Equivalence Relation : (ER)

A Relation R is ER iff

$\left\{\begin{array}{l}\text{Ref} \\ \text{Sym} \\ \text{Trans}\end{array}\right\}$ RST properties satisfies.

For Antisym:

most of the times use following Definition:

Assume aRb and bRa

then prove that $a=b$

For Antisym:

Sometimes use following Definition:

Take $a \neq b$ then show that

$\boxed{(aRb \text{ and } bRa)}$ can never happen.

Consider these relations on the set of integers:

Base set

$$R_1 = \{(a, b) \mid a \leq b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

$R_1 : Z \to Z$; $(a,b) \in R_1$ iff $a \leq b$.

① Ref ✓ $a \leq a$; $\forall a \in Z$

② Sym ✗ $2 \leq 3$ But $3 \not\leq 2$

③ Antisym: ✓ $\forall_{a,b} \left(aRb, bRa \longrightarrow a = b \right)$

$a R_1 b$ and $b R_1 a$

$a \leq b$ and $b \leq a \longrightarrow a = b$

(4) Asym : ✗ $2 \leq 2$

(5) Trans : ✓

R_1 : Not Eq Rel.
(Not Sym)

aR_1b , bR_1c

$\boxed{a \leq b \;;\; b \leq c}$

$\boxed{a \leq c} \longrightarrow aR_1c$

$R_2 : \{(a,b) \mid a > b\}$

① Ref : ✗ $\quad 2 \not> 2$

② Irref : ✓ $\quad \forall a \in Z \quad a \not> a$

③ Sym : ✗ $\quad 2 > 1$ but $1 \not> 2$

④ Antisym ✓ $\quad$ Asym ✓

Trans ✓

Antisym:

Assume $a \neq b$

then

$\boxed{a > b \text{ and } b > a}$

Can Never happen

Trans:

aRb , bRc

$a > b$ $b > c$

$\boxed{a > c} \longrightarrow aRc$

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \leq b\},$$
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$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \leq 3\}.$$

Consider these relations on the set of integers:

R4 ; R✓; IR✗; Sym✓; Antisym✓; Asym✗;
Trans✓; ER✓

$R_1 = \{(a, b) \mid a \leq b\}$,

$R_2 = \{(a, b) \mid a > b\}$,

$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\}$,

$R_4 = \{(a, b) \mid a = b\}$, — Identity Rel

$R_5 = \{(a, b) \mid a = b + 1\}$, R✗; Ir✓; Sym✗;
Antisym✓; Asym✓;
Trans✗

$R_6 = \{(a, b) \mid a + b \leq 3\}$.

$R_3 : \{(a,b) \mid a=b \text{ OR } a=-b\}$

$|||$

$R_3 = \{(a,b) \mid |a| = |b|\}$

$2R_3^2 \; ; \; 2R_3^{-2} \; ; \; -2R_3^2 \; ; \; -2R_3^{-2}$

R_3 : Ref ✓ ; Irref ✗ ; Sym ✓ ; Antisym ✗

$a = a$

$|a| = |b|$

$\downarrow$

$|b| = |a|$

$2R-2$

$-2R2$

Asym ✗ ;

Trans ✓

$aRb \wedge bRc \rightsquigarrow aRc$

$|a| = |b| = |c| \rightarrow |a| = |c|$

Eq. Rel

R_6 : Not Ref $\quad 4 \cancel{R_6} 4$

Not Irref $\quad 1 R_6 1$

Sym ✓

$a+b \leq 3 \longrightarrow b+a \leq 3$

Antisym: ✗ $\quad 1 R_6 2 \; , \; 2 R_6 1$

Asym ✗

Trans: ✗

ER ✗

$$\begin{tikzcd} 2 \arrow[r, "R_6"] \arrow[rr, bend right, "/" description] & 1 \arrow[r, "R_6"] & 2 \end{tikzcd}$$

Properties of Relations

Definitions

A relation R is called **reflexive** on a set S if for all $x \in S$, $(x, x) \in R$.

A relation R is called **irreflexive** on a set S if for all $x \in S$, $(x, x) \notin R$.

A relation R is **symmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ then $(y, x) \in R$.

A relation R is **antisymmetric** on a set S if for all $x \in S$ and for all $y \in S$, if $(x, y) \in R$ and $(y, x) \in R$, then x = y

A relation R is **transitive** on a set S if for all $x, y, z \in S$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Equivalence Relations

The properties of relations are sometimes grouped together and given special names. A particularly useful example is the equivalence relation.

Definitions

A relation that is reflexive, symmetric, and transitive on a set S is called an **equivalence relation** on S.

Equivalence Relations

- Some relations are reflexive, symmetric, and transitive:
  - $x = y$
  - $u \leftrightarrow v$
  - $x \equiv_k y$
- Definition: An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

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Solution.
R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

Definition 1. Let A and B be sets. A *relation from A to B* is a set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. In other words, a relation from A to B is a subset of $A \times B$.

If A is a set then a *relation on A* means a relation from A to A. We often write aRb to mean $(a, b) \in R$.

Definition 2. Suppose that R is a relation on a set A.

We say that R is ...	if ...
reflexive	$\forall\, x \in A,\ (x, x) \in R$
irreflexive	$\forall\, x \in A,\ (x, x) \notin R$
symmetric	$\forall\, x, y \in A,\ (x, y) \in R \implies (y, x) \in R$
antisymmetric	$\forall\, x, y \in A,\ (x, y) \in R \wedge (y, x) \in R \implies x = y$
transitive	$\forall\, x, y, z \in A,\ (x, y) \in R \wedge (y, z) \in R \implies (x, z) \in R$

Properties of Binary Relations

A binary relation $R \subseteq A \times A$ is called

- Reflexive iff $\forall x\ (x, x) \in R$
- Symmetric iff $\forall x, y\ ((x, y) \in R \rightarrow (y, x) \in R)$
- Antisymmetric iff $\forall x, y\ ((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$
- Transitive iff $\forall x, y, z\ ((x, y) \in R \wedge (y, z) \in R \rightarrow (x, z) \in R)$.

Examples:

- $\leq$ and $=$ are reflexive, but $<$ is not.
- $=$ is symmetric, but $\leq$ is not.
- $\leq$ is antisymmetric.
  Note: $=$ is also antisymmetric, i.e., $=$ is symmetric and antisymmetric.
  $<$ is also antisymmetric, since the precondition of the implication is always false.
  However, $R = \{(x, y) \mid x + y \leq 3\}$ is not antisymmetric, since $(1, 2), (2, 1) \in R$.
- All three, $=$, $\leq$ and $<$ are transitive.
  $R = \{(x, y) \mid y = 2x\}$ is not transitive.