Graph Theory :

Lecture 1 :

Introduction, Basic Definitions

Website : https://www.goclasses.in/

GATE:

"Graph Theory" — $\geq$ 3-4 Questions

$\geq$ 4-6 marks

- Algo
- DSA
- OS
- CD
- ...

2022 GATE: 4-5 Questions

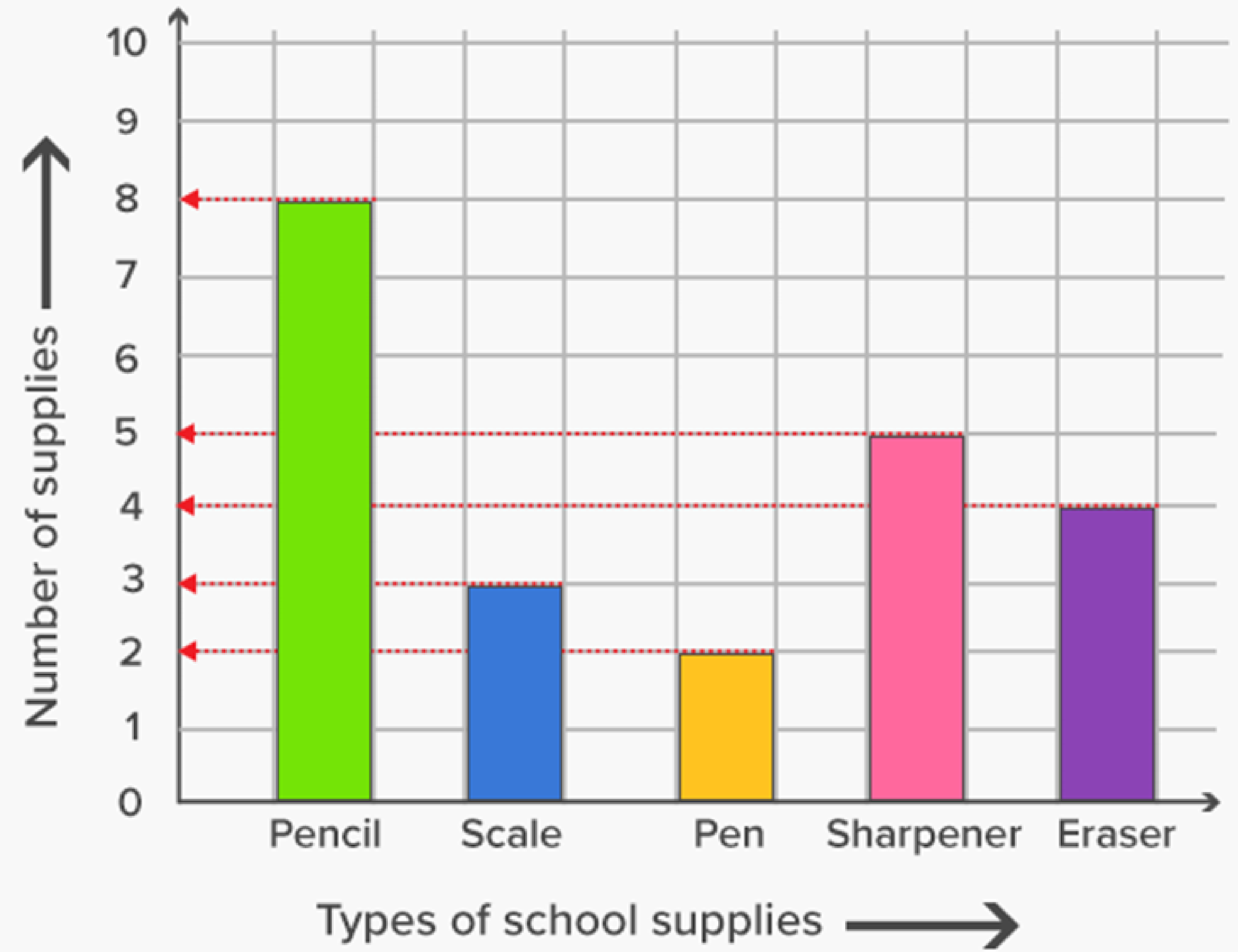
Graph $\rightarrow$ 6-7 marks

Graph Theory :

Topic 1 :

What is Graph in Graph Theory?

Website : https://www.goclasses.in/


10
9
8
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4
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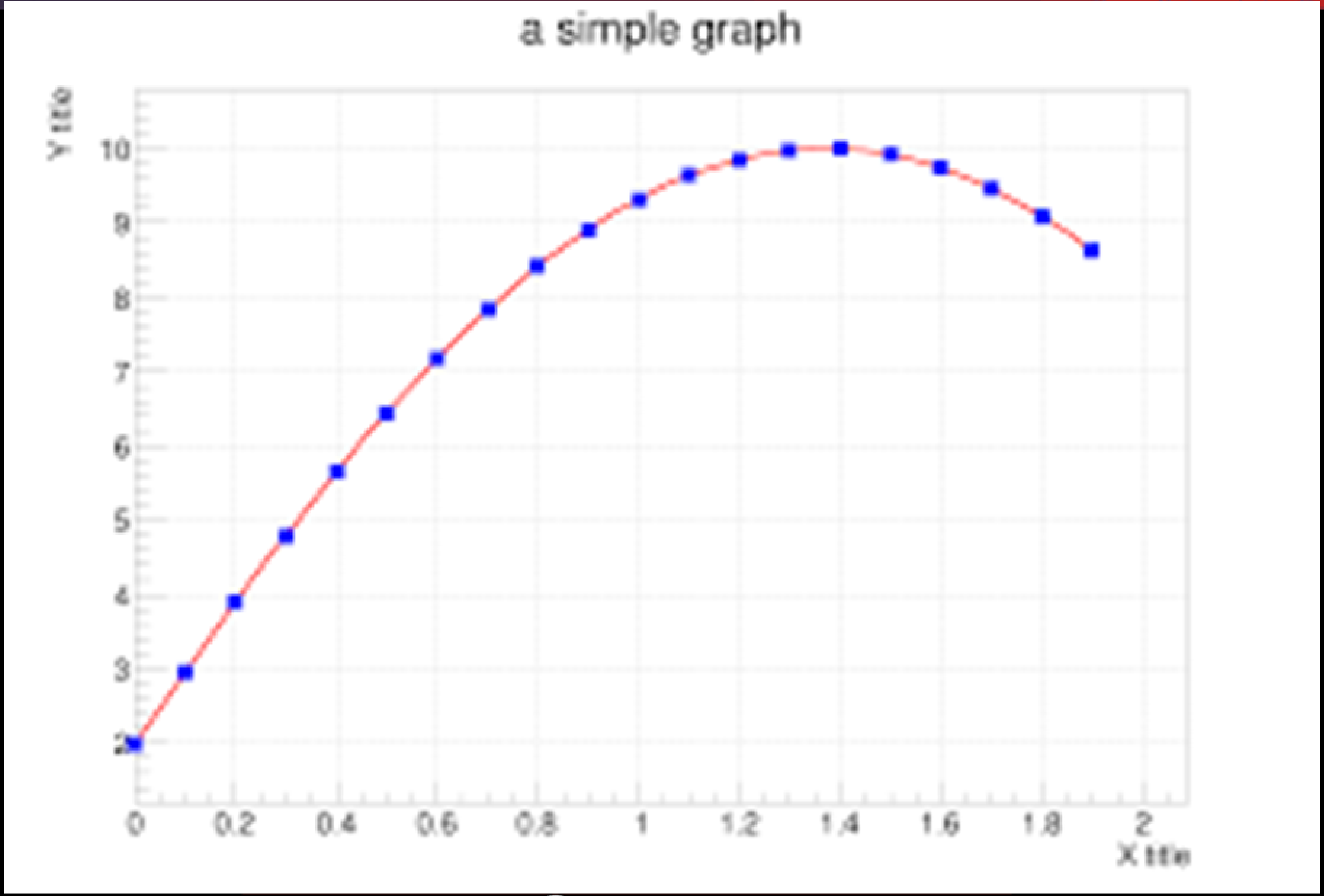
1
0
Number of supplies
Pencil
Scale
Pen
Sharpener
Eraser
Types of school supplies


a simple graph
Y title
10
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8
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4
3
2
0
0.2
0.4
0.6
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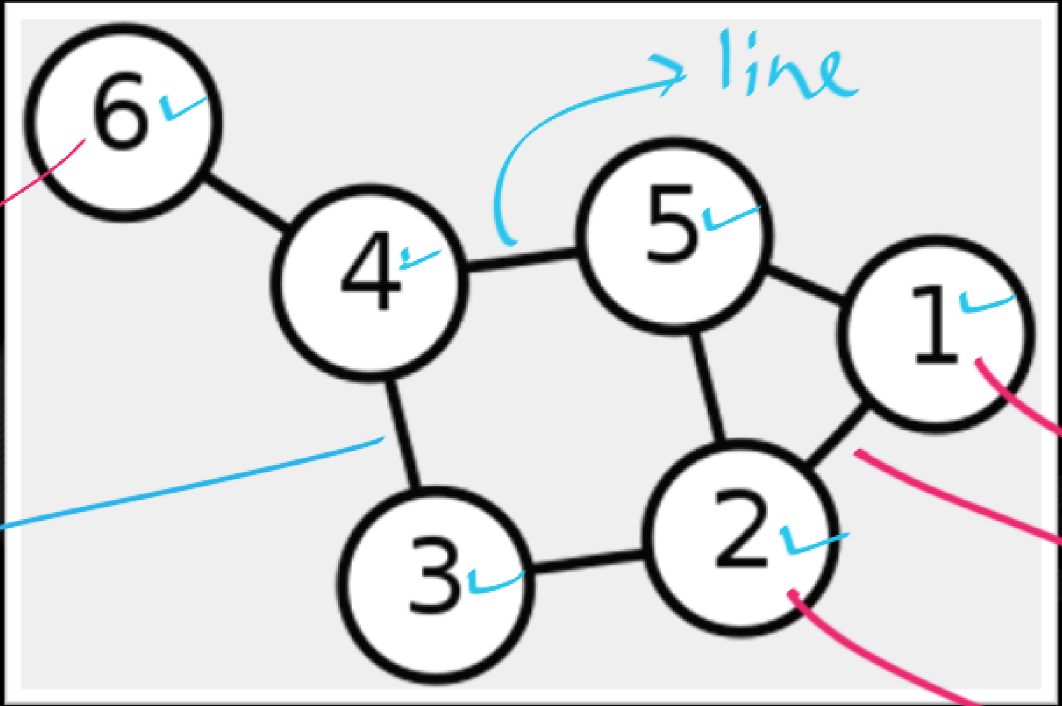
1
1.2
1.4
1.6
1.8
2
X title

Plot


"We don't do that here"

Graph:



dots/node

Edge/line

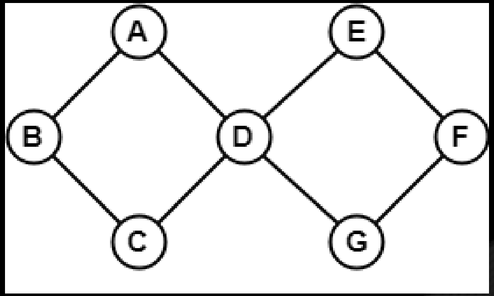
node

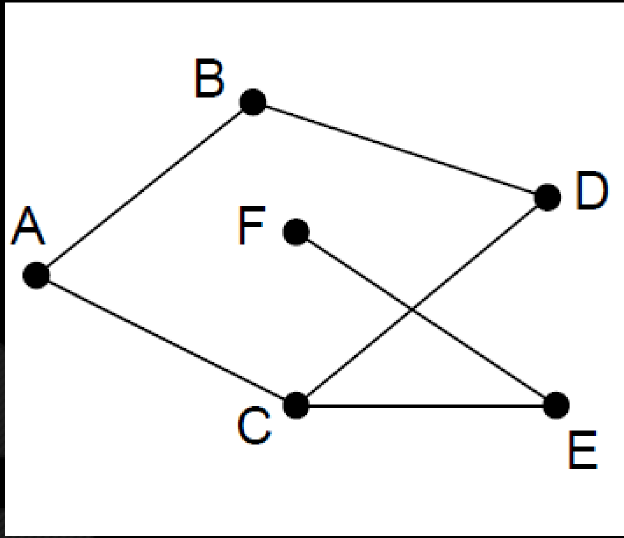
line

node

#nodes = 6

#Edges = 7


A
E
B
D
F
C
G


B
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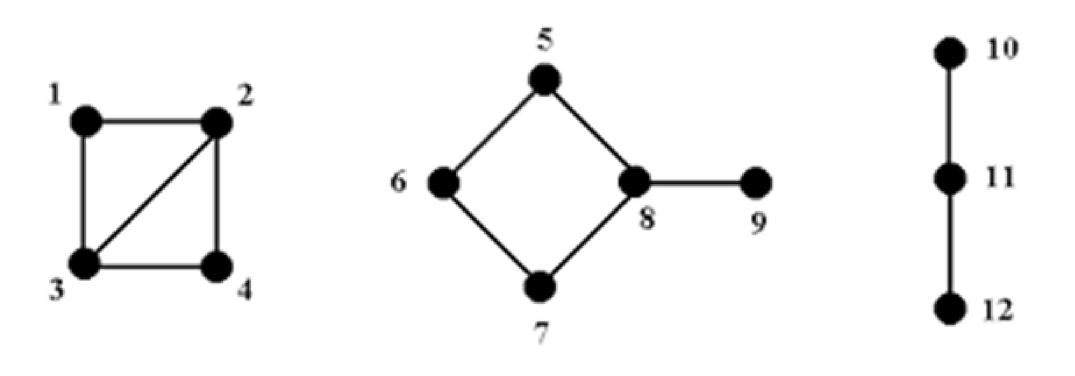
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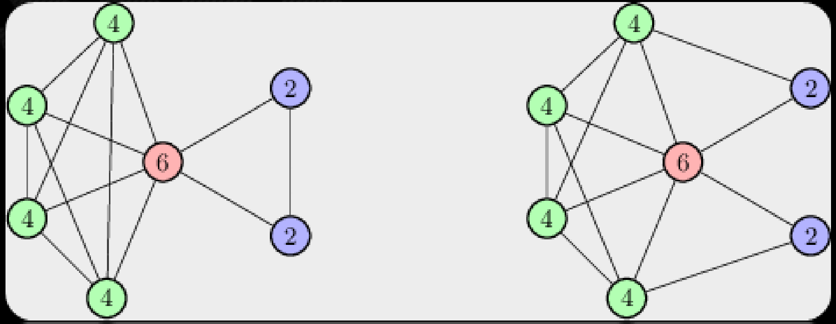
8
9
10
11
12


4
4
2
6
4
2
4
4
4
2
6
4
2
4

What is Graph ?

Simplest Definition :

Some dots(nodes) and some lines(edges) connecting those dots.

WHY Study Dots, Lines(Graphs) ?

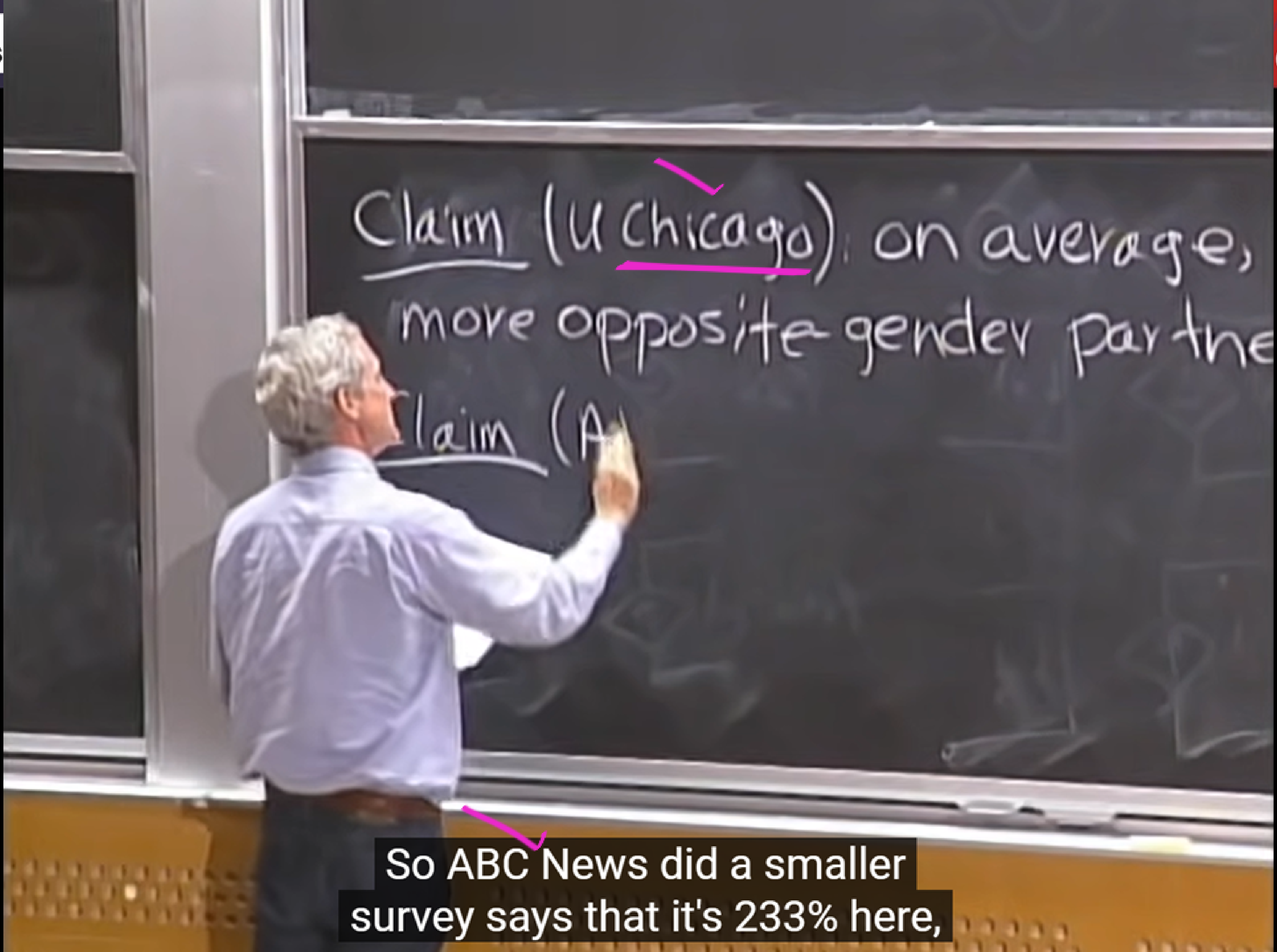
WHY Study Dots, Lines ?

1. Because it is in the Syllabus....Duh!!

Do Your Assumptions Reflect Reality?

Example 1:

- On average, who has more opposite-gender partners, men or women?


Claim (U Chicago): on average,
more opposite gender partne
laim (A
So ABC News did a smaller
survey says that it's 233% here,

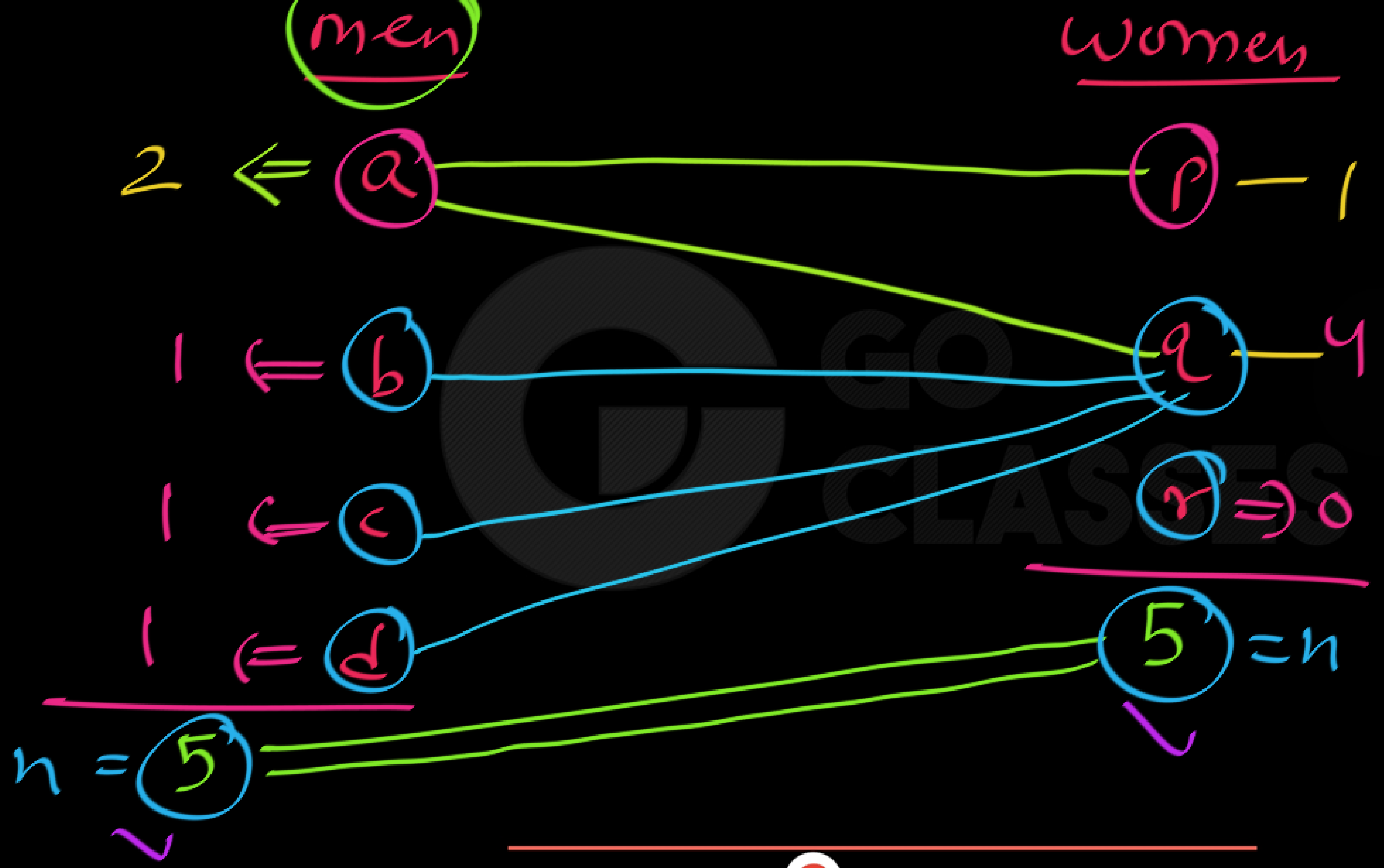
men		women	
2 ⇐ a		p	— 1
b		q	— 1
c		r	
d			
2		2	

On Average

women $\vee$ men

Average #partners men have $= \frac{2}{4}$

" # " women " $= \frac{2}{3}$



Avg # partners for women $= \frac{n}{3}$

for men $\frac{n}{4}$

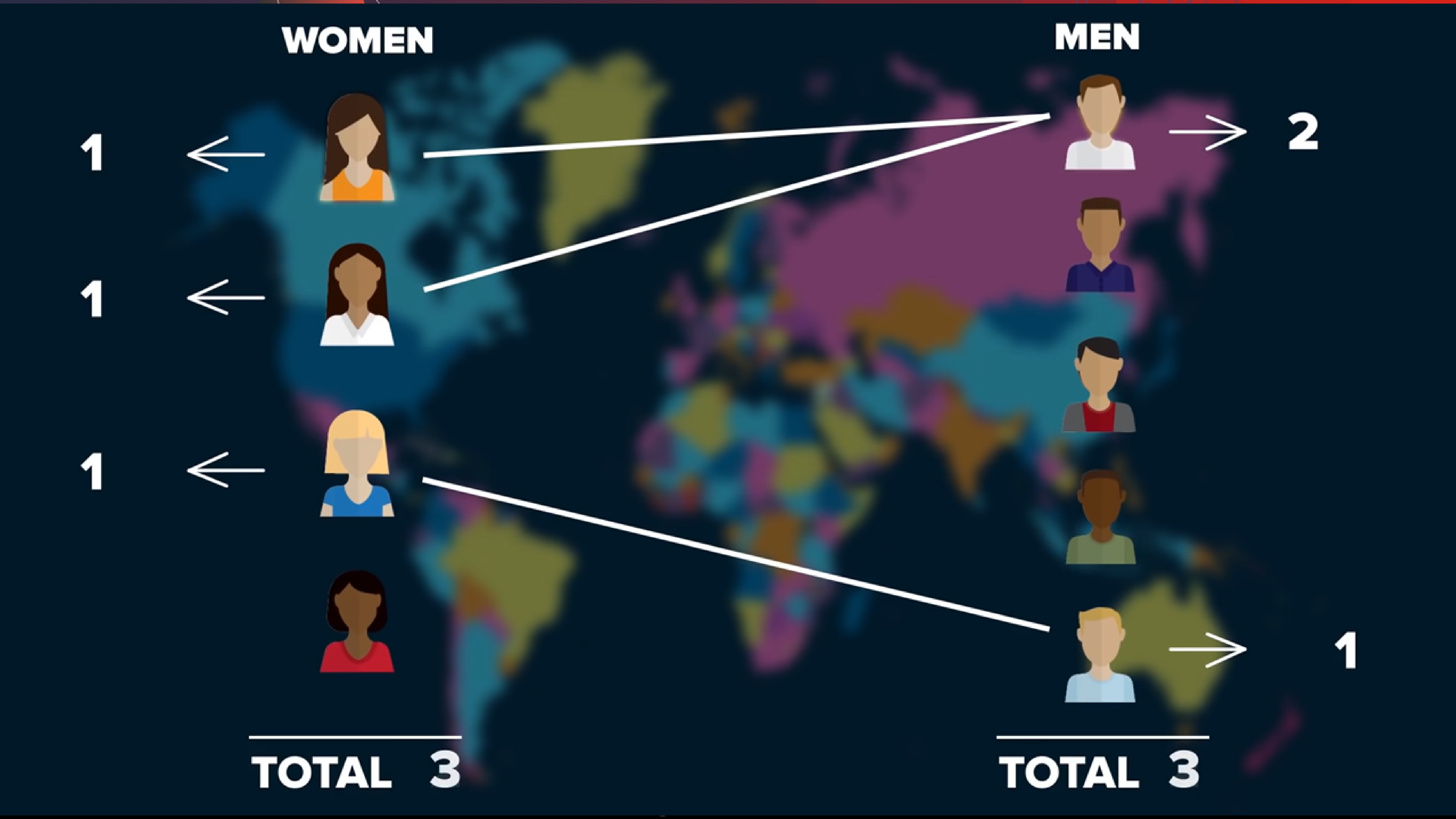
India:

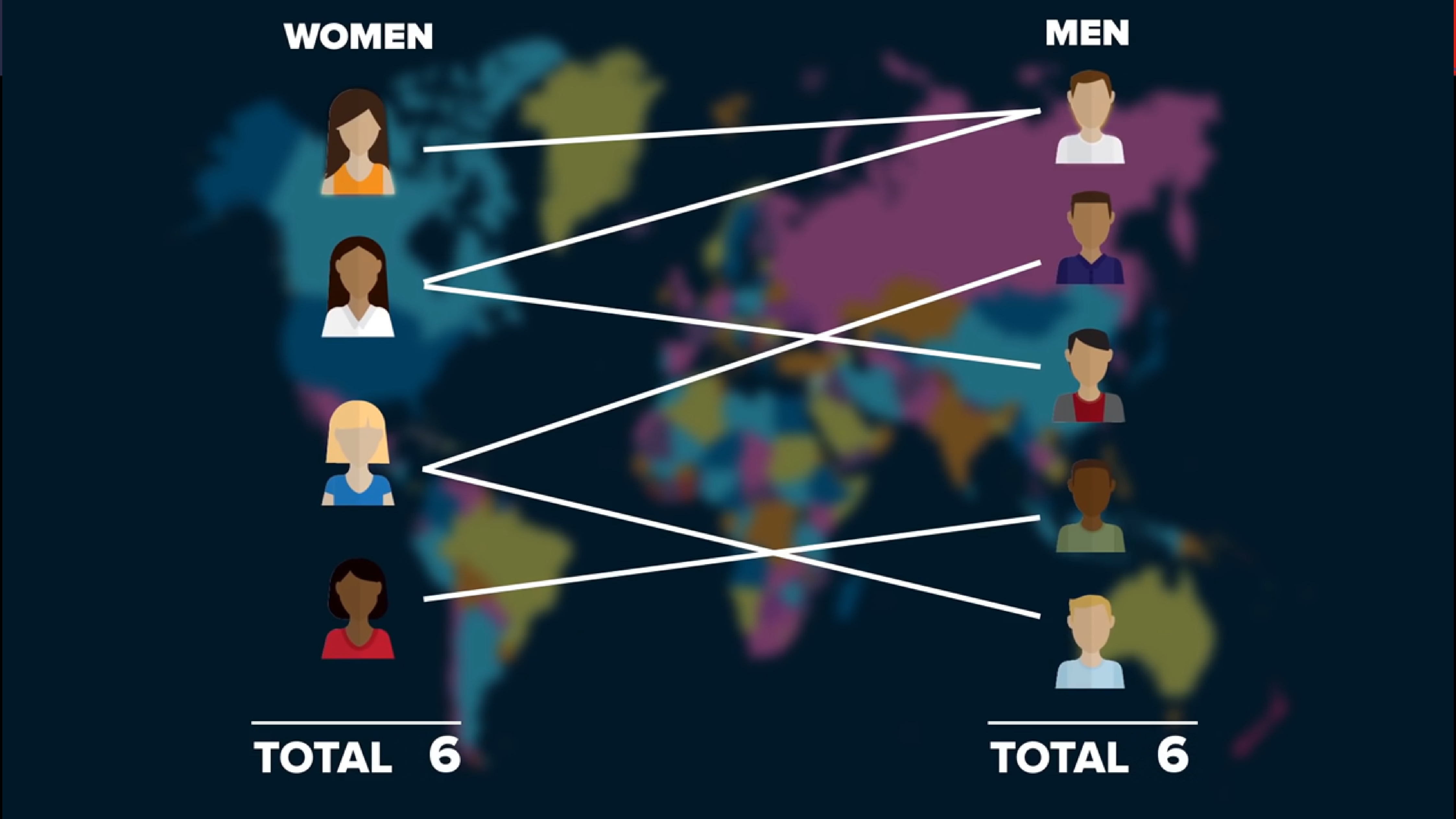
$\underbrace{\text{Population of women}}_{w}$ < $\underbrace{\text{Population of men}}_{m}$

$w < m$

Ratio of Avg. Partner for women to men:

$$\frac{\cancel{n}/w}{\cancel{n}/m} = \frac{m}{w} > 1 \ldots$$


WOMEN
MEN
1
1
1
2
1
TOTAL 3
TOTAL 3


WOMEN
MEN
TOTAL 6
TOTAL 6

WOMEN

TOTAL

$$\frac{6}{4} = 1.5$$

MEN

TOTAL

$$\frac{6}{5} = 1.2$$

$$\frac{1.5}{1.2} = 1.25$$

**on average*

WOMEN

TOTAL

$$\frac{6}{4} = 1.5$$

MEN

TOTAL

$$\frac{6}{5} = 1.2$$

$$\frac{1.5}{1.2} = \boxed{1.25 \quad \frac{5}{4}}$$

= Ratio of Men to Women

in America

- On average, who has more opposite-gender partners, men or women?
- The University of Chicago interviewed a "random sample" of 2500 people over several years to try to get an answer to this question. Their study, published in 1994, and entitled The Social Organization of Sexuality found that on average men have 74% more opposite gender partners than women.
- ABC News "Primetime Live" found through a survey in 2004 that the average man has 20 partners, and women 6, over their lifetimes, with a claimed 2.5% margin of error. (That's 233%)
- What do you think about this? (mathematically)

Do Your Assumptions Reflect Reality?

Answer: NO

Proof: By Graph Theory

People → Nodes / Dots

modelled / Representes using

Actual Entity

Abstraction

Relationship → Lines / Edges

modelled / Representes using

Actual Entity

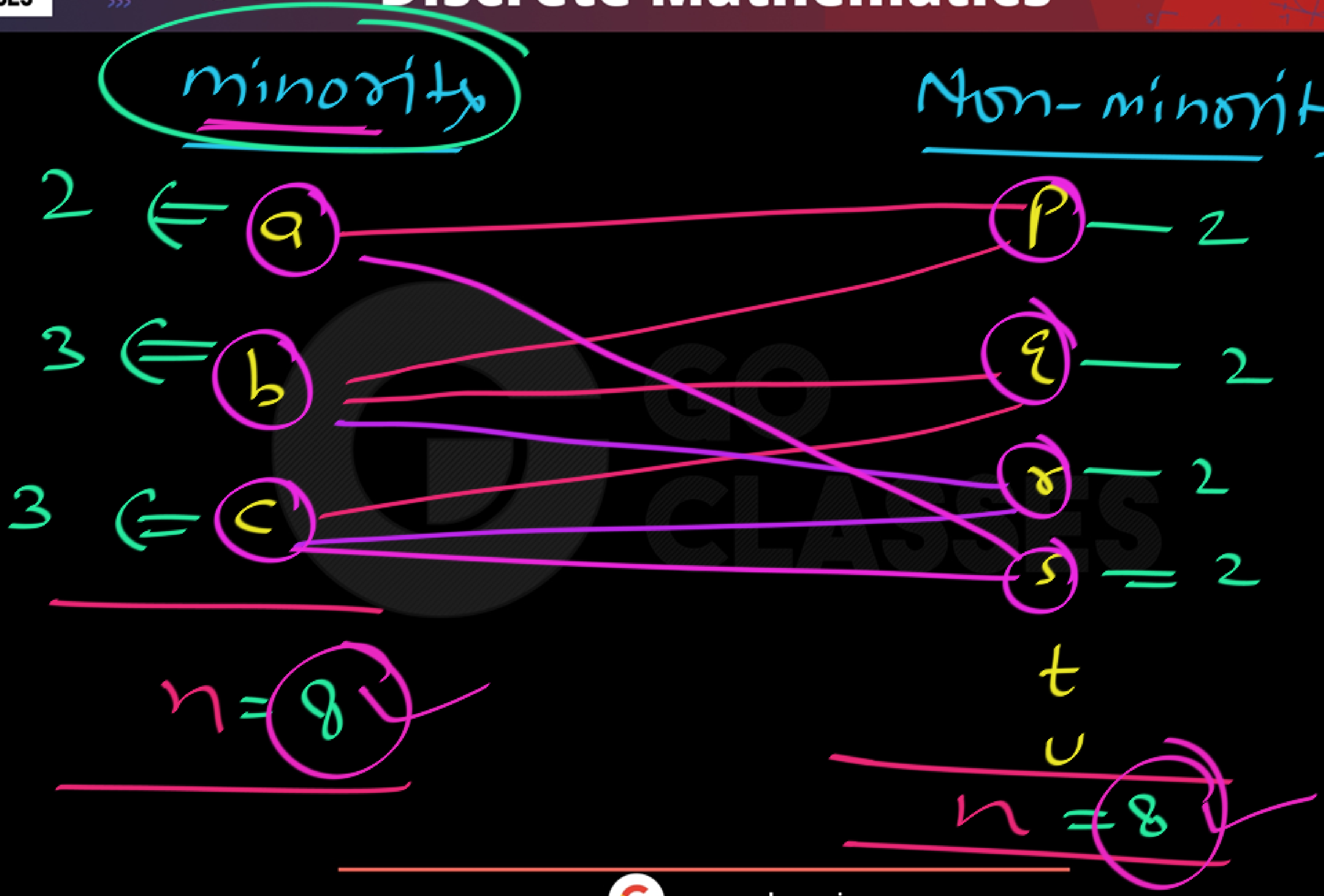
Connecting the Nodes.

Abstraction

How easy it is to manipulate public perception??

Boston Study :

"On average, minority students study with non minority students more than the other way around"



minority
Non-minority
2
a
3
b
3
c
p
q
r
s
t
u
2
2
2
2
n = 8
n = 8

murder:

Assume
2018

↓

Brutal 1000 per day

↓

TV

2022 → Actually Better

↓

1 per day

↓

TV

Perception

⇓

2018 was good.

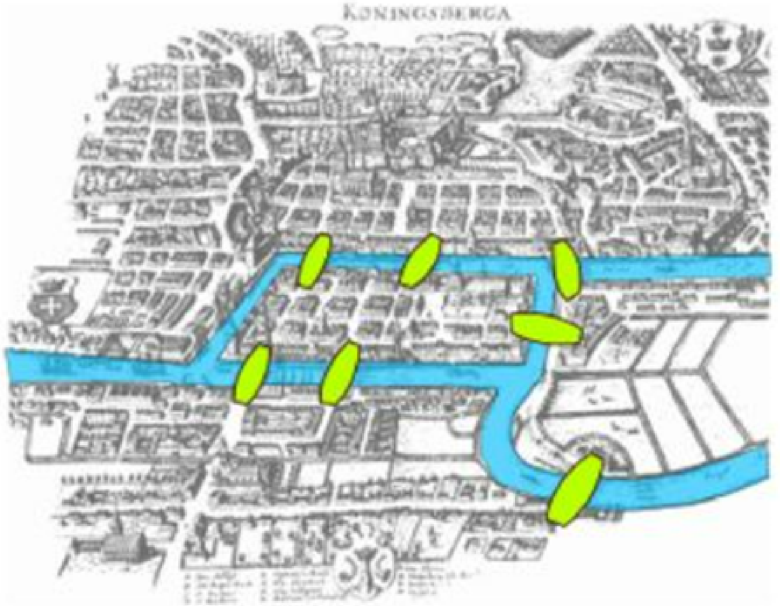
Do Your Assumptions Reflect Reality?

Answer: NO

Proof: By Graph Theory

Seven Bridges of Königsberg

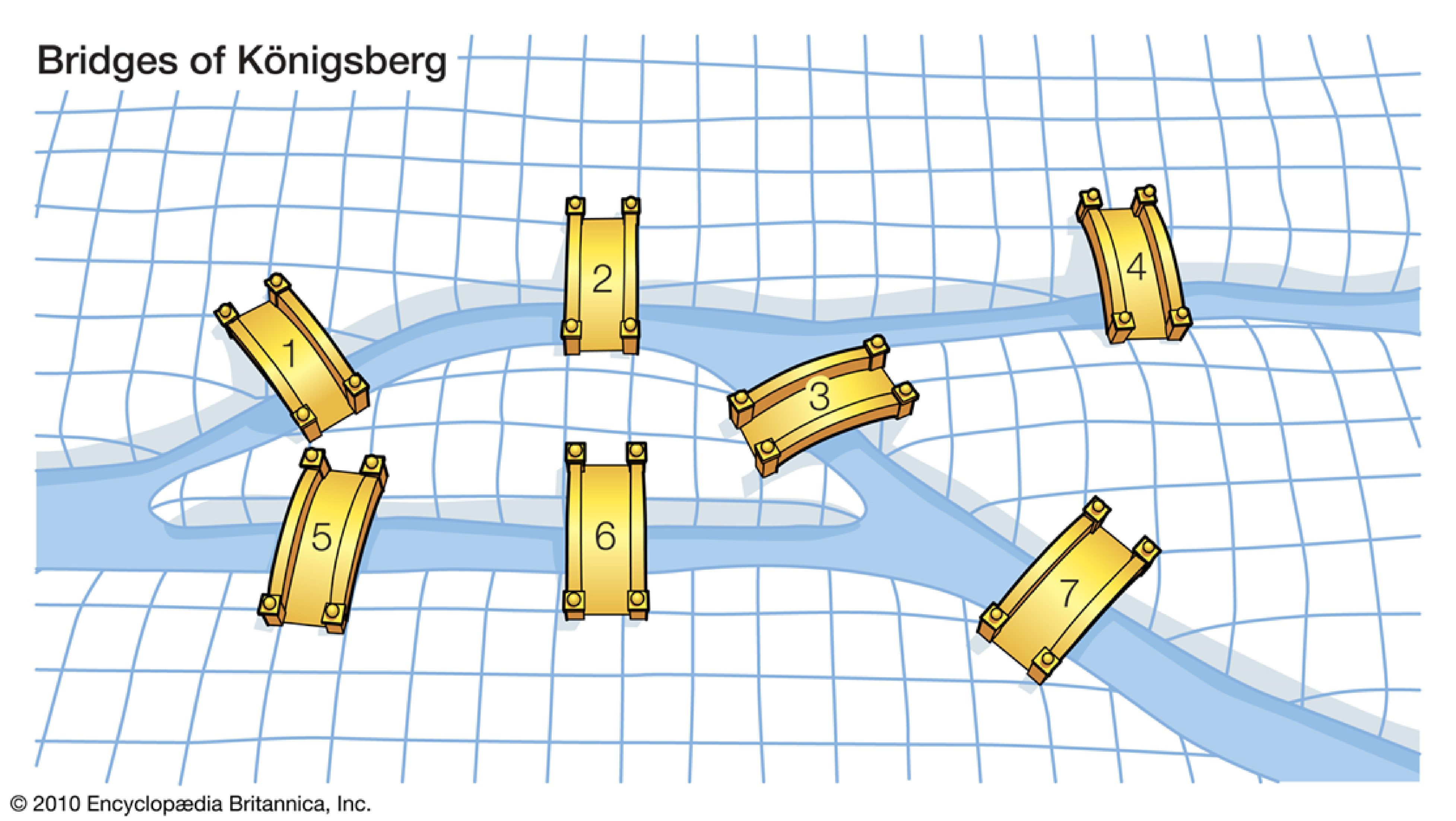
- Leonhard Euler in 1736 laid the foundations of graph theory.
- The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges.
- He wished to devise a walk through the city that would cross each bridge once and only once.



10.1 Euler paths and circuits

10.1.1 The Konisberg Bridge Problem

Konisberg was a town in Prussia, divided in four land regions by the river Pregel. The regions were connected with seven bridges as shown in figure 10.1. The problem is to find a tour through the town that crosses each bridge exactly once. Leonhard Euler gave a formal solution for the problem and -as it is believed- established the graph theory field in mathematics.


Bridges of Königsberg
1
2
3
4
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7
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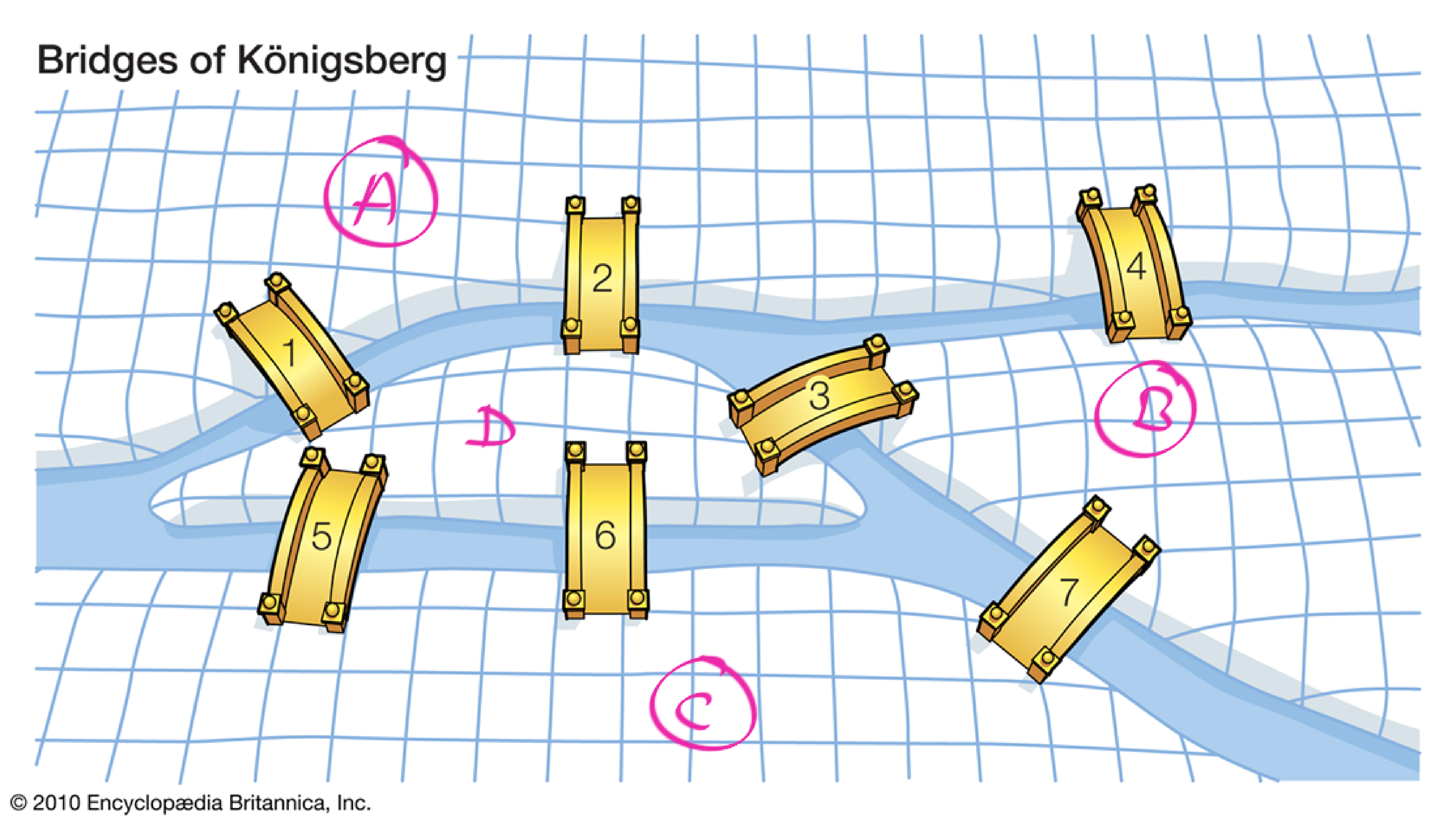
Bridges of Königsberg
A
1
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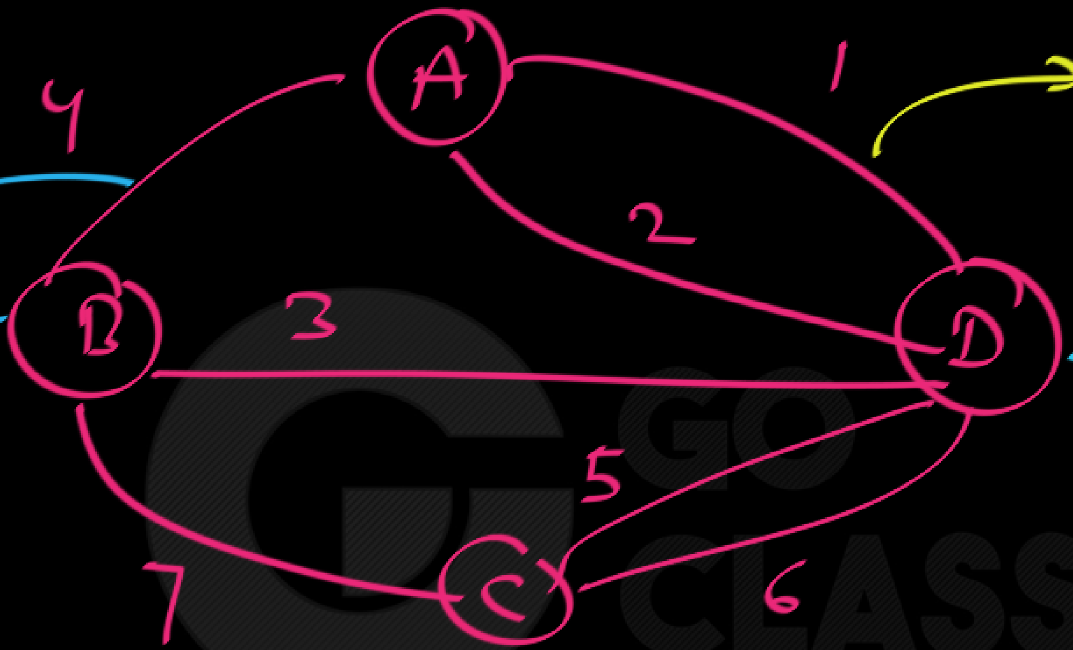
Euler :

Edge

node

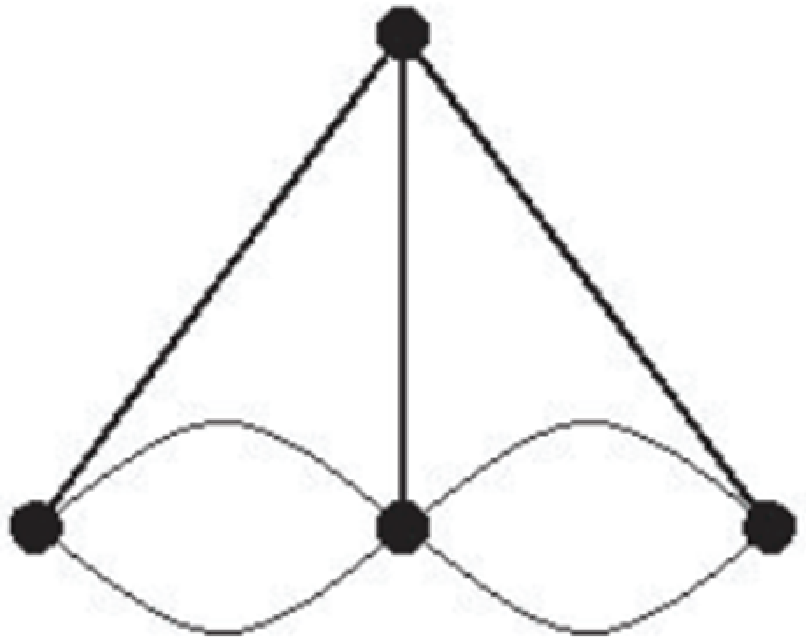


Abstraction of Bridge

Abstraction of Land

$\boxed{\text{Abstract structure}} \equiv$ Graph

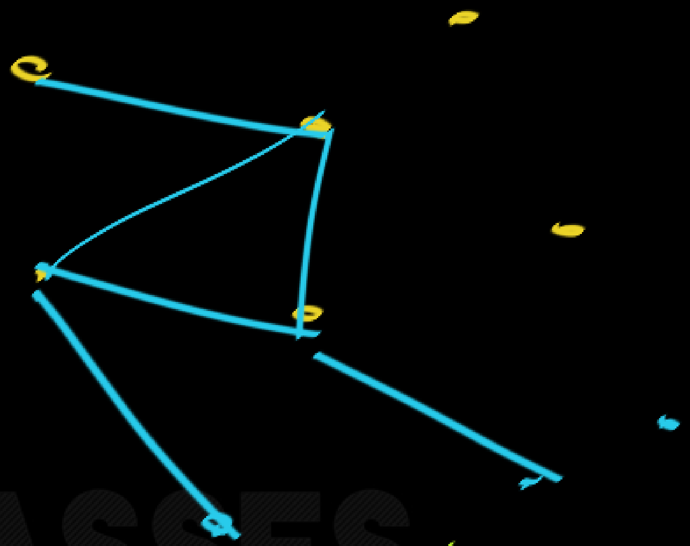
Konisberg bridge and the graph induced



(b) respective graph

Graph Theory →

1. Google maps:



2. CS

maths

Internet:

So, What is Graph Theory?

Theory

Graph is a field of mathematics in which we use a structure of Dots and Lines, called Graph, to model real world problems and then solve them.

WHY Study Dots, Lines ?

1. Because it is in the Syllabus....Duh!!

2. Many(Many many many) real life problems can be Modelled(represented) using Graphs(Dots and Lines) and then solved. So, Graphs are Tools to solve problems.

3. Extremely important in computer science and mathematics. Example: Google Maps, Facebook and many more.
4. Numerous important applications, in many different fields.
5. Modelling the concept of binary relation.

Here we introduce basic mathematical view on graphs.

Graph — Abstract Structure

Graph — Model of Real world

|||

Representation

Nodes
Edges

Abstractions of Real world Entities

What is Graph (Formal Definition) ?

In mathematics, and more specifically in graph theory,
A graph is a structure consisting of a set of objects in which some pairs of the objects are in some sense "related".

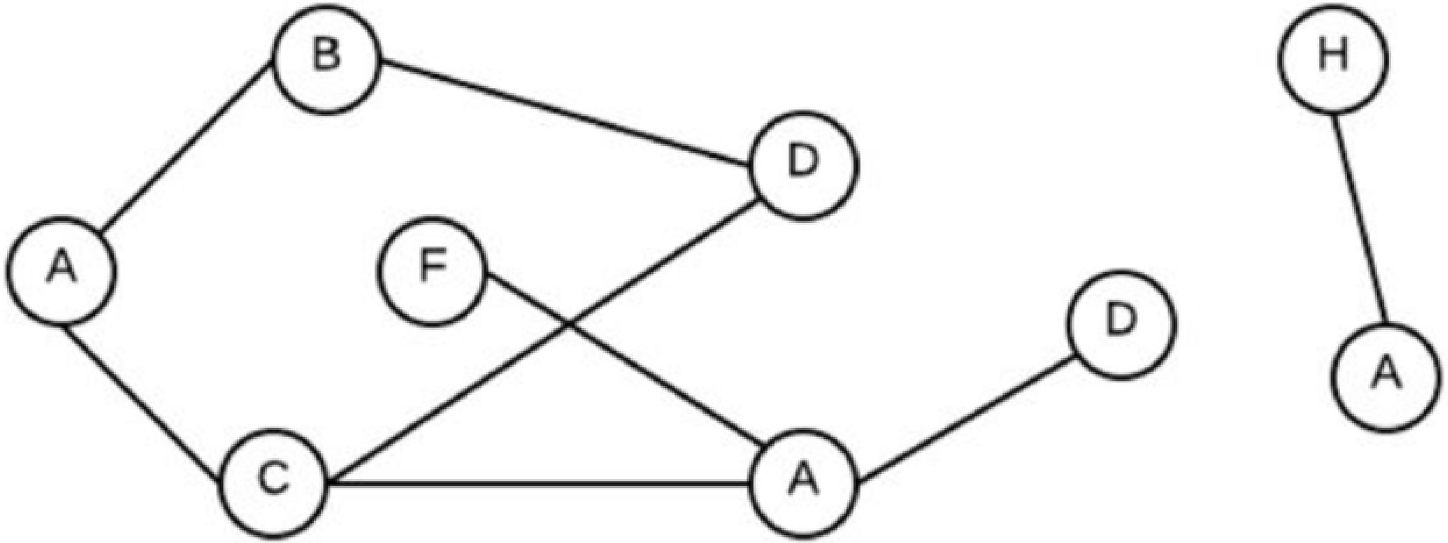
The objects correspond to mathematical abstractions called vertices(also called nodes or points) and each of the related pairs of vertices is called an edge (also called link or line or branch).

Pictures like the dot and line drawing are called ***graphs***. Graphs are made up of a collection of dots called ***vertices*** and lines connecting those dots called ***edges***.

Typically, a graph is depicted in diagrammatic form as a set of dots or circles for the vertices, joined by lines or curves for the edges.

Introduction

- Informally, a *graph* is a bunch of dots connected by lines



Dots / Nodes / Points / Vertices / Vertex

Lines / Edges / Branch

Graph $G(V, E)$

V → Set of Nodes

E → Set of Edges

NOTE:

Before we start further, NOTE that in Graph Theory many definitions are Author-Dependent like Walk, Path, Train, Euler Graph, Empty graph, Null graph, Empty graph as Bipartite graph and so on..

SO, Please ONLY follow our Definitions as those are taken from the Most Referred Standard resources in IIT/IISc courses.

IISc : MTech

Pool A — Theory Pool Algo, Graph

Pool B — System — OS, CD

Pool C — ML, Game theory, DS

Graph Theory

1 Introduction

A graph $G = (V, E)$ consists of two sets V and E. The elements of V are called the vertices and the elements of E the edges of G. Each edge is a pair of vertices. For instance, the sets $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$ define a graph with 5 vertices and 4 edges.

Graphs have natural visual representations in which each vertex is represented by a point and each edge by a line connecting two points.

G :

Figure 1: Graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}$

G: $1 \xrightarrow[e_1]{} 2 \xrightarrow[e_2]{} 3$

$2 \xrightarrow[e_2]{} 3$

2 → End point of e_2

3 → End point of e_2

$G(V, E)$

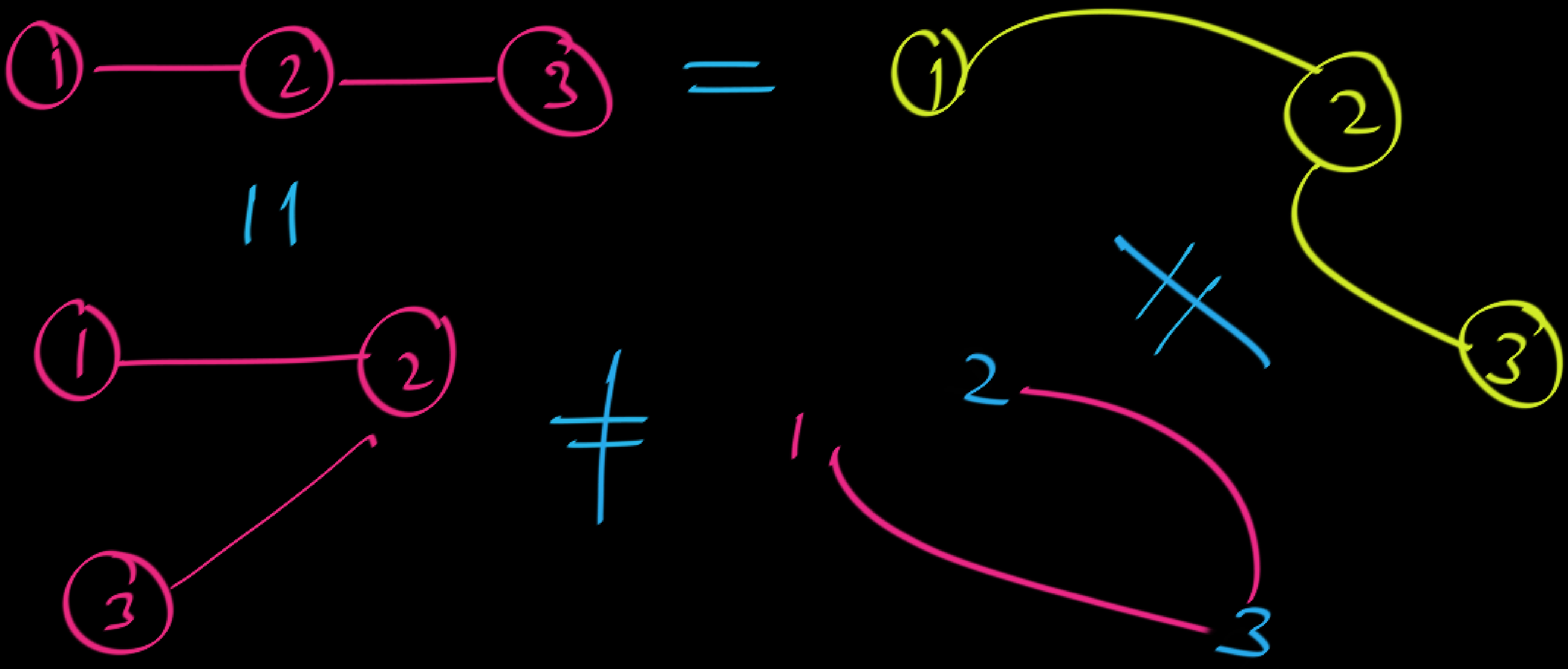
$V = \{1, 2, 3\}$

$|V| = 3$

$|E| = 2$

$E = \{ \underbrace{\{1, 2\}}_{e_1}, \underbrace{\{2, 3\}}_{e_2} \}$

End points of $e_1 = 1, 2$



<u>Drawing</u>, Really, Doesnot matter.

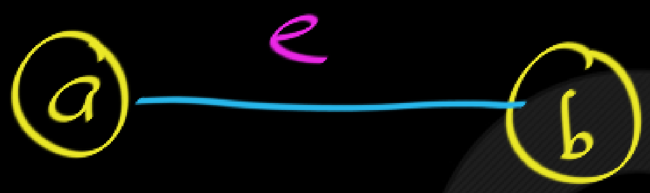
Graph Theory :

Next Topic :

Types of Edges

Website : https://www.goclasses.in/

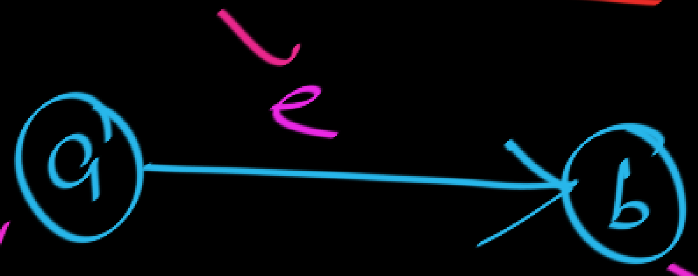
① Undirected Edge



$e = \underbrace{\{a,b\}}_{\text{set}}$

End points of e $= \{a,b\}$

Directed Edge



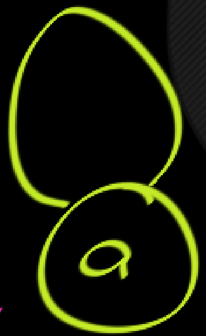
Source

Terminal

$e = \underbrace{(a,b)}_{\text{ordered Pair}}$

2 self loop

Loop → Edge whose both End points Same.



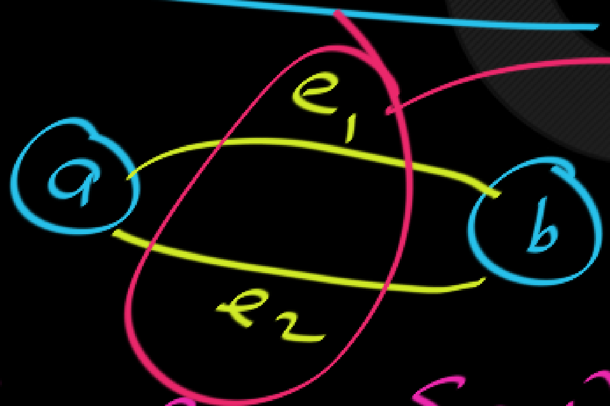
self loop but undirected

→ Directed self loop

3 multiEdges / Parallel Edges :

Basically, Exactly same Edges.

Undirected



Parallel Edges

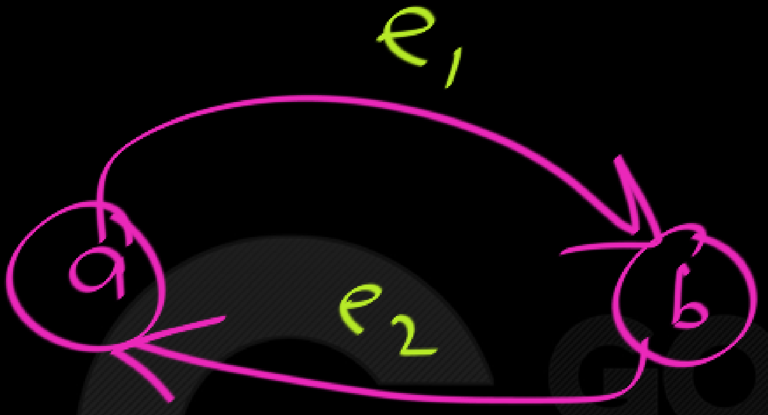
$e_1 = e_2 = \{a, b\}$

Directed

multi Edges

$e_1 = e_2 = (a, b)$

NOTE:



e_1, e_2 are **NOT** multi-edges.

$e_1 \neq e_2$

(a,b)

(b,a)

Definitions

Definition

A **graph** $G = (V, E)$ consists of a set V of **vertices** (also called **nodes**) and a set E of **edges**.

Definition

If an edge connects to a vertex we say the edge is **incident** to the vertex and say the vertex is an **endpoint** of the edge.

Definition

If an edge has only one endpoint then it is called a **loop edge**.

Definition

If two or more edges have the same endpoints then they are called **multiple** or **parallel** edges.

Graph Theory :

Next Topic :

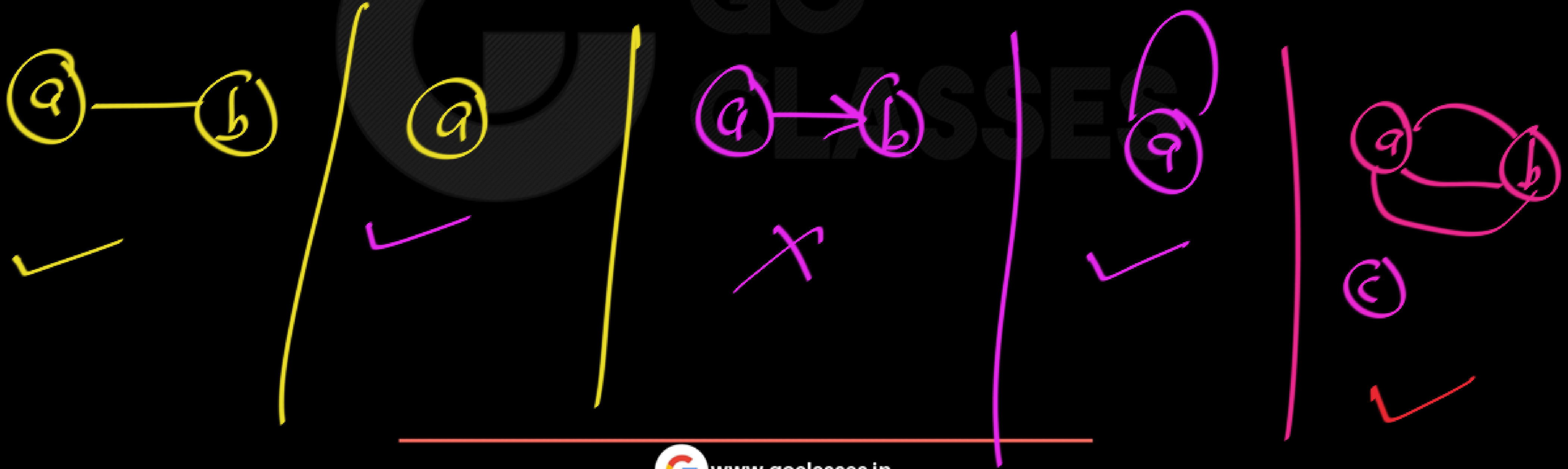
Types of Graphs

(Based on Edge Types)

Website : https://www.goclasses.in/

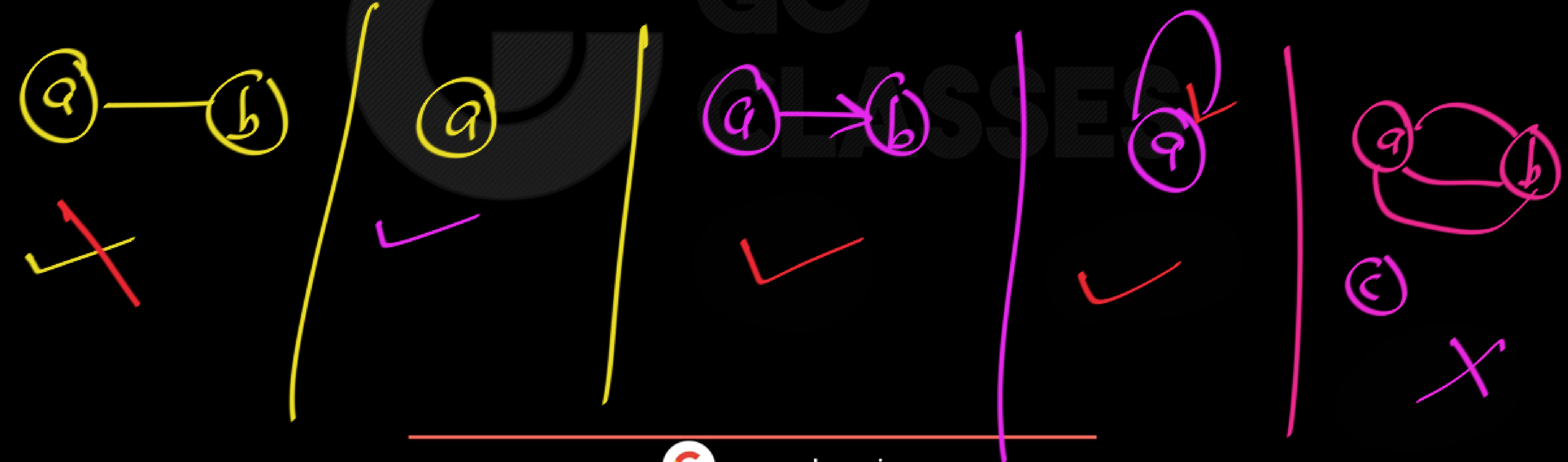
① Undirected Graph :

Every Edge is undirected.



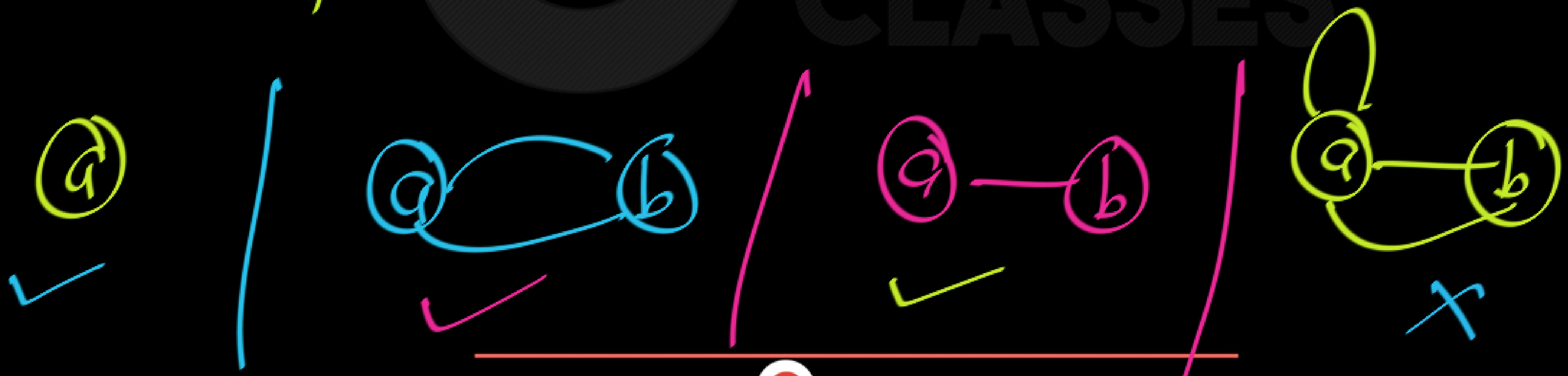
(2) directed Graph :

Every Edge is directed.



③ multiGraph: → multiedges Allowed ✓

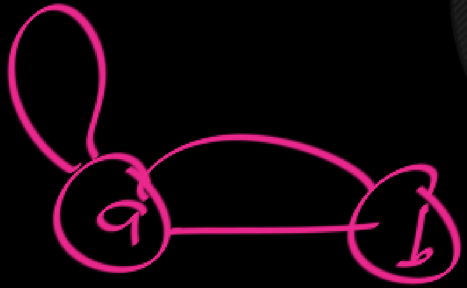
- Undirected Graph
- No Self loops



(4) PsudoGraph : — Self loops Allowed

— multiEdges "

Any "Undirected" Graph



(5) Directed multiGraph :

Author Dependent

Any Directed Graph

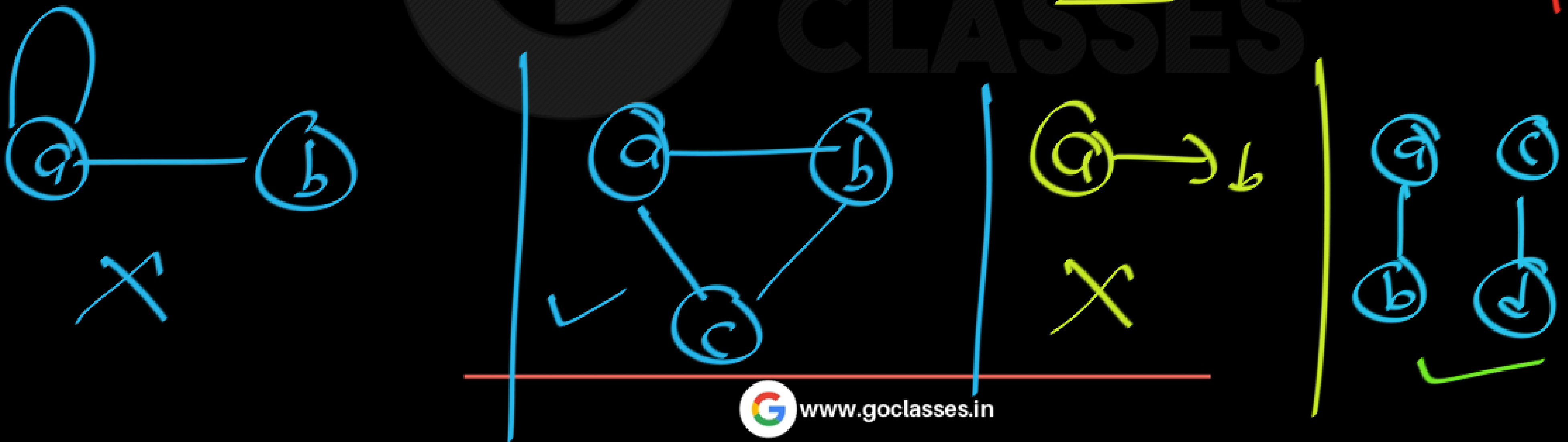
Self loops ✓

multiEdges ✓

6 ** "Simple" Graph:

Undirected ✓

No self loops, No multiEdges



7 Simple Directed Graph;

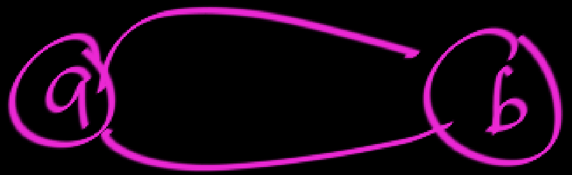
Directed Graph;

No self loops; No multiEdges

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Q: multigraph



multigraph

⇓

undirected ✓

G(V, E)

set of vertices

multiset of Edges

$G = (V, E)$

$\{a, b\}$

$\{\{a,b\}, \{a,b\}\}$

multiset ✓

Directed Simple Graph

ϕ :

$G = (V, E)$ → Set of Edges But Every Edge is Ordered Pair.

V: $\{a, b\}$

E: $\{(a, b), (b, a)\}$

for Convenience

Undirected Graph

$e = \{a, b\}$

$= ab$ for Convenience

$ab = \{a, b\}$

$ba = \{a, b\}$

Directed Graph

$e = ab = (a, b)$

$ba = (b, a)$

NOTE: 99% time we will only study "Simple Graphs"

- Undirected Graph
- No self loops
- No multi Edges

By altering the definition, we can obtain different types of graphs. For instance,

- by replacing the set E with a set of *ordered pairs* of vertices, we obtain a *directed graph* or *digraph*, also known as *oriented graph* or *orgraph*. Each edge of a directed graph has a specific orientation indicated in the diagram representation by an arrow (see Figure 2). Observe that in general two vertices i and j of an oriented graph can be connected by *two* edges directed opposite to each other, i.e. (i, j) and (j, i).
- by allowing E to contain both directed and undirected edges, we obtain a *mixed graph*.
- by allowing repeated elements in the set of edges, i.e. by replacing E with a multiset, we obtain a *multigraph*.
- by allowing edges to connect a vertex to itself (a loop), we obtain *pseudographs*.

- by allowing V and E to be infinite sets, we obtain *infinite graphs*.

Figure 2: An oriented graph $G = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{(1, 2), (3, 2), (3, 4), (4, 5)\}$

Definition 1. A simple graph is a finite undirected graph without loops and multiple edges.

All graphs in these notes are simple, unless stated otherwise.

2 Terminology, notation and introductory results

- The sets of vertices and edges of a graph G will be denoted $V(G)$ and $E(G)$, respectively.
- For notational convenience, instead of representing an edge by $\{a, b\}$ we shall denote it by ab.

13. The **intersection graph** of a collection of sets $A_1, A_2, \ldots, A_n$ is the graph that has a vertex for each of these sets and has an edge connecting the vertices representing two sets if these sets have a nonempty intersection. Construct the intersection graph of these collections of sets.

Assume No self loops

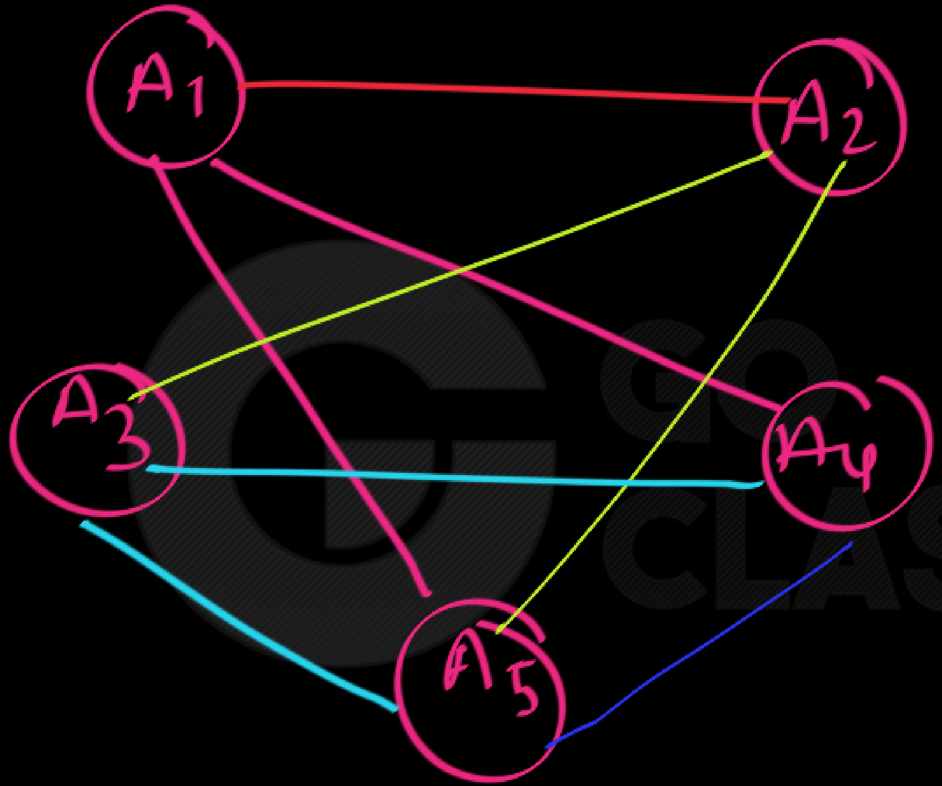
a) $A_1 = \{0, 2, 4, 6, 8\}$, $A_2 = \{0, 1, 2, 3, 4\}$, $A_3 = \{1, 3, 5, 7, 9\}$, $A_4 = \{5, 6, 7, 8, 9\}$, $A_5 = \{0, 1, 8, 9\}$

$\boxed{A \cap B = B \cap A}$

HW

b) $A_1 = \{\ldots, -4, -3, -2, -1, 0\}$, $A_2 = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $A_3 = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, $A_4 = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$, $A_5 = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

9



Graph Theory :

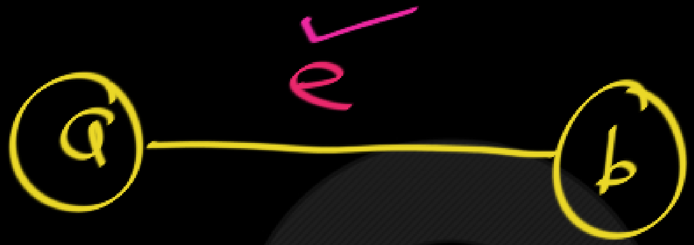
Next Topic :

Adjacency, Degree of a vertex

Degree Summation Formula

Website : https://www.goclasses.in/

Simple Graph:



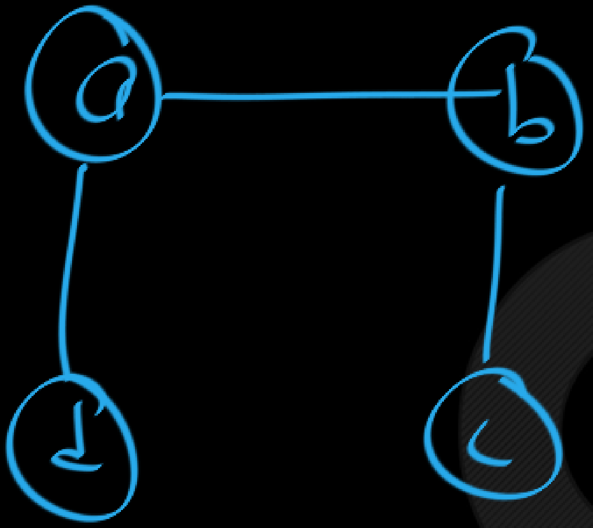
$\{a, b\} \in E$

So, a, b are Adjacent / Neighbours

Endpoints of e = a, b

"e" is falling on / Incident on a, b.

Neighbourhood of a vertex = set of Neighbours



a, b — Adjacent

a, c — Non "

a, d — "

d, c — Non "

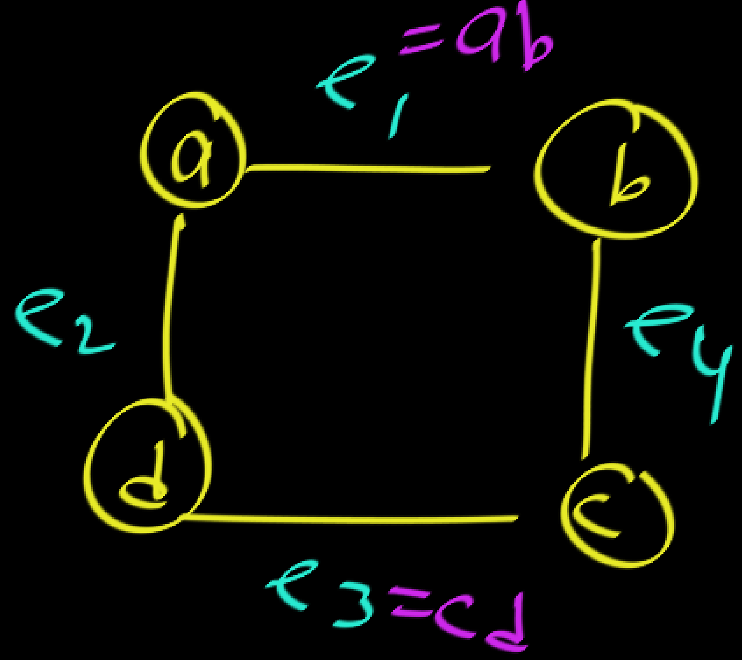
b, d — " "

c, b — Adjacent / Neighbours

$N(b) = \{a, c\}$

$N(c) = \{b\}$

Adjacent Edges :



$e_2 = ad$; $e_4 = bc$

e_1, e_4 are Adjacent

have some Common End Point

e_2, e_4 are Non-Adjacent

No Common End point

Degree of a vertex := Number of Edges incident on a vertex

$Deg(v) = 4$

Definition 2. [ADJACENCY, NEIGHBOURHOOD, VERTEX DEGREE]

Let u, v be two vertices of a graph G.

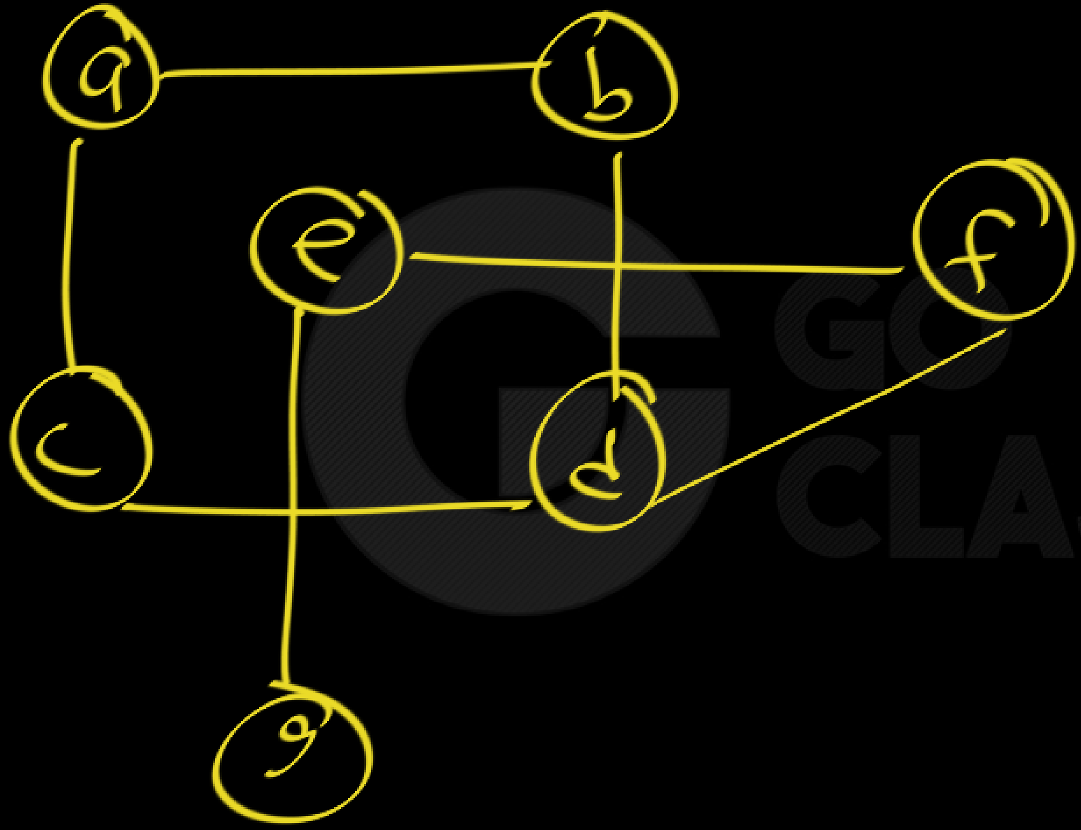
- If $uv \in E(G)$, then u, v are said to be *adjacent*, in which case we also say that [redacted] u is a *neighbour* of v. If $uv \notin E(G)$, then u and v are *nonadjacent* ([redacted] *non-neighbours*).

- The *neighbourhood* of a vertex $v \in V(G)$, denoted $N(v)$, is the set of vertices adjacent to v, i.e. $N(v) = \{u \in V(G) \mid vu \in E(G)\}$. The *closed neighbourhood* of v is denoted and defined as follows: $N[v] = N(v) \cup \{v\}$.

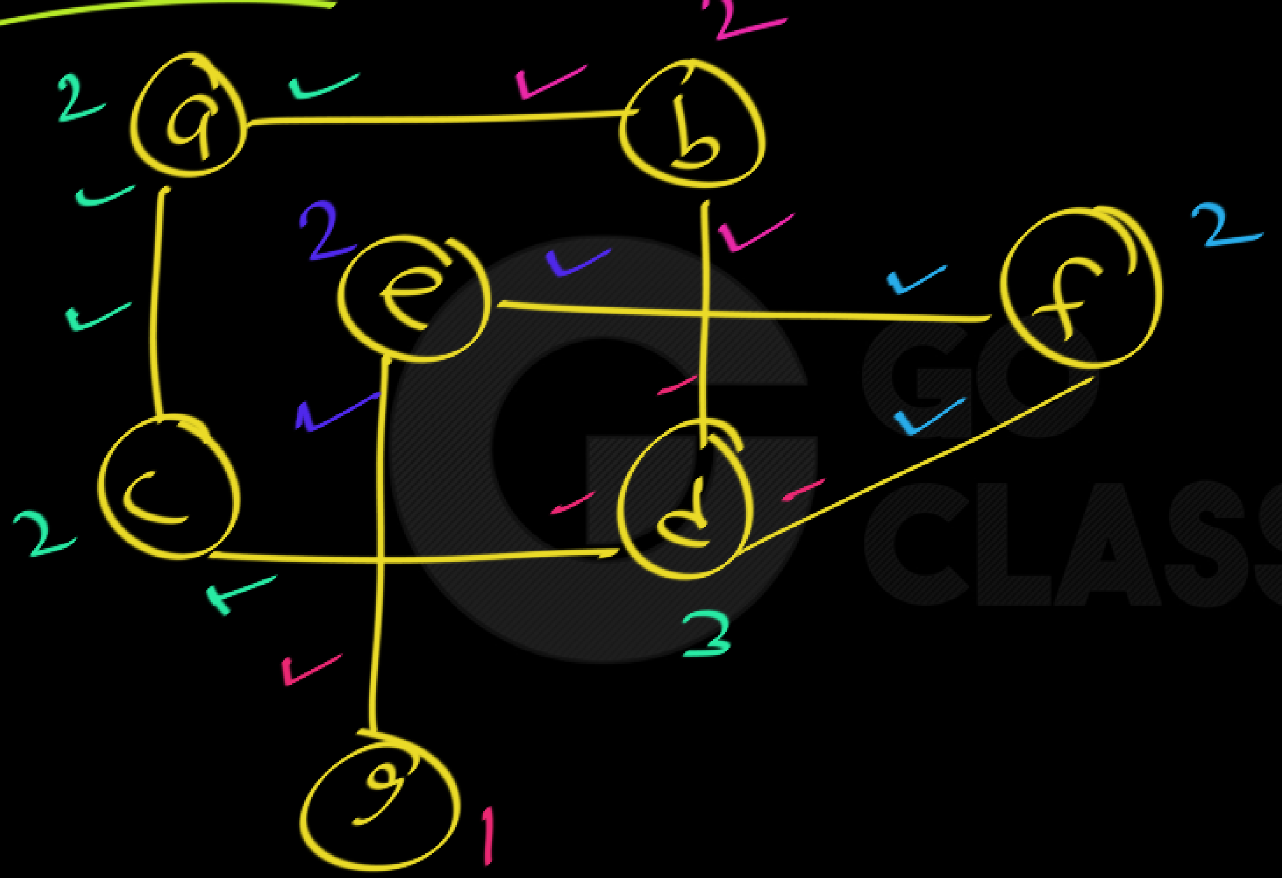
- If $e = uv$ is an edge of G, then e is *incident* to u and v. We also say that u and v are the *endpoints* of e.

- The *degree* of $v \in V(G)$, denoted $deg(v)$, is the number of edges incident to v. Alternatively, $deg(v) = |N(v)|$. If $deg(v) = 0$, then vertex v is called *isolated*. If $deg(v) = 1$, then vertex v and the only edge incident to v are called *pendant*. The maximum vertex degree and the minimum vertex degree in a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively.

find Degree:



find Degree:

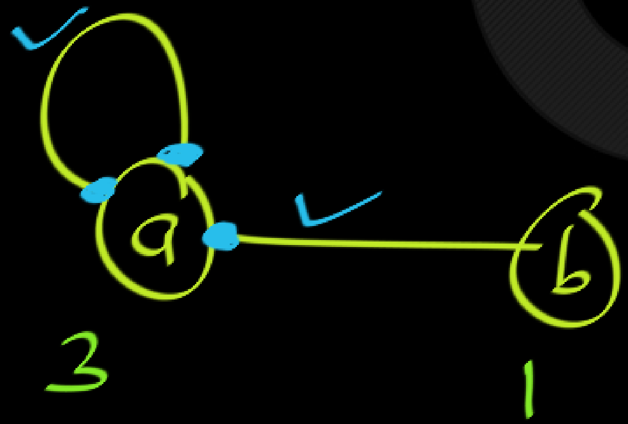


Even Deg vertices = a, b, c, e, f

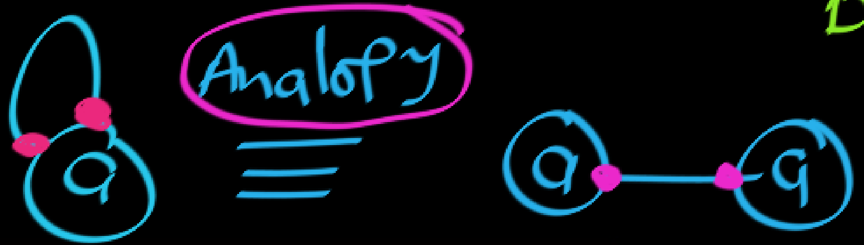
Odd Deg vertices = g, d

In Psudograph ⟹

Degree(v) = Number of times Edges are incident on v.



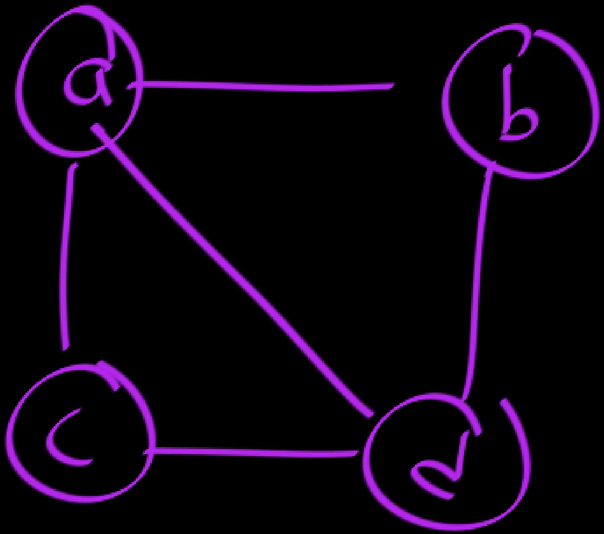
Self loop: → gives a Deg of 2.



NOTE:

Concepts: $\boxed{\text{Adjacency, Degree}}$

Only Defined for "UnDirected" Graphs



$N(a) = \{b, c, d\}$

Neighbourhood

$N(d) = \{a, c, d\}$

$N(b) = \{a, d\}$

Neighbour $\equiv$ Adjacent

for Simple Graph: → No self loop
→ No multiEdges

Deg(v) = Number of Neighbours.

= $|N(v)|$

= Number of Edges incident on v.

The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

Solution: In G, $\deg(a) = 2$, $\deg(b) = \deg(c) = \deg(f) = 4$, $\deg(d) = 1$, $\deg(e) = 3$, and $\deg(g) = 0$. The neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \emptyset$. In H, $\deg(a) = 4$, $\deg(b) = \deg(e) = 6$, $\deg(c) = 1$, and $\deg(d) = 5$. The neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$. ◀

Even Deg vertex

pendant vertex

#Even Deg vertices = 5

#odd Deg vertices = 2

odd Deg vertex

Isolated vertex

FIGURE 1 **The Undirected Graphs G and H.**

Isolated Vertex: Deg 0 vertex

Eg: g

Pendant Vertex: Deg 1 vertex