Homework 5
Questions and Solutions

Please join our public telegram group by clicking here –

https://t.me/GATECSE_Goclasses

If you are an Enrolled Student, Please drop us a message or mail to join Private Telegram groups.

Q1

(a) $7 \cdot 9 \pmod{36}$.

(b) $8 - 21 \pmod{31}$.

(c) $68 \cdot 69 \cdot 71 \pmod{72}$.

(d) $108! \pmod{83}$.

(e) $60^{59} \pmod{61}$.

https://math.mit.edu/~roed/courses/magic/exams/08mid2PS.pdf

5. **Do the following calculations. As always, when working mod n, leave your answer in the range 0, 1, . . . , $n - 1$.**

(a) $7 \cdot 9 \pmod{36}$.

This is straight-forward: $7 \cdot 9 \equiv 63 \equiv \boxed{27} \pmod{36}$.

(b) $8 - 21 \pmod{31}$.

Again, this is an easy computation: $8 - 21 \equiv -13 \equiv \boxed{18} \pmod{31}$.

(c) $68 \cdot 69 \cdot 71 \pmod{72}$.

If we note that $68 \equiv -4$, $69 \equiv -3$, and $71 \equiv -1$ (all of these are taken $\pmod{72}$), then we get

$$68 \cdot 69 \cdot 71 \equiv -4 \cdot -3 \cdot -1 \equiv -12 \equiv \boxed{60} \pmod{72}.$$

(d) $108! \pmod{83}$.

Note that 83 divides 108!. Therefore, $108! \equiv \boxed{0} \pmod{83}$.

(e) $60^{59} \pmod{61}$.

Observe that $60 \equiv -1 \pmod{61}$. Thus

$$60^{59} \equiv (-1)^{59} \equiv -1 \equiv \boxed{60} \pmod{61}$$

https://math.mit.edu/~roed/courses/magic/exams/08mid2PS.pdf

Q2 What is the last decimal digit of 3^{2010}?

Solution

To find last digit we can take mod 10

$$
\begin{aligned}
&3^{2010}\ mod\ 10 \\
&= (3^2)^{1005}\ mod\ 10 \\
&= (-1)^{1005}\ mod\ 10 \\
&= -1\ mod\ 10 \\
&= 9
\end{aligned}
$$

Q3: True/False

If $a \equiv b \ (mod\ n)$ then $a^3 + a \equiv b^3 + b \ (mod\ n)$

Solution

True

We can easily prove using properties discussed in class.

Suppose we have the following two congruence relations:

$$\begin{aligned} a &\equiv c \pmod{m} \\ b &\equiv d \pmod{m}. \end{aligned}$$

YES

Are we able to combine these to obtain

$$\begin{aligned} a+b &\equiv c+d \pmod{m}, \\ a-b &\equiv c-d \pmod{m}, \\ a\times b &\equiv c\times d \pmod{m}? \end{aligned}$$

If $a \equiv b \bmod n$ then $a^3 + a \equiv b^3 + b \pmod n$

$\Rightarrow$ $a \equiv b \bmod n$ — (1)

$a \equiv b \bmod n$ — (2)

$a \equiv b \bmod n$ — (3).

Multiply (1), (2), (3)

$a^3 \equiv b^3 \bmod n$ — (4).

Add (3) & (1). $\Rightarrow$ $a^3 \equiv b^3 \bmod n$

$a \equiv b \bmod n$

$a^3 + a \equiv b^3 + b \bmod n$

Question 4

Suppose x has digits a, b, c, d: that is,

$$x = 1000a + 100b + 10c + d.$$

What is $x \bmod 9$?

For example, $5776 = 5000+700+70+6$.

Solution

We have 10 $9 + 1 \equiv 1 \pmod{9}$, $100 = 99 + 1 \equiv 1 \pmod{9}$, $1000 = 999 + 1 \equiv 1 \pmod{9}$, etc. So

$$x \bmod 9 = a + b + c + d \pmod{9}.$$

Q5 True/False

An integer is divisible by 9 if and only if the sum of its digits is divisible by 9.

Solution

True

In base 10, every number can be written as a sum of ones, tens, hundreds, thousands, and so forth. For example, $5776 = 5000+700+70+6$.

$$
\begin{aligned}
&5776 \; mod \; 9 \\
&= \quad (5000 + 700 + 70 + 6) \; mod \; 9 \\
&= \quad (5\times10^3 + 7\times10^2 + 7\times10 + 6) \; mod \; 9 \\
&= \quad (5 + 7 + 7 + 6) \; mod \; 9
\end{aligned}
$$

Q5: True/False

An integer is divisible by 11 if and only if the alternating sum (add the first digit, subtract the second, add the third, subtract the fourth, etc.) of its digits is divisible by 11.

Solution

True

$$5776 \; mod \; 11$$
$$= \; (5000 + 700 + 70 + 6\,) \; mod \; 11$$
$$= \; (5 \times 10^3 + 7 \times 10^2 + 7 \times 10 + 6\,) \; mod \; 11$$
$$= \; (5 \times (-1)^3 + 7 \times (-1)^2 + 7 \times (-1) + 6\,) \; mod \; 11$$
$$= \; -5 + 7 + (-7) + 6 \;\; mod \; 11$$
$$= \; -5 + 7 - 7 + 6 \;\; mod \; 11$$

Q6. True/False ?

$-7 \equiv -57 \pmod{10}$

If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m}$.

If $ab \equiv 0 \ (m)$, then $a \equiv 0 \ (m)$ or $b \equiv 0 \ (m)$.

Solution

$-7 \equiv -57 \pmod{10}$ True

Add 10 to -7 to check remainder. Remainder is 3
Add 60 to -57 to check remainder. Remainder is 3

If $ac \equiv bc \pmod{m}$, then $a \equiv b \pmod{m}$. False

We can not arbitrarily cancel numbers in modulus arithmetic.

If $ab \equiv 0 \ (m)$, then $a \equiv 0 \ (m)$ or $b \equiv 0 \ (m)$. False

If m divides ab then it does not mean it divides
either a or b. Take m =6, a = 2, b=3

Class homework

Example 1: Find the remainder when $25^{100} + 11^{500}$ is divided by 3.

Class homework

Example 1: Find the remainder when $25^{100} + 11^{5\,00}$ is divided by 3.

We observe that $25 \equiv 1$ (mod 3) and $11 \equiv -1$ (mod 3). Raising these to the approopriate powers, $25^{100 \equiv} 1^{100}$ (mod 3) and $11^{500} \equiv (-1)^{500}$ (mod 3). That is,

$25^{100 \equiv} 1$ (mod 3) and $11^{500} \equiv 1$ (mod 3). Adding these congruecies, we get $25^{100} + 11^{500} \equiv 2$ (mod 3).

Thus the remainder is 2.

https://www.math.toronto.edu/rosent/Mat246Y/PDF/cong.pdf

Class homework

Example 2: What is the remainder when 3^{5555} is divided by 80?

Class homework

Example 2: What is the remainder when 3^{5555} is divided by 80?

We notice that $3^4 = 81 \equiv 1 \pmod{80}$. That is, we have $3^4 \equiv 1 \pmod{80}$ ------------ (1)

We also know that 5555 when divided by 4, gives a quotient of 1388 and the remainder 3.

Hence, $3^{5555} = (3^4)^{1388} . 3^3$. Now raising congruence (1) to the power of 1388, we have

$(3^4)^{1388} \equiv 1 \pmod{80}$.

Multiplying this by 3^3 we get $(3^4)^{1388} . 3^3 \equiv 3^3 \pmod{80}$.

Which means, $3^{5555} \equiv 27 \pmod{80}$.

Thus the required remainder is 27. Unfortunately you cannot verify this by using your pocket calculator!

https://www.math.toronto.edu/rosent/Mat246Y/PDF/cong.pdf

Bonus question (Tough)

If $17! = 355687ab8096000$, where a and b are two missing digits, find a and b.

Definition of factorial:-

$n! = n.(n-1).(n-2) \dots \dots .1$

If we expand 17! then we must have 9 and 11 in multiplication hence 9 and 11 both divides 17 !

We know 17! is divisible both by 9 and by 11, so:

$$\begin{cases} 3+5+5+6+8+7+a+b+8+0+9+6+0+0+0 \\ \qquad \equiv a+b+3 \equiv 0 \pmod{9}, \\ 3-5+5-6+8-7+a-b+8-0+9-6+0-0+0 \\ \qquad \equiv a-b-2 \equiv 0 \pmod{11}. \end{cases}$$

The only pair (a, b) that satisfies both conditions is
$a = 4,\ b = 2.$