Combinatorics

Topics:

Binomial Theorem

Bijective Proofs,

Permutation with Repetition

Combinatorics

Next Topic:

Some More Combinatorial Arguments

Website : https://www.goclasses.in/

Create a Combinatorial Argument

$$\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$$

Story:

$[n] = \{1, 2, --, n\}$

Create Subsets

$$\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$$

Story: Create subset of a set $[n] = \{1, 2, \ldots, n\}$

Side 1: $\#\text{subsets} = 2^n$ ✓

Side 2: #Subset of size 0 + #subset of size 1 + #subset of size 2 + ... + #subset of size n

$$^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$$

Side 1 = Side 2

$$2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n$$

$$\boxed{2^n = \sum_{r=0} {}^nC_r}$$

binomial coefficient

Theorem 4: For any nonnegative integer n

$$\Sigma_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

LHS tells you all the ways of picking a subset of n people(a subcommittee) and designating one of its members as special (subcommittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Theorem 4: For any nonnegative integer n

Subsets of n People

one Designated element

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

LHS tells you all the ways of picking a subset of n people(a subcommittee) and designating one of its members as special (subcommittee chairman).

What's another way of doing this? Pick the chairman first, and then the rest of the subcommittee!

Story: n People ; Select subset from n People and make one of them President.

Side 1: first make a subset from n People then from that subset make one President.

$$\sum_{k=0}^{n} \binom{n}{k} * k$$

Side 1: Select 0 People and make one of them president OR Select 1 People and make one of them President OR ... Select n People and one President

$$0 \cdot \binom{n}{0} + \binom{n}{1} 1 + \binom{n}{2} 2 + \cdots + \binom{n}{n} n$$

$$= \sum_{k=0}^{n} \binom{n}{k} * k$$

Story: n people ; Select subset from n people and make one of them President.

Side2: first select one President then select a subset of Remaining People.

$$\binom{n}{1} * 2^{n-1}$$

Side 1 = Side 2

$$\sum_{k=0}^{n} k\binom{n}{k} = n(2)^{n-1}$$ ✓

$$\sum_{k=1}^{n} k\binom{n}{k} = n(2)^{n-1}$$ ✓

Story:

$\underbrace{\text{subset from n People}}_{1}$ AND $\underbrace{\text{make one of them from secles subset a President}}_{2}$

Side1: 1 $\longrightarrow$ 2

Side2: 2 $\longrightarrow$ 1

Q: n People ; $\Longrightarrow$ Committee with Captain

Side 2 $\rightarrow$ Captain

$$\underbrace{\left({}^{n}C_{1}\right)}_{\text{Rohit Sharma}} * \underbrace{2^{n-1}}_{\text{remaining} = n-1 \text{ People}}$$

2^{n-1} $\rightarrow$ any subset of Remaining n-1 People

take

Don't take

Q: $n = 3$ → V, R, P

$3 \times 2^2 = 12$

make a IPL team with Captain

Auction

Cap		Cap		Cap	
V		R	V	P	
V	R	R	P	P	R
V	P	R	V, P	P	V
V	R P	R		P	R, V

$\Rightarrow 12$

Prove that

$$nC_r + (n-1)C_r + (n-2)C_r + \ldots\ldots\ldots + rC_r$$
$$= (n+1)C_{r+1}$$

Prove that

$$nC_r + (n-1)C_r + (n-2)C_r + \ldots\ldots. + rC_r = (n+1)C_{r+1}$$

selecting r+1 numbers from n+1 numbers

[n+1] = $\boxed{1, \quad 2, \quad 3, \ldots r}$, r+1 ... , n-2, n-1, n, n+1

Story: from n+1 numbers, select r+1 numbers.

Side 1: $\binom{n+1}{r+1}$ ← select r+1

$[n+1] = \{1, 2, 3, --, n+1\}$

Story: from n+1 numbers, select r+1 numbers.

Side 2: Select r+1 But first you select what is the maximum number in your subset of r+1 numbers.

Side 2: $\longrightarrow$ Target = subset of $r+1$ numbers.

$1 \quad 2 \quad 3 \quad \ldots \quad r \quad r+1 \quad r+2 \quad \ldots \quad n-2 \quad n-1 \quad n \quad n+1$

maximum no. selected $= n+1$ OR max no. selected n OR ... OR max no. selected $r+1$

$$^{n}C_r + {}^{n-1}C_r + {}^{n-2}C_r + \ldots + {}^{r}C_r$$

side 1 = side 2 → first you Decide on max. no. that you will select

$$\binom{n+1}{r+1} = {}^{r}C_{r} + {}^{r+1}C_{r} + \cdots + {}^{n}C_{r}$$

max = r+1

max = n+1

$$\boxed{\binom{n+1}{r+1} = \sum_{k=r}^{n} {}^{k}C_{r}}$$

Note: All stories that you need to be familiar with (all "standard" stories) we have covered. No new variation you need to worry about.

Conclusion:

① $\binom{n}{r} = \binom{n}{n-r}$

Story: select r ≡ Reject n−r

② $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$

specific person is selected or Rejected

Recursive formula for Bin. Coeff

Variation:

$$\binom{n-1}{r-1} = \binom{n}{r} - \binom{n-1}{r}$$

$$\checkmark \quad \binom{n}{r} = \binom{n+1}{r+1} - \binom{n}{r+1}$$

$$\binom{20}{14} = \binom{21}{15} - \binom{20}{15}$$

Q:

$${}^{5}C_{5} + {}^{6}C_{5} + {}^{7}C_{5} + {}^{8}C_{5} + {}^{9}C_{5} = ?$$

Eg:

$$ {}^{5}C_5 + {}^{6}C_5 + {}^{7}C_5 + {}^{8}C_5 + {}^{9}C_5 = ? $$

$$ \left(\cancel{{}^{6}C_6} - {}^{5}C_6\right) + \left(\cancel{{}^{7}C_6} - \cancel{{}^{6}C_6}\right) + \left(\cancel{{}^{8}C_6} - \cancel{{}^{7}C_6}\right) + \left(\cancel{{}^{9}C_6} - \cancel{{}^{8}C_6}\right) $$

$$ 0 \qquad + \left({}^{10}C_6 - \cancel{{}^{9}C_6}\right) = {}^{10}C_6 \checkmark $$

Eg:

$$\underbrace{{}^5C_5 + {}^6C_5 + {}^7C_5 + {}^8C_5 + {}^9C_5}_{} = ?$$

$$= {}^{10}C_6$$

$$\boxed{{}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \cdots + {}^nC_r}$$

$$= {}^{n+1}C_{r+1}$$

(3) $$\sum_{k=r}^{n} {}^{k}C_{r} = \binom{n+1}{r+1}$$ ✓

(4) $$\binom{n+m}{r} = \sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k}$$

Vandermond identity

5 $$\sum_{r=0}^{n} {}^{n}C_{r} = 2^{n}$$

6 $$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \Longrightarrow$$

Story

subset from n people AND one president

7 $\underbrace{\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2}$ – Variation of Vandermonde identity

$\boxed{\text{n men, n women}}$

Select n People

8

$$\underbrace{\binom{2n}{2} = 2\binom{n}{2} + n^2}_{\text{n boys, n girls}}$$

$$\underbrace{\text{n boys, n girls}} \to \underline{\text{2 people}}$$

Combinatorics

Next Topic:

Binomial Theorem

Website : https://www.goclasses.in/

Binomial : $\underbrace{a+b}$ $a, b \in \text{Real}$

Binomial theorem: $n \in N$

$(a+b)^n$

; $(a+b)^{10}$?

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^2 = \overset{1}{(a+b)}\ \overset{2}{(a+b)}$$

$$aa + \underbrace{ab + ba}_{} + bb$$

$$= \underbrace{a^2 + 2ab^1 + b^2}_{} \longrightarrow \underline{3 \text{ terms}}$$

$\overset{1}{(a+b)}\ \overset{2}{(a+b)} = a^2 + 2ab + b^2$

${}^2C_2\, b^2$

1

$a^k\, b^{2-k}$

${}^2C_1\, a^1\, b^1$

How many ways we can create this term

${}^2C_2\, a^2$

1

2 ways

$$(a+b)^3 = \overset{1}{(a+b)}\overset{2}{(a+b)}\overset{3}{(a+b)}$$

$$\boxed{{}^3C_1 \underbrace{a^2 \, b}} + 0\, a b^3 + 1\, a^3$$

$$+ \; {}^3C_1 \, a \, b^2$$

$(a+b)^n$

$$\underbrace{\overset{1}{(a+b)}\overset{2}{(a+b)}\overset{3}{(a+b)} - - - \overset{n}{(a+b)}}$$

r a's, $(n-r)$ b's

$${}^{n}C_{r}\left(a^{r}\, b^{n-r}\right)$$

Give me a's

Every term is like this.

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} \underline{\underline{a^r b^{n-r}}} \checkmark$$

$$= \sum_{r=0}^{n} \binom{n}{n-r} a^r b^{n-r} \checkmark$$

The Binomial Theorem

In algebra a sum of two terms, such as $a+b$, is called a binomial. The binomial theorem gives an expression for the powers of a binomial $(a+b)^n$, for each positive integer n and all real numbers a and b.

Consider what happens when you calculate the first few powers of $a+b$. According to the distributive law of algebra, you take the sum of the products of all combinations of individual terms:

$$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b)\\ &= aaa + aab + aba + abb + baa + bab + bba + bbb\end{aligned}$$

$$\begin{aligned}(a+b)^4 &= (a+b)(a+b)(a+b)(a+b)\\ &= aaaa + aaab + aaba + aabb + abaa + abab + abba + abbb + baaa + baab + baba + babb + bbaa + bbab + bbba + bbbb\end{aligned}$$

$$(a+b)^4 = {}^4C_0\, a^0 b^4 + {}^4C_1\, a^1 b^3 + {}^4C_2\, a^2 b^2$$

$$+ {}^4C_3\, a^3 b^1 + {}^4C_4\, a^4 b^0$$

5 terms

$$= \sum_{r=0}^{4} \binom{4}{r} a^r b^{4-r} \checkmark$$

When we look at these expansions of $(a + b)^n$ for $n = 1, 2, 3, 4$, and 5, several patterns become apparent.

1. There are $n + 1$ terms, from a^n to b^n.
2. Every term has essentially the same form: some coefficient times the product of a power of a times a power of b.
3. In each term the sum of the exponents on a and b is always n.
4. The powers (exponents) on a decrease, term by term, from n down to 0 where the last term is given by $b^n = a^0 b^n$, and the exponents on b increase from 0 to n.

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$\#$terms $= n+1$

THE BINOMIAL THEOREM Let x and y be variables, and let n be a nonnegative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n.$$

The binomial theorem

We will motivate the following theorem with an example:

$$\begin{aligned}(x+y)^6 &= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 \\ &= \tbinom{6}{0}x^6 + \tbinom{6}{1}x^5y + \tbinom{6}{2}x^4y^2 + \tbinom{6}{3}x^3y^3 + \tbinom{6}{4}x^2y^4 + \tbinom{6}{5}xy^5 + \tbinom{6}{6}y^6.\end{aligned}$$

Theorem

For any x, y and $n \geq 1$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof

Multiply out, or "FOIL" the product $\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$.

This results in 2^n terms, all distinct length-n words in x and y. E.g., for $n = 6$:

$$xxxxxx + xxxxxy + \cdots + xyxyxy + \cdots + xxxyyy + \cdots + yyyyyy$$

There are $\binom{n}{k}$ words with exactly k instances of x, so this is the coefficient of $x^k y^{n-k}$.

What is the expansion of $(x + y)^4$?

Solution: From the binomial theorem it follows that

$$
\begin{aligned}
(x+y)^4 &= \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j \\
&= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 \\
&= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.
\end{aligned}
$$

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

$\binom{25}{12} x^{12} y^{13}$

$12 + 13 = 25$

$= \binom{25}{13} x^{12} y^{13}$

Ans: $\binom{25}{12} = \binom{25}{13}$

What is the coefficient of $x^{12}y^{14}$ in the expansion of $(x + y)^{25}$?

0 $x^{12} y^{14}$

$12 + 14 \neq 25$

Ans: 0

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

$a = 2x$
$b = -3y$

$(a+b)^{25} \Rightarrow {}^{25}C_{12}\, a^{12} b^{13}$

$\Leftarrow {}^{25}C_{12}(2x)^{12}(-3y)^{13}$

$2^{12} \times (-3)^{13} \times \binom{25}{12} \boxed{x^{12} y^{13}}$

Ans: $-(2)^{12}(3)^{13}\binom{25}{12}$

Q: $(x^3 + 2y^2)^{10}$

Coeff of $\underbrace{x^{12} y^{14}}_{?}$

Q: $(x^3 + 2y^2)^{10}$

Coeff of $x^{12} y^{14}$

$(x^3)^4 \; (2y^2)^{10-6}$

0 ___
___

Q: $(x^3 + 2y^2)^{10}$

Coeff of $\underbrace{x^{12} y^{12}}_{?}$

Q: $\left(x^3 + 2y^2\right)^{10}$ Coeff of $\underbrace{?\, x^{12} y^{12}}$

$$\binom{10}{4}\left(x^3\right)^{4^{\,r}}\left(2y^2\right)^{10-4}$$

$$= \boxed{\binom{10}{4} 2^5} \cdot 2\, \underbrace{x^{12} y^{12}}$$

Ans.

Note:

$$(1+x)^n = {}^nC_0\, x^0 + {}^nC_1\, x^1 + {}^nC_2\, x^2 + \cdots + {}^nC_n\, x^n$$

$$(1+x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \binom{n}{3} x^3 + \cdots + \binom{n}{n} x^n$$

Put $x=1$;

$$\boxed{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots \binom{n}{n}}$$

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n$$

Let n be a nonnegative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Put $x = -1$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots$$

$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} \cdots$$

Let n be a positive integer. Then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0.$$

$$\underbrace{\binom{n}{0}+\binom{n}{2}+\binom{n}{4}+\cdots}=\underbrace{\binom{n}{1}+\binom{n}{3}+\binom{n}{5}+\cdots}.$$

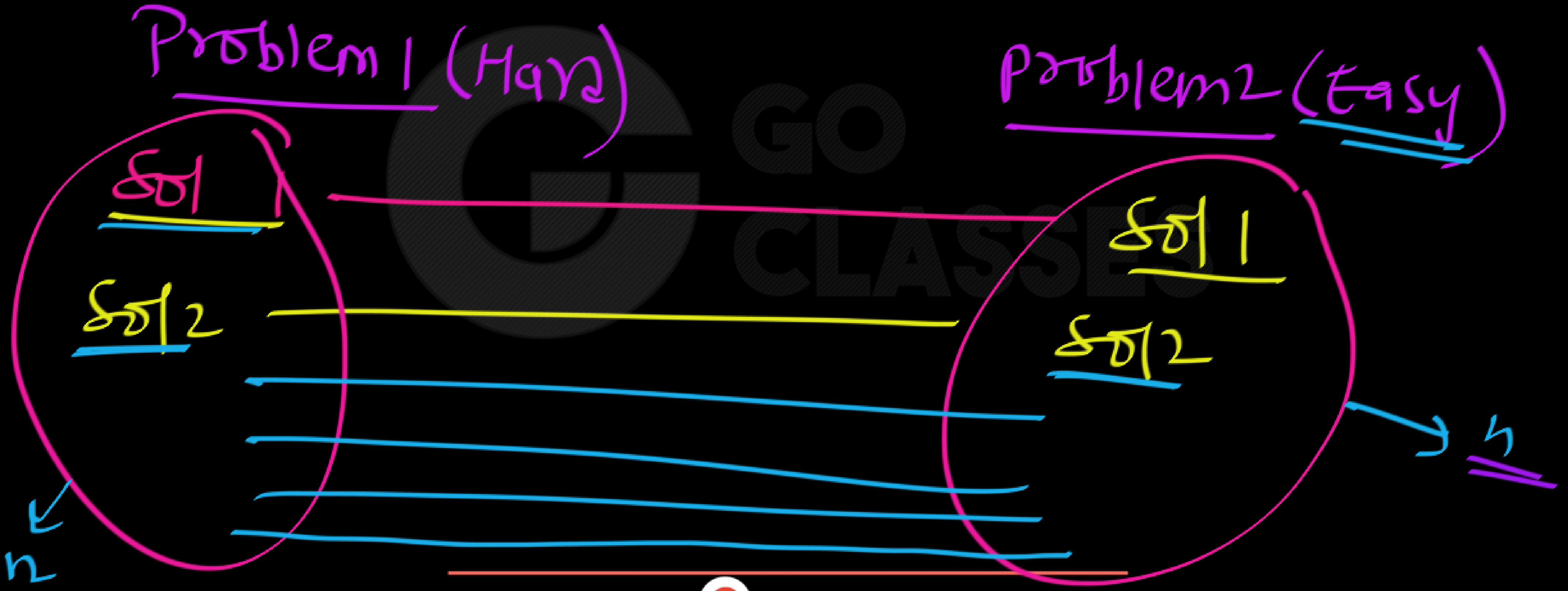
Combinatorics

Next Topic:

Bijective Proofs

Website : https://www.goclasses.in/

A bijective proof in combinatorics just means that you transfer one counting problem that seems "difficult" to another "easier" one by putting the two sets into exact correspondence.



Example. Prove that $\binom{n}{k} = \binom{n}{n-k}$. Verify this for $n = 5$ and $k = 2$.

Solution. Recall that $\binom{n}{k}$ denotes the number of ways of choosing k balls from a collection of n distinguishable balls, where the order of your choice does not matter. In a similar fashion we know that $\binom{n}{n-k}$ denotes the number of ways of forming a collection of $n - k$ balls from the total n.

But forming a collection of k balls is the same as *not forming* a collection with the other $n - k$ balls. In other words, the number of ways of forming collections of k balls is exactly the same as the number of ways of *not forming* collections of $n - k$ balls. Hence, the two expressions are the same, which finishes the proof. □

Eg: $n=4$; select 1 person a, b, c, d

Problem 1 ⇒ select 1 Person

Problem 2 ⇒ Reject 3 person

Set of solutions

a	b, c, d
b	a, c, d
d	a, b, c
c	a, b, d

4

Ans: 4

Q:

Which of the following is Easier(for you)?

Counting bit-strings of length n.

Counting number of subsets of a set of size n.

$[4] = \{1, 2, 3, 4\}$ Prob 1 $\longleftrightarrow$ Prob 2

Prob 1 → subset

Prob 2 → 4 length bitstring

Subsets

$\{1, 2\} \rightarrow 1100$

$\{2, 3, 4\} \rightarrow 0111$

$\{1, 3, 4\} \leftarrow 1011$

$\{\} \longleftrightarrow 0000$

Set = $S = \{a, b, c, d\}$ → fixed order

Subset =

$\{a, b\} = 1100$

$\{a, c\} = 1010$

$\{b, c, d\} = 0111$

$\{a, b, c\}$

Order Doesn't matter

So I can "FIX" a order

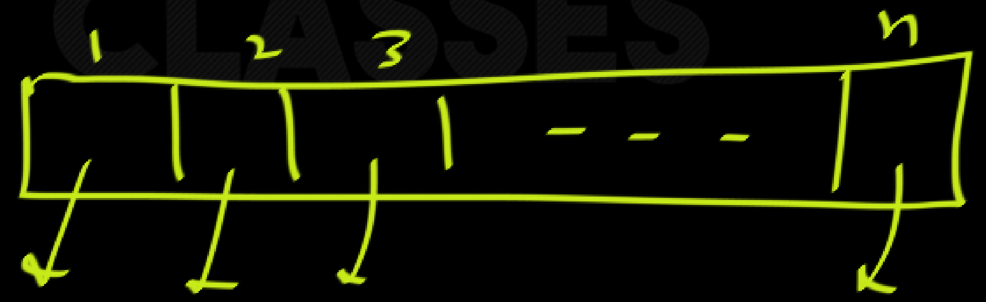
$\{a, b, c\}$

$\{1, 2, 3, 4\} \longrightarrow$ subset $= \{2, 1\} = \{1, 2\}$

no. of subsets from a set of size n

= no. of n length bit-strings.

$= 2^n$



$2 \times 2 \times 2 - - \times 2$

Combinatorics

Next Topic:

Permutations with Repetition

Website : https://www.goclasses.in/

two digits ⇒ 1, 2

three Digit number ⇒

Order matter (Permutation)

Repetition Allowed

1 1 1

1 1 2

1 2 1

2 1 1

2 2 1

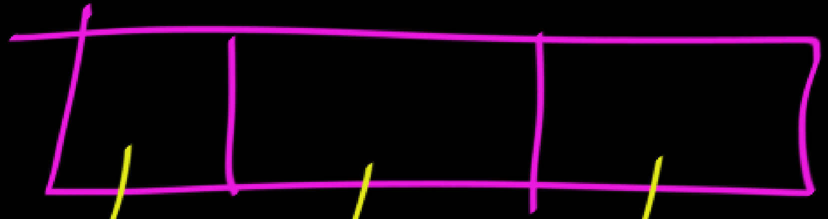
2 1 2

1 2 2

2 2 2

Ans ⇒ 8

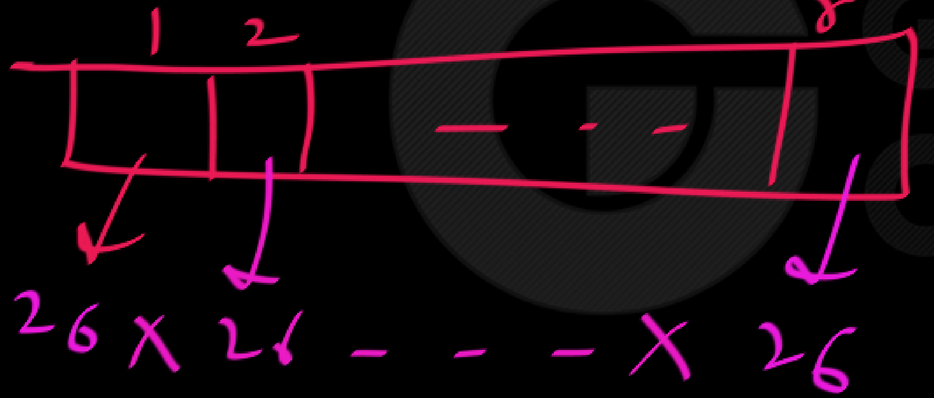
two digits 1, 2



$$2 \times 2 \times 2 = 2^3$$

Permutations with Repetition

How many strings of length r can be formed from the uppercase letters of the English alphabet?



$(26)^r$

Note: n Distinct objects

Permute r of them
with "Repetition Allowed"

1 2 ... r

$n \times n \times - - \times n$

$= (n)^r$

Note: n Distinct objects

Permute r of them
with $\boxed{\text{Repetition Allowed}}$

$^{n}P_{r}$

"wrong"

The number of r-permutations of a set of n objects with repetition allowed is n^r.

Proof: There are n ways to select an element of the set for each of the r positions in the r-permutation when repetition is allowed, because for each choice all n objects are available. Hence, by the product rule there are n^r r-permutations when repetition is allowed. ◁

Combinatorics

Next Topic:

Permutations with Repeated Elements

Website : https://www.goclasses.in/

$\underbrace{ab}$ → Permutations ⇒ $2!$

$\left.\begin{matrix} ab \\ ba \end{matrix}\right\} 2$

$\underbrace{aa}$ → Permutations → 1 ✓

$\underbrace{aab}$ → Permutations

$\left.\begin{matrix} aab \\ aba \\ baa \end{matrix}\right\} 3$

Permutations with Repeated Elements :

A permutation of a set of objects is an ordering of those objects. When some of those objects are identical, the situation is transformed into a problem about permutations with repeated elements.

Q: How many permutations of "Cook" are there?

Method 1:

1	2	3	4
	o		o

$$\underbrace{{}^{4}C_{2}}_{\text{for o's}} * 1 * \underbrace{2!}_{\text{for c, k}}$$

Q: How many permutations of "Cook" are there?

method 2: Division Rule: → life is good if O_1, O_2 were there.

$C O_1 O_2 K$

Easy → $4!$

2 to 1 correspon.

Actual

$O_1 C K O_2$ → $O C K O$

$O_2 C K O_1$ → $O C K O$

$C O_1 O_2 K$ → $C O O K$

$C O_2 O_1 K$ → $C O O K$

→ $\frac{4!}{2}$

Q: How many permutations of "Cook" are there?

Method 3:

1	2	3	4

$\Rightarrow$ Distinct Positions = $\{1, 2, 3, 4\}$

C $\longrightarrow$ 1 Pos

o $\longrightarrow$ 2 Pos

K $\longrightarrow$ 1 Pos

$\Rightarrow {}^4C_1 \times {}^3C_2 \times {}^1C_1$

c, o, K

C → Pos 4
O → Pos 1, 3
K → Pos 2

⇒ OKOC

C → 2
O → 3, 4
K → 1

KCOO

Q: Success — 3 s, 2 c, 1 u, 1 e

method 1:

1	2	3	4	5	6	7
c	s	s	u	c	s	

$$\underbrace{{}^{7}C_{3}}_{\text{to } s} * \underbrace{{}^{4}C_{2}}_{\text{for } c} * \underbrace{{}^{2}C_{1}}_{\text{for } u} * \underbrace{{}^{1}C_{1}}_{\text{for } e}$$

$^7C_3 * {}^4C_2 * {}^2C_1 * {}^1C_1$

35, 2C, 1U, 1E

$$= \frac{7!}{3!\,\cancel{4!}} \cdot \frac{\cancel{4!}}{\cancel{2!}\,2!} \cdot \frac{\cancel{2!}}{1!\,1!} = \frac{7!}{3!\,2!}$$

$$\frac{7!}{3!\,2!}$$

Q: Success

method 2: Division method:

Easy → $S_1UC_1C_2ES_2S_3$ → $7!$

12 to 1 correspondence

USSECSC

$US_1S_2EC_1S_3C_2$

$US_2S_1EC_1S_2C_2$

$12 = 3! \times 2!$

S C

$\frac{7!}{3!2!}$

$s_1 s_2 s_3 \Rightarrow 3!$

$c_1 c_2 \rightarrow 2!$

$s_1 s_2 s_3 u c_1 e c_2$ — sssucec

$s_2 s_1 s_3 u c_1 e c_2$

$s_3 s_1 s_2 u c_1 e c_2$

$s_1 s_2 s_3 u c_2 e c_1$

$s_3 s_1 s_2 u c_2 e c_1$

$3! \times 2!$ to 1

Correspondance

SUCCESS

Positions = {1, 2, - -, 7}

Distinct

S → 3 Pos
U → 1 Pos
C → 2 Pos
E → 1 Pos

$\underbrace{{}^{7}C_{3}}_{\text{for } S} * \underbrace{{}^{4}C_{1}}_{\text{for } U} * \underbrace{{}^{3}C_{2}}_{\text{for } C} * \underbrace{{}^{1}C_{1}}_{\text{for } e}$

$= 7! / 3!\,2!$

Note: Total n elements

n_1 of type 1 n_2 of Type 2   - - n_k of type k

* $\boxed{n_1 + n_2 + \cdots + n_k = n}$

$$\#\text{Permutations} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

$n! \rightarrow$ when life is Easy

Note: Total n elements

n_1 of type 1 $\quad$ n_2 of Type 2 $\quad$ - - n_k of type k

* $\boxed{n_1 + n_2 + \cdots + n_k = n}$

$$\#\text{Permutations} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

5.3.1. Permutations with Repeated Elements. Assume that we have an alphabet with k letters and we want to write all possible words containing n_1 times the first letter of the alphabet, n_2 times the second letter,..., n_k times the kth letter. How many words can we write? We call this number $P(n; n_1, n_2, \ldots, n_k)$, where $n = n_1 + n_2 + \cdots + n_k$.

Example: With 3 a's and 2 b's we can write the following 5-letter words: *aaabb*, *aabab*, *abaab*, *baaab*, *aabba*, *ababa*, *baaba*, *abbaa*, *babaa*, *bbaaa*.

Permutations with Indistinguishable Objects

Theorem 2. *The number of different permutations of n objects, where there are n_1 indistinguishable objects of type 1, n_2 indistinguishable objects of type 2, ..., and n_k indistinguishable objects of type k, is*

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

Proof. To determine the number of permutations, first note that the n_1 objects of type one can be placed among the n positions in $\binom{n}{n_1}$ ways, leaving $n - n_1$ positions free. Then the objects of type two can be placed in $\binom{n-n_1}{n_2}$ ways, leaving $n - n_1 - n_2$ positions free. Continue placing the objects of type three, ..., type $k-1$, until at the last stage, n_k objects of type k can be placed in $\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$ ways. Hence, by the product rule, the total number of different permutations is

$$\begin{aligned}\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-\cdots-n_{k-1}}{n_k} &= \frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdots\frac{(n-n_1-\cdots-n_{k-1})!}{n_k!0!}\\ &= \frac{n!}{n_1!n_2!\cdots n_k!}.\end{aligned}$$

□

How many different strings can be made by reordering the letters of the word $SUCCESS$?

SUCCESS → Total $n = 7$

1 of Type 4 → U

3 of Type 1 → S, 2 of type 2 → C

1 of Type 3 → e

$$\boxed{1+3+2+1=7}$$

$$\frac{7!}{3!\,2!\,1!\,1!}$$

How many different strings can be made from the letters in MISSISSIPPI, using all the letters?

$11!$ → when all Different

$$\frac{11!}{4!\ 4!\ 2!\ 1!}$$

4 I's can move in their Position

I's S P M