Set Theory

Summary Lecture

Equivalence Relation

Equivalence Classes

Website : https://www.goclasses.in/

Equivalence relations are important throughout mathematics and computer science.

Equivalence Relations

The properties of relations are sometimes grouped together and given special names. A particularly useful example is the equivalence relation.

Definitions

A relation that is reflexive, symmetric, and transitive on a set S is called an **equivalence relation** on S.

Equivalence Relations

- Some relations are reflexive, symmetric, and transitive:
  - $x = y$
  - $u \leftrightarrow v$
  - $x \equiv_k y$
- Definition: An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.

Equivalence Relations

A binary relation R over a set A is called an **equivalence relation** if it is

- **reflexive**,
- **symmetric**, and
- **transitive**.

"x and y have the same color"

"x and y have the same shape"

"x = y"

Same value

"x and y have the same area"

"x and y are programs that produce the same output"

In Eq. Relations, there is Some kind of Equality involved.

Eg: Same value
Same color
Same remainder when divided by m
Same shape
Same length
- - -

Informally

An **equivalence relation** is a relation that indicates when objects have some trait in common.

Do <u>not</u> use this definition in proofs!
It's just an intuition!

Set S = set of all shapes

R on S

$R : S \to S$

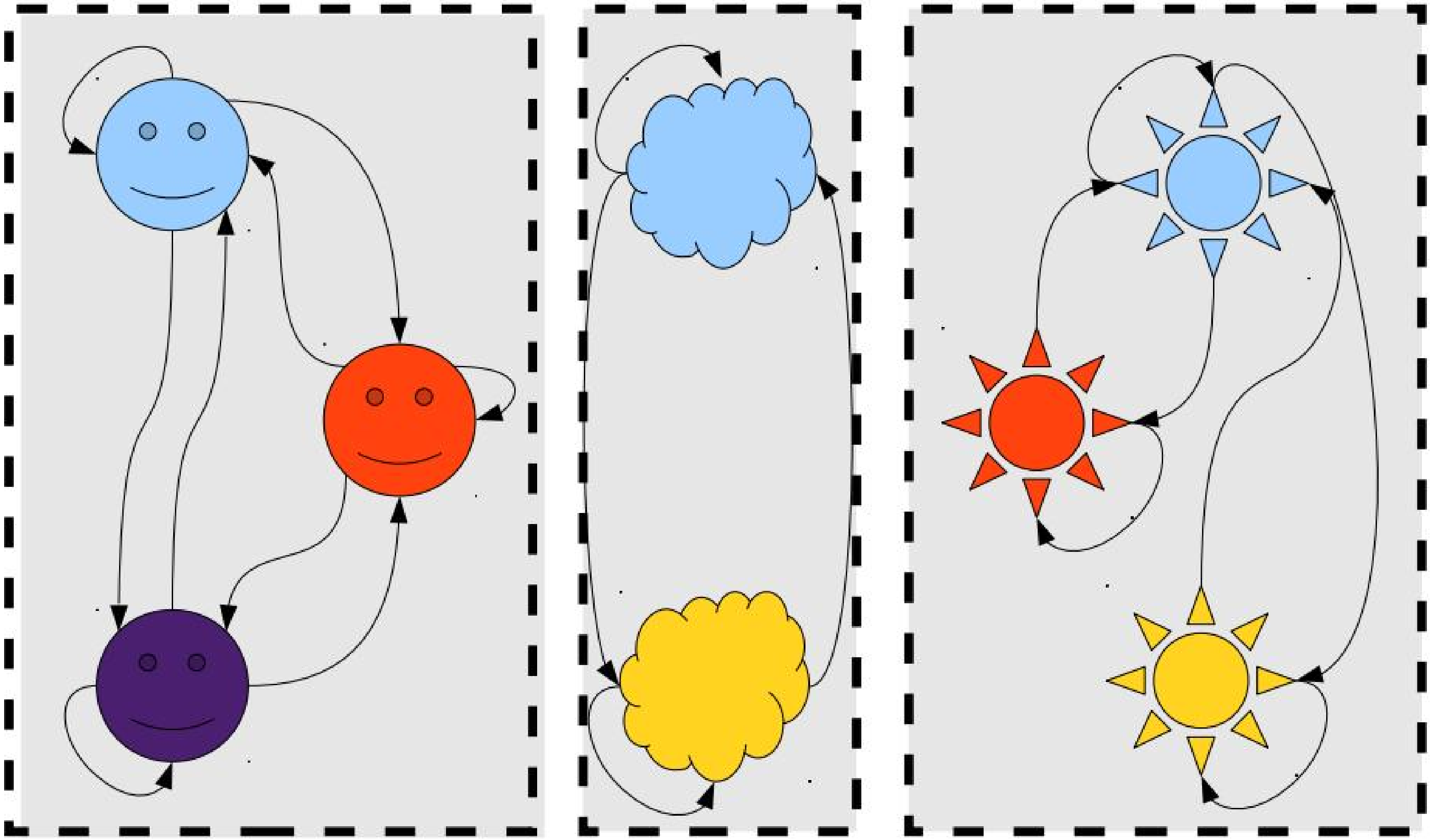
$R \subseteq S \times S$

Same

R on S

xRy iff x, y have same shape.

S

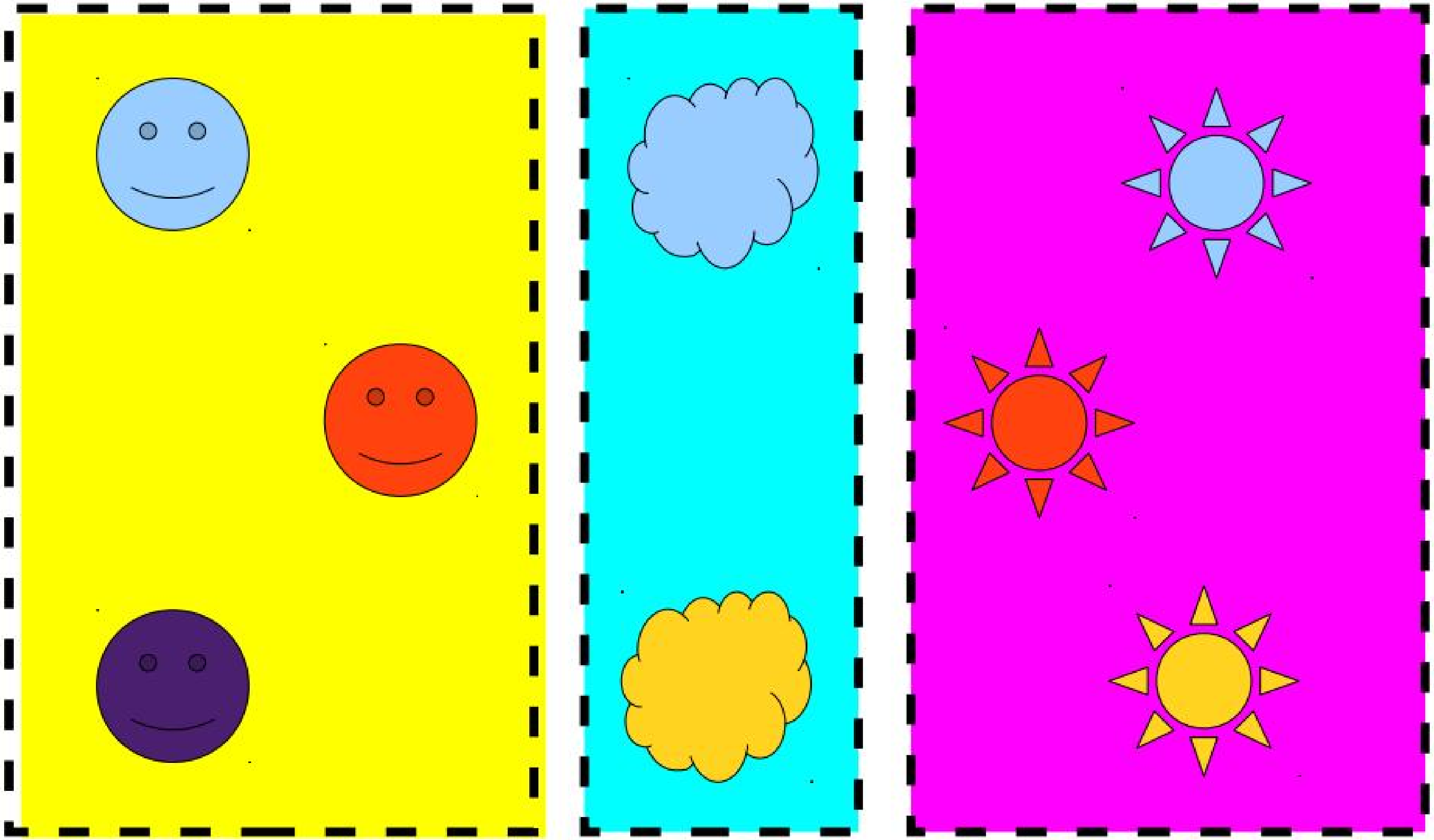


$xRy \equiv x$ and y have the same shape.

GO CLASS

GO Classes

S



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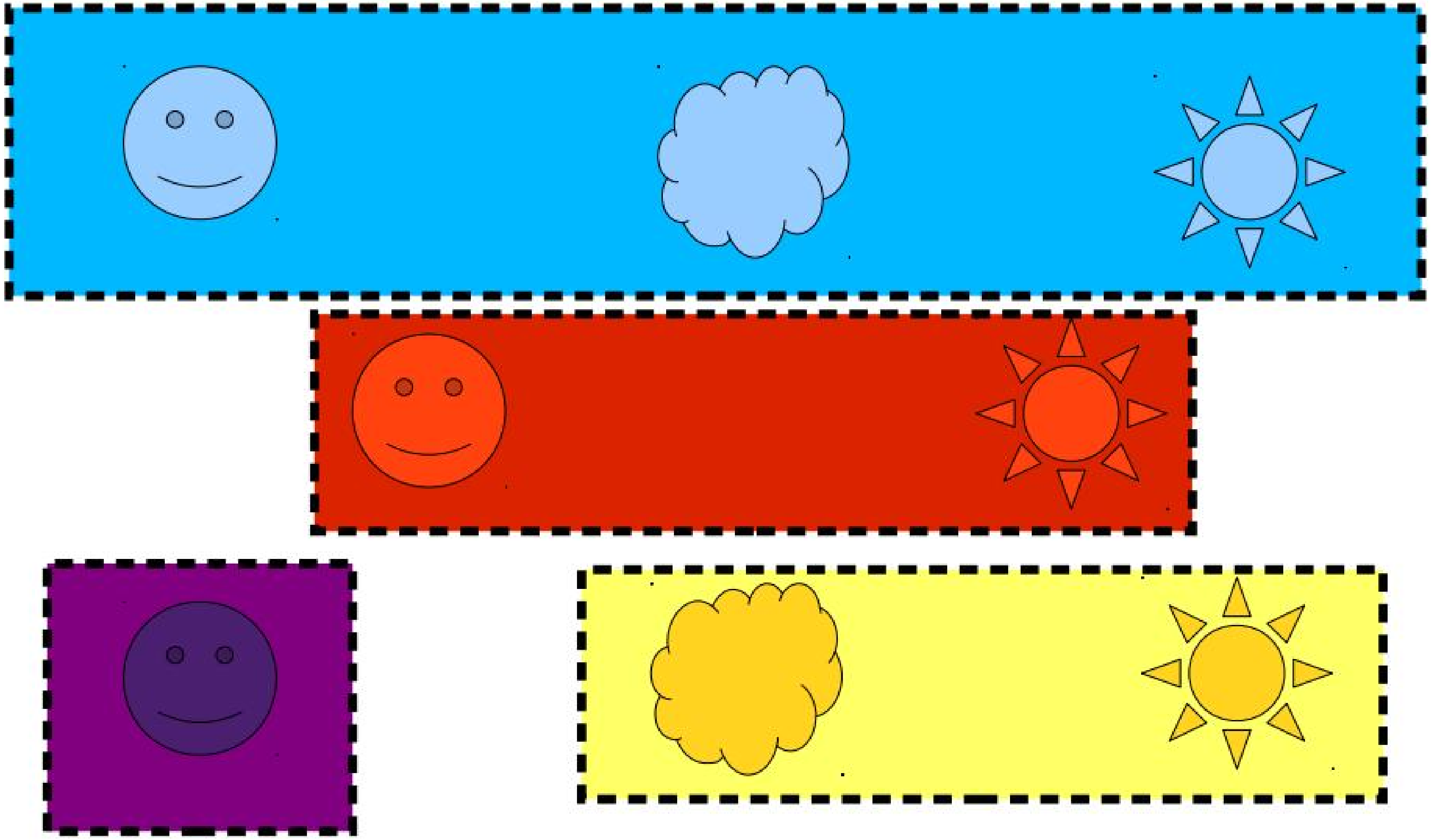


$xRy \equiv x$ and y have the same **color**.

Two elements a and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

$a R b$ then $\boxed{a \sim b}$ $\boxed{b \sim a}$

$b R a$

Equivalence Classes

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a. The equivalence class of a with respect to R is denoted by $[a]_R$. When only one relation is under consideration, we can delete the subscript R and write $[a]$ for this equivalence class.

In other words, if R is an equivalence relation on a set A, the equivalence class of the element a is

$$[a]_R = \{s \mid (a, s) \in R\}.$$

If $b \in [a]_R$, then b is called a **representative** of this equivalence class. Any element of a class can be used as a representative of this class. That is, there is nothing special about the particular element chosen as the representative of the class.

Set A

$a \in A$

R on A

ER

$[a]_R = \{ y \mid aRy \}$

Equivalence

Class of a wrt R

ER R on A ; $a \in A$

$$[a]_R = \{y \mid aRy\} = \{y \mid yRa\}$$

$$[a]_R = \{y \mid a \sim y\} = \{y \mid y \sim a\}$$

Eg 1 :

Is the following relation on $\{0, 1, 2, 3\}$ an equivalence relations?

If yes, find the equivalence classes.

$R = \{(0, 0),(1, 1),(2, 2),(3, 3)\}$ — Identity Relation

ERL

$[0]_R = \{y \mid 0 \sim y\} = \{y \mid 0 R y\}$

$$[0]_R = \{0\}$$

$$[1]_R = \{1\}$$

$$[2]_R = \{2\}$$

$$[3]_R = \{3\}$$

Partition of Base set

Eg 2 :

Is the following relation on $\{0, 1, 2, 3\}$ an equivalence relations?

If yes, find the equivalence classes.

$R = \{(0, 0),(0, 2),(2, 0),(2, 2),(2, 3),(3, 2)(3, 3)\}$ — Not Ref

not ER

Eg 3 :

Is the following relation on $\{0, 1, 2, 3\}$ an equivalence relations?

Ref
Sym
Trans

If yes, find the equivalence classes.

$R = \{(0, 0),(1, 1),(1, 2),(2, 1),(2, 2),(3, 3)\}$ — Ref; Sym; Trans

ER

$[0]_R = \{0\}$; $[1]_R = \{1,2\} = [2]_R$

$[3]_R = \{3\}$

$= \{0\}$

$= \{1, 2\}$

Partition of Base set

Part of partition

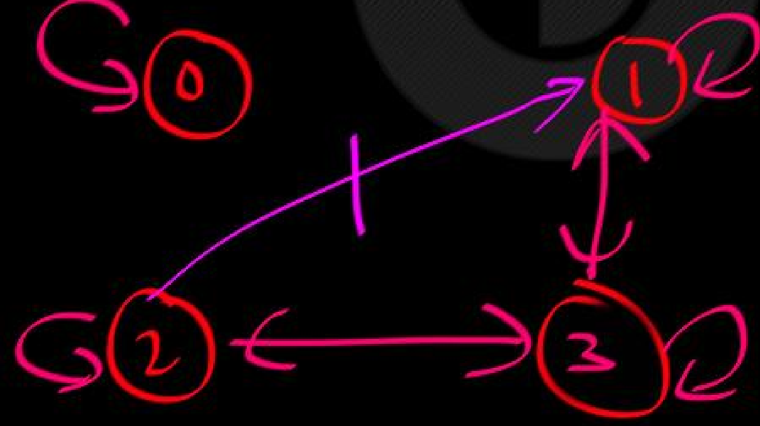
Eq class

Eg 4 :

Is the following relation on {0, 1, 2, 3} an equivalence relations?

If yes, find the equivalence classes.

$R = \{(0, 0),(1, 1),(1, 3),(2, 2),(2, 3),(3, 1),(3, 2),(3, 3)\}$ Sym, Ref,

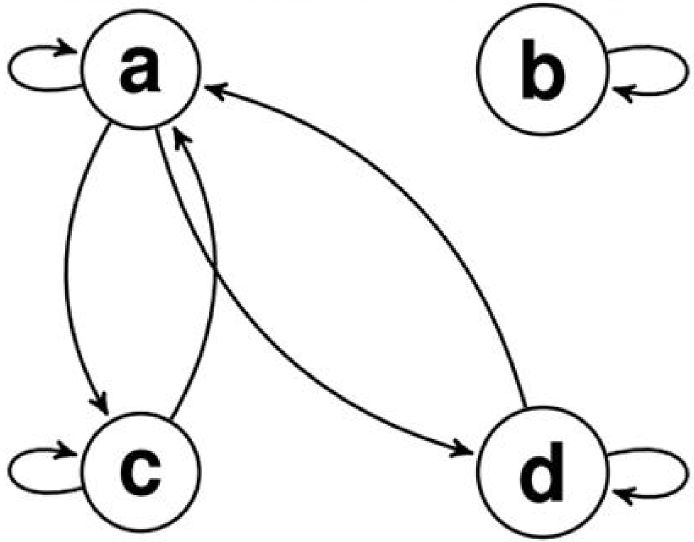


Not Trans

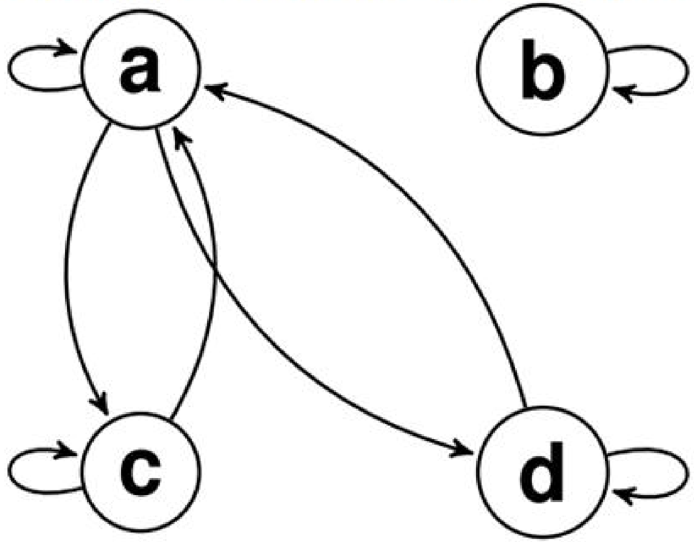
$2R3$, $3R1$ But $2 \not R 1$

not ER

Determine whether the relation with the directed graph shown is an equivalence relation.



Determine whether the relation with the directed graph shown is an equivalence relation.



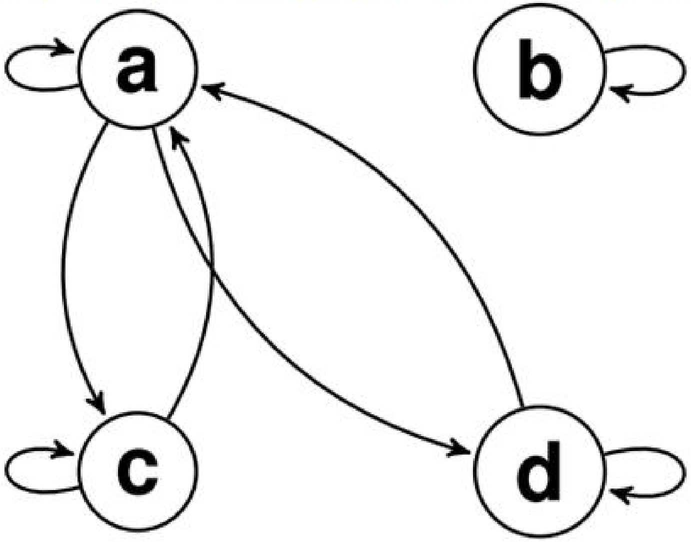
Ref ✓

Sym ✓

Trans X

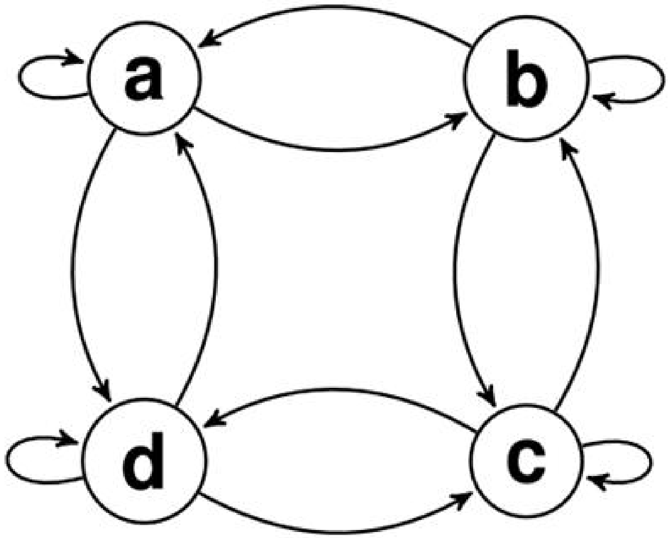
dRa, aRc But $d \not R c$.

Determine whether the relation with the directed graph shown is an equivalence relation.

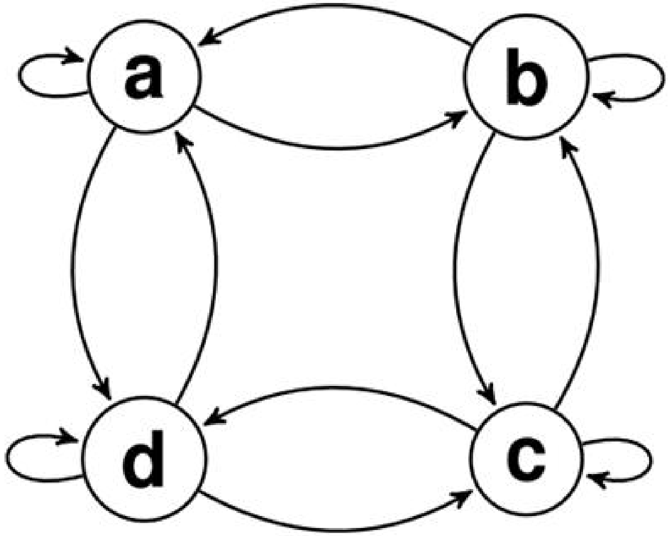


Not an equivalence relation because we are missing the edges (c, d) and (d, c) for transitivity.

Determine whether the relation with the directed graph shown is an equivalence relation.



Determine whether the relation with the directed graph shown is an equivalence relation.



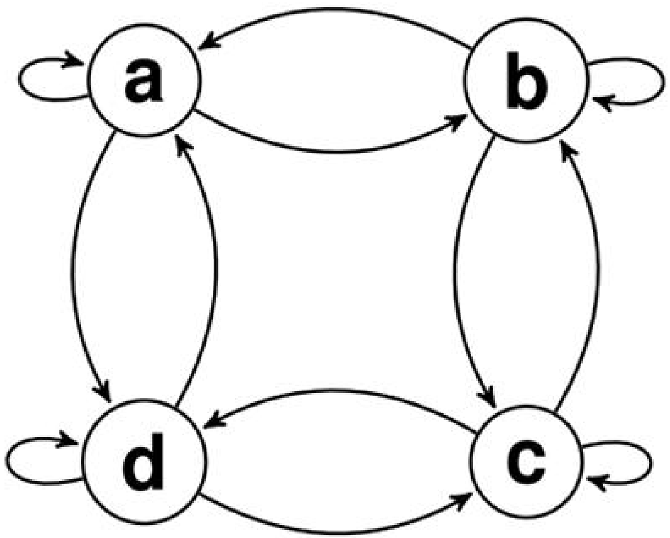
Refl ✓

Sym ✓

Trans ✗

$d R c$, $c R b$ but $d \not R b$

Determine whether the relation with the directed graph shown is an equivalence relation.



Not an equivalence relation because we are missing the edges $(a, c), (c, a), (b, d)$, and (d, b) for transitivity.

Q1 :

Let R be the relation on the set of integers such that aRb if and only if a = b or a = −b. Is R an equivalence relation?

$R : Z \to Z$ $\quad aRb$ iff $a=b$ or $a=-b$

means

$|a|=|b|$

$2R2 \qquad -2R2$

$2R-2 \qquad -2R-2$

aRb iff a, b have same absolute value.

Ref: aRa $a = a$

Sym: $|a| = |b|$ then $|b| = |a|$

Trans: $|a| = |b| = |c| \longrightarrow |a| = |c|$

aRc

ER

$[2]_R = \{2, -2\} = [-2]_R$

$2 \sim 2$, $\boxed{2 \sim -2}$ $[3]_R = \{3, -3\}$

$[0]_R = \{0\}$; for any $x \in Z$

$[x]_R = \{x, -x\}$

$$\left.\begin{array}{l} \{0\} \\ \{1, -1\} \\ \{2, -2\} \\ \{3, -3\} \\ \vdots \end{array}\right\} \text{Partition of } \underbrace{\text{Base set}}_{\mathbb{Z}}$$

Q2 :

Let S be the relation on the set of real numbers such that aSb if and only if a – b is an integer. Is S an equivalence relation?

Base set: Real numbers.

$S : R \rightarrow R$

(1) Refl ✓ $a - a = 0 \in Z$

(2) Sym ✓ $a - b \in Z \Rightarrow b - a \in Z$

(3) Trans: $a - b \in Z, b - c \in Z \Rightarrow$
$a - c \in Z$

S on Reals

S → ER

$1 \sim 2$ $\quad 1 S 2$

$1 \sim -2$ $\quad 1 S -2$

$$[1]_S = \{1, 0, -1, -2, 2, -3, 3, ---\}$$

$$[1]_S = Z = [2]_S = [\text{integer}]_S$$

$$[0.5]_S = \{x.5 \mid x \text{ is integer}\} = [3.5]_S$$

$$= [-3.5]_S$$

Q3 : Congruence Modulo 3

Show that the relation

$R = \{(a, b) \mid a \equiv b \pmod{3}\}$ means a, b have same Remainder when Divided by 3.

is an equivalence relation on the set of integers.

What are the equivalence classes of 0 and 1 for this relation?

Recall that : $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

$$[0]_R = \{ y \mid 0 R y \} = \{ y \mid y R 0 \}$$

$$[0]_R = \{ 0, 3, -3, 6, -6, --- \} = \{ 3k \mid_{k \in Z} \}$$

$0 \sim 3$, $0 \sim 6$, $0 \sim -3$, $0 \sim 9$

$$[1]_R = \{1, 4, 7, -2, -5, \ldots \}$$

$1 \sim 4$; $1 \sim 7$; $1 \sim 10$; $1 \sim -2$

$1 \sim 1$

$$[1]_R = \{3k+1 \mid k \in Z\}$$

$[2]_R = \{2, 5, 8, 11, ---\} = \{3k+2 \mid k \in Z\}$

$$\left.\begin{array}{l} \{3k \mid k \in Z\} \\ \{3k+1 \mid k \in Z\} \\ \{3k+2 \mid k \in Z\} \end{array}\right\} \text{Partition of } Z$$