First Order Logic

Next Chapter:

Quantifiers

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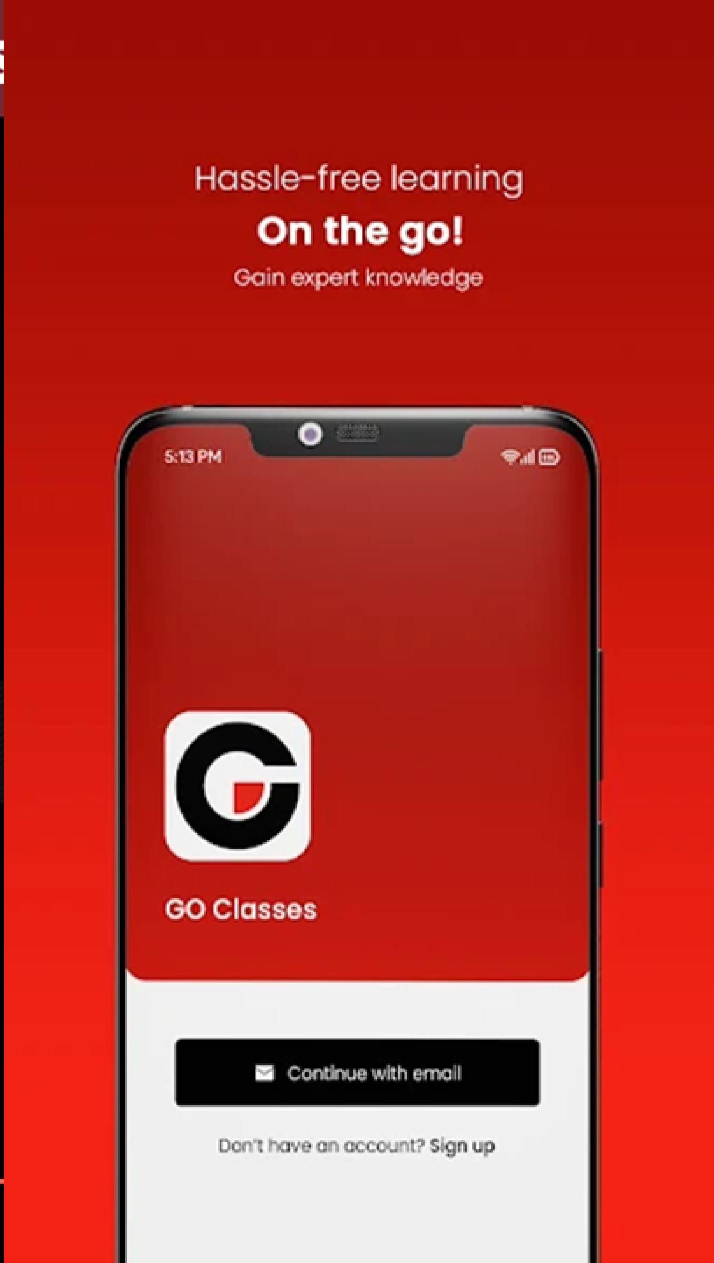

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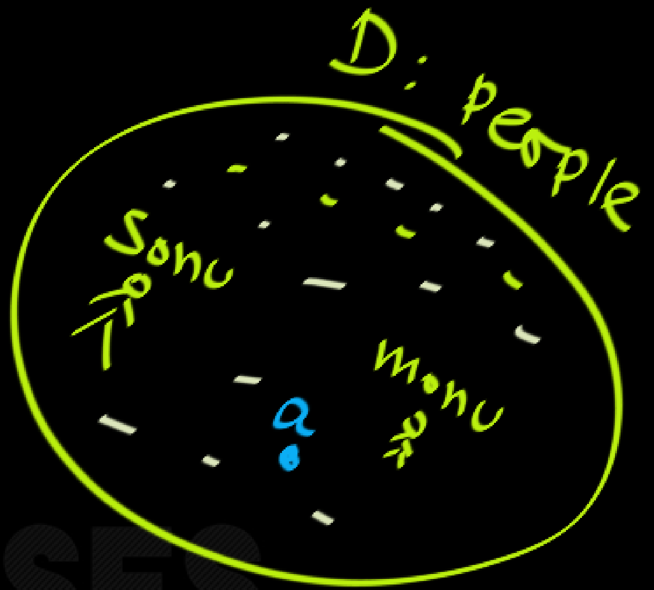
First Order Logic

Next Topic:

Quantifiers

Assume:



Domain: Set of all people

Predicate: Tall(x) : x is tall.

Tall(monu) : false ; Tall(sonu) : True

What does it mean when we say "Quantify the property Tall(x)" ??

Quantification Words in English Language:

Few people are tall.

All people are tall.

Many people are tall.

Some(at least one) people are tall.

No people are tall.

Most people are tall.

Quantify the property (predicate)

Tall(x)

Quantifiers:

Quantification of a Property…i.e. Talking about if a property is satisfied by Multiple Objects..

Quantification

- Statements like
  - Some birds are angry.
  - On the internet, no one knows who you are.
  - The square of any real number is nonnegative.

Quantification Words in English Language:

Few elements in the domain satisfy a property P

All elements in the domain satisfy a property P

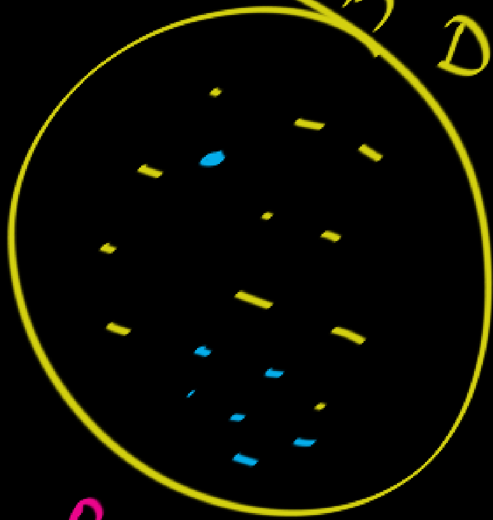
Many elements in the domain satisfy a property P

Some(at least one) elements in the domain satisfy a property P

No elements in the domain satisfy a property P

Most elements in the domain satisfy a property P

Domain D



Predicate

P(x)

Quantification Words in Mathematics:

All elements in the domain satisfy a property P

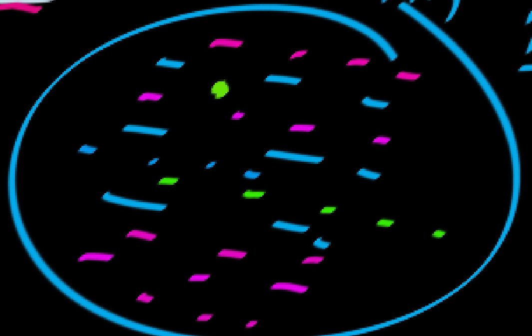
Universal Quantification

Some(at least one) elements in the domain satisfy a property P

Existential Quantification

Universe ≡ Domain D

Predicate P



Quantification words in English:

few, All, many, some, any ...

Quantification words in logic:

All ≡ every

"some" ≡ "at least one"

Although there are many types of *quantifiers* in English (e.g., many, few, most, etc.) in mathematics we, for the most part, stick to two: existential and universal.

First Order Logic

Next Topic:

Universal Quantifier

(for all)

Universal Quantification:

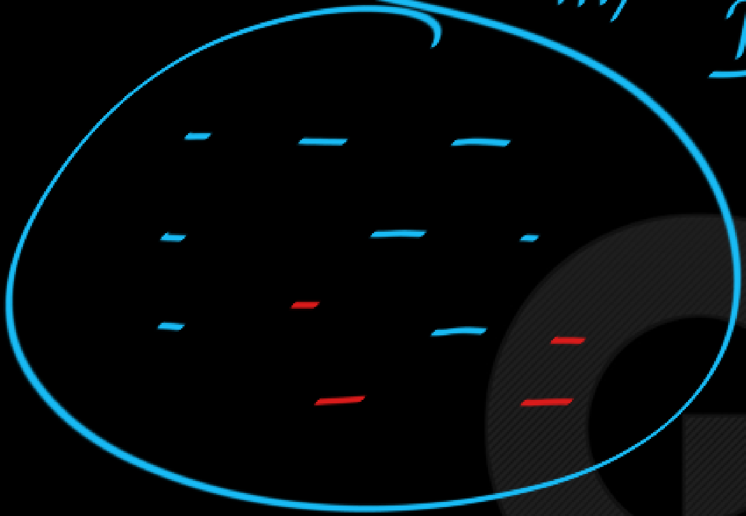
Universal Quantification means: Saying that a property P is satisfied by ALL elements in the domain.

Domain ≡ Universe

All elements in the domain satisfy a property P.

→ Universal Quantification of prop. P.

Domain D



Predicate P(x)

Universal Quantification of P(x) :

Every element in Domain, Satisfies P(x).

$\forall$: Universal Quantifier Symbol

Read as "for all"

$\in$: mean "in"

Universal Quantification:

English:

For all elements x in the domain, P(x) is true.

FOL:

$\forall x \in D \; (P(x))$

in

Domain D

Predicate P(x)

English:

for every x in the Domain, P(x)

$\equiv$

$$\boxed{\forall_x P(x)} \equiv \boxed{\forall_{x \in D} P(x)}$$

FOL

implicit

Domain: D:

x

Universal Quantification:

English:

For all elements x in the domain, P(x) is true.

FOL: $\forall_x P(x)$

Domain

All elements in the domain satisfy a property P.

$\equiv \forall_x P(x)$

$$\boxed{\begin{aligned} &\underline{A \text{ is True}} \equiv A \\ &\underline{A \text{ is false}} \equiv \neg A \end{aligned}}$$

Universal Quantification:

For all elements x in the domain, P(x) is true.

$\forall_x P(x)$

implicit

For all x, P(x) is true.

P(x)

$\forall_x P(x)$

x

Domain

For all x, P(x). $\equiv$ $\forall_x P(x)$

Universal Quantification:

For all elements x in the domain, P(x) is true.

All elements in the domain satisfy a property P.

For all elements x in the domain, P(x) is true.

For all x, P(x) is true. $\boxed{\forall_x P(x)}$

For all x, P(x).

Same

Predicate $P(x)$

Expression

$\forall_x P(x)$

means

for every x in the Domain $P(x)$ is True

The Universal Quantifier

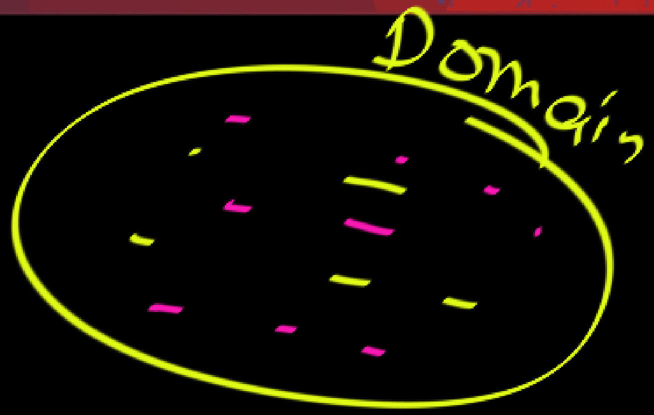
- A statement of the form

$$\forall x.\ \boxed{\textit{some-formula}}$$

is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.

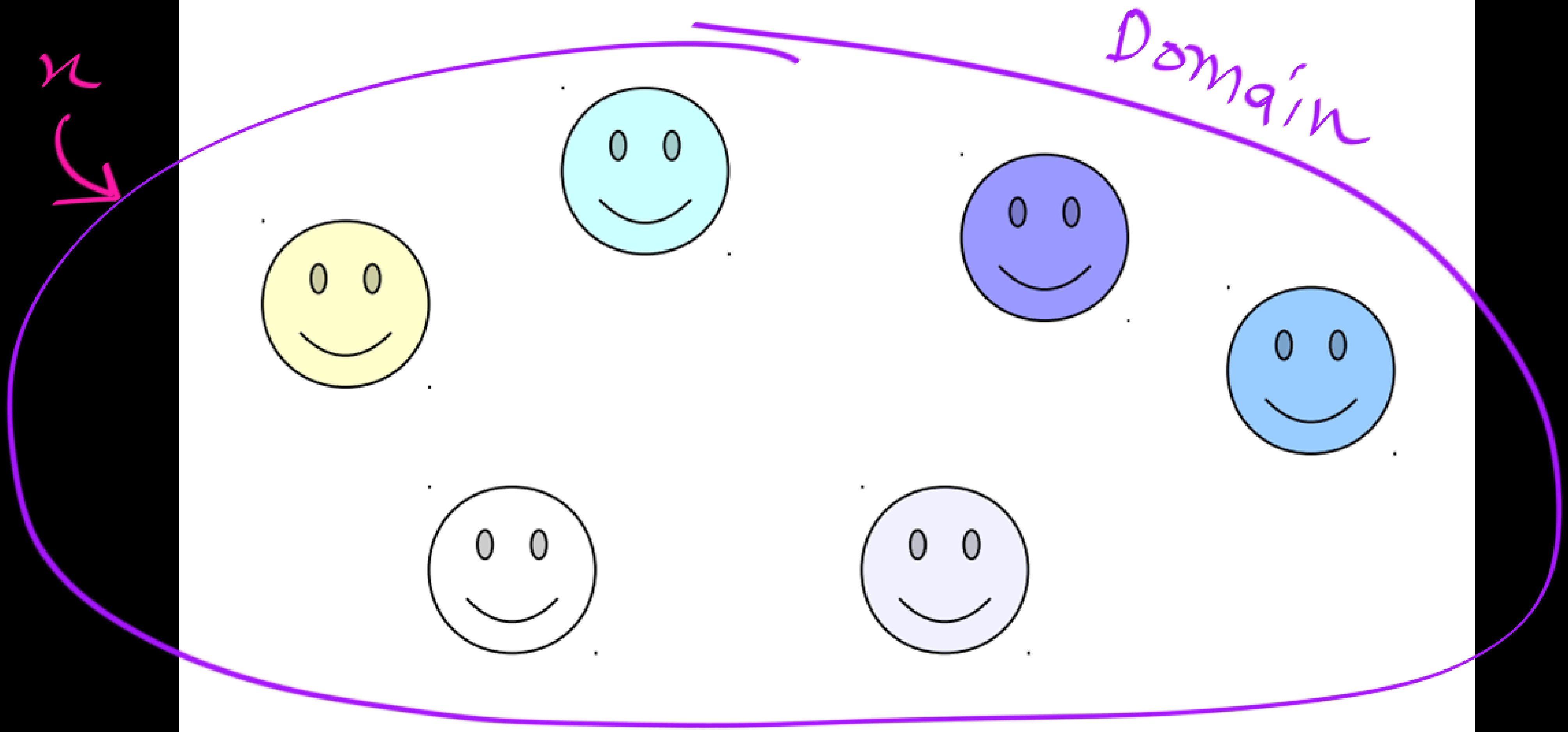
Box

$\forall x \left(\boxed{\text{- - - - - -}} \right)$



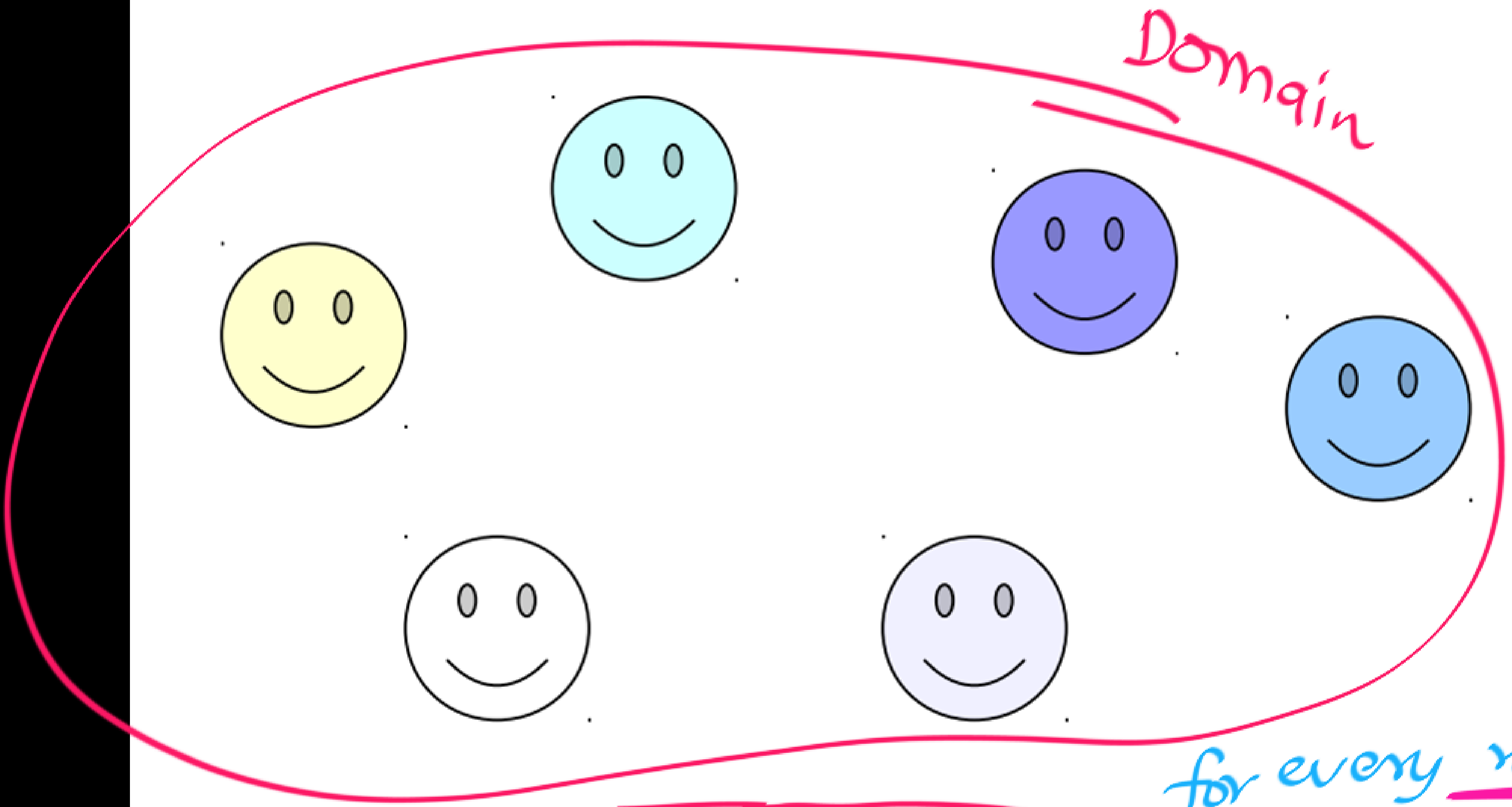
means : for every element in the Domain, Box is True.

The Universal Quantifier



$\forall x.\ Smiling(x)$

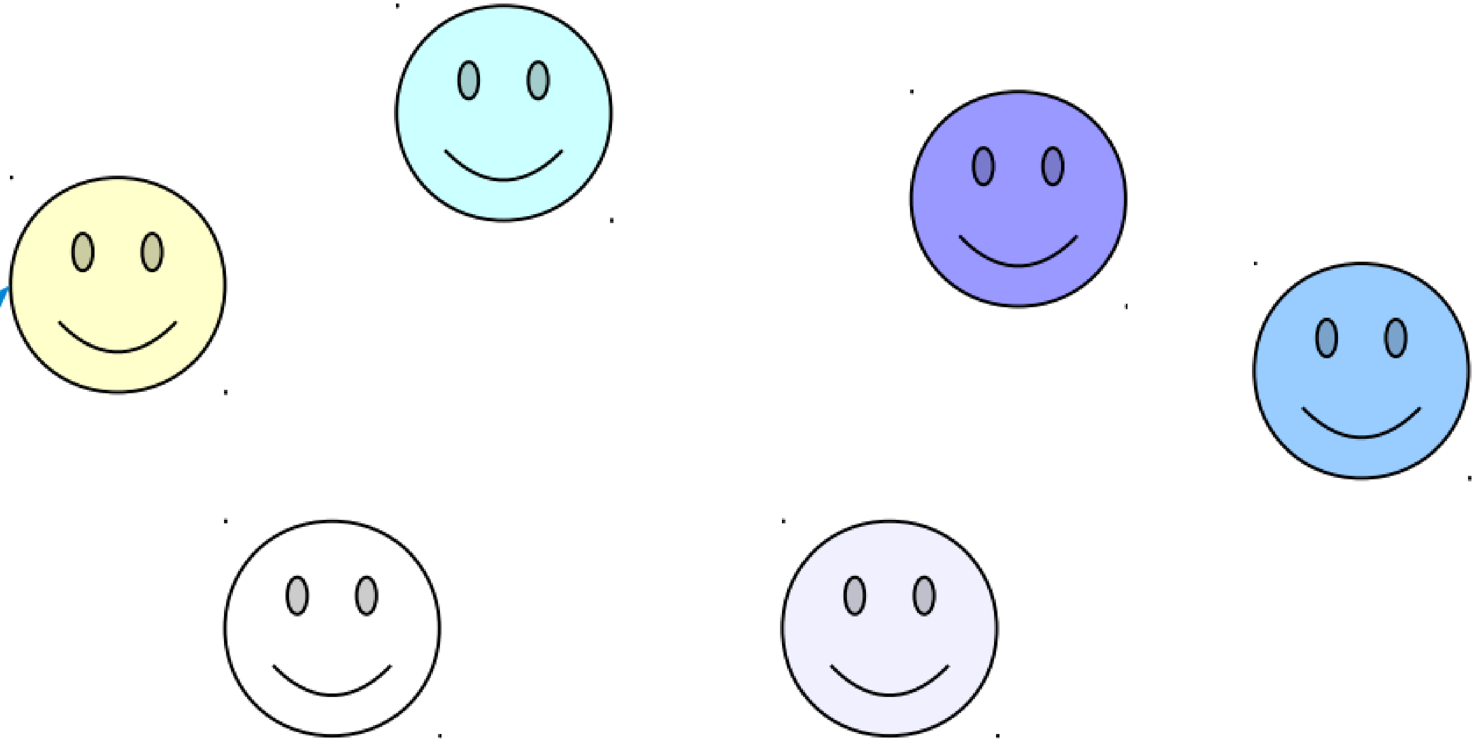
The Universal Quantifier



for every x in Domain,

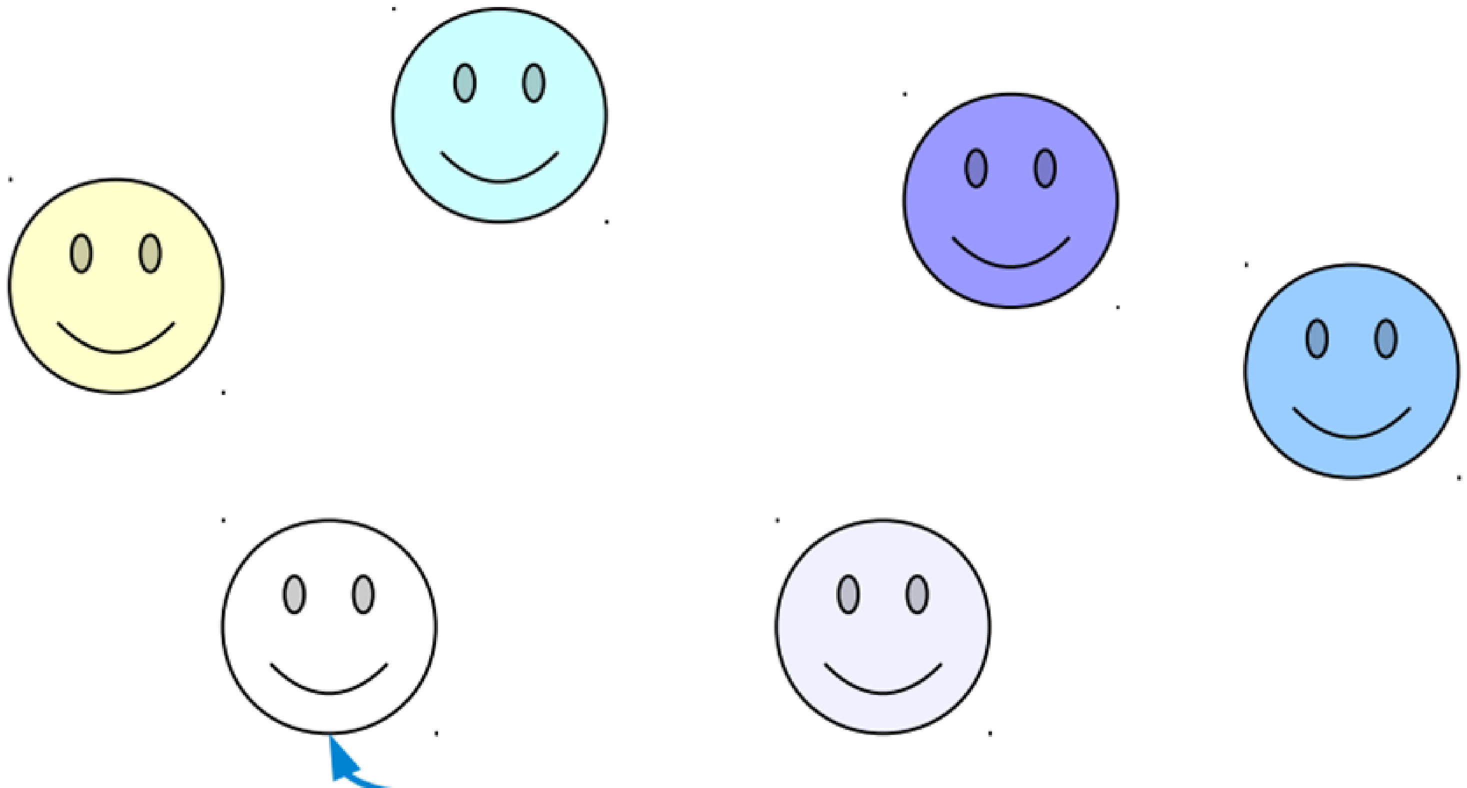
forall $\boxed{\forall x.\ Smiling(x)}$ $\equiv$ x is smiling.

The Universal Quantifier



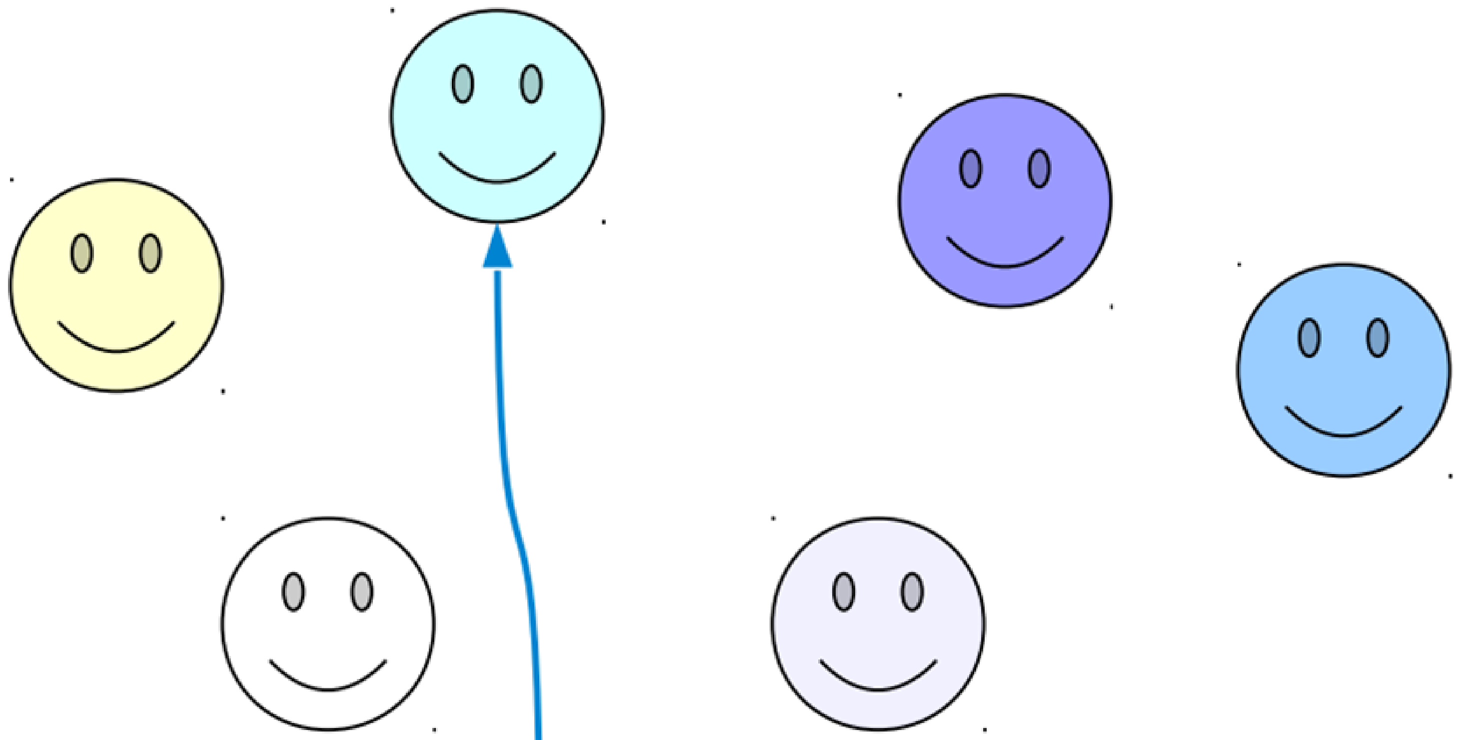
$\forall x.\ Smiling(x)$

The Universal Quantifier



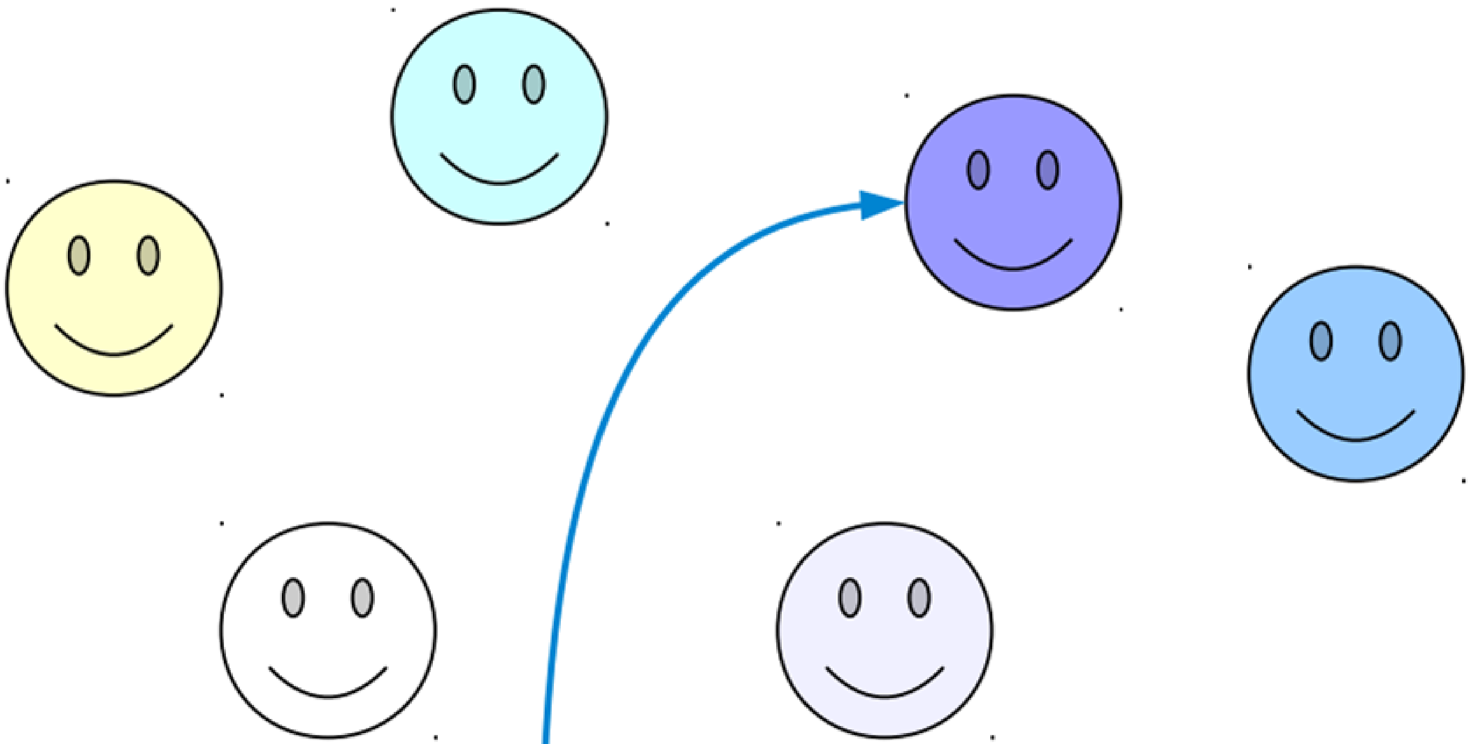
$\forall x.\ Smiling(x)$

The Universal Quantifier



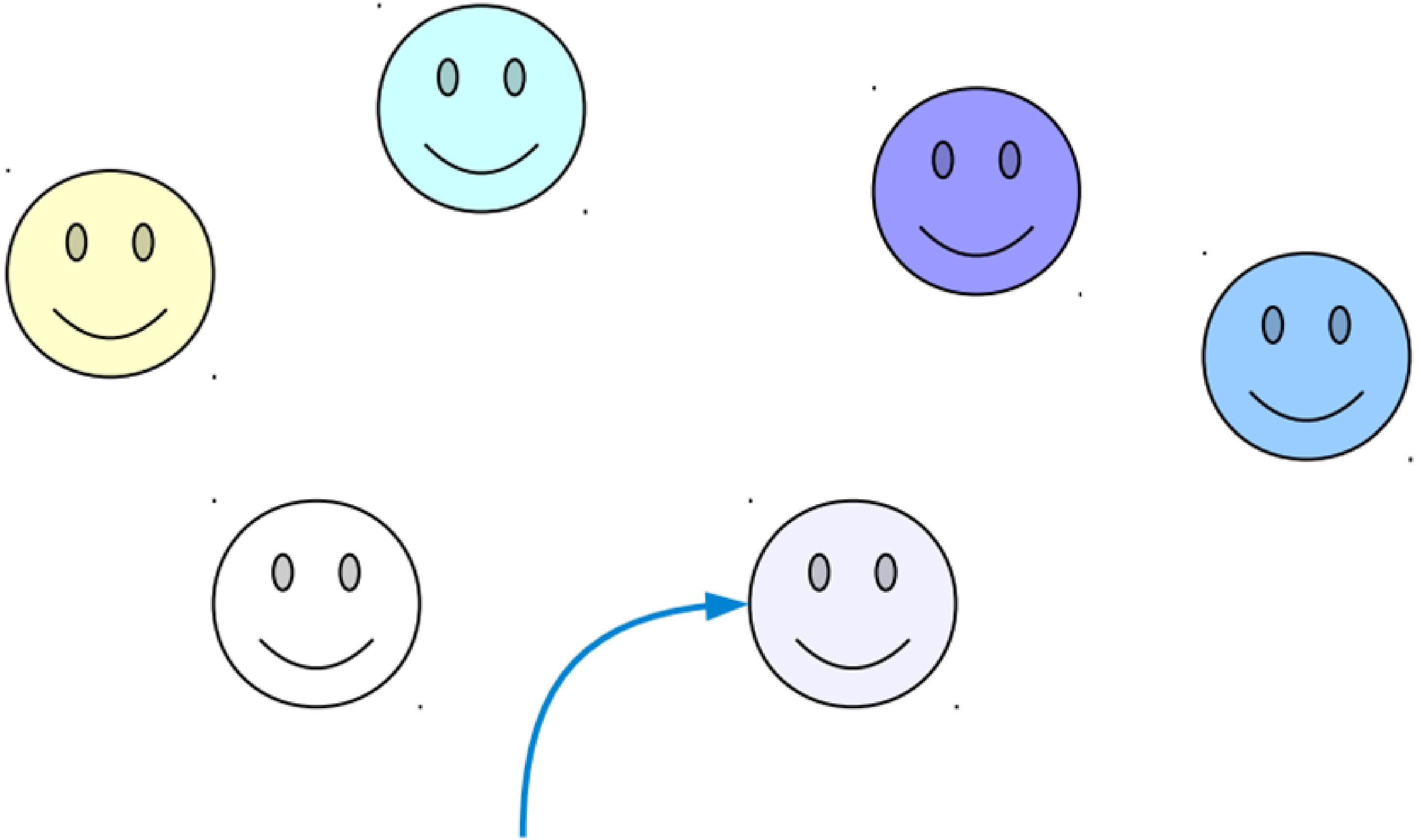
$\forall x.\ Smiling(x)$

The Universal Quantifier



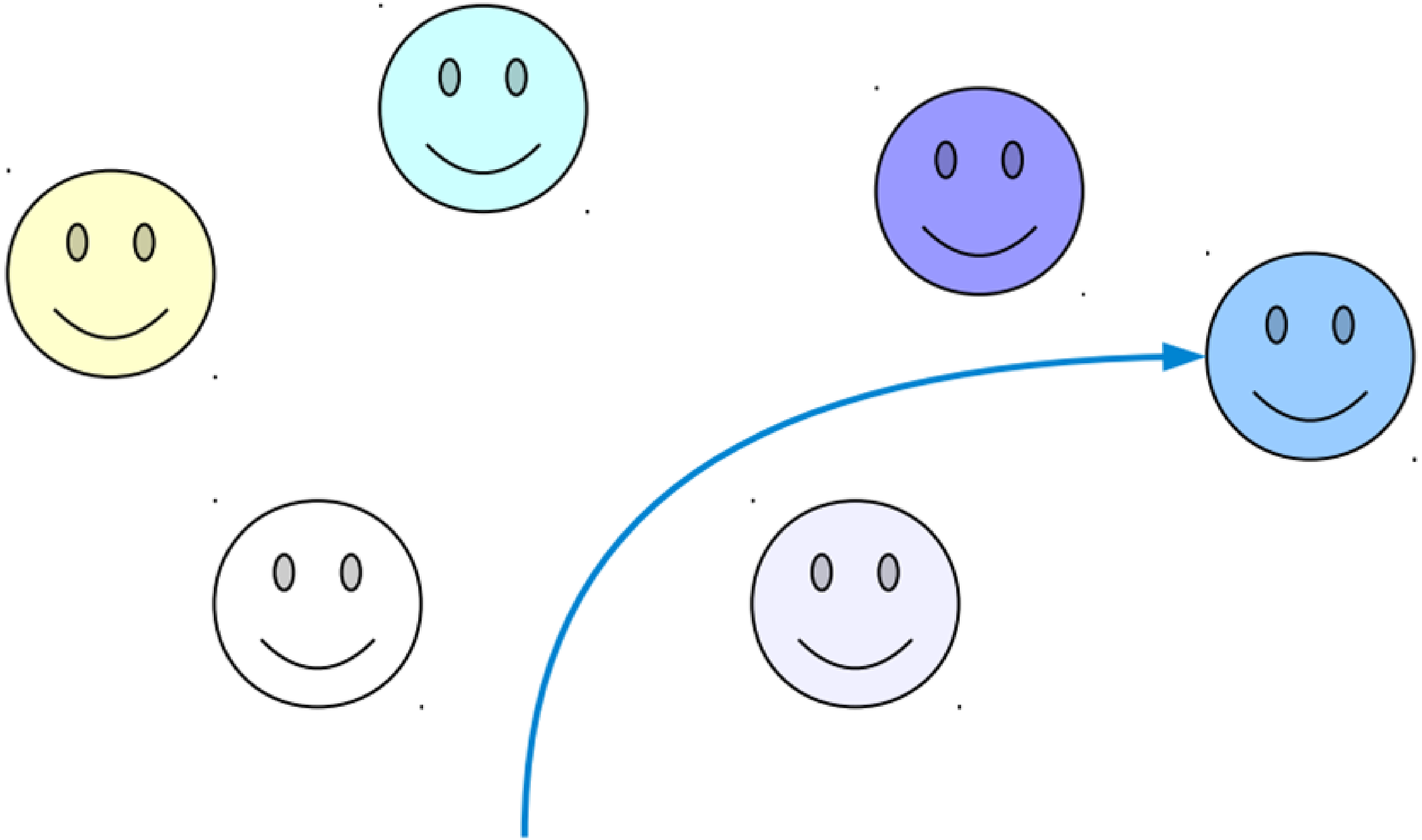
$\forall x.\ Smiling(x)$

The Universal Quantifier



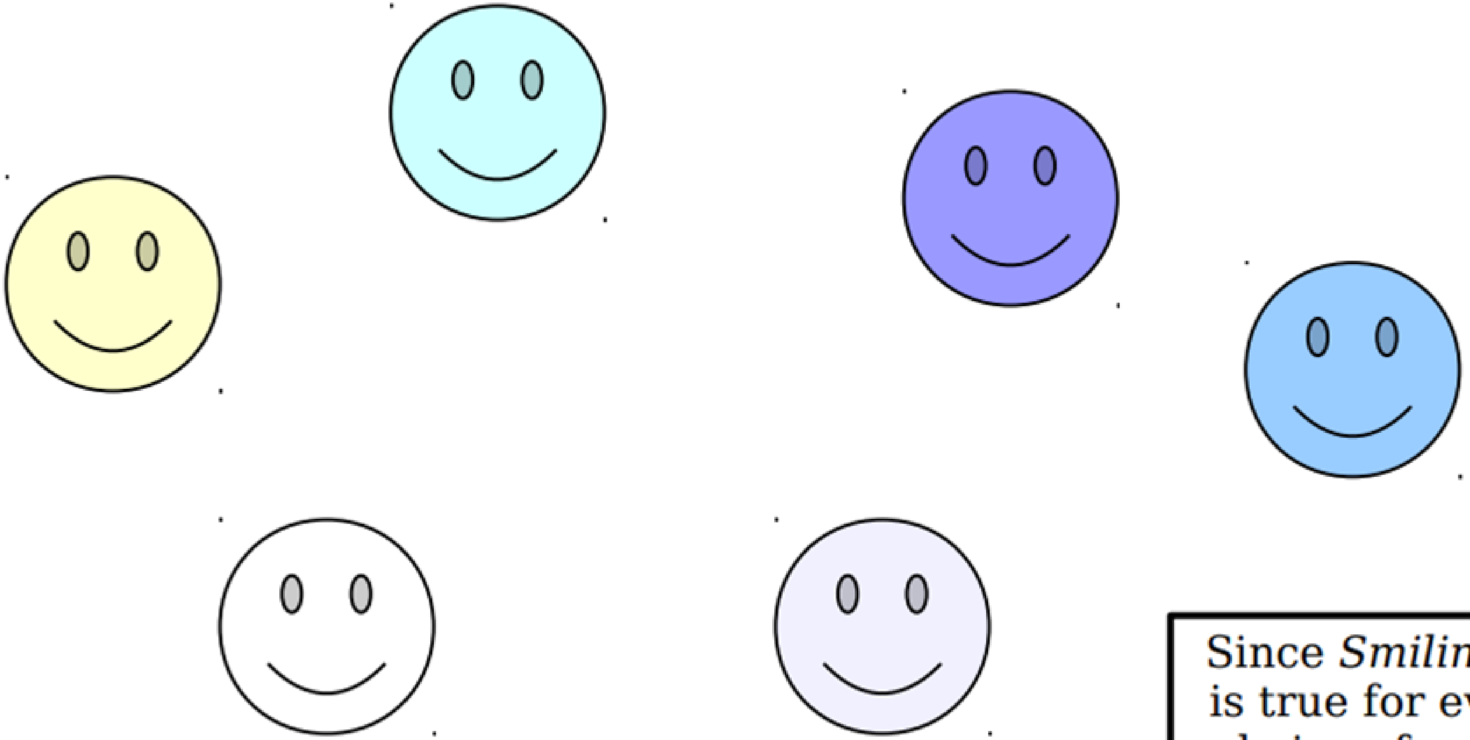
$\forall x.\ Smiling(x)$

The Universal Quantifier



$\forall x.\ Smiling(x)$

The Universal Quantifier

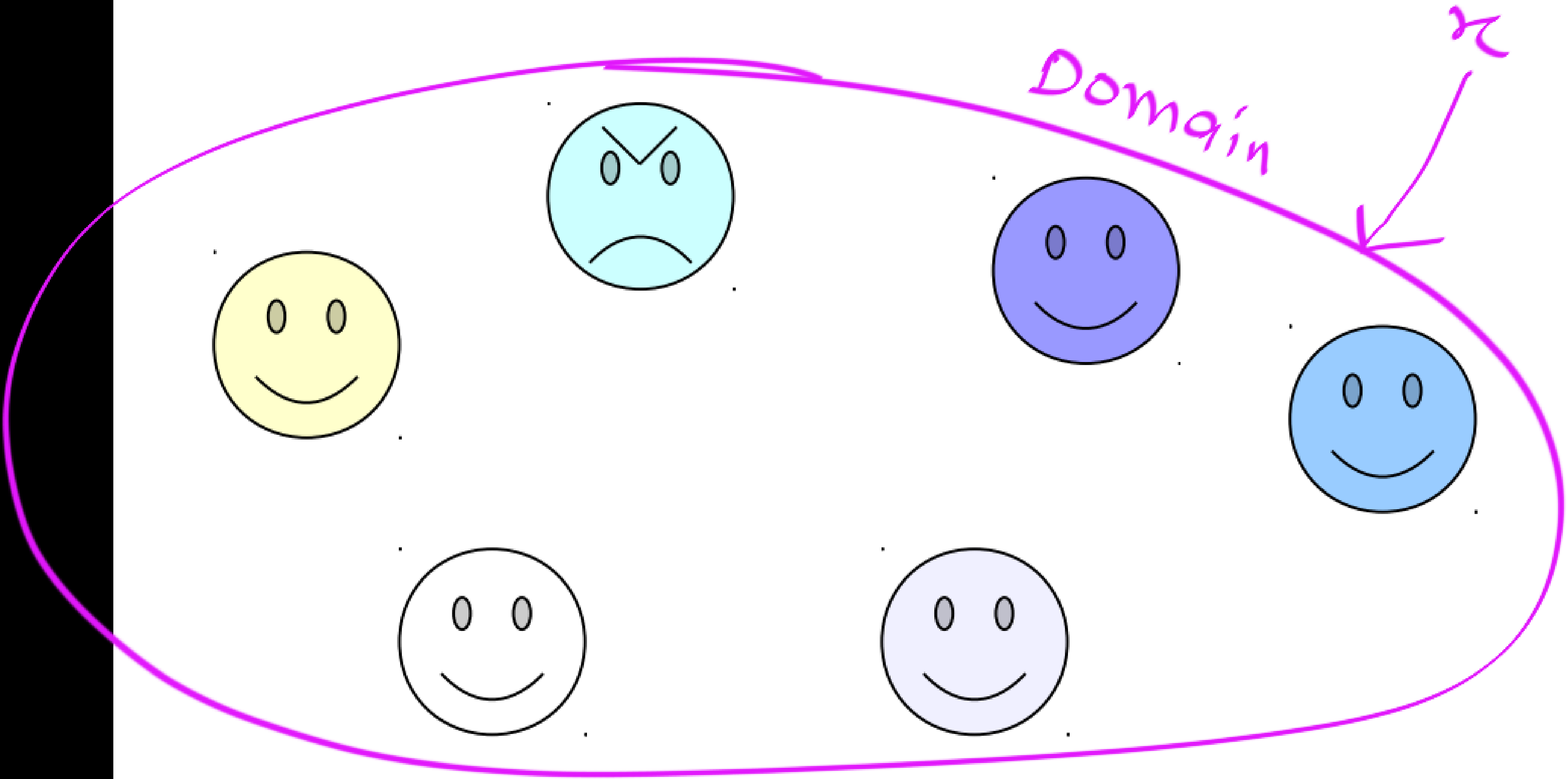


Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

True

$\boxed{\forall x.\ Smiling(x)}$

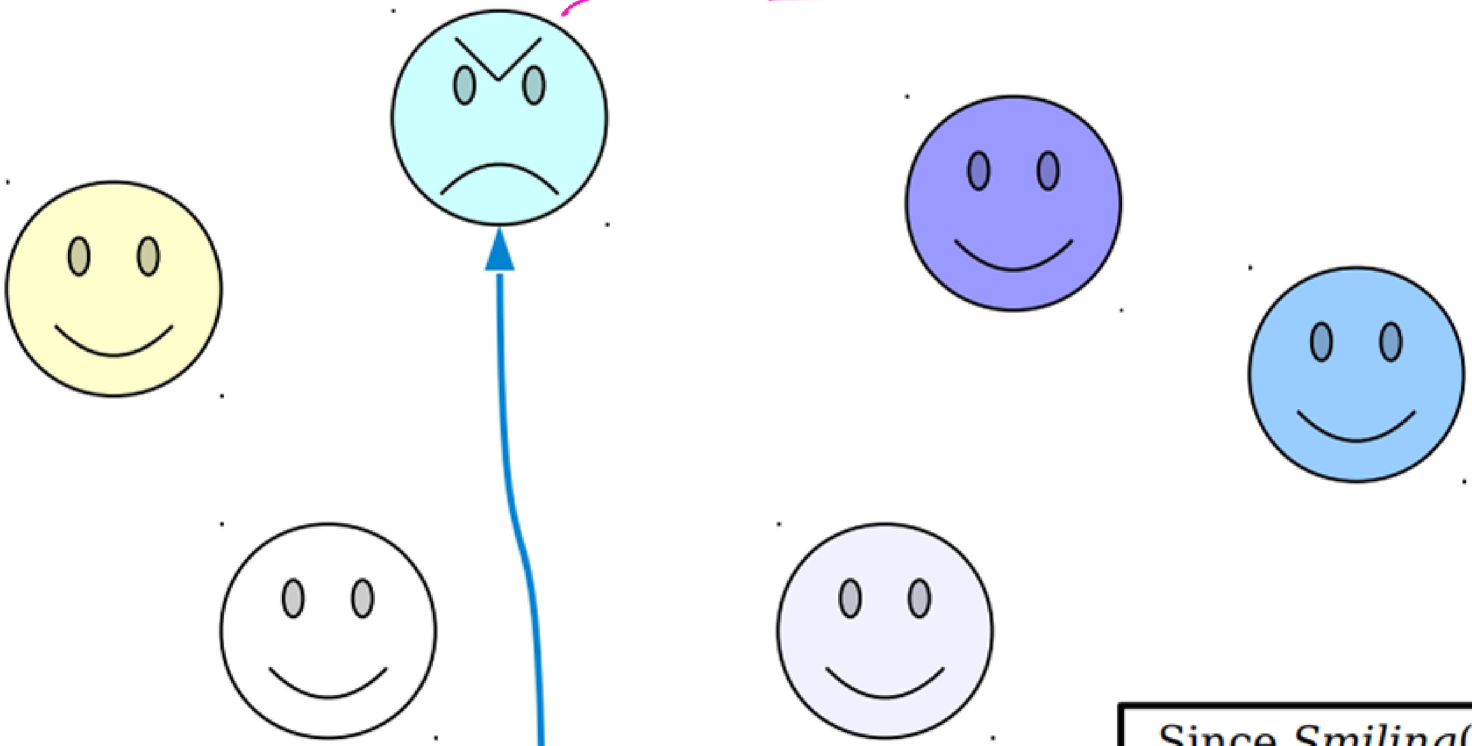
The Universal Quantifier



$$\boxed{\forall x.\ Smiling(x)}$$

The Universal Quantifier

Counterexample



false

$\forall x.\ \mathit{Smiling}(x)$

Since *Smiling*(x) is false for this choice x, this statement evaluates to false.

DEFINITION 3.3.1. *The* **universal quantification** *of $P(x)$ is the proposition "$P(x)$ is true for all values x in the universe of discourse."*

Notation: "For all x $P(x)$" or "For every x $P(x)$" is written

$$\forall x P(x).$$

for all x, $P(x)$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

$\forall_x\, p(x)$? T/F? $\forall_x\, r(x)$? T/F?

$\forall_x\, q(x)$? T/F? $\forall_x\, s(x)$? T/F?

Example:

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$\forall_x\, p(x)$? T/F ? $\forall_x\, r(x)$? T/F ?

$\forall_x\, q(x)$? T/F ? $\forall_x\, s(x)$? T/F ?

Domain: R

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

$\forall_x P(x)$

Counterexample : -10

for every real no x, $x \geq 0$: false

$\forall_x P(x)$ is false iff

there is at least one

Counterexample.

An element for which $P(x)$ is false.

Domain : R

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

$\forall x\, q(x)$: True

No Counterexample

for every real no x, $x^2 \geq 0$: True

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x):\quad x \geq 0 \qquad\qquad r(x):\quad \boxed{x^2 - 3x - 4 = 0}$$
$$q(x):\quad x^2 \geq 0 \qquad\qquad s(x):\quad x^2 - 3 > 0.$$

$\forall_x\ r(x)$ → false

Counterexample:

$x = 2$

for every real no x, $x^2 - 3x - 4 = 0$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

$\forall_x\, s(x)$

false

Counterexample:

$x = 1$

$x = 0$

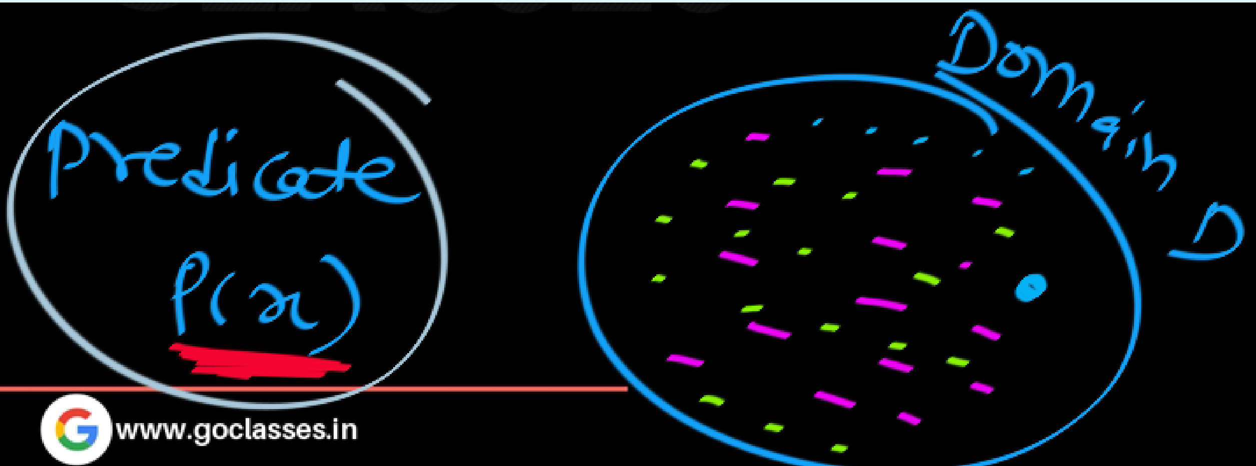
$x = -1$

$\boxed{\text{for every real no } x,\ x^2 - 3 > 0}$

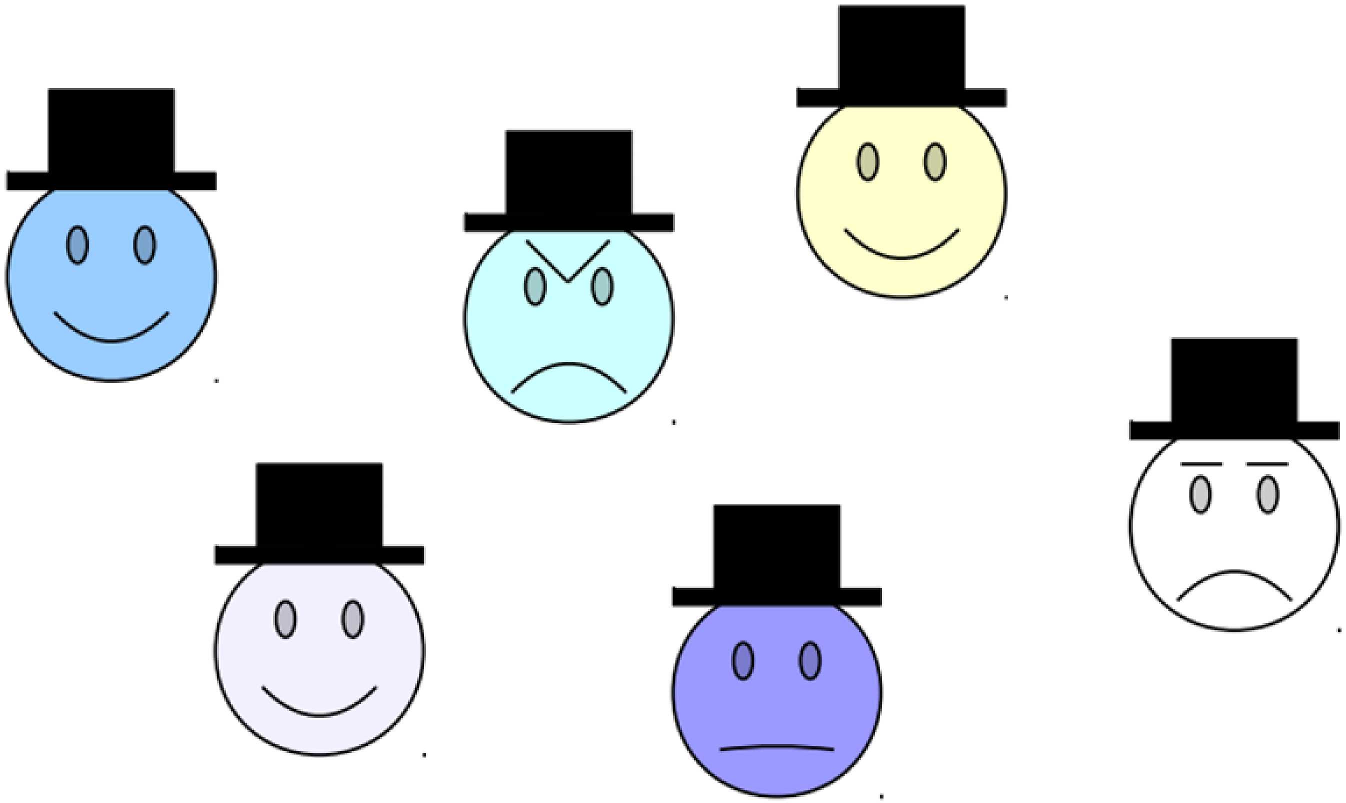
The *universal quantification* of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain." $\equiv$ $\boxed{\forall_x P(x)}$

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$. Here $\forall$ is called the **universal quantifier.** We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$." An element for which $P(x)$ is false is called a **counterexample** of $\forall x P(x)$.



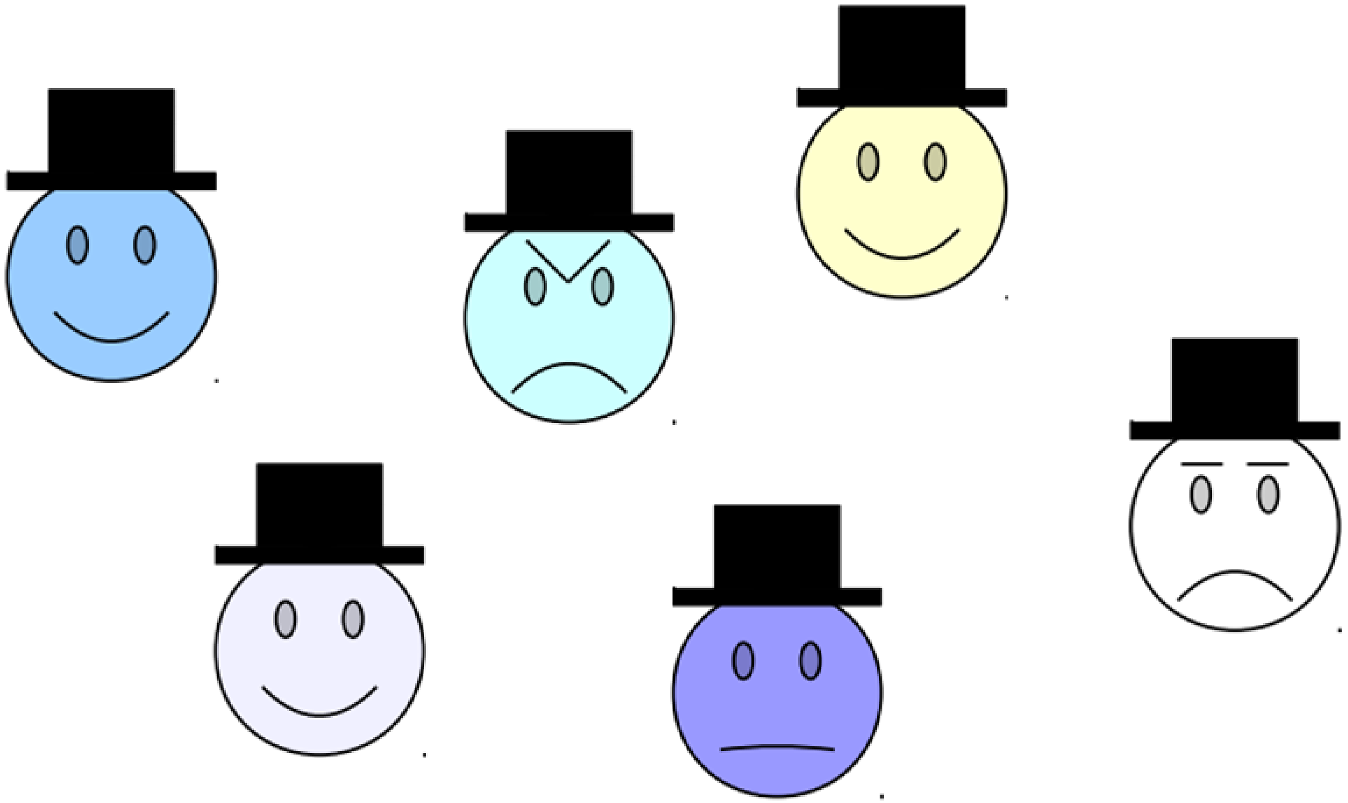
The Universal Quantifier



True / false ??

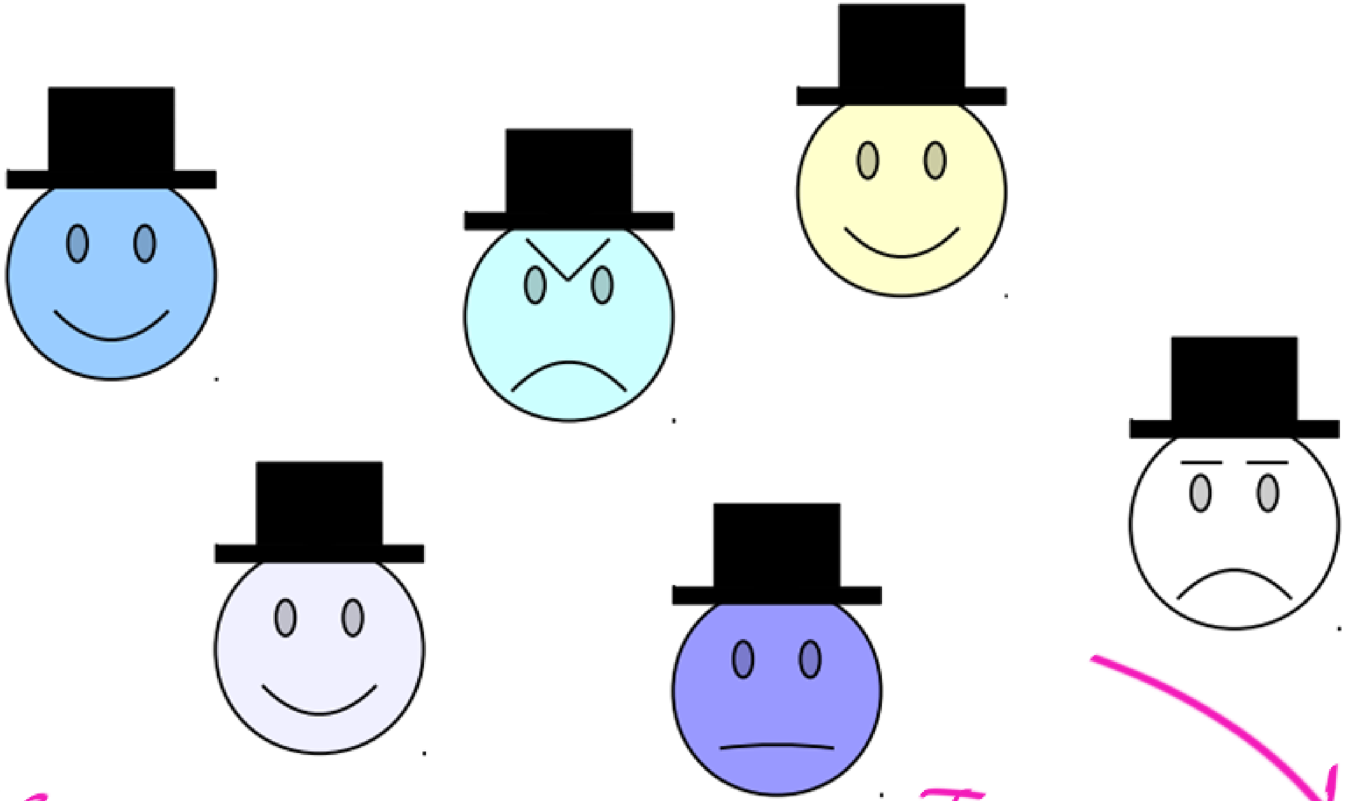
$$\boxed{(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))}$$

The Universal Quantifier



$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

The Universal Quantifier



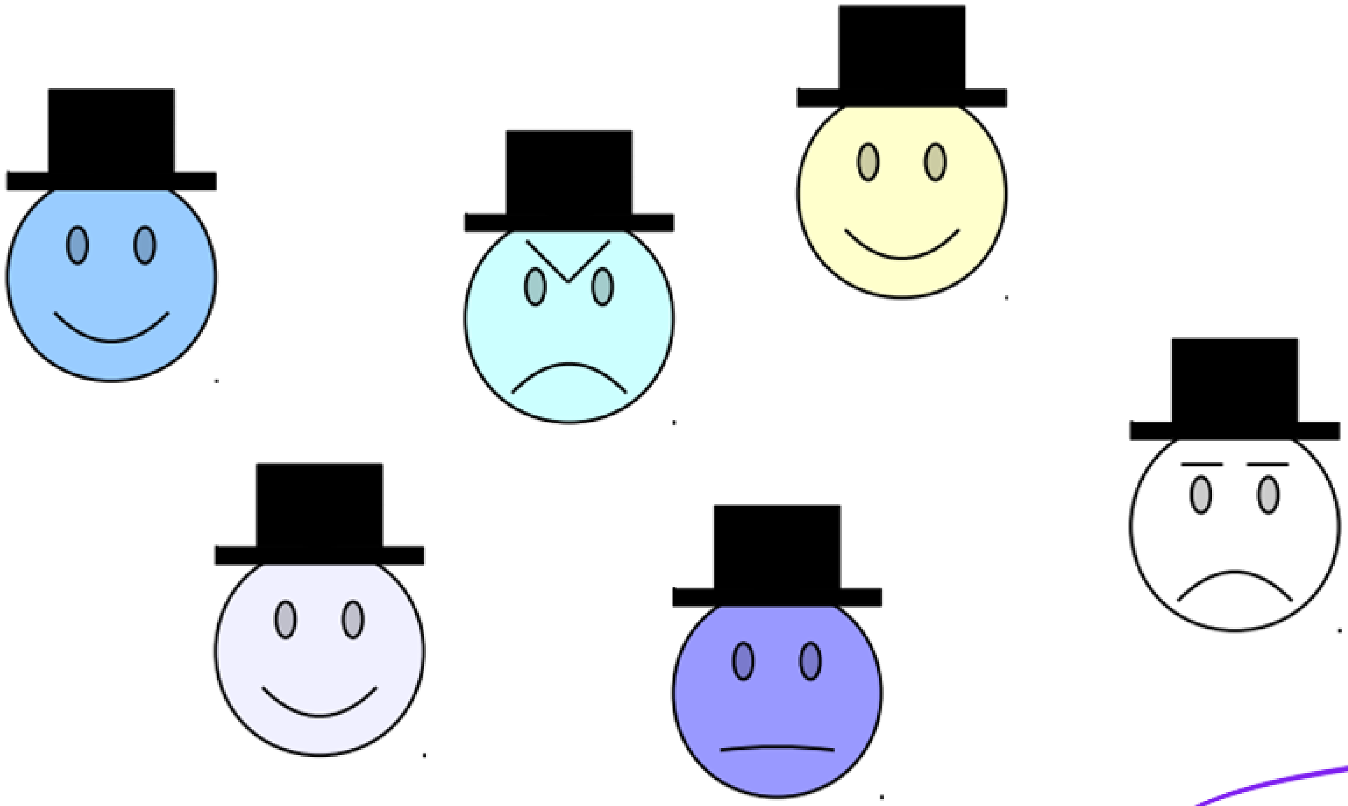
False

True

True / false ??

$$\boxed{(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))}$$

The Universal Quantifier



$$(\forall x.\ Smiling(x)) \rightarrow \underbrace{(\forall y.\ WearingHat(y))}_{B}$$

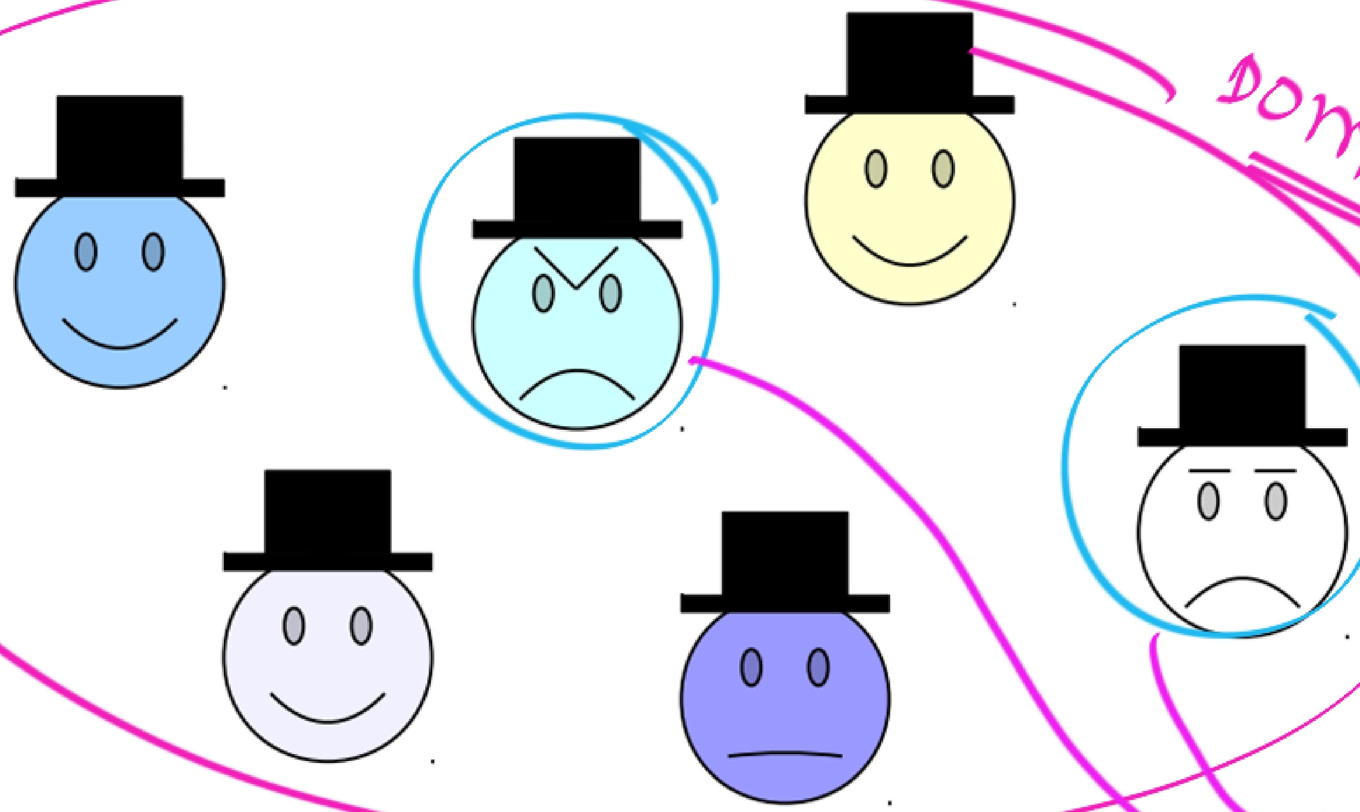
B

True

The Universal Quantifier

x

y



Domain

false

Counterexamples for A

A

$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

$$\underbrace{\left(\forall_x \text{ Smiling}(x)\right)}_{A} \rightarrow \underbrace{\left(\forall_y \text{ wearinghat}(y)\right)}_{B}$$

$$A \rightarrow B$$

$$\underbrace{\left(\forall_x \; smiling(x)\right)}_{A} \rightarrow \underbrace{\left(\forall_y \; wearinghat(y)\right)}_{B}$$

false

True

$A \rightarrow B \equiv$ True

false

True

Universal Quantification

when Domain is finite set:

Like a Conjunction

Predicate

$P(x)$

Domain D

a, b, d, c

finite set

$\forall_x P(x)$

every element in the Domain, satisfies P.

Predicate $P(x)$

$\forall_x P(x)$

Domain D

a, b, c, d

$\equiv P(a) \wedge P(b) \wedge P(c) \wedge P(d)$

finite set

$\forall_x P(x)$

$\equiv P(a_1) \wedge P(a_2) \wedge P(a_3)$

$- - - - - \wedge P(a_n)$

Conjunction

Domain D

$a_1, a_2, a_3, \ldots, a_n$

finite Domain

n elements

Universal Quantifier

Definition

Definition

The *universal quantification* of a predicate $P(x)$ is the proposition "$P(x)$ is true for all values of x in the universe of discourse" We use the notation

$$\forall x P(x)$$

which can be read "for all x"

If the universe of discourse is finite, say $\{n_1, n_2, \ldots, n_k\}$, then the universal quantifier is simply the conjunction of all elements:

$$\forall x P(x) \iff P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k)$$

First Order Logic

Next Topic:

Existential Quantifier

(there exists)

Existential Quantification:

Existential Quantification means: Saying that a property P is satisfied by at least one element in the domain.

Some $\equiv$ At least one

Some element in the domain satisfy a property P.

$\exists$: Existential Quantifier Symbol

Read as "there exists"

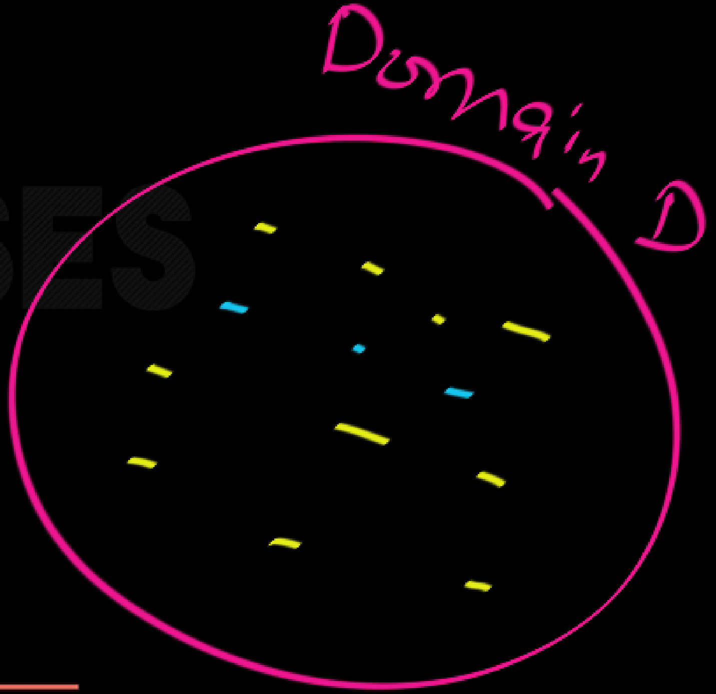
Existential Quantification:

English:

There exists an element x in the domain, P(x) is true.

FOL:

$$\boxed{\exists_x P(x)}$$

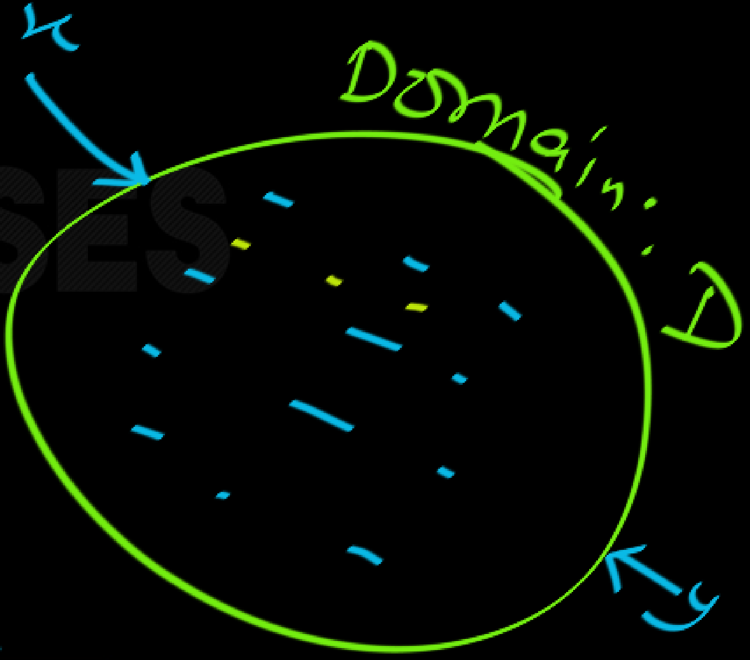


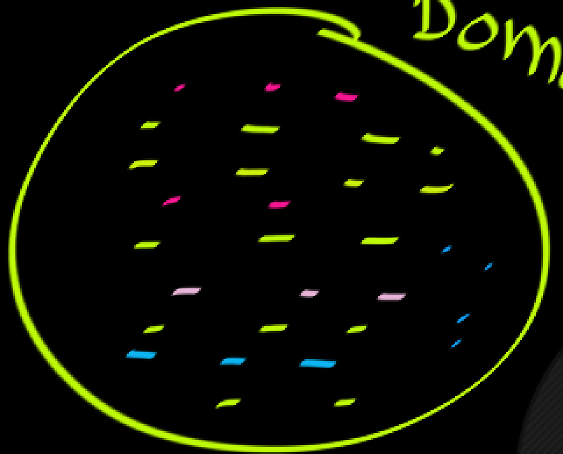
Existential Quantification:

There exists an element x in the domain, P(x) is true.

$$\exists_x P(x) \equiv \exists_{x \in D} P(x)$$

implicit





Predicate $P(x)$

Existential Quantification of P : Searching for existence of an element x such that $P(x)$ True

There exists an element x in the Domain, for which $P(x)$ is True.

Existential Quantification:

There exists an element x in the domain, P(x) is true.

$\exists_x (P(x))$

For at least one element x in the domain, P(x) is true.

$\exists_x P(x)$

= at least one

$\exists_x P(x)$

Some element in the domain satisfy a property P.

Existential Quantification:

There exists x in the domain, P(x).

$\exists_x P(x)$

For some x, P(x) is true.

$\exists_x P(x)$

There exists x, P(x). :

$\exists_x P(x)$

Existential Quantification:

There exists x in the domain, P(x).

For some x, P(x) is true.

There exists x, P(x).

There exists an element x in the domain, P(x) is true.

For at least one element x in the domain, P(x) is true.

Some element in the domain satisfy a property P.

$\boxed{\exists_x P(x)}$

Same

The Existential Quantifier

- A statement of the form

$$\exists x.\ \boxed{\textit{some-formula}}$$

is true if, for *some* choice of x, the statement ***some-formula*** is true when that x is plugged into it.

Domain: $N = \{1, 2, 3, 4, \ldots\}$

- Examples:

$\exists x.\ (Even(x) \land Prime(x))$

$$\exists x \, \underbrace{\Big(\overbrace{\boxed{\;- \;-\; -\; -\;}}^{\text{Box}} \Big)}$$

means there is at least one element in the Domain, for which Box is True.

Domain: $\mathbb{N}$

Box

$$\exists_x \left(\boxed{\text{even}(x) \wedge \text{prime}(x)} \right) \rightarrow \text{True}$$

$\mathbb{N}$

1, 2, 3, ...

$\underbrace{\text{Even}(1)}_{}^{F} \wedge \underbrace{\text{prime}(1)}_{}^{F} = \text{false}$

$\overset{T}{\text{Even}(2)} \wedge \overset{T}{\text{prime}(2)} = \text{True}$

witness

$\exists x \; P(x)$

witness: an element for which P is True.

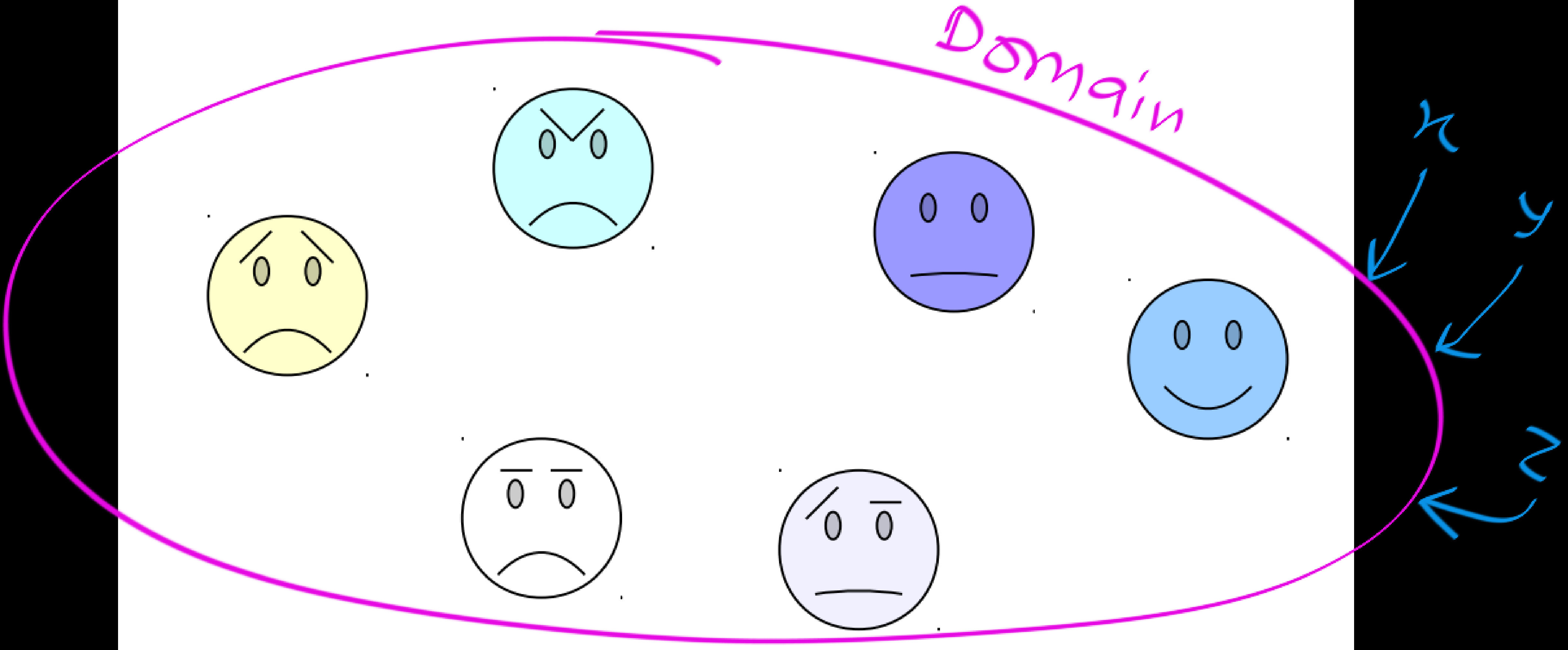
$\exists x \, P(x)$ is True iff there is at least one witness

Domain: N

$$\exists_x \left(\boxed{\text{even}(x) \wedge \text{prime}(x)} \right) = \underline{\underline{\text{True}}}$$

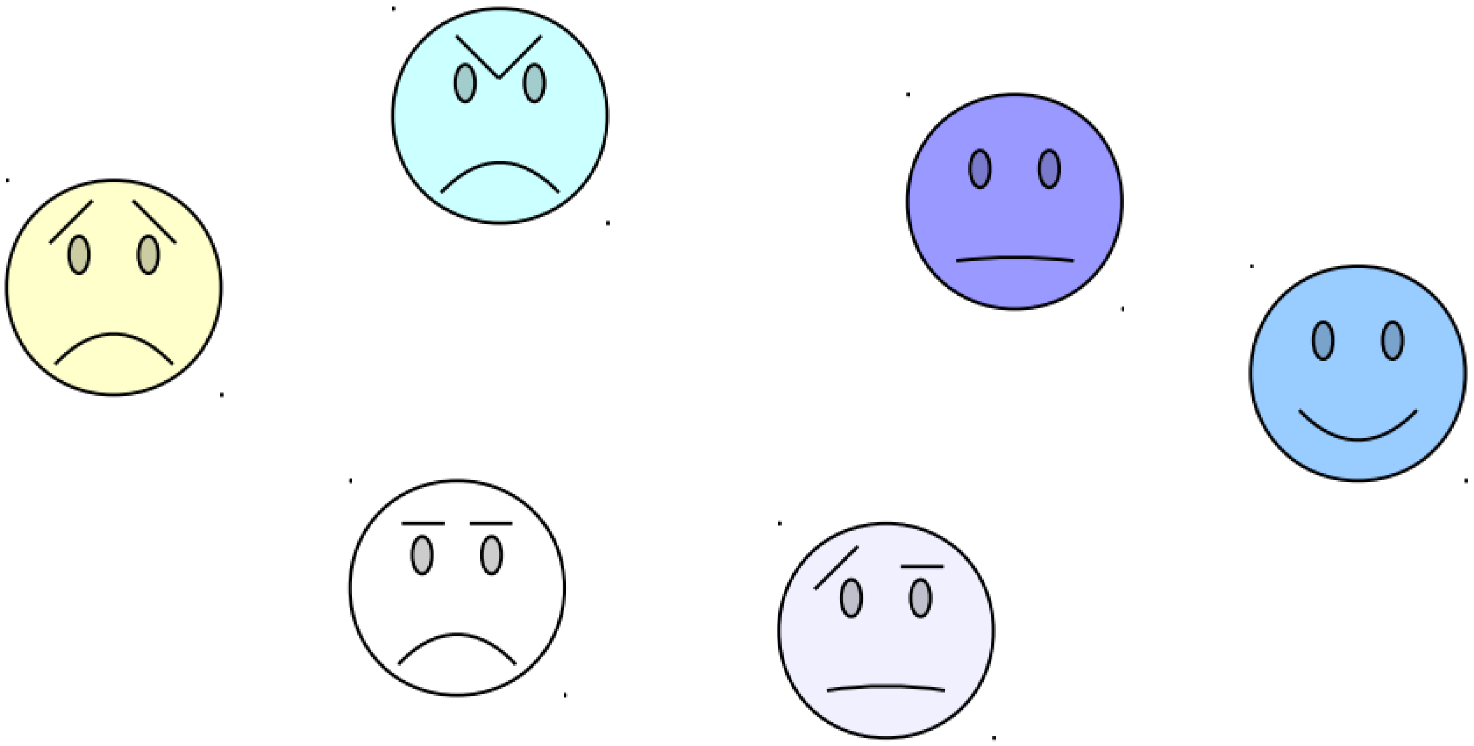
witness: $\underline{\underline{x = 2}}$

The Existential Quantifier



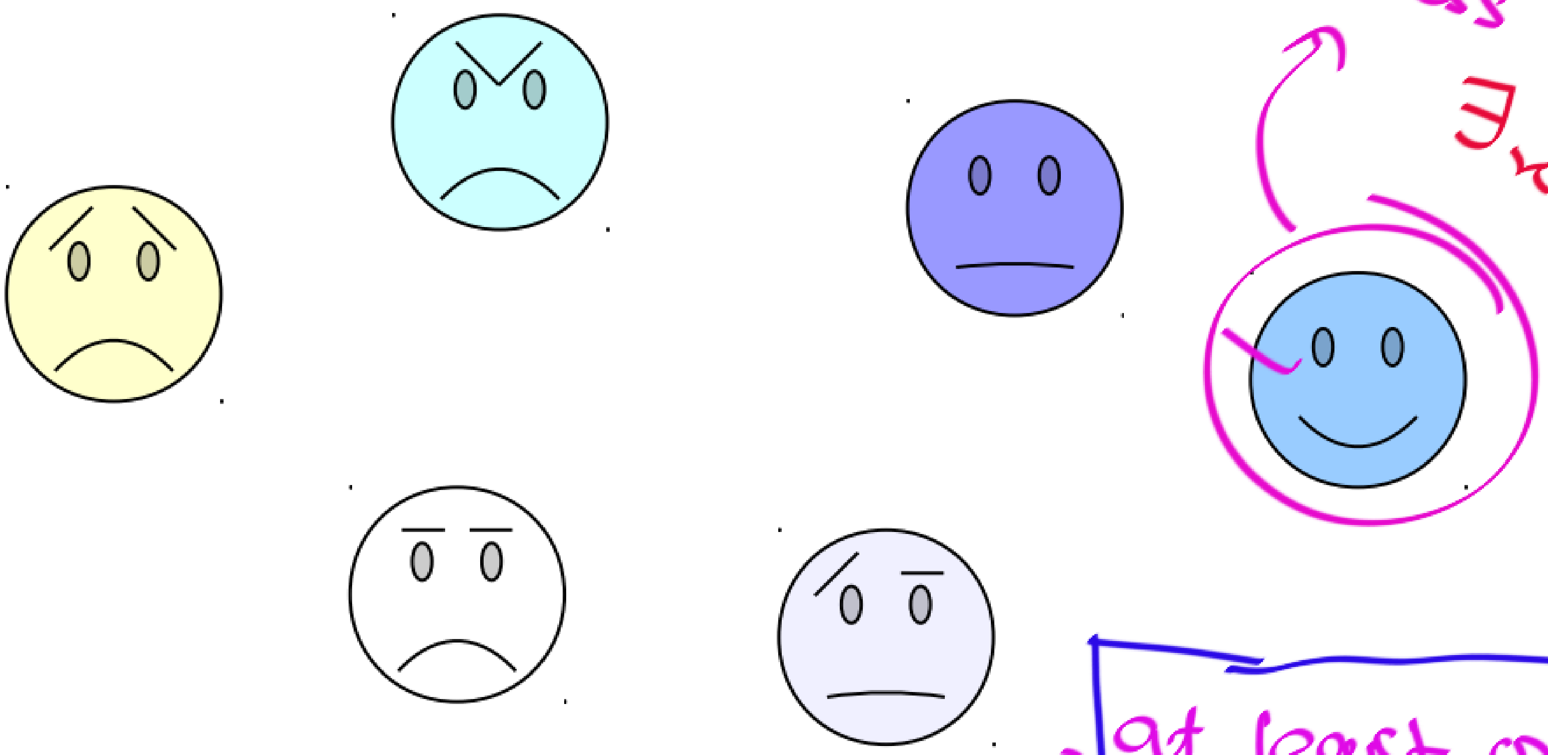
$\boxed{\exists x.\ Smiling(x)}$ = True/false ??

The Existential Quantifier



$\exists x.\ Smiling(x)$

The Existential Quantifier



witness

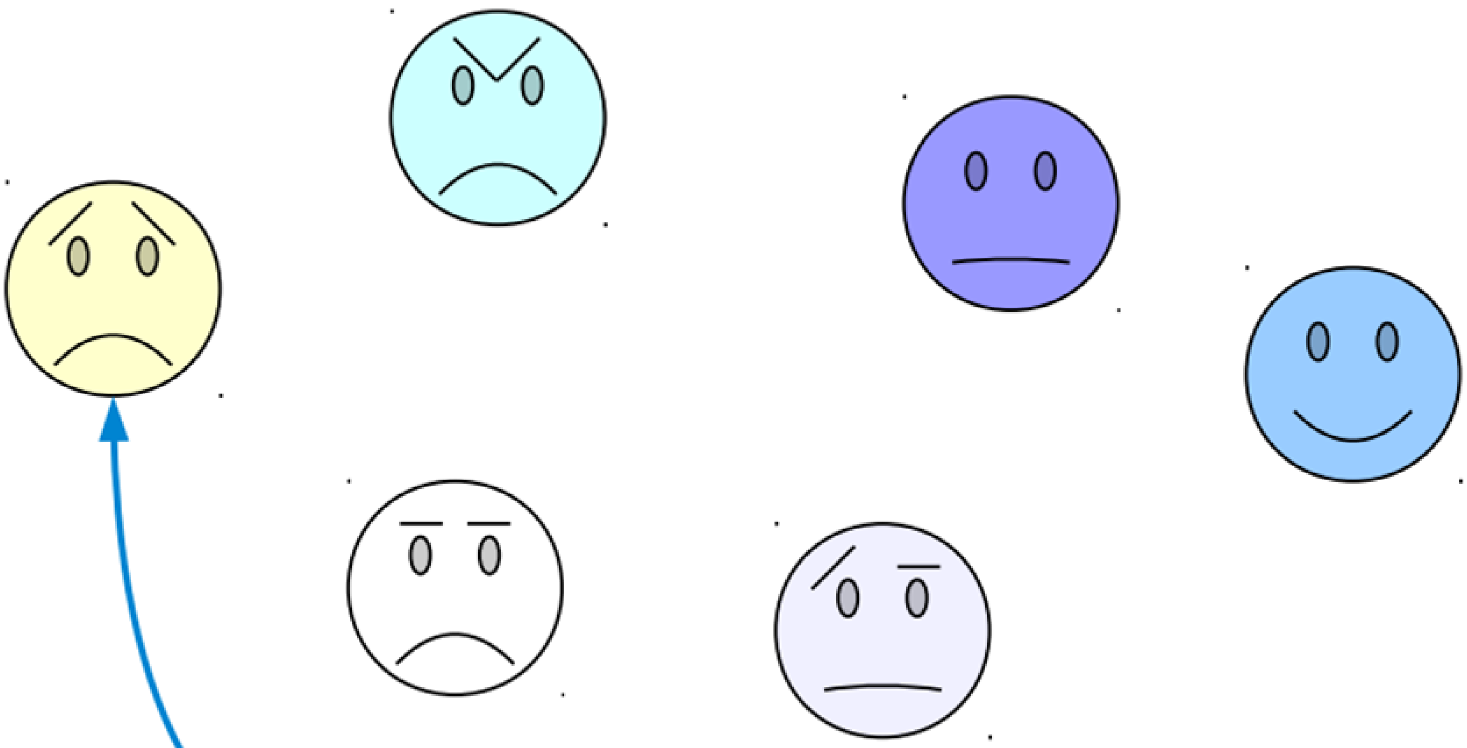
for $\exists x$ smiling(x)

there exists

$$\boxed{\exists x.\ Smiling(x)}$$

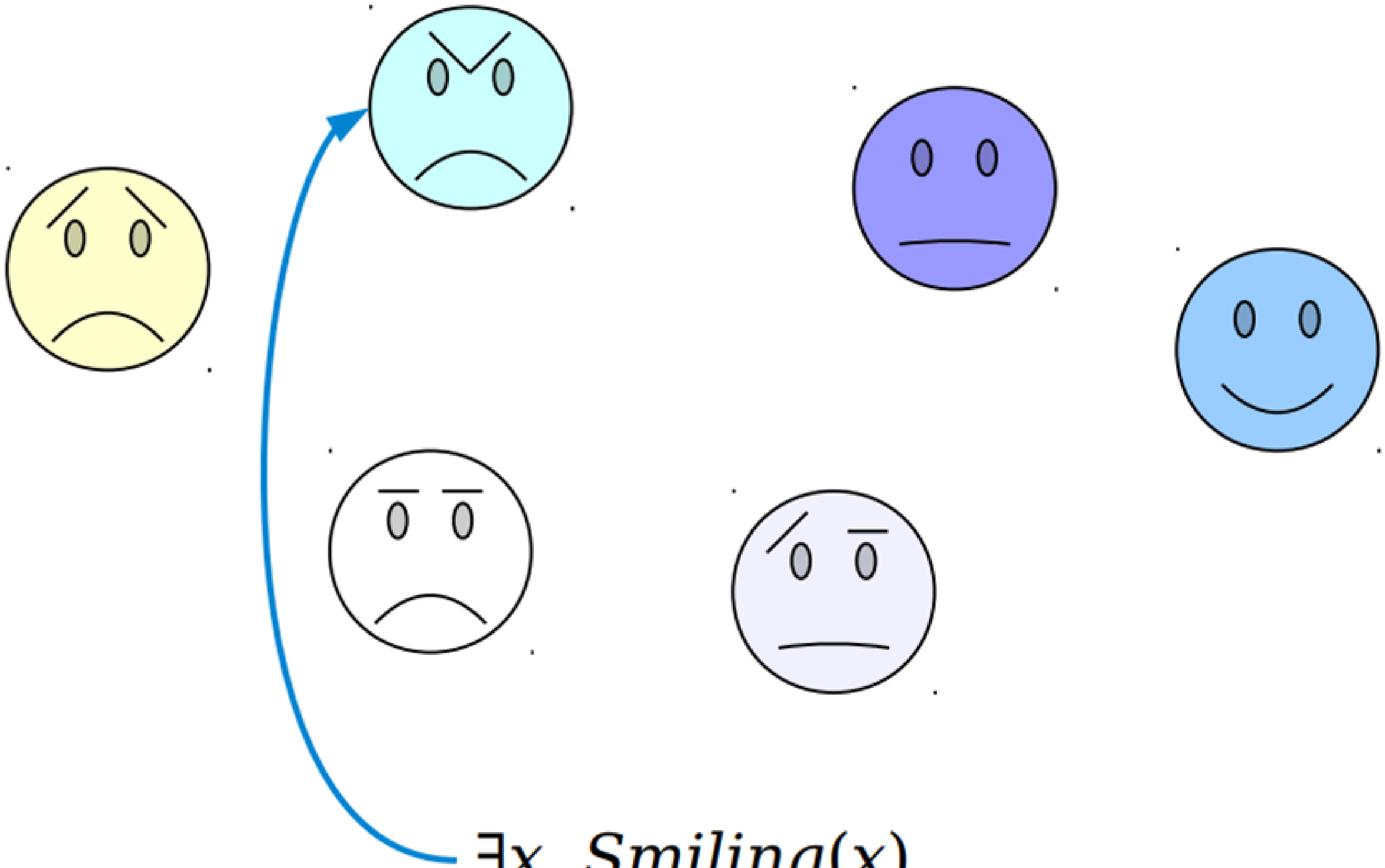
at least one element is smiling.

The Existential Quantifier



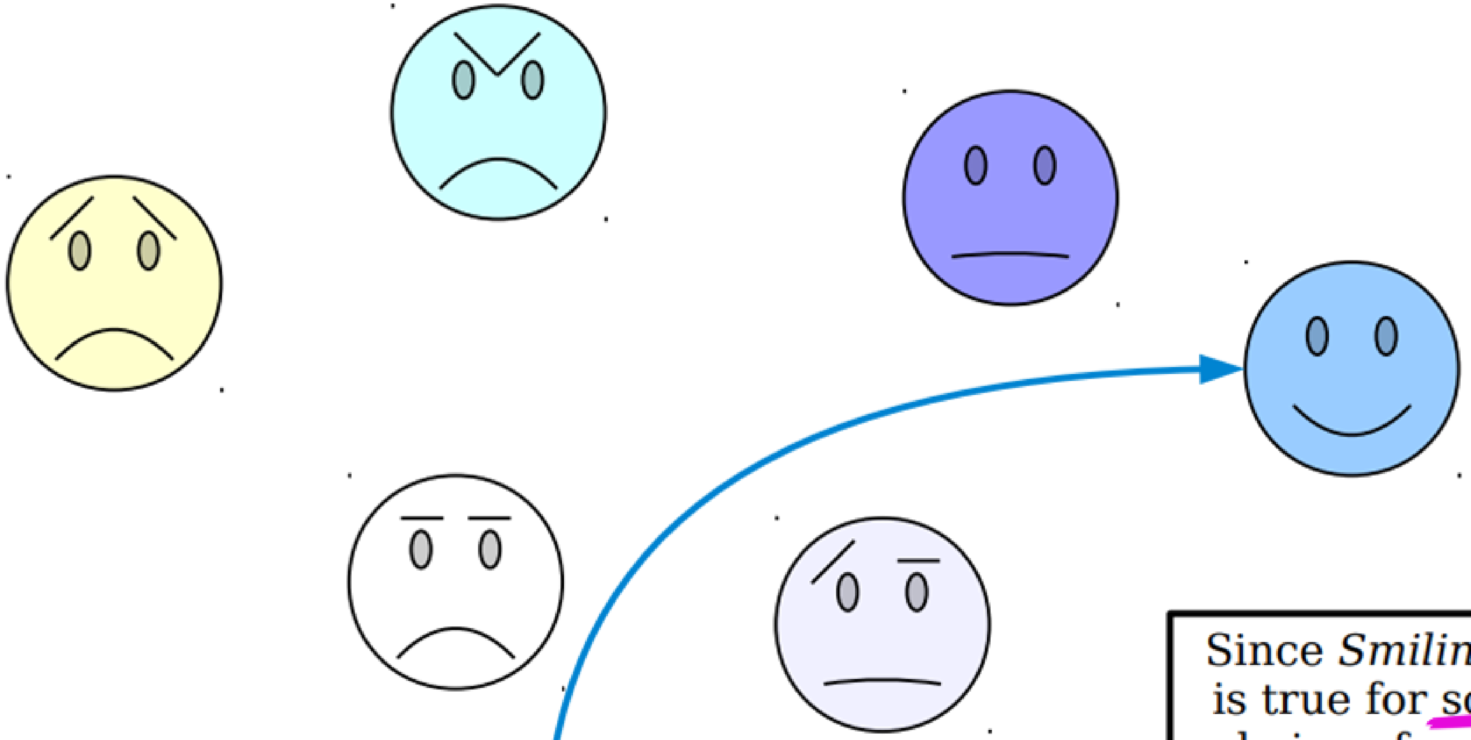
$\exists x.\ Smiling(x)$

The Existential Quantifier



$\exists x.\ Smiling(x)$

The Existential Quantifier



Since *Smiling*(x) is true for some choice of x, this statement evaluates to true.

True

$$\boxed{\exists x.\ Smiling(x)}$$

Domain: $Z = \{0, +1, -1, +2, -2, \ldots\}$

$\exists_x \; even(x) = True$

witness: 10

Domain: $\mathbb{Z}$

$0, 1, -1, 2, -2, \ldots$

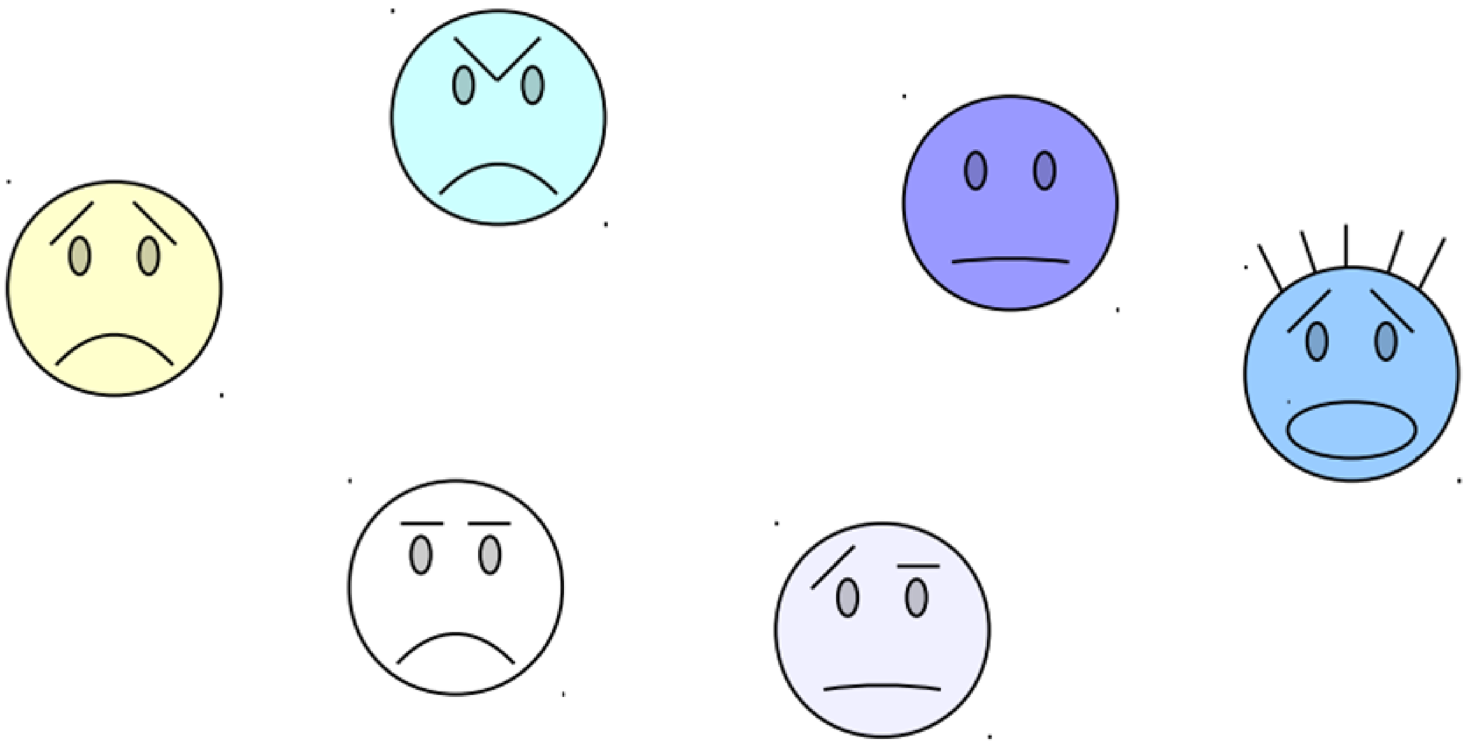
$\exists_x (x^2 = -1)$

$= \text{false}$

because No witness

for $\exists x (x^2 = -1)$

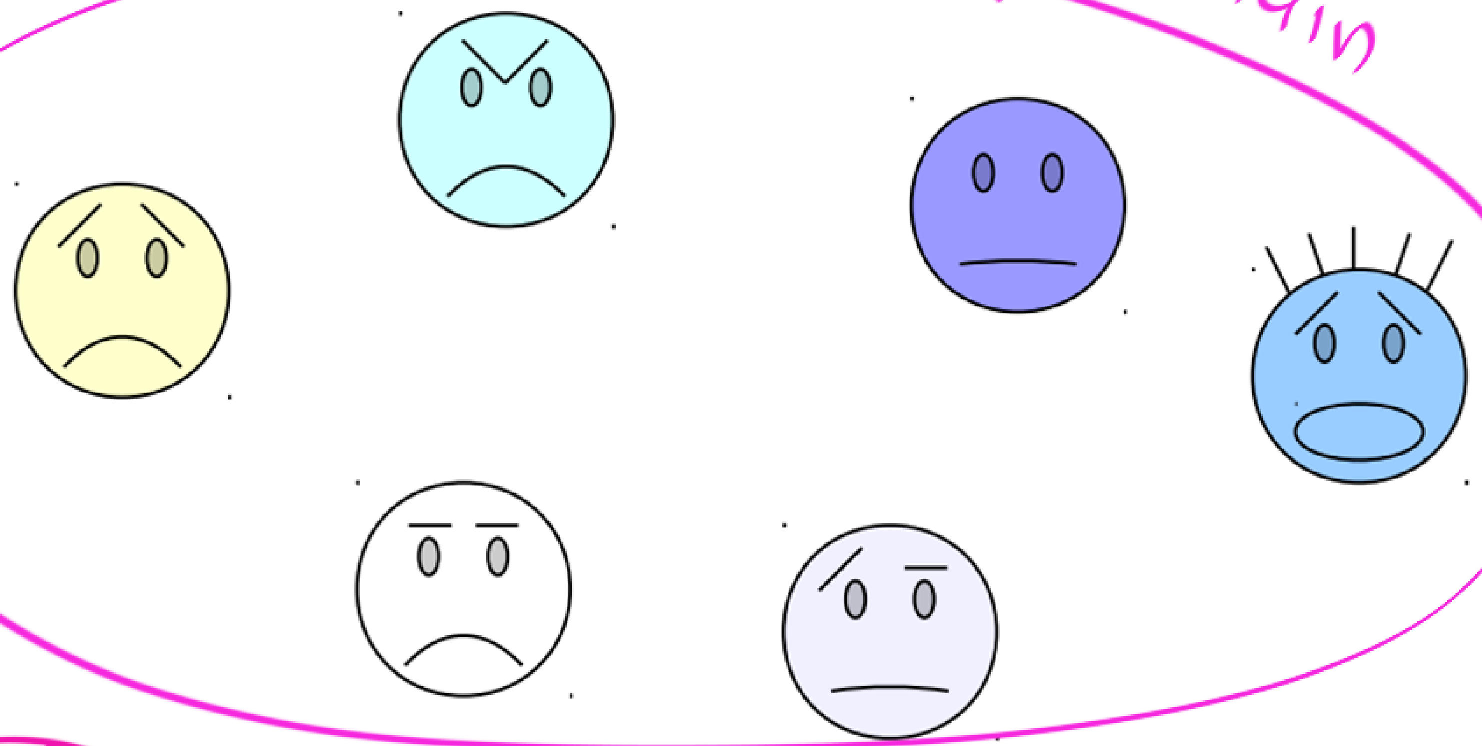
The Existential Quantifier



$\exists x.\ Smiling(x)$

The Existential Quantifier

Domain



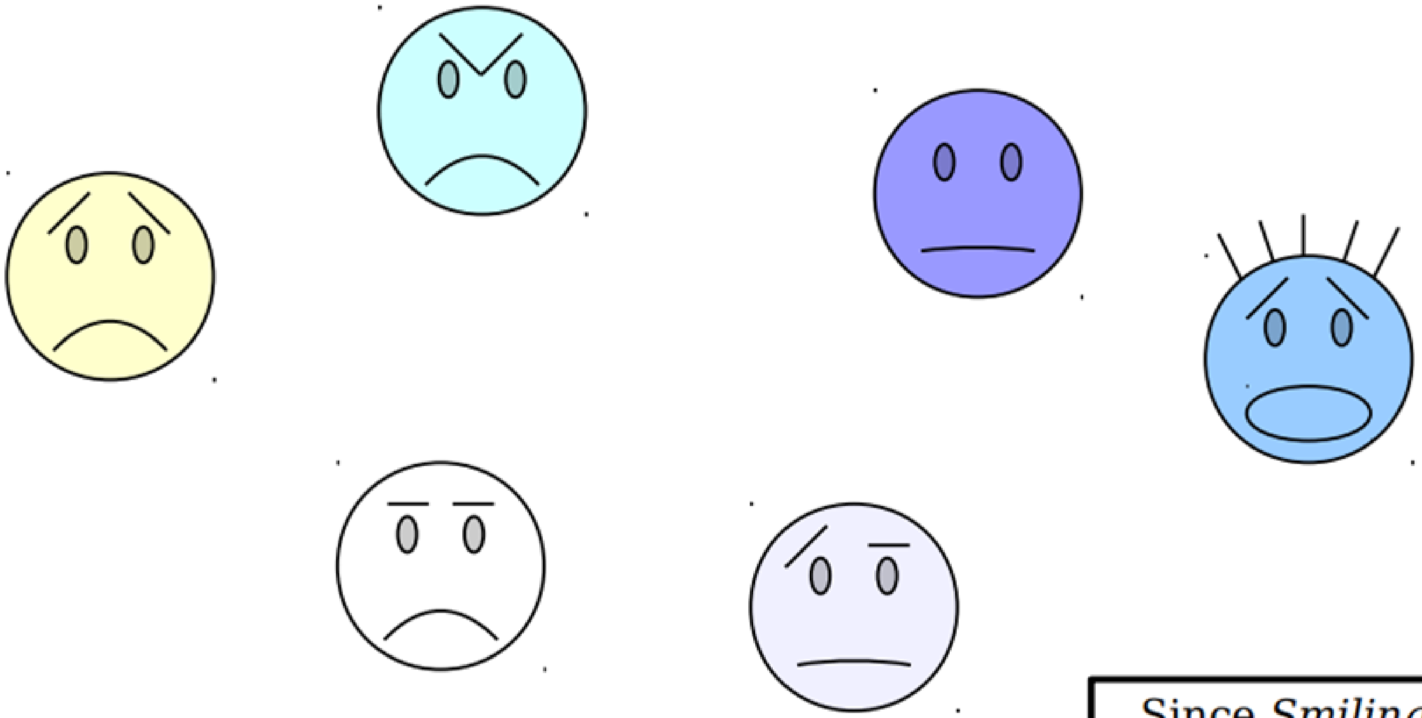
x

y

No witness

$\boxed{\exists x.\ \mathit{Smiling}(x)}$ = false

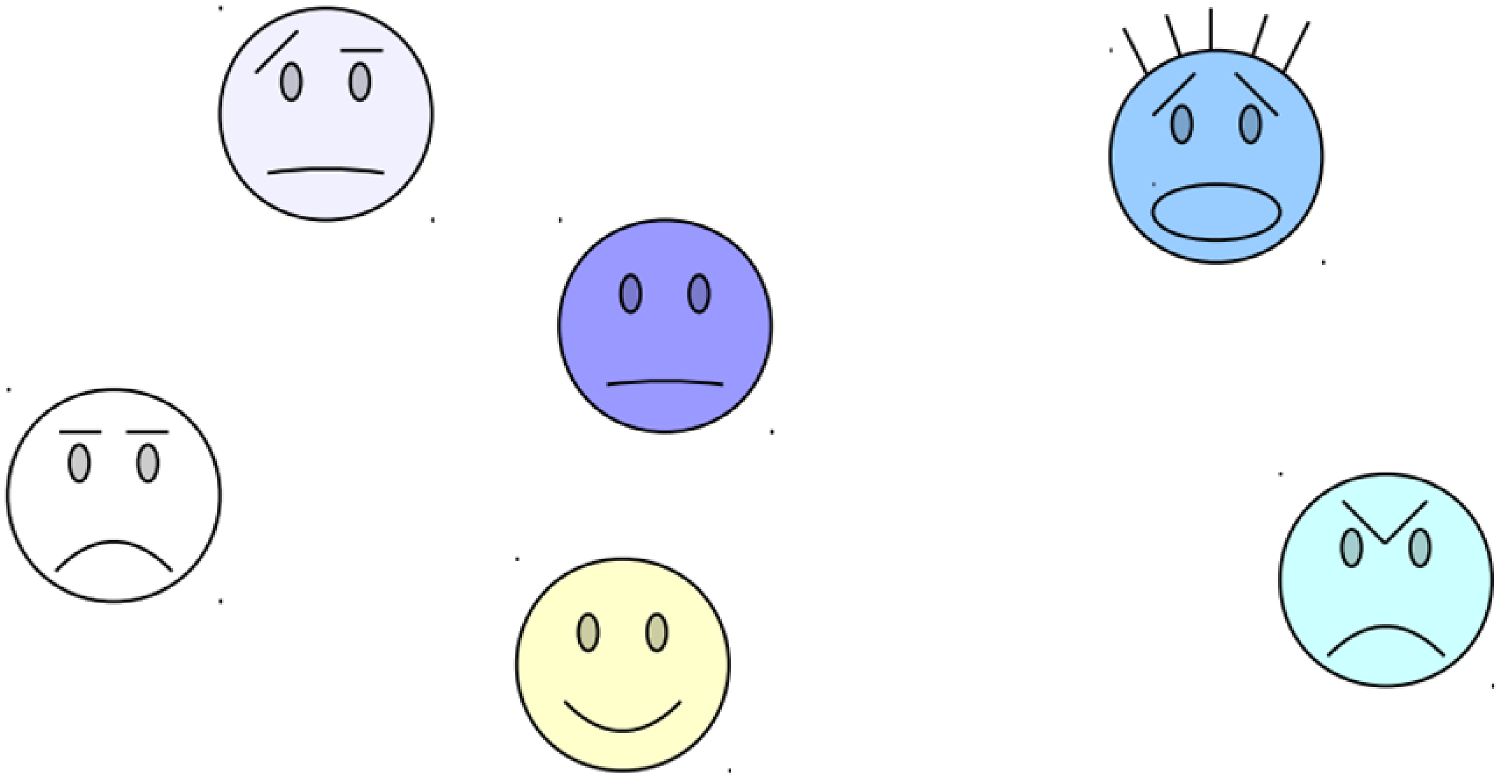
The Existential Quantifier



$\exists x.\ Smiling(x)$

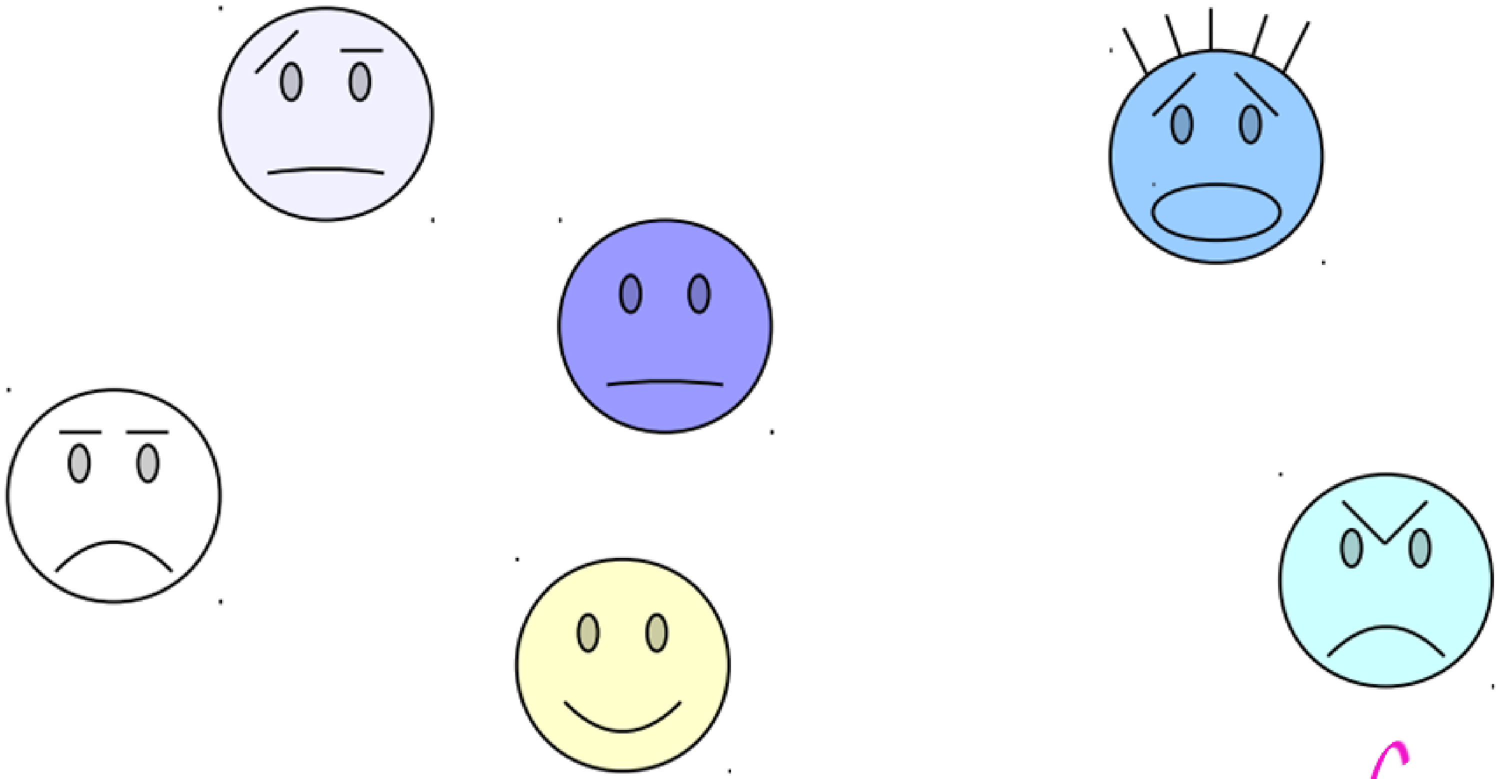
Since *Smiling*(x) is not true for any choice of x, this statement evaluates to false.

The Existential Quantifier



$$(\exists x.\ Smiling(x)) \to (\exists y.\ WearingHat(y))$$

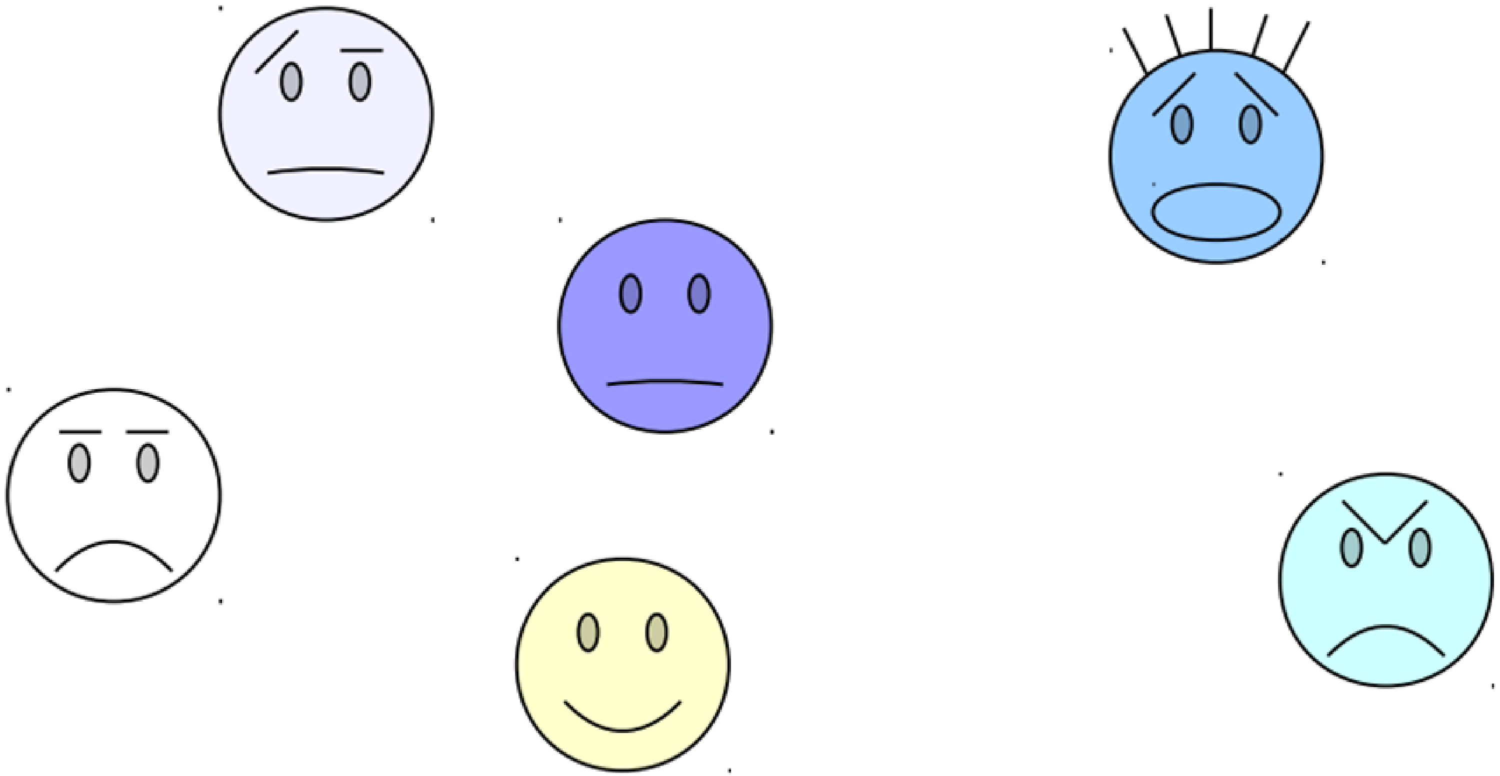
The Existential Quantifier



false

$$\boxed{(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))}$$

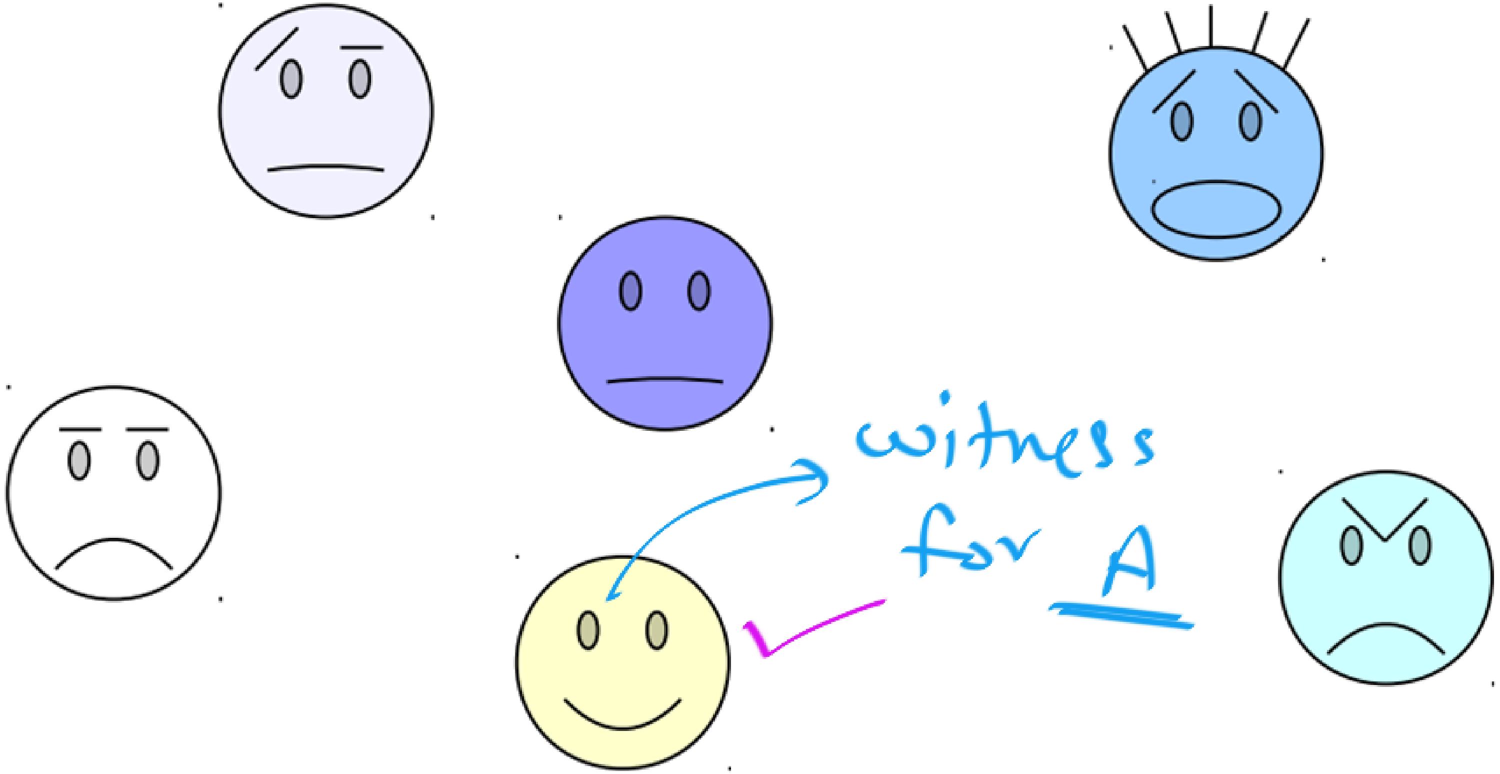
The Existential Quantifier



false

$$(\exists x.\ Smiling(x)) \to (\exists y.\ WearingHat(y))$$

The Existential Quantifier



witness for $\underline{A}$

True

A

$$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$$

$$\underbrace{\left(\exists_x \text{ smiling}(x) \right)}_{A} \rightarrow \underbrace{\left(\exists_y \text{ wearinghat}(y) \right)}_{B}$$

$$A \longrightarrow B$$

$$\underbrace{\left(\exists_x \text{ smiling}(x) \right)}_{A} \rightarrow \underbrace{\left(\exists_y \text{ wearinghat}(y) \right)}_{B}$$

True

false

$\boxed{A \rightarrow B}$

True

false

false

Definition 3.3.2. *The* **existential quantification** *of $P(x)$ is the proposition*

"There exists an element x in the universe of discourse such that $P(x)$ is true."

Notation: "There exists x such that $P(x)$" or "There is at least one x such that $P(x)$" is written

$$\exists x P(x).$$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

$\exists_x P(x)$: T/F ?

$\exists_x r(x)$: T/F ?

$\exists_x q(x)$: T/F ?

$\exists_x s(x)$: T/F ?

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad s(x): \quad x^2 - 3 > 0.$$

$\exists_x P(x)$: T/F ?

$\exists_x r(x)$: T/F ?

$\exists_x q(x)$: T/F ?

$\exists_x s(x)$: T/F ?

Domain: R

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x):\quad x \geq 0 \qquad\qquad r(x):\quad x^2 - 3x - 4 = 0$$
$$q(x):\quad x^2 \geq 0 \qquad\qquad s(x):\quad x^2 - 3 > 0.$$

$\exists x\, P(x)$ → True

witness 5

$\boxed{\text{there exists a real no. } x,\ x \geq 0}$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x):\quad x \geq 0 \qquad\qquad r(x):\quad x^2 - 3x - 4 = 0$$
$$q(x):\quad x^2 \geq 0 \qquad\qquad s(x):\quad x^2 - 3 > 0.$$

$\exists x\, q(x)$: True   witness : π

$\boxed{\text{there is some Real no } x,\ x^2 \geq 0}$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad \boxed{x^2 - 3x - 4 = 0}$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

$\exists x \; r(x)$: True

witness : -1, 4

there is atleast one Real no x, $x^2 - 3x - 4 = 0$

$$\boxed{1 \cdot x^2 - 3x - 4 = 0}$$

find x :

$$x = \frac{3 \pm \sqrt{9+16}}{2}$$

5

$$x = 4 ; -1$$ ✓

$$\boxed{ax^2 + bx + c = 0}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x):\quad x \geq 0 \qquad\qquad r(x):\quad x^2 - 3x - 4 = 0$$
$$q(x):\quad x^2 \geq 0 \qquad\qquad s(x):\quad \boxed{x^2 - 3 > 0.}$$

$\exists x\, S(x)$: True

witness: $x = -5$

The *existential quantification* of $P(x)$ is the proposition

"There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$. Here $\exists$ is called the *existential quantifier*.

There exists

Besides the phrase "there exists," we can also express existential quantification in many other ways, such as by using the words "for some," "for at least one," or "there is." The existential quantification $\exists x P(x)$ is read as

"There is an x such that $P(x)$,"
"There is at least one x such that $P(x)$,"

or

"For some $x P(x)$."

existential quantification

Existential Quantification

when Domain is finite set:

Disjunction (OR)

$\exists x\, P(x)$

$\equiv P(a) \vee P(b) \vee P(c) \vee P(d)$

Domain

$\{a, b, c, d\}$

finite Domain

$\exists x\, P(x)$

$\equiv P(a_1) \vee P(a_2) \vee$

$- - - - \vee P(a_n)$

Domain

$a_1, a_2, a_3,$

$- - - - - -$

a_n

Existential Quantifier

Definition

Definition

The *existential quantification* of a predicate $P(x)$ is the proposition "There exists an x in the universe of discourse such that $P(x)$ is true." We use the notation

$$\exists x P(x)$$

which can be read "there exists an x"

Again, if the universe of discourse is finite, $\{n_1, n_2, \ldots, n_k\}$, then the existential quantifier is simply the disjunction of all elements:

$$\exists x P(x) \iff P(n_1) \vee P(n_2) \vee \cdots \vee P(n_k)$$

First Order Logic

Next Topic:

Quantifier Practice

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n+1>n)$

b) $\exists n(2n=3n)$

c) $\exists n(n=-n)$

d) $\forall n(3n \leq 4n)$

13. Determine the truth value of each of these statements if the domain consists of all integers.

Variable a) $\forall n(n + 1 > n)$ = True

b) $\exists n(2n = 3n)$

c) $\exists n(n = -n)$

d) $\forall n(3n \leq 4n)$

Domain: $\mathbb{Z}$ = {0, 1, −1, 2, −2, ... } ← n

(a) for every integer n, n+1 > n → True

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n + 1 > n)$

b) $\exists n(2n = 3n)$ = True

c) $\exists n(n = -n)$

d) $\forall n(3n \leq 4n)$

(b) for some integer n, $2n = 3n$; True

witness: 0

13. Determine the truth value of each of these statements if the domain consists of all integers.

a) $\forall n(n+1>n)$

b) $\exists n(2n=3n)$

c) $\exists n(n=-n)$ = True

d) $\forall n(3n \leq 4n)$: false

witness for (c) : 0

for every integer n, $3n \leq 4n$

Counterexample : $n=-1$

$-3 \geq -4$

$\forall x \left(\underbrace{\boxed{}_{\text{Box}}} \right)$ means for every element in the domain, Box must be True.

$\exists x \left(\underbrace{\boxed{}}_{\text{Box}} \right)$ means for at least one element in the Domain, Box is True.

$\exists x \left(\boxed{Box(x)} \right) \equiv$ for some x, Box(x) must be True.

$\exists x \boxed{\neg P(x)} \equiv$ for some x, $\neg P(x)$ must be True

box

$\equiv$ for some x, P(x) is False.

$\exists_x \boxed{\neg P(x)}$: for some element, Box is True.

Box

$\equiv$ for some element, $\boxed{\neg P(x) \text{ is True}}$.

$\equiv$ for some element, $P(x)$ is false.

$\exists x \; \boxed{\neg P(x)}$: for some element, $P(x)$ is false.

$\forall x \; \boxed{\neg P(x)}$: for every element, $P(x)$ is false.

$\forall x \boxed{\neg P(x)}$: for all x, $\neg P(x)$ is True.

box

$\equiv$ for all x, $P(x)$ is false.

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

A. $\forall x P(x)$

B. $\exists x P(x)$

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$

E. $\neg \exists x P(x)$

F. $\neg \forall x P(x)$

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

A. $\forall x P(x)$

B. $\exists x P(x)$

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$

E. $\neg \exists x P(x)$

F. $\neg \boxed{\forall x P(x)}$

=

True

false

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

A. $\forall x P(x)$

B. $\exists x P(x)$

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$

E. $\neg \boxed{\exists x P(x)}$

True

false

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

A. $\forall x P(x)$

B. $\exists x P(x)$

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$: $\boxed{\text{for someone, } P(x) \text{ false}}$ → True

E. $\neg \exists x P(x)$

F. $\neg \forall x P(x)$

1, 2, 3

witness = 3

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

$P(x)$

A. $\forall x P(x)$

B. $\exists x P(x)$

C. $\forall x \boxed{\neg P(x)}$: false

E. $\neg \exists x P(x)$

F. $\neg \forall x P(x)$

Domain

1, 2, 3

for everyone, $P(x)$ is false. → false

→ Counterexample : 2

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $x + 2 = 2x$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

$P(x)$

A. $\forall x P(x)$

B. $\exists x P(x)$: True

Witness: 2

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$

E. $\neg \exists x P(x)$

F. $\neg \forall x P(x)$

Domain

1, 2, 3

EXAMPLE 3.4.1. *Suppose $P(x)$ is the predicate $\boxed{x + 2 = 2x}$, and the universe of discourse for x is the set $\{1, 2, 3\}$. Then...*

$P(x)$

A. $\forall x P(x)$: false

B. $\exists x P(x)$

C. $\forall x \neg P(x)$

D. $\exists x \neg P(x)$

E. $\neg \exists x P(x)$

F. $\neg \forall x P(x)$

Domain

1, 2, 3

Counterexample : 1

Example 3.4.1. *Suppose $P(x)$ is the predicate $x+2=2x$, and the universe of discourse for x is the set $\{1,2,3\}$. Then...*

- $\forall x P(x)$ *is the proposition "For every x in $\{1,2,3\}$ $x+2=2x$." This proposition is false.*
- $\exists x P(x)$ *is the proposition "There exists x in $\{1,2,3\}$ such that $x+2=2x$." This proposition is true.*

EXERCISE 3.4.1. *Let $P(n, m)$ be the predicate $mn > 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*

(1) $P(-3, 2)$
(2) $\forall m P(0, m)$
(3) $\exists n P(n, -3)$

EXERCISE 3.4.1. *Let $P(n, m)$ be the predicate $mn > 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*

(1) $P(-3, 2)$: $\boxed{(-3)(2) > 0}$ false

[redacted]

n m

false

EXERCISE 3.4.1. *Let $P(n, m)$ be the predicate $mn > 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*

[redacted]

(2) $\forall m P(0, m)$ $:\equiv$ $\forall_m \left(0(m) > 0 \right)$: false

[redacted]

n

Counterexample :

Domain

$\mathbb{Z}$

$P(0,m)$: $\boxed{(0)(m) > 0}$ $m = 3$

EXERCISE 3.4.1. *Let $P(n, m)$ be the predicate $mn > 0$, where the domain for m and n is the set of integers. Which of the following statements are true?*

(1) $P(-3, 2)$
(2) $\forall m P(0, m)$
(3) $\exists n P(n, -3)$ $= \exists n \left(-3n > 0 \right)$: True

True

witness : $n = -2$

$P(n, -3) = \boxed{(-3)(n) > 0}$

m

Z

0, 1, −1, 2, −2, ...

Domain

$P(x)$: x is prime

$Q(x)$: x is even

Domain of x

2, 4, 9

$\forall_x (P(x))$ = false

Counterexample : 4

$\forall_x (Q(x))$: false

for every x

Counterexample : 9

$\forall_x (\boxed{P(x) \to Q(x)})$: True

$\overset{T}{P(2)} \to \overset{T}{Q(2)}$: T

$\underset{F}{P(4)} \to \underset{T}{Q(4)}$: T

$\underset{F}{P(9)} \to \underset{F}{Q(9)}$: T

$\boxed{\text{Open statement}} \equiv \underline{\text{Predicate}}$

Question:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

Then the following statements are

1) $\exists x\, [p(x) \wedge r(x)]$

2) $\forall x\, [p(x) \rightarrow q(x)]$

Question:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & \boxed{x \geq 0} & r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & s(x): & x^2 - 3 > 0. \end{array}$$

Then the following statements are

1) $\exists x\, [p(x) \wedge r(x)]$

for some real x

$\boxed{p(x) \wedge r(x)}$

True

witness: 4

$r_{(x)}$; $\boxed{1.x^2 - 3x - 4 = 0}$

$$x = \frac{3 \pm \sqrt{9+16}}{2}$$

$x = 4, -1$

$\boxed{ax^2 + bx + c = 0}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Question:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

Then the following statements are

[redacted] [redacted]

2) $\forall x \left[p(x) \rightarrow q(x)\right]$

$\forall x \left(P(x) \rightarrow Q(x) \right)$

→ implicit

for every x $\boxed{\text{in the Domain}}$, $P(x) \rightarrow Q(x)$

for every real no. x, $\boxed{\text{if } P(x) \text{ True}}$ then $Q(x)$ True.

$P(x): x \geq 0$

$q(x): x^2 \geq 0$

for every Real no. x,

if $P(x)$ True

$q(x)$ True.

True

Question:

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x): \quad x \geq 0 \qquad\qquad r(x): \quad x^2 - 3x - 4 = 0$$
$$q(x): \quad x^2 \geq 0 \qquad\qquad s(x): \quad x^2 - 3 > 0.$$

Then the following statements are

1) $\exists x\, [p(x) \wedge r(x)]$: True

witness : 4

2) $\boxed{\forall x\, [p(x) \rightarrow q(x)]}$: True

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$\begin{array}{llll} p(x): & x \geq 0 & \qquad r(x): & x^2 - 3x - 4 = 0 \\ q(x): & x^2 \geq 0 & \qquad s(x): & x^2 - 3 > 0. \end{array}$$

Then the following statements are true.

1) $$\exists x\, [p(x) \wedge r(x)]$$

This follows because the real number 4, for example, is a member of the universe and is such that both of the statements $p(4)$ and $r(4)$ are true.

2) $$\forall x\, [p(x) \rightarrow q(x)]$$

If we replace x in $p(x)$ by a negative real number a, then $p(a)$ is false, but $p(a) \rightarrow q(a)$ is true regardless of the truth value of $q(a)$. Replacing x in $p(x)$ by a nonnegative real number b, we find that $p(b)$ and $q(b)$ are both true, as is $p(b) \rightarrow q(b)$. Consequently, $p(x) \rightarrow q(x)$ is true for all replacements x taken from the universe of all real numbers, and the (quantified) statement $\forall x\, [p(x) \rightarrow q(x)]$ is true.

$\exists x\, P(x)$

One element can make it True.

Such element we call "witness"

$\forall x\, P(x)$

One element can make it false

Such element we call "Counterexample"

First Order Logic

Next Topic:

Quantifiers

when True, when false??

Domain : D

Predicate : $P(x)$

$\forall x \, P(x)$ when become false.

$\exists x \, P(x)$ " " True.

Domain : D ✓	when True	when false
$\forall x P(x)$	for every $x \in D$ $P(x) = \text{True}$	when there is a Counter example.
$\exists x P(x)$	for atleast one $x \in D$ $P(x) = \text{True}$.	when there is No witness

Domain : D	when True	when false
$\forall x P(x)$	for every $x \in D$ $P(x) = \text{True}$	for at least one $x \in D$; $P(x)$: false
$\exists x P(x)$	for at least one $x \in D$ $P(x) = \text{True}$.	for every $x \in D$; $P(x) = \text{false}$.

Domain: D	when True	when false
$\forall_x \boxed{\neg P(x)}$	for every $x \in D$; $P(x) = \text{false}$	for some $x \in D$; $P(x) = \text{True}$
$\exists_x \boxed{\neg P(x)}$	for some $x \in D$; $P(x) = \text{false}$	for every $x \in D$; $P(x) = \text{True}$.

Universal Quantifier ≡ Conjunction

Existential " ≡ Disjunction

$\forall_x P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge P(d)$

$\exists_x P(x) \equiv P(a) \vee P(b) \vee P(c) \vee P(d)$

Domain

a, b, c, d

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Truth Value of Quantified Statements

GO Classes

Statement	When true	When false
$\forall x \in D, P(x)$	P(x) is true for every x.	There is one x for which P(x) is false.
$\exists x \in D, P(x)$	There is one x for which P(x) is true.	P(x) is false for every x.

Assume that D consists of $x_1, x_2, \ldots, x_n$

- $\forall x \in D,\ P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- $\exists\ x \in D,\ P(x) \equiv P(x_1) \vee P(x_2) \vee \ \ldots \vee P(x_n)$

- $(\forall\ x \in D,\ P(x))$ is true exactly when $P(x)$ is true for every $x \in D$. Thus it is false whenever there is at least one x for which $P(x)$ is false. Formally, for $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:

$$(\forall\ x \in D,\ P(x)) \equiv (P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)).$$

- $(\exists\ x \in D,\ P(x))$ is true exactly when $P(x)$ is true for at least one $x \in D$. Thus it is false when $P(x)$ is false for all $x \in D$. Formally, for $D = \{x_1, \ldots, x_n\}$, we have the following equivalence:

$$(\exists\ x \in D,\ P(x)) \equiv (P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)).$$

First Order Logic

Next Topic:

Quantifier Tricky Notes