Set Theory : Lattices

Some Questions related to

Maximal, Minimal, Greatest, Least Elements

in POSET.

Website : https://www.goclasses.in/

Special elements in a poset $(A, \preceq)$ (book notation) are: maximal, minimal, greatest (largest) and smallest (least) and are defined below.

Smallest (least) $a_0 \in A$ is a smallest (least) element in the poset $(A, \preceq)$ iff $\forall a \in A\ (a_0 \preceq a)$.

Greatest (largest) $a_0 \in A$ is a greatest (largest) element in the poset $(A, \preceq)$ iff $\forall a \in A\ (a \preceq a_0)$.

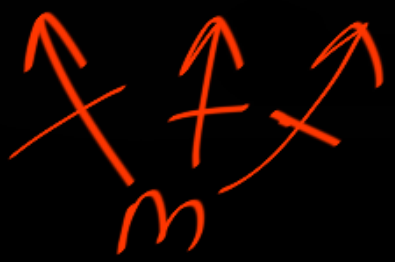
Maximal (formal) $a_0 \in A$ is a maximal element in the poset $(A, \preceq)$ iff $\neg\ \exists a \in A\ (a_0 \preceq a\ \cap\ a_0 \neq a)$.

Maximal (informal) $a_0 \in A$ is a maximal element in the poset $(A, \preceq)$ iff on the diagram of $(A, \preceq)$ there is no element placed above a_0.

Minimal $a_0 \in A$ is a minimal element in the poset $(A, \preceq)$ iff $\neg\ \exists a \in A\ (a \preceq a_0\ \cap\ a_0 \neq a)$.

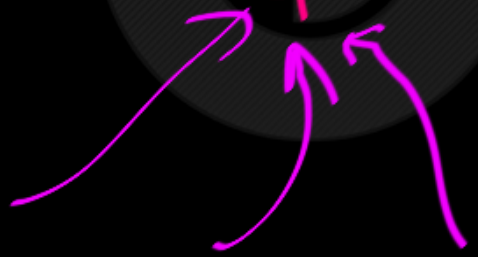
Minimal (informal) $a_0 \in A$ is a minimal element in the poset $(A, \preceq)$ iff on the diagram of $(A, \preceq)$ there is no element placed below a_0.

maximal

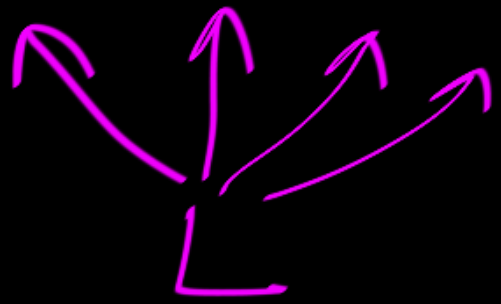


minimal

Greatest / Largest / maximum



least / Smallest / minimum



Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

(A, R) poset

Q 1:

Every maximal element is greatest.
Every greatest element is maximal.

Every minimal element is least.
Every least element is minimal.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

(A, R) poset

Q 1:

Every maximal element is greatest. ✗

Every greatest element is maximal. ✓

Every minimal element is least. ✗

Every least element is minimal. ✓

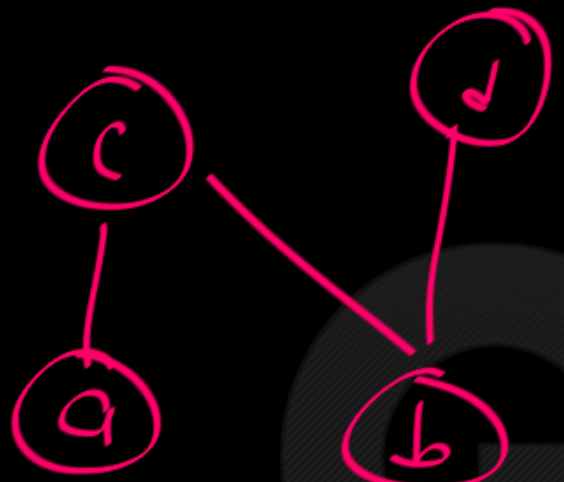
You are *maximal* when there is nobody above you.

You are *greatest* when you are above everyone else.

Examples:

- If nobody has eaten you, it doesn't follow that you have eaten everyone else.
- If nobody is standing on your head, it doesn't follow that you are standing on everyone else's head.
- If you live on the top floor of your apartment building, it doesn't follow that you live above everyone else in the city.

So maximal elements need not be greatest.



maximal : c, d

minimal : a, b

Greatest: DNE

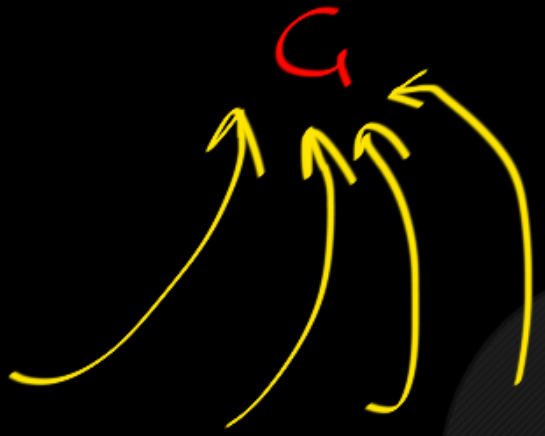
Least: DNE

$c \not R a$

$d \not R b$

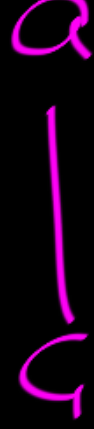
$a \not R b$

$b \not R a$



Assume G is Greatest and

GRa



Since G is Greatest,

So, aRG

by assumption, GRa

Because Poset is Antisym,

So, $G = a$

If G is Greatest then

$G \not\preceq a$, for all $a \neq G$

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 2:

(A,R) has at least one minimal element and at least one maximal element.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 2:

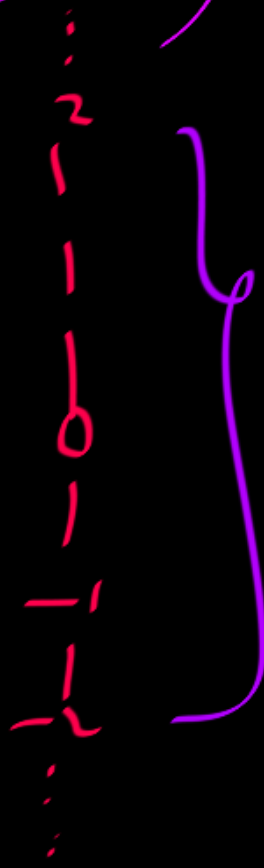
(A,R) has at least one minimal element and at least one maximal element.

false.

Eg: $(N, \leq)$ → minimal = 1

→ maximal = DNE

$(Z, \leq)$ — minimal – DNE
maximal – DNE



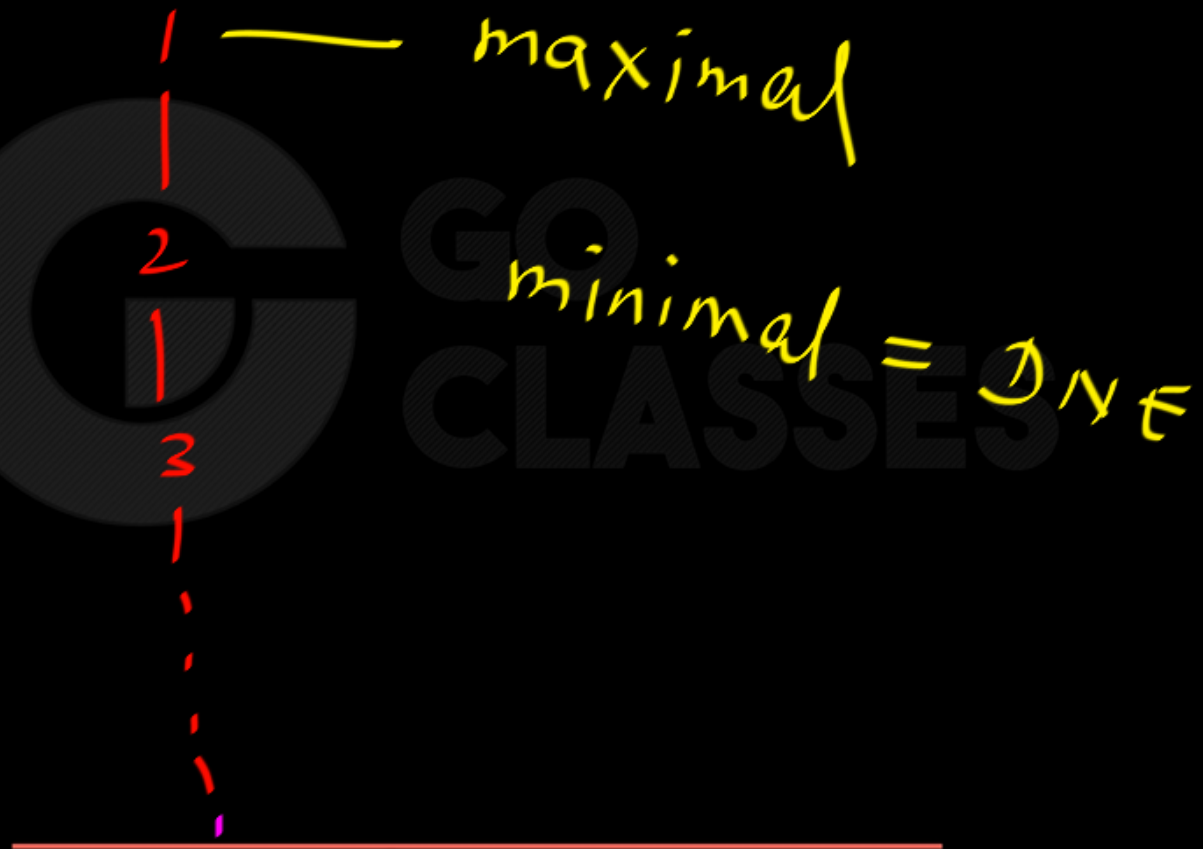
$n \; R \; n+1, \forall n \in Z$

$n-1 \; R \; n, \forall n \in Z$

$9 \; R_{10} \; ; \; 2 \; R_{10}$

$(\mathbb{N}, \geq)$

$2 R 1 \Leftrightarrow 2 \geq 1$



Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 3:

(A,R) has at least one maximum element and at least one minimum element.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 3:

(A,R) has at least one maximum element and at least one minimum element. — false

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 4:

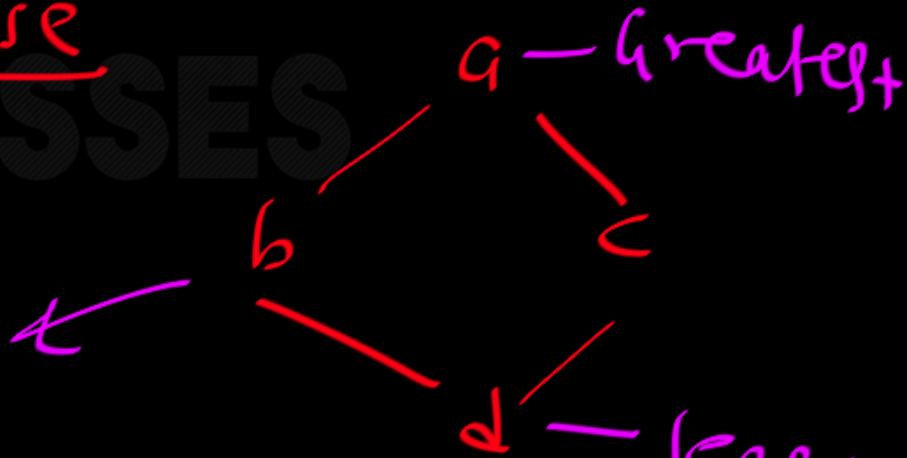
If (A,R) has a smallest(least) and largest(greatest) elements, then every two element of A are comparable.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 4:

If (A,R) has a smallest(least) and largest(greatest) elements, then every two element of A are comparable.

— false



Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 5:

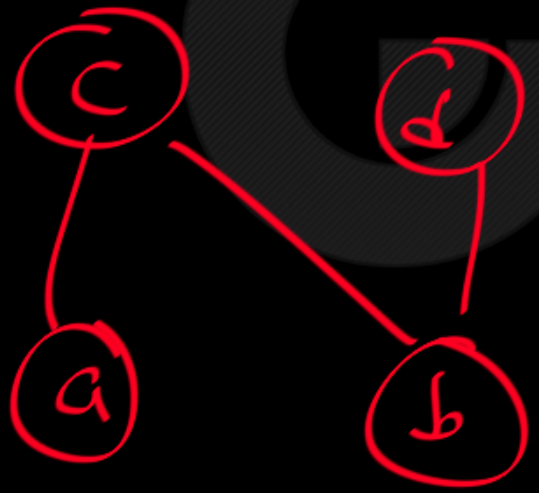
If (A,R) has no maximum elements, then A is infinite.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 5:

If (A,R) has no maximum elements, then A is infinite. — false

greatest



maximum : DNE

minimum : DNE

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 6:

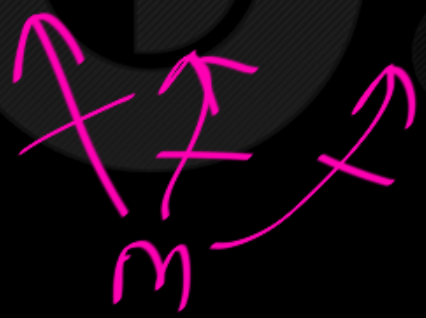
If (A,R) has no maximal elements, then A is infinite.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 6:

If (A,R) has no maximal elements, then A is infinite. — True

maximal:



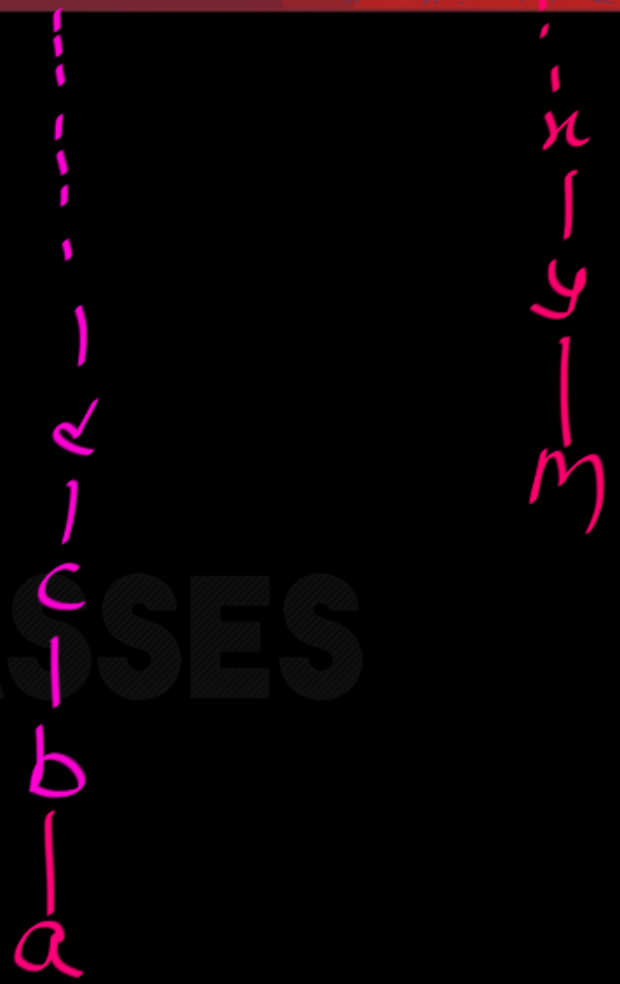
If "a" is not maximal :

means: $a \, R \, b$, for some $b \neq a$

No maximal element:

↓

Poset is infinite.



If Poset (A, R) has no minimal element, then Poset is infinite.

"a" is not minimal : for some $b \neq a$

$b R a$

no minimal element :

↓

Poset infinite

no maximal element ⟶ Poset is infinite

Q: Is Converse True?

Infinite Poset ⟶ no maximal

false

Eg: $(\mathbb{N}, \geq)$ — maximal = 1

$2 R 1 \iff 2 \geq 1$

no maximal element $\longrightarrow$ Poset is infinite

Contrapositive:

Poset is finite $\longrightarrow$ at least one maximal element.

no minimal element $\longrightarrow$ Poset is infinite

Contrapositive:

Poset is finite $\longrightarrow$ at least one minimal element.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 7:

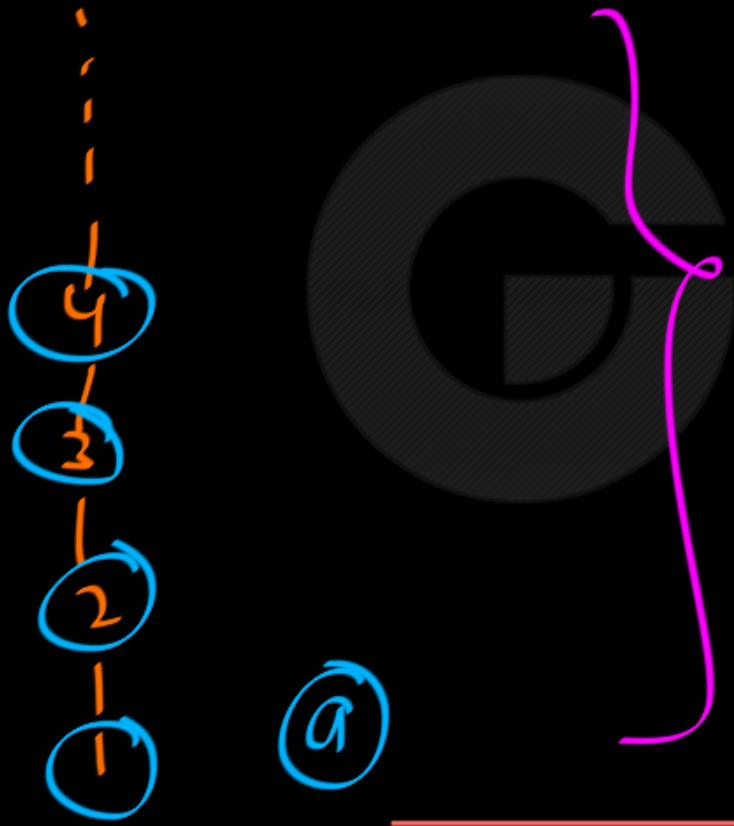
If (A,R) has a single(unique) maximal element, then it is the greatest element.

Q: R is a partial order relation on some set A (which can be either finite or infinite). Which of the following statements are true?

Q 7:

If (A,R) has a single(unique) maximal element, then it is the greatest element. — false

Eg: $(N, \leq) \cup \{(a,a)\}$ — Poset



Maximal : a — Unique

Greatest : DNE

$(N, \leq) \cup \{(9,9)\}$ —— maximal = a
Greatest = DNE

$(N, \geq) \cup \{(9,9)\}$ —— minimal = a
Least = DNE

In Poset (infinite Poset)

Unique maximal $\not\longrightarrow$ Greatest

" minimal $\not\longrightarrow$ Least

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 8:

Let base set A be finite.
If (A,R) has a single(unique) maximal element, then it is the greatest element.

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 8:

Let base set A be finite.
If (A,R) has a single(unique) maximal element, then it is the greatest element. — True

finite Poset :

Unique maximal ⟶ Greatest

" minimal ⟶ least

finite Poset:

Assume "m" is Unique maximal.

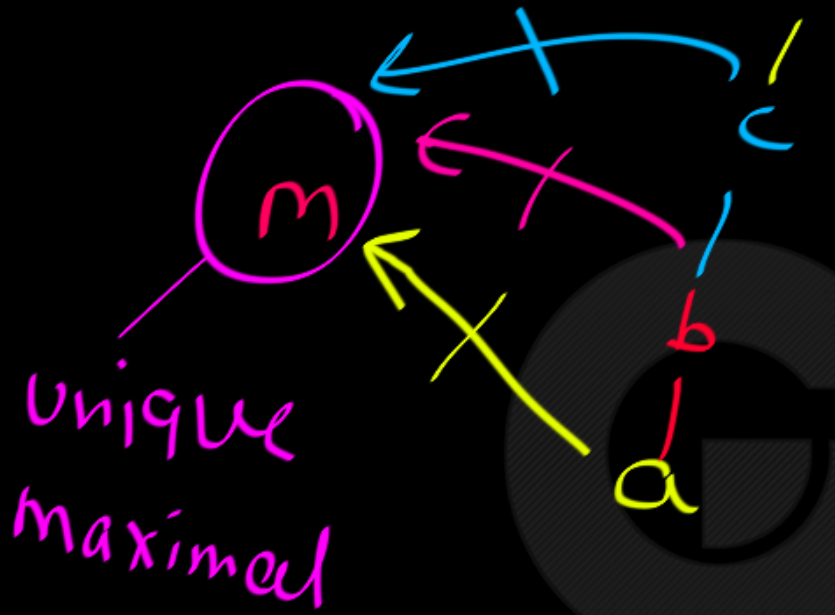
Then m will be Greatest because every element will be Relates to m. Why ??

In finite Poset :

Assume : m — unique maximal

But for Contradiction, assume m is not Greatest.

means: for some $a \neq m$, $a \not\leq m$.



unique maximal

$\Rightarrow a \mathcal{R} m$

$\Rightarrow a =$ not maximal

$\Rightarrow a R b \;,\; a \neq b$

$\Rightarrow b \mathcal{R} m$

$b =$ not maximal

$\Rightarrow b R c \;,\; b \neq c$

and so on ---

So, poset is not finite.

Contradiction

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 9:

If every two elements are comparable, then there is a smallest and largest elements.

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 9:

If every two elements are comparable, then there is a smallest and largest elements.

false

Total order / Chain

Eg: $(N, \leq)$ — Total order

— Maximal = DNE

— Greatest = DNE

Eg: $(Z, \leq)$ — Minimal = DNE

Maximal = "

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 10:

Let base set A be finite.
If every two elements are comparable, then there is a smallest and largest elements.

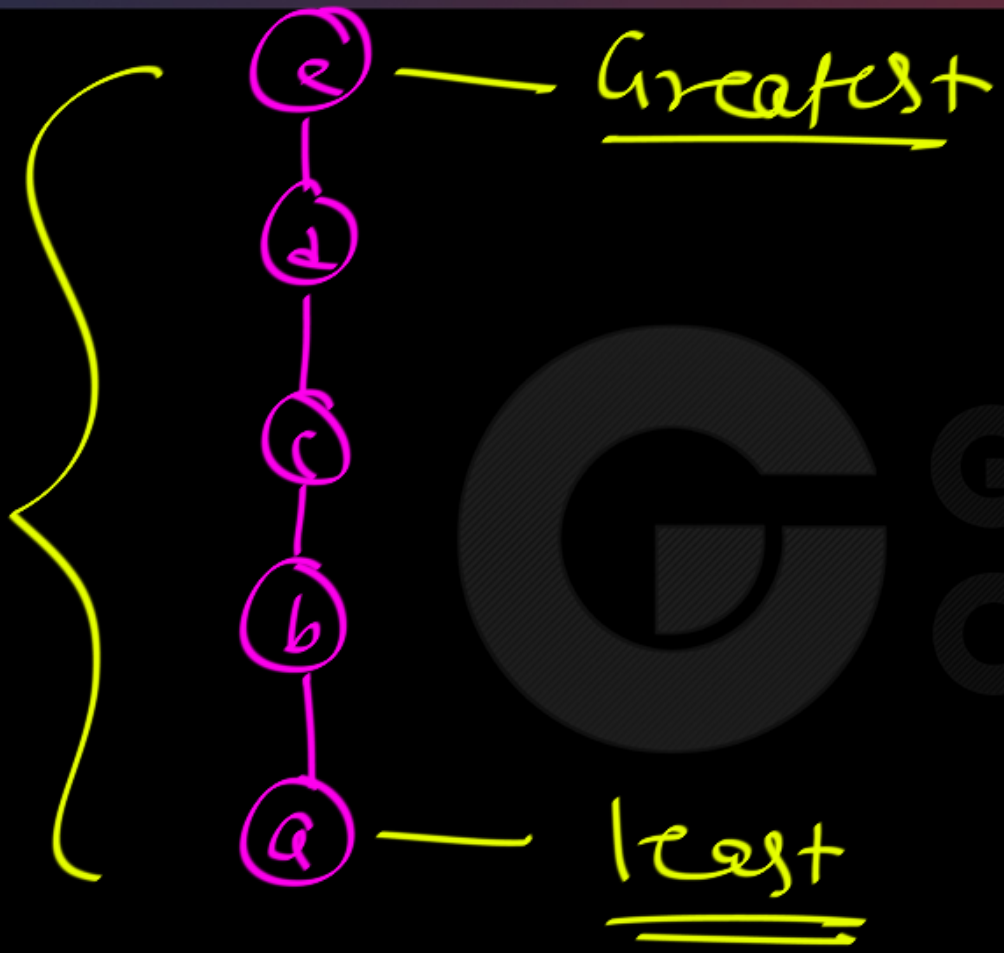
Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 10:

finite Total order / Chain

Let base set A be finite.
If every two elements are comparable, then there is a smallest and largest elements.

True.



Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 11:
Unique maximal element is the greatest element. false

i.e. If M is unique maximal element then it is the greatest element.

false

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 12:
Unique minimal element is the least element. false

i.e. If m is unique minimal element then it is the least element.

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 13:
In a Finite Poset, Unique maximal element is the greatest element.

i.e. In a finite poset, If M is unique maximal element then it is the greatest element.

Q: R is a partial order relation on some set A.
Which of the following statements are true?

Q 14:

In a Finite Poset, Unique minimal element is the least element.

i.e. In a finite poset, If m is unique minimal element then it is the least element.

Conclusions:

Let R be a partial order relation on some set A. (A, R) Poset

1. Every greatest element is maximal. Not vice verse.
2. Every least element is minimal. Not vice verse.

3. Every finite poset has at least one maximal, at least one minimal element.
4. A poset (finite or infinite) Need Not have greatest, least elements.

Conclusions:

Let R be a partial order relation on some set A.

5. In a poset, unique maximal element doesn't imply greatest element.
i.e. In a poset, if "M" is unique maximal element then "M" may not be the greatest element.

6. In a poset, unique minimal element doesn't imply least element.
i.e. In a poset, if "m" is unique minimal element then "m" may not be the least element.

Conclusions:

Let R be a partial order relation on some set A.

7. In a finite poset, unique maximal element implies greatest element.
i.e. In a finite poset, if "M" is unique maximal element then "M" is the greatest element.

8. In a finite poset, unique minimal element implies least element.
i.e. In a finite poset, if "m" is unique minimal element then "m" is the least element.

Finite Poset :-

Non-Empty finite Poset.

Note: In Relations, Base set is, by default, non-empty.