Set Theory

Next Topic

Equivalence Relation

RST Properties

Website : https://www.goclasses.in/

Equivalence Relations

The properties of relations are sometimes grouped together and given special names. A particularly useful example is the equivalence relation.

Definitions

A relation that is reflexive, symmetric, and transitive on a set S is called an **equivalence relation** on S.

Equivalence Relations

- Some relations are reflexive, symmetric, and transitive:
  - $x = y$
  - $u \leftrightarrow v$
  - $x \equiv_k y$
- Definition: An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

R : Not Ref , $2 \not R 2$

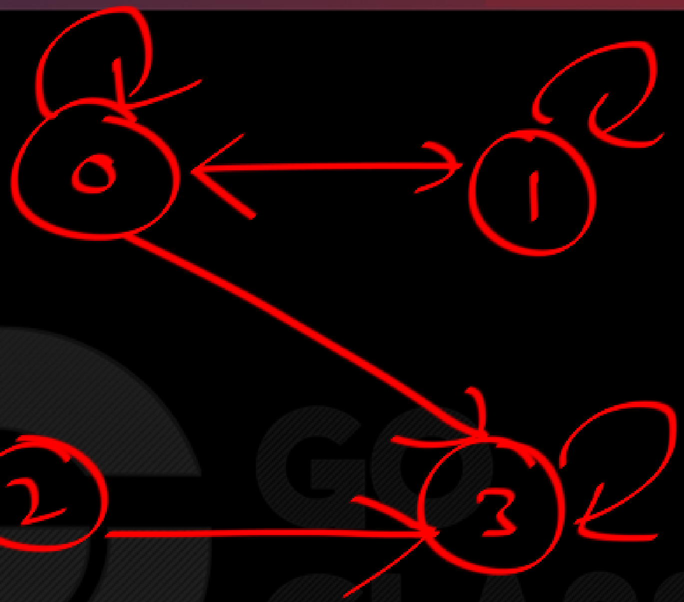
R : Not Sym ; $0 R 3$ but $3 \not R 0$

R : Not Antisym : $0 R 1$ AND $1 R 0$

R : " Asym : because not Antisym ; $0 R 0$

Not Eq. Rel

Transitive:



Not Trans.

$1 \to 0 \to 3$

missing

R not irref

$0 R 0$

Exercise 1. Let $A = \{0, 1, 2, 3\}$ and R a relation over A:

$$R = \{(0,0), (0,1), (0,3), (1,1), (1,0), (2,3), (3,3)\}$$

Draw the directed graph of R. Check whether R is an equivalence relation. Give a counterexample in each case in which the relation does not satisfy one of the properties of being an equivalence relation.

Solution.
R is not reflexive because $(2,2) \notin R$. It is not symmetric because $(3,2) \notin R$. It is not transitive because $(1,0)$ and $(0,3)$ are in R but $(1,3) \notin R$.

xRy iff $\underbrace{x - y = 3n, \text{ for some } n \in \mathbb{Z}}_{\text{means}}$

xRy iff $3 \mid (x - y)$

xRy iff $x \equiv y \pmod 3$

xRy iff x, y give same remainder when Divided by 3.

Exercise 17. Let $A = \{2, 3, 4, 5, 6, 7, 8\}$ and R a relation over A. Draw the directed graph of R, after realizing that xRy iff $x - y = 3n$ for some $n \in \mathbb{Z}$. Check that R is an equivalence relation.

Base set: $A = \{2, 3, 4, 5, 6, 7, 8\}$

R :

Ref: ✓ $a \equiv a \pmod 3$ ✓

$3 \mid (a-a)$ ✓

✓
Sym: $a \equiv b \pmod 3 \longrightarrow b \equiv a \pmod 3$

Antisym: ✗ $3R6,\ 6R3$

Asym: ✗ $\quad 3 R 3$

Trans: $\boxed{a R b \;,\; b R c}$

$a \bmod 3 = r$, $b \bmod 3 = r$, $c \bmod 3 = r$

$a R c$

Base set: $S = \{2, 3, 4, 5, 6, 7, 8\}$

R on S ; xRy iff same rem when Div by 3.

① R is ER — RST

$2R5$, $5R2$

$2R8$, $8R2$

$2R2$

$[2]_R$ Class 2

2, 5, 8

$[3]_R$

3, 6

$[4]_R$

4, 7

$[5]_R$ = 2, 5, 8

$[6]_R$ = 6, 3

$[7]_R$ = 7, 4

$$[a]_R = \{ b \mid aRb \}$$

Eq. Relation

ER → sym

$$[x]_R = \{ y \mid xRy \} = \{ y \mid yRx \}$$

Eq. Relation

Base set S

↓

R on S is ER then R will partition the Base set.

Each part is called Equivalence class

$S = \{2, 3, 4, 5, 6, 7, 8\}$

R : xRy iff $x \equiv y \pmod 3$

Eq Rel

Rem 2   rem 0   rem 1

$\{2, 5, 8\}, \{3, 6\}, \{4, 7\}$

Exercise 17. Let $A = \{2,3,4,5,6,7,8\}$ and R a relation over A. Draw the directed graph of R, after realizing that xRy iff $x - y = 3n$ for some $n \in \mathbb{Z}$. Check that R is an equivalence relation.

Solution. The relation is:

$$\begin{aligned} R = \{&(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),\\ &(8,5),(8,2),(7,4),(6,3),(5,2)\\ &(5,8),(2,8),(4,7),(3,6),(2,5)\}. \end{aligned}$$

Question 3 (20%)

Determine whether the binary relation is:

- Reflexive,
- Symmetric,
- Transitive.

Support your answer with reason or proper example.

a) The relation R on $\{w, x, y, z\}$ where

$$R = \{(w,w),(w,x),(x,w),(x,x),(x,z),(y,y),(z,y),(z,z)\}$$

b) The relation R on $\mathbb{Z}$ where aRb means $a^2 = b^2$.

c) The relation R on $\mathbb{Z}$ where aRb means $a \neq b$. — RX, TX, S✓

d) The relation R on the set of all people where aRb means that a is at least as tall as b.

e) The relation R on the set $\{(a,b) \mid a, b \in \mathbb{Z}\}$ where $(a,b)R(c,d)$ means $a = c$ or $b = d$.

(20 marks)

(9) Refl ✓

Sym ✗ xRz, $z \cancel{R} x$

Antisym ✗ xRw, wRx

Trans: ✗ wRx, xRz but $w \cancel{R} z$

(b) $R ; Z \to Z$; xRy iff $x^2 = y^2$

Refl ✓ $x^2 = x^2$

Sym ✓ $x^2 = y^2$ then $y^2 = x^2$

Antisym: X $2R-2$, $-2R2$

Trans: ✓ aRb, bRc

$a^2 = b^2 = c^2 \longrightarrow a^2 = c^2 \to aRc$

R is ER

Complete Analysis of ER R;

$R: Z \to Z$; xRy iff $x^2 = y^2$

↳ ER ✓

① $|[2]_R| = ? = 2$ $[2]_R = \{2, -2\}$

$|[0]_R| = ? = 1$

If R is ER then

$$[x]_R = \{y \mid xRy\} = \{y \mid yRx\}$$

$$[0]_R = \{0\}$$

$$[2]_R = \{2, -2\}$$

$$[-2]_R = [2]_R$$

$$[-2]_R = \{-2, 2\}$$

Same

$2 \longrightarrow \{2, -2\}$

$3 \longrightarrow \{3, -3\}$

$4 \longrightarrow \{4, -4\}$

$5 \longrightarrow \{5, -5\}$

$0 \longrightarrow \{0\}$

$\vdots$

Partition of $\mathbb{Z}$

$[3]_R = [-3]_R$

Each part is called Eq. class

R has created a partition of base set $\mathbb{Z}$.

$\{0\}, \{2, -2\}, \{3, -3\}, \{4, -4\}, \ldots$

EC

EC

(e) R on $\{(a,b) \mid a,b \in Z\}$

$(a,b) R (c,d)$ iff $a=c$ OR $b=d$.

Base set: $Z \times Z$

$R : Z \times Z \longrightarrow Z \times Z$

$(a,b) R (c,d)$ iff $a=c$ OR $b=d$

R : Refl ✓ $(a,b)\ R\ (a,b)$

Sym ✓ If $(a,b)\ R\ (c,d)$ then $(c,d)\ R\ (a,b)$

Antisym ✗ $(1,2)\ R\ (1,3)$ and $(1,3)\ R\ (1,2)$

Trans:

$$\overbrace{(2, 3) \longrightarrow (2,}\underbrace{4) \longrightarrow (5, 4)}$$

But $(2,3) \not R (5,4)$

So Not Trans.

R is not ER.

Binary Relations: Example

Let

$$R = \{(x, y) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid \exists k \in \mathbb{Z}^+ y = kx\}$$

Clicker: Is R

1. reflexive, symmetric, transitive
2. not reflexive, antisymmetric, not transitive
3. reflexive, not antisymmetric, transitive
4. reflexive, symmetric, not transitive
5. reflexive, antisymmetric, transitive
6. reflexive, not symmetric, not transitive

Base set: $Z^+ = N$

$\cancel{Z^+ \times Z^+}$

$R : Z^+ \to Z^+$

xRy iff $x | y$

N N

R Not ER.

Ref ✓ $a|a$

Sym: ✗ $2|4$ ✓ $4 \nmid 2$

Antisym: ✓

$a|b$, $b|a$ then $a=b$

Trans: ✓ $a|b$, $b|c$ then $a|c$

Base set A ; $|A| \geq 2$

Complete/universal/full Relation on A:

$A \times A$

- Refl ✓
- Sym ✓
- Trans ✓
- Antisym ✗

Asym ✗

ER ✓

Base set A ; $|A|=1$

Complete / universal / full Relation on A:

$A \times A$

- Refl ✓
- Sym ✓
- Trans ✓
- Antisym ✓

Asym ✗

ER ✓

Example 1. *Are these equivalence relations on* $\{0, 1, 2\}$? $= S$

(a) $\{(0,0),(1,1),(0,1),(1,0)\}$

(b) $\{(0,0),(1,1),(2,2),(0,1),(1,2)\}$

(c) $\{(0,0),(1,1),(2,2),(0,1),(1,2),(1,0),(2,1)\}$

(d) $\{(0,0),(1,1),(2,2),(0,1),(0,2),(1,0),(1,2),(2,0),(2,1)\}$ $= S \times S$

(e) $\{(0,0),(1,1),(2,2)\}$

Solution. (a) R is not reflexive: $(2,2) \notin R$. Thus, by definition, R is not an equivalence relation.

(b) R is not symmetric: $(1,2) \in R$ but $(2,1) \notin R$. Thus R is not an equivalence relation.

(c) R is not transitive: $(0,1),(1,2) \in R$, but $(0,2) \notin R$. Thus R is not an equivalence relation.

(d) R is reflexive, symmetric, and transitive. Thus R is an equivalence relation.

(e) R is reflexive, symmetric, and transitive. Thus R is an equivalence relation.

Base set $= \{0, 1, 2\} = S$

(D): $R = S \times S$ full relation on S

$\hookrightarrow$ ER

$$[0]_R = \{0, 1, 2\} = [1]_R = [2]_R$$

Base set $\{0,1,2\}$

Partition created by R

Set of all Eq. Classes $= \{ \{0,1,2\} \}$

R has one Eq. Class

$$[0]_R = \{y \mid 0 R y\} = \{y \mid y R 0\}$$

Equivalence class of 0

Containing 0

(e) $S = \{0, 1, 2\}$

$R = \{(0,0), (1,1), (2,2)\}$ — Identity Rel.

Eq. Rel

$[0]_R = \{0\}$; $[1]_R = \{1\}$; $[2]_R = \{2\}$

Eq. Rel

Base Set $S = \{0,1,2\}$

Partition of S by R

$$\{\{0\}, \{1\}, \{2\}\}$$

3

Eq. Classes

$$\left| [0]_R \right| = \left| \{ 0 \} \right| = 1$$

$$\left| [1]_R \right| = 1$$

Example 3.2. Let $S = \{1, 2, 3, \ldots, 19, 20\}$ and define an equivalence relation R on S by

$$\boxed{xRy \iff 4|(x-y)}$$

Determine the equivalence classes of R.

$xRy \iff 4 \mid (x-y)$

<u>means</u>

$xRy \iff \boxed{x \equiv y \pmod 4}$

$$[1]_R = \{1, 5, 9, 13, 17\}$$

$$[2]_R = \{2, 6, 10, 14, 18\}$$

$$[3]_R = \{3, 7, 11, 15, 19\}$$

$$[4]_R = \{4, 8, 12, 16, 20\}$$

Partition of Base set

$$[3]_R = [19]_R = [7]_R = [11]_R$$

Eq. class of 3 = Eq. class of 9 = Eq. class of 7

$3 \sim 19$; $19 \sim 7$; $3 \sim 11$

$S = \{1, \ldots, 20\}$

Partition of S by R

4 Eq. classes

$\{\{1, 5, 9, 13, 17\}, \{2, 6, 10, 14, 18\}, \{3, 7, 11, 15, 19\}, \{4, 8, 12, 16, 20\}\}$

In Eq Rel R;

If a R b then we say that a, b are Equivalent.

Notation: $a \sim b$

$|R| = ?$

Hint: within a Eq. Class, Every element is Relates to every element.

R has 4 Eq. Classes :

$E_2 = \{2, 6, 10, 14, 18\}$

E_1 $\rightarrow$ 5 elements

E_2 $\rightarrow$ 5 elements

E_3 $\rightarrow$ 5 "

E_4 $\rightarrow$ 5 "

$E_1 = \{1, 5, 9, 13, 17\}$

$|E_2| = 5$

$R = \{ (1,1), (1,5), (1,9), (1,13), (1,17)$
$(5,1), (5,9), (5,13), (5,17)$ - - - - - -

25 by E_1

$(2,2), (2,6), (2,10), (2,14), (2,18)$
$(6,2), (6,6), (6,10),$ - - - -
- - - - - -

25 by E_2

$$R \longrightarrow E_1,\ E_2,\ E_3,\ E_4$$

$$|R| = |E_1|^2 + |E_2|^2 + |E_3|^2 + |E_4|^2$$

$$|R| = 25 + 25 + 25 + 25 = 100 \checkmark$$

Example 3.2. Let $S = \{1, 2, 3, \ldots, 19, 20\}$ and define an equivalence relation R on S by

$$xRy \iff 4|(x - y)$$

Determine the equivalence classes of R.

In order to determine equivalence classes, we simply need to group the elements together according to which ones are equivalent to each other. We start with 1 and move onward:

$$[1] = \{1, 5, 9, 13, 17\}$$

$$[2] = \{2, 6, 10, 14, 18\}$$

$$[3] = \{3, 7, 11, 15, 19\}$$

$$[4] = \{4, 8, 12, 16, 20\}$$

Notice that any other equivalence class we construct will be the same as one of these e.g $[1] = [5]$. Thus we have found all the different equivalence classes of R.

Q: $S = \{2, 3, 5, 6, 11, 12\}$ = Base Set

R on S ; xRy iff $x \equiv y \pmod 3$

Eq. Relation (R S T)

$R = \{(2,2), (2,5), (5,5), (5,2), (3,6) \ldots\}$

$$[2]_R = \{2, 5, 11\}$$

$$[3]_R = \{3, 6, 12\}$$

2 Equi Classes

Base set = $\{2, 3, 5, 6, 11, 12\}$

Partitioned by R

$\Pi = \{ \{2, 5, 11\}, \{3, 6, 12\} \}$

$|R| = ?$

R has 2 ECs;

$E_1 = \{2, 5, 11\}$; $E_2 = \{3, 6, 12\}$

$R = \{(2,2), (2,5), (2,11), (5,2), (5,5), (5,11), (11,2), (11,5), (11,11), (3,3) \ldots\}$

$$|R| = |E_1|^2 + |E_2|^2$$

$$R = \underbrace{(E_1 \times E_1)}_{} \cup \underbrace{(E_2 \times E_2)}_{}$$

$$|R| = |E_1 \times E_1| + |E_2 \times E_2|$$

Disjoint

Conclusion:

ER R on set A gives unique Partition of A.

Each part of this Partition is called Eq. Class of R.

ER $R \longrightarrow$ n Eq. Classes

$$\boxed{E_1 \,,\; E_2 \,,\; - - - \; E_n}$$ — Partition of Base set.

$$R = (E_1 \times E_1) \cup (E_2 \times E_2) \cup \cdots (E_n \times E_n)$$

Disjoint

$$|R| = |E_1|^2 + |E_2|^2 + \cdots + |E_n|^2$$

Note:

for every partition of set A, we have unique ER on A corresponding to that partition.

Eg: $A = \{a, b, c\}$

$\Pi_1 = \{a, bc\}$ — Partition of A

If we assume each Part as a EC of some ER R then what is R ?

E_1 a E_2 b, c

$$R = \{(a,a), (b,b), (b,c), (c,b), (c,c)\}$$

$$|R| = 5 = |E_1|^2 + |E_2|^2$$

$$R = (E_1 \times E_1) \cup (E_2 \times E_2)$$

Eg: $A = \{a, b, c\}$

$\Pi_1 = \{ abc \}$ — Partition of A

If we assume each part as a EC of some ER R then what is R?

Simple EC —— a, b, c $= E_1$

$$|R| = |E_1|^2 = 9$$

$$R = \underbrace{E_1 \times E_1}_{} = \underbrace{A \times A}_{}$$

Eg: $A = \{a, b, c\}$

$\Pi_1 = \{a, b, c\}$ — Partition of A

If we assume each Part as a EC of some ER R then what is R?

E_1 a $\quad E_2$ b $\quad E_3$ c

$$R = (E_1 \times E_1) \cup (E_2 \times E_2) \cup (E_3 \times E_3)$$

$$R = \{(a,a), (b,b), (c,c)\}$$

$$|R| = 3 = |E_1|^2 + |E_2|^2 + |E_3|^2$$

Equivalence Classes

- Given an equivalence relation R over a set A, for any $x \in A$, the ***equivalence class of x*** is the set

$$[x]_R = \{ \ y \in A \mid xRy \ \}$$

- $[x]_R$ is the set of all elements of A that are related to x.
- ***Theorem:*** If R is an equivalence relation over A, then every $a \in A$ belongs to exactly one equivalence class.

Equivalence Relations

The properties of relations are sometimes grouped together and given special names. A particularly useful example is the equivalence relation.

Definitions

A relation that is reflexive, symmetric, and transitive on a set S is called an **equivalence relation** on S.

Equivalence Relations

- Some relations are reflexive, symmetric, and transitive:
  - $x = y$
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- Definition: An ***equivalence relation*** is a relation that is reflexive, symmetric and transitive.