Group Theory

Next Topic

Cayley Table(Multiplication Table)

Operation Table

Website : https://www.goclasses.in/

$(\{T, F\}, \rightarrow) \longrightarrow$ Groupoid, not even semigroup

$\rightarrow$	T	F
T	T	F
F	T	T

$T \rightarrow F$

$F \rightarrow T$

1.4 Cayley tables

A binary operation $*$ on a *finite* set S can be displayed in the form of an array, called the *Cayley* table.

If S has n elements, then the Cayley table is an $n \times n$ array, with each row and each column labelled (uniquely) by an element of S.

The entry of the table in row x and column y is the element $x * y \in S$.

Here is a simple example: $S = \{0, 1\}$, and $*$ is just multiplication of numbers.

*	0	1
0	0	0
1	0	1

1.3 Remark Sometimes a binary operation $*$ is given by a table of the form

$$\begin{array}{c|ccc} * & \cdots & y & \cdots \\ \hline \vdots & & \vdots & \\ x & \cdots & x * y & \\ \vdots & & \vdots & \end{array}$$

For instance, the binary operation "and" on the set $\{\text{true}, \text{false}\}$ can be depicted as

$\wedge$	true	false
true	true	false
false	false	false

Thus, $(\{\text{true}, \text{false}\}, \wedge)$ is a commutative monoid with identity element true.

Cayley Tables

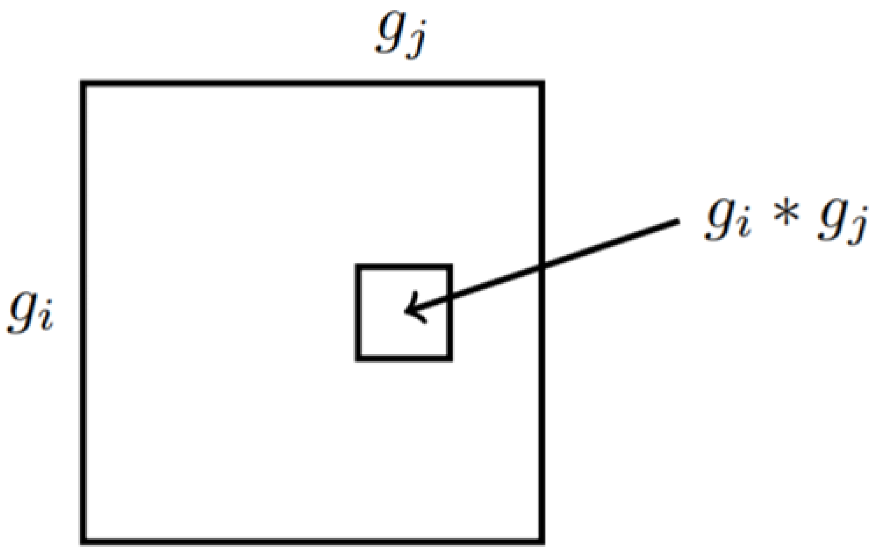
A (binary) operation on a finite set can be represented by a table. This is a square grid with one row and one column for each element in the set. The grid is filled in so that the element in the row belonging to x and the column belonging to y is x*y. Example:

This is a table for a binary operation on the set {A,B,C,D,E,F}

	A	B	C	D	E	F
A	A	B	C	D	E	F
B	B	C	A	F	D	E
C	C	A	B	E	F	D
D	D	E	F	A	B	C
E	E	F	D	C	A	B
F	F	D	E	B	C	A

Notice that B*E = D while E*B = F so this operation is not commutative.

Let $(G, *)$ be a group where $G = \{g_1, g_2, \ldots, g_n\}$. Consider the multiplication table of $(G, *)$.



Group (Abelian)

The set of complex numbers $G = \{1, i, -1, -i\}$ under multiplication.
The multiplication table for this group is:

$*$	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

finding "e" from Cayley Table :

Id

	a	b	c	d	e
Id — a	a	b	c	d	e
b	b				
c	c				
d	d				
e	e				

Finding Commutative property from Cayley Table: $\Leftrightarrow$ Symmetric Table (Symmetric matrix)

	a	b
a		$a*b$
b	$b*a$	


a b c d e
a
b
c
d
e
diagonal

Group Theory

Next Topic

Properties of Cayley Table of Groups

For Groups, Cayley table has some interesting properties.

Website : https://www.goclasses.in/

Cayley Table is created for finite Structures.

Cayley tables of groups

If $*$ is a binary operation on a finite set S, then properties of $*$ often correspond to properties of the Cayley table.

If (G,*) is a Finite Group then:

1. Each element $g \in G$ appears exactly once in each row and in each column.
2. So, Every row/column is simply a permutation of all elements i.e. every element appears exactly once.
3. Row and Column of Identity element is EXACTLY same as the Header.

In Cayley Table of a finite Group:

In any Row, no element can repeat.

In any Column, " " " ".

Proof : Group $(G, *)$

	b	c
a	g	g

$\cancel{a}b = \cancel{a}c$

so

$\underline{b = c}$

Proof : Group $(G, *)$

	b
a	g
c	g

$a\cancel{b} = c\cancel{b}$

So

$a = c$

G

	a	b	c	d
a				
b				
c	b	a	c	d
d				

1. For Finite Groups, Every row of the multiplication table contains every element of G exactly once:

Every row of the multiplication table contains every element of G exactly once:

Suppose first that g appears twice in row x. Then there are two distinct elements $y, z \in G$ such that $x * y = g = x * z$. Let $\overline{x}$ be the inverse of x in $(G, *)$. Then

$$y = e_G * y = (\overline{x} * x) * y = \overline{x} * (x * y) = \overline{x} * g = \overline{x} * (x * z) = (\overline{x} * x) * z = e_G * z = z,$$

contrary to the assumption that y, z are distinct.

Hence g cannot appear twice in any row of the Cayley table. A similar argument applies to any other element of the group, so **no** element appears twice in the same row.

Q: Is G a Group?

Here is an example of a multiplication table for a binary operation $*$ on the set $G = \{a, b, c, d\}$.

$*$	a	b	c	d
a	a	b	c	a
b	a	c	d	d
c	a	b	d	c
d	d	a	c	b

Q: Is G a Group? ⇒ No

Here is an example of a multiplication table for a binary operation $*$ on the set $G = \{a, b, c, d\}$.

e = DNE

$*$	a	b	c	d
a	a	b	c	a
b	a	c	d	d
c	a	b	d	c
d	d	a	c	b

Example 1.2: A latin square of side 6 in standard form with respect to the sequence $e, g_1, g_2, g_3, g_4, g_5$.

	e	g_1	g_2	g_3	g_4	g_5
e	e	g_1	g_2	g_3	g_4	g_5
g_1	g_1	e	g_3	g_4	g_5	g_2
g_2	g_2	g_3	e	g_5	g_1	g_4
g_3	g_3	g_4	g_5	e	g_2	g_1
g_4	g_4	g_5	g_1	g_2	e	g_3
g_5	g_5	g_2	g_4	g_1	g_3	e

Asso: $(g_1 * g_2) * g_3 \neq g_1 * (g_2 * g_3)$

Example 1.2: A latin square of side 6 in standard form with respect to the sequence $e, g_1, g_2, g_3, g_4, g_5$.

	e	g_1	g_2	g_3	g_4	g_5
e	e	g_1	g_2	g_3	g_4	g_5
g_1	g_1	e	g_3	g_4	g_5	g_2
g_2	g_2	g_3	e	g_5	g_1	g_4
g_3	g_3	g_4	g_5	e	g_2	g_1
g_4	g_4	g_5	g_1	g_2	e	g_3
g_5	g_5	g_2	g_4	g_1	g_3	e

① Closed ✓

② Id $= e$ ✓

③ $e^{-1} = e$

$g_1^{-1} = g_1$

$g_2^{-1} = g_2$

$g_3^{-1} = g_3$

$g_4^{-1} = g_4$

$g_5^{-1} = g_5$

Inverse ✓

Comm

The above latin square is not the multiplication table of a group, because for this square:

$$\begin{aligned}(g_1 * g_2) * g_3 &= g_3 * g_3 = e \\ \text{but} \\ g_1 * (g_2 * g_3) &= g_1 * g_5 = g_2\end{aligned}$$

So, NOT Group

Note:

Cayley Table of Group

No Repetition in any Row, any Column

Note: Checking Associtivity from Cayley table is Very Hard.

So, Small Cayley tables will be given in Questions.

To Check Associativity, never include Identity element.

$(e\ a)\ b = e\ (a\ b)$

$a\ b = ab$

$$\underline{\underline{(a\ e)}}\ b \stackrel{?}{=} a\ (\underline{\underline{e\ b}})$$

$$ab = ab \checkmark$$

$$(a\ b)\ e \stackrel{?}{=} a\ (b\ e)$$

$$= \underline{\underline{ab}} = ab$$

To Check Commutative :

$\boxed{i^{th} \text{ Row} = i^{th} \text{ Column}}$

Symmetric matrix

The following Latin square is the Cayley table of a group:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Verify associativity for the non-identity elements in the group.

Is the group abelian? Why, or why not?

What are the inverses of a, b, and c, respectively?

Is the inverse of ab equal to ba? Why, or why not?

Commutative : ✓

The following Latin square is the Cayley table of a group:

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Identity $= e = e$ ✓

Verify associativity for the non-identity elements in the group.

Is the group abelian? Why, or why not? ✓

What are the inverses of a, b, and c, respectively?

Is the inverse of ab equal to ba? Why, or why not?

$a^{-1} = a$ $\quad e^{-1} = e$

$b^{-1} = b$

$c^{-1} = c$

$(ab)^{-1} =$

$c^{-1} = c$
$= ba$

(i) Verify associativity for the non-identity elements in the group.

$$(ab)c = cc = e \qquad a(bc) = aa = e$$

$$(aa)b = eb = b \qquad a(ab) = ac = b$$

$$(aa)a = ea = a \qquad a(aa) = ae = a$$

(ii) Is the group abelian? Why, or why not?

The Cayley table is symmetric. Thus, the group is abelian.

(iii) What are the inverses of a, b, and c, respectively?

$$a^{-1} = a, \quad b^{-1} = b, \quad c^{-1} = c.$$

(iv) Is the inverse of ab equal to ba? Why, or why not?

$$(ab)^{-1} = b^{-1}a^{-1} = ba$$