First Order Logic

Next Topic:

Validity of FOL Expressions Involving Implication

Website : https://www.goclasses.in/

$$\underbrace{\forall x\, P(x)}_{\alpha} \longrightarrow \underbrace{\exists x\, P(x)}_{\beta} \equiv \text{Valid}$$

Domain: make $\forall x P(x)$ = True ; Can you make β = false

a —— P = T

b —— P = T

c —— P = T

⋮ ⋮

$\Downarrow$

No

Already $P(a)$ = True

α, β = FOL Expressions

$$(\alpha) \rightarrow (\beta)$$

To make it false:

α = True and

β = false

Simultaneously

Procedure: α, β = FOL Expression

$(\alpha) \longrightarrow (\beta)$; valid ??

Step 1: Assume Domain $D = \{a, b, c, d, \ldots\}$

Step 2: Try to make $\boxed{\alpha \longrightarrow \beta \text{ false}}$

App1: $\alpha = T \longrightarrow \beta = F$

OR

App2: $\beta = F \longrightarrow \alpha = T$

If we "can" make $\alpha \to \beta$ false then $\alpha \to \beta$ is Invalid.

If we "cannot" make $\alpha \to \beta$ false, then $\alpha \to \beta$ is Valid.

Q: $\underbrace{\forall x (P(x) \wedge Q(x))}_{\alpha} \xrightarrow{?} \underbrace{\forall x P(x)}_{\beta} \wedge \forall x Q(x)$

Domain: $\{a, b, c, d, \_, \_, \_\}$

Approach 1: make $\alpha = \text{True}$

then Try to make $\beta = \text{false}$.

Domain:

$\forall_x \left(P(x) \wedge Q(x) \right)$ = True

a = $P \wedge Q$ = T

b = $P \wedge Q$ = T

c = $P \wedge Q$ = T

⋮

a ⇒ P = T, Q = T

b ⇒ P = T, Q = T

c ⇒ P = T, Q = T

⋮

$\forall_x P(x) \wedge \forall_x Q(x)$

True ∧ True

= True

Cannot make β false.

So

$$\forall x \left(P(x) \wedge Q(x) \right) \longrightarrow \left((\forall x P(x)) \wedge (\forall x Q(x)) \right)$$

Valid

$$\underbrace{\forall x\,(P(x) \wedge Q(x))}_{\beta} \longleftarrow (\forall x\, P(x)) \wedge (\forall x\, Q(x))$$

Can we make β = false?

$a \Rightarrow P(a) \wedge Q(a) = \text{True}$

$b \Rightarrow P(b) \wedge Q(b) = \text{True}$

$c \Rightarrow \quad = \text{"}$

$d \Rightarrow \quad \text{"}$

make α = True

Domain:

$P = T \quad Q = T \Leftarrow a$

$P = T \quad Q = T \Leftarrow b$

$P = T \quad Q = T \Leftarrow c$

$P = T \quad Q = T \quad \vdots$

So After making α = True, cannot make β = false,

So $\alpha \rightarrow \beta$ is valid.

$$\forall x (P(x) \wedge Q(x)) \underset{\checkmark}{\overset{\checkmark}{\rightleftarrows}} \forall x P(x) \wedge \forall x Q(x)$$

So After making $\alpha = $ True, Cannot make $\beta = $ false,

So $\alpha \to \beta$ is valid.

$$\forall x\,(P(x) \land Q(x)) \equiv \forall x P(x) \land \forall x Q(x)$$

$$\underbrace{\forall x\left(P(x) \wedge Q(x)\right)}_{\text{Scope of } \forall x}$$

a : $P(a) \wedge Q(a)$

b : $P(b) \wedge Q(b)$

e : $P(e) \wedge Q(e)$

$$\left(\forall x\, P(x)\right) \wedge \left(\forall x\, Q(x)\right)$$

$$\left.\begin{matrix} P(a) \\ P(b) \\ P(e) \end{matrix}\right\} \qquad \left\{\begin{matrix} Q(a) \\ Q(b) \\ Q(e) \end{matrix}\right.$$

An example of a valid formula is $\forall x(P(x) \wedge Q(x)) \Rightarrow \forall y P(y)$. An example of an unsatisfiable formula is $\forall x P(x) \wedge \exists y \neg P(y)$.

Scope

$\forall x(P(x) \wedge Q(x)) \xrightarrow{?\checkmark} \forall y P(y)$: Valid

α β

Domain:

make α = True

$a \Rightarrow P(a) \wedge Q(a)$ = True

$b \Rightarrow P(b) \wedge Q(b)$ = True

$c \Rightarrow P(c) \wedge Q(c)$ = True

⋮

Can we make β = false ? $\Rightarrow$ No

Already, for all, P = True.

$\underbrace{\forall x (P(x) \wedge Q(x))}_{\beta} \longleftarrow \underbrace{\forall y\, P(y)}_{\alpha}$; Invalid

Can you make β false?

$a \Rightarrow P(a) = T$; $Q(a) = F$

$P(a) \wedge Q(a) = \text{false}$

make $\alpha = \text{True}$

Domain $\begin{cases} a \Rightarrow P(a) = T \\ b \Rightarrow P(b) = T \\ c \Rightarrow P(c) = T \\ \vdots \end{cases}$

$G: \forall x P(x) \wedge \exists y \neg P(y)$

for everyone P = True and
for someone P = false

$\Rightarrow$ can never happen

First Order Logic

Next Topic:

Logical Equivalences Involving Quantifiers

Website : https://www.goclasses.in/

α, β = FOL expressions

If $\alpha \longrightarrow \beta$ = Valid

and $\beta \longrightarrow \alpha$ = Valid

$\alpha \equiv \beta$

so $\boxed{\alpha \longleftrightarrow \beta = \text{Valid}}$

$\underbrace{\alpha \equiv \beta}$

iff

$\alpha \longrightarrow \beta$ Valid

and

$\beta \longrightarrow \alpha$ Valid

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value for…
  - every predicate substituted into these statements

    and

  - every domain used for the variables in the expressions.

- The notation $S \equiv T$ indicates that S and T are logically equivalent.

- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

$$\forall x\, P(x) \equiv \underbrace{\neg \exists x\, \neg P(x)}_{\equiv\ \forall x\, P(x)}$$

$$\forall x\,(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

$$\forall x\,(P(x) \vee Q(x)) \overset{?}{\equiv} \forall x P(x) \vee \forall x Q(x)$$

$$\xrightarrow{?}$$

$$\xleftarrow{?}$$

Scope

$\underbrace{\forall x\,(P(x) \vee Q(x))}_{\alpha} \xrightarrow{?=\times} \underbrace{(\forall x\,P(x)) \vee (\forall x\,Q(x))}_{\beta}$

Domain:

make α = True

$a = P(a)^{=F} \vee Q(a)^{=T} = T$

$b = P(b)^{T} \vee Q(b)^{T} = T$

$c = P(c) \vee Q(c) = T$

T F

Can we make β = false?

To make β = false

$\forall x\,P(x)$ = false ✓

and

$\forall x\,Q(x)$ = false ✓

$\underbrace{\forall x\,(P(x) \lor Q(x))}_{\alpha} \;\not\rightarrow\; \forall x P(x) \lor \forall x Q(x)$

Domain: N

$P(x) = x$ is even
$Q(x) = x$ is odd

$\alpha = \text{True}$

$\forall x P(x) \lor \forall x Q(x)$

everyone is even
or
everyone is odd

$\forall x\,(P(x) \vee Q(x))$

Everyone is even or odd

$\Downarrow$

True

$\forall x\,P(x) \vee \forall x\,Q(x)$

Everyone is even = F

or

everyone is odd. = F

$F \vee F = F$

$$\underbrace{\forall x\,(P(x) \vee Q(x))}_{\beta} \longleftarrow \underbrace{(\forall x\, P(x)) \vee (\forall x\, Q(x))}_{\alpha}$$

To make α = True

β = True $\longleftarrow$ $\forall x\, P(x)$ = True

or

β = True $\longleftarrow$ $\forall x\, Q(x)$ = True

$$\forall x\,(P(x) \lor Q(x)) \;\underset{\longleftarrow}{\xrightarrow{\times}}\; (\forall x\,P(x) \lor \forall x\,Q(x))$$

Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions. We use the notation $S \equiv T$ to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

First Order Logic

Next Topic:

Distributive Properties of Quantifiers

(Distribution of Quantifiers over logical connectives)

Website : https://www.goclasses.in/

$$\underline{\forall x}\,(P(x)\ \underline{\wedge}\ Q(x)) \ \overset{\checkmark}{\underset{\checkmark}{\rightleftarrows}}\ \forall x P(x) \wedge \forall x Q(x)$$

$$\forall x\,(P(x) \vee Q(x)) \ \overset{\times}{\underset{\checkmark}{\rightleftarrows}}\ \forall x P(x) \vee \forall x Q(x)$$

Distribution of Quantifier over logical Connectives :

$\Pi \longrightarrow$ Quantifier $(\exists, \forall, \exists!)$

$$\Pi_x \left(P(x) \# Q(x) \right) \underset{?}{\overset{?}{\rightleftarrows}} \left(\Pi_x P(x) \# \Pi_x Q(x) \right)$$

$\wedge, \vee, \oplus, \rightarrow, \leftrightarrow, \uparrow, \downarrow$

$\Pi_x (P(x) \# Q(x))$

Compact Exp

$\Pi_x P(x) \# \Pi_x Q(x)$

Expanded Exp

$\forall$ with $\wedge$:

Compact $\xrightarrow{\checkmark}$ Expanded

Compact $\xleftarrow{\checkmark}$ Expanded

$\forall$ with $\vee$:

Compact $\xleftarrow{\checkmark}$ Expanded

Compact $\xrightarrow{\times}$ Expanded

$\exists$ with $\wedge$:

Compact $\xrightarrow{?\ \checkmark}$ Expanded

$\xleftarrow{?\ \times}$

$\exists x (P(x) \wedge Q(x)) \longrightarrow \exists x P(x) \wedge \exists x Q(x)$ ✓

there is someone y for whom $P \wedge Q = $ True

There is someone → y for whom $P = $ True

AND

There is someone → y for whom $Q = $ True

logical thinking method

$$\underbrace{\exists x \overbrace{(P(x) \land Q(x))}^{\text{scope}}}_{\alpha} \longrightarrow \underbrace{(\exists x P(x)) \land (\exists x Q(x))}_{\beta}$$

Domain:

a

b $\Longrightarrow \underbrace{P(b)}_{=T} \land \underbrace{Q(b)}_{=T} = \text{True}$

c

⋮

Make $\alpha = \text{True}$

Can we make β false? $\Rightarrow$ No

$P(b) = T$ so

$\exists x P(x) = T$

$Q(b) = T$ so $\exists x Q(x) = T$

$\exists x(P(x) \wedge Q(x)) \longleftarrow \exists x P(x) \wedge \exists x Q(x)$

for someone $P \wedge Q = \text{True}$

Cannot guarrentee

for someone M $P = \text{True}$

and

for someone N $Q = \text{True}$

$\exists x (P(x) \wedge Q(x)) \longleftarrow \underbrace{\exists x P(x)}_{\text{witness} = 2} \wedge \underbrace{\exists x Q(x)}_{\text{witness} = 3}$

$\exists x (P(x) \wedge Q(x))$ → false

$P(x)$ = evenprime

$Q(x)$ = odd

Noone is evenprime AND odd

witness = 2

witness = 3

$$\underbrace{\exists x (P(x) \wedge Q(x))}_{\beta} \xleftarrow{\times} \underbrace{\exists x P(x) \wedge \exists x Q(x)}_{}$$

make α True

Can we make β = false

Domain:

a — $P = T$ ✓

b

c — $Q = T$

d

⋮

$a \Rightarrow Q(a) = F$

$b \Rightarrow P(b) = F$

$c \Rightarrow P(c) = F$

$d \Rightarrow P(d) = F$

β = false

$\alpha \longrightarrow \beta$

Domain: $\{a, b, c, d, - - -\}$

$\exists$ with $\forall$:

Compact ⟶ Expanded

Compact ⟵ Expanded

$\exists x(P(x) \vee Q(x)) \xrightarrow{\checkmark} \exists x P(x) \vee \exists x Q(x)$

Someone m

$P(m) \vee Q(m) = \text{True}$

$P(m)$ or $Q(m)$ is True.

Someone m $P = \text{True}$

OR

Someone m $Q = \text{True}$

$\underbrace{\exists x (P(x) \lor Q(x))}_{\beta} \leftarrow (\exists x P(x) \lor \exists x Q(x))$

β = True ← Someone M P = True

OR

β = True ← Someone N Q = True

$$\underbrace{\exists x \left(P(x) \oplus Q(x) \right)}_{\alpha} \;\not\longrightarrow\; \underbrace{\exists x P(x) \oplus \exists x Q(x)}_{\beta}$$

Domain:
a
b
c

make α = True

$\underline{P(b) \oplus Q(b)}$ = True, $P = T$, $Q = F$

Can we make $\underline{\beta \text{ false}}$?

$\exists x P(x) = \text{True}$ ($P(b) = T$)

$\exists x Q(x) = \text{True}$

$Q(a) = \text{True}$ ✓

$$\underbrace{\exists x\,(P(x) \oplus Q(x))}_{\beta} \leftarrow \underbrace{\exists x\,P(x) \oplus \exists x\,Q(x)}_{\alpha}$$

Can make β = false !

Domain: a, b, c, ...

$\boxed{P(a) = T,\ Q(a) = F}$

a → witness for β

make α = True

$\boxed{\exists x\,P(x) = \text{True}}$ $P(a) = T$

AND

$\boxed{\exists x\,Q(x) = \text{False}}$

→ for everyone Q = false

$\exists$ with $+$:

Compact $\xrightarrow{\times}$ Expanded

Compact $\xleftarrow{\checkmark}$ Expanded

$\rightarrow$ means Compact $\rightarrow$ Expanded

$\leftarrow$ means Compact $\leftarrow$ Expanded

28 Question

10 Done

HW: 18 Q.

	$\forall$	$\exists$
$\wedge$	$\leftrightarrow$ (proven)	$\rightarrow$ (proven)
$\vee$	$\leftarrow$ (proven)	$\leftrightarrow$ (proven)
$\rightarrow$	$\rightarrow$	$\leftarrow$
$\leftrightarrow$	$\rightarrow$	×
$\oplus$	×	$\leftarrow$ (proven)
$\uparrow$	$\rightarrow$	$\leftarrow$
$\downarrow$	$\rightarrow$	$\leftarrow$

Q:
msq

(A) $\forall(\wedge) \equiv \forall \wedge \forall$

(B) $\exists(\vee) \equiv \exists \vee \exists$

(C) $\forall(\rightarrow) \equiv \forall \rightarrow \forall$

(D) $\forall(\oplus) \equiv \forall \oplus \forall$

8 Questions

Note: Don't blindly by-heart this table.

Prove Complete table then Remember this table.

	$\forall$	$\exists$
$\wedge$	$\Leftrightarrow$	$\rightarrow$
$\vee$	$\leftarrow$	$\Leftrightarrow$
$\rightarrow$	$\Rightarrow$	$\leftarrow$
$\Leftrightarrow$	$\rightarrow$	$\times$
$\oplus$	$\times$	$\leftarrow$
$\uparrow$	$\rightarrow$	$\leftarrow$
$\downarrow$	$\rightarrow$	$\leftarrow$

Q: which is valid?

1. $[\forall x\, P(x) \rightarrow \forall x\, Q(x)] \rightarrow \forall x\, (P(x) \rightarrow Q(x))$

2. $\exists x\, (P(x) \uparrow Q(x)) \rightarrow \exists x\, P(x) \uparrow \exists x\, Q(x)$

3. $\exists x\, (P(x) \leftrightarrow Q(x)) \rightarrow \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$

Q: which is valid?

① $\left[\forall x\, P(x) \longrightarrow \forall x\, Q(x)\right] \underset{\times}{\overset{\checkmark}{\rightleftarrows}} \forall x\left(P(x) \rightarrow Q(x)\right)$

② $\exists x\left(P(x) \uparrow Q(x)\right) \underset{\times}{\overset{\checkmark}{\rightleftarrows}} \exists x\, P(x) \uparrow \exists x\, Q(x)$

③ $\exists x\left(P(x) \leftrightarrow Q(x)\right) \underset{\times}{\overset{\times}{\rightleftarrows}} \exists x\, P(x) \leftrightarrow \exists x\, Q(x)$

Q: Which is valid?

1. $[\forall x\, P(x) \oplus \forall x\, Q(x)] \rightarrow \forall x\,(P(x) \oplus Q(x))$

2. $\exists x\,(P(x) \downarrow Q(x)) \leftarrow \exists x\, P(x) \downarrow \exists x\, Q(x)$

3. $\forall x\,(P(x) \leftrightarrow Q(x)) \longrightarrow \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$

Q: which is valid?

~~1~~ $\left[\forall x\, P(x) \oplus \forall x\, Q(x)\right] \underset{\times}{\overset{\times}{\rightleftarrows}} \forall x \left(P(x) \oplus Q(x)\right)$

✓ 2 $\exists x \left(P(x) \downarrow Q(x)\right) \underset{\times}{\overset{✓}{\leftrightarrows}} \exists x\, P(x) \downarrow \exists x\, Q(x)$

✓ 3 $\forall x \left(P(x) \leftrightarrow Q(x)\right) \underset{\times}{\overset{✓}{\rightleftarrows}} \forall x\, P(x) \leftrightarrow \forall x\, Q(x)$

Q:

Invalid

$T \oplus T = F$

$$\underbrace{\forall x\,(P(x) \leftrightarrow Q(x))}_{\alpha} \xrightarrow{?} \underbrace{\exists x\,P(x) \oplus \forall x\,Q(x)}_{\beta}$$

make α = True

a — $P = Q$

b — $P = Q$

c — $P = Q$

Can we make β = false?

a — $Q = T$

b — $Q = T$

c — $Q = T$

$\Rightarrow$ $P(a) = T$, $P(b) = T$, $P(c) = T$

Q:

$$\forall x\,(P(x) \leftrightarrow Q(x)) \xleftarrow{?} \exists x\, P(x) \oplus \forall x\, Q(x)$$

Q:

$$\underbrace{\forall x\,(P(x) \leftrightarrow Q(x))}_{\beta} \longleftarrow \underbrace{\exists x P(x) \oplus \forall x Q(x)}_{\alpha}$$

Invalid

make β = false

$a \rightarrow Q(a) = F$

d : $P(d) \neq Q(d)$, $P = T$, $Q = F$

$b \longrightarrow Q(b) = F$

$c \rightarrow Q(c) = F$

Can we make α = True

$\exists x P(x)$ = True ($P(d) = T$)

$\forall x Q(x)$ = false $\Rightarrow$ for everyone Q = false

Which of the following semantic entailments are valid in predicate logic?

1. $\forall x\,(P(x) \vee Q(x)) \models \forall x\, P(x) \vee \forall x\, Q(x)$
2. $\forall x\,(P(x) \rightarrow Q(x)) \models \forall x\, P(x) \rightarrow \forall x\, Q(x)$
3. $\forall x\, P(x) \rightarrow \forall x\, Q(x) \models \forall x\,(P(x) \rightarrow Q(x))$
4. $\neg \forall x\,(P(x) \wedge Q(x)) \models \exists x\, \neg P(x) \wedge \exists x\, \neg Q(x)$
5. $\exists x\, P(x) \wedge \exists x\, Q(x) \models \exists x\,(P(x) \wedge Q(x))$