Set Theory

Practice Set - 2

Relations

30 Standard Questions

Instructor:

Deepak Poonia
MTech, IISc Bangalore

GATE CSE AIR 53; AIR 67; AIR 107; AIR 206; AIR 256

Discrete Mathematics Complete Course:

https://www.goclasses.in/courses/Discrete-Mathematics-Course

GO CLASSES

GO Classes

Test Series

On “GATE Overflow” Website

Join **GO+ GO Classes Combined Test Series** for BEST quality tests, matching GATE CSE Level:

Visit www.gateoverflow.in website to join Test Series.

1. **Quality Questions:** No Ambiguity in Questions, All Well-framed questions.
2. Correct, **Detailed Explanation**, Covering Variations of questions.
3. **Video Solutions.**

https://gateoverflow.in/blog/14987/gate-overflow-and-go-classes-test-series-gate-cse-2024

Join GO Classes **GATE CSE Complete Course** now:

https://www.goclasses.in/s/pages/gatecompletecourse

1. Quality Learning: No Rote-Learning. Understand Everything, from basics, In-depth, with variations.
2. Daily Homeworks, Quality Practice Sets, Weekly Quizzes.
3. Summary Lectures for Quick Revision.
4. Detailed Video Solutions of Previous ALL GATE Questions.
5. Doubt Resolution, Revision, Practice, a lot more.

Download the GO Classes Android App:

https://play.google.com/store/apps/details?id=com.goclasses.courses

Search "GO Classes" on Play Store.

NOTE :

Complete Discrete Mathematics & **Complete Engineering Mathematics** Courses, by GO Classes, are **FREE** for ALL learners.

Visit here to watch : https://www.goclasses.in/s/store/

SignUp/Login on Goclasses website for free and start learning.

We are on Telegram. **Contact us** for any help.

Link in the Description!!

Join GO Classes **Doubt Discussion** Telegram Group :

Username:

@GATECSE_GOCLASSES

We are on Telegram. **Contact us** for any help.

Join GO Classes Telegram Channel, Username: @GOCLASSES_CSE

Join GO Classes **Doubt Discussion** Telegram Group :

Username: @GATECSE_Goclasses

(Any doubt related to Goclasses Courses can also be asked here.)

Join GATEOverflow Doubt Discussion Telegram Group :

Username: @GateOverflow_CSE

Q 1:

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Solution 1:

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Solution.
R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive						
Symmetric						
Transitive						

Q 2: Fill "Yes or No" in the above table.

Solution 2:

Relations on $\mathbb{Z}$:	$<$	$\leq$	$=$	$\mid$	$\nmid$	$\neq$
Reflexive	no	yes	yes	yes	no	no
Symmetric	no	no	yes	no	no	yes
Transitive	yes	yes	yes	yes	no	no

3. For each of these relations on the set $\{1, 2, 3, 4\}$, decide whether it is reflexive, whether it is symmetric, whether it is antisymmetric, and whether it is transitive.

a) $\{(2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

b) $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4)\}$

c) $\{(2, 4), (4, 2)\}$

d) $\{(1, 2), (2, 3), (3, 4)\}$

e) $\{(1, 1), (2, 2), (3, 3), (4, 4)\}$

f) $\{(1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4)\}$

Q 4: Find the Types of Relation.

Problem 1. The following table describes a binary relation. Find the set of ordered pairs that is this relation, as in the definition of a binary relation.

$\sim$	1	2	3	4	5	6
1	*					*
2		*				
3				*	*	
4			*		*	
5			*	*		
6	*					*

Q 5:

Problem 5. Let A be the set of all ordered pairs of integers, that is, $A = Z \times Z$. Define a binary relation R on A as follows: for all $(a, b), (c, d) \in A$,

$$(a, b)R(c, d) \quad \Leftrightarrow \quad a \leq c \text{ and } b \leq d.$$

(a) Is R reflexive?

(b) Is R symmetric?

(c) Is R antisymmetric?

(d) Is R transitive?

(e) Is R an equivalence relation, a partial order, neither, or both?

Q 6:

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

(a) $\mathcal{R}$ is symmetric but not reflexive.

(b) $\mathcal{R}$ is transitive and symmetric but not reflexive

(c) $\mathcal{R}$ is reflexive and transitive but not symmetric

(d) None of the above

Question 7:

Let A be any set.

Subset Relation on P(A) is Anti-symmetric??

Given a relation R on a set A , R is called antisymmetric if for all $a, b \in A$, (aRb and bRa) $\Rightarrow a = b$.

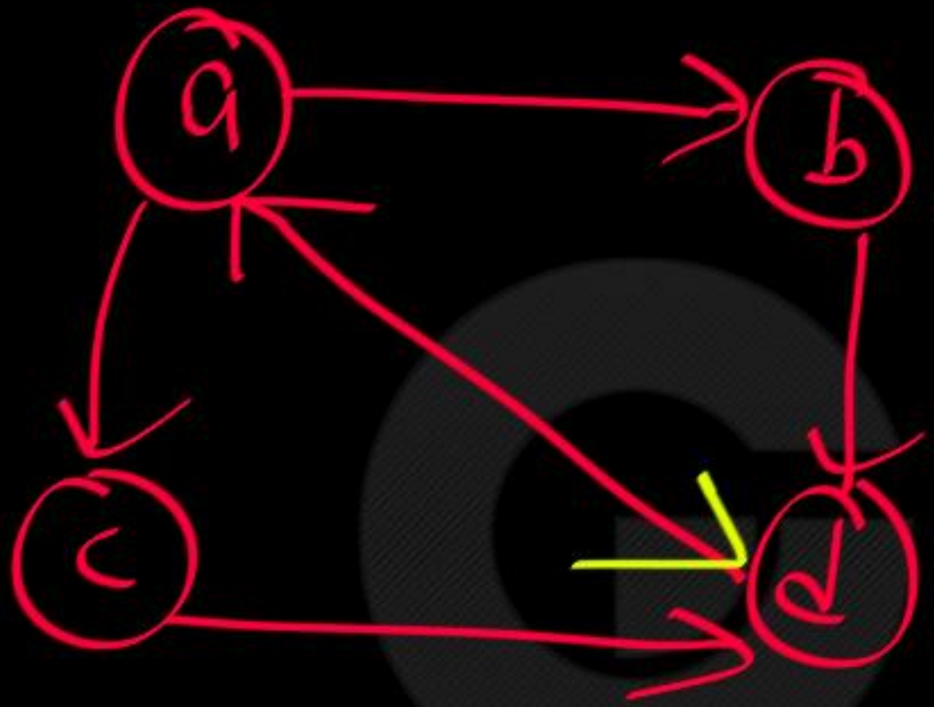
Example (7.11)

For a universe U, define the relation R on $P(U)$ by $(A, B) \in R$ if $A \subseteq B$, for $(A, B) \subseteq U$.

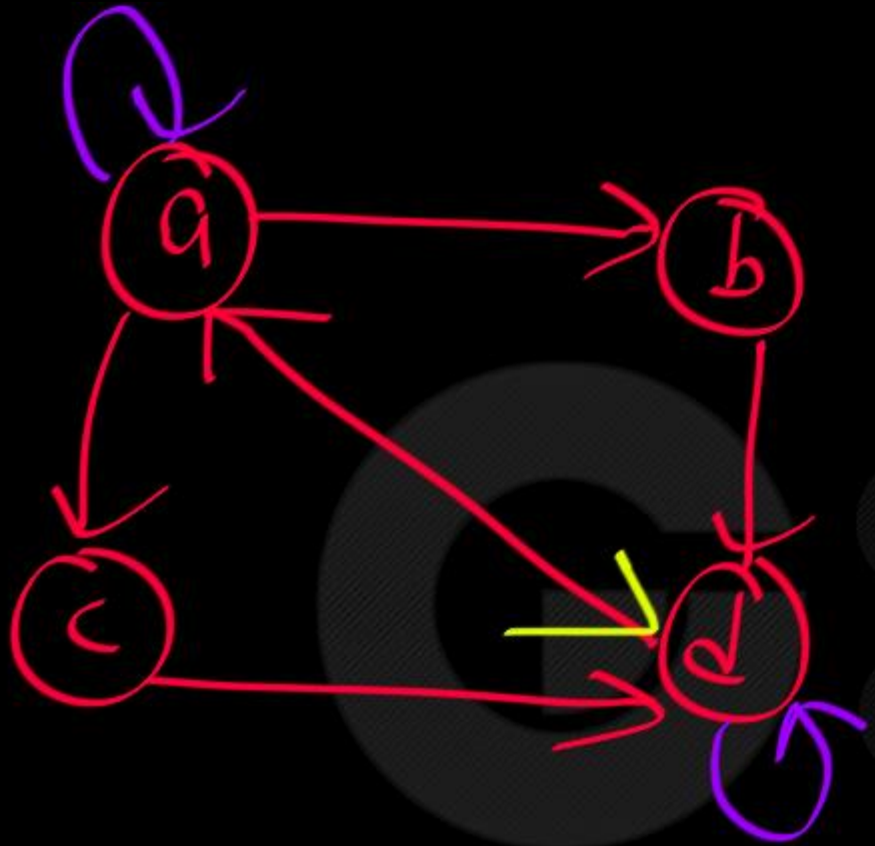
So R is the subset relation of Chapter 3 and if ARB and BRA, then we have $A \subseteq B$ and $B \subseteq A$, which gives us $A = B$.

Consequently, this relation is antisymmetric, as well as reflexive and transitive, but it is not symmetric.

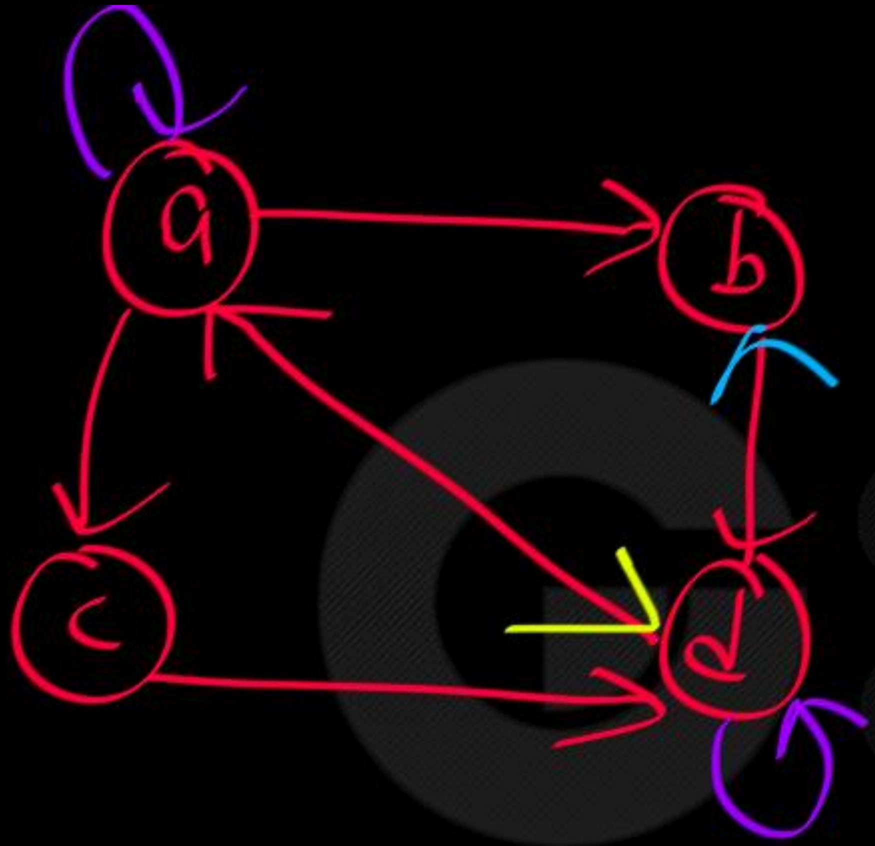
Q 8: Consider the Graph Representation below of a Relation R over the set {a,b,c,d}.
Is Relation R Transitive?



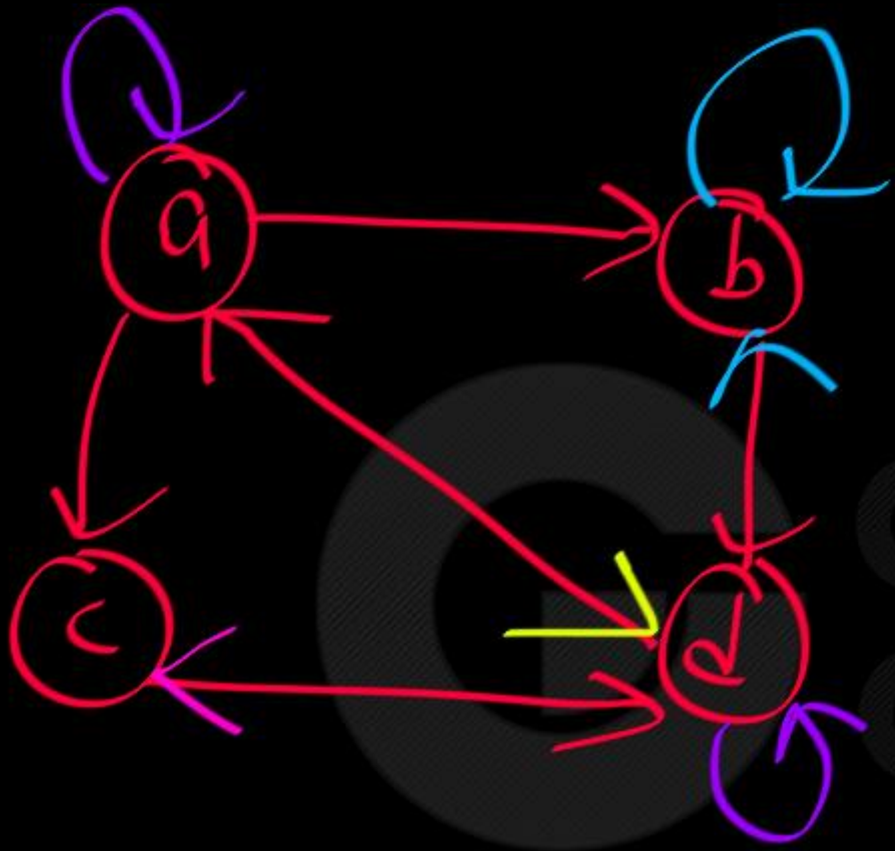
Q 9: Consider the Graph Representation below of a Relation R over the set {a,b,c,d}.
Is Relation R Transitive?



Q 10: Consider the Graph Representation below of a Relation R over the set {a,b,c,d}.
Is Relation R Transitive?



Q 11: Consider the Graph Representation below of a Relation R over the set {a,b,c,d}.
Is Relation R Transitive?



Q 12:

9. Define $\mathcal{R}$ the binary relation on $\mathbb{N} \times \mathbb{N}$ to mean $(a, b)\mathcal{R}(c, d)$ iff $b|d$ and $a|c$

Q 13:

P2.8.2 Consider the following binary relations on the naturals (non-negative integers). Which ones are reflexive? Symmetric? Anti-symmetric? Transitive? Partial orders? Justify your claims.

(a) $A(x, y)$, defined to be true if and only if y is even.

(b) $B(x, y)$, defined to be true if and only if $x < y$.

(c) $C(x, y)$, defined to be true if and only if $x + 2 \geq y$.

(d) $D(x, y)$, defined to be true if and only if $x \neq y$.

(e) $E(x, y)$, defined to be true if and only if the English *name* of x comes no later than the name of y in alphabetical order. (So, for example, $E(8, 81)$ is true because `eight` comes before `eighty-one`, and $E(8, 8)$ is true because `eight` comes no later than `eight`.)

Q 14:

Show that the relation R = ∅ on a nonempty set S is symmetric and transitive, but not reflexive.

8. Show that the relation $R = \emptyset$ on a nonempty set S is symmetric and transitive, but not reflexive.

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications Seventh Edition

Q 15:

(a) The relation $\mathcal{R}$ on $\mathbb{Q}$ with $\forall x, y \in \mathbb{Q} : x \sim y$ if $xy = 0$.

(b) The relation "has the same mother" on the set of [redacted] students.

(c) The relation $\sim$ on $\mathbb{Q}$ where $a, b \in \mathbb{Q}$ have $a\mathcal{R}b$ if $ab > 0$.

(d) The relation of division on the integers.

(e) The relation $\{(a, b), (b, c), (a, c)\}$ on the set $\{a, b, c\}$.

(f) The relation $\{(x, x), (y, y)\}$ on $\{x, y\}$

Q 16:

Are the following relations reflexive, symmetric, transitive, antisymmetric? Explain.

1. Let R be a relation on $\mathbb{Z}$ such that $(a, b) \in R$ iff $b = a$ or $b = -a$.
2. Let R be a relation on $\mathbb{R}$ such that $(a, b) \in R$ iff $ab \leq 0$.
3. Let R be a relation on $\mathbb{R}$ such that $(a, b) \in R$ iff $a + 2b = 17$.
4. Let R be a relation on $\mathbb{R}$ such that $(a, b) \in R$ iff $a + b$ is a rational number, that is can be represented by a fraction.

Q 17:

(8) Let $A \neq \varnothing$ be a set. Consider the following statements: (1) $\varnothing$ is a reflexive binary relation on A; (2) $\varnothing$ is a symmetric binary relation on A; (3) $\varnothing$ is a transitive binary relation on A; Which of the following is correct?

(a) Only (1) and (3) are correct.
(b) Only (1) and (2) are correct.
(c) Only (2) and (3) are correct.
(d) None is correct.
(e) All are correct.

Q 18:

Question 2

1 point

Define the binary relation R on the set $A := \{-4, -3, -2, -1, 1, 2, 3, 4\}$ as follows:

$$(x, y) \in R \iff |x^2 - y^2| \leqslant 5$$

for all $x, y \in A$. Which of the following statements are true? Tick all of the correct options; there may be more than one, or none at all.

- [ ] R is reflexive.
- [ ] R is irreflexive.
- [ ] R is transitive.
- [ ] R is symmetric.
- [ ] R is asymmetric.
- [ ] R is antisymmetric.

Q 19:

Q1 (10 points)

$$R = \{(x, y) \in \mathbb{N}^2 : \exists n \in \mathbb{N}, x^n = y\}$$

is a binary relation on the set of natural numbers $\mathbb{N}$.

Determine which of the following properties R satisfies:

(a) Reflexive

(b) Symmetric

(c) Anti-symmetric

(d) Transitive

For each property, either justify that the property always holds or show by a counterexample that the property does not hold.

Q 20:

1. Determine if each of the following relations is reflexive, symmetric, antisymmetric, or transitive. Indicate if the relation is an equivalence relation.

(a) $R_1 = \{(a, b) | -1 \leq a - b \leq 1\}$ on $\mathbf{R}$

(b) $R_2 = \{(1, 1), (1, 3), (1, 4), (2, 2), (2, 4), (3, 1), (3, 3), (4, 1), (4, 2)\}$ on $\{1, 2, 3, 4\}$

(c) $R_3 = \{(X, Y) | X \cap Y = \emptyset\}$ on $\mathcal{P}(\{a, b, c\})$

(d) $R_4 = \{(a, b) |$ "is a friend of"$\}$ on the set of all people

(e) $R_5 = \{((a, b), (c, d)) | ad = bc\}$ on $\mathbf{Z} \times \mathbf{Z}$

(f) $R_6 = \{(a, b) | \frac{a}{b} \in \mathbf{Z}\}$ on $\mathbf{N}$

(g) $R_7 = \{(a, b) | \frac{a}{b} \in \mathbf{Z}\}$ on $\mathbf{Z}$

Q 21:

Let $R \subseteq \mathbb{N} \times \mathbb{N}$ be a relation (a binary relation) on the set of natural numbers defined as follows:

$$(x, y) \in R \Leftrightarrow x + y \geq 18.$$

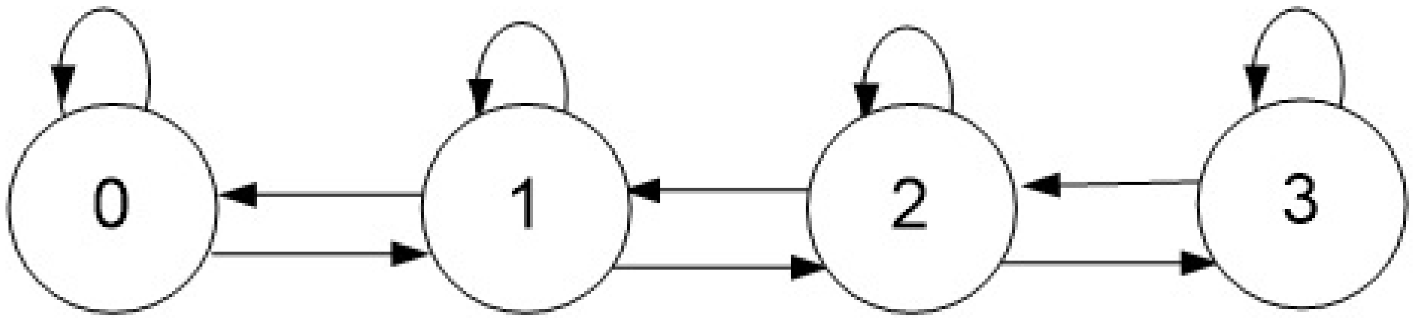
a) Is R reflexive? Prove your answer. {1 point}

b) Is R symmetric? Prove your answer. {1 point}

c) Is R antisymmetric? Prove your answer. {1 point}

d) Is R transitive? Prove your answer. {1 point}

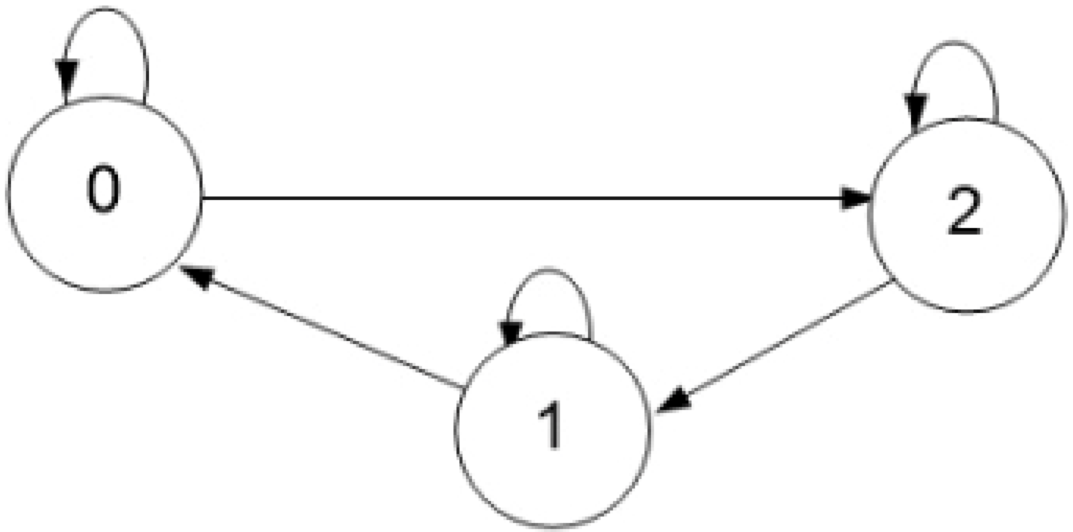
Question 5 (10 points)



Among reflexive, symmetric, antisymmetric and transitive, which of those properties are true of the above relation?

- ◯ It is both reflexive and symmetric
- ◯ It is only reflexive
- ◯ It is only antisymmetric
- ◯ It is both reflexive and transitive

Question 3 (10 points) ✓ *Saved*



Among reflexive, symmetric, antisymmetric and transitive, which of those properties are true of the above relation?

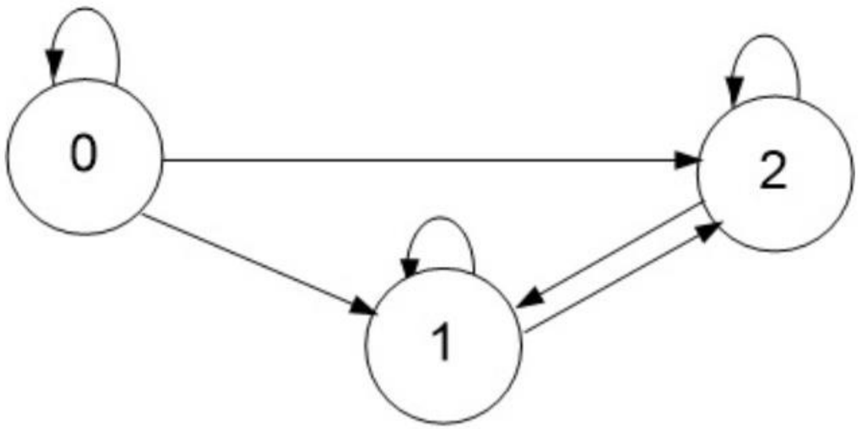
- ○ It is both symmetric and transitive
- ◉ It is both reflexive and transitive
- ○ It is reflexive, antisymmetric and transitive
- ○ It is both reflexive and antisymmetric

GO CLASSES

GO Classes

Q 24:

Question 2 (10 points)



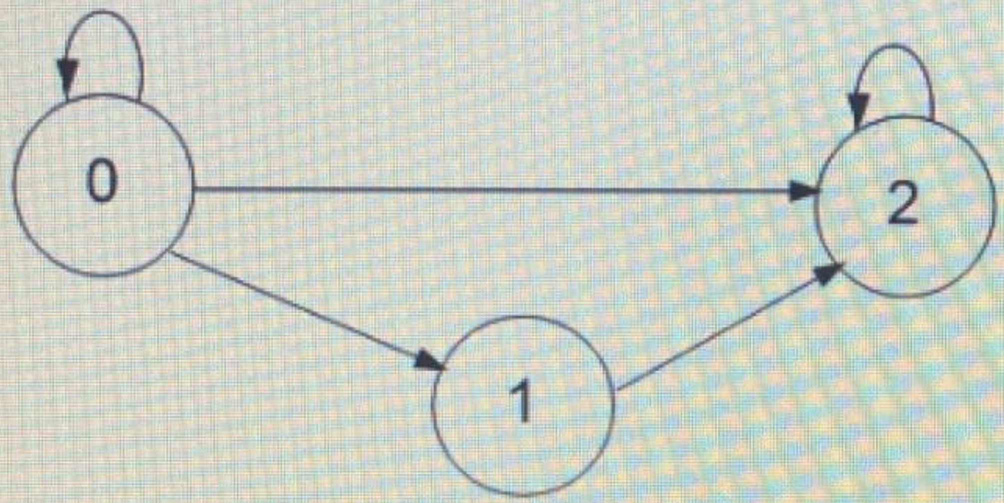
Among reflexive, symmetric, antisymmetric and transitive, which of those properties are true of the above relation?

- It is only reflexive
- It is reflexive, symmetric and transitive
- It is both reflexive and antisymmetric
- It is both reflexive and symmetric

GO CLASSES

Q 25:

Question 2 (10 points)



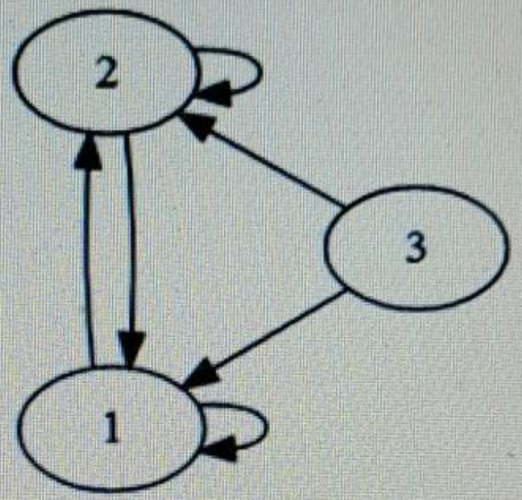
Among reflexive, symmetric, antisymmetric and transitive, which of those properties are true of the above relation?

- It is only transitive
- It is both antisymmetric and transitive
- It is both reflexive and transitive
- It has none of those properties

GO CLASSES

Q 26:

Let R be the relation on $M = \{1,2,3\}$ with the following diagraph representation:



Then

- R is not reflexive, not symmetric, and not transitive
- R is transitive but not reflexive
- R is an equivalence relation
- R is symmetric but not transitive
- R is reflexive but not symmetric

Define the relation O on Z as follows:

$$\forall m, n \in Z,\ m\, O\, n \longleftrightarrow \exists k \in Z \mid (m - n) = 2k + 1$$

Which one of the following statements about the relation O is true?

- The relation O is reflexive, not symmetric, and transitive.
- The relation O is reflexive, symmetric, and transitive.
- The relation O is not reflexive, not symmetric, and transitive.
- The relation O is not reflexive, symmetric, and not transitive.

Q 28:

Given the relation $R = \{(n, m) \mid n, m \in \mathbb{Z}, |n| \neq |m|\}$. Which of the following statements about R is correct?

- R is not an equivalence relation because it is not reflexive or transitive
- R is not an equivalence relation because it is not antisymmetric
- R is not an equivalence relation because it is not symmetric
- R is an equivalence relation

Q 29:

1 Relations

Determine whether the following relations are reflexive, symmetric, antisymmetric, and/or transitive:

a) The empty relation $R = \{\}$ defined on the natural numbers.

b) The complete relation $R = \mathbf{N} \times \mathbf{N}$ defined on the natural numbers.

c) The relation R on the positive integers where aRb means $a \mid b$.

d) The relation R on $\{w, x, y, z\}$ where $R = \{(w, w), (w, x), (x, w), (x, x), (x, z), (y, y), (z, y), (z, z)\}$.

e) The relation R on the integers where aRb means $a^2 = b^2$.

Q 30:

7.5.3. For each of the following relations R on **the set of real numbers**, decide whether it is reflexive, symmetric and/or transitive? Justify your arguments. Is the relation an equivalence relation? Explain.

(a) $(x, y) \in R$ if and only if $|x - y| \leq 3$.

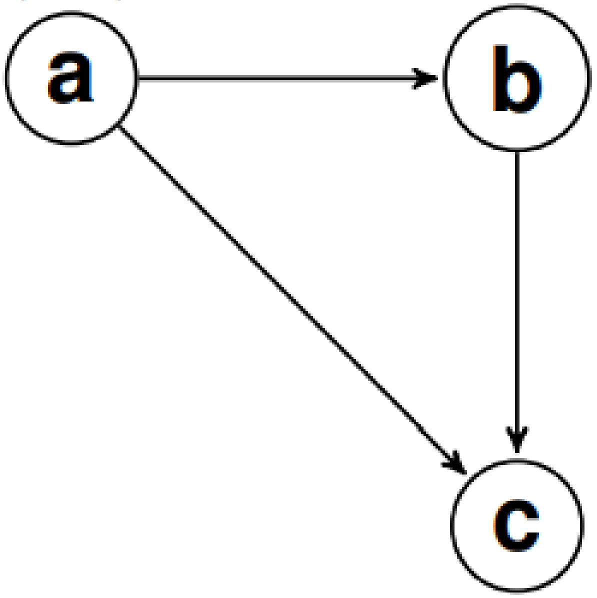
(b) $(x, y) \in R$ if and only if $x \cdot y > 0$.

(c) $(x, y) \in R$ if and only if $x^2 - y = y^2 - x$.

(d) $(x, y) \in R$ if and only if $(x - y)(x^2 + y^2 - 1) = 0$.

(e) $(x, y) \in R$ if and only if $|x + y| = |x| + |y|$.

Transitive: A relation R on a set A is called *transitive* if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.



If there is a path from one vertex to another, there is an edge from the vertex to another.

Properties of Relations

Definitions

A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.

A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.

A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.

A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then x = y

A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

Properties of Binary Relations

A binary relation $R \subseteq A \times A$ is called

- Reflexive iff $\forall x\ (x, x) \in R$
- Symmetric iff $\forall x, y\ ((x, y) \in R \to (y, x) \in R)$
- Antisymmetric iff $\forall x, y\ ((x, y) \in R \wedge (y, x) \in R \to x = y)$
- Transitive iff $\forall x, y, z\ ((x, y) \in R \wedge (y, z) \in R \to (x, z) \in R)$.

Examples:

- $\leq$ and $=$ are reflexive, but $<$ is not.
- $=$ is symmetric, but $\leq$ is not.
- $\leq$ is antisymmetric.
  Note: $=$ is also antisymmetric, i.e., $=$ is symmetric and antisymmetric.
  $<$ is also antisymmetric, since the precondition of the implication is always false.
  However, $R = \{(x, y) \mid x + y \leq 3\}$ is not antisymmetric, since $(1, 2), (2, 1) \in R$.
- All three, $=$, $\leq$ and $<$ are transitive.
  $R = \{(x, y) \mid y = 2x\}$ is not transitive.

Relation R on set A is called symmetric if $(x, y) \in R \Rightarrow (y, x) \in R$, for all $x, y \in A$.

Example (7.6)

With $A = \{1, 2, 3\}$, we have:

a) $R_1 = \{(1, 2), (2, 1), (1, 3), (3, 1)\}$ a symmetric, but not reflexive, relation on A;

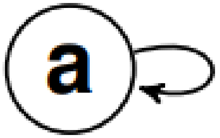
b) $R_2 = \{(1, 1), (2, 2), (3, 3), (2, 3)\}$ a reflexive, but not symmetric, relation on A;

c) $R_3 = \{(1, 1), (2, 2), (3, 3)\}$ and $R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$, two relations on A that are both reflexive and symmetric; and

d) $R_5 = \{(1, 1), (2, 2), (3, 3)\}$, a relation on A that is neither reflexive nor symmetric.

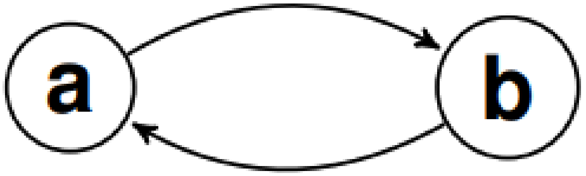
Properties

Reflexive: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.



Every vertex has a self-loop.

Symmetric: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.



If there is an edge from one vertex to another, there is an edge in the opposite direction.

Antisymmetric: A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called *antisymmetric*.

There is at most one edge between distinct vertices.