Set Theory

Proofs involving Sets

Set Equality

Power Set, Subset, Set Identities

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Symbols commonly used with Sets –

$\in \rightarrow$ *indicates an object is an* ***element*** *of a set.*

$\notin \rightarrow$ *indicates an object is* ***not*** *an element of a set.*

$\subseteq \rightarrow$ *indicates a set is a* ***subset*** *of another set.*

$\subset \rightarrow$ *indicates a set is a* ***proper subset*** *of another set.*

$\cap \rightarrow$ *indicates the* ***intersection*** *of sets.*

$\cup \rightarrow$ *indicates the* ***union*** *of sets.*

If A and B are sets, then:

$$
\begin{aligned}
A \times B &= \{(x, y) : x \in A,\ y \in B\}, \\
A \cup B &= \{x : (x \in A) \vee (x \in B)\}, \\
A \cap B &= \{x : (x \in A) \wedge (x \in B)\}, \\
A - B &= \{x : (x \in A) \wedge (x \notin B)\}, \\
\overline{A} &= U - A.
\end{aligned}
$$

Recall that $A \subseteq B$ means that every element of A is also an element of B.

Subsets - For Sets A and B, Set A is a **Subset** of Set B if every element in Set A is also in Set B. It is written as $\boldsymbol{A} \subseteq \boldsymbol{B}$.

Proper Subsets - For Sets A and B, Set A is a **Proper Subset** of Set B if every element in Set A is also in Set B, ***but*** *Set A does* ***not*** *equal Set B.* $(\boldsymbol{A} \neq \boldsymbol{B})$ It is written as $\boldsymbol{A} \subset \boldsymbol{B}$.

Example: $Set\ A = \{2, 4, 6\}$ $\qquad$ $Set\ B = \{0, 2, 4, 6, 8\}$

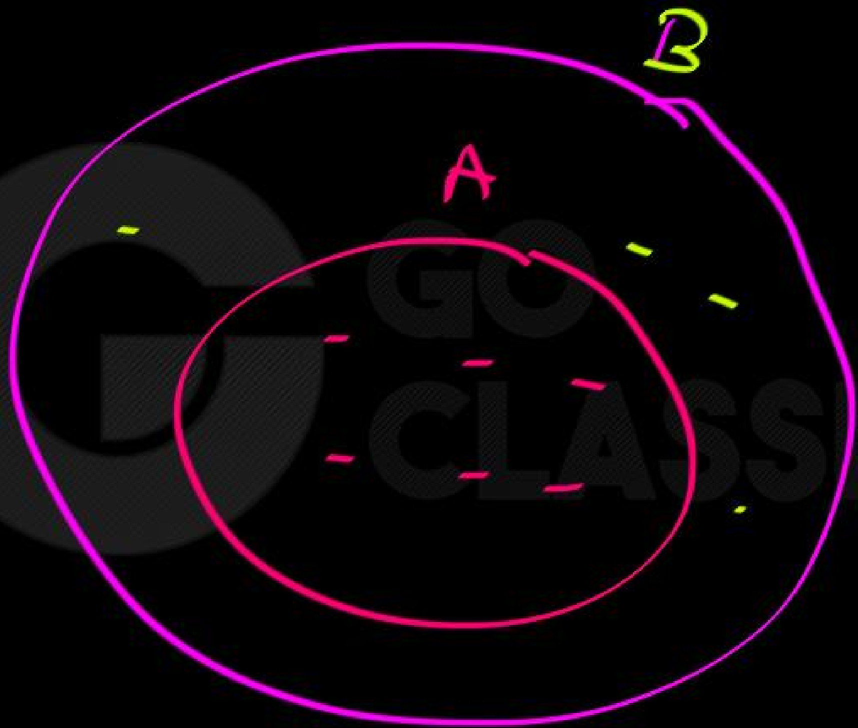
$A \subseteq B$

$\{2, 4, 6\} \subseteq \{0, 2, 4, 6, 8\}$ **and** $\{2, 4, 6\} \subset \{0, 2, 4, 6, 8\}$

Set A is a **Subset** of Set B because every element in A is also in B. $\boldsymbol{A} \subseteq \boldsymbol{B}$

Set A is a **Proper Subset** of Set B because every element in A is also in B, but $A \neq B$. $\boldsymbol{A} \subset \boldsymbol{B}$

$$\underbrace{A \subseteq B}$$



$A = \{a, b\}$

$B = \{a, b\}$

$A = B$

$A \subseteq B$

$X \subset Y$ proper subset

iff $X \subseteq Y$ and $X \neq Y$

Proper subset of a set S, is any subset other than S.

Note: *The Empty Set is a Subset of every Set.*
The Empty Set is also a Proper Subset of every Set <u>except</u> the Empty Set.

<u>Number of Subsets</u> – The number of distinct subsets of a set containing n elements is given by $\mathbf{2^n}$.

<u>Number of Proper Subsets</u> – The number of distinct proper subsets of a set containing n elements is given by $\mathbf{2^n - 1}$.

Example: How many Subsets and Proper Subsets does Set *A* have?

$$Set\ A = \{bananas, oranges, strawberries\}$$
$$n = 3$$

Subsets = $2^n = 2^3 = 8$ **Proper Subsets** = $2^n - 1 = 7$

Example: How many Subsets and Proper Subsets does Set A have?

$$Set\ A = \{bananas, oranges, strawberries\}$$
$$n = 3$$

Subsets = $2^n = 2^3 = 8$ **Proper Subsets** = $2^n - 1 = 7$

Example: List the **Proper Subsets** for the Example above.

1. $\{bananas\}$
2. $\{oranges\}$
3. $\{strawberries\}$
4. $\{bananas, oranges\}$
5. $\{bananas, strawberries\}$
6. $\{oranges, strawberries\}$
7. $\emptyset$

To prove $S \subseteq T$:

$$\forall_x \left(x \in S \rightarrow x \in T \right)$$

Assume some $x \in S$.

knowledge that we already have

Show that $x \in T$

then $S \subseteq T$

(8) $$A \subseteq B \Leftrightarrow \text{If } x \in A, \text{ then } x \in B.$$

(9) $$A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A.$$

Proving a subset relation "$S \subseteq T$"

Using the definition (8) of a subset relation yields a proof of the following structure:

> Let $x \in S$.
> ...
> [Logical deductions]
> ...
> Therefore $x \in T$.
> This proves that $S \subseteq T$.

How to Prove A ⊆ B (Direct approach)

Proof. Suppose $a \in A$.
$\vdots$
Therefore $a \in B$.
Thus $a \in A$ implies $a \in B$,
so it follows that $A \subseteq B$. ■

How to Prove A ⊆ B (Contrapositive approach)

Proof. Suppose $a \notin B$.
$\vdots$
Therefore $a \notin A$.
Thus $a \notin B$ implies $a \notin A$,
so it follows that $A \subseteq B$. ■

To prove $S \subseteq T$:

1. $x \in S \longrightarrow x \in T$ ✓

most used

or

2. $x \notin T \longrightarrow x \notin S$ ✓

Set Equality:

$$\underbrace{S = T}$$

$$\boxed{x \in S \longrightarrow x \in T} \qquad S \subseteq T$$

and

$$x \in T \longrightarrow x \in S \qquad T \subseteq S$$

☞ Theorem: If S and T are sets, then $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

☞ To prove that $S = T$, prove that $S \subseteq T$ and $T \subseteq S$. ☜

☞ If $S = T$, you can conclude that $S \subseteq T$ and $T \subseteq S$. ☜

How to Prove A = B

Proof.

[Prove that $A \subseteq B$.]

[Prove that $B \subseteq A$.]

Therefore, since $A \subseteq B$ and $B \subseteq A$, it follows that $A = B$. ■

☞ To prove $S \subseteq T$, pick an arbitrary $x \in S$, then prove that $x \in T$. ☜

☞ Two sets S and T are equal ($S = T$) if for every object x, we have $x \in S$ if and only if $x \in T$. ☜

☞ If $S = T$ and $x \in S$, you can conclude that $x \in T$.
If $S = T$ and $x \notin S$, you can conclude that $x \notin T$. ☜

☞ If you know $x \in S$ and $S \subseteq T$, you can conclude $x \in T$. ☜

Q :

For any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, Show that we have $A \subseteq D \cup E$.

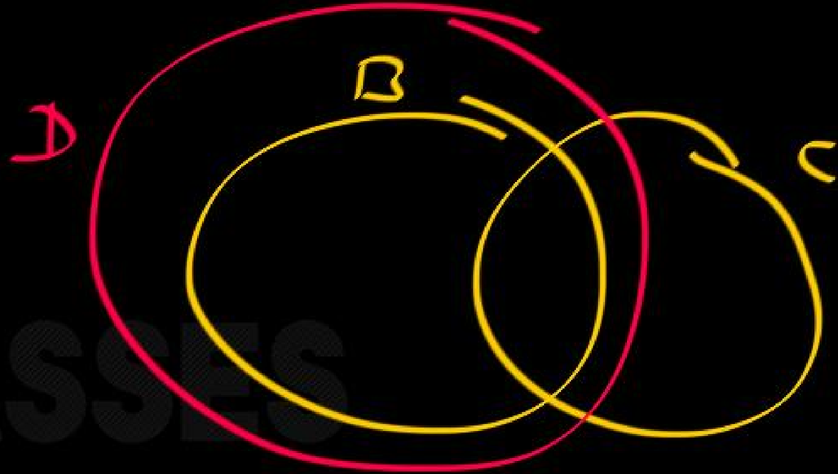
Q :

For any sets A, B, C, D, and E where $A \subseteq B \cup C$, $B \subseteq D$, and $C \subseteq E$, Show that we have $A \subseteq D \cup E$.

$A \subseteq B \cup C$

$B \subseteq D$

$A \subseteq C \cup D$



$x \in A \longrightarrow x \in B \cup C \longrightarrow x \in D \cup C$

$A \subseteq C \cup D$

$C \subseteq E$

$A \subseteq E \cup D$

$x \in A \rightarrow x \in C \cup D$ means

$x \in C \rightarrow x \in E$
or
$x \in D$

$x \in E$ or $x \in D$

$x \in D \cup E$

fact:

$$\boxed{\begin{array}{l} A \subseteq B \quad \text{and} \quad B \subseteq C \\ \text{then} \quad A \subseteq C \end{array}}$$

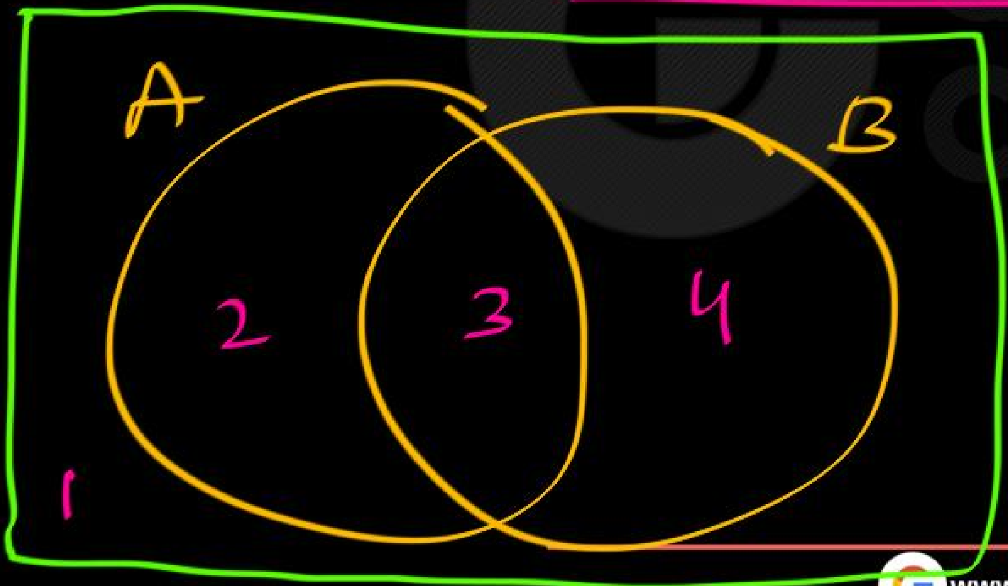
$$\begin{tikzcd} A \arrow[r, "\subseteq"] \arrow[rr, bend right, "\subseteq"'] & B \arrow[r, "\subseteq"] & C \end{tikzcd}$$

Example 1. For any set A and B we have

$$A \cap B \subseteq A \cup B$$

Proof. Take $x \in A \cap B$ arbitrary. So $x \in A$ and $x \in B$. In particular, $x \in A$, so $x \in A \cup B$. $\square$

Method 1: Venn Diagram



$\boxed{A \cap B = 3 \quad A \cup B = 2,3,4}$

$\boxed{A \cap B \subseteq A \cup B}$

method2:

$$\underbrace{A \cap B}_{S} \subseteq \underbrace{A \cup B}_{T}$$

$x \in A \cap B$

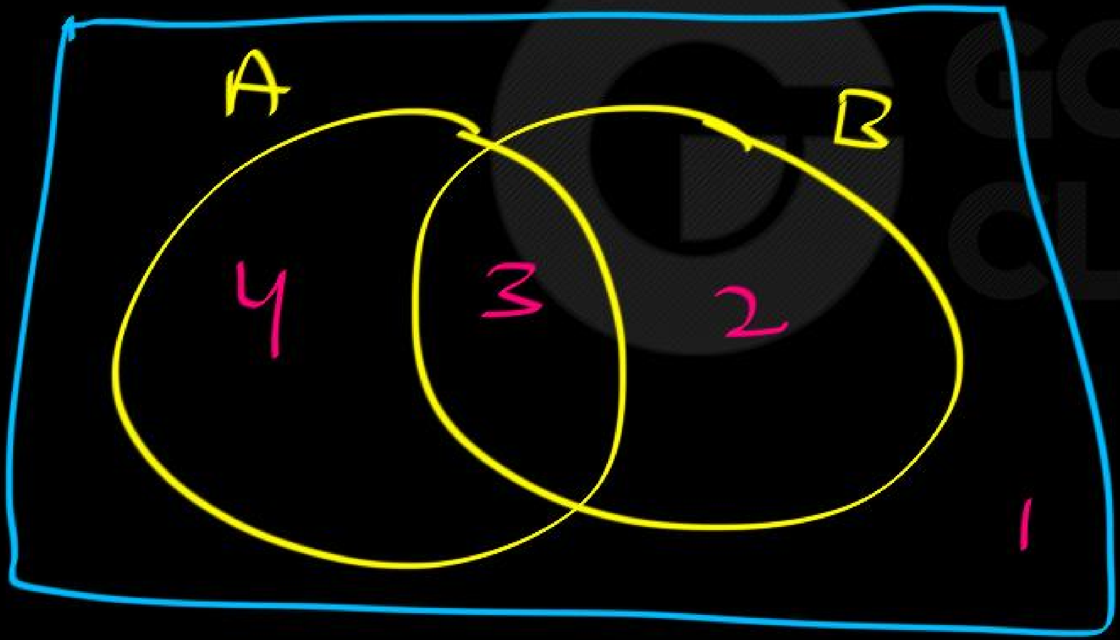
means $x \in A$ and $x \in B$

$\hookrightarrow \boxed{x \in A \cup B}$

So $S \subseteq T$ ✓

Example 1: Proof of $A - (A - B) \subseteq B$ (where A and B are arbitrary sets)

Method 1: Venn Diagram:



$A = 3, 4$

$\boxed{B = 2, 3}$

$A - B = 4$

$\boxed{A - (A - B) = 3}$

$3, 4 - (4)$

$A - (A - B) \subseteq B$

Example 1: Proof of $A - (A - B) \subseteq B$ (where A and B are arbitrary sets)

method2 :

$$\underbrace{A - (A - B)}_{S} \subseteq \underbrace{B}_{T}$$

Assume $x \in S$

$x \in A - (A - B)$

$x \in A$, $x \notin A - B$

$x \in A$, $\boxed{x \notin A - B}$

$x \in A$, $\left(\underline{x \notin A} \text{ OR } \underline{x \in B} \right)$

$x \in A$, $\boxed{x \in B}$

So $\quad S \subseteq T$; $\quad A - (A - B) \subseteq B$ ✓

$x \in A - B$ then $x \in A$, $x \notin B$

$x \notin A - B$ then $x \notin A$ <u>OR</u> $x \in B$

Example 1: Proof of $A - (A - B) \subseteq B$ (where A and B are arbitrary sets)

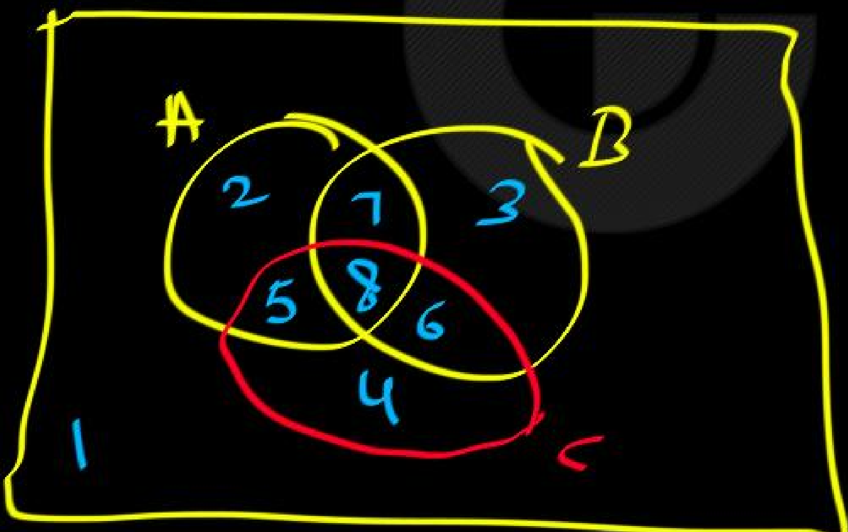
- Let $x \in A - (A - B)$.
- Then $x \in A$ and $x \notin A - B$, by the definition of a set difference (see (5)).
- By the definition of a set difference (in the negated form (6)), "$x \notin A - B$" is equivalent to "$x \notin A$ or $x \in B$".
- Therefore we have $x \in A$ and ($x \notin A$ or $x \in B$).
- Since $x \in A$, the first of the two alternatives in "$x \notin A$ or $x \in B$" is impossible, so the second alternative must hold, i.e., $x \in B$.
- Thus we have $x \in A$ and $x \in B$.
- Hence $x \in B$.
- This proves that $A - (A - B) \subseteq B$.

Remark: The reverse inclusion, $B \subseteq A - (A - B)$, does not hold in general. Therefore the sets $A - (A - B)$ and B are, in general, not equal. To prove this, we exhibit a **counterexample**: Let $A = \{1, 2\}$, $B = \{2, 3\}$. Then $A - B = \{1\}$, $A - (A - B) = \{2\}$, while $B = \{2, 3\}$. Thus, $A - (A - B)$ is a *proper* subset of B, but not equal to B.

Example 2: Proof of $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (where A, B, C are arbitrary sets)

$A \cup (B \cap c) = (A \cup B) \cap (A \cup c)$

Venn Diagram:



$B \cap c = 6, 8$

$A = 2, 5, 7, 8$

$A \cup (B \cap c) = \underline{2, 5, 6, 7, 8}$

$A \cup B = 2, 3, 5, 6, 7, 8$

$A \cup C = 2, 4, 5, 6, 7, 8$

$$\underbrace{(A \cup B)} \cap \underbrace{(A \cup C)}$$

$$(2, 3, 5, 6, 7, 8) \cap (2, 4, 5, 6, 7, 8)$$

$$= \boxed{2, 5, 6, 7, 8}$$

$$\boxed{A \cup (B \cap C)} - 2, 5, 6, 7, 8$$

Example 2: Proof of $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (where A, B, C are arbitrary sets)

$$\underbrace{A \cup (B \cap C)}_{S} \subseteq \underbrace{(A \cup B) \cap (A \cup C)}_{T}$$

Method 2:

To prove $S \subseteq T$

Assume $\boxed{x \in S}$; $x \in A \cup (B \cap C)$

$x \in A$ OR $x \in B \cap C$

$x \in A$ OR $x \in B \cap C$

Case 1: $\boxed{x \in A} \Rightarrow x \in A \cup B$ and $x \in A \cup C$ $\Big\}$ $x \in (A \cup B) \cap (A \cup C)$

$\rightarrow x \in T \rightarrow$ SST

Case 2: $x \in B \cap C \rightarrow x \in B,\ x \in C$

$x \in T \leftarrow x \in B \cup A$ and $x \in C \cup A$

$\rightarrow$ SST

1. $x \in A$ then $x \in A \cup B$ ✓
2. $x \notin A$ then $x \notin A \cap B$
3. $x \notin A \cup B$ then $x \notin A$ and $x \notin B$
4. $x \in A - B$ iff $x \in A$, $x \notin B$
5. $x \notin A - B$ then $x \notin A$ OR $x \in B$

Example 2: Proof of $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ (where A, B, C are arbitrary sets)

- Let $x \in A \cup (B \cap C)$.
- Then $x \in A$ or $x \in B \cap C$, by the definition of a union (see (1)).
- Therefore $x \in A$ or ($x \in B$ and $x \in C$), by the definition of an intersection (see (3)).
  - In the first case (i.e., the case "$x \in A$"), we have $x \in A \cup B$ and $x \in A \cup C$, by the definition of a union. By the definition of an intersection it follows that $x \in (A \cup B) \cap (A \cup C)$.
  - In the second case (i.e., the case "$x \in B$ and $x \in C$"), we have $x \in A \cup B$ and $x \in A \cup C$, by the definition of a union. As before, by the definition of an intersection it follows that $x \in (A \cup B) \cap (A \cup C)$.
- Thus in either case we have $x \in (A \cup B) \cap (A \cup C)$.
- This proves that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Remark: In this example, the reverse inclusion relation does hold, and it can be proved in much the same way as above. Therefore, we have the set equality $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, for any sets A, B, C. (This shows that the union and intersection of sets satisfy a distributive law.)

Example for proving set identities

Example:

Show that if A, B, C are sets, then:

$$(\overline{A} - B) - C = (\overline{A} - C) - (B - C)$$

Proof: First we show $x \in$ LHS $\rightarrow x \in$ RHS:

$x \in (\overline{A} - B) - C$ (by the definition of "set difference")
$\Rightarrow x \in (\overline{A} - B)$ but $x \notin C$. Hence $x \in \overline{A}, x \notin B$ and $x \notin C$.
So $x \in \overline{A} - C$ and $x \notin B - C$. This means $x \in (\overline{A} - C) - (B - C) =$ RHS.

Next we show $x \in$ RHS $\rightarrow x \in$ LHS:

$x \in (\overline{A} - C) - (B - C)$ (by definition of "set difference")
$\Rightarrow x \in (\overline{A} - C)$ but $x \notin (B - C)$. Hence: $x \in \overline{A},\ x \notin C$ and $x \notin (B - C)$.
Here: $x \notin B - C$ means either $x \notin B$ or $x \in C$.
Since the latter contradicts $x \notin C$, we must have $x \notin B$.
This implies $x \in (\overline{A} - B) - C =$ LHS.

Q: Prove that $\underbrace{(\bar{A}-C)-(B-C)}_{S} \subseteq \underbrace{(\bar{A}-B)-C}_{T}$

Prove that $\boxed{S \subseteq T}$

Assume $x \in S$

$x \in (\bar{A}-C)-(B-C)$

$x \in (\overline{A} - C) - (B - C)$

means

$x \in \overline{A} - C$ and $x \notin B - C$

means $x \in \overline{A}$, $x \notin C$, $(x \notin B$ OR $x \in C)$

$\Rightarrow x \in \overline{A}$, $x \notin C$, $x \notin B$

$x \in \overline{A}, \; x \notin C, \; x \notin B$

$\boxed{x \in \overline{A}, \; x \notin B} \rightarrow \boxed{x \in \overline{A} - B \quad x \notin C}$

$\boxed{x \in (\overline{A} - B) - C} \rightarrow x \in T \rightarrow SST$

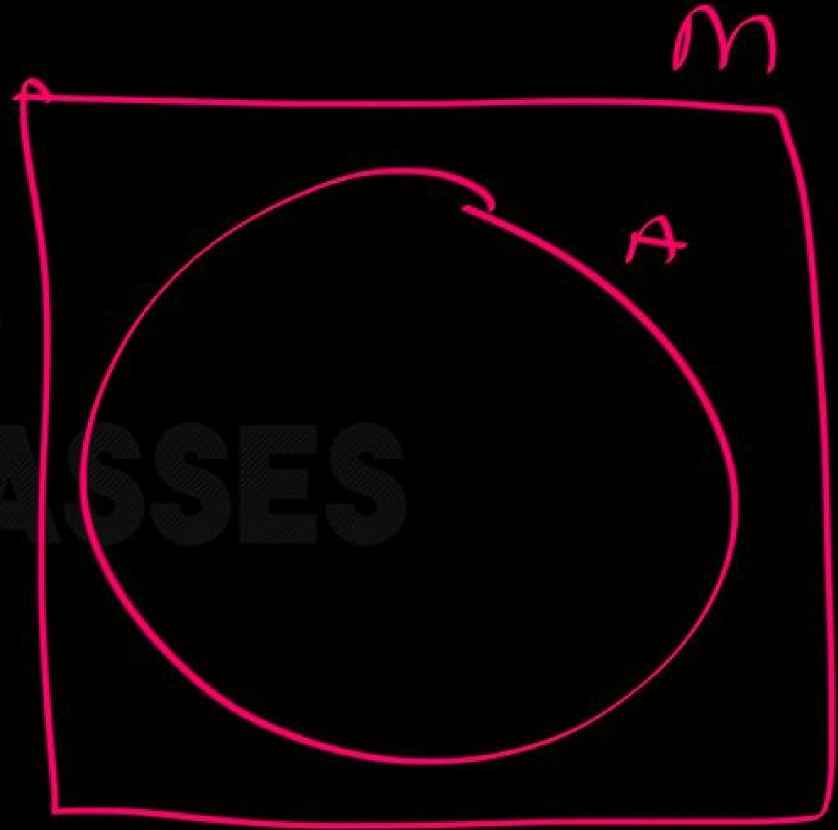
$x \in A - B$ iff $x \in A, x \notin B$

$x \notin A - B$ iff $x \notin A$ OR $x \in B$

$x \in A, x \notin B \longrightarrow x \in A - B$

Universal set : M

$A \cup M = M$



Set Identities

- Identity laws

$$A \cup \emptyset = A \quad A \cap U = A$$

- Domination laws

$$A \cup U = U \quad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \quad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

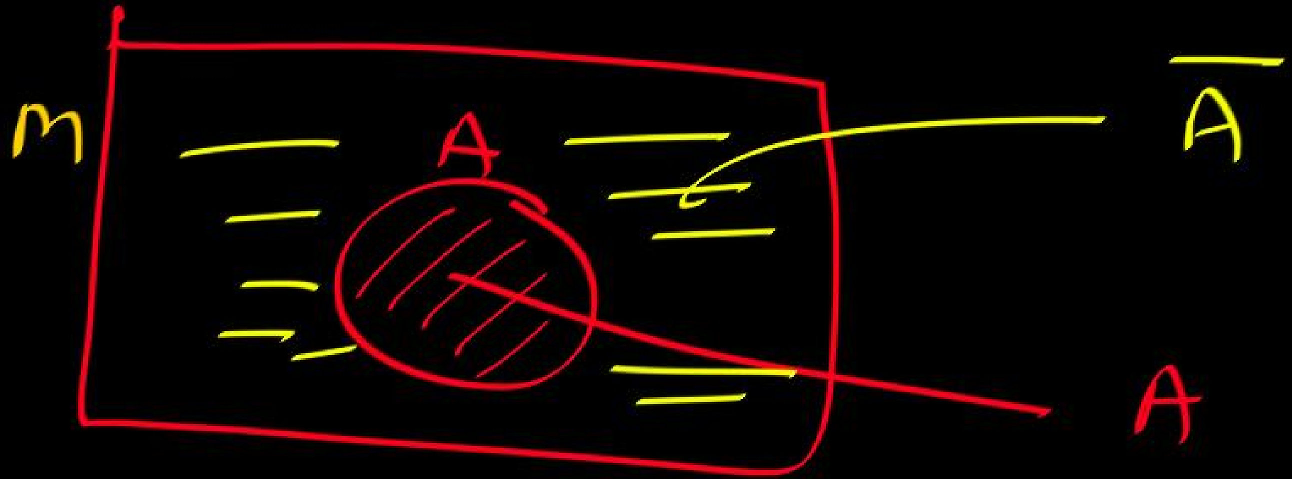
$$\overline{(\overline{A})} = A$$

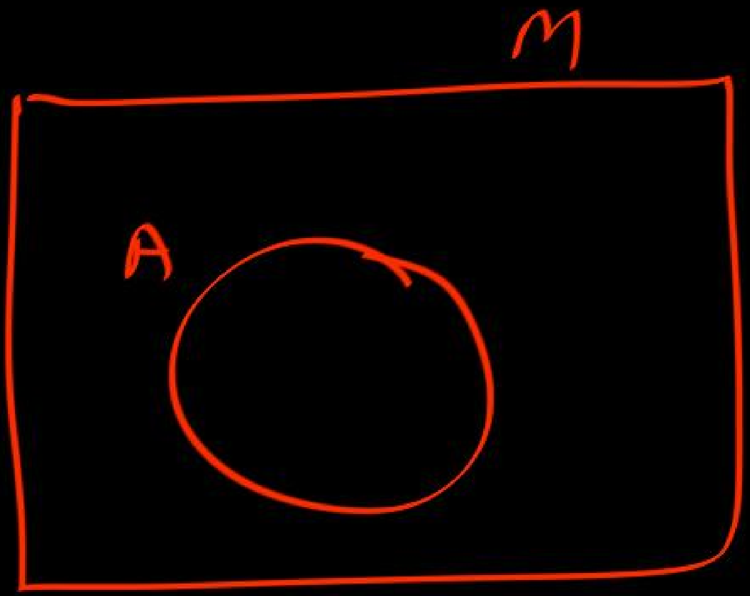
- Complement laws

$$A \cap \overline{A} = \emptyset \quad A \cup \overline{A} = U$$

$$A \cap \bar{A} = \phi$$

$$A \cup \bar{A} = m$$





$$\left\{\begin{array}{l} \left.\begin{array}{l} A \cup M = M \\ A \cap \phi = \phi \end{array}\right\} \text{Domination laws} \\ A - \phi = A \\ A \Delta \phi = A \\ \phi - A = \phi \\ A \times \phi = \phi \end{array}\right.$$

③ Idempotent laws:

operation # is Idempotent iff

$$\boxed{x \# x = x}$$

Additional is idempotent : NO

$x + x \neq x$; $3+3 \neq 3$

mul is Not idempotent;

$$3 \times 3 \neq 3$$

Set A, B

Union is idempotent? Yes

$$\boxed{A \cup A = A}$$

$$\boxed{A \cap A = A}$$

Set difference is idempotent? No

$\boxed{A - A \neq A}$; $A - A = \phi$

Symmetric Difference is idempotent? No

$A \Delta A = \phi$; $A \Delta A \neq A$

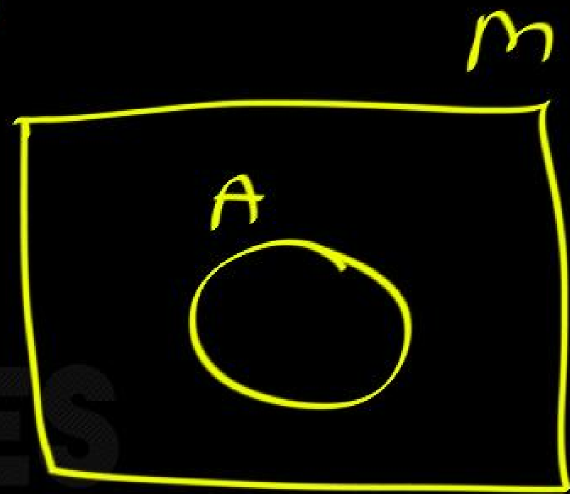
Cross product is idempotent? No

$A \times A \neq A$; $A = \{a\}$ $A \times A = \{(a, a)\}$

① Identity law:

Set A

$$\begin{cases} A \cup \phi = A \\ A \cap M = A \end{cases}$$



① Identity law:

In Number theory,

$n + 0 = n$ Identity element of addition

$n \times 1 = n$ Identity element for mul.

Identity element "e" for operation

$\#$:

$$\boxed{\begin{array}{l} x \# e = x \\ \text{and} \\ e \# x = x \end{array}}$$

$$\left.\begin{array}{l} n \times 1 = n \\ n + 0 = n \\ A \cup \phi = A \\ A \cap M = A \end{array}\right\}$$

Q: Identity element for Symmetric Difference? $\longrightarrow \phi$

$$A \,\Delta\, X = A$$

$$X \,\Delta\, A = A$$

$X = ?$

$X = \phi$

$$A \Delta \phi = (A - \phi) \cup (\phi - A)$$

$$= A \cup \phi$$

$$\boxed{A \Delta \phi = A}$$

$$\boxed{\phi \Delta A = A}$$

$$\boxed{A \Delta B = A \oplus B = (A - B) \cup (B - A)}$$

Q: Identity element for Set Difference?

$$A - \phi = A$$

$$\phi - A = \phi \neq A$$

NO Identity element exists for set different.

$$\boxed{\begin{aligned} A - \phi &= A \\ \phi - A &= \phi \end{aligned}}$$

$$\left\{\begin{aligned} A \,\Delta\, \phi &= A \\ \phi \,\Delta\, A &= A \end{aligned}\right\}$$

Q: Identity element for Cross product?

$\Rightarrow$ No Identity element

$A \times \bigcirc = A$

$A \neq A \times B$

$A = \{a, b\}$

$A \times B = \{(x, y) \mid x \in A, y \in B\}$

$$A \times \phi = \phi$$

$$A \times B = \left\{ \boxed{(a, b)} \;\middle|\; a \in A, b \in B \right\}$$

$$A \times B \neq A$$

$$A \times B = A \text{ iff } A = \phi$$

Set Identities (cont.)

- Commutative laws

$$A \cup B = B \cup A \quad A \cap B = B \cap A$$

- Associative laws
  $A \cup (B \cup C) = (A \cup B) \cup C$
  $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive laws
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption laws

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

Distributive property:

✓ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

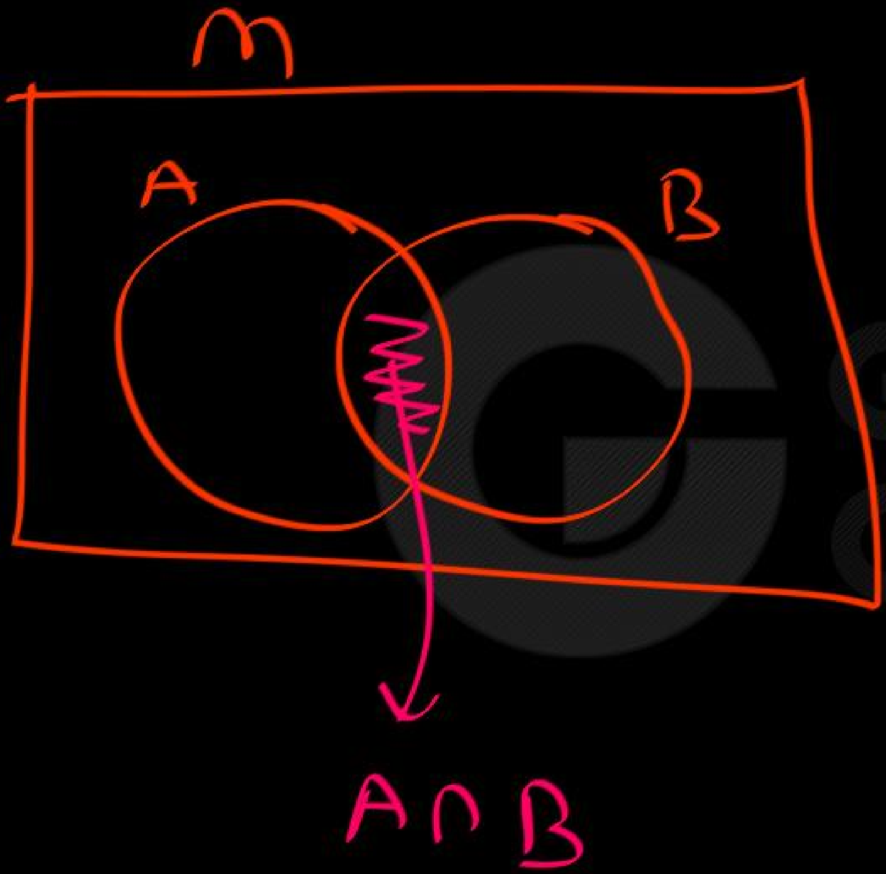
✓ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Absorption law:

$$B \cap (A \cup B) = B$$

$$\left\{ \begin{array}{l} A \cup (A \cap B) = A \\ A \cap (A \cup B) = A \end{array} \right\}$$

$$B \cup (A \cap B) = B$$



$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

Demorgan law:

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \checkmark$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B} \checkmark$$

Proof: $x \in \overline{(A \cup B)}$ then

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in \overline{A}$ and $x \in \overline{B}$

$\Rightarrow x \in \overline{A} \cap \overline{B}$

$\Rightarrow \overline{(A \cup B)} \subseteq \overline{A} \cap \overline{B}$

$x \in \overline{A}$ then $x \notin A$

$x \notin A$ then $x \in \overline{A}$

$x \in \overline{A} \cap \overline{B}$

$\Rightarrow x \in \overline{A},\ x \in \overline{B}$

$\Rightarrow x \notin A,\ x \notin B$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \in \overline{A \cup B}$

$\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$

So

$\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ ✓

$\overline{A \cap B} = \overline{A} \cup \overline{B}$ ✓

Demorgan law:

If expression E contains only

$\cup$, $\cap$, $\overline{A}$, A then

How to find $\overline{E}$ = ? $\Rightarrow$ Apply Demorgan law

To find $\overline{E}$:

$$\left.\begin{array}{l} \cap \longrightarrow \cup \\ \cup \longrightarrow \cap \\ \overline{A} \longrightarrow A \\ A \longrightarrow \overline{A} \end{array}\right\}$$

$$\overline{\left(A \cup (\overline{B} \cap C)\right)} =$$

$$E = \boxed{A \cup (\overline{B} \cap C)}$$

$$\overline{E} = ? = \boxed{\overline{A} \cap (B \cup \overline{C})}$$

Important: Precedence of Set operations:

$$\boxed{\overline{A} > A \cap B > A \cup B}$$

Eg: $A \cup B \cap C$ — $(A \cup B) \cap C$ / $A \cup (B \cap C)$ ✓

Eg: $E = A \cup B \cap C$

$\overline{E} = ?$

$\boxed{(\overline{A} \cap \overline{B}) \cup \overline{C}}$

wrong Answer

Eg:

$E = A \cup B \cap C$

$\overline{E} = ?$ $= A \cup (B \cap C)$

$\overline{E}$

$\overline{E} = \boxed{\overline{A} \cap (\overline{B} \cup \overline{C})}$

Commutative laws:

Operation # is called Commutative iff $\boxed{x \# y = y \# x}$

$$\left.\begin{array}{l} 2+3=3+2 \\ a+b=b+a \end{array}\right\}$$

$a \times b = b \times a$

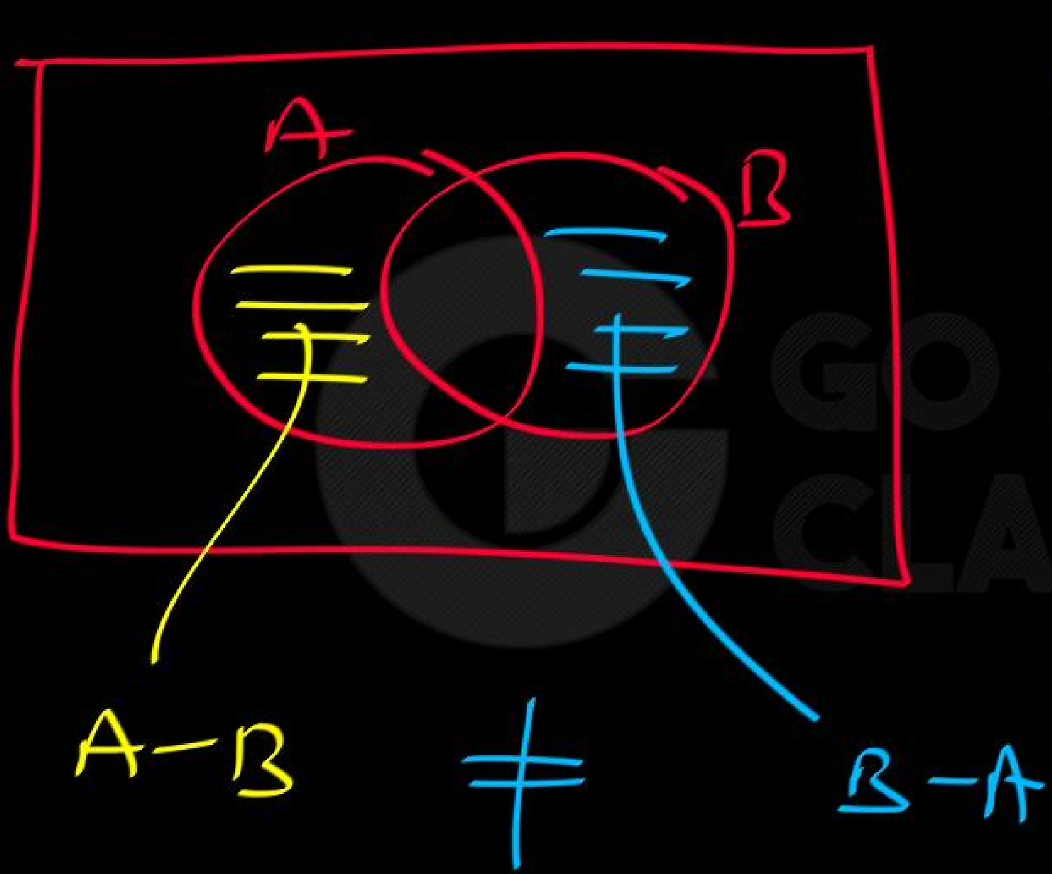
Set theory ; set A , B

$A \cup B = B \cup A$ ✓

$A \cap B = B \cap A$ ✓

$A - B = B - A$ ✗

$A \Delta B = B \Delta A$ ✓



$$A \Delta B = (A - B) \cup (B - A)$$

$$B \Delta A = A \Delta B$$

$$A \times B \neq B \times A$$

$$A = \{a\} \quad ; \quad B = \{b\}$$

$$A \times B = \{(a,b)\}$$

$$B \times A = \{(b,a)\}$$

not equal

Associative law:

operation # is Associative iff

$$(a \# b) \# c = a \# (b \# c)$$

$\forall a, b, c$

$$(n + m) + t = n + (m + t)$$

$$(n \times m) \times t = n \times (m \times t)$$

Set theory : set A, B, C

$$(A \cup B) \cup C = A \cup (B \cup C) \checkmark$$

$$(A \cap B) \cap C = A \cap (B \cap C) \checkmark$$

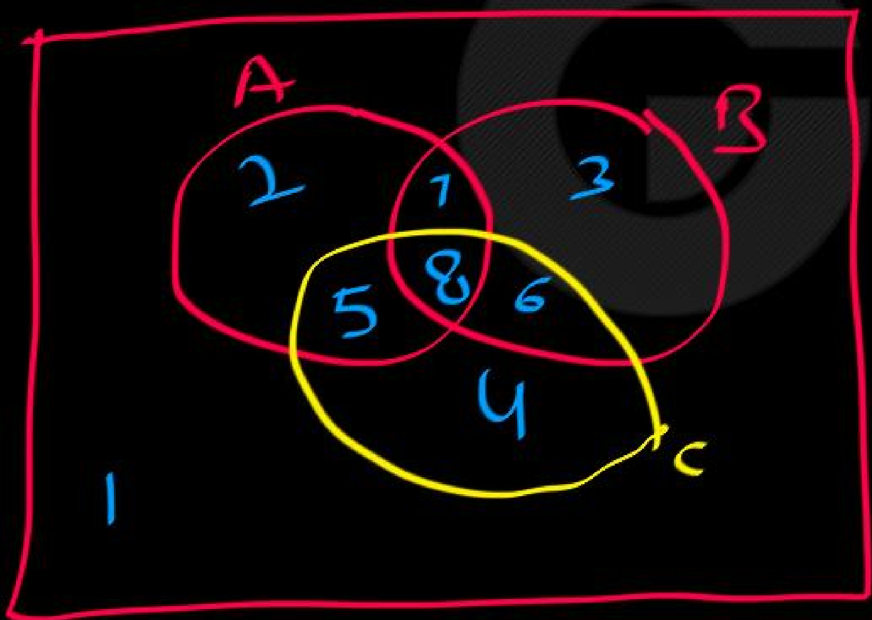
$$(A - B) - C \neq A - (B - C)$$

$A = \{1\}$, $B = \{1\}$, $C = \{1\}$

$$\left.\begin{array}{l}(A - B) - C = \phi - C = \phi \\ A - (B - C) = A - \phi = A = \{1\}\end{array}\right\} \neq$$

$$\underbrace{(A \Delta B) \Delta C}_{\text{LHS}} = \underbrace{A \Delta (B \Delta C)}_{\text{RHS}}$$

Proof:



$A \Delta B = 2, 5, 3, 6$

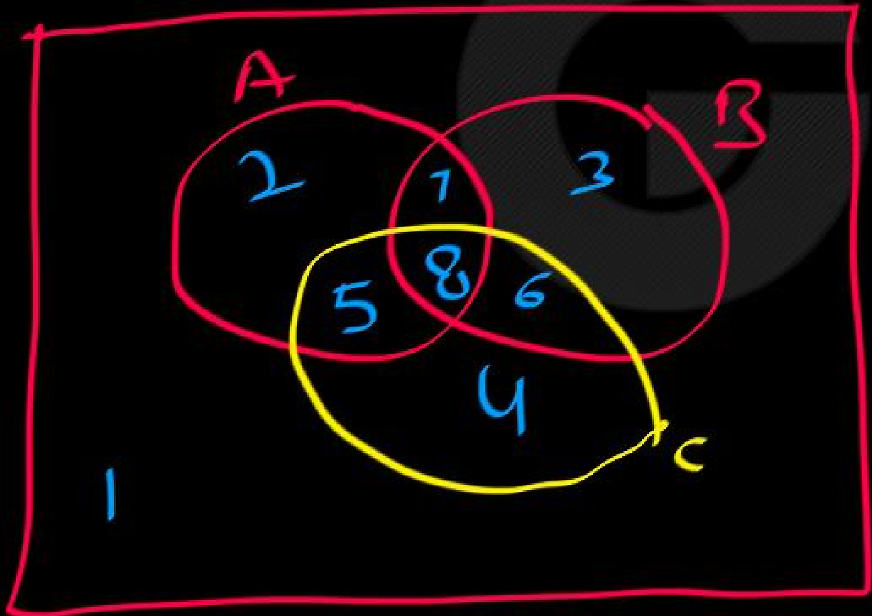
$(A \Delta B) \Delta C =$

$(2, 5, 3, 6) \Delta (4, 5, 6, 8)$

$= (2, 3, 4, 8) =$ LHS

$$\underbrace{(A \Delta B) \Delta C}_{\text{LHS}} = \underbrace{A \Delta (B \Delta C)}_{\text{RHS}}$$

Proof:



$B \Delta C = 3, 7, 4, 5$

$A \Delta (B \Delta C) =$

$(2, 5, 7, 8) \Delta (3, 4, 5, 7)$

$= (2, 8, 4, 3) = \text{RHS}$

Δ is Associative.

$$(A \,\Delta\, B) \,\Delta\, C = A \,\Delta\, (B \,\Delta\, C)$$

Set identities: DeMorgan law

DeMorgan laws:

Just like the DeMorgan laws in logic, we have:

$$\begin{aligned}\overline{A \cap B} &= \overline{A} \cup \overline{B}\\ \overline{A \cup B} &= \overline{A} \cap \overline{B}\end{aligned}$$

Example:

Let the universe be $\{0,1,2,3\}$.

$\overline{\{0,1\} \cap \{1,2\}} = \overline{\{1\}} = \{0,2,3\} = \{2,3\} \cup \{0,3\} = \overline{\{0,1\}} \cup \overline{\{1,2\}}$.

For each Law of Logic, there is a corresponding Law of Set Theory.

- Commutative: $A \cup B = B \cup A, \quad A \cap B = B \cap A.$
- Associative: $A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  and also on the right: $(B \cap C) \cup A = (B \cup A) \cap (C \cup A), \quad (B \cup C) \cap A = (B \cap A) \cup (C \cap A)$
- Double Complement: $(A^c)^c = A$
- DeMorgan's Laws: $(A \cup B)^c = A^c \cap B^c, \quad (A \cap B)^c = A^c \cup B^c$
- Identity: $\emptyset \cup A = A, \quad \mathcal{U} \cap A = A$
- Idempotence: $A \cup A = A, \quad A \cap A = A$
- Dominance: $A \cup \mathcal{U} = \mathcal{U}, \quad A \cap \emptyset = \emptyset$

Comparing Prop. logic and set theory:

Prop. logic	set theory
Propositions	Sets
T	Universal set
F	ϕ
$P \wedge Q$	$P \cap Q$

Comparing Prop. logic and set theory:

Prop. logic	set theory
$P \vee Q$	$P \cup Q$
$\neg P$	$\overline{P}$
Exor $\oplus$	$\oplus$ Sym Diff

Comparing Prop. logic and set theory:

Prop logic	Set theory
P, Q, R are propositions	P, Q, R are set
$P \vee T = T$	$P \cup M = M$
$P \wedge F = F$	$P \cap \phi = \phi$

$P \vee F = P$	$S \cup \phi = S$
$P \wedge T = P$	$S \cap M = S$
$P \wedge F = F$	$S \cap \phi = \phi$
$P \vee T = T$	$S \cup M = M$
$P \wedge (P \vee Q) = P$	$S \cup (S \cap T) = S$
$P \vee (P \wedge Q) = P$	$S \cap (S \cup T) = S$

$$A - B = A \setminus B = A \cap \overline{B}$$

$$A - B = \overline{B} - \overline{A}$$

$$A \oplus B = (A - B) \cup (B - A)$$

$$= (A \cup B) - (A \cap B)$$

Set equalities of note:

- $A \setminus B = A \cap B^c$
- $A \oplus B = (A \cup B) \setminus (A \cap B)$

Set Theory

Proofs involving Power Sets

Power Set, Subset

Website : https://www.goclasses.in/

First, let's recap the formal definition of the power set. The power set of a set S is the set of all subsets of S:

$$\wp(S) = \{\ T \mid T \subseteq S\ \}$$

The above definition is wonderfully useful. For example, if you want to show that an object belongs to $\wp(S)$, you need to show that that object obeys the set-builder notation. Specifically:

☞ **To prove that $T \in \wp(S)$, prove that $T \subseteq S$.** ☜

You can also run this definition the other way:

☞ **If you know $T \in \wp(S)$, you can conclude $T \subseteq S$.** ☜

Set A

$$P(A) = \{ S \mid S \subseteq A \}$$

$$A = \{a, b\}$$

$$P(A) = \{ \phi, \{a\}, \{b\}, \{a, b\} \}$$

Set A, $|A| = n$

$\#$subsets of $A = 2^n$

$|P(A)| = 2^n$

① Every element in $P(A)$ is a subset of A.

$$P(A) = \{ X \mid X \subseteq A \}$$

If $X \in P(A)$ then $X \subseteq A$.

If $X \subseteq A$ then $X \in P(A)$.

$X \subseteq A$ iff $X \in P(A)$. ✓

(2) $\phi \in \text{Powerset}(A)$; $\forall$ sets A.

$\boxed{\phi \subseteq A}$ $\forall$ set A.

for Any set A, $A \subseteq A$.

$\boxed{A \in P(A)}$ ✓

③ Every element in $P(A)$ is a set.

$$P(A) = \{ x \mid x \subseteq A \}$$

If $x \in P(A)$ then $x = \{ - - - - - - \}$

$x \neq 1$, $x = \{1\}$, $x = a$, $x = \{a\}$

(4) $X \in P(A)$

$X \subseteq A$

$X = \{ \dots \}$

To prove $X \in P(A)$ then show that $X \subseteq A$

☞ To prove that $T \in \wp(S)$, prove that $T \subseteq S$. ☜

☞ If you know $T \in \wp(S)$, you can conclude $T \subseteq S$. ☜

Q :

For any sets A and B, Show that we have $A \cap B = A$ if and only if $A \in \wp(B)$.

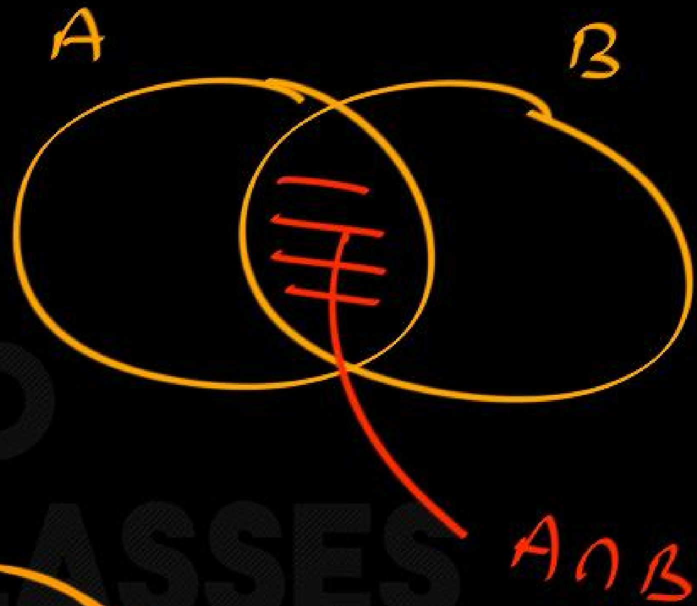
To prove:

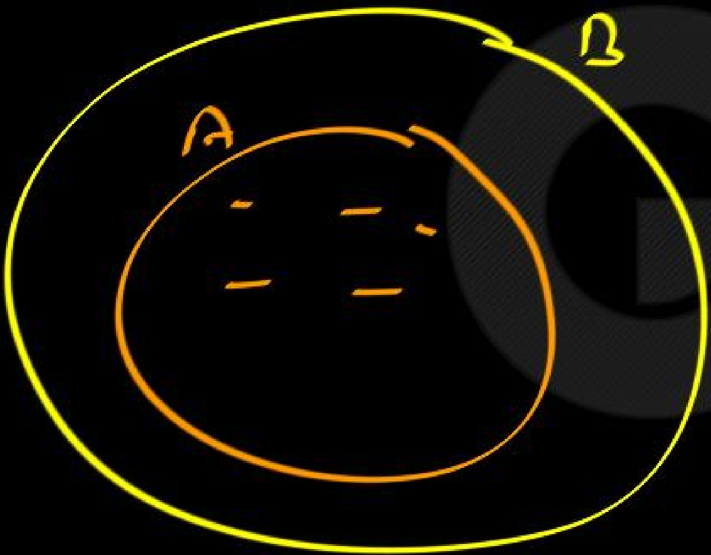
$A \cap B = A$ iff $A \in P(B)$

① $A \cap B = A \longrightarrow \boxed{A \in P(B)} \equiv A \subseteq B$

$A \cap B = A \longrightarrow A \subseteq B$

$A \cap B = A$



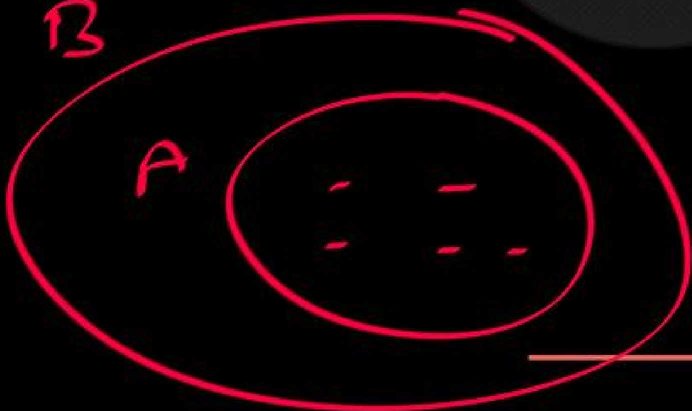


$A \subseteq B \longrightarrow A \in P(B)$

(2) $\boxed{A \in P(B)}$ then $A \cap B = A$

means

$\boxed{A \subseteq B} \rightarrow \boxed{A \cap B = A}$



1. Suppose A and B are sets. Show that $\mathscr{P}(A) \cap \mathscr{P}(B) = \mathscr{P}(A \cap B)$.

2. Show that if A and B are sets, and $\mathscr{P}(A) \subseteq \mathscr{P}(B)$, then $A \subseteq B$.