Propositional Logic

Next Chapter:

Logical Identities (Logical Laws)

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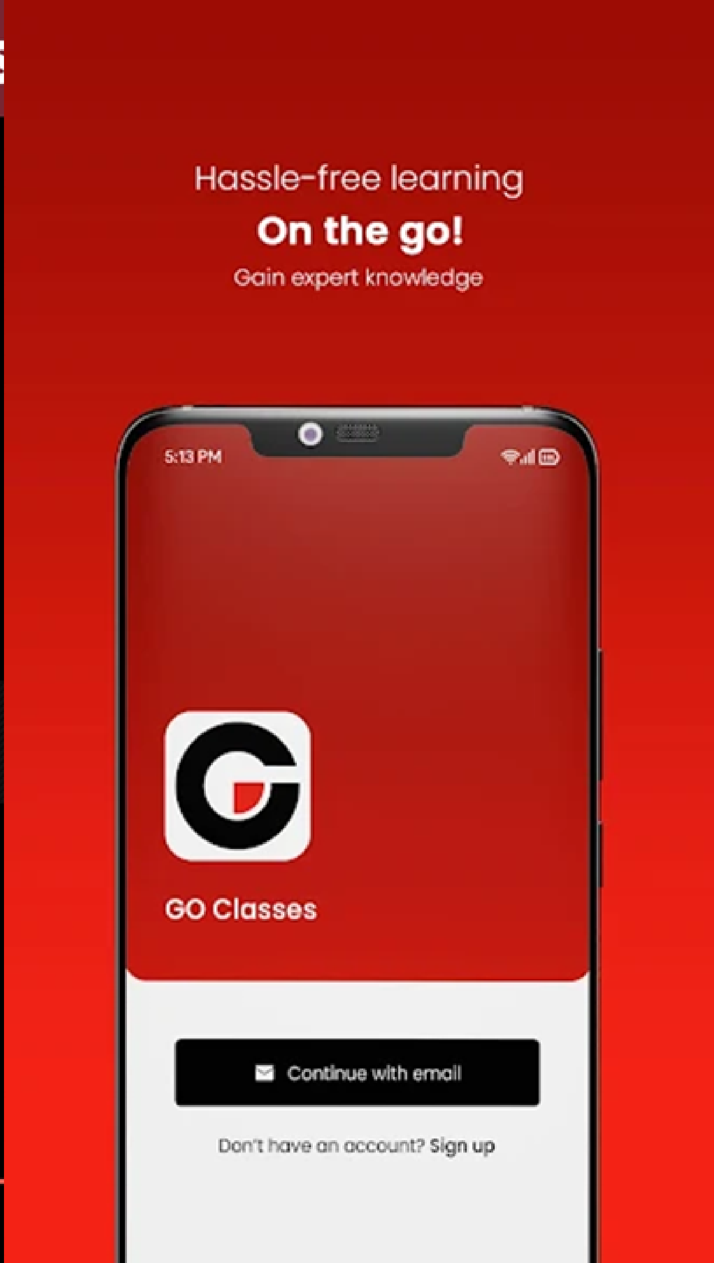

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Next Topic:

Recap

Logical Equivalence

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

2.2. Logically Equivalent.

DEFINITION 2.2.1. *Propositions* r *and* s *are* **logically equivalent** *if the statement* $r \leftrightarrow s$ *is a tautology.*

Notation: If r and s are logically equivalent, we write

$$r \Leftrightarrow s.$$

$\underbrace{\alpha \equiv \beta}$ iff $\underbrace{\alpha \leftrightarrow \beta}$ is Tautology.

P	Q	R	α	β	$\alpha \leftrightarrow \beta$
			F	F	T
			F	F	T
			T	T	T
			T	T	T
					T
					T

Tautology

① $\alpha \equiv \beta$ iff $\alpha \longleftrightarrow \beta$ is Tautology.

② $\alpha \equiv \beta$ iff α, β have Same Truth Table

③ $\alpha \equiv \beta$ iff {
Whenever α True, β True
" β True, α True

Next Topic:

Logical Laws

in Propositional Logic

Logical Laws/Logical Identities:

Standard(Frequently Used) Logical Equivalences

Next Topic:

Logical Laws

0. Some Basic Laws

① $\neg(\neg p) = p$

$\overline{(\overline{p})} = p$

Double Negation Rule

$\overline{(\overline{p})} = p$

Proof:

p	$y = \overline{p}$	$\overline{y} = \overline{(\overline{p})}$
T	F	T
F	T	F

$\boxed{\overline{(\overline{p})} = p}$

6. Negations: The $\neg$ operator cancels itself, which means

$$\neg(\neg A) = A.$$

(In arithmetic, we know $-(-5) = 5$.)

$-(-5) = 5$

$\overline{\overline{A}} = A$

(2) Domination Law:

$p \vee T = T$

$p \wedge F = F$

$T \vee \text{anything} = T$

$F \wedge \text{anything} = F$

3

$A \vee \bar{A} = T$

$A \wedge \bar{A} = F$

$A, \bar{A}$;

$A \vee \bar{A} = T$

$A \wedge \bar{A} = F$

④ Identity law:

$p \vee F = p$

$p \wedge T = p$

$P \vee F = \boxed{?}$ → P

Constant (fixed)

T —— $\boxed{T}$

F —— $\boxed{F}$

$$\boxed{P \vee F = P}$$

$$\boxed{P \vee F = P}$$

P	$P \vee F$
T	T
F	F

$P \wedge T = \boxed{?}$

Constant (fixes)

T —— $\boxed{T}$

F —— $\boxed{F}$

P

P	$P \wedge T$
F	F
T	T

$$\boxed{P \wedge T = P}$$

$$\boxed{P \wedge T = P}$$

Conclusion:

1. $\neg(\neg p) = p$
2. $p \vee T = T$; $p \wedge F = F$
3. $p \vee \overline{p} = T$; $p \wedge \overline{p} = F$
4. $p \vee F = p$; $p \wedge T = p$

Some Rules (logical laws)

$p \vee T = \boxed{?}$

Constant

p		$p \vee T$
T	—	T
F	—	T

$\rightarrow T$

$\boxed{p \vee T = T}$

$\boxed{p \vee T = T}$

Always True

p	$p \vee T$
F	T
T	T

Tautology

So far, Name of laws was Not important.

Now, <u>Names</u> <u>are also important.</u>

Next Topic:

Logical Laws

1. Commutative Law

Number theory:

addition

$$\boxed{a+b=b+a}$$

for all a, b

→ Commutative law

→ Addition operation in Arithmetic is Commutative.

$5+4=4+5$

$$\boxed{r+s=s+r}$$

Number theory:

$\boxed{a \times b = b \times a}$ → for all a, b

→ mul is Also Commutative.

$\boxed{a - b = b - a}$ ✗ Subtraction is NoT Commutative.

$4 - 5 \neq 5 - 4$

$\boxed{2 - 2 = 2 - 2}$

Number theory (Arithmatic)

Commutative operations:

Addition ✓

multiplication ✓

Not Comm. op:

Subtraction, Division

$4/2 \neq 2/4$

Commutative operation :

Operator # .

is Commutative iff $\boxed{a \# b = b \# a}$

for all a, b

In prop. logic :

$\wedge$ is Commutative ;

$$\boxed{a \wedge b = b \wedge a}$$

Always

proof :

a	b	$a \wedge b$	$b \wedge a$
F	F	F	F
F	T	F	F
T	F	F	F
T	T	T	T

$$\boxed{a \wedge b = b \wedge a}$$

Always

1. Commutativity: The $\wedge$ and $\vee$ operators are commutative, which means

$$A \wedge B = B \wedge A \quad \text{and} \quad A \vee B = B \vee A.$$

(In arithmetic, we know $2+3=3+2$ and $2 \times 3 = 3 \times 2$.)

In Prop logic, which Connective is Commutative?

1. $\wedge$
2. $\vee$
3. $\rightarrow$
4. $\leftrightarrow$
5. $\oplus$
6. $\uparrow$
7. $\downarrow$

In Prop. logic, which Connective is Commutative?

1. $\wedge$ ✓
2. $\vee$ ✓
3. $\rightarrow$ ✗
4. $\leftrightarrow$ ✓
5. $\oplus$ ✓
6. $\uparrow$ ✓ → NAND
7. $\downarrow$ ✓ → NOR

$P \wedge Q = Q \wedge P$ ✓

$P \vee Q = Q \vee P$ ✓

$\underline{\underline{P \rightarrow Q}} \neq \underline{\underline{Q \rightarrow P}}$

P	Q	$P \rightarrow Q$	$Q \rightarrow P$
F	F	T	T
F	T	T	F
T	F	F	T
T	T	T	T

" $\rightarrow$ " is NOT Commutative.

$\boxed{P \rightarrow Q \neq Q \rightarrow P}$

$$\boxed{P \longleftrightarrow Q = Q \longleftrightarrow P}$$ ✓

$$P \oplus Q = Q \oplus P$$ ✓

$$P \uparrow Q = \overline{(P \wedge Q)} = \overline{(Q \wedge P)} = Q \uparrow P$$

$$Q \uparrow P = \overline{Q \wedge P}$$

Next Topic:

Logical Laws

2. Associative Law

In Number theory:

Addition

$(a+b)+c = a+(b+c)$

$(5+3)+10 = 5+(3+10)$

Associative property

$(a \times b) \times c = a \times (b \times c)$ for all a, b, c

$+, \times$ are Associative.

In Number theory:

$$(a-b)-c \neq a-(b-c)$$

$$(5-3)-2 \qquad 5-(3-2)$$

$$0 \neq 4$$

Subtraction operation is NOT Associative.

Associative operation:

$a \# b \# c$

Operation $\#$

Operation $\#$ is Associative iff

$$(a \# b) \# c = a \# (b \# c)$$

for all a, b, c.

In prop. logic,

$\wedge$ is Associative.

proof: To prove:

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

proved

$$\underbrace{(a \wedge b) \wedge c}_{\alpha} \overset{?}{=} \underbrace{a \wedge (b \wedge c)}_{\beta}$$

$T \wedge y = y$

By Case method:

Case 1: $a = F$

$\alpha = F$
$\beta = F$ } Same

Case 2: $a = T$

$\alpha = b \wedge c$
$\beta = b \wedge c$ } Same

Similarly; $\vee$ is Also Associative.

2. Associativity: The $\wedge$ and $\vee$ operators are associative, which means

$$\Big((A \wedge B) \wedge C\Big) = \Big(A \wedge (B \wedge C)\Big) \quad \text{and} \quad \Big((A \vee B) \vee C\Big) = \Big(A \vee (B \vee C)\Big).$$

(In arithmetic, we know $(3+4)+2 = 3+(4+2)$ and $(3 \times 4) \times 2 = 3 \times (4 \times 2)$.) Together, commutativity and associativity essentially mean "order doesn't matter."

In prop. logic, which connective is Associative?

1 $\wedge$ 2 $\vee$ 3 $\rightarrow$ 4 $\leftrightarrow$

5 $\oplus$ 6 $\uparrow$ 7 $\downarrow$

In Prop. logic, which Connective is Associative?

1. $\wedge$
2. $\vee$
3. ~~$\rightarrow$~~
4. $\leftrightarrow$
5. $\oplus$
6. ~~$\uparrow$~~
7. ~~$\downarrow$~~

Note: Implication is NOT Associative.

$$(a \to b) \to c \neq a \to (b \to c)$$

$a = F$
$b = T$
$c = F$

False

True

Note: Implication is NOT Associative.

$$\underbrace{(a \to b) \to c}_{\alpha} \neq \underbrace{a \to (b \to c)}_{\beta}$$

Case 1: $a = F$

$\alpha = c$

$\beta = T$

NOT Same

Implication is **Not Associative**

Case 2: $a = T$

Note: Implication is NOT Associative.

$$(a \rightarrow b) \rightarrow c \not\equiv a \rightarrow (b \rightarrow c)$$

$\underbrace{\leftrightarrow, \oplus}$

Associative

$$(a \leftrightarrow b) \leftrightarrow c \overset{HW}{\equiv} a \leftrightarrow (b \leftrightarrow c)$$

$$\boxed{(a \oplus b) \oplus c = a \oplus (b \oplus c)}$$

proved

To prove: $\underbrace{(a \oplus b) \oplus c}_{\alpha} = \underbrace{a \oplus (b \oplus c)}_{\beta}$

Proof: by case method:

Case 1: $a = F$

$\alpha = b \oplus c$

$\beta = b \oplus c$

Same

Case 2: $a = T$

$\alpha = \bar{b} \oplus c$

$\beta = \overline{b \oplus c}$

equivalent

$$\underbrace{\bar{b} \oplus c}_{M} \equiv \underbrace{\overline{b \oplus c}}_{N}$$

Case 1: $b = T$

$M = c$
$N = c$ } Same

Case 2: $b = F$

$M : \bar{c}$
$N : \bar{c}$ } Same

$F \oplus b = \boxed{?} \rightarrow b$

Constant

T — $\boxed{T}$

F — $\boxed{F}$

$$\boxed{F \oplus Y = Y}$$

$T \oplus Y = \boxed{?} \rightarrow \overline{Y}$

Constant

T — $\boxed{F}$

F — $\boxed{T}$

$$\boxed{T \oplus Y = \overline{Y}}$$

NANA, NoR are NOT Associative.

$$(a \uparrow b) \uparrow c \neq a \uparrow (b \uparrow c)$$

$$(a \downarrow b) \downarrow c \neq a \downarrow (b \downarrow c)$$

$a = F$; $b = T$; $c = T$

$$(a \uparrow b) \uparrow c \neq a \uparrow (b \uparrow c)$$

$$(\overline{a \wedge b}) \uparrow c \qquad F \uparrow F \qquad \overline{b \wedge c} = F$$

$$T \uparrow c = \overline{T \wedge c} = F \qquad \overline{F \wedge F} = T$$

$$\boxed{a = F;\ b = T;\ c = T}$$

$$(a \uparrow b) \uparrow c \neq a \uparrow (b \uparrow c)$$

$\uparrow$ is Not Associative.

In prop logic:

Associative

$\wedge$

$\vee$

$\leftrightarrow$

$\oplus$

NOT Asso:

$\rightarrow$

$\uparrow$

$\downarrow$

Next Topic:

Logical Laws

3. Idempotence Law

Idempotent operator:

Operator # is called Idempotent

iff $\boxed{a \# a = a}$ for all a.

$\boxed{a \# a = a}$ for all a $\Rightarrow$ # Idempotent

Number theory:

$+, \times, -, \div$ are NOT Idempotent.

$+$ is NOT idempotent.

$a + a \neq a$

$5 + 5 \neq 5$

$5 \times 5 \neq 5$

$5 - 5 \neq 5$

$5 / 5 \neq 5$

In prop. logic:

$$\boxed{p \wedge p = p}$$

Conjunction "$\wedge$" is Idempotent.

proof:

p	$p \wedge p$
T	T
F	F

$p \wedge p = p$

$p \wedge p \wedge p \wedge p$

$p \wedge p \wedge p$

$p \wedge p$

p

$$\boxed{p \wedge p \wedge p \wedge \cdots \wedge p = p}$$

$$\boxed{p \wedge p = p}$$

idempotent

In prop. logic:

$$\boxed{p \vee p = p}$$

$\vee$ is also Idempotent

Proof:

p	$p \vee p$
T	T
F	F

$p \vee p = p$

In prop. logic:

$$p \rightarrow p \overset{?}{=} p \quad \text{No}$$

$$\boxed{p \# p = p}$$

Definition of Idempotent operator

$p = F$

$$p \rightarrow p = T$$

p	$p \rightarrow p$
F	T
T	T

$$p \rightarrow p \neq p$$

3. Idempotence: The $\wedge$ and $\vee$ operators are idempotent, which means

$$A \wedge A = A \text{ and } A \vee A = A.$$

(This rule has no analogy in arithmetic: $3+3$ is not 3, and neither is 3×3.)

In Prop. logic, which Connective is Idempotent?

(1) $\wedge$ (2) $\vee$ (3) $\rightarrow$ (4) $\leftrightarrow$

(5) $\oplus$ (6) $\uparrow$ (7) $\downarrow$

In Prop. logic, which Connective is Idempotent?

1 $\wedge$ 2 $\vee$ ~~3~~ $\rightarrow$ ~~4~~ $\leftrightarrow$

~~5~~ $\oplus$ ~~6~~ $\uparrow$ ~~7~~ $\downarrow$

$$\boxed{p \rightarrow p = T}$$

F	F	T
T	T	T

$$p \rightarrow p \neq p$$

$\rightarrow$ is NOT Idempotent.

$$p \leftrightarrow p = \boxed{T}$$

F	F	T
T	T	T

$$\boxed{p \leftrightarrow p = T}$$

$$p \leftrightarrow p \neq p$$

$$\boxed{p \oplus p = F}$$

T	T	F
F	F	F

$$p \oplus p \neq p$$

$$p \uparrow p = \overline{p \wedge p}$$

$$= \overline{p}$$

$$p \uparrow p = \overline{p}$$

$$p \uparrow p \neq p$$

Next Topic:

Logical Laws

4. Distributive Law

In Number theory:

$$\boxed{a\cdot(b+c) = ab+ac}$$

multiplication is Distributive over $+$.

$$8(5+2) = 8\times5 + 8\times2$$

$$a+(b\cdot c) \neq (a+b)(a+c)$$

$$\underbrace{2+3\times2}_{8} \neq \underbrace{(2+3)(2+2)}_{20}$$

Number theory:

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$a + (b \times c) \neq (a + b)(a + c)$$

In Number theory :

$\times$ is Dist. over $+$ ✓

$$a \times (b+c) = (a \times b) + (a \times c)$$

$+$ is Not Dist over $\times$.

$$a + (b \times c) \neq (a+b)(a+c)$$

General Definition of Distributive property :

two operators $\#$, $*$

$\#$ is Distributive over $*$ iff

$$a \# (b * c) = (a \# b) * (a \# c)$$

General Definition of Distributive property :

two operators #, *

* is Distributive over # iff

$$a * (b \# c) = (a * b) \# (a * c)$$

In prop. logic:

$\wedge$ is Dist over $\vee$.

$\vee$ is Dist over $\wedge$.

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

In prop. logic:

Prove: $\wedge$ is Dist. over $\vee$.

To prove: $\underbrace{p \wedge (q \vee r)}_{\alpha} = \underbrace{(p \wedge q) \vee (p \wedge r)}_{\beta}$

Proof: by case method:

$p = T$

$\left.\begin{array}{l} \alpha = q \vee r \\ \beta = q \vee r \end{array}\right\}$ same

$p = F$

$\left.\begin{array}{l} \alpha = F \\ \beta = F \end{array}\right\}$ same

In prop. logic:

prove: $\vee$ is Dist. over $\wedge$.

To prove: $\underbrace{p \vee (q \wedge r)}_{\alpha} = \underbrace{(p \vee q) \wedge (p \vee r)}_{\beta}$

Proof: by case method:

$p = T$	$p = F$
$\left.\begin{array}{l}\alpha = T \\ \beta = T\end{array}\right\}$ same ✓	$\left.\begin{array}{l}\alpha = q \wedge r \\ \beta = q \wedge r\end{array}\right\}$ same ✓

In prop. logic :

$\wedge$ is Dist. over $\vee$.
$\vee$ " " " $\wedge$.

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

proved

5. Distributivity: The operators $\wedge$ and $\vee$ distribute over each other, which means

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C) \quad \text{and} \quad A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C).$$

(In arithmetic, the analogy only applies in one case: $3 \times (2 + 5) = 3 \times 2 + 3 \times 5$, but the statement is not true if we swap the $\times$ and $+$ symbols.)

HW: Prove OR Disprove: True/false?

1. $\rightarrow$ is Dist over $\wedge$ $\rightarrow$ True
2. ~~$\wedge$ is " "~~ $\rightarrow$ false
3. $\rightarrow$ is Dist over $\vee$
4. $\vee$ " " " $\rightarrow$

HW

True / false?

→ is Dist. over $\wedge$. T/F ??

means:

$$p \rightarrow (q \wedge r) \overset{?}{=} (p \rightarrow q) \wedge (p \rightarrow r)$$

Check: $\underbrace{p \rightarrow (q \wedge r)}_{\alpha} \overset{?}{\equiv} \underbrace{(p \rightarrow q) \wedge (p \rightarrow r)}_{\beta}$

by case method:

$P = F$

$\left.\begin{array}{l}\alpha = T \\ \beta = T\end{array}\right\}$ Same

$P = T$

$\left.\begin{array}{l}\alpha = q \wedge r \\ \beta = q \wedge r\end{array}\right\}$ Same

(2) $\wedge$ is Dist. over $\rightarrow$. T/F?

$$\underbrace{p \wedge (q \rightarrow r)}_{\alpha} \stackrel{??}{=} \underbrace{(p \wedge q) \rightarrow (p \wedge r)}_{\beta}$$

$P = F$

$\alpha = \underline{F}$

$\beta = \underline{T}$

$\alpha \not\equiv \beta$

$P = T$ $\rightarrow$ No need to check

$$p \rightarrow (q \wedge r) = (p \rightarrow q) \wedge (p \rightarrow r)$$

$\rightarrow$ is Dist over $\wedge$. True

$\wedge$ is Dist over $\rightarrow$ False

$$p \wedge (q \rightarrow r) \neq (p \wedge q) \rightarrow (p \wedge r)$$

HW: Prove/Disprove

5 $+$ is Dist over $\to$

6 $\to$ is Dist over $+$

HW

both are not true

Next Topic:

Logical Laws

5. De Morgan's Law

α : John is both Rich and Humble.

R H T

Negation of α :

$\overline{\alpha}$ = John is Not Rich OR Not humble.

$\alpha = R \wedge H$; $\boxed{\overline{\alpha} = \overline{R} \vee \overline{H}}$

$\alpha : R \wedge H$

$\overline{\alpha} = \overline{R} \vee \overline{H}$

$$\boxed{\neg(R \wedge H) = \neg R \vee \neg H}$$

De-Morgan Law.

$$\boxed{\overline{(P \wedge Q)} = \overline{P} \vee \overline{Q}}$$

Negation of $\overline{P \text{ and } Q}$

$\equiv \overline{P}$ OR $\overline{Q}$

α : John is intelligent OR lucky.

I L

$\neg\alpha$

Negation of α :

$\neg\alpha$ = John is Not int & Not lucky.

$\alpha = I \vee L$; $\neg\alpha = \neg I \wedge \neg L$

$\alpha = I \vee L$

$\overline{\alpha} = \neg I \wedge \neg L$

$$\boxed{\overline{I \vee L} = \neg I \wedge \neg L}$$

$$\neg(p \vee q) = \neg p \wedge \neg q$$

Negation of $p \vee q$:

$$\neg p \wedge \neg q$$

De-morgan laws:

$$\left.\begin{aligned} \overline{P \land Q} &= \overline{P} \lor \overline{Q} \\ \overline{P \lor Q} &= \overline{P} \land \overline{Q} \end{aligned}\right\}$$

De-morgan :

Expression G : only $\wedge$, $\vee$, P, Q, $\bar{P}$, $\bar{Q}$

find $\bar{G}$:

$\wedge \longrightarrow \vee$

$\vee \longrightarrow \wedge$

$P \longrightarrow \bar{P}$

$\bar{P} \longrightarrow P$

Eg: $G : (P \wedge \overline{Q}) \vee (\overline{P} \vee Q)$

$\overline{G} = ?$

Eg: $G : (p \wedge \overline{q}) \vee (\overline{p} \vee q)$

$\overline{G} = \boxed{(\overline{p} \vee q) \wedge (p \wedge \overline{q})}$

$\overline{G}$

De-morgan law:

Expression $G(\wedge, \vee, P, Q, \overline{P}, \overline{Q})$

$\overline{G}(\vee, \wedge, \overline{P}, \overline{Q}, P, Q)$

8. De Morgan's Laws: Named after Augustus De Morgan, these rules allow us to distribute the $\neg$ operator over $\wedge$ and $\vee$. More specifically, we have

$$\neg(A \vee B) = \neg A \wedge \neg B \quad \text{and} \quad \neg(A \wedge B) = \neg A \vee \neg B.$$

In other words, the $\neg$ gets distributed and the corresponding operator "flips." (There is no illustrative example in arithmetic.)

Next Topic:

Logical Laws

6. Absorption Law

$$\boxed{P \vee (PQ) = P}$$

Absorption Law.

$$P + PQ = PT + PQ$$

$$= P(T + Q) = P$$

$$P \wedge T = P$$

OR

$$\boxed{p + pq = p}$$

True

OR

$$p + pq = p(\underline{1 + q}) = p$$

1

$$\boxed{p + pq = p}$$

Absorption law

$$\boxed{p \vee pq = p}$$

Note:

$$\boxed{p + \cancel{\bar{p}}\, q = p + q}$$

Useful property

Note:

LHS RHS

$$\boxed{p + \overline{p}q = p + q}$$

Prove: by case:

$p = T$

LHS = T ; RHS = T

same

$p = F$

LHS = q
RHS = q

same

Useful properties:

$p \lor pq = p$ Absorption Law

$p \lor \overline{p}q = p \lor q$

$$p + \cancel{\bar{p}}\alpha = p + \alpha$$

$$p + p\alpha = p$$

Next Topic:

Logical Laws

7. Implication Laws

$$\boxed{P \to Q \equiv \bar{P} \vee Q}$$

P	Q	$P \to Q$	$\equiv \bar{P} \vee Q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

$\boxed{\text{P is false}} \equiv \overline{P}$

$\boxed{\text{p is true}} \equiv p$

$\boxed{\alpha \text{ is true} \equiv \alpha}$

$\boxed{\alpha \text{ is false} \equiv \neg\alpha}$

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

$P \to Q$ is True iff P is false OR Q is True

$$\boxed{P \to Q \equiv \bar{P} \vee Q}$$

$\underline{P \to Q}$ is True $\equiv$ P is false ($\bar{P}$)

OR ($\vee$)

Q is true. (Q)

$\Downarrow$

$$P \to Q \equiv \bar{P} \vee Q$$

$$\boxed{p \rightarrow q \equiv \neg p \vee q}$$

Very Important

$$\boxed{p \rightarrow q \equiv \neg q \rightarrow \neg p}$$

$$\boxed{P \rightarrow Q \equiv \overline{P} \vee Q}$$

Very Important

2.4. Important Logical Equivalences. The logical equivalences below are important equivalences that should be memorized.

Identity Laws: $p \wedge T \Leftrightarrow p$

$p \vee F \Leftrightarrow p$

Domination Laws: $p \vee T \Leftrightarrow T$

$p \wedge F \Leftrightarrow F$

Idempotent Laws: $p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Double Negation Law: $\neg(\neg p) \Leftrightarrow p$

Commutative Laws: $p \vee q \Leftrightarrow q \vee p$

$p \wedge q \Leftrightarrow q \wedge p$

Associative Laws: $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Distributive Laws: $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

De Morgan's Laws:

$$\neg(p \land q) \Leftrightarrow \neg p \lor \neg q$$

$$\neg(p \lor q) \Leftrightarrow \neg p \land \neg q$$

Absorption Laws:

$$p \land (p \lor q) \Leftrightarrow p$$

$$p \lor (p \land q) \Leftrightarrow p$$

Implication Law: $(p \to q) \Leftrightarrow (\neg p \lor q)$

Contrapositive Law: $(p \to q) \Leftrightarrow (\neg q \to \neg p)$

Tautology: $p \lor \neg p \Leftrightarrow T$

Contradiction: $p \land \neg p \Leftrightarrow F$

Equivalence: $(p \to q) \land (q \to p) \Leftrightarrow (p \leftrightarrow q)$

Some useful Rules:

1. $p \rightarrow q \equiv \bar{p} \vee q$
2. $p \rightarrow q \equiv \bar{q} \rightarrow \bar{p}$
3. $p + pq \equiv p$
4. $p + \bar{p}q = p + q$

OR

$$p + \overline{p}q = (p+\overline{p})(p+q)$$

$$T = p+q$$

$$pq = p \wedge q$$

$$T \wedge y = y$$

Next Topic:

Simplification Method

(Using Logical Laws to solve questions in propositional Logic)

In Propositional Logic, we have Several Method to Solve questions:

1. Truth Table
2. “By Case” Method
3. Simplification using Logical laws
4. Some More we will see in the Digital Logic Subject

Question 3

6 pts

Using the laws of propositional logic, prove the following statement is a tautology.

No justification is needed about the laws you used, but make sure to show your steps clearly, one per line.

$$\neg p \wedge (p \vee q) \rightarrow q$$

Suggestion: you can use these abbreviations:

NOT: for the negation

AND: for the conjunction

OR: for the disjunction

-->: for the implication

=: for equivalent

GO Classes

Question 3

6 pts

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No justification is needed about the laws you used, but make sure to show your steps clearly, one per line.

$$\neg p \wedge (p \vee q) \rightarrow q$$

Suggestion: you can use these abbreviations:

NOT: for the negation

AND: for the conjunction

OR: for the disjunction

-->: for the implication

=: for equivalent

$\alpha : \boxed{\neg p \wedge (p \vee q) \longrightarrow q}$

α is Tautology iff α is Always True.

$\boxed{\alpha \text{ must be True in every Case.}}$

$\longleftrightarrow$ α Tautology

α : $\boxed{\bar{p} \wedge (p \vee q) \rightarrow q}$

Method 1: By case:

$\boxed{q = F}$	$q = T$
α : True	α : True

α : Tautology

$p \vee F = p$ Identity Law

$\boxed{p \wedge \bar{p} = F}$

Contradiction

α : $\bar{p} \wedge (p \vee q) \longrightarrow q$

method 2 : Simplification using logical laws.

$$(\bar{p} \wedge (p \vee q)) \longrightarrow q$$

Distributive law

$$((\underbrace{\bar{p} \wedge p}_{F}) \vee (\bar{p} \wedge q)) \longrightarrow q$$

$$(\bar{p} \wedge q) \rightarrow q$$

$$\overline{(\bar{p} \wedge q)} \vee q \equiv$$

$$(p \vee \bar{q}) \vee q = p \vee (\bar{q} \vee q) \equiv T$$

T

$F \vee Y = Y$

$\boxed{X \rightarrow Y \equiv \overline{X} + Y}$

$\alpha \rightarrow \beta \equiv \overline{\alpha} \vee \beta$

$A \rightarrow B$

$\equiv \overline{A} + B$

$\overline{AB} = \overline{A} \vee \overline{B}$

GO CLASSES

10. Using the laws of logic to determine whether the following logical expressions are tautologies, contradictions or neither. Be sure to specify the law you use for each step. You can use the Laws of Propositional Logic sheet as reference.

(a) $(p \to q) \vee p$

Simplification method:

$(p \to q) \vee p$

$\boxed{p \to q \equiv \bar{p} \vee q}$

$\bar{p} \vee q \vee p \equiv q \vee \overbrace{\bar{p} \vee p}^{T}$

$\equiv$ True

$(p \to q) \vee p \equiv$ True $\to$ Tautology

(b) $[\neg q \wedge (p \to q)] \to \neg p$

GO Classes

(b) $\left(\overline{q} \wedge (p \rightarrow q)\right) \rightarrow \overline{p}$

$\boxed{p \rightarrow q \equiv \overline{p} \vee q}$

$\boxed{F \vee y \equiv y}$

$\left(\overline{q} \wedge (\overline{p} + q)\right) \rightarrow \overline{p}$

$\left((\overline{q} \wedge \overline{p}) \vee (\underbrace{\overline{q} \wedge q}_{F})\right) \rightarrow \overline{p}$

$$(\overline{p} \wedge \overline{q}) \rightarrow \overline{p}$$

$$\overline{(\overline{p} \wedge \overline{q})} \vee \overline{p}$$

$$\boxed{x \rightarrow y \equiv \overline{x} \vee y}$$

$$p \vee q \vee \overline{p} \equiv q \vee \overset{T}{p \vee \overline{p}} \equiv \text{True}$$

(b) $\boxed{\left(\overline{q} \wedge (p \to q)\right) \to \overline{p}} \equiv$ True

Tautology

GO CLASSES

10. Using the laws of logic to determine whether the following logical expressions are tautologies, contradictions or neither. Be sure to specify the law you use for each step. You can use the Laws of Propositional Logic sheet as reference.

(a) $(p \to q) \lor p$

By case method:

a : $(p \to q) \lor p$

$p = T$

a : True

$p = F$

a : True

a : Tautology

(b) $[\neg q \land (p \to q)] \to \neg p$

GO Classes

(b) $\left(\overline{q} \wedge (p \rightarrow q)\right) \rightarrow \overline{p}$ → Tautology

By case method:

$P = T$	$P = F$
(b) : True	(b) : True

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $(p \wedge q) \rightarrow (p \vee r)$

(b) $p \rightarrow (r \rightarrow p)$

(c) $\neg r \vee (\neg r \rightarrow p)$

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $\underbrace{(p \wedge q)}_{A} \to \underbrace{(p \vee r)}_{B}$ $\equiv \overline{p \wedge q} \vee (p \vee r)$

$\equiv \bar{p} \vee \bar{q} \vee p \vee r$

$\equiv \underbrace{\bar{p} \vee p}_{T} \vee \bar{q} \vee r \equiv$ True

(a): Tautology

[redacted]

$A \to B \equiv \bar{A} \vee B$

[redacted]

$\overline{(p \wedge q)} = \bar{p} \vee \bar{q}$

Use the laws of propositional logic to prove that each statement is a tautology.

(a) $(p \wedge q) \rightarrow (p \vee r)$

By Case method:

$p = T$

$p = F$

[redacted]

a : True

a : True

[redacted]

a : Tautology

Use the laws of propositional logic to prove that each statement is a tautology.

$A \to B \equiv \overline{A} \vee B$

[redacted]

$b \equiv \text{True}$

Tautology

(b) $p \to (r \to p)$ $\equiv \overline{p} \vee (r \to p) \equiv \overline{p} \vee (\overline{r} \vee p)$

A B

$\equiv \overline{p} \vee \overline{r} \vee p \equiv T \vee \overline{r}$

$\equiv \text{True}$

[redacted]

Use the laws of propositional logic to prove that each statement is a tautology.

By Case method:

[redacted]

p = F	p = T
b : True	b : True

(b) $p \to (r \to p)$

b : Tautology

[redacted]

Use the laws of propositional logic to prove that each statement is a tautology.

[redacted]

$\bar{r} \to p \equiv r \vee p$

$a \vee (b \vee c) \equiv (a \vee b) \vee c$

[redacted]

(c) $\neg r \vee (\neg r \to p)$ $\equiv \bar{r} \vee (r \vee p) \equiv \overbrace{(\bar{r} \vee r)}^{T} \vee p \equiv$ True

(c) $\bar{r} \vee (\bar{r} \rightarrow p)$

By Case:

$r = T$

(c) : True

$r = F$

(c) : True

(c) : Tautology

4. Propositions which are "always true" (respectively, "always false") are called tautologies (respectively, contradictions). More precisely:

Definition 1.1.20. A *tautology* is a proposition that is true for every possible assignment of truth values to the statement letters that occur in it. A *contradiction* is a proposition that is false for every possible assignment of truth values to the statement letters that occur in it.

Let P and Q be statements. Determine whether each of the following statements is a tautology, a contradiction, or neither.

(a) $P \Leftrightarrow \neg(\neg P)$.
(b) $P \wedge \neg P$.
(c) $P \vee \neg P$.
(d) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
(e) $P \Rightarrow (Q \Rightarrow P)$.

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Let P and Q be statements. Determine whether each of the following statements is a tautology, a contradiction, or neither.

(a) $P \Leftrightarrow \neg(\neg P)$. Tautology
(b) $P \wedge \neg P$. Contradiction
(c) $P \vee \neg P$.
(d) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
(e) $P \Rightarrow (Q \Rightarrow P)$.

(a) $p \Longleftrightarrow \neg(\neg p)$

$\equiv \boxed{p \leftrightarrow p \equiv \text{True}}$

$\neg(\neg p) \equiv p$

$p \Longleftrightarrow p \equiv \boxed{?} \to$ True

$T \longleftrightarrow T$ — T

$F \longleftrightarrow F$ — T

(b)

$p \wedge \overline{p} \equiv$ false → Contradiction

$p \wedge \overline{p} = \boxed{?}$ → false

F ∧ T — F

T ∧ F — F

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Let P and Q be statements. Determine whether each of the following statements is a tautology, a contradiction, or neither.

(a) $P \Leftrightarrow \neg(\neg P)$.
(b) $P \wedge \neg P$.
(c) $P \vee \neg P$.
(d) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$. Contingency
(e) $P \Rightarrow (Q \Rightarrow P)$.

Tautology

$$p \vee \bar{p} = \boxed{?}$$

True

$T \vee F$ — T

$F \vee T$ — T

$$\boxed{p \vee \bar{p} \equiv \text{True}}$$

(d) $(p \wedge q) \vee (\bar{p} \wedge \bar{q})$ Contingency

By Case:

$p = F$ | $p = T$ No need to check

(d) : $\bar{q}$

$\boxed{(d) = \text{false}}$ when $q = T,\ p = F$

(d) = True when $q = F = p$

(e) $P \rightarrow (Q \rightarrow P)$

$A \rightarrow B \equiv \bar{A} \vee B$

$P \rightarrow (\bar{Q} \vee P)$

$\bar{P} \vee P \vee \bar{Q}$

$T \vee \bar{Q} \equiv$ True

$\boxed{e \equiv \text{True}}$

Tautology

4. Propositions which are "always true" (respectively, "always false") are called tautologies (respectively, contradictions). More precisely:

Definition 1.1.20. A *tautology* is a proposition that is true for every possible assignment of truth values to the statement letters that occur in it. A *contradiction* is a proposition that is false for every possible assignment of truth values to the statement letters that occur in it.

Let P and Q be statements. Determine whether each of the following statements is a tautology, a contradiction, or neither.

(a) $P \Leftrightarrow \neg(\neg P)$.
(b) $P \wedge \neg P$.
(c) $P \vee \neg P$.
(d) $(P \wedge Q) \vee (\neg P \wedge \neg Q)$.
(e) $P \Rightarrow (Q \Rightarrow P)$.

(1 pt) Match the following propositions with their types below.

[redacted] 1. $(p \wedge q) \rightarrow \neg p$

2. $(p \wedge q) \vee (q \leftrightarrow p)$

[redacted] 3. $\neg p \wedge \neg(p \rightarrow q)$

[redacted] 4. $(p \vee \neg p) \rightarrow (p \wedge \neg p)$

A. Tautology

B. Contradiction

C. Contingency

GO Classes

(1 pt) Match the following propositions with their types below.

[redacted]

$4 \equiv \text{False}$

[redacted] **4.** $(p \vee \neg p) \rightarrow (p \wedge \neg p)$

$T \longrightarrow F \equiv F$

A. Tautology

B. Contradiction

C. Contingency

(1 pt) Match the following propositions with their types below.

[redacted] 1. $(p \wedge q) \rightarrow \neg p$ $\equiv \overline{p \wedge q} \vee \overline{p}$

[redacted]

$A \vee A = A$

$\equiv \overline{p} \vee \overline{q} \vee \overline{p}$

[redacted]

$\equiv \overline{p} \vee \overline{p} \vee \overline{q}$

[redacted]

$\overline{p}$

$\equiv \overline{p} \vee \overline{q}$

True ($p = F$)

false ($p = T$; $q = T$)

A. Tautology

B. Contradiction

✓ **C.** Contingency

(1 pt) Match the following propositions with their types below.

[redacted]

2. $(p \wedge q) \vee (q \leftrightarrow p)$

[redacted]

By Case :

$p = T$

2 : $q \vee q$

$\equiv q$

False

True

$p = F$

No need to check

A. Tautology

B. Contradiction

C. Contingency

$A \leftrightarrow T \equiv \boxed{?}$

Constant

$T \quad \boxed{T}$

$F \quad \boxed{F}$

A

$\boxed{A \leftrightarrow T \equiv A}$

(1 pt) Match the following propositions with their types below.

$\boxed{3 \equiv \text{false}}$

[redacted]

3. $\neg p \land \neg(p \to q)$ $\equiv \bar{p} \land \overline{(\bar{p} \lor q)}$

$\equiv \bar{p} \land (p \land \bar{q})$

$\equiv (\bar{p} \land p) \land \bar{q} \equiv F \land$

$\equiv$

$\bar{q} \equiv F$

[redacted]

A. Tautology

B. Contradiction ✓

C. Contingency

GO Classes

$a \# (b * c)$: Distributive law

$a \# (b \# c)$: Associative law

Which of the following propositions are tautologies? For tautologies, use equivalences to show it (state the names of the equivalences used in your proof). If it is not a tautology, give a counterexample. Do NOT use truth tables.

(a) $(p \wedge q) \vee (\neg p \vee q)$

(b) $(p \wedge q) \rightarrow (p \vee q)$

HW

by case

simplification

Determine whether or not the following statement is a tautology or not. If you need to, you can build a truth table to answer this question.

$$(q \to p) \vee (\sim q \to \sim p)$$

- This is a tautology.
- This is not a tautology.

Determine whether or not the following statement is a tautology or not. If you need to, you can build a truth table to answer this question.

$$(q \to p) \lor (\sim q \to \ \sim p)$$

$$(\bar{q} \lor p) \lor (q \lor \bar{p})$$

$A \to B \equiv \bar{A} + B$

- This is a tautology.
- This is not a tautology.

$\equiv \bar{q} \lor p \lor q \lor \bar{p}$

$\equiv T \lor A$

$\equiv$ True

GO Classes

Question 7.

Which of the following logical statements is a tautology, which is a contradiction, and which is neither?

A. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

B. $\neg(\neg P \vee P)$

C. $P \to (Q \to P)$

D. $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

Answers.

1. A and D are tautologies, B is a contradiction and C is neither.
2. A and C are tautologies, B is a contradiction and D is neither.
3. C is a tautology, B is a contradiction, A and D are neither.
4. C and D are tautologies, B is a contradiction and A is neither.

Question 7.

Which of the following logical statements is a tautology, which is a contradiction, and which is neither?

A. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

B. $\neg(\neg P \vee P)$

C. $P \rightarrow (Q \rightarrow P)$

D. $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

$\neg(P \vee \overline{P}) \equiv \neg(T) \equiv$ False

$P \vee \overline{P} \rightarrow T$

$\boxed{B \equiv \text{False}}$

contradiction

Answers.

1. A and D are tautologies, B is a contradiction and C is neither.
2. A and C are tautologies, B is a contradiction and D is neither.
3. C is a tautology, B is a contradiction, A and D are neither.
4. C and D are tautologies, B is a contradiction and A is neither.

Question 7.

Which of the following logical statements is a tautology, which is a contradiction, and which is neither?

A. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$

B. $\neg(\neg P \vee P)$

C. $P \rightarrow (Q \rightarrow P)$

D. $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

$P \rightarrow (Q \rightarrow P)$

$\equiv \bar{P} \vee \bar{Q} \vee P$

$\equiv \underbrace{\bar{P} \vee P}_{T} \vee \bar{Q}$

$\equiv T \vee \bar{Q} \equiv \text{True}$

Answers.

1. A and D are tautologies, B is a contradiction and C is neither.
2. A and C are tautologies, B is a contradiction and D is neither.
3. C is a tautology, B is a contradiction, A and D are neither.
4. C and D are tautologies, B is a contradiction and A is neither.

$C \equiv \text{True}$

Question 7.

Which of the following logical statements is a tautology, which is a contradiction, and which is neither?

A. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ — Contingency

B. $\neg(\neg P \vee P)$

C. $P \to (Q \to P)$

D. $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ — Tautology

Answers.

1. A and D are tautologies, B is a contradiction and C is neither.
2. A and C are tautologies, B is a contradiction and D is neither.
3. C is a tautology, B is a contradiction, A and D are neither.
4. C and D are tautologies, B is a contradiction and A is neither.

(D) $PQ \vee \bar{P}Q \vee P\bar{Q} \vee \bar{P}\bar{Q}$

$Q(P \vee \bar{P}) \vee \bar{Q}(P \vee \bar{P})$

T T

$Q \vee \bar{Q} \equiv \underline{\text{True}}$

$T \wedge y = y$

Next Topic:

Analysis of Implication

"Analysis of Implication" is **VERY Important** to solve MANY Question Very Quickly.

Next Topic:

$\boxed{\text{logical Arguments}}$ → Analysis of Implication

Q: "$\alpha \rightarrow \beta$" is Tautology or not?

Implication Statement

New method to solve

1. Truth Table
2. By Case
3. Simplification using logical laws

Q : "$\alpha \to \beta$" is Tautology or not?

Implication Statement

$\alpha \to \beta$: Tautology iff we Cannot make $\alpha \to \beta$ false.

Q: "$\alpha \rightarrow \beta$" is Tautology or not?

Implication Statement

$\alpha \rightarrow \beta$

"only way" to make Implication false:

$\boxed{\text{make } \alpha \text{ True \& } \beta \text{ false.}}$

Q: "$\alpha \to \beta$" is Tautology or not?

Implication Statement

If $\boxed{\begin{array}{l}\alpha = \text{True} \\ \beta = \text{false}\end{array} \&}$ is possible Simultaneously

then $\alpha \to \beta$ is Not Tautology.

Q: "$\alpha \to \beta$" is Tautology or not?

$\alpha \to \beta$ → Implication Statement

If $\boxed{\begin{array}{l}\alpha = \text{True} \\ \& \\ \beta = \text{false}\end{array}}$ is Never possible Simultaneously

then $\alpha \to \beta$ is Tautology.

Q: "$\alpha \to \beta$" is Tautology or not?

Implication Statement

Simultaneously { α = True and β = false }

Possible → $\alpha \to \beta$ NOT Tautology

NOT possible → $\alpha \to \beta$ Tautology

Q: "$\alpha \to \beta$" Tautology or not?

$\boxed{\alpha = T \;\&\; \beta = F}$

Possible

$\alpha \to \beta$: NOT Tautology

NOT possible

$\alpha \to \beta$: Tautology

Q: " $\alpha \rightarrow \beta$ " Tautology or not?

$\boxed{\alpha = T \ \& \ \beta = F}$

Approach 1:

first $\alpha = T$ then

$\beta = \text{false}$

Approach - 2

first $\beta = \text{false}$

then $\alpha = \text{True}$

Checking whether a Implication Statement is Tautology or not ??

"$\alpha \longrightarrow \beta$" Is it Tautology ??

$\alpha, \beta \equiv$ propositional Expression

Tautology: when $\alpha = T$; $\beta = F$ can NEVER happen Together.

Not Tautology: when It is possible to make $\alpha = T$ and $\beta = F$ simultaneously.

To check if "$\alpha \to \beta$" is Tautology?

Approach 1

make α True
then I will Try
to make β = false

If possible: Not Tautology
In Not possible: Tautology

Apprach 2

make β = false
then Try your best to
make α = True

If possible $\equiv$ Not Tautology
If Not possible $\equiv$ Tautology

Checking Tautology for Conditional Statement:

Check if given implication statement "$X \rightarrow Y$" is Tautology or Not ?

Checking Tautology for Conditional Statement:

Check if given implication statement "X→Y" is Tautology or Not ?

Logical Method:

We know that Implication statement is ALWAYS TRUE Except when X=T, Y=F. (The ONLY way to make implication false is to make Antecedent True and Consequent False.)

So, Try to make X→Y False.

If you can, then X→Y is Not Tautology.

If you can NOT, then X→Y is Tautology.

Checking Tautology for Conditional Statement:

Check if given implication statement "X→Y" is Tautology or Not ?

Approach 1:

Make X True.

Now, TRY to Make Y False.

If you can, then X→Y is Not Tautology.

If you can NOT, then X→Y is Tautology.

(Basically, we are trying to make X→Y False. If we can then Not Tautology; If we cannot then Tautology.)

Checking Tautology for Conditional Statement:

Check if given implication statement "X→Y" is Tautology or Not ?

Approach 2:

Make Y False.

Now, TRY to Make X True.

If you can, then X→Y is Not Tautology.

If you can NOT, then X→Y is Tautology.

(Basically, we are trying to make X→Y False. If we can then Not Tautology; If we cannot then Tautology.)

Q: $\underbrace{(p \wedge q)}_{\alpha} \rightarrow \underbrace{(p \vee q)}_{\beta}$ Tautology or Not?

$\boxed{\alpha = T \text{ and } \beta = F}$

Simultaneously

possible → NOT Tautology

NOT possible → Tautology

Q: $(\underbrace{p \wedge q}_{\alpha}) \rightarrow (\underbrace{p \vee q}_{\beta})$ Tautology or Not?

Approach 1:

$\alpha = T$ then $\beta = F$ → Not possible → Tautology

$\alpha = T$ → $\boxed{p \wedge q = \text{True}}$ → $p = T, q = T$

β cannot be false now ← $p \vee q$

Q: $\underbrace{(p \wedge q)}_{\alpha} \rightarrow \underbrace{(p \oplus q)}_{\beta}$ Tautology or Not?

$\alpha = T$
and
$\beta = \text{false}$ → Possible → NOT Tautology

$\alpha = T$ → $p \wedge q = \text{True}$ → $p = T$, $q = T$

$\beta = \underset{T}{p} \oplus \underset{T}{q} = \text{false}$

Q:

$$\underbrace{(p \oplus q)}_{\alpha} \longrightarrow \underbrace{(p \vee q)}_{\beta}$$

Tautology or Not?

$\alpha = T$ and $\beta = F$ → NOT Possible → Tautology ✓

$\alpha = T$ → $\boxed{p \oplus q = \text{True}}$ → $p \neq q$

$\beta = p \vee q$ cannot be false now

(c) $[(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow r.$ Tautology or Not?

① Truth Table : Dumb.

② By case method :

③ Simplification using logical laws

(c) $[(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)] \Rightarrow r.$ Tautology

By Case:

Case 1: $r = F$

$\left((p \vee q) \wedge \bar{p}\,\bar{q} \right) \rightarrow F$

$\left((\bar{p}\,\bar{q}\,p) \vee (\bar{p}\,\bar{q}\,q) \right) \rightarrow F$

$F \vee F$

$F \rightarrow F \equiv$ True (c)

Case 2: $r = T$

(c): True

$$\boxed{A \rightarrow F = \overline{A}}$$ ✓

$$P\overline{P} = F$$

$$FA = F$$

$$A \rightarrow F = \boxed{?}$$

$\overline{A}$

T	F
F	T

(c) $\underbrace{[(p \vee q) \wedge (p \Rightarrow r) \wedge (q \Rightarrow r)]}_{\alpha} \Rightarrow \underbrace{r}_{\beta}$.

Tautology or Not?

$\alpha \rightarrow \beta$

$\alpha = T$
and
$\beta = F$

Approach 2: make β = false.

r = false.

Now make α = True.

$\alpha : \quad (p \vee q) \wedge (\underbrace{p \Rightarrow F}_{p'}) \wedge (\underbrace{q \Rightarrow F}_{q'})$

$\alpha : \quad (p \vee q) \wedge \bar{p}\,\bar{q}$

$= \underbrace{\bar{p}\,\bar{q}\,p}_{F} \vee \bar{p}\,\bar{q}\,q^{\,F} = \text{False}$

$A \to F \equiv A'$

So $\boxed{\alpha = T \text{ and } \beta = F}$ $\underline{\text{Not possible}}$ → Tautology

$\varphi: \big((P \to Q) \wedge (Q \to R)\big) \to (P \to R)$

Tautology or Not?

① Truth Table

② By Case

③ Simplification

HW

$$\varphi: \underbrace{\big((P \to Q) \wedge (Q \to R)\big)}_{\alpha} \to \underbrace{(P \to R)}_{\beta}$$

$\alpha = T$

and

$\beta = F$

Approach 2: Make β = false

then $\alpha = T$

$$\varphi : \underbrace{\big((P \to Q) \land (Q \to R)\big)}_{\alpha} \to \underbrace{(P \to R)}_{\beta}$$

$\alpha = T$
and
$\beta = F$

App2 : $\beta = F$; $\boxed{P \to R}$: false

$\boxed{P = T \; ; \; R = F}$

Now, make $\alpha = \text{True}$

α : $(\overset{T}{P} \rightarrow Q) \wedge (Q \rightarrow \overset{F}{R})$

$\equiv$ $P \wedge \bar{Q}$ = false

$T \rightarrow A \equiv A$

$A \rightarrow F \equiv \bar{A}$

So, if $\beta = F$ then α cannot be True.

So $\boxed{\alpha = T \;\&\; \beta = F}$ NOT Possible $\Rightarrow$ Tautology

Use a truth table to determine whether the following statement is a contradiction, a tautology or neither. If it is a contradiction or a tautology, verify your answer using logical equivalences.

$$\underbrace{((p \vee q) \wedge (p \rightarrow r) \wedge (\sim r))}_{\alpha} \rightarrow \underbrace{q}_{\beta}$$

$\alpha = T$
and
$\beta = F$

Approach 2 : $\boxed{\text{make } \beta = \text{false}}$

$q = \text{false}$

then make $\alpha = \text{True}$

q :

$$(p \vee \overset{F}{q}) \wedge (p \to r) \wedge \overline{r}$$

$p \vee F \equiv p$

$$= p \wedge (\overline{p} \vee r) \wedge \overline{r}$$

$$\equiv p\overline{r}(\overline{p} \vee r)$$

$$\equiv \overset{F}{p\overline{r}\overline{p}} \vee \overset{F}{p\overline{r}r} \equiv \text{false}$$

So;

$\boxed{\beta = F \;\&\; \alpha = T}$

Not possible

Tautology

$$[(P \Rightarrow (Q \Rightarrow R)) \wedge (P \wedge Q)] \Rightarrow R$$

Tautology OR Not?

HW

Next Topic:

Logical Arguments