Group Theory

Next Chapter:

Some Points in Group Theory

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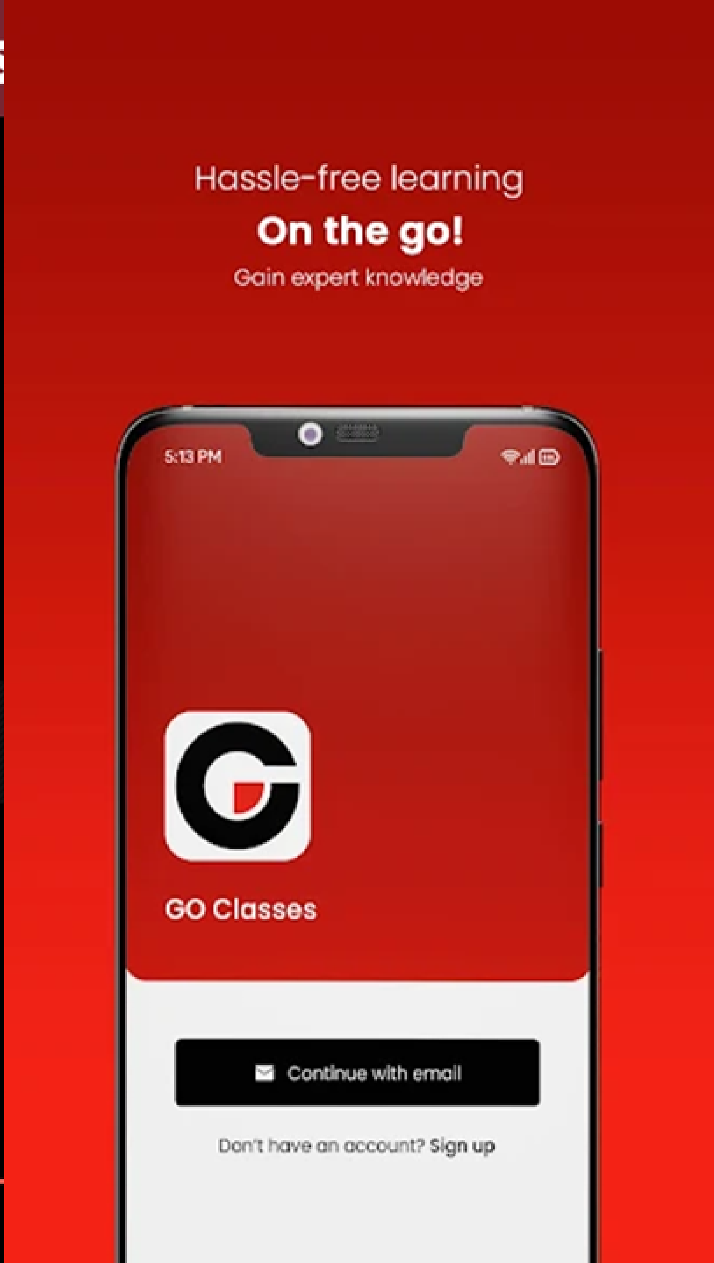

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Next Topic:

Associativity & Parentheses

In propositional logic :

$F \wedge T \wedge F$

$= F$ ✓

Associative

$(F \wedge T) \wedge F = F \wedge (T \wedge F)$

NOT Asso.

$F \to T \to F$

$(F \to T) \to F$

$T \to F$

F ✓

$F \to (T \to F)$

$F \to F$

T ✓

$\rightarrow$ is NOT Asso.

$a \rightarrow b \rightarrow c$ $\rightarrow$ Ambiguous

$(a \rightarrow b) \rightarrow c \neq a \rightarrow (b \rightarrow c)$

$(a \rightarrow b) \rightarrow c$ $\rightarrow$ Unambiguous

$a \rightarrow (b \rightarrow c)$

In Number theory:

Associative

$2 + 3 + 2$

$= 7$

$(2+3)+2 = 7$

$7 \quad 2 + (3+2)$

NOT Associative

$2 - 3 - 2$

$(2-3)-2$

$= -3$

$2 - (3-2)$

$= 1$

NOT Asso.

$a - b - c$

Ambiguous

$(a - b) - c$ ✓

$a - (b - c)$ ✓

Associative

$$\underbrace{2 \times 3 \times 2}$$

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$$(2 \times 3) \times 2$$

$$= 2 \times (3 \times 2)$$

Power

$$2 \uparrow 3 \uparrow 2$$

NOT Asso.

$$(2 \uparrow 3) \uparrow 2$$

$$= \left(2^3\right)^2 = 8^2$$

$$= 64$$

$$2 \uparrow (3 \uparrow 2)$$

$$= (2)^{\left(3^2\right)}$$

$$= 2^9 = \underline{\underline{512}}$$

What properties of numbers allow us to remove parentheses from expressions?

What properties of numbers allow us to remove parentheses from expressions?

Ans: Associativity

Asso : $+$, $\times$, $\wedge$

$$a+b+c+d = (a+b)+(c+d)$$

$$= ((a+b)+c)+d = (a+(b+c))+d$$

$$= a+[b+(c+d)] = \ldots$$

Asso : $+$, $\times$, $\wedge$

$a + b + c + d$ — fine

$a - b - c - d$ — Ambiguous

$a \rightarrow b \rightarrow c$ — Ambiguous

Q: What properties of numbers allow us to remove parentheses from expressions?

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The point is that no matter how you parenthesize $a + b + c$, the result is the same, so the expression without parentheses is unambiguous. Thus, it doesn't matter whether you define $a + b + c$ to be $(a + b) + c$ or $a + (b + c)$. (Note that some such definition is ultimately required, since $+$ is originally defined only as a binary operation.)

Compare this with $a - b - c$: in general $(a - b) - c \neq a - (b - c)$, so the expression $a - b - c$ is uninterpretable without some convention, e.g., work from left to right. When the operation is associative, no such convention is required.

: binary op^n

Q:

a#b#c

for

For which of the following the above expression makes sense (i.e. is unambiguous) ?

1. Groupoid
2. Semi Group
3. Monoid
4. Group

: binary opⁿ

Q:

a#b#c

For which of the following the above expression makes sense (i.e. is unambiguous) ?

for

1. Groupoid → may not be Associative.
2. Semi Group
3. Monoid
4. Group

→ Associative prop.

$(Z, +)$ – Group

Groupoid

$(Z, -)$ – Groupoid

Q:

$$(a\#a)\#a = a\#(a\#a)$$

For which of the following the above expression is Always True ?

1. Groupoid
2. Semi Group
3. Monoid
4. Group

: binary op^n

Q:

$$\boxed{(a\#a)\#a = a\#(a\#a)}$$

For which of the following the above expression is Always True ?

1. Groupoid
2. Semi Group
3. Monoid
4. Group

$\rightarrow$ Associative

$(Z, -)$ — Groupoid; but not semigroup

$a = 2$

$(a - a) - a \overset{?}{=} a - (a - a)$

$-2 \neq 2$

What properties of numbers allow us to remove parentheses from expressions?

Ans: Associativity

If operation # is associative then:

a#b#c#d is unambiguous expression

Next Topic:

Cayley Table =

= Operation Table =

= Multiplication Table

1.4 Cayley tables

A binary operation $*$ on a *finite* set S can be displayed in the form of an array, called the *Cayley* table.

finite set $S = \{a, b, c\}$; binary op^n #

Header

#	a	b	c
a	a#a	a#b	a#c
b	b#a	b#c	b#c
c	c#a		c#c

c#b → Row → Column

Binary Algebraic Structure $(\{T,F\}, \wedge)$

$\wedge$	F	T
F	F	F
T	F	T

Cayley Table

Groupoid ✓
Semigroup ✓
monoid ✓ ($e = T$)
Group ✗ (no inverse prop)

$\boxed{T^{-1} = T}$ ✓

$F \wedge \bigcirc = T$

DNE

$F^{-1} = \text{DNE}$

Binary Algebraic Structure $(\{T,F\}, \vee)$

$\vee$	T	F
T	T	T
F	T	F

multiplication Table /

Cayley Table

monoid

$(e = F)$

not a Group

$F^{-1} = F$

$T \vee ? = F$

DNE

T^{-1}: DNE

Binary Algebraic Structure $(\{0,1\}, \times)$

$\times$	0	1
0	0	0
1	0	1

multiplication

monoid ($e = 1$)

not a Group

$1^{-1} = 1$

$0 \times \underline{?} = 1$

0^{-1} : DNE

Binary Algebraic Structure $(\{1,-1\}, \times)$

$\times$	-1	1
-1	1	-1
1	-1	1

Assoc

Group

$e = 1$

$1^{-1} = 1$

$e^{-1} = e$

$(-1) \times \underline{(-1)} = 1$

$(-1)^{-1} = -1$

1.4 Cayley tables

A binary operation $*$ on a *finite* set S can be displayed in the form of an array, called the *Cayley* table.

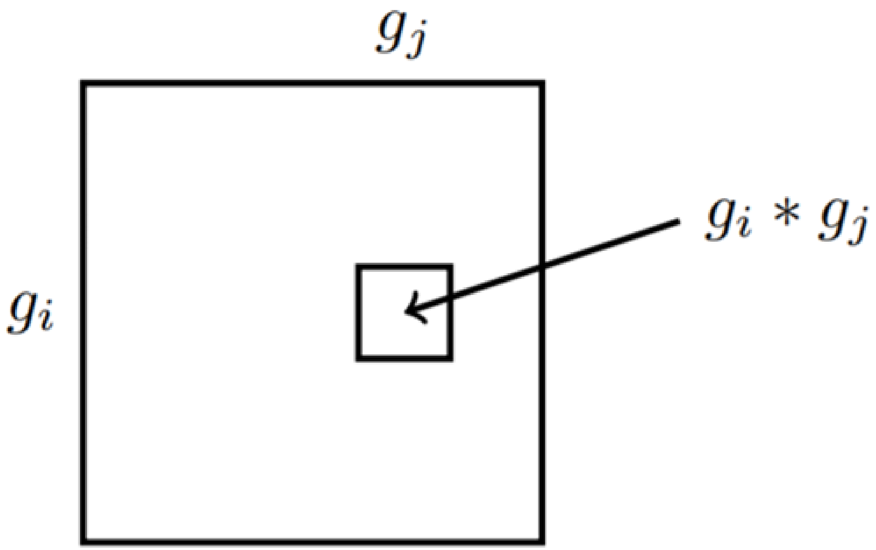
If S has n elements, then the Cayley table is an $n \times n$ array, with each row and each column labelled (uniquely) by an element of S.

The entry of the table in row x and column y is the element $x * y \in S$.

Here is a simple example: $S = \{0, 1\}$, and $*$ is just multiplication of numbers.

$*$	0	1
0	0	0
1	0	1

Let $(G, *)$ be a group where $G = \{g_1, g_2, \ldots, g_n\}$. Consider the multiplication table of $(G, *)$.



1.3 Remark Sometimes a binary operation $*$ is given by a table of the form

$$\begin{array}{c|ccc} * & \cdots & y & \cdots \\ \hline \vdots & & \vdots & \\ x & \cdots & x * y & \\ \vdots & & \vdots & \end{array}$$

For instance, the binary operation "and" on the set $\{\text{true}, \text{false}\}$ can be depicted as

$\wedge$	true	false
true	true	false
false	false	false

Thus, $(\{\text{true}, \text{false}\}, \wedge)$ is a commutative monoid with identity element true.

Cayley Tables

A (binary) operation on a finite set can be represented by a table. This is a square grid with one row and one column for each element in the set. The grid is filled in so that the element in the row belonging to x and the column belonging to y is x*y. Example:

This is a table for a binary operation on the set {A,B,C,D,E,F}

	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	C	A	F	D	E
C	C	A	B	E	F	D
D	D	E	F	A	B	C
E	E	F	D	C	A	B
F	F	D	E	B	C	A

Notice that B*E = D while E*B = F so this operation is not commutative.

The set of complex numbers $G = \{1, i, -1, -i\}$ under multiplication. The multiplication table for this group is:

$*$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Let G be a group with operation $\star$ and Cayley table given below.

Find:

$b*d = ?$

$db = ?$

$bbabc = ?$

$bbbb = ?$

$eeee = ?$

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	c	e	a	d
c	c	d	e	b	f	a
d	d	c	f	a	e	b
f	f	b	a	d	a	e

Group opⁿ

Let G be a group with operation $\star$ and Cayley table given below.

Associative

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	c	e	a	d
c	c	d	e	b	f	a
d	d	c	f	a	e	b
f	f	b	a	d	a	e

Find:

$b*d = ? = a$

$db = ? = f$

$bbabc = ?$

$bbbb = ?$

$eeee = ?$

$b \star d = a$

$db = d \star b = d.b$

$d \star b = f$

$\boxed{a\ b\ c}$ — unambiguous → fine because $*$ is Associative.

$(ab)c = a(bc)$

$abc = a * b * c$

$(a * b) * c$

$=$

$a * (b * c)$

Let G be a group with operation $\star$ and Cayley table given below.

Find:

b*d = ?

db = ?

bbabc = ?

bbbb = ?

eeee = ?

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	c	e	a	d
c	c	d	e	b	f	a
d	d	c	f	a	e	b
f	f	b	a	d	a	e

$abc = a$

$a * b * c$

$(\underbrace{a * b}_{d}) * c = d * c = a$

$a * (\underbrace{b * c}_{e}) = a * e = a$

Let G be a group with operation $\star$ and Cayley table given below.

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	c	e	a	d
c	c	d	e	b	f	a
d	d	c	f	a	e	b
f	f	b	a	d	a	e

Find:

$b*d = ?$

$db = ?$

$bbabc = ? = a$

$bbbb = ?$

$eeee = ?$

$c*a$

$bbabc$

$=$

$b*b*a*b*c$

$c*a*b*c$

$d*b*c =$

$f*c$

$f*c = a$

Let G be a group with operation $\star$ and Cayley table given below.

$\star$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	c	e	a	d
c	c	d	e	b	f	a
d	d	c	f	a	e	b
f	f	b	a	d	a	e

$\boxed{bb = b * b}$

Find:

$b*d = ?$

$db = ?$

$bbabc = ?$

$bbbb = ? = b$

$eeee = ? = e$

$bbbb$

$c\,bb$

$eb = b$

$e * b = eb$

$eeee$

$e\,e$

e

$= e$

Group $(G, \#)$ → Asso

ab means $a \# b$

$$abc = (ab)c = a(bc)$$

fine because $\#$ is Asso.