Next Topic:

$(N, |)$ Lattice

$$GLB = GCD$$
$$LUB = LCM$$

The poset $\mathbb{N}_+$ under the divisibility ordering is a lattice! Indeed, it turns out that the meet operation corresponds to *greatest common divisor* and the join operation corresponds to *least common multiple*.

$(N, |)$ — Lattice ✓

Poset ✓

TOR X

$$\left\{\begin{array}{l} a \vee b = \text{Lcm}(a,b) \\ a \wedge b = \text{GCD}(a,b) \end{array}\right\}$$

$7 \wedge 18 = 1$

$10 \wedge 12 = 2$

$7 \vee 18 = \text{Lcm}(7,18) = 54$

Next Topic:

Properties of Lattice

(Properties that are satisfied by EVERY Lattice)

Proposition 5.2.2 *If X is a lattice, then the following identities hold for all $a, b, c \in X$:*

L1	$a \vee b = b \vee a,$	$a \wedge b = b \wedge a$
L2	$(a \vee b) \vee c = a \vee (b \vee c),$	$(a \wedge b) \wedge c = a \wedge (b \wedge c)$
L3	$a \vee a = a,$	$a \wedge a = a$
L4	$(a \vee b) \wedge a = a,$	$(a \wedge b) \vee a = a.$

Properties (L1) correspond to commutativity, properties (L2) to associativity, properties (L3) to idempotence and properties (L4) to absorption. Furthermore, for all $a, b \in X$, we have

$$a \leq b \quad \text{iff} \quad a \vee b = b \quad \text{iff} \quad a \wedge b = a,$$

called consistency.

for Every Lattice :

① $a \vee a = a$, $a \wedge a = a$ } Idempotent Property

② $a \vee b = \text{LUB}\{a, b\} = \text{LUB}\{b, a\} = b \vee a$

② $a \vee b = b \vee a$
$a \wedge b = b \wedge a$ } Commutative Property

③ $(a \vee b) \vee c = a \vee (b \vee c)$
$= a \vee b \vee c =$
$LUB\{a, b, c\}$

③ $(a \vee b) \vee c = a \vee (b \vee c)$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

Absorption Property:

$$a \vee (a \wedge b) = a$$

$$a \wedge (a \vee b) = a$$

Absorption Property:

$$a \vee (a \wedge b) = a$$

Case1: aRb

$a \wedge b = a$

$a \vee a = a$

$\boxed{a = a}$

Case2: bRa

$a \wedge b = b$

$a \vee b = a$

$a = a$

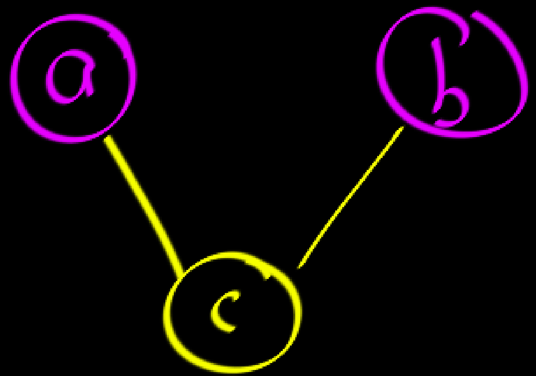
Case3: a, b Not Comparable.

Case 3: a, b are not Comparable:

$$\boxed{a \vee \underbrace{(a \wedge b)}_{c}}$$

because it is lattice, $a \wedge b = c$

$$\boxed{a \vee c = a}$$



Prove that $\boxed{a \wedge (a \vee b) = a}$

proof:

① Case 1: aRb

$a \wedge (a \vee b)$

$a \wedge b = \boxed{a = a}$

② Case 2: bRa

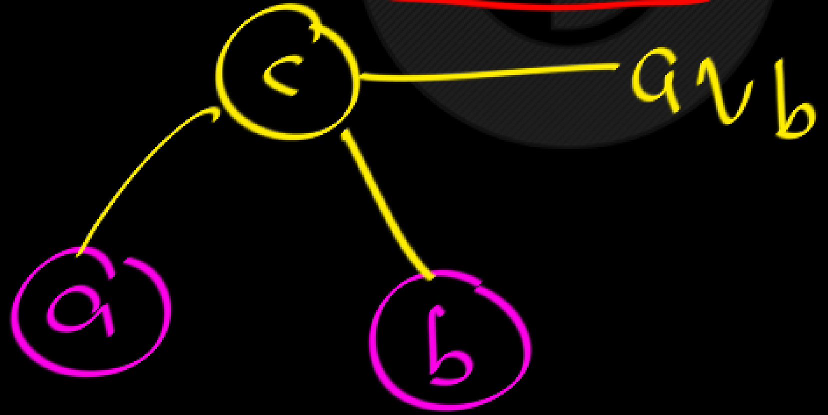
$a \wedge (a \vee b) = a$

$a \wedge a = \boxed{a = a}$

③ **Case 3:** a, b are not Comparable.

$a \wedge (a \vee b)$

$a \wedge c = \boxed{a = a}$



because it is a Lattice, $a \vee b$ exists.

Proposition

Any lattice has the following properties:

1. *Commutativity:* $a \cap b = b \cap a$ *and* $a \cup b = b \cup a$.
2. *Associativity:* $(a \cap b) \cap c = a \cap (b \cap c)$ *and* $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law:* $a \cap a = a$ *and* $a \cup a = a$.
4. *Absorption law:* $(a \cup b) \cap a = a$ *and* $(a \cap b) \cup a = a$.

Let L be a lattice. Define the **meet** ($\wedge$) and **join** ($\vee$) operations by $x \wedge y = \text{glb}(x, y)$ and $x \vee y = \text{lub}(x, y)$.

39. Show that the following properties hold for all elements x, y, and z of a lattice L.

a) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$ **(commutative laws)**

b) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ and $(x \vee y) \vee z = x \vee (y \vee z)$ **(associative laws)**

c) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$ **(absorption laws)**

d) $x \wedge x = x$ and $x \vee x = x$ **(idempotent laws)**

Proposition Every Lattice Satisfies these:

Any lattice has the following properties:

1. *Commutativity:* $a \cap b = b \cap a$ *and* $a \cup b = b \cup a$.
2. *Associativity:* $(a \cap b) \cap c = a \cap (b \cap c)$ *and* $(a \cup b) \cup c = a \cup (b \cup c)$.
3. *Idempotent law:* $a \cap a = a$ *and* $a \cup a = a$.
4. *Absorption law:* $(a \cup b) \cap a = a$ *and* $(a \cap b) \cup a = a$.

Identity
Complement
De-morgan
Distributive

Only satisfied by some lattices, Not by All lattices

Identity element of LUB := e

$$\left\{\begin{array}{l} a \vee e = a \\ e \vee a = a \end{array}\right\} \forall a$$

fixes for all a

$x \vee y = y \vee x$

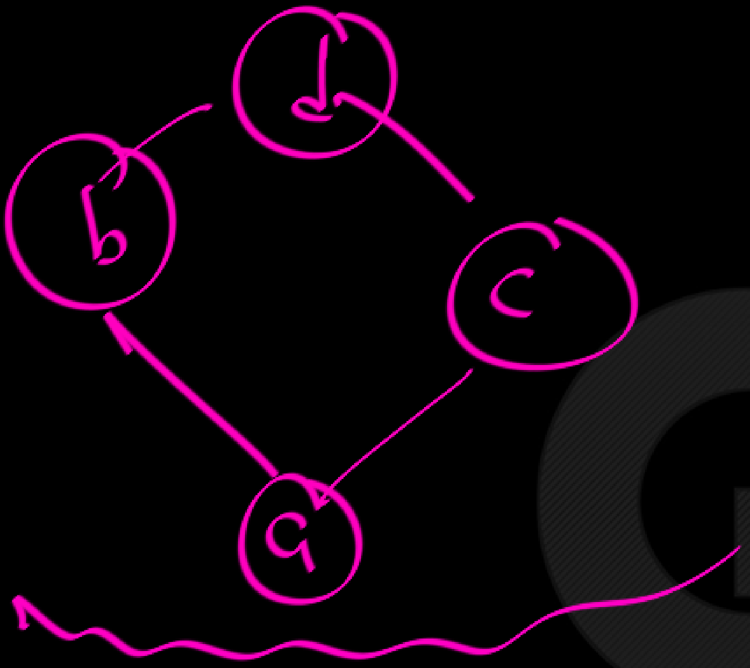
for all lattices

Identity element of GLB : $= e$

$$\left\{ \begin{array}{l} a \wedge e = a \\ e \wedge a = a \end{array} \right\} \forall a$$

fixes for all a

$x \wedge y = y \wedge x$



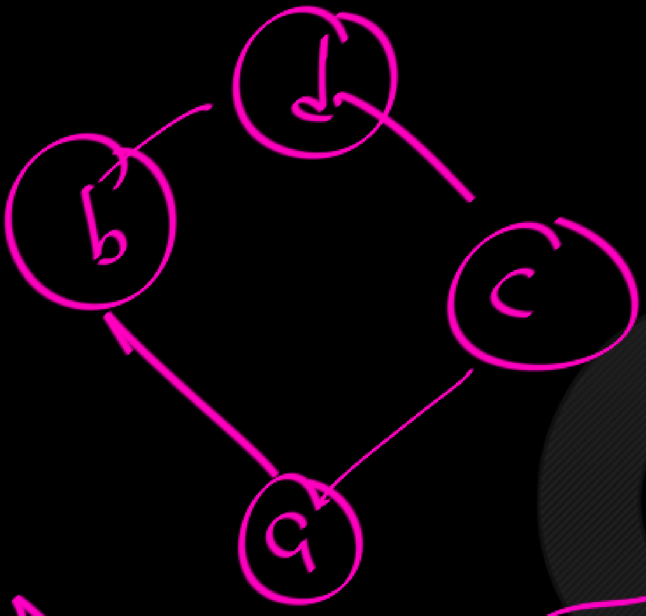
$c \vee a = c$

$b \vee a = b$

$a \vee a = a$

$d \vee a = d$

ID for LUB
$= a$



$$c \wedge d = c$$

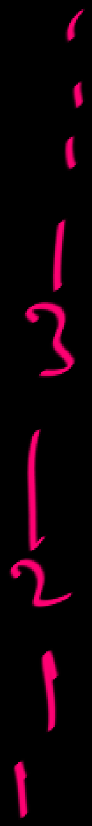
$$b \wedge d = b$$

$$a \wedge d = a$$

$$d \wedge d = d$$

ID for GLB $= d$

$(N, \leq)$ —— Id element for LUB $= 1$



$1 \vee 1 = 1$

$1 \vee 2 = 2$

$1 \vee 3 = 3$

$1 \vee 4 = 4$

$1 \vee a = a, \ \forall a$

$(N, \leq)$ — Id element for GLB = **DNE**

Lattice

$2 \wedge \bigcirc = 2$

$3 \wedge \bigcirc = 3$

$4 \wedge \bigcirc = 4$

$5 \wedge \bigcirc = 5$

⋮

3

2

1

Every total order is a Lattice : Yes

No pair of Non-Comparable elements.

$\forall_{a,b}$, a, b are Comparable

$a \vee b$ = exists ; $a \wedge b$ = exists

Lattice in which we don't have

Identity element for LUB : DNE

$(Z^-, \le)$

-1
|
-2
|
-3
⋮

$-1 \wedge a = a$

$-1 \wedge -100 = 100$

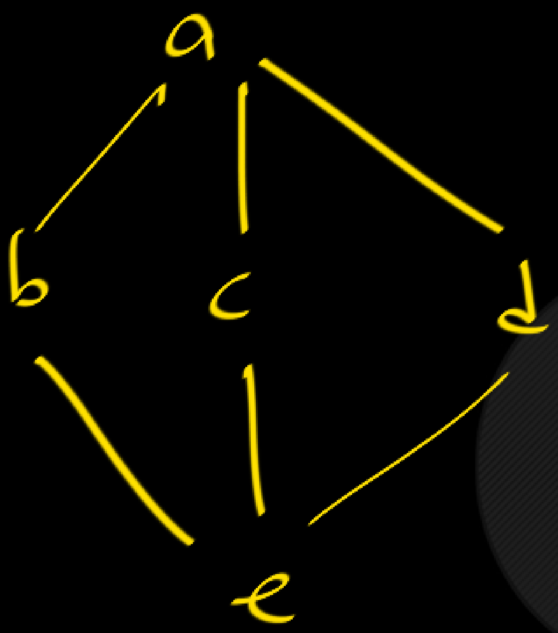
$-1 \wedge -10000 = -10000$

Id element for GLB = -1

Lattice in which we don't have Identity element for LUB, GLB:

$(\mathbb{Z}, \leq)$, $(\mathbb{R}, \leq)$

So, Identity property is not satisfied by some lattices.



Lattice

$$b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$$

$$b \wedge a \qquad\qquad e \vee e$$

$$b \neq e$$