Relations

Recap

Equivalence Relation

Website : https://www.goclasses.in/

Equivalence Relations

$R : A \to A$

Base set

A binary relation R over a set A is called an **equivalence relation** if it is

- **reflexive**,
- **symmetric**, and
- **transitive**.

A Relation which satisfies Reflexive, Symmetric, Transitive properties:

Why do we call such relations "Equivalence Relation" ??

Why do we use the word "Equivalence" for such relations ??

A Relation which satisfies Reflexive, Symmetric, Transitive properties:
Why do we call such relations “Equivalence Relation” ??

Because in every equivalence relation, there is some sense/type of “equality” involved.

Normally, an equivalence relation R on a set A has the following structure:
For all a,b in A; aRb iff a,b have same *******.

"x and y have the same color"

xRy

"x and y have the same shape"

"$x = y$"

Same value

"x and y have the same area"

"x and y are programs that produce the same output"

In every equivalence relation, there is some sense/type of "equality" involved.

Normally, an equivalence relation R on a set A has the following structure:

For all a,b in A; aRb iff a,b have same *******.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

$$\left.\begin{matrix} aRb \\ bRa \end{matrix}\right\} \quad a \sim b$$

Parity of an Integer : even / odd

Parity of 2 = even

" " 3 = odd

" " 0 = even

" " −2 = even

" " −1 = odd

Definition

An ***equivalence relation*** on a set S, is a relation on S which is *reflexive, symmetric* and *transitive*.

Examples:

Let $S = \mathbb{Z}$ and define $R = \{(x,y) \mid x \text{ and } y \text{ have the same parity}\}$ i.e., x and y are either both even or both odd.

The parity relation is an equivalence relation.

1. For any $x \in \mathbb{Z}$, x has the same parity as itself, so $(x,x) \in R$.
2. If $(x,y) \in R$, x and y have the same parity, so $(y,x) \in R$.
3. If $(x,y) \in R$, and $(y,z) \in R$, then x and z have the same parity as y, so they have the same parity as each other (if y is odd, both x and z are odd; if y is even, both x and z are even), thus $(x,z) \in R$.

R on Z. $\qquad R : Z \to Z$;

aRb iff a, b have same parity.

a ∈ Z, b ∈ Z

① Refl — a, a have same parity.

② Sym

③ Trans

Equality of Parity

If R is ER on set A;

then for $a \in A$

$$[a]_R = \{ y \mid a \sim y \}$$

Eq. class of R containing a

Base Set A = { 2,3,4,5,6,7,8 } ; Relation R on A. ER

R = {(x,y) | x and y have the same parity} i.e., x and y are either both even or both odd.

$[2]_R = \{2, 4, 6, 8\}$

Eq. class of 2 (Eq. class of R containing 2)

$[3]_R = \{3, 5, 7\} = [5]_R = [7]_R$

for R

Eq. Class of R

Eq. Classes of R

$\{2, 4, 6, 8\}$

$\{3, 5, 7\}$

Partition of Base set $\{2, 3, --, 8\}$

2 Parts

2 Eq classes of R

$A = \{2, 3, 4, 5, 6, 7, 8\}$

R : Parity Relation on A.

2, 3, 4, 8

3, 7, 5

Group / Part

Eq. Classes

2 ECs of R

2R2, 3R2
2R3, 3R4

3R3, 3R5, 5R3
3R7, 5R5

Base Set A = { 2,4,6,8 } ; Relation R on A.

R = {(x,y) | x and y have the same parity} i.e., x and y are either both even or both odd.

$\#ECs$ of $R = 1$

$[2]_R = \{2, 4, 6, 8\}$

Very Important:

Equivalence Relation R on a set A will create partition of base set A.

The elements of base set A will form groups. Eq. Classes

Each element is in exactly one group.

Eq. Class

Within a group, every element is related to every element.

Eq. Class

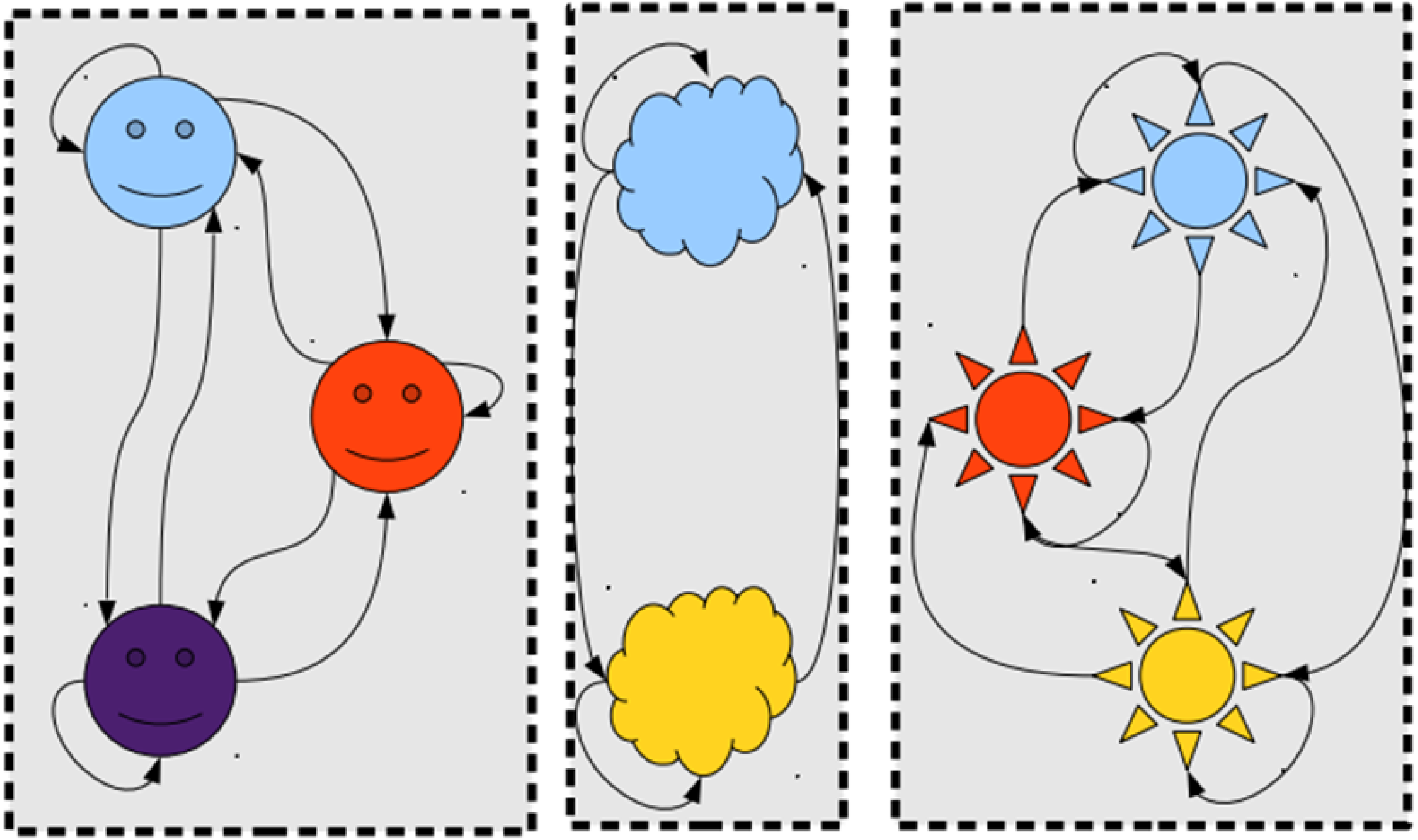
No element of a group is related to any element of other group.

All Eq. Classes are Disjoint.

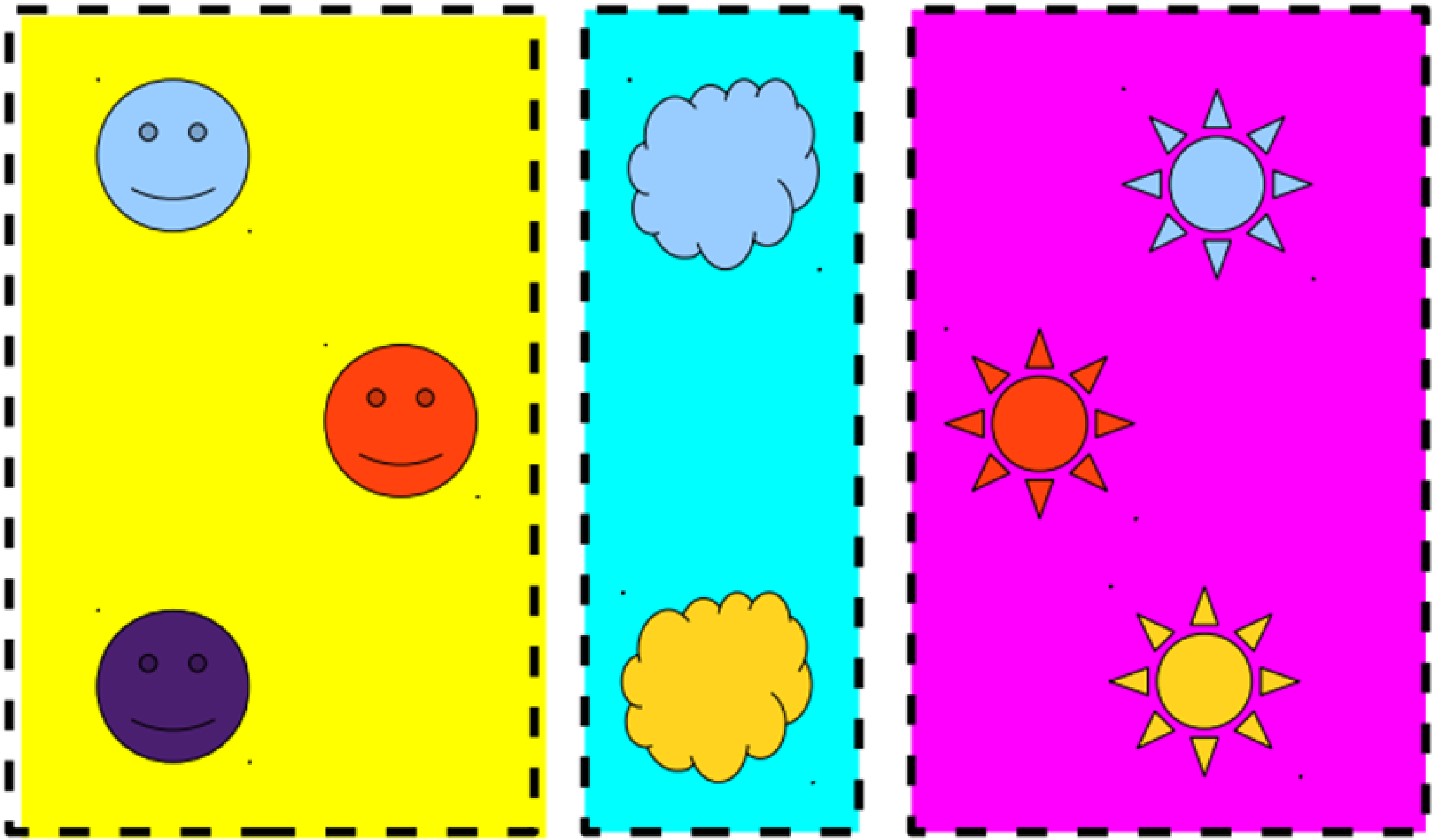
Each group is called an Equivalence Class of relation R.

Base set

set of Shapes



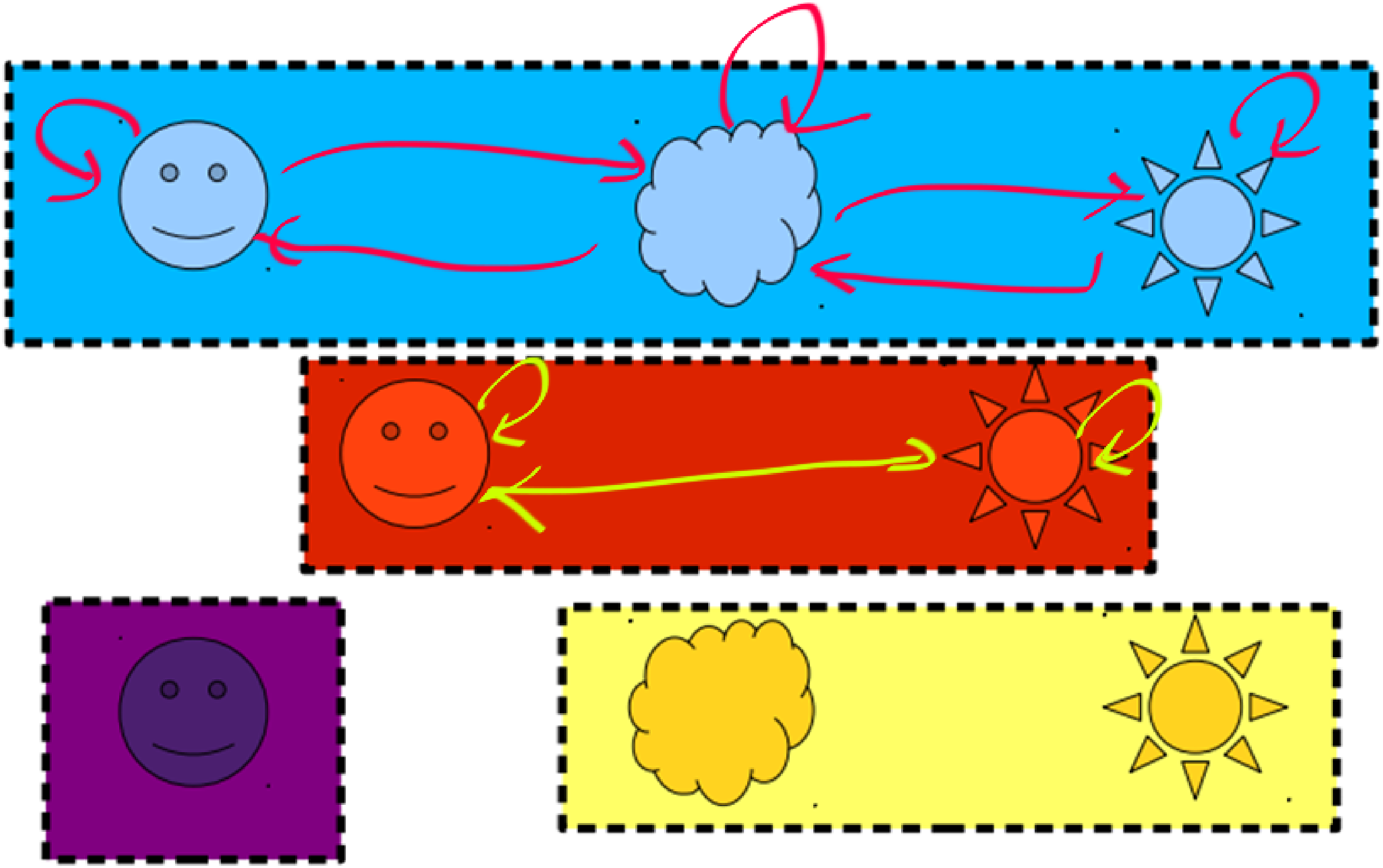
Let R be "has the same shape as"



Let R be "has the same shape as"

Base set

Set of all Pictures



Let T be "is the same color as"

On set of Reals $\mathbb{R}$; Relation S

xSy iff $x^2 = y^2$

xSy iff x, y have same square.

ER
- Refl ✓
- Sym ✓
- Trans ✓

Equality of Square

$[2]_s = \{2, -2\}$

$2 \sim -2$

$x \in$ Reals

$[3]_s = \{3, -3\}$

$[x]_s = \{x, -x\}$

$[0]_s = \{0\}$

$[0.3]_s = \{0.3, -0.3\}$

Examples

Let $S = \mathbb{R}$ and define the "square" relation $R = \{(x,y) \mid x^2 = y^2\}$.

The square relation is an equivalence relation.

1. For all $x \in \mathbb{R}$, $x^2 = x^2$, so $(x,x) \in R$.
2. If $(x,y) \in R$, $x^2 = y^2$, so $y^2 = x^2$ and $(y,x) \in R$.
3. If $(x,y) \in R$ and $(y,z) \in R$ then $x^2 = y^2 = z^2$, so $(x,z) \in R$.

For any set S, the identity relation on S, $I_S = \{(x,x) \mid x \in S\}$.

This is an equivalence relation for rather trivial reasons.

1. obvious
2. If $(x,y) \in R$ then $y = x$, so $(y,x) = (x,x) \in R$.
3. If $(x,y) \in R$ and $(y,z) \in R$ then $x = y = z$, so $(x,z) = (x,x) \in R$.

Modular Arithmetic

Let $S = \mathbb{Z}$. For each positive integer m, we define the ***modular relation*** $\equiv_m$,by $x \equiv_m y$ iff $m \mid (x-y)$, i.e. $\equiv_m = \{(x,y) : m \mid x - y \}$.

Examples:

$7 \equiv_5 2$, $11 \equiv_5 1$, $10 \equiv_5 0$, $-12 \equiv_5 3$

$7 \equiv_3 1$, $11 \equiv_3 2$, $10 \equiv_3 1$, $-12 \equiv_3 0$

Another way to think about the modular relation is:

$x \equiv_m y$ iff x and y have the same remainder when divided by m.

By the division algorithm, $x = mq_1 + r_1$, $y = mq_2 + r_2$, so $x - y = m(q_1-q_2) + (r_1-r_2)$ so, $m \mid x-y$ iff $m \mid r_1 - r_2$. Since $|r_1-r_2| < m$, $m \mid r_1 - r_2$ iff $r_1 - r_2 = 0$ iff $r_1 = r_2$.

Modular Arithmetic

Theorem: For any natural number m, the modular relation $\equiv_m$ is an equivalence relation on $\mathbb{Z}$.

Pf: For any x in $\mathbb{Z}$, since $x - x = 0$ and $m \mid 0$, $x \equiv_m x$. (Reflexitivity)

If $x \equiv_m y$ then $m \mid x - y$. Since $y - x = -(x-y)$, $m \mid y - x$, and so, $y \equiv_m x$. (Symmetry)

If $x \equiv_m y$ and $y \equiv_m z$ then $m \mid x - y$ and $m \mid y - z$. Since

$$x - z = (x - y) + (y - z)$$

we have $m \mid x - z$, so $x \equiv_m z$. (Transitivity)

Equivalence Classes

Given an equivalence relation R on a set S, we define the ***equivalence class containing an element x*** of S by:

$$[x]_R = \{y \mid (x,y) \in R\} = \{y \mid x\ R\ y\}.$$

The text uses the notation x/R (which I am not fond of) for what I have called $[x]_R$.

Examples:

Let $S = \{1, 2, 3\}$ and $R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$.
Then $[1] = \{1,2\}$ $[2] = \{1,2\}$ $[3] = \{3\}$.

Let $S = \mathbb{R}$ and $R = \{(x,y) \mid x^2 = y^2\}$.
Then $[0] = \{0\}$, $[1] = \{1,-1\}$, $[\frac{1}{4}] = \{\frac{1}{4}, -\frac{1}{4}\}$, $[x] = \{x, -x\}$.

Equivalence Classes

More Examples:

Let $S = \mathbb{Z}$ and $R = \{(x,y) \mid x \text{ and } y \text{ have the same parity}\}$.

$[0] = [2] = \ldots = [2k] = \{0, \pm 2, \pm 4, \pm 6, \ldots, \pm 2k, \ldots\}$

$[-1] = [1] = \ldots = [2k+1] = \{\pm 1, \pm 3, \pm 5, \ldots, \pm 2k+1, \ldots\}$

For any set S, $I_S = \{(x,x) \mid x \in S\}$.

$[a] = \{a\}$ for all $a \in S$.

Let $S = \mathbb{Z}$ and $R = \text{"} \equiv_5 \text{"}$.

$[0] = \{0, \pm 5, \pm 10, \pm 15, \ldots, 5k\}$ $(k \in \mathbb{Z})$

$[1] = \{\ldots, -9, -4, 1, 6, 11, \ldots, 5k+1\}$

$[2] = \{\ldots, -8, -3, 2, 7, 12, \ldots, 5k+2\}$

$[3] = \{\ldots, -7, -2, 3, 8, 13, \ldots, 5k+3\}$

$[4] = \{\ldots, -6, -1, 4, 9, 14, \ldots, 5k+4\}$

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

Q: ER R on set A.

let $x, y \in A$.

1. If xRy then $[x]_R = [y]_R$

2. If $[x]_R = [y]_R$ then xRy.

xRy iff $[x]_R = [y]_R$

3) $x \not R y$ then $[x]_R \neq [y]_R$

4) $[x]_R \neq [y]_R$ then $x \not R y$

$x \not R y$ iff $[x]_R \neq [y]_R$

5. $\forall x \in A, \quad x \in [x]_R$

Reason: Reflexive

6. If $y \in [x]_R$ then $x \in [y]_R$

Reason: Symmetric

7. If $[x]_R \neq [y]_R$ then $[x]_R \cap [y]_R = \phi$

8 for any $x, y \in A$

$[x]_R = [y]_R$ OR $[x]_R \cap [y]_R = \phi$

iff xRy

iff $x \not R y$

9 If $[x] \cap [y] \neq \phi$ then

$x \sim y$

xRy, yRx

$[x] = [y]$

Properties of Equivalence Classes

Let R be an equivalence relation on the set S.

I. ***For all $x \in S$, $x \in [x]$.***

Since R is reflexive, $(x,x) \in R$ for all $x \in S$.

II. ***If $y \in [x]$ then $x \in [y]$, and $[x] = [y]$.***

Since R is symmetric, if $y \in [x]$ then $(x,y) \in R$ so $(y,x) \in R$ and we have $x \in [y]$. If $s \in [x]$, then $(x,s) \in R$, so $(s,x) \in R$ and then $(s,y) \in R$ (by transitivity) and finally $(y,s) \in R$, so $s \in [y]$. Similarly, if $t \in [y]$ then $t \in [x]$ and so, $[x] = [y]$.

III. ***For any x and $y \in S$, either $[x] = [y]$ or $[x] \cap [y] = \varnothing$.***

If there is a $z \in [x]$ which is also in $[y]$, then $(x,z) \in R$ and $(y,z) \in R$. By symmetry, $(z,y) \in R$ as well. By transitivity, $(x,y) \in R$, so $y \in [x]$. By II, $[x] = [y]$.

Let R be an equivalence relation on a set A. These statements for elements a and b of A are equivalent:

(i) aRb (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

aRb

$[a] = [b]$

$[a] \cap [b] \neq \phi$

same

Partitions and Equivalences

- ***Theorem:*** Any equivalence relation R over a set A defines a partition of A.
- ***Theorem:*** Any partition of A defines an equivalence relation over A.

Theorem 3.3. *Suppose that S is a set. Then:*

(i) *Suppose R is an equivalence relation of S. Then the classes of R form a partition of S.*

(ii) *Suppose that $P = \{S_1, \ldots, S_k \ldots\}$ is a partition of the set S. Then we can define an equivalence relation on S by*

$$xRy \iff x, y \in S_i \text{ for some } i$$

i.e. two elements are equivalent if they are in the same set.

Moreover, this correspondence between equivalence relations and partitions is a one-to-one correspondence i.e. the number of equivalence relations on a set is equal to the number of partitions of that set.

List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

Base Set A

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$ Partition of A

If we consider this Partition as Set of Eq. Classes of some Eq. Relation R then what is R?

ER $R \longrightarrow$ Eq Classes $E_1, E_2, --, E_n$

$$R = (E_1 \times E_1) \cup (E_2 \times E_2) \cup \dots \cup (E_n \times E_n)$$

$$|R| = |E_1|^2 + |E_2|^2 + \dots + |E_n|^2 \checkmark$$

List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

a) $\{0\}, \{1, 2\}, \{3, 4, 5\}$

$R = \{(0,0), (1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4)(5,5)\}$

$|R| = 1^2 + 2^2 + 3^2$

List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$ — Partition of Base set

set of Eq. Classes of R

b) $\{0, 1\}, \{2, 3\}, \{4, 5\}$

$R = \{(0,0), (0,1), (1,0), (1,1), (2,2), (2,3), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$

$|R| = 2^2 + 2^2 + 2^2 = 12$

List the ordered pairs in the equivalence relations produced by these partitions of $\{0, 1, 2, 3, 4, 5\}$.

c) $\underbrace{\{0, 1, 2\}, \{3, 4, 5\}}_{\text{set of Eq. Classes of } R}$ — partition

$|R| = ?$

c) $\{0, 1, 2\}, \{3, 4, 5\}$

$R = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,3), (3,4), (3,5),$
$(4,3), (4,4), (4,5), (5,3), (5,4), (5,5)\}$

$|R| = 3^2 + 3^2 = 18$

Q: (Some Real Life, Non-mathematical examples of ER)
Relation R on set of all Indian people.
aRb iff a,b are from same state. → Equality of State
What are the equivalence classes?

States + UT

State — Eq Class

Base set

R: India → India

Q: (Some Real Life, Non-mathematical examples)

Relation R on set of all people in the world.

aRb iff a,b are have same gender. — ER ✓

What are the equivalence classes?

R

3 Eq classes

males

females

special

Q :

Let R be an equivalence relation on Set A.

Which of the following is/are true?

1. Every equivalence class of R is subset of R.
2. Every equivalence class of R is subset of A.
3. Set of equivalence classes of R is subset of A.
4. Set of equivalence classes of R is subset of P(A).
5. Two different equivalence classes are disjoint.
6. Set of equivalence classes of R is partition of A.
7. Set of equivalence classes of R is partition of P(A).
8. Set of equivalence classes of R is partition of R.

Q :

Base set

Let R be an equivalence relation on Set A.

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Base set $A = \{2, 3, 4, 5, 6, 7\}$

R on A ; xRy iff $\boxed{x \equiv_4 y}$

means

$x \equiv y \pmod 4$

① $R \rightarrow$ ERL ✓

② Eq. Classes of R :

$E_1 : \{2, 6\}$ $E_3 = \{4 \}$

$E_2 : \{3, 7\}$ $E_4 = \{5\}$

4 Eq. Classes of R =

Eq. Classes of R

$\{2, 6\}$

$\{3, 7\}$

$\{4\}$

$\{5\}$

Subsets of Base set

a Partition of Base set.

$R = \{(2,2), (2,6), (6,2), (6,6), (3,3), (3,7), (7,3), (7,7), (4,4), (5,5)\}$

Set of ECs of R =

$$\{ \{2,6\}, \{3,7\}, \{4\}, \{5\} \}$$

Subset of Base set

is Subset of P(Base set)

Set of ECs of R =

$$\{ \{2,6\}, \{3,7\}, \{4\}, \{5\} \}$$

is Partition of Base set.

Eq. Class of R :

a Group of equivalent elements of Base set

Eq Class $\subseteq$ Base set

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example 3.2. Let $S = \{1, 2, 3, \ldots, 19, 20\}$ and define an equivalence relation R on S by

$$xRy \iff 4|(x-y)$$

Determine the equivalence classes of R.

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Determine the equivalence classes of R.

In order to determine equivalence classes, we simply need to group the elements together according to which ones are equivalent to each other. We start with 1 and move onward:

$$[1] = \{1, 5, 9, 13, 17\}$$

$$[2] = \{2, 6, 10, 14, 18\}$$

$$[3] = \{3, 7, 11, 15, 19\}$$

$$[4] = \{4, 8, 12, 16, 20\}$$

Notice that any other equivalence class we construct will be the same as one of these e.g $[1] = [5]$. Thus we have found all the different equivalence classes of R.

GO Classes

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$$xRy \iff 4|(x-y)$$

Determine the equivalence classes of R.

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Eq. Classes of R

$$\begin{aligned}
[1] &= \{1, 5, 9, 13, 17\} \\
[2] &= \{2, 6, 10, 14, 18\} \\
[3] &= \{3, 7, 11, 15, 19\} \\
[4] &= \{4, 8, 12, 16, 20\}
\end{aligned}$$

subset of Base set

Partition of Base set.

Notice that any other equivalence class we construct will be the same as one of these e.g $[1] = [5]$. Thus we have found all the different equivalence classes of R.

Q: Set A.

Partition of A is subset of P(A). ✓

Partition of A belongs to P(A) ✗

Set $A = \{a, b\}$

part

part/cell

Partition $\Pi = \underbrace{\{\{a\}, \{b\}\}}_{\text{partition}}$

belongs to $P(A)$

$$\boxed{\Pi \subseteq P(A)}$$

Example 3. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ such that*

$$((a, b), (c, d)) \in R \Leftrightarrow a + d = b + c.$$

Show that R is an equivalence relation.

Solution. R is reflexive: Suppose (a, b) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$. [We must show that $(a, b)\, R\, (a, b)$.] We have $a + b = a + b$. Thus, by definition of R, $(a, b)\, R\, (a, b)$.

R is symmetric: Suppose (a, b) and (c, d) are two ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ and $(a, b)\, R\, (c, d)$. [We must show that $(c, d)\, R\, (a, b)$.] Since $(a, b)\, R\, (c, d)$, $a + d = b + c$. But this implies that $b + c = a + d$, and so, by definition of R, $(c, d)\, R\, (a, b)$.

R is transitive: Suppose $(a, b), (c, d)$, and (e, f) are elements of $\mathbb{Z} \times \mathbb{Z}$, $(a, b)\, R\, (c, d)$, and $(c, d)\, R\, (e, f)$. [We must show that $(a, b)\, R\, (e, f)$.] Since $(a, b)\, R\, (c, d)$, $a + d = b + c$, which means $a - b = c - d$, and since $(c, d)\, R\, (e, f)C$, $c + f = d + e$, which means $c - d = e - f$. Thus $a - b = e - f$, which means $a + f = b + e$, and so, by definition of R, $(a, b)\, R\, (e, f)$. □

Example 3. *Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ such that*

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Show that R is an equivalence relation.

$(a, b)\, R\, (c, d)$

Solution. R is reflexive: Suppose (a, b) is an ordered pair in $\mathbb{Z} \times \mathbb{Z}$. [We must show that $(a, b)\, R\, (a, b)$.] We have $a + b = a + b$. Thus, by definition of R, $(a, b)\, R\, (a, b)$.

R is symmetric: Suppose (a, b) and (c, d) are two ordered pairs in $\mathbb{Z} \times \mathbb{Z}$ and $(a, b)\, R\, (c, d)$. [We must show that $(c, d)\, R\, (a, b)$.] Since $(a, b)\, R\, (c, d)$, $a + d = b + c$. But this implies that $b + c = a + d$, and so, by definition of R, $(c, d)\, R\, (a, b)$.

R is transitive: Suppose (a, b), (c, d), and (e, f) are elements of $\mathbb{Z} \times \mathbb{Z}$, $(a, b)\, R\, (c, d)$, and $(c, d)\, R\, (e, f)$. [We must show that $(a, b)\, R\, (e, f)$.] Since $(a, b)\, R\, (c, d)$, $a + d = b + c$, which means $a - b = c - d$, and since $(c, d)\, R\, (e, f)C$, $c + f = d + e$, which means $c - d = e - f$. Thus $a - b = e - f$, which means $a + f = b + e$, and so, by definition of R, $(a, b)\, R\, (e, f)$. □

Base set : $Z \times Z$

R ; $Z \times Z \rightarrow Z \times Z$

$(2,3) \cancel{R} (3,2)$

$(2,3)\, R\, (2,3)$

$[2]_R$ = Non sense

not element in Base set.

$$[(2,3)]_R = \{(2,3), (3,4), (-2,-1), \ldots \ldots\}$$

$$[(2,3)]_R = \{ (y-1, y) \mid y \in \mathbb{Z} \}$$

$(2,3)\ R\ (x,y)$

$2 + y = x + 3$

$x = y - 1$

$$[(0,0)]_R = \{(y,y) \mid y \in Z\}$$

$$[(2,-2)]_R = \{(y,-y) \mid y \in Z\}$$

$$[(2,-2)]_R \subseteq Z \times Z$$

Base set

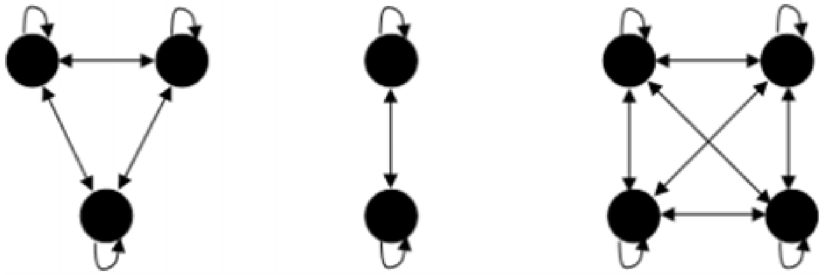
Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{\ y \in A \mid xRy\ \}$$

- $[x]_R$ is the set of all elements of A that are related to x.
- ***Theorem:*** If R is an equivalence relation over A, then every $a \in A$ belongs to exactly one equivalence class.

Examples of equivalence relations on a set of people include "same age as" and "born in the same month as." If we have nine people, with three born in March, two born in May, and four born in December, the graph of the relation "born in the same month as" would appear as follows:



E_1 E_2 E_3

Q : Congruence Modulo 3

The relation $R = \{(a, b) \mid a \equiv b \pmod 3\}$

is an equivalence relation on the set $\{1,3,4,6,10,12\}$. Base set

How many equivalence classes does R have? 2

$$\left.\begin{array}{l} [1]_R = \{1, 4, 10\} \\ [3]_R = \{3, 6, 12\} \end{array}\right\} \text{Partition of Base set}$$

Q :

Base set

Suppose that R is the relation on the set of English words such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x.

Is R an equivalence relation? → yes

room $\sim$ gate

gate $\sim$ book

gate $\not\sim$ study

room $\in [\text{gate}]_R$

mo $\in [\text{go}]_R$

Note: Set A

Eq. Relation on A $\longleftrightarrow$ Partition of A

#ER on A = #Partitions of A

Example 3.4. Determine the total number of equivalence relations on a set with three elements.

$A = \{a, b, c\}$

Partitions? — 5

5 Eq. Rel. on A.

Q: Let $\sim$ be an equivalence relation on $A = \{v, w, x, y, z\}$ with three equivalence classes. Suppose $v \sim y$ and $z \in [x]$, where $[x]$ denotes the equivalence class of x. Write $\sim$ as a subset of $A \times A$ and find the partition of A determined by the equivalence classes.

R

Base set

Q: Let $\sim$ be an equivalence relation on $A = \{v, w, x, y, z\}$ with three equivalence classes. Suppose $v \sim y$ and $z \in [x]$, where $[x]$ denotes the equivalence class of x. Write $\sim$ as a subset of $A \times A$ and find the partition of A determined by the equivalence classes.

R —— E_1, E_2, E_3

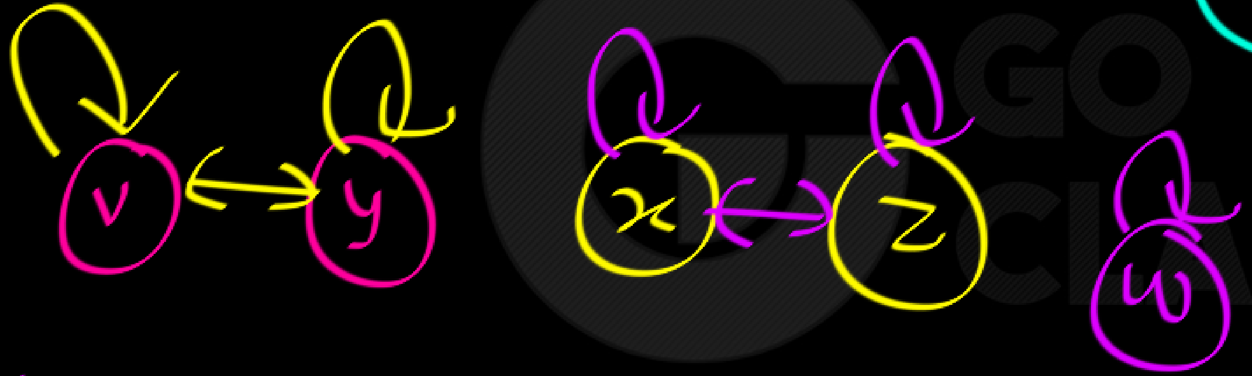
$\{ \{v, y\}, \{z, x\}, \{w\} \}$

$|R| = 2^2 + 2^2 + 1^2$
$= 9$

$\{\{v, y\}, \{x, z\}, \{w\}\}$ — set of Eq. classes

— Partition of A

— subset of P(A)



$|R| = 9 = \{(v,v), (v,y), (y,v), (y,y) \ldots\}$

Observation: $z \notin [v]$

because if $z \in [v]$ then

$\{v, y, z, x\}$ $\{w\}$

So we cannot have 3 Eq classes.

Equivalence relations. A relation that is reflexive, symmetric, and transitive is called an *equivalence relation*. Clearly, the common relation $=$ on the set $\mathbb{R}$, say, is an equivalence relation. Also, we have seen earlier that the congruence relation $\equiv_n$ on the set $\mathbb{Z}$ is reflexive, symmetric, and transitive, thus it is also an equivalence relation. The similarity relation on the set of triangles in the plane is another example.

Equivalence relations are special in that they naturally partition the underlying set into *equivalence classes*. For example, the relation $\equiv_2$ partitions the integers into even and odd ones. These are, respectively, the integers that are related (by $\equiv_2$) to 0, and the ones related to 1. Let's formalize these concepts.

Equivalence Classes

- Given an equivalence relation R over a set A, for any $a \in A$, the **equivalence class of *a*** is the set

$$[a]_R \equiv \{ x \mid x \in A \text{ and } a\mathrm{R}x \}$$

- Informally, the set of all elements equal to a.
- R **partitions** the set A into a set of equivalence classes.

Partial Orders

Partial order Relation (POR)