Xor, Xnor Functions & Their Properties

As we have previously observed, some of the theorems of Boolean algebra are not true for ordinary algebra. Similarly, some of the theorems of ordinary algebra are *not* true for Boolean algebra. Consider, for example, the cancellation law for ordinary algebra:

$$\text{If } x + y = x + z, \quad \text{then} \quad y = z \tag{3-31}$$

The cancellation law is *not* true for Boolean algebra. We will demonstrate this by constructing a counterexample in which $x + y = x + z$ but $y \neq z$. Let $x = 1$, $y = 0$, $z = 1$. Then,

$$1 + 0 = 1 + 1 \text{ but } 0 \neq 1$$

In ordinary algebra, the cancellation law for multiplication is

$$\text{If } xy = xz, \quad \text{then} \quad y = z \tag{3-32}$$

This law is valid provided $x \neq 0$.

Q: In boolean Alg^n:

$$\underbrace{ab = ac}_{} \implies b = c \ ?$$

Ans: false! what if $a=0$; $b=1$; $c=0$.

$\underbrace{ab = ac}$ but $b \neq c$.

In Boolean algebra, the cancellation law for multiplication is also *not* valid when $x = 0$. (Let $x = 0, y = 0, z = 1$; then $0 \cdot 0 = 0 \cdot 1$, but $0 \neq 1$). Because $x = 0$ about half of the time in switching algebra, the cancellation law for multiplication cannot be used.

Even though Statements (3-31) and (3-32) are generally false for Boolean algebra, the converses

$$\text{If } y = z, \qquad \text{then} \qquad x + y = x + z \tag{3-33}$$

$$\text{If } y = z, \qquad \text{then} \qquad xy = xz \tag{3-34}$$

Q: let a, b, c be boolean variables; which is True?

(1) $ab = ac \implies b = c$

(2) $a + b = a + c \implies b = c$

(3) $(ab = ac) \wedge (a = 1) \implies b = c$

Q: let a, b, c be boolean variables; which is True?

1. $ab = ac \implies b = c$
2. $a + b = a + c \implies b = c$
3. $(ab = ac) \wedge (a = 1) \implies b = c$

$\times$

4. $(a+b=a+c) \wedge (a=1) \Rightarrow b=c$

5. $(a+b=a+c) \wedge (a=0) \Rightarrow b=c$

Exclusive-OR and Equivalence Operations

The *exclusive-OR* operation ($\oplus$) is defined as follows:

$$0 \oplus 0 = 0 \qquad 0 \oplus 1 = 1$$
$$1 \oplus 0 = 1 \qquad 1 \oplus 1 = 0$$

The truth table for $X \oplus Y$ is

X	Y	$X \oplus Y$
0	0	0
0	1	1
1	0	1
1	1	0

From this table, we can see that $X \oplus Y = 1$ iff $X = 1$ or $Y = 1$, but *not* both. The ordinary OR operation, which we have previously defined, is sometimes called inclusive OR because $X + Y = 1$ iff $X = 1$ or $Y = 1$, or both.

Exclusive OR can be expressed in terms of AND and OR. Because $X \oplus Y = 1$ iff X is 0 and Y is 1 or X is 1 and Y is 0, we can write

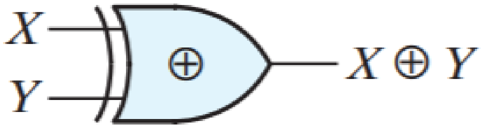
$$X \oplus Y = X'Y + XY' \tag{3-6}$$

The first term in (3-6) is 1 if $X = 0$ and $Y = 1$; the second term is 1 if $X = 1$ and $Y = 0$. Alternatively, we can derive Equation (3-6) by observing that $X \oplus Y = 1$ iff $X = 1$ or $Y = 1$ *and* X and Y are not both 1. Thus,

$$X \oplus Y = (X + Y)(XY)' = (X + Y)(X' + Y') = X'Y + XY' \tag{3-7}$$

In (3-7), note that $(XY)' = 1$ if X and Y are not both 1.

We will use the following symbol for an exclusive-OR gate:



Any of these theorems can be proved by using a truth table or by replacing $X \oplus Y$ with one of the equivalent expressions from Equation (3-7). Proof of the distributive law follows:

$$
\begin{aligned}
XY \oplus XZ &= XY(XZ)' + (XY)'XZ = XY(X' + Z') + (X' + Y')XZ \\
&= XYZ' + XY'Z \\
&= X(YZ' + Y'Z) = X(Y \oplus Z)
\end{aligned}
$$

The *equivalence* operation ($\equiv$) is defined by

$$(0 \equiv 0) = 1 \qquad (0 \equiv 1) = 0$$
$$(1 \equiv 0) = 0 \qquad (1 \equiv 1) = 1 \tag{3-16}$$

The truth table for $X \equiv Y$ is

X	Y	$X \equiv Y$
0	0	1
0	1	0
1	0	0
1	1	1

From the definition of equivalence, we see that $(X \equiv Y) = 1$ iff $X = Y$. Because $(X \equiv Y) = 1$ iff $X = Y = 1$ or $X = Y = 0$, we can write

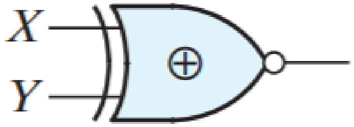
$$(X \equiv Y) = XY + X'Y' \tag{3-17}$$

Equivalence is the complement of exclusive OR:

$$\begin{aligned}(X \oplus Y)' &= (X'Y + XY')' = (X + Y')(X' + Y) \\ &= XY + X'Y' = (X \equiv Y)\end{aligned} \tag{3-18}$$

Just as for exclusive OR, equivalence is commutative and associative.

Because equivalence is the complement of exclusive OR, an alternate symbol for the equivalence gate is an exclusive-OR gate with a complemented output:



$(X \oplus Y)' = (X \equiv Y)$

The equivalence gate is also called an exclusive-NOR gate.

ExNOR opⁿ / Equivalence opⁿ / Coincidence opⁿ

$\odot$ $\quad \Longleftrightarrow$

$$a \odot b = \overline{a \oplus b}$$

The following theorems apply to exclusive OR:

$$X \oplus 0 = X$$

$$X \oplus 1 = X'$$

$$X \oplus X = 0$$

$$X \oplus X' = 1$$

$$X \oplus Y = Y \oplus X \text{ (commutative law)}$$

$$(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z) = X \oplus Y \oplus Z \text{ (associative law)}$$

$$X(Y \oplus Z) = XY \oplus XZ \text{ (distributive law)}$$

$$(X \oplus Y)' = X \oplus Y' = X' \oplus Y = XY + X'Y'$$

$a \oplus b = \bar{a}b + a\bar{b}$

$\bar{a} \oplus b = ab + a'b'$

x

$= x'b + xb'$

$a \oplus \bar{b} = ab + a'b' = a \odot b$

$a' \oplus b' = ab' + a'b$

$\overline{a \oplus b} = a \odot b$

$\bar{a} \oplus b = a \odot b$

$a \oplus b' = a \odot b$

$a' \oplus b' = a \oplus b$

$a' \odot b' = a \odot b$

$a' \oplus b = \left\{ \begin{array}{c} a' \text{ is } 0 \text{ AND } b \text{ is } 1 \\ \text{OR} \\ a' \text{ is } 1 \text{ AND } b \text{ is } 0 \end{array} \right\}$

$\underbrace{a' \oplus b = ab + a'b'}$

How is an XOR with more than 2 inputs supposed to work?

$$a \oplus b \left\{ \begin{array}{ll} 1 & \text{when } a \neq b \\ 0 & \text{when } a = b \end{array} \right\}$$

$a \oplus b \oplus c \oplus d - - - \oplus n$

$0 \oplus 0 \oplus 1 \oplus 0 = 1$

0

1

$\oplus \Rightarrow$ Comm, Asso

$0 \oplus 1 = 1$

$0 \oplus 1 \oplus 1 = 0$

$0 \oplus 1 \oplus 1 \oplus 1 = 1$

$0 \oplus 0 \oplus 0 = 0$

Claim: XOR is 1 iff we have odd number of 1's:

Proof:

$0 \oplus 1 \oplus 1 \oplus 0 \oplus 1$

Comm; Assoc

$\Rightarrow 1 \oplus 1 \oplus 1 \oplus 0 \oplus 0$

$1 \oplus (0 \oplus 1) \oplus 1$ (Asso)

Comm

$1 \oplus 1 \oplus 0 \oplus 1$

Comm

$1 \oplus 1 \oplus 1 \oplus 0$

odd no. of 1's :

$\underbrace{1 \oplus 1}_{0} \oplus 1$

$0 \oplus 0 = 0$

$0 \oplus 0 \oplus 0 = 0$

$\underbrace{\boxed{1 \oplus 1 \oplus \cdots \oplus 1}}_{\text{odd no. of 1's}} \oplus \underbrace{\boxed{0 \oplus 0 \cdots \oplus 0}}_{} \Rightarrow 0$

no. of 0 Doesn't matter

$= 1 \oplus 0 \Rightarrow 1$ ✓

EXOR is 1 iff Number of 1's is odd.

Q: $\underbrace{a \oplus a \oplus a \oplus \cdots \oplus a}_{\text{even no. of } a\text{'s}} = 0$

Because we can't have odd no. of 1's

Can we consider an XNOR operation as an even number of 0 checker?

→ Yes ✓

XNOR is 1 iff even no. of 0's.

Proof:

$\underbrace{0\ 0\ 0\ 0\ 0 - - - 0\ 0}_{\text{even}}$ $\underbrace{0\ 1\ 0 - - 0\ 1}_{\text{no. of 1's Doesn't matter.}}$

$1 \odot 1 = 1$

$1 \odot 1 \odot 1 = 1$

$0 \odot 0 = 1$

$0 \odot 0 \odot 0 \odot 0 \odot 0 = 1$

$\underbrace{0 \odot 0 \odot - - \odot 0}_{\text{even no. of 0's}} \odot \underbrace{1 \odot - - - \odot 1}_{\text{any no. of 1's}}$

$\Downarrow \qquad \Downarrow$

$1 \odot 1 = 1$

XOR = 1 iff #1 = odd

XNOR = 1 iff #0 = even

Is XOR equal to XNOR when odd number of inputs are considered?

Yes

If odd no. of inputs:

$\underbrace{a \oplus b \oplus c}_{}$ $=$ $\underbrace{a \odot b \odot c}_{}$

1 iff odd #1

iff even #0

0 1

$$a \oplus b \oplus c = a \odot b \odot c$$ ✓

$$a_1 \oplus a_2 \oplus a_3 \oplus a_4 \oplus a_5 = a_1 \odot a_2 \cdots \odot a_5$$

If even no. of inputs:

then EXOR, EXNOR complement of each other.

$$a \oplus b = \overline{a \odot b} \quad ✓$$

$$a \oplus b \oplus c \oplus c = \overline{a \odot b \odot c \odot d} \quad ✓$$