Logic: another thing that penguins aren't very good at.

Mathematical Logic

Next Chapter

First Order Logic, Predicate, Quantifiers, Domain

Website : https://www.goclasses.in/

First Order Logic/ (FOL)

Predicate Calculus/

Predicate Logic/

Quantificational Logic

Same

Propositional logic: $\Rightarrow$ world of T, F.

set of all propositional formulas.

Variable < T or F

Propositional Variable (Atomic proposition)

$P, Q, R, S, \_\_\_\_, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \uparrow, \oplus, \downarrow$

Logical connectives

Propositional formulas:

$P \rightarrow Q$, $P \wedge \overline{P}$, $P \vee \overline{P}$, _ _ _

P, Q, _ _

Several types of $\boxed{\text{logics}}$:

① Propositional logic
② FOL

in GATE Syllabus

③ 2^{nd} order logic
④ 3^{rd} " "
⋮
⑤ fuzzy logic ⑥ temporal logic

WHY we need another type of logic when we already have Propositional Logic??

N

G

"If you have access to the network, then you can change your grade."

"You have access to the network."

N

∴ "You can change your grade."

G

Using prop. logic ;

Argument form (logical form)

$$\begin{array}{l} N \to G \\ N \\ \hline \therefore G \\ \hline \end{array}$$

logically Valid Argument

modus ponens (Valid Arg)

Is the following argument Valid?

T — P

T — Q

$\therefore$ R — F

invalid Argument

③ $P, Q \not\models R$

② $P \wedge Q \rightarrow R$ is Not Tautology.

All men are mortal. Socrates is a man.

$\underbrace{\text{All men are mortal.}}_{P}$ $\underbrace{\text{Socrates is a man.}}_{S}$ } logically valid

Hence, $\underbrace{\text{Socrates is mortal.}}_{R}$

what prop. logic will do? ⇒ Argument form

logical form

$$\begin{array}{c} P \\ S \\ \hline \therefore R \\ \hline \end{array}$$

Invalid ⇒ so prop. logic can not express our Arg correctly.

All men are mortal. Socrates is a man.

Hence, Socrates is mortal.

logically valid

Argument

Unfortunately, there is really no way to express this in propositional logic. Propositional logic has very limited expressive power.

All men are mortal. Some aliens are men.

M A

Hence, some aliens are mortal. P

How prop. logic will treat it ?

Prop. logic has no feature to express them.

$$\begin{array}{l} M \\ A \\ \hline \therefore P \end{array}$$

Invalid

$M, A \not\models P$

Unfortunately, there is really no way to express this in propositional logic. Propositional logic has very limited expressive power.

All men are mortal. Some aliens are men.

Hence, some aliens are mortal.

logically Valid Arg

Unfortunately, there is really no way to express this in propositional logic. Propositional logic has very limited expressive power.

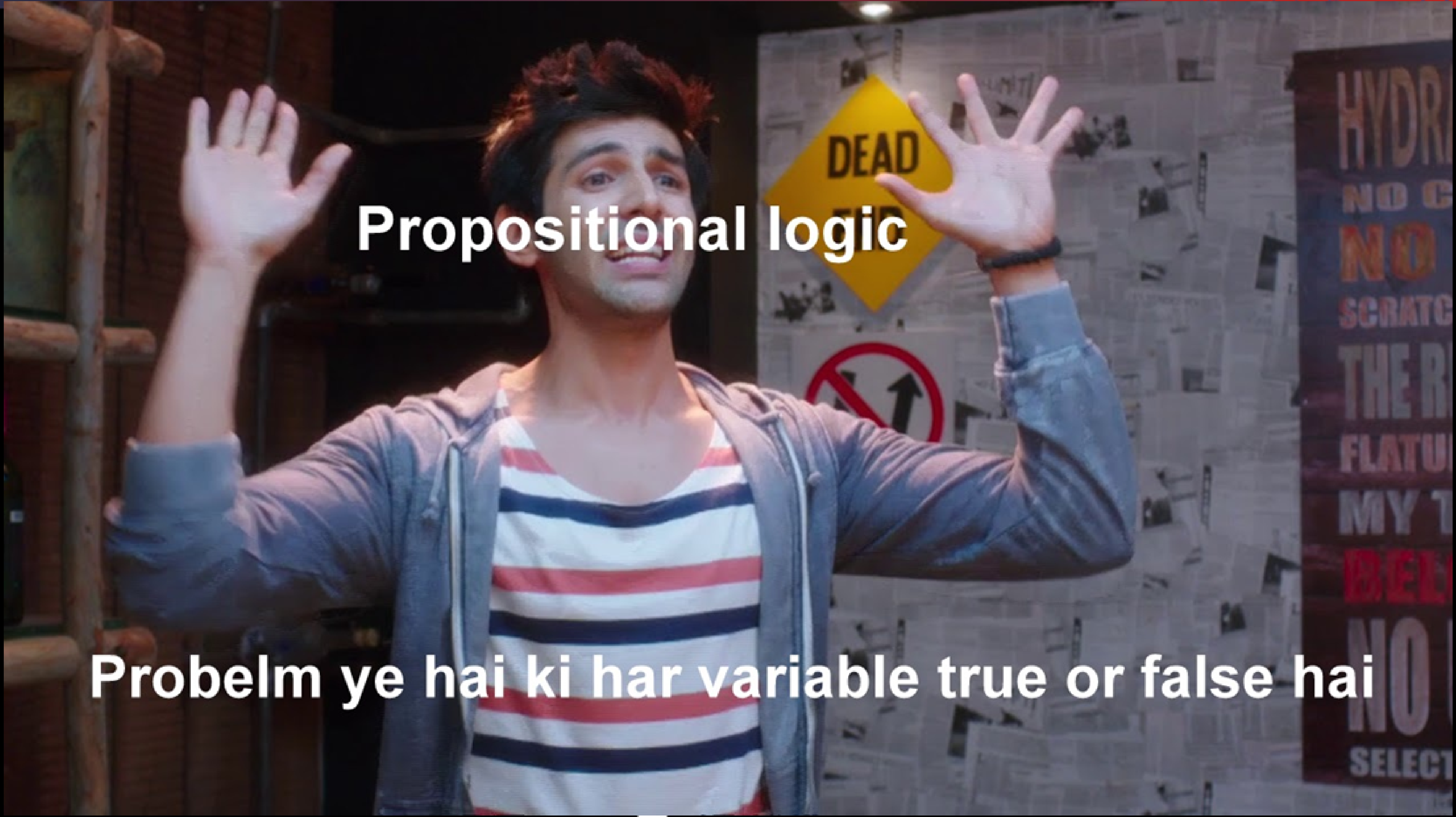
Toh Problem kya hai??
(So, What is the problem with propositional logic?)

→ deals with facts/propositions

It treats everything as True or false.

Variable — T or f

Variable — ~~Sun~~, ~~5~~, ~~chair~~, ~~Mary~~

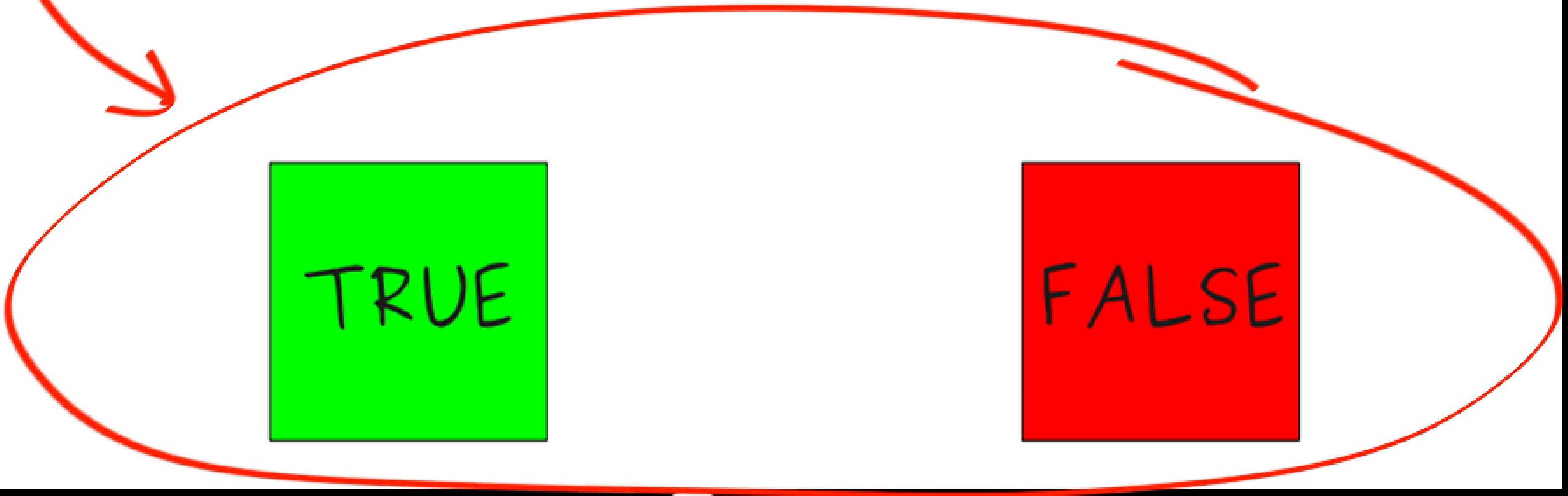

Propositional logic
Probelm ye hai ki har variable true or false hai

The Universe of Propositional Logic

p, q, r, s

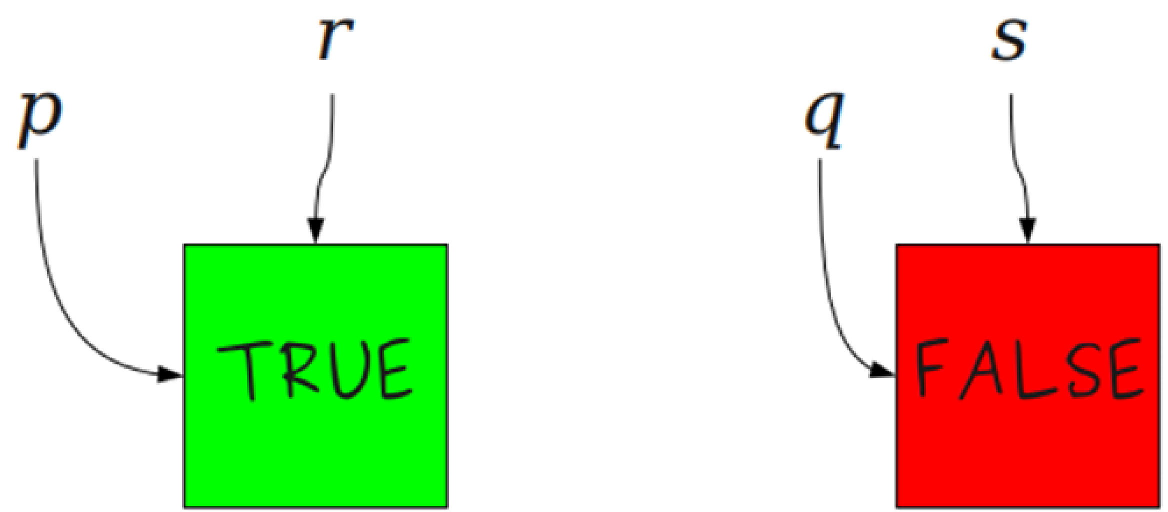
variable

$$p \land q \to \neg r \lor \neg s$$



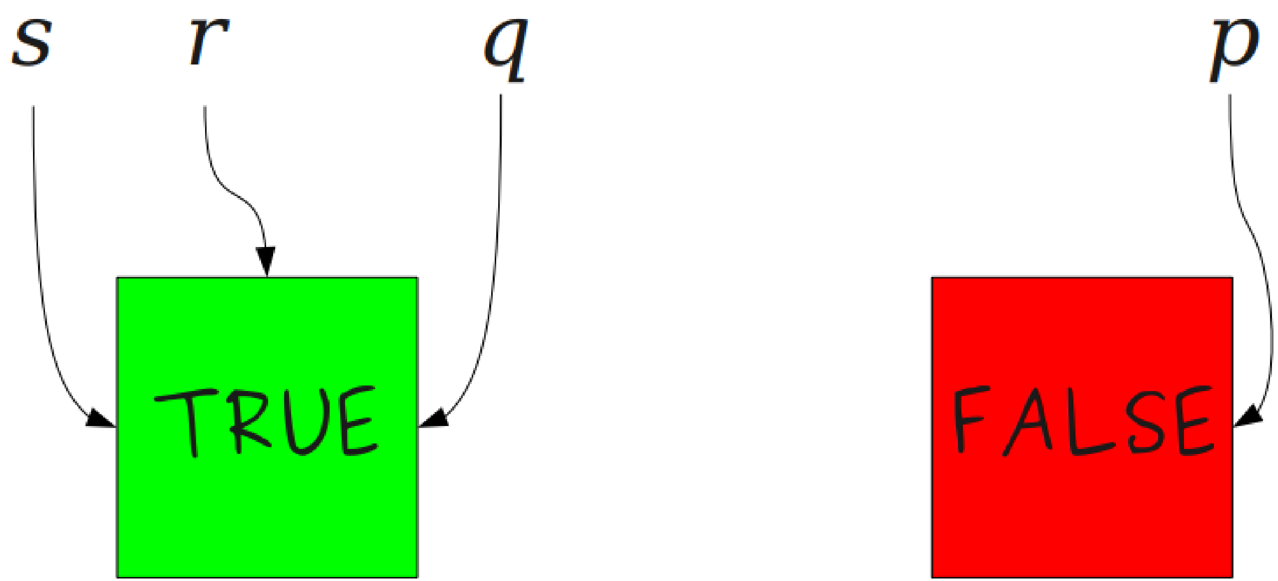
The Universe of Propositional Logic

$$\boxed{p \land q \rightarrow \neg r \lor \neg s} = \text{True}$$



The Universe of Propositional Logic

$$p \wedge q \rightarrow \neg r \vee \neg s \;=\; \text{True}$$



Propositional Logic

- In propositional logic, each variable represents a **proposition**, which is either true or false.
- Consequently, we can directly apply connectives to propositions:
  - $p \to q$
  - $\neg p \land q$
- The truth or falsity of a statement can be determined by plugging in the truth values for the input propositions and computing the result.
- We can see all possible truth values for a statement by checking all possible truth assignments to its variables.

A better logic for real world :

(1)

Price

Variable

2, _ , _ 2.5, _ _ _ _ _ 120.25, _ _ _

set of non-negative real numbers

set of values "price" variable can take.

Our variable should be able to take some specific set of values of our choice.

Domain / universe of that variable

Child_Count

Variable

0, 1, 2, ---

Set of whole numbers.

A better logic for real world :

(2) we want to talk about objects.

Raj, mary, deepak, - - -

movie, mobile, - - -

1, 2, 3, - - -

T, F, - - -

and their properties

2

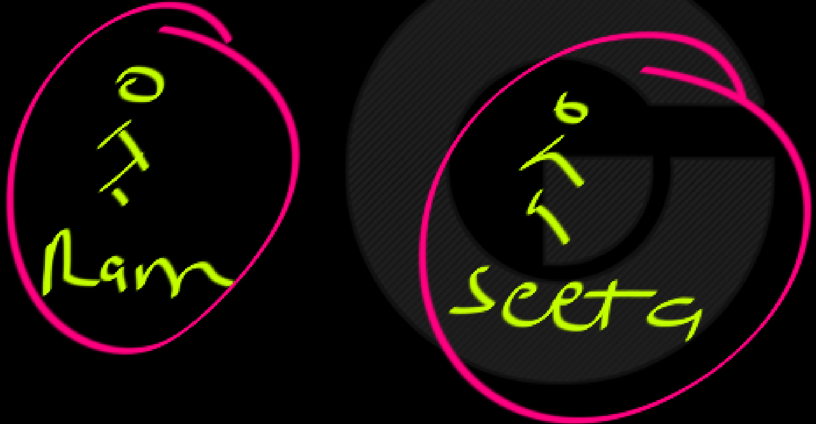
Object/Constant

properties

- is even
- is prime
- < 3
- > 0

A better logic for real world ;

3 Relationships b/w objects.



marriage(x,y) $\equiv$ x marries y

2, 1, 3, 4, ---

less_than(x,y) : $\boxed{x<y}$

sum(x, y, z) : $x + y = z$

Domain

-2, -1, 0, 1, 2

integers

object/constant

object/constant

sum(2, 3, 5) : True

sum(2, 3, 4) : false

A better logic for real world:

(4) Transformation / function!

function / Transformation

$f(x) = x^2$

$2 \xrightarrow{f} 4$

$f(2) = 4$; $f(3) = 9$

$3 \xrightarrow{f} 9$

father_of(x)

father_of(Ram) = Dasrath

$$\text{Ram} \xrightarrow{\text{father\_of}} \text{Dasrath}$$

Object

Transformation function

Object

A better logic for real world:

final feature: we want to be able to talk about multiple objects.

5 Quantification:

all, some, none

A better logic for real world :

FOL

Frege and Pierce noted the need for more than propositional logic, and they started developing First-Order Logic (or Predicate Calculus).

So, What does FOL have, that propositional logic doesn't ??

First Order Logic:

A World of Objects, their properties, their relationships, their transformation(function)

In FOL, we notice that the FOL world is blessed with objects, some of which are related to other objects, and in which we endeavor to reason about them.

So, What does FOL have, that propositional logic doesn't ??

1. First-Order Logic speaks about objects, which are the domain of discourse or the universe.

First-Order Logic

- In first-order logic, each variable refers to some object in a set called the **domain of discourse**.

$\mathbb{N}$

x — 1, 2, 3, - - -

Domain / universe of variable x

$x = 5$ ✓

$x = 100$ ✓

$x = \infty$ ✗ — not a number

$x = 0$ ✗

$x = -2$ ✗

vowel — a, e, i, o, u

y

z — T, F

p — T, F

x — 2

$x = 2$ ✓

$x = 3$ ✗

$x = 4$ ✗

$x =$ mobile ✗

$x =$ mary ✗

$x = 0$ ✗

① Every variable has a Domain.

Domain / Universe / Domain of Discourse / Universe of Discourse

② Different Variables can have Different Domains.

③ Every Variable can refer to Objects from its own Domain.

set of Possible Values that a variable can take

Object ≡ elements / values in the Domain.

≡ Constant

$x \rightarrow$ 1, 2, 3, 4 – – – – – – – .

objects / constant

So, What does FOL have, that propositional logic doesn't ??

2. First-Order Logic is also concerned about Properties of these objects.

x — 1, 2, 3, 4 ... $\mathbb{N}$

Even(x) : $\boxed{x \text{ is even}}$

Even(5) : false

Even(2) : True ; Even(3) : false

y

a, b, c, - -
— — —
— —
— . .

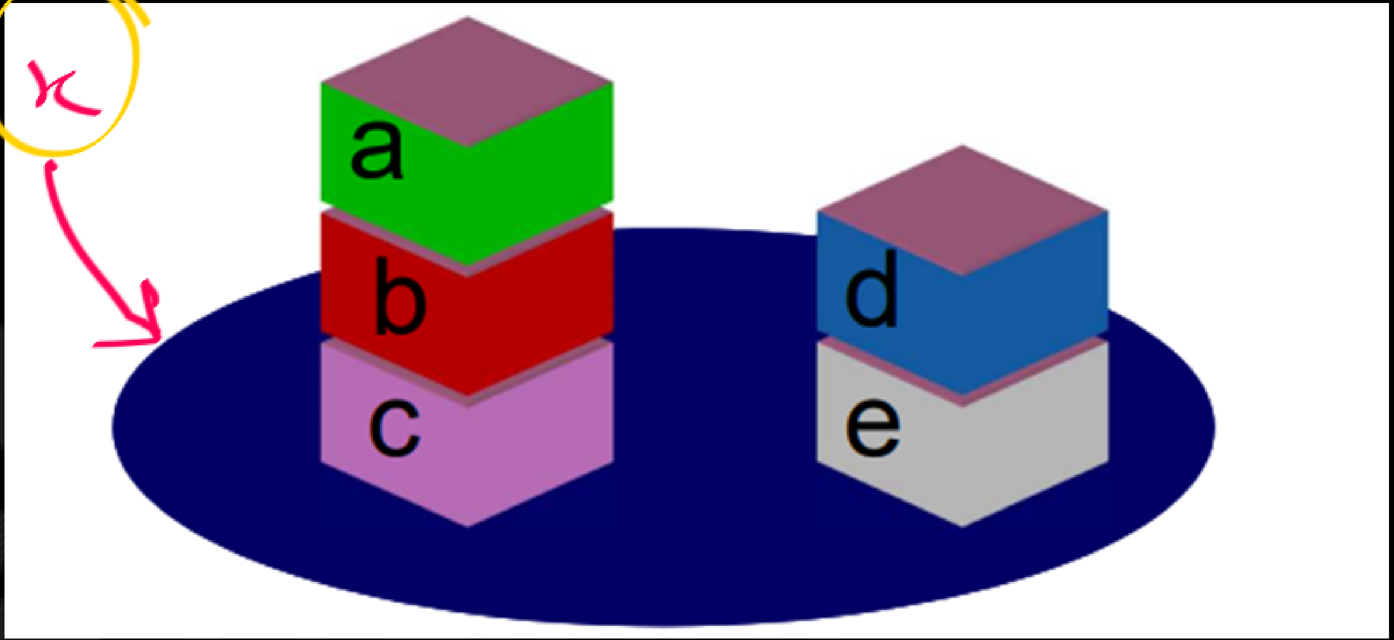
English Alphabet

Vowel(y) : y is vowel.

Vowel(a) : True

Vowel(b) = false

Vowel(U) : True



Green(x) : x is green

Green(a) : True

Green(b) = false

Green(g) = Invalid (because x can never take value g)

Highest(x) : x is Highest

Highest(d) : True

Two Different elements

1, 1 ✗

2, 3 ✓

2 ✗

2, 3, 1 ✗

Two elements

$\left(\begin{array}{l} 2,3,4 \times \\ 2 \times \end{array} \right.$

1, 1 ✓

2, 2 ✓

2, 3 ✓

1 ×

Same or Different

Two Same elements :

1, 1

1, 2

1

1, 1, 1

So, What does FOL have, that propositional logic doesn't ??

3. In FOL, we also have Relations over/between/among objects(called Predicates)

unary relations/predicate are called properties.

x, y — $\{1, 2, 3, \ldots\}$ N

$P(x, y)$: $\boxed{x < y}$

$P(\underset{x}{\cancel{\ }}, \underset{y}{\cancel{\ }})$

$P(1, 3)$: True

$P(3, 1)$: false

$P(1, 1)$: false

$P(0, 2)$: Invalid

x — 1, 2, 3

y — 2, 3, 5

z — 2, 3, 4, 5, 6, 7, 8

sum(x, y, z) : $x + y = z$

sum(2, 3, 6) : false

sum(4, 2, 6) : Invalid

sum(3, 3, 6) : True

Relation / Predicate :

$P(x, y)$: $x < y$; Predicate over 2 variable

$Sum(x, y, z)$; Predicate over 3 variables (relationships among 3 objects)

even(x) ; x is even ;
green(x) ; x is green. → Unary Predicate → called Property

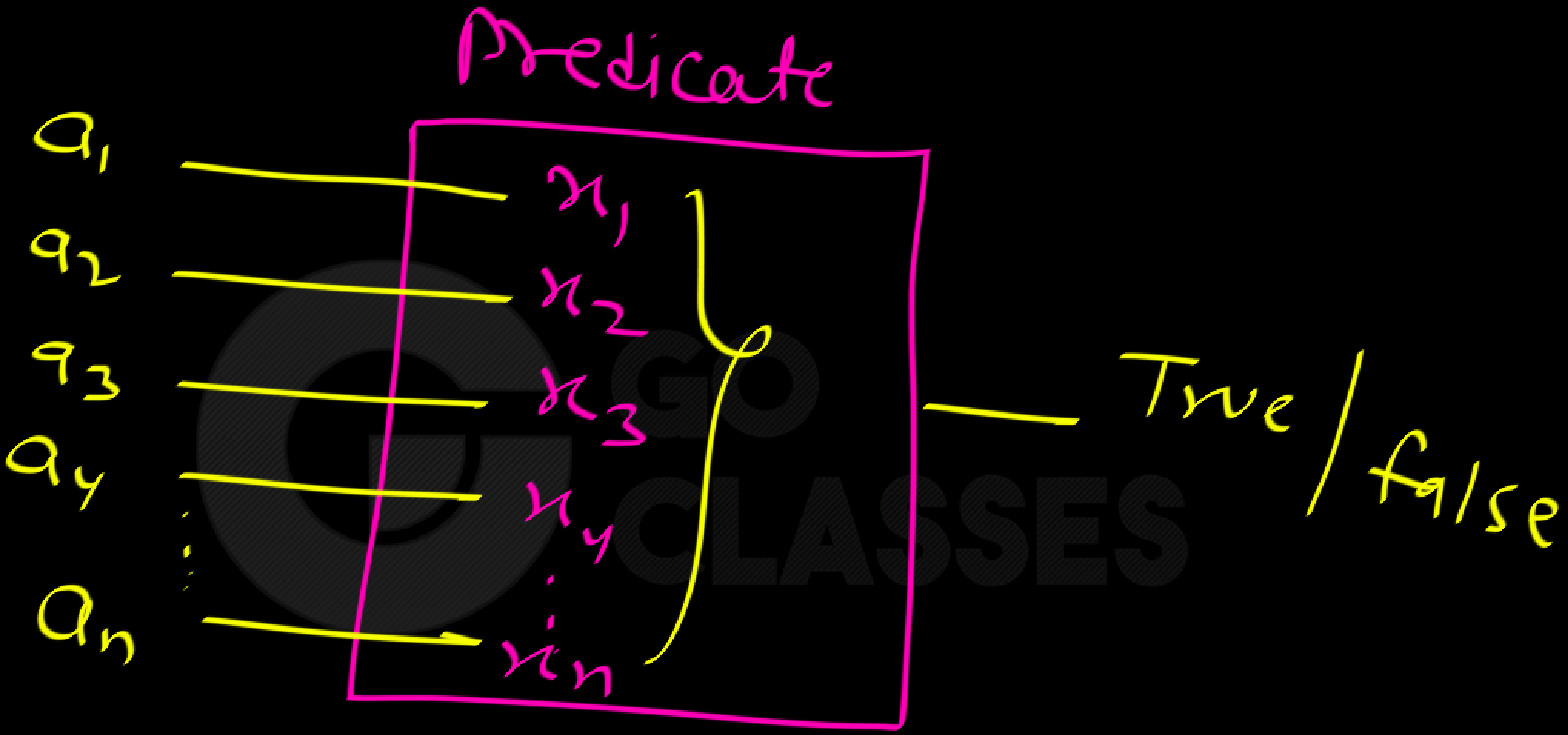
Predicate :

D_1 D_2 D_3 D_n

$P(x_1, x_2, x_3, ----, x_n)$ n-ary Predicate

Predicate over n variables

Once we replace every variable with some value from their domain, then it becomes proposition (True, False)



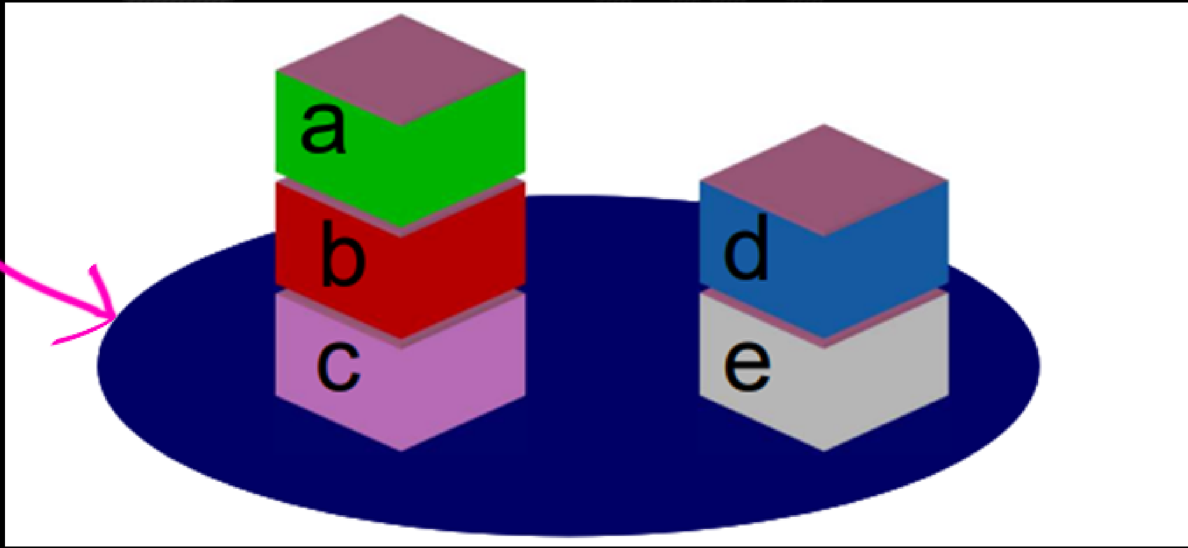
n-ary Predicate

Relations (Predicates)

- Unary relation, binary relation, …

- Example: set of blocks {a, b, c, d, e}

x, y, z



Unary predicate :

$\boxed{\text{red}(x) : x \text{ is red.}}$ predicate over one variable

red(a) : $\boxed{a \text{ is red}}$ ≡ false

proposition

binary predicate : (Relation b/w two objects)

→ predicate over two variables

On(x,y) : x is on y

On(a,b) : a is on b : True

On (c, b) : c is on b ≡ false

On (a, d) : a is on d = false

On (d, a) : d " " a = "

Predicate with no variable

0-ary predicate = proposition

P : a is green : True

proposition

predicate over 0 variables

P : $\boxed{\text{b is green}}$: false

0-ary predicate

In English,

Quantification words:

few, all, many, some, any ---

In logic,

All, Some

for all

at least one

So, What does FOL have, that propositional logic doesn't ??

5. Another significant new concept in first-order logic is quantification:

the ability to assert that a certain property holds for all elements or that it holds for some element.

all elements in Domain of x are Young,

$x \rightarrow \{a, b, c, d\}$ $D(x)$

$\forall x \in D(x)$

for all

$Young(x) \equiv \forall x \; Young(x)$

$\forall \equiv$ for all

for all x, P(x) is True.

hidden (implicit) (in the Domain of x)

$$\boxed{\forall_x P(x)}$$

x

$\{a, b, c, d\}$

$\forall_x P(x)$

$\equiv P(a) \wedge P(b) \wedge P(c) \wedge P(d)$

$P(x)$: $\boxed{x \text{ is even}}$

Predicate

$D(x)$: $\mathbb{N}$

$\forall x\, P(x)$: Every natural is even

false

false

$D(x)$: $\{2, 4, 10\}$

$\forall x\, P(x)$: True

for all elements in Domain of x, $P(x)$ is True'

$P(x)$: x is even

$D(x)$: $\mathbb{N}$

$\boxed{\forall_x P(x)}$ = false

Counter Example:

19

$D(x)$: $\{2, 4, 6, 8, 11\}$

$\boxed{\forall_x P(x)}$ = false

Counter Example:

11

Counter Example:

Someone is against you.

Someone who makes a statement false.

at least one student is male.

Some " " " .

same

$\exists x$ Male(x)

There exist

set of students

x

$\exists x \; P(x)$

there exist x, in Domain of x, $P(x)$ is True.

there is at least one

there is some

Witness : Someone who can vouch for you.

Someone who supports.

Someone who makes a statement true.

mad max fury road

"Witness me"

$P(x)$: x is prime.

$D(x) = \mathbb{N}$

$\exists x P(x)$

True

witness $= 5$

$D(x)$: $\{3, 4, 5\}$

$\exists x P(x)$

$=$ True

witness: 3

$D(x)$: $\{2, 3, 4\}$

$\exists x P(x)$: Yes

witness $= 2$

$P(x)$: x is prime

witness $= 6$

$D(x)$: $\{4, 6, 9, 21\}$

$\exists x\, P(x) = \text{false}$

$\exists x \boxed{\neg P(x)} \equiv \text{True}$

at least one element in Domain of x, $P(x)$ is false

$\exists x \, \neg P(x)$

$\exists x$, $\boxed{\neg P(x) \text{ is True}}$

$\exists x$, $P(x)$ is false

So, What does FOL have, that propositional logic doesn't ??

1. First-Order Logic speaks about objects, which are the domain of discourse or the universe.
2. First-Order Logic is also concerned about Properties of these objects
3. Also we have Relations over/between/among objects(called Predicates)

Predicate

4. In FOL, we also have Functions of objects
5. Another significant new concept in first-order logic is quantification: the ability to assert that a certain property holds for all elements or that it holds for some element.

Prop. Logic vs. FOL

- Propositional logic assumes that there are facts that either hold or do not hold
- FOL assumes more: the world consists of objects with certain relations among them that do or do not hold

First-Order Logic (FOL)

- Whereas propositional logic assumes the world contains facts

- First-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus, …

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

First-Order Logic (FOL or FOPC) Syntax

Domain / universe

- User defines these primitives:
  - **Constant symbols** (i.e., the "individuals" in the world) E.g., Mary, 3
  - **Function symbols** (mapping individuals to individuals) E.g., father-of(Mary) = John, color-of(Sky) = Blue
  - **Predicate symbols** (mapping from individuals to truth values) E.g., greater(5,3), green(Grass), color(Grass, Green)

$$\text{Sky} \longrightarrow \boxed{\text{color-of}} \longrightarrow \text{blue}$$

Q :

For Domain/Universe as set of natural numbers, find:

Objects/Constant Symbols

Some properties of them

Some relations/predicates among them

Some functions on them

Q :

For Domain/Universe as set of natural numbers, find:

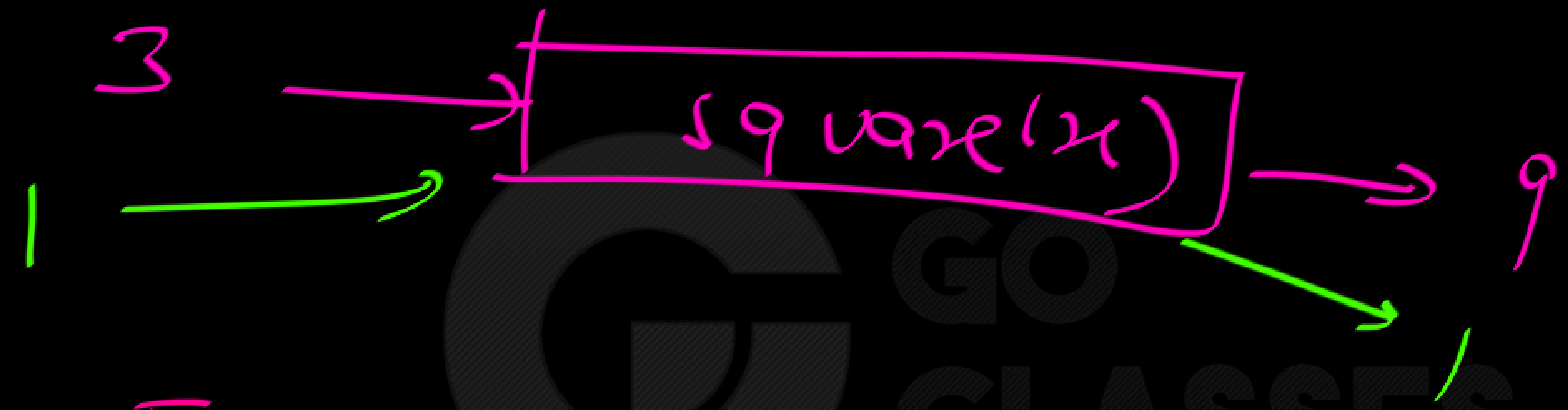
Objects/Constant Symbols → 1, 2, 3, 4, 5, - - - - -

Some properties of them → Even (x) ; prime(x) ;

Some relations/predicates among them → less_than(x, y) ; x < y

Some functions on them → add(x, y) ; square(x)

Property ≡ Predicate over one variable



Even(x) : x is even

Even(2) : True ; Even(3) : false

- The domain of interest is the natural numbers, $\mathbb{N}$.
- There are objects, $0, 1, 2, 3, \ldots$.
- There are functions, addition and multiplication, as well as the square function, on this domain.
- There are predicates on this domain, "even," "odd," and "prime."
- There are relations between elements of this domain, "equal," "less than", and "divides."

For our logical language, we will choose symbols 1, 2, 3, *add*, *mul*, *square*, *even*, *odd*, *prime*, *lt*, and so on, to denote these things. We will also have variables x, y, and z ranging over the natural numbers. Note all of the following.

- Functions can take different numbers of arguments: if x and y are natural numbers, it makes sense to write $mul(x, y)$ and $square(x)$. So *mul* takes two arguments, and *square* takes only one.
- Predicates and relations can also be understood in these terms. The predicates $even(x)$ and $prime(x)$ take one argument, while the binary relations $divides(x, y)$ and $lt(x, y)$ take two arguments.
- Functions are different from predicates! A function takes one or more arguments, and returns a *value*. A predicate takes one or more arguments, and is either true or false. We can think of predicates as returning propositions, rather than values.

FOL = Proposition logic

+

Predicate, Quantifiers, function symbols, Domain for Variables

Tomorrow:

All About Quantifier

$\forall$, $\exists$

Universal Quantifier

Existential Quantifier