Types of Lattices:

1. Bounded Lattices
2. Complemented Lattice
3. Distributive Lattice
4. Boolean lattice

Every Lattice Satisfies these:

Proposition

Any lattice has the following properties:

1. *Commutativity:* $a \cap b = b \cap a$ *and* $a \cup b = b \cup a$.
2. *Associativity:* $(a \cap b) \cap c = a \cap (b \cap c)$ *and* $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law:* $a \cap a = a$ *and* $a \cup a = a$.
4. *Absorption law:* $(a \cup b) \cap a = a$ *and* $(a \cap b) \cup a = a$.

Properties Satisfied by EVERY Lattice:

PROPOSITION 7.2-2: ***Basic Operational Properties of Meet and Join***

Let $\mathcal{A}$ be a lattice with order relation $\leq$. Then the following properties hold:

a) ***Commutativity***: $x \wedge y = y \wedge x$; $\quad x \vee y = y \vee x$.

b) ***Associativity***: $(x \wedge y) \wedge z = x \wedge (y \wedge z)$; $\quad (x \vee y) \vee z = x \vee (y \vee z)$.

c) ***Idempotence***: $x \wedge x = x$; $\quad x \vee x = x$.

d) ***Absorption***: $x \wedge (x \vee y) = x$; $\quad x \vee (x \wedge y) = x$.

Properties Satisfied by EVERY Lattice:

Proposition

Any lattice has the following properties:

1. *Commutativity:* $a \cap b = b \cap a$ *and* $a \cup b = b \cup a$.
2. *Associativity:* $(a \cap b) \cap c = a \cap (b \cap c)$ *and* $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law:* $a \cap a = a$ *and* $a \cup a = a$.
4. *Absorption law:* $(a \cup b) \cap a = a$ *and* $(a \cap b) \cup a = a$.

Proposition 5.2.2 *If X is a lattice, then the following identities hold for all $a, b, c \in X$:*

$L1$	$a \vee b = b \vee a$,	$a \wedge b = b \wedge a$
$L2$	$(a \vee b) \vee c = a \vee (b \vee c)$,	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
$L3$	$a \vee a = a$,	$a \wedge a = a$
$L4$	$(a \vee b) \wedge a = a$,	$(a \wedge b) \vee a = a$.

Properties (L1) correspond to commutativity, properties (L2) to associativity, properties (L3) to idempotence and properties (L4) to absorption. Furthermore, for all $a, b \in X$, we have

$$a \leq b \quad \text{iff} \quad a \vee b = b \quad \text{iff} \quad a \wedge b = a,$$

called consistency.

Let L be a lattice. Define the **meet** ($\wedge$) and **join** ($\vee$) operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y, and z of a lattice L.

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ **(commutative laws)**

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ **(associative laws)**

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ **(absorption laws)**

d) $x \wedge x = x$ and $x \vee x = x$ **(idempotent laws)**

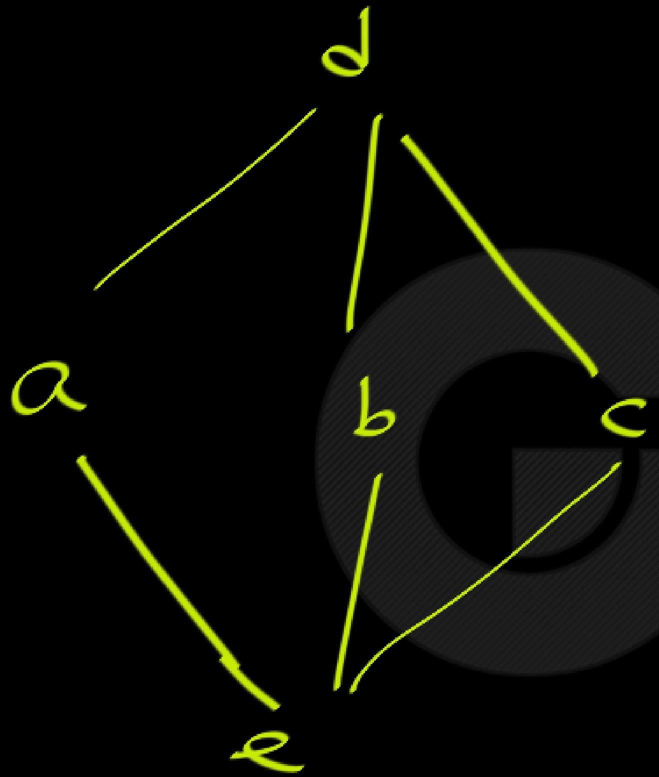
Are these Properties Satisfied by EVERY Lattice??

$a \vee 0 = a$	$a \wedge 1 = a$	identity
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements

Distributive Prop :

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$



$$a \wedge (b \vee c) \overset{?}{=} (a \wedge b) \vee (a \wedge c)$$

$$a \wedge d \qquad\qquad e \vee e$$

$$a \neq e$$

Lattice

Are these Properties Satisfied by EVERY Lattice?? NO.

$a \vee 0 = a$	$a \wedge 1 = a$	identity
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements

Some Lattices Satisfy them, Some Don't.

Identity
Complement
De-morgan
Distributive

Only satisfied by some lattices, Not by All lattices

Some Lattices Satisfy the following, Some Don't.

$a \vee 0 = a$	$a \wedge 1 = a$	identity
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$	$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$	distributivity
$a \vee \neg a = 1$	$a \wedge \neg a = 0$	complements

Special types of lattices:

1. Bounded lattice — Identity prop
2. Dist lattice — Distributive prop
3. Complemented " — Complement "

Types of Lattices:

1. Bounded Lattices — identity prop
2. Complemented Lattice — Complement prop
3. Distributive Lattice — Dist. prop
4. Boolean lattice — identity Dist Compl

Types of Lattices:

1. Bounded Lattices
2. Complemented Lattice
3. Distributive Lattice
4. Boolean lattice

Set Theory : Lattices

Types of Lattices:

1. Bounded Lattices

DEFINITION 7.2 - 4: ***Bounded Lattices***

A lattice $\langle \mathcal{A}, \leq \rangle$ is ***bounded*** *iff it has a minimum element and a maximum element. These are denoted by 0 and 1 respectively.*

Lattice (L, R) is bounded Lattice

G

L

DEFINITION 7.2 - 4: ***Bounded Lattices***

A lattice $\langle \mathcal{A}, \leq \rangle$ is ***bounded*** *iff it has a minimum element and a maximum element. These are denoted by 0 and 1 respectively.*

Lattice (L, R) is bounded iff

there is Greatest & least

element.

Q: Is every lattice bounded?

Q: Is every lattice bounded ? **No**

$(Z, \leq)$

TOR $\rightarrow$ lattice

⋮
2
1
0
−1
−2
⋮

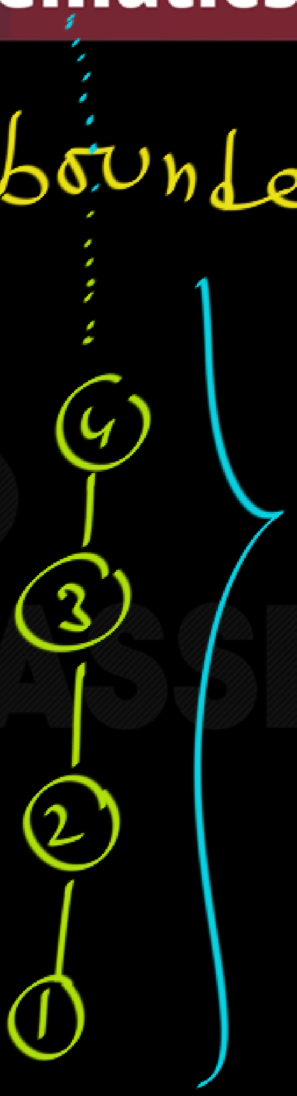
No Greatest

No least

Q: Is every lattice bounded? **No**

$(N, \leq)$ → TOR → lattice

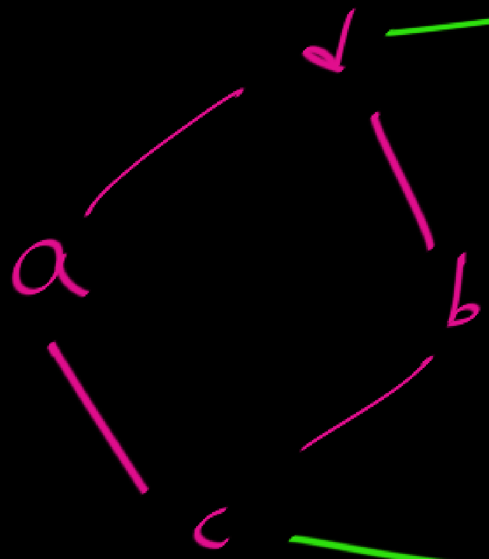
$\infty \notin N$



No Greatest

Least = 1

Q: No lattice is bounded? **No**



Greatest

Least

Lattice ✓

Bounded Lattice

Lattices

Some Bounded

Others NOT bounded

Infinite Lattice which is NOT Bounded:

$(\mathbb{Z}, \leq)$ $(\mathbb{Z}, \geq)$

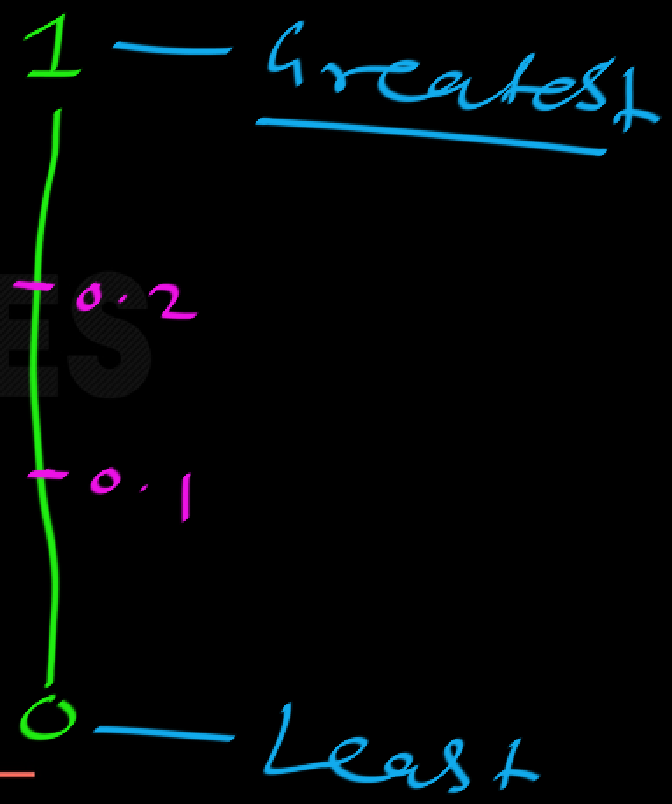
$(\mathbb{N}, \leq)$

$(\mathbb{R}, \leq)$

Infinite Lattice which is Bounded:

$([0,1], \leq) \implies$ Bounded Lattice

$\leq$: less than or equal to



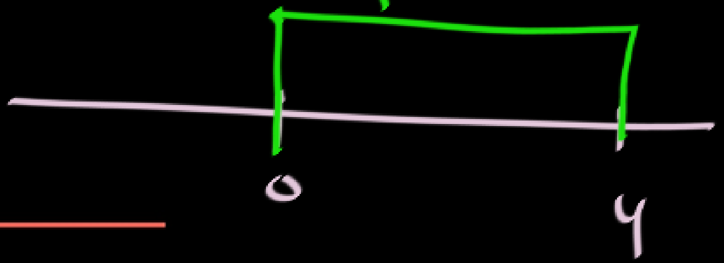
$\underbrace{[a, b]}$

Real interval from a to b including a, b

including 0

$[0, 4]$

including 4



$$\underbrace{[a, b)}_{} = \left\{ x \in R \;\middle|\; a \leq x < b \right\}$$

Real interval from a to b including a

But not including b

$$\underbrace{(a, b)}_{\text{Real interval from } a \text{ to } b} = \{x \in R \mid a < x < b\}$$

not inclusive

a, b

NOTE:

We have already seen(proven that)

Every Finite Lattice has a Greatest & a Least element…

So, Every Finite Lattice is Bounded.

NOTE:

Every Finite Lattice is Bounded.

Finite Lattice which is NOT Bounded:

NOT Possible.

finite lattice → Greatest & least — Bounded Lattice.

Finite Lattice which is Bounded:

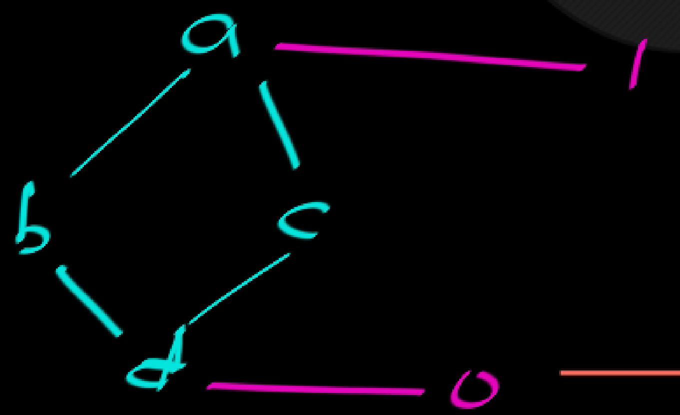
Ans: Every Finite Lattice.

Every Finite Lattice is Bounded.

Bounded Lattice :

Greatest element $\equiv 1 \equiv G$

Least " $\equiv 0 \equiv L$



$$\underbrace{(\{1,2,3,4\}, \le)}$$

Bounded Lattice

$$
\left.
\begin{tikzcd}[column sep=small]
 & 1 \arrow[dl, dash] \arrow[dr, dash] & \\
b \arrow[dr, dash] & & c \arrow[dl, dash] \\
 & 0 &
\end{tikzcd}
\right\}
\left\{
\begin{array}{l}
a \equiv 1 \\
L \equiv 0
\end{array}
\right\}
$$

Bounded Lattice : Greatest G, Least L

Greatest G ; $\forall_x \; x R G$

Least L ; $\forall_x \; L R x$

Bounded Lattice: Greatest G, Least L

$\forall x \quad x R G$

$\forall x;$

$$x \vee G = G$$
$$x \wedge G = x$$

$$G = x \vee G$$

$$x = x \wedge G$$

Bounded Lattice ; Greatest G
Least L

$\forall x \quad L R x$

$\forall x$;

$x = x \vee L$

$L = x \wedge L$

x

$= x \vee L$

L

$= x \wedge L$

Bounded Lattice : Greatest G, Least L

$G \equiv 1$; $L \equiv 0$

$$\forall x \left\{ \begin{array}{l} x \vee 1 = 1 \\ x \wedge 1 = x \\ \hline x \vee 0 = x \\ x \wedge 0 = 0 \end{array} \right\}$$

DEFINITION 7.2 - 4: ***Bounded Lattices***

A lattice $\langle \mathcal{A}, \leq \rangle$ is ***bounded*** *iff it has a minimum element and a maximum element. These are denoted by* 0 *and* 1 *respectively.*

The extreme elements of bounded lattices interact with other elements of the lattice in the obvious ways, captured by the next proposition.

PROPOSITION 7.2 - 3: ***Extreme Elements in a Bounded Lattice***

Suppose $\langle \mathcal{A}, \leq \rangle$ is a bounded lattice having minimum 0 *and maximum* 1*, and let x be any element in $\mathcal{A}$. Then*

a) $0 \vee x = x = x \vee 0$; $\qquad 1 \wedge x = x = x \wedge 1$

b) $0 \wedge x = 0 = x \wedge 0$; $\qquad 1 \vee x = 1 = x \vee 1$

$$\boxed{\begin{aligned} a \vee b &= b \vee a \\ a \wedge b &= b \wedge a \end{aligned}}$$

Every lattice

A lattice L is **bounded** if it has both an **upper bound**, denoted by 1, such that $x \preccurlyeq 1$ for all $x \in L$ and a **lower bound**, denoted by 0, such that $0 \preccurlyeq x$ for all $x \in L$.

41. Show that if L is a bounded lattice with upper bound 1 and lower bound 0 then these properties hold for all elements $x \in L$.

a) $x \vee 1 = 1$ **b)** $x \wedge 1 = x$

c) $x \vee 0 = x$ **d)** $x \wedge 0 = 0$

42. Show that every finite lattice is bounded.

Note:

Lattice L

$\bigvee L$ = Greatest

$\bigwedge L$ = Least

Lattice L is bounded iff entire lattice L has UB & LB

Infinite Lattice which is Not Bounded:

$N = \{1,2,3,\ldots\}$

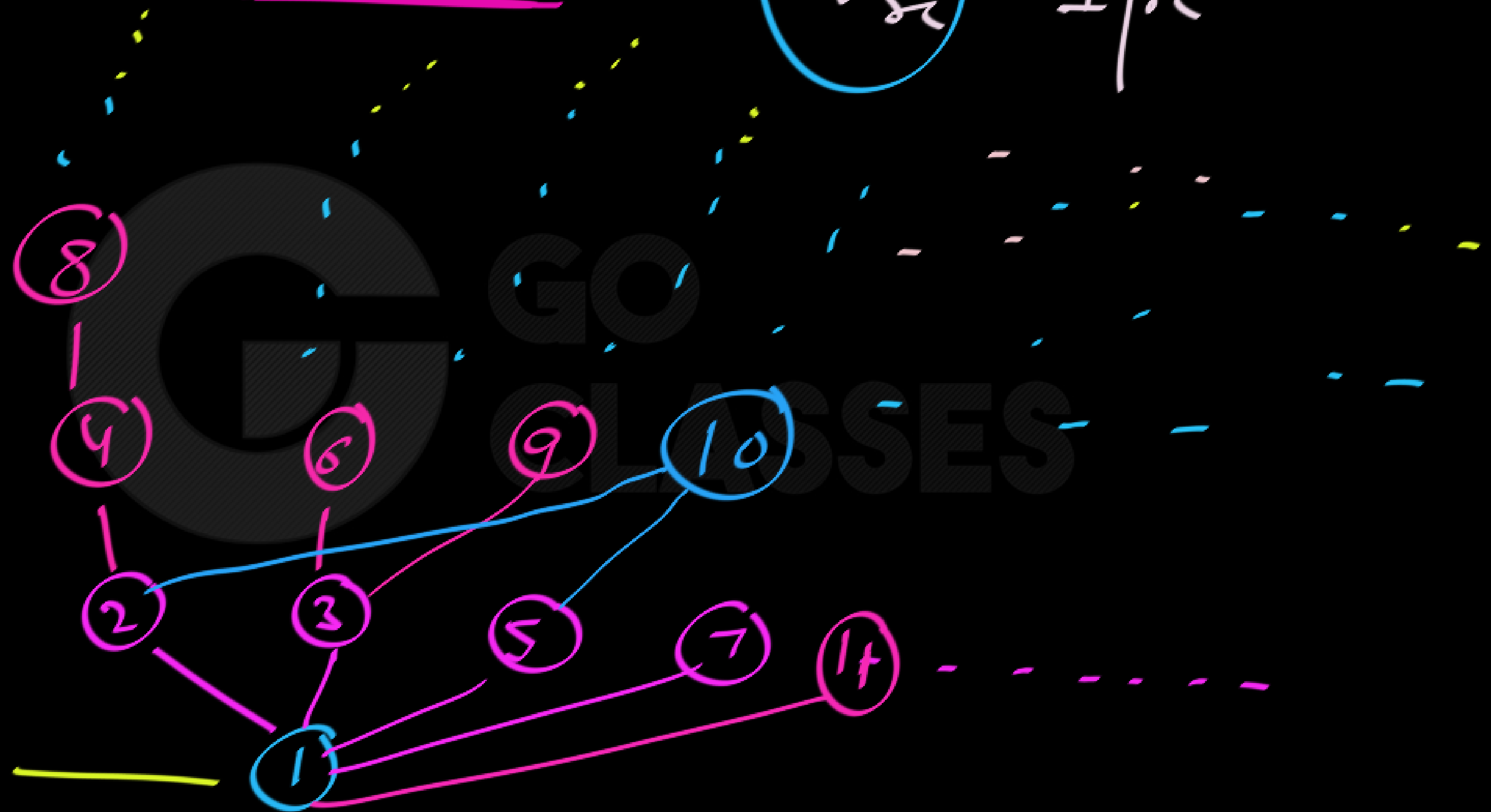
$(N, |)$ — Not Bounded

Division

$(N, |) \rightarrow$ 1 Least

$\forall x \quad 1|x$

No Greatest



Least

Infinite Lattice which is Bounded:

$(P(N), \subseteq)$

$N = \{1, 2, 3, \ldots\}$

P(N) — Base Set

$\subseteq$ — Subset

$P(N) = \{\phi, \{1\}, \{2\}, \{3\}, \ldots, \{1,2\}, \ldots, N, \ldots\}$

$N = \{1, 2, 3, ---\}$ — ∞ set

$P(N) = \{ x \mid x \subseteq N \}$

∞ set

Base set : $P(N) = \{ X \mid X \subseteq N \}$

$(P(N), \subseteq)$

least : ϕ

for All $A \in P(N)$

$\phi \subseteq A$

Greatest : N

$\forall_x ; x \subseteq N$

N

$(P(N), \subseteq)$

Lattice

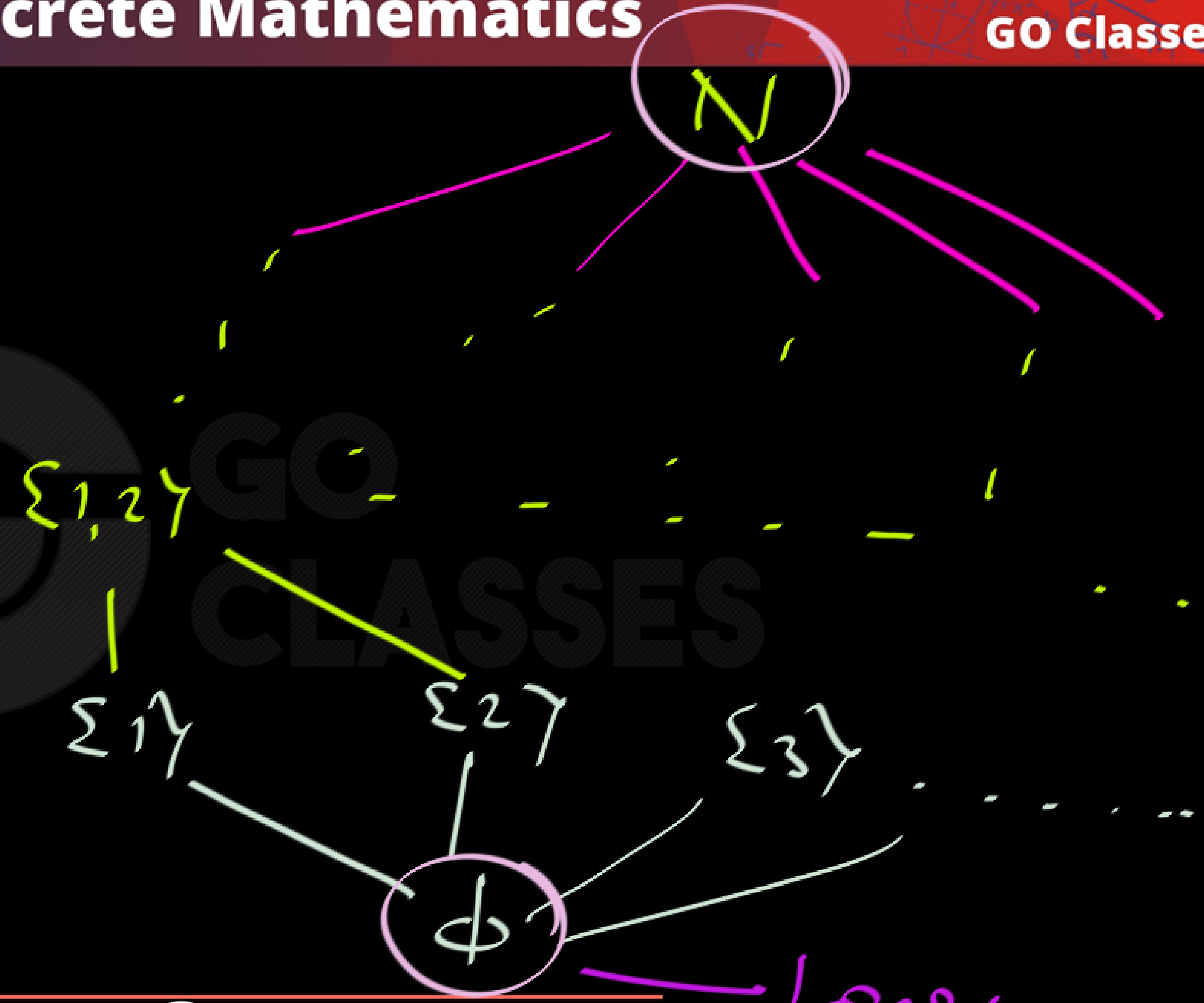
Least $= \phi$

Greatest $= N$

Bounded lattice

$(P(\mathbb{N}), \subseteq)$

Bounded Lattice



Least

Infinite Lattice which is Bounded:

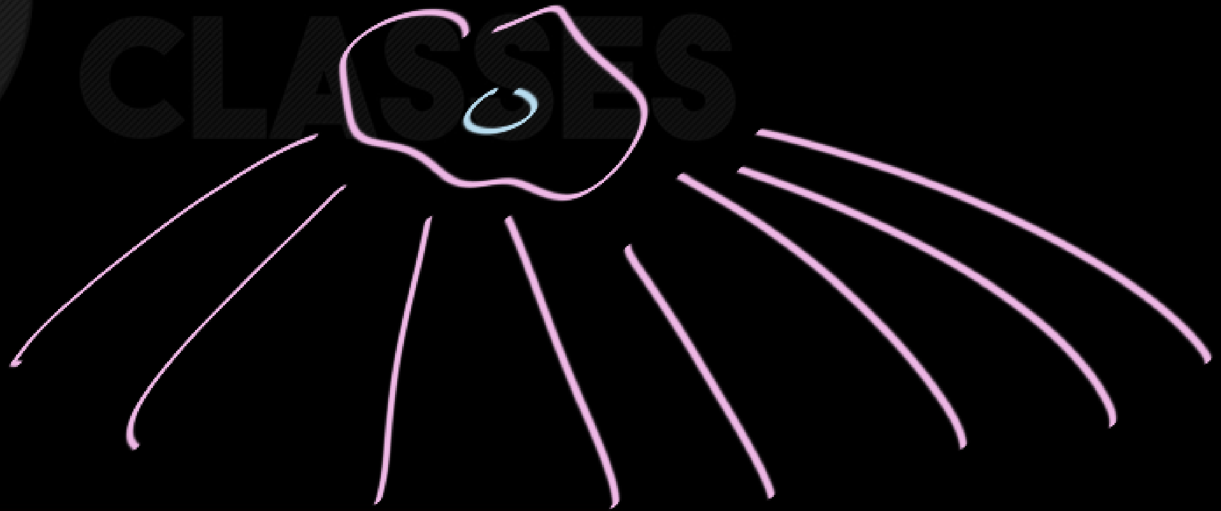
$W = \{0,1,2,3,\ldots..\}$

$(W, |)$

Divides

$(W, |)$ $\quad W = \{0, 1, 2, 3, \ldots\}$

Least: $\rightarrow 1$ $\quad 1 \mid x \; ; \; \forall x \in W$



Greatest: $\rightarrow 0$

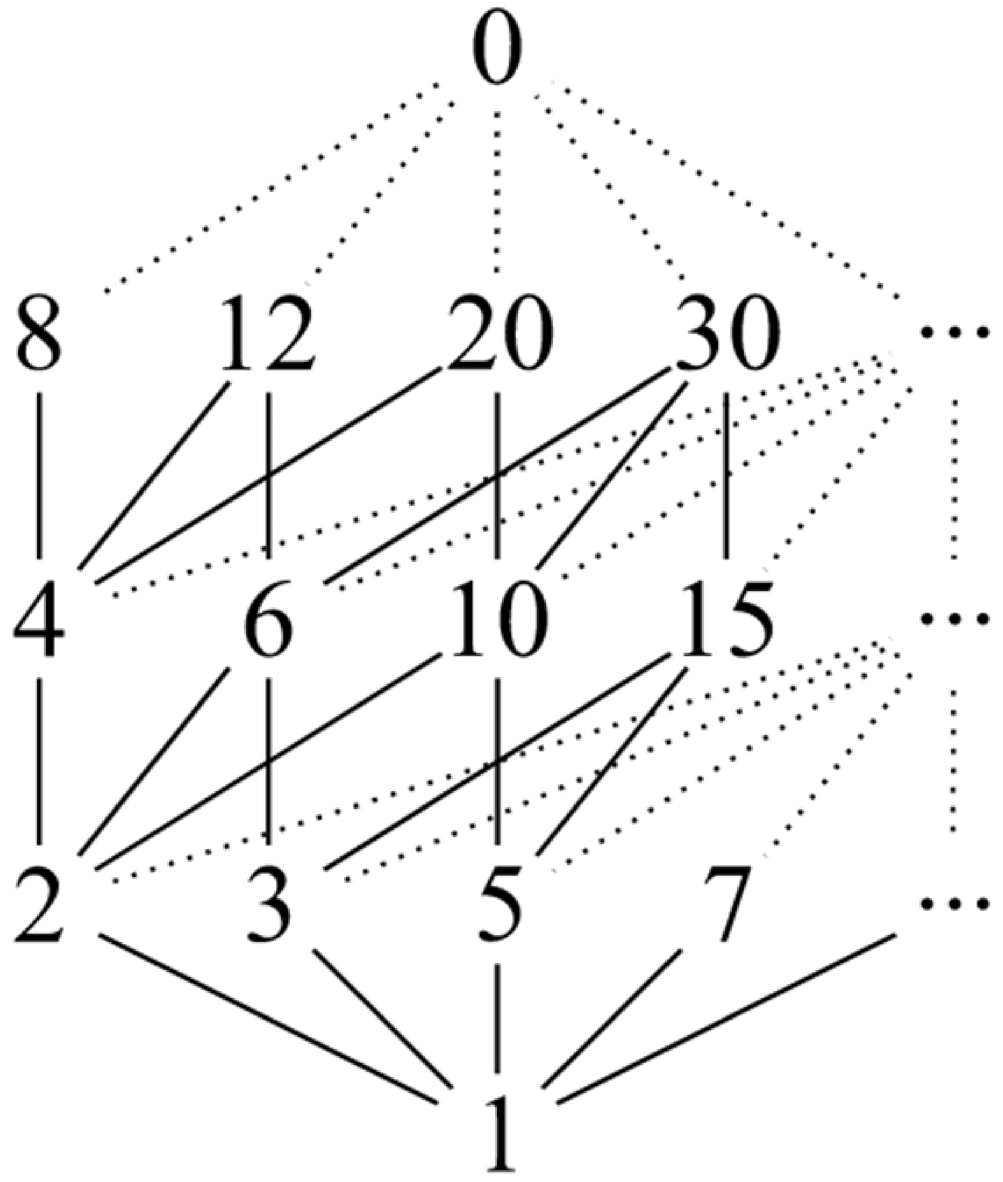
$(W, |)$

$W = \{0, 1, 2, 3, \ldots\}$

Bounded Lattice

Greatest

Least



0
8
12
20
30
...
4
6
10
15
...
2
3
5
7
...
1