



Relations

Recap

Properties of Relation

Website : <https://www.goclasses.in/>



Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A . We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A .

We say that R is ...	if ...
<i>reflexive</i>	$\forall x \in A, (x, x) \in R$
<i>irreflexive</i>	$\forall x \in A, (x, x) \notin R$
<i>symmetric</i>	$\forall x, y \in A, (x, y) \in R \implies (y, x) \in R$
<i>antisymmetric</i>	$\forall x, y \in A, (x, y) \in R \wedge (y, x) \in R \implies x = y$
<i>transitive</i>	$\forall x, y, z \in A, (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$



Properties of Relations

Definitions

A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.

A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.

A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.

A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$

A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Properties of Binary Relations

A binary relation $R \subseteq A \times A$ is called

- Reflexive iff $\forall x (x, x) \in R$
- Symmetric iff $\forall x, y ((x, y) \in R \rightarrow (y, x) \in R)$
- Antisymmetric iff $\forall x, y ((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$
- Transitive iff $\forall x, y, z ((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$.

Examples:

- \leq and $=$ are reflexive, but $<$ is not.
- $=$ is symmetric, but \leq is not.
- \leq is antisymmetric.

Note: $=$ is also antisymmetric, i.e., $=$ is symmetric and antisymmetric.

$<$ is also antisymmetric, since the precondition of the implication is always false.

However, $R = \{(x, y) \mid x + y \leq 3\}$ is not antisymmetric, since $(1, 2), (2, 1) \in R$.

- All three, $=$, \leq and $<$ are transitive.

$R = \{(x, y) \mid y = 2x\}$ is not transitive.



Set Theory

Next Topic

“Partition” of a Set

Website : <https://www.goclasses.in/>



Apple

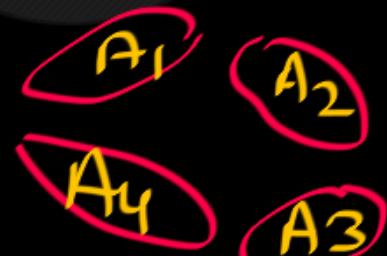


↓
Partial
1
(Bachelors)



2 parts

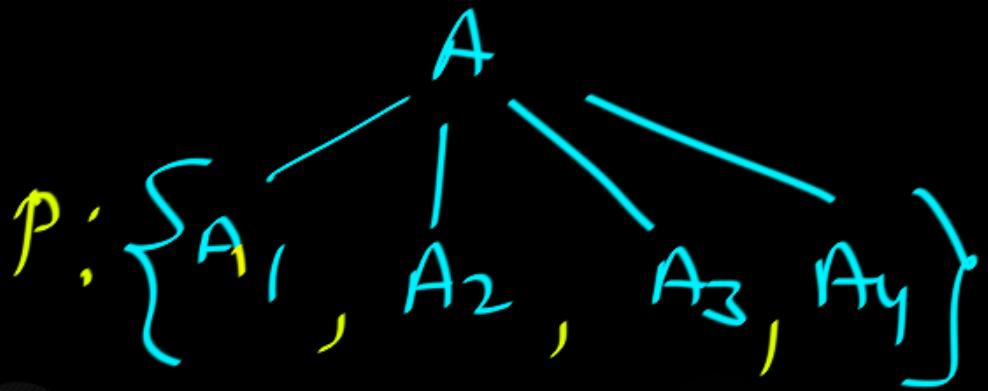
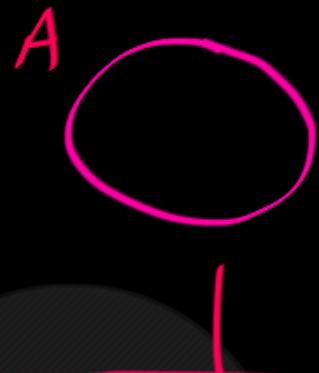
(mom)
Partition 2



4 parts

(child)
Partition 3



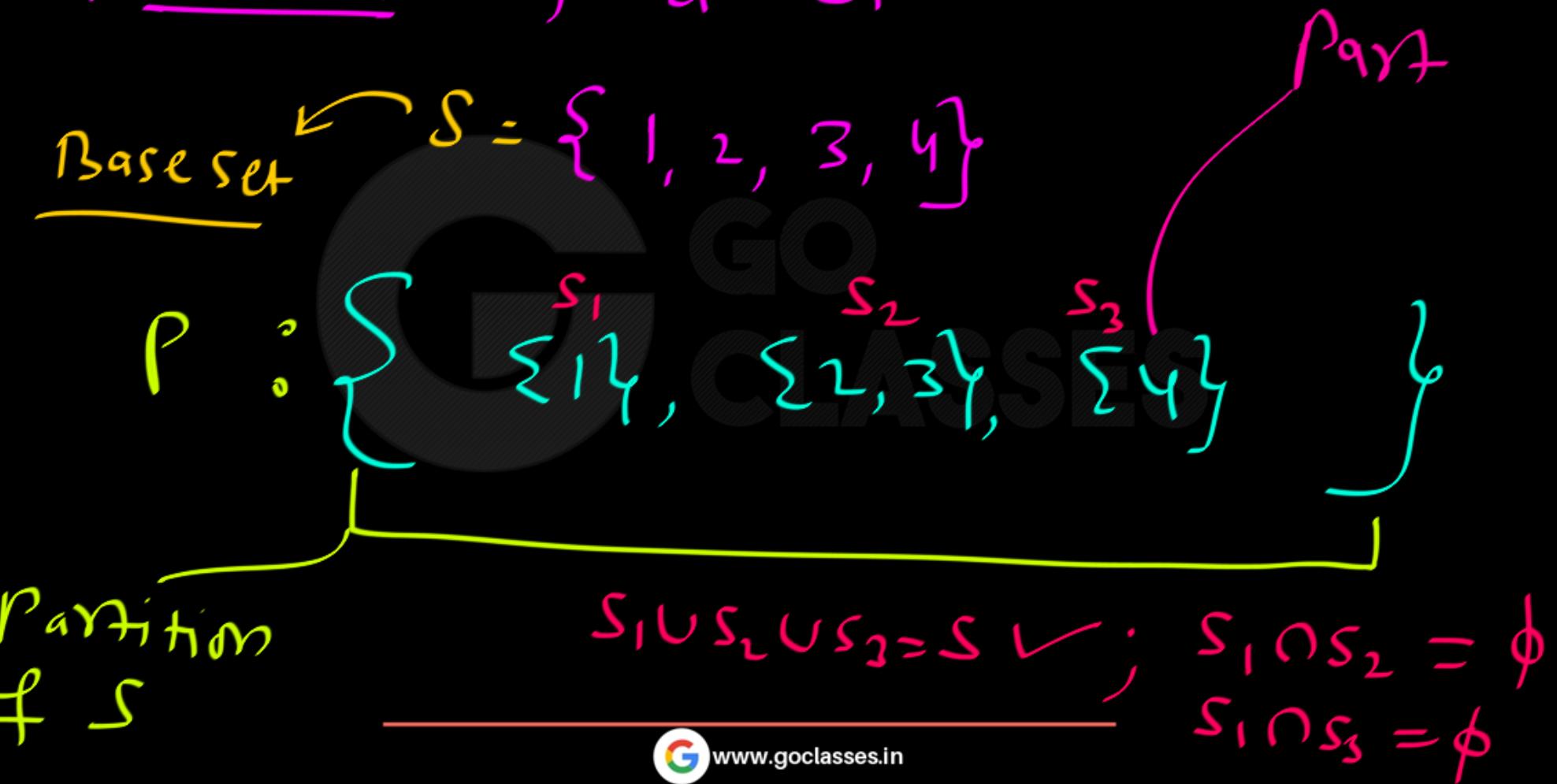
Apple

- Part $\{1\} A_1 \cap A_2 = \emptyset$
- Part $\{2\} A_1 \cup A_2 \cup A_3 \cup A_4 = A$
- Part $\{3\} A_i \neq \emptyset$

Properties of Partition :



"Partition" of a set





Disjoint Sets and Partitions

- Two sets are disjoint if their intersection is the empty set
- A partition is a collection of disjoint sets



A set S is **partitioned** into k nonempty subsets A_1, A_2, \dots, A_k if:

1. Every pair of subsets in disjoint: that is $A_i \cap A_j = \emptyset$ if $i \neq j$.
2. $A_1 \cup A_2 \cup \dots \cup A_k = S$.

Base set should be non-empty.

Base set S

$P = \{ s_1, s_2, s_3, \dots, s_k \}$

Part

① $s_i \subseteq S$

② $s_1 \cup s_2 \cup s_3 \cup \dots \cup s_k = S$

③ $s_i \neq \emptyset$ ④ $s_i \cap s_j = \emptyset ; i \neq j$



$$\mathcal{S} : \{1, 2\}$$

partitions of $\mathcal{S} = ? = 2$

① $P_1 = \{ \{1, 2\} \}$ } → one part

② $P_2 = \{ \{1\}, \{2\} \}$ } → 2 parts



$$\mathcal{S} = \left\{ \begin{smallmatrix} \text{R} \\ \text{VK} \end{smallmatrix}, \begin{smallmatrix} \text{G} \\ \text{MSD} \end{smallmatrix} \right\}$$

Partition:

- ① $P_1 = \left\{ \left\{ \begin{smallmatrix} \text{R} \\ \text{VK} \end{smallmatrix} \right\}, \left\{ \begin{smallmatrix} \text{G} \\ \text{MSD} \end{smallmatrix} \right\} \right\}$ — IPL Partition
- ② $P_2 = \left\{ \left\{ \text{VK, MSD} \right\} \right\}$ — Team India Partition



Base set

Definition 1.5.8 (Partition of a set). *Let A be a non-empty set. Then a partition Π of A , into m -parts, is a collection of non-empty subsets A_1, A_2, \dots, A_m , of A , such that*

1. $A_i \cap A_j = \emptyset$ (empty set), for $1 \leq i \neq j \leq m$ and

2. $\bigcup_{i=1}^m A_i = A$.



"Partition" of a set S : Base set
a Collection (set) of Subsets of S which are Pairwise Disjoint, non-empty, and they Cover ALL Elements of Base set.



$$S = \{1\}$$

#Partitions = ? = 1 ✓

$$P = \left\{ \begin{array}{c} \{1\} \\ \text{Partition} \end{array} \right\}$$

$$S = \{1, 2, 3\} \quad \# \text{Partitions} = 7$$

① one part Partitions: $\{S\} = 1$

② Two part Partitions: $\{\{1\}, \{2, 3\}\}$ ✓

short notation
 $\begin{cases} 1, 23 \\ 2, 13 \\ 3, 12 \end{cases}$ → $\{\{2\}, \{1, 3\}\}$ ✓
→ $\{\{3\}, \{1, 2\}\}$ ✓



③ 3 Part Partitions:

$$\left\{ \{1\}, \{2\}, \{3\} \right\} \checkmark$$

④ 4 Part Partitions: ~~none~~

Total Partitions of 5 = 5 ✓

Set S ;

Every Partition is a \checkmark set.

Every part in a partition is also
a set.  is a subset of S .



for "Convenience";

a part $\{a, b, c\}$ we will write

abc .

ab means

$\{a, b\}$

a means

$\{a\}$

Partition =

$\{\{a\}, \{b, c\}\}$

$\{a, b, c\}$
shows

$$\text{Q: } S = \{a, b, c, d, e\}$$

which is a Partition of S?

① $\{a, b, c, d, e\}$ ✓

② $\{a, bc, cd, de\}$ ✗ bc, de not Disjoint.

③ $\{a, bc, cd\}$ ✗ Doesn't cover all elements.

Q: $S = \{a, b, c, d, e\}$ base set

which is a Partition of S?

- ① $\{a, b, c, def\} \times$ & not in base sets,
- ② $\{a, bcde\} \checkmark$
- ③ $\{a, bc, d, e\} \checkmark$

$$Q: S = \{a, b, c, d, e\}$$

which is a Partition of S?

① $\{a, b, c, d, e\}$ ✓

② $\{a, b, c, d, e\}$ ✓

③ $\{a, b, c, d, e, \phi\}$ X No part should be empty.



The idea of a partition is that you take a whole (the set X) and you divide it to parts.

Now if I cut off an apple into slices (and one core) I have several pairwise disjoint parts of the apple, but if I reassemble the parts I get a whole apple again.

Similarly we require this from a partition of a set. We want that the union of *all* the parts give us the entire set we partitioned.





The examples will help. Examples of partitions of $\{1, 2, 3\}$ are

$\{1\}, \{2\}, \{3\}$

$\{1, 2\}, \{3\}$

$\{1\}, \{2, 3\}$

$\{1, 2, 3\}$

$\{2\}, \{1, 3\}$



Example: Let $S = \{a,b,c,d,e\}$ and consider the following four partitions of S .

$$P_1 = \{ \{a,b,c,d,e\} \}$$

$$P_2 = \{ \{a,b\}, \{c,d,e\} \}$$

$$P_3 = \{ \{a\}, \{b\}, \{c\}, \{d,e\} \}$$

$$P_4 = \{ \{a\}, \{b\}, \{c\}, \{d\}, \{e\} \}$$

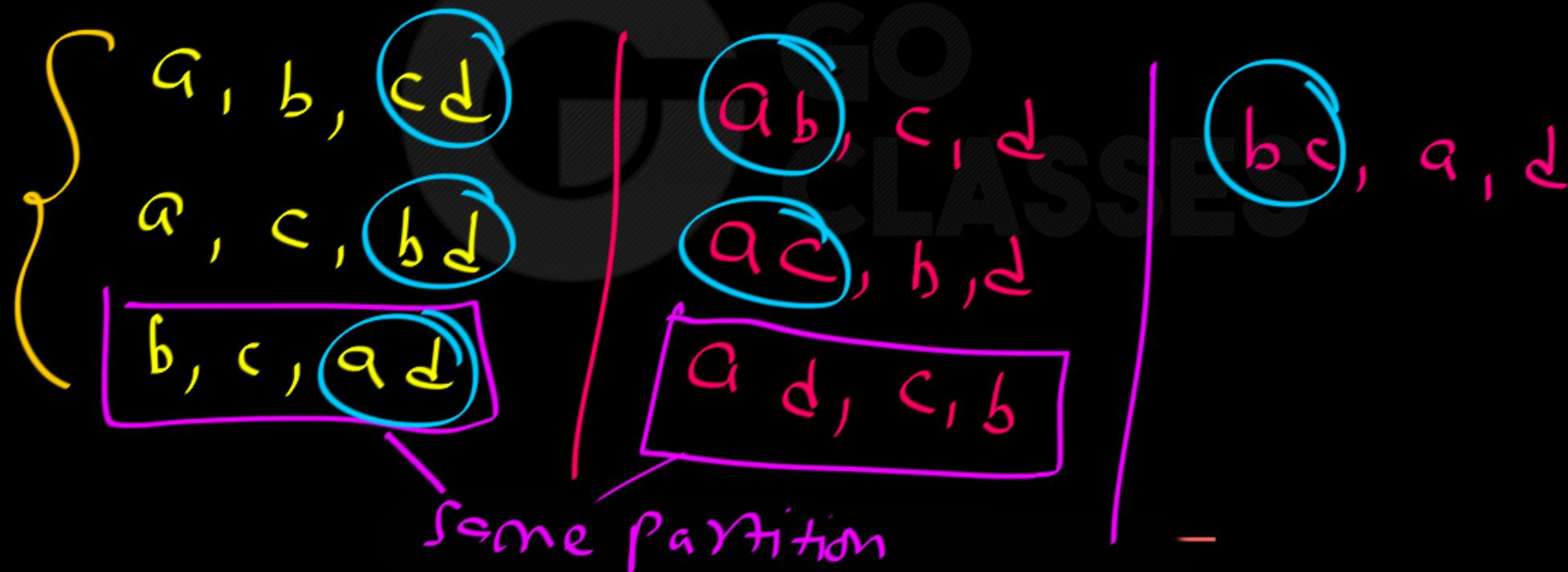


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$$Q: A = \{a, b, c, d\}$$

- , - , -

Partitions with 3-parts 1 — 6





Example 1.5.9. 1. The partitions of $A = \{a, b, c, d\}$ into

- (a) 3-parts are $a| b| cd$, $a| bc| d$, $ac| b| d$, $a| bd| c$, $ad| b| c$, $ab| c| d$, where the expression $a|bc|d$ represents the partition $A_1 = \{a\}$, $A_2 = \{b, c\}$ and $A_3 = \{d\}$.
- (b) 2-parts are

$$a| bcd, \quad b| acd, \quad c| abd, \quad d| abc, \quad ab| cd, \quad ac| bd \text{ and } ad| bc.$$



0
not +ve }
not -ve }

1
+ not Composite
- not prime

 \mathbb{Z}

① 2 Part Partition: $\{ \text{even, odd} \}$ ✓

not a partition of \mathbb{Z} ← $\{ \text{+ve, -ve} \}$ X
because 0 missing.



2. Let $A = \mathbb{Z}$ and define

- (a) $A_0 = \{2x : x \in \mathbb{Z}\}$ and $A_1 = \{2x + 1 : x \in \mathbb{Z}\}$. Then $\Pi = \{A_0, A_1\}$ forms a partition of \mathbb{Z} into odd and even integers.
- (b) $A_i = \{10n + i : n \in \mathbb{Z}\}$, for $i = 1, 2, \dots, 10$. Then $\Pi = \{A_1, A_2, \dots, A_{10}\}$ forms a partition of \mathbb{Z} .



Z
L

ω - Part Partition: G mod 10 Partition

{rem₀, rem₁, rem₂, ..., rem_q} }

Note: let $n \in \mathbb{Z}^+$

then Possible remainders when

divides

by n

$0, 1, 2, \dots, n-1$

Divide by 3
possible rem

0, 1, 2



$\{ \text{Prime, Composite} \}$ a Partition of \mathbb{N} !

No.





Let $A = \{a, b, c, d\}$. Then $\Pi = \{\{a\}, \{b, d\}, \{c\}\}$ is a partition of A .

{ Prime, Composite, {1} } is a partition of N.

$$A = \{a, b, c, d, e\}$$

T₁ is partition of A consisting 3 Parts. And a, b belong to same Part ; c, d belong to same Part
How many such partitions are possible?



$$A = \{a, b, c, d, e\}$$

$$\pi =$$



Answer: /

Observation:

q, c cannot be in same part,

$$A = \{a, b, c, d, e, f\}$$

T₁ is partition of A consisting 3 parts. And a, b belong to same part ; c, d belong to same part
How many such partitions are possible?

$$A = \{a, b, c, d, e, f\}$$

T₁ is partition of A consisting 3 parts. And a, b belong to same part ; c, d belong to same part
How many such partitions are possible?

Answer: 6 ✓

$$A = \{a, b, c, d, e, f\}$$

$$\pi = \{\text{ }, \text{ }, \text{ }\}$$

a, b c, d

Case I: a, b, c, d in a Partition | such
Partition

a, b, c, d , e, f

Case 2: $\{ab\}$, $\{cd\}$ in diff parts.

ab | cd | ef ✓ — |
 ab | cd | ef — — |
2 possible
partitions

ab | cd | e — — |
2 possible
partitions.



Congruence Modulo m

- We say that two integers a and b are *congruent modulo m* if $a - b$ is a multiple of m ($m > 0$).
- Write $a \equiv b \pmod{m}$.
- For example, $2 \equiv 7 \pmod{5}$.


$$a \equiv b \pmod{n}$$

iff

$$\text{rem}(a,n) = \text{rem}(b,n)$$



Definitions

Two sets A and B are **disjoint** if they have no common elements; that is, if $A \cap B = \emptyset$.

A family of sets A_1, A_2, \dots, A_n is **pairwise disjoint** if for all $i, j \leq n$, if $i \neq j$, then A_i and A_j are disjoint.

A **partition** of a non-empty set S is a collection of pairwise disjoint non-empty subsets of S that have S as their union.



The last definition can also be written like this:

Let S be a non-empty set. A **partition** of S is a collection D of non-empty subsets of S such that

- (i) if $P, Q \in D$ and $P \neq Q$, then $P \cap Q = \emptyset$; and
- (ii) $\bigcup_{P \in D} P = S$.

The notation in condition (ii) means that the union of all the sets in the partition D is S . Given these definitions, we see that if R is an equivalence relation on S , the equivalence classes of R form a partition of S .