Group Theory

Summary Lecture 1

Introduction to Abstract Algebra

Group Theory

Website : https://www.goclasses.in/


GO CLASSES
GATE CSE 2023
(LIVE + RECORDED COURSE)
GATE OVERFLOW
GO Classes
SPECIAL
Revision Course
GATE PYQs Video Solutions
Standard Resources Practice Course
Enroll Now
Best Mentorship and Support
GO Test Series Available Now
Free
Discrete Mathematics classes Free
C language classes Free
Weekly Quizzes
Sachin Mittal
(CO-Founder GOCLASSES)
MTech IISc Bangalore
Ex Amazon scientist
GATE AIR 33
Deepak Poonia
(CO-Founder GOCLASSES)
MTech IISc Bangalore
GATE AIR 53; 67
Dr. Arjun Suresh
(Mentor)
Founder GATE Overflow
Ph.D. INRIA France
ME IISc Bangalore
Postdoc The Ohio -
-State University
Recorded Lectures:
Watch Anytime.
Any number of times.
COURSE FEE
₹22,000/-
Course Features
Quality Learning.
Interactive Classes & Doubt Resolution.
Summary Lectures.
Daily Homeworks & Solutions.
Standard Practice Sets & Video Solutions.
ALL GATE PYQs Video Solutions.
No Prerequisites.
Doubts Resolution by Faculties on Telegram.
SIGN UP
www.goclasses.in
+91 - 6302536274
www.goclasses.in/s/pages/schedule

ABSTRACT ALGEBRA

INTRODUCTION TO GROUP THEORY

Group Theory

Topic 1

What is Algebraic/Abstract Structure?

Website : https://www.goclasses.in/

An algebraic structure (Abstract Structure) :

(A Non-Empty Set + One or more Closed Operations)

Algebraic Structure : (AS)

$$\underbrace{(\text{Base set}, \underbrace{*, \#, \ldots}_{\text{Operations}})}_{\text{AS}}$$

Algebraic structure:

✓ $(Z, +)$ — GATE

$(Z, +, \times, -)$

Base set

operations

$(N, +, \times)$

✓ (set of matrices, product) — GATE

$(\mathbb{Z}, +) \rightarrow$ Alg. Structure

binary opⁿ

$2 + 3 = 5$

$2 + 2 = 4$

An algebraic structure (Abstract Structure) :

(A Non-Empty Set + One or more Closed Operations)

(N, f) ; $f(a,b,c) = ab+c$

(N, f) ; $f(a,b,c) = ab + c$

$f(1,1,2) = 1 \times 1 + 2 = 3$

$f(2,3,5) = 2 \times 3 + 5 = 11$

$f(1,1,1) = 1 \times 1 + 1 = 2$

An algebraic structure (Abstract Structure) :

(A Non-Empty Set + One or more n-ary Closed Operations)

Abstract Algebra:
Study of these Algebraic Structures.

In mathematics, an **algebraic structure** consists of a nonempty set *A*, a collection of operations on *A* of finite arity, and a finite set of identities, known as axioms, that these operations must satisfy.

An algebraic structure is understood to be an arbitrary non-empty set, with one or more operations defined on it. And algebra, then, is defined to be the study of algebraic structures.

In GATE syllabus:

We have algebraic structures(Abstract Structure) with single binary operation.

(A Non-Empty Set + One Binary Operation)

Algebraic Structures : (Abstract Structures)

Examples of algebraic structures with a single binary operation are:

- Magma
- Quasigroup
- Monoid
- Semigroup
- Group

GATE

Examples involving several operations include:

- Ring
- Field
- Module
- Vector space
- Algebra over a field
- Associative algebra
- Lie algebra
- Lattice
- Boolean algebra

GATE

Lattice: $\left(\underbrace{L}_{\text{Set}}, \underbrace{\wedge, \vee}_{\text{two binary operations}} \right)$

GO CLASSES

GO Classes

Lattice

- The **lattice** is an algebraic system $\langle A, \vee, \cdot \rangle$, given a,b,c in A, the following axioms are satisfied:
  1. Idempotent laws: $a \vee a = a$, $a \cdot a = a$;
  2. Commutative laws: $a \vee b = b \vee a$, $a \cdot b = b \cdot a$
  3. Associative laws: $a \vee (b \vee c) = (a \vee b) \vee c$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
  4. Absorption laws: $a \vee (a \cdot b) = a$, $a \cdot (a \vee b) = a$

Algebraic Structures with Single Binary Operation:

Magma/Groupoid

Semi-Group

Monoid

Group

Abelian Group

GATE

Algebraic Structures with Single Binary Operation:

Magma/Groupoid (Closure Property)

Semi-Group (Closure + Associative Property)

Monoid (Closure + Associative + Identity Property)

Group(Closure + Associative + Identity + Inverse Property)

Abelian Group (Group + Commutative Property)

§1.1 Group Theory?

You may be wondering what it is. Here's a definition from Wikipedia:

> **Definition 1.1** (Group Theory) — **Group theory** studies the algebraic structures known as groups.

The concept of a group is central to abstract algebra: other well-known algebraic structures, such as rings, fields,

Group Theory

Summary Lecture 2

Binary Operation(Closed Operation)

(The Closure Property)

Group Theory

Next Topic:

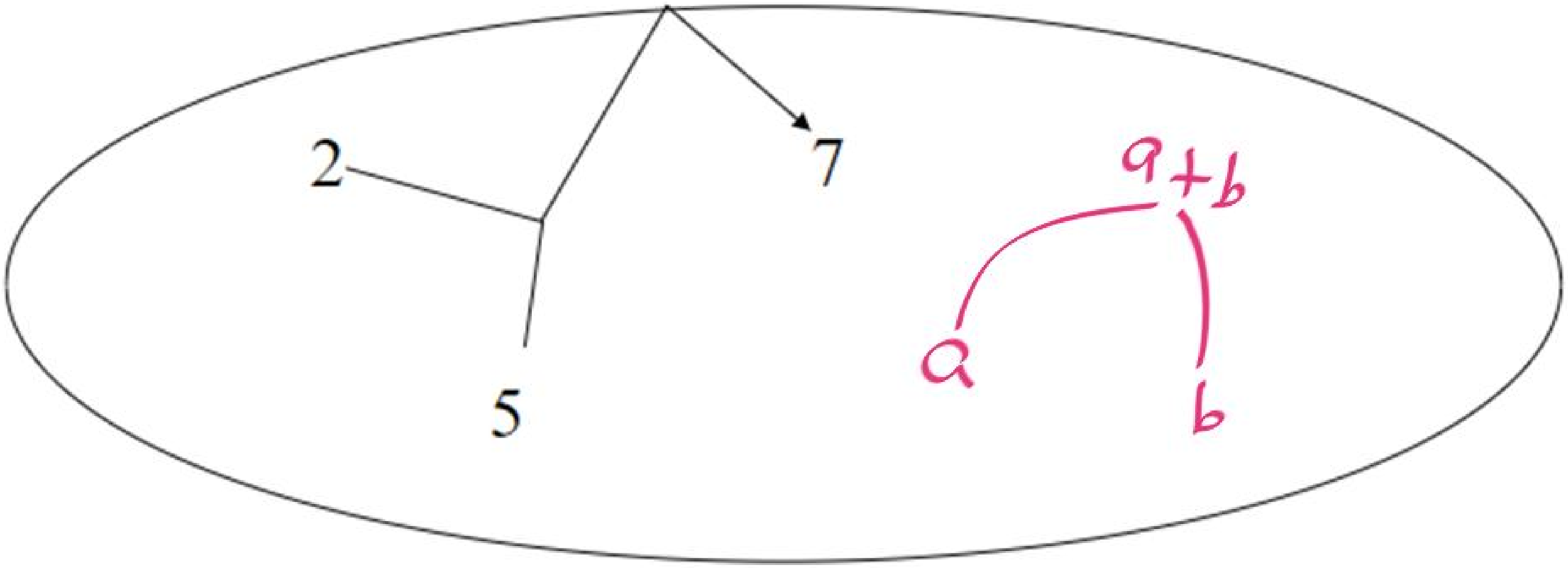
The Closure Property

What happens inside a set, should remain inside the set.

Closure Properties

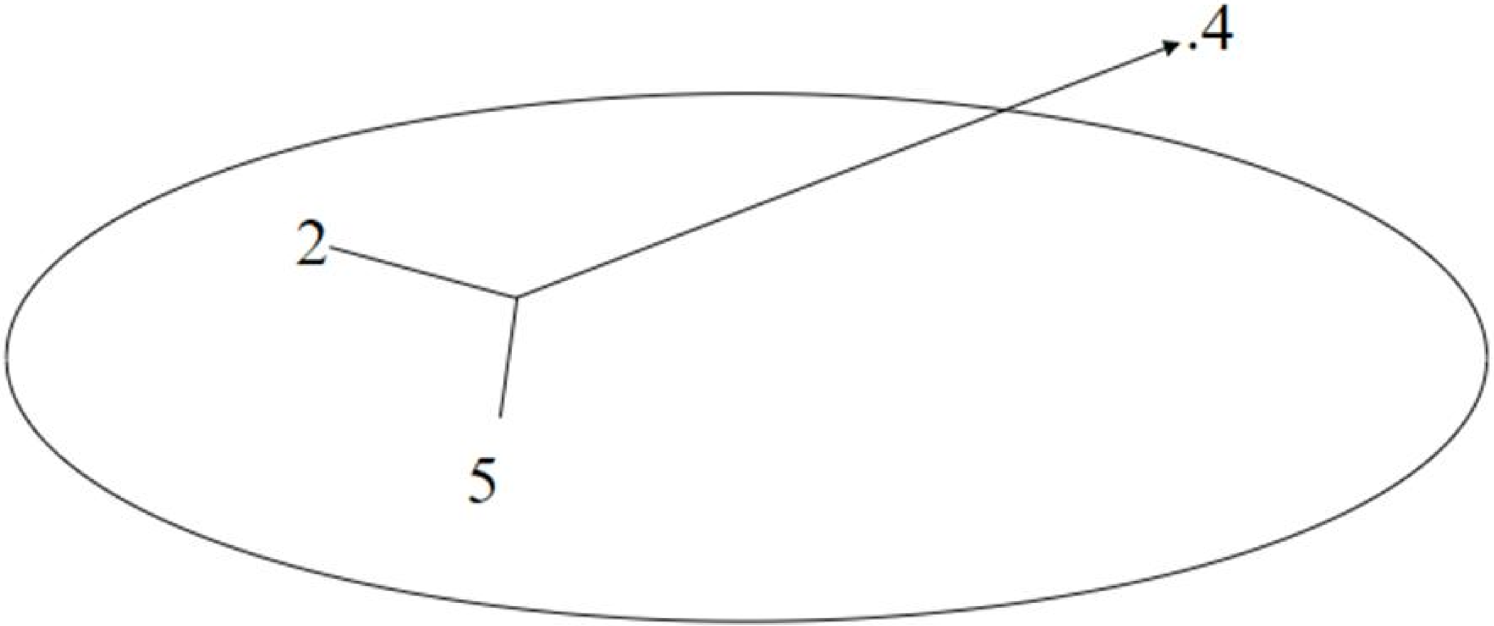
- A set is closed under an operation if applying the operation to elements of the set produces another element of the set
- Example/Counterexample
  - set of integers and addition
  - set of integers and division

Integers and Addition



Integers

Integers and Division



Integers

Closure property :

Set S is closed under operation $*$

iff $\forall a, b \in S$

$$\boxed{a * b \in S}$$

NOT Closed :

Set S is NOT Closed under operation $*$

iff $\exists a, b \in S$

$a * b \notin S$

$(\{0,1\}, \text{Addition})$

S

0

1

$a=1, b=1$

$1+1=2 \notin S$

$1+1=2$

The Property of Closure

A set has the **closure property** under a particular **operation** if the result of the operation is always an element in the set. If a set has the **closure property** under a particular **operation**, then we say that the set is **"closed under the operation."**

Group Theory

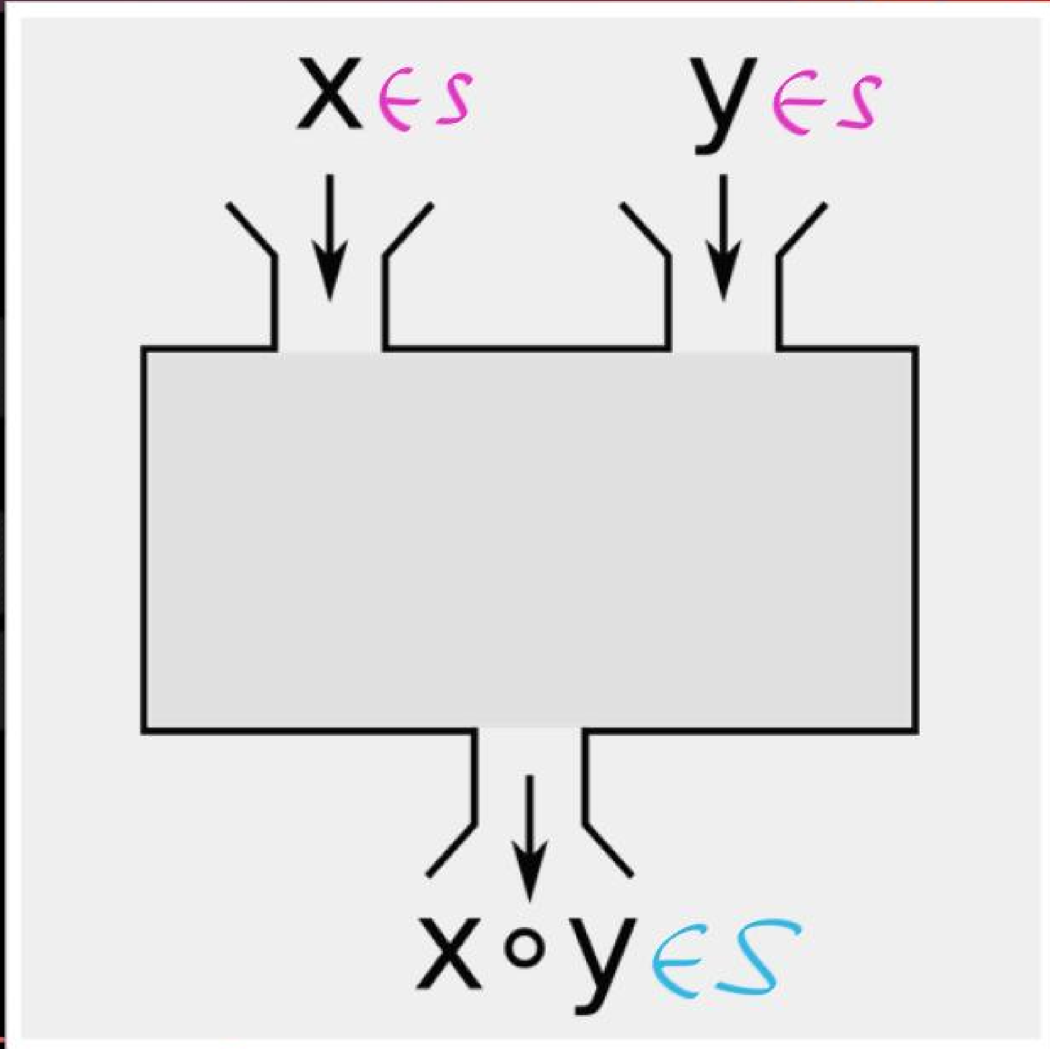
Next Topic:

The Binary Operation

Applies on two elements(same or different), and must be closed.

Sets, $\circ$

$\left.\begin{array}{l} x \in S \\ y \in S \end{array}\right\}$




A
x
y
Binary operation (*)
A
x*y

Definition:

- A ***binary operation*** on a nonempty set A is a mapping defined on $A \times A$ to A, denoted by $f : A \times A \to A$.

$(\underline{\underline{N}}, +)$

$+ : \underbrace{N \times N} \to N$

$+(\underline{1, 2}) = 3$

$+(\underline{2, 2}) = 4$

$+(1, 100) = 101$

$\underbrace{+(1,2)}$ ✓

$1+2$ ✓

$+(a, b) = \underline{\underline{a+b}}$

$a \in N$

$b \in N$

$\Downarrow$

$\in N$

The set $\{1,2,3,4\}$ is **not** **closed** under the **operation** of addition because $2 + 3 = 5$, and 5 is **not** an element of the set $\{1,2,3,4\}$.

4.1: Binary Operations

Closed operation

DEFINITION 1. *A* **binary operation** $*$ *on a nonempty set A is a function from $A \times A$ to A.*

Addition, subtraction, multiplication are binary operations on $\mathbf{Z}$.

$a, b \in Z$; $a+b \in Z$, $a-b \in Z$, $a \times b \in Z$

Addition is a binary operation on $\mathbf{Q}$ because

$\frac{a}{b} + \frac{c}{d} = \frac{ad + cb}{bd} \in Q$

Division is NOT a binary operation on $\mathbf{Z}$ because

$2, 3$; $2/3 \notin Z$

Division is a binary operation on

Set of nonzero Rational numbers.

$(\mathbb{Q}, \div)$ — NOT closed

$a = 2$; $b = 0$; $a \div b = 2 \div 0 \notin \mathbb{Q}$

$(\mathbb{Q}^*, \div)$ closed

$\mathbb{Q}^* = \mathbb{Q} - \{0\}$

$$\frac{a/b}{c/d} = \frac{ad}{bc} \in \mathbb{Q}^*$$

§2.3 Binary Operation

A **Binary Operation** on a nonempty set is a map $f : A \times A \to A$ such that $f(a_1, a_2)$ is defined for every pair of elements $a_1, a_2 \in A$ and $f(a_1, a_2)$ is an element in A.

Example 2.3

Arithmetic operations such as addition and multiplication are binary operations on the set of integers. Division is not a binary operation on the set of integers because the result of the division of two integers a, b is not always an integer for every pair of a, b. Division is a binary operation for the set of rationals excluding 0 because the division of two rational numbers always produces a rational number. (We exclude 0 because dividing by 0 causes problems).

But really, a binary operation can be anything we define it to be, as long as it takes 2 inputs and generates 1 output that is in the same set.[2]

$(N, \#)$

$\underbrace{a \# b = \max(a, b)}$

$\# : \underline{N \times N} \rightarrow N$

$\boxed{\#(2, 3)} = \max(2, 3) = 3$

$2 \# 3 = \max(2, 3) = 3$

$\#(2, 2)$

$= 2$

$3 \# 3 = 3$

Binary Operation

A binary operation $f(x, y)$ is an operation that applies to two quantities or expressions x and y.

A binary operation on a nonempty set A is a map $f : A \times A \to A$ such that

1. f is defined for every pair of elements in A, and
2. f uniquely associates each pair of elements in A to some element of A.

Examples of binary operation on A from $A \times A$ to A include addition $(+)$, subtraction $(-)$, multiplication $(\times)$ and division $(\div)$.

Definition. *Let G be a set. A* ***binary operation*** *is a map of sets:*

$$* : G \times G \to G.$$

For ease of notation we write $*(a, b) = a * b \ \forall a, b \in G$. Any binary operation on G gives a way of *combining* elements. As we have seen, if $G = \mathbb{Z}$ then $+$ and $\times$ are natural examples of binary operations. When we are talking about a set G, together with a fixed binary operation $*$, we often write $(G, *)$.

So, Binary Operation is same as Closed Operation.

Operation # on Set S is binary iff S is closed under operation #.

Binary Operation is a function from $S \times S \rightarrow S$

Group Theory

Summary Lecture 3:

Questions on Binary Operation

(Closure Property)

Question

Let S be the set of positive integers. If a,b are in S, which of the following is a binary operation?

- ○ a*b=a/b
- ○ a*b=a-2b
- ○ $a*b=a^3+b$
- ○ a*b=ln(a+b)

Question

Let S be the set of positive integers. If a,b are in S, which of the following is a binary operation?

- a*b=a/b
- a*b=a-2b
- $a*b=a^3+b$
- a*b=ln(a+b)

$a=2, b=3$; $a*b = \frac{2}{3} \notin S$

$a=1, b=2$; $a*b = a-2b = 1-2\times 2 = -3 \notin S$

$a, b \in N$, $a^3 \in N$, $a^3+b \in N$

$a=1, b=1$; $a*b = \ln(1+1) = \ln 2$

$= 0.693$

$\notin S$

$\ln = \log_e$

Which of the following is a binary operation if a and b are in S?

a. $a*b = ab + b^2$, $S =$ *set of negative numbers*

b. $a*b = \sqrt{a+b}$; $S =$ *set of positive integers*

c. $a*b = ab - a$; $S =$ *set of real numbers except zero*

d. $a*b = a^2 - b^2$; $S =$ *set of real numbers except zero*

○ a

○ b

○ c

○ d

Which of the following is a binary operation if a and b are in S?

✗ a. $a*b = ab + b^2$, S = set of negative numbers

✗ b. $a*b = \sqrt{a+b}$; S = set of positive integers

✗ c. $a*b = ab - a$; S = set of real numbers except zero

✗ d. $a*b = a^2 - b^2$; S = set of real numbers except zero

$a = -1$
$b = -2$

$a*b = ab + b^2$
$= 2 + 4$
$= 6 \notin S$

(b) NOT closed $\equiv$ NOT binary op^n

$a = 1$, $b = 2$

$a*b = \sqrt{1+2} = \sqrt{3} \notin S$

$\underbrace{(N, *)}_{\text{Closed}}$ $a*b = \underbrace{ab + b^2}$

$a, b \in N$; $ab \in N$; $b^2 \in N$

$ab + b^2 \in N$

(c) $(R^*, *)$ $a*b = ab - a$

$R^* = R - \{0\}$ $a=1, b=1$; $a*b = 0 \notin R^*$

A binary operation Δ is defined on the set of real numbers by $a\Delta b = a^b$. Find the value of $3 \Delta - 2$.

A. -9 B. $-\frac{1}{9}$ C. $\frac{1}{9}$ D. -6

[SSSCE 1993 Qu 6]

A binary operation Δ is defined on the set of real numbers by $a\Delta b = a^b$. Find the value of $3 \, \Delta -2$.

A. -9 B. $-\frac{1}{9}$ C. $\frac{1}{9}$ D. -6

[SSSCE 1993 Qu 6]

$(\mathbb{R}, \Delta)$

$a \Delta b = a^b$

$3 \Delta 3 = 3^3 = 27$

$\underbrace{3 \Delta -2} = 3^{-2} = \frac{1}{9}$

A binary operation $*$ is defined on the set of real numbers, R, by $x * y = x^2 - y^2 + xy$, where $x, y \in R$. Evaluate $(\sqrt{3}) * (\sqrt{2})$.

A. $1 + \sqrt{6}$

B. $\sqrt{6}$

C. $\sqrt{6} - 1$

D. $1 - \sqrt{6}$

A binary operation $*$ is defined on the set of real numbers, R, by $x * y = x^2 - y^2 + xy$, where $x, y \in R$. Evaluate $(\sqrt{3}) * (\sqrt{2})$.

A. $1 + \sqrt{6}$

B. $\sqrt{6}$

C. $\sqrt{6} - 1$

D. $1 - \sqrt{6}$

$x * y = x^2 - y^2 + xy$

$(\sqrt{3}) * (\sqrt{2}) =$

$(\sqrt{3})^2 - (\sqrt{2})^2 + \sqrt{3}\sqrt{2}$

$= 3 - 2 + \sqrt{6} = 1 + \sqrt{6}$

Q 2 : Which of the following mappings are binary operations in P. (P is the set of Positive Integers)

Which of the following are binary operations in P (throughout $(i, j) \in P^2$)?

(i) $\alpha : (i, j) \rightarrow i + j$

(ii) $\alpha : (i, j) \rightarrow i - j$

(iii) $\alpha : (i, j) \rightarrow i \div j$

(iv) $\alpha : (i, j) \rightarrow i + j + i^2$

(v) $\alpha : (i, j) \rightarrow j$

binary op^n $\equiv$ function

$\#(a, b) = s$

$(S, \#)$

$\# : S \times S \rightarrow S$

Q 2 : Which of the following mappings are binary operations in P. (P is the set of Positive Integers)

Which of the following are binary operations in P (throughout $(i, j) \in P^2$)?

(i) $\alpha : (i, j) \to i + j$ (iii) $\alpha : (i, j) \to i \div j$ (v) $\alpha : (i, j) \to j$

(ii) $\alpha : (i, j) \to i - j$ (iv) $\alpha : (i, j) \to i + j + i^2$

binary opn ≡ function

$\#(a, b) = s$

$(S, \#)$

$\# : S \times S \to S$

$(\mathbb{N}, \alpha)$

$$\boxed{\alpha : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}}$$

(i) $\alpha(i, j) = i + j$ ✓

$i, j \in \mathbb{N}$, $i + j \in \mathbb{N}$

(v) $\alpha(i, j) = j \in \mathbb{N}$ ✓

Group Theory

Summary Lecture 4:

Questions on Binary Operation

(Closure Property) PART 2

- The following are attempts to define a binary operation on a set, but are they actually binary operations on the given set? If so, simply write *YES*, and if not, write *NO* and provide a brief explanation.

  (a) $a * b = a - 2020$ on the set $\mathbb{Z}$ of integers

  (b) $a * b = a \log b$ on the set $\mathbb{R}^+$ of positive real numbers

  (c) $a * b = |a - b|$ on the set $\mathbb{N}$ of non-negative integers

  (d) $a * b = \sqrt{|ab|}$ on the set $\mathbb{Q}$ of rational numbers

  (e) $x * y = \sqrt{x^2 + y^2}$ on the set $\mathbb{R}$ of real numbers

  (f) $x * y = |x + y - 1|$ on the set $\mathbb{R}$ of real numbers

  (g) $x * y = \max(x, y)$ on the set $\mathbb{R}$ of real numbers
  (The notation $\max(x, y)$ refers to the maximum of the numbers x and y.)

  (h) $x * y = \frac{xy}{x+y+1}$ on the set $\mathbb{R}$ of real numbers

- The following are attempts to define a binary operation on a set, but are they actually binary operations on the given set? If so, simply write *YES*, and if not, write *NO* and provide a brief explanation.

(a) $a * b = a - 2020$ on the set $\mathbb{Z}$ of integers

(b) $a * b = a \log b$ on the set $\mathbb{R}^+$ of positive real numbers

(c) $a * b = |a - b|$ on the set $\mathbb{N}$ of non-negative integers

(d) $a * b = \sqrt{|ab|}$ on the set $\mathbb{Q}$ of rational numbers

(e) $x * y = \sqrt{x^2 + y^2}$ on the set $\mathbb{R}$ of real numbers

(f) $x * y = |x + y - 1|$ on the set $\mathbb{R}$ of real numbers

(g) $x * y = \max(x, y)$ on the set $\mathbb{R}$ of real numbers
(The notation $\max(x, y)$ refers to the maximum of the numbers x and y.)

(h) $x * y = \frac{xy}{x+y+1}$ on the set $\mathbb{R}$ of real numbers

(d) $\underbrace{(\mathbb{Q}, *)}_{\text{NOT Closed}}$ $\underbrace{a*b = \sqrt{|ab|}}$ → NOT a binary oph on $\mathbb{Q}$

$\mathbb{Q}$ → Rational Numbers

$\left.\begin{array}{l} a=1 \\ b=2 \end{array}\right\}$ $a*b = \sqrt{2} \notin \mathbb{Q}$

(h) $(R, *)$ $\quad a * b = \dfrac{ab}{a+b+1}$

NOT a binary opⁿ

$a = -1$
$b = 0$

$a * b = \dfrac{0}{0} \notin R$

(a) $a, b \in \mathbb{Z}$; $\underbrace{a * b} = a - 2020 \in \mathbb{Z}$

$=$

$(\mathbb{Z}, *)$ $\boxed{a * b = a - 2020}$ ✓

$-2 * 2 = -2 - 2020 = -2022$

$3 * 3 = 3 - 2020$

$3 * 5 = 3 - 2020$

$(N, *)$ $\qquad \underline{a * b = a - 2020}$

$\underline{\text{NOT binary op}^n}$

$a = 2, b = 3$

$\underline{a * b} = \underbrace{2 - 2020} \notin N$

(b) $(R^+, *)$

$a * b = a \log_2 b$

$\log \frac{m}{n} = \log m - \log n$

NOT binary opⁿ

$a = 1; \quad b = \frac{1}{2}$

$a * b = a \log_2 b$

$= \log_2 \left(\frac{1}{2}\right)$

$= \log_2 1 - \log_2 2 = 0 - 1 = -1 \notin R^+$

(c) $(W, *)$ $a * b = |a-b|$

$a, b \in W$

$|a-b| \in W$

$|-2.5| = 2.5$

$|-3| = 3$

$|3| = 3$

$|-1| = 1$

Absolute Value function

$|0| = 0$

Group Theory

Summary Lecture 5

The Associative Property

The Commutative Property

Classification of binary operations by their properties

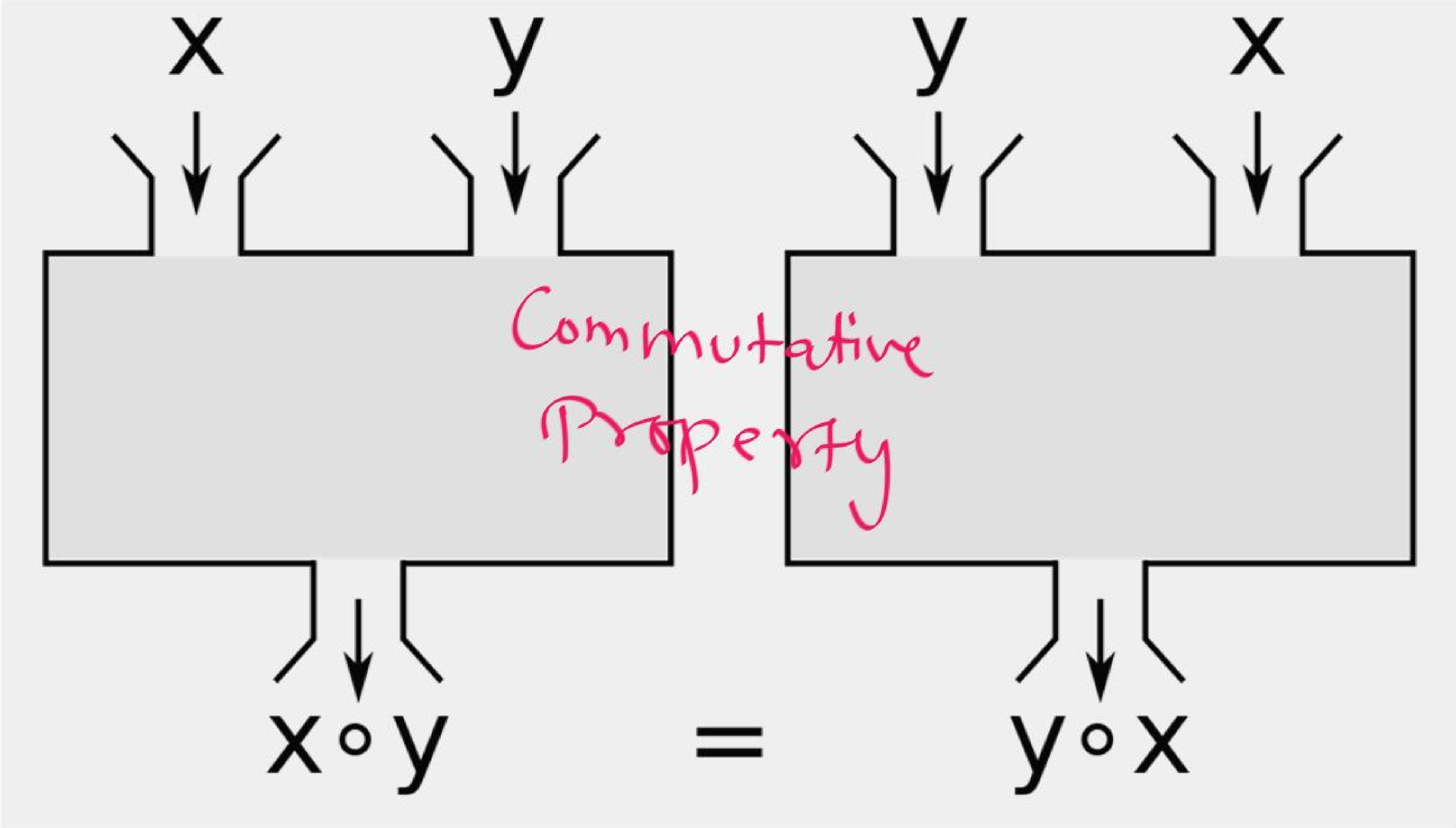
Associative and Commutative Laws

DEFINITION . *A binary operation* $*$ *on* A *is* **associative** *if*

$$\forall a, b, c \in A, \quad (a * b) * c = a * (b * c).$$

A binary operation $*$ *on* A *is* **commutative** *if*

$$\forall a, b \in A, \quad a * b = b * a.$$


x
y
y
x
Commutative Property
x∘y
=
y∘x

$(S, *)$

Base Set

binary opn

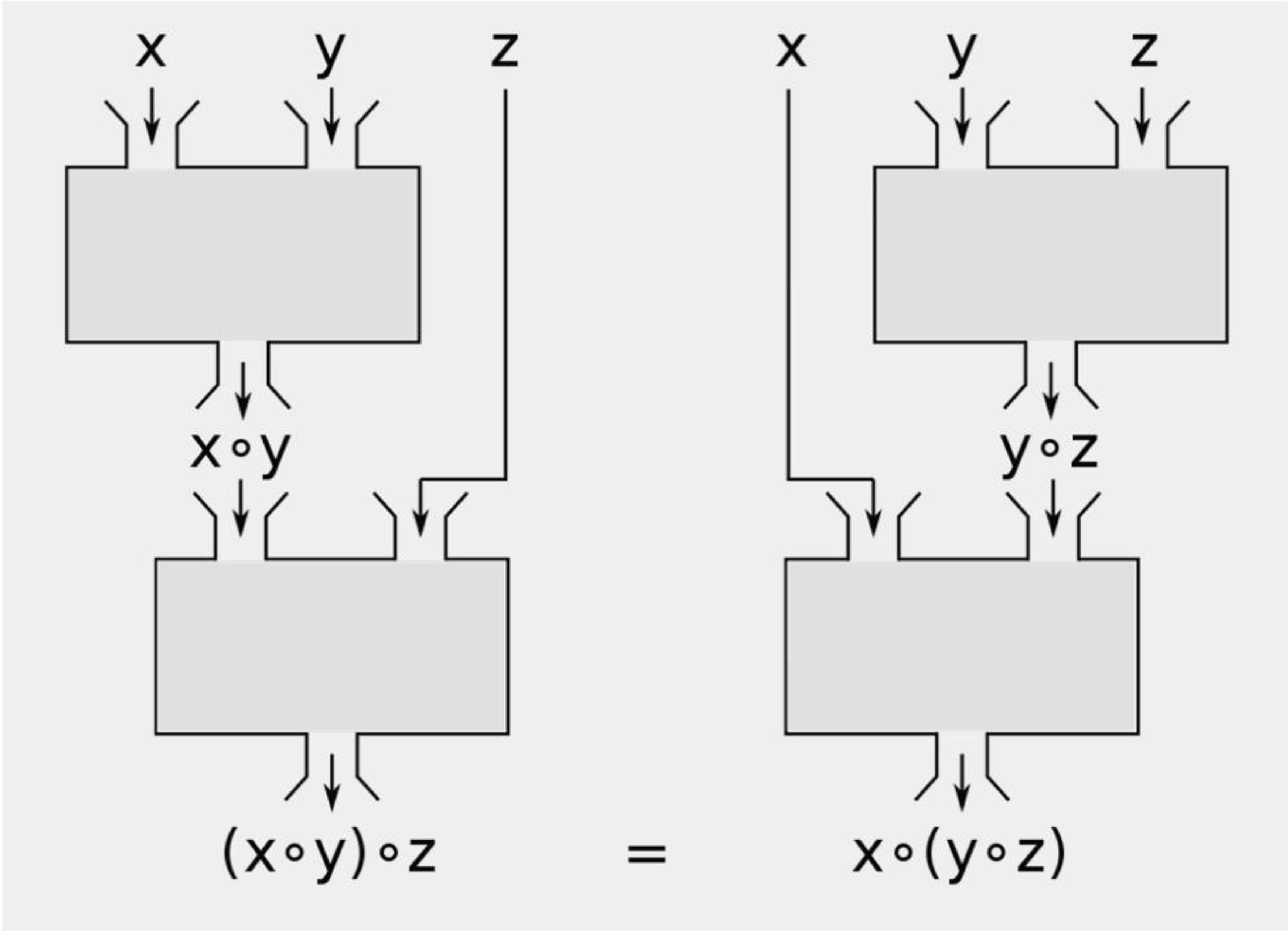
$*$ is Commutative iff

$\underbrace{\forall a, b}$ $\boxed{a * b = b * a}$

Associative

$(S, \circ)$

$\forall x, y, z$



if x, y and z are variables that represent **any 3 arbitrary elements** in the set we are looking at (let's call the set we are looking at A), and the symbol # represents our operation, then the **associative property** for A with the operation # would be:

$$(x\#y)\#z = x\#(y\#z).$$

This means that the **associative property** only holds for the set A and the operation # if, **no matter what elements we take from A and put in place of *x*, *y* and *z***, $(x\#y)\#z$ will always give us the same result as $x\#(y\#z)$.

Remember that the parentheses just tell us **which pair to do first**.

Non-associative operation [edit]

A binary operation $*$ on a set S that does not satisfy the associative law is called **non-associative**. Symbolically,

$$(x * y) * z \neq x * (y * z) \qquad \text{for some } x, y, z \in S.$$

For such an operation the order of evaluation *does* matter. For example:

- Subtraction

$$(5-3)-2 \neq 5-(3-2)$$

- Division

$$(4/2)/2 \neq 4/(2/2)$$

- Exponentiation

$$2^{(1^2)} \neq (2^1)^2$$

Non-[redacted] Commutative operation [edit]

Commutative

A binary operation $*$ on a set S that does not satisfy the [redacted] law is called **non-**[redacted]. Symbolically,

Commutative

[redacted] $a * b \neq b * a$, for some $a, b \in S$

For such an operation the order of evaluation *does* matter. For example:

- Subtraction

[redacted] $5 - 4 \neq 4 - 5$

- Division

[redacted] $4 \div 2 \neq 2 \div 4$

- Exponentiation

[redacted] $2^1 \neq 1^2$

$(\mathbb{Z}, +)$ $\quad \forall a, b, c$

$(a + b) + c = a + (b + c)$

$(2 + 2) + 3 = 2 + (2 + 3)$

$\underbrace{+, \times}$ — Assoc, Comm

$a + b = b + a$

$a \times b = b \times a$

Q.

Let S be any set. We defined binary operation “*” on S.

defined by *(a,b) = a.

function

$S \times S \rightarrow S$

Is (S,*) Associative?

Is * Associative on S?

Is * Commutative on S?

Q.

$\boxed{a * b = a}$

Let S be any set. We defined binary operation "*" on S.

defined by *(a,b) = a.

function

$S \times S \rightarrow S$

Is (S,*) Associative?

Is * Associative on S?

same

Is * Commutative on S? ✗

$1 * 2 \neq 2 * 1$

$1 \neq 2$

$$(a * b) * c \stackrel{?}{=} a * (b * c)$$

$$\underbrace{a * c} \qquad \underbrace{a * b}$$

$$a \qquad \rule{3cm}{0.4pt} \qquad a$$

NOTE:

When operation "*" is NOT Associative, we cannot write:

$$a*b*c$$

Because it is Ambiguous. We should put parentheses to dis-ambiguate.

$(a*b)*c$ ✓ $a*(b*c)$ ✓

Subtraction : $\longrightarrow$ Not Asso

$\boxed{4 - 5 - 7}$ Ambiguous

$$(4 - 5) - 7 \neq 4 - (5 - 7)$$

Addition $\rightarrow$ Asso ✓

$$\boxed{4 + 5 + 7}$$ — NOT Ambiguous

$$(4 + 5) + 7 = 4 + (5 + 7)$$

NOTE:

When operation “*” is Associative, we can write:

a*b*c

So, we can drop/remove parentheses in case of Associative Operation.

Asso : $a * b * c$ ✓

NOT Asso : $a * b * c$ Ambiguous

$2 \times 3 \times 6$ ✓

✓

$2 \div 4 \div 6$

Ambiguous

Group Theory

Summary Lecture 6:

Questions on Associative, Commutative Property

Q.

Give an Operation which is Associative and Commutative.

Q.

Give an Operation which is Associative and Commutative.

$\underbrace{+\ ,\ \times}_{\text{Number theory}}$; $\underbrace{\wedge, \vee, \oplus, \leftrightarrow}_{\text{Prop. logic}}$; $\underbrace{\cup, \cap, \oplus}_{\text{Set theory}}$

Q.

Give an Operation which is Associative But Not Commutative.

Q.

Give an Operation which is Associative But Not Commutative.

matrix multiplication

$(A \times B) \times C = A \times (B \times C)$

$AB \neq BA$

$(N, *)$ $a * b = a$; function Composition

$(f \circ g) \circ h = f \circ (g \circ h)$

$(Z, *)$

$a * b = \max(a, b)$

Comm ✓ ; Asso ✓

Q.

Give an Operation which is Commutative But Not Associative.

Q.

Give an Operation which is Commutative But Not Associative.

Consider the operation $(x, y) \mapsto xy + 1$ on the integers.

$(\mathbb{Z}, *)$; $x * y = xy + 1$

$x * y = \underbrace{y * x}$

$xy + 1 = yx + 1$ ✓ } Comm ✓

$(\mathbb{Z}, *)$ $\quad x * y = xy + 1$

$$(a * b) * c \overset{?}{=} a * (b * c)$$

$$(ab+1) * c \qquad a * (bc+1)$$

$$(ab+1)c + 1 \neq a(bc+1) + 1$$

$$abc + c + 1 \neq abc + a + 1$$

Q. Give an Operation which is Commutative But Not Associative.

The NAND is commutative but not associative.

A	B		A nand A	$(A$ nand $A)$ nand B		A nand B	A nand $(A$ nand $B)$
0	0		1	1		1	1
0	1		1	0		1	1
1	0		0	1		1	0
1	1		0	1		0	1

$$a \uparrow b = \overline{(ab)}$$

NAND

$$\underline{\underline{b \uparrow a}} = \overline{(ba)} = \overline{(ab)} = \underline{\underline{a \uparrow b}}$$

Q. Give an Operation which is Commutative But Not Associative.

For $x, y \in \mathbb{Z}$, define $x * y = -x - y$.

Commutativity is clear.

$x * y = -x - y$
$y * x = -y - x$ } same

For associativity,

$$a * (b * c) = a * (-b - c) = -a - (-b - c) = -a + b + c$$

$$(a * b) * c = (-a - b) * c = -(-a - b) - c = a + b - c$$

so associativity fails if $a \neq c$.

Q. Give an Operation which is Commutative But Not Associative.

$$a * b = \frac{a+b}{2}$$, Q

$$a * b = \sqrt{ab}$$, R

$(N, *)$

NOT Closed

$a * b = \frac{a+b}{2}$

NOT a binary op^n on N.

$a=1, b=2$

$a * b = \frac{1+2}{2} = \frac{3}{2} \notin N$

$(\mathbb{Q}, *)$

$a * b = \sqrt{ab}$

NOT closed

Not a binary op on $\mathbb{Q}$.

$a = 1$

$b = 2$

$a * b = \sqrt{2} \notin \mathbb{Q}$

Q. Give an Operation which is Commutative But Not Associative.

$$x * y = |x - y|.$$

Z

$x * y = |x - y|$
$y * x = |y - x|$ } Same

$x = 2$
$y = 3$

$|2-3| = 1$
$|3-2| = 1$

$(\mathbb{Z}, *)$ $\quad x * y = |x - y|$

$(2 * 3) * 4 \overset{?}{=} 2 * (3 * 4)$

$(|2-3|) * 4 \qquad 2 * 1$

$1 * 4 \qquad |2-1| = 1$

$= |1-4| = 3$

Q.

Give an Operation which is Neither Associative Not Commutative.

$-$, $\div$

Number theory

$\Rightarrow$

Prop logic

Set Difference

Set theory

Decide whether the statement is an example of the commutative, associative, identity, inverse, or distributive property.

$$-6 + (14 + 13) = (14 + 13) + (-6)$$

The statement is an example of which of the following properties?

- A. Distributive
- B. Associative
- C. Inverse
- D. Identity
- E. Commutative

Decide whether the statement is an example of the commutative, associative, identity, inverse, or distributive property.

$$-6 + (14 + 13) = (14 + 13) + (-6)$$

The statement is an example of which of the following properties?

- A. Distributive
- B. Associative
- C. Inverse
- D. Identity
- E. Commutative ✓

$(14+13) = b$; $-6 = a$

$a + b = b + a$

$$\boxed{a * b = b * a}$$

Identify the rule(s) of algebra illustrated by the statement. (Select all that apply.)

$$x + (y + 12) = (x + y) + 12$$

- ☐ Identity (Addition)
- ☐ Inverse (Addition)
- ☐ Distributive Property
- ☐ Identity (Multiplication)
- ☐ Inverse (Multiplication)
- ☐ Associative (Addition)
- ☐ Commutative (Addition)
- ☐ Associative (Multiplication)
- ☐ Commutative (Multiplication)

Identify the rule(s) of algebra illustrated by the statement. (Select all that apply.)

$$x + (y + 12) = (x + y) + 12$$

- [ ] Identity (Addition)
- [ ] Inverse (Addition)
- [ ] Distributive Property
- [ ] Identity (Multiplication)
- [ ] Inverse (Multiplication)
- [ ] Associative (Addition)
- [ ] Commutative (Addition)
- [ ] Associative (Multiplication)
- [ ] Commutative (Multiplication)

Name the property illustrated by the true statement.

$$(x + 9) + 5 = (9 + x) + 5$$

Choose the correct answer below.

- A. associative property of multiplication
- B. associative property of addition
- C. commutative property of addition
- D. commutative property of multiplication

Name the property illustrated by the true statement.

$$(x + 9) + 5 = (9 + x) + 5$$

Choose the correct answer below.

- A. associative property of multiplication
- B. associative property of addition
- C. commutative property of addition
- D. commutative property of multiplication

Which Property of Addition is shown? (y + z) + (6) = (6) + (y + z)

Select one:

- a. distributive property
- b. identity property
- c. commutative property
- d. associative property

Which Property of Addition is shown? (y + z) + (6) = (6) + (y + z)

$(y+z) = a$

$(6) = b$

$a+b = b+a$

Select one:

- a. distributive property
- b. identity property
- c. commutative property
- d. associative property