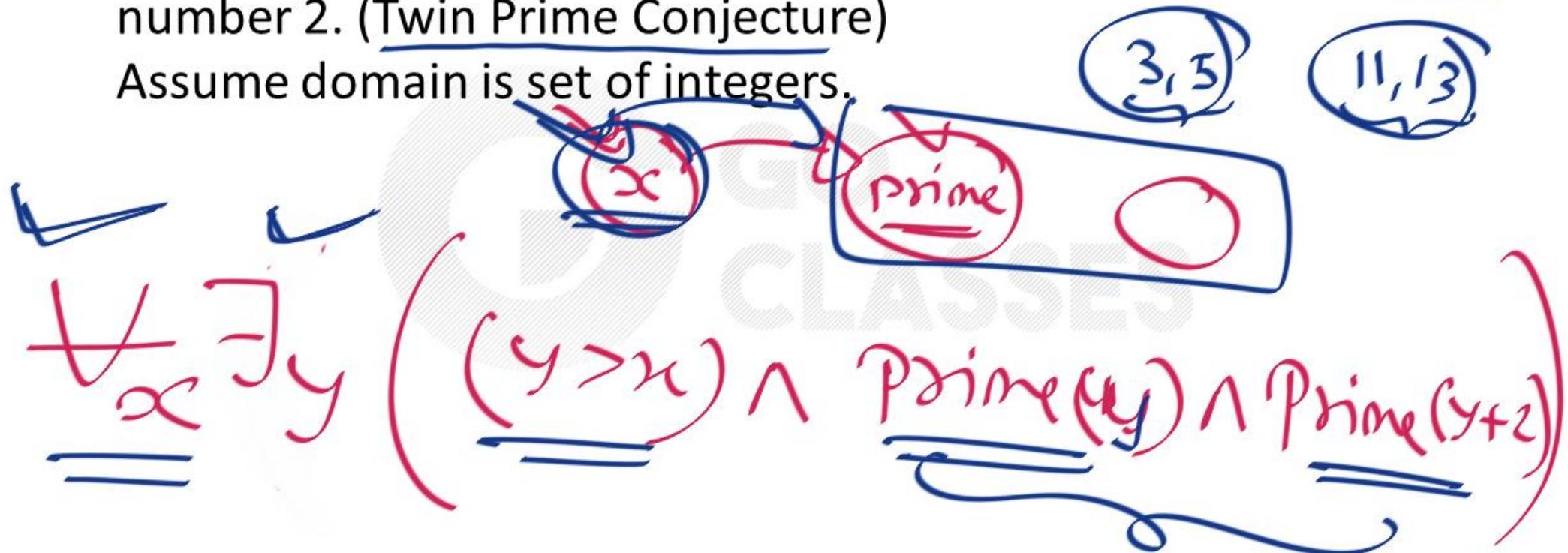
There are an infinite number of primes.

Assume domain is set of integers.



"There are an infinite number of pairs of primes that differ by the number 2. (Twin Prime Conjecture)

Assume domain is set of integers.





There are an infinite number of pairs of primes that differ by the number 2. (Twin Prime Conjecture)

Assume domain is set of integers.

$$\forall \underline{x} \exists \underline{y} \left( (\underline{x} < \underline{y}) \wedge \underline{\text{prime}}^*(\underline{y}) \wedge \underline{\text{prime}}^*(\underline{y} + 2) \right)$$



"All even numbers greater than two are the sum of two primes.  
(Goldbach's Conjecture)

Assume domain is set of integers.

$$\forall \underline{x} \left( \text{even}(x) \wedge (x > 2) \right) \rightarrow \exists \underline{y} \exists \underline{z} \left( \text{Prime}(y) \wedge \text{Prime}(z) \wedge x = y + z \right)$$



All even numbers greater than two are the sum of two primes.  
(Goldbach's Conjecture)

Assume domain is set of integers.

$$\forall x \left( ((2|x) \wedge (2 < x)) \rightarrow \right.$$

$$\left. \exists y \exists z \left( prime^*(y) \wedge prime(z) \wedge (x \approx y + z) \right) \right)$$