Partial Order Relations

Next Topic:

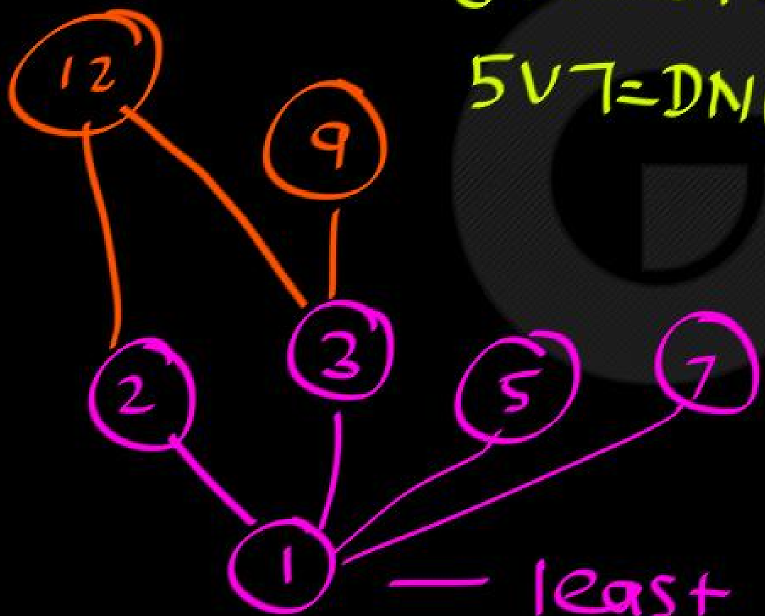
Complete Analysis of Divisibility Relations

Website : https://www.goclasses.in/

$(\{1, 2, 3, 5, 7, 9, 12\}, |)$ — Poset $\left\{ \begin{array}{l} \text{Ref} \\ \text{Antisym} \\ \text{Trans} \end{array} \right\}$ $a|a$

$9 \vee 12 = \text{DNE}$

$5 \vee 7 = \text{DNE}$



— least

Greatest : DNE

maximal : 12, 9, 5, 7

minimal : 1

Least : 1

$1 | a, \forall a$

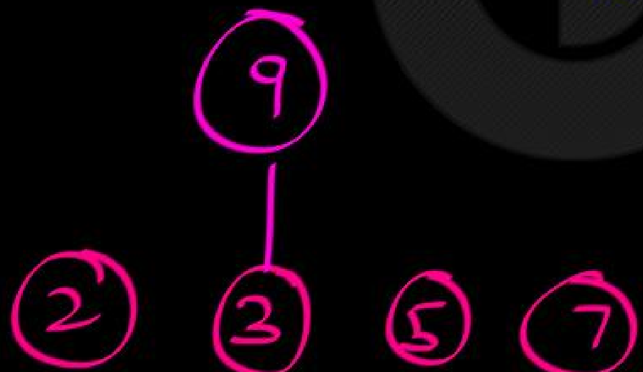
lattice? No

$(\{2, 3, 5, 7, 9\}, |)$ — poset

Ref, Antisym, Trans

$5 \vee 7 = DNE$

$2 \wedge 3 = DNE$



Greatest : DNE

Least : DNE

maximal : 9, 2, 5, 7

minimal : 2, 3, 5, 7

lattice

No

Divisibility "Relation" : Symbol : $|$ → Divides

$a|b$: a Divides b

$2|4 \equiv$ 2 Divides 4 = True

$4|2$ = No

Divisibility Relation $\Rightarrow$ True or false

Symbol : |

$\rightarrow$ Divides

$a | b \equiv$ a Divides b

2|4 = True
4/2 = false

Division $\rightarrow$ value

$\div$, /

$4 \div 2 = 2$

$a \div b = \dfrac{a}{b}$ value

$4 \div 2 = \dfrac{4}{2} = 2$ ✓

$8 \mid 4$ = false

means 8 Divides 4

$0 \mid 0$ = True / False ?

0 Divides 0

$8 \div 4 = 8/4 = \frac{8}{4} = 2$

$0 \div 0$ = Indeterminate

Value when 0 is Divided 0

$\frac{0}{0}$ = Indeterminate

Definition of Divisibility Relation :

$a, b, q \in Z$

$a \mid b$ means a Divides b

$a \mid b$ iff $b = aq$; $\exists q \in Z$

$a|b$ iff b is some integer multiple of a.

$a|b$ iff $\exists$ integer q, $b = qa$

$4 \mid 8 = \text{True}$ $\quad 8 = 4(2)$

$8 \mid 4 = \text{false}$ $\quad 4 = 8(?)$ → No such integer

$-4 \mid 8 = \text{True}$ $\quad 8 = -4(-2)$ integer

$-4 \mid -4 = \text{True}$ $\quad -4 = -4(1)$

$4 \mid -13 = \text{false}$ $\quad -13 = 4(?)$ → No such integer

$0 | 0$ = True $\qquad 0 = 0(3)$

$0 | 0$ = True because $\exists$ integer q $\to 3$

$0 = 0(3)$

Divides

$0 | 0$ = True

Divisibility Relation

$4 | 8$ = True

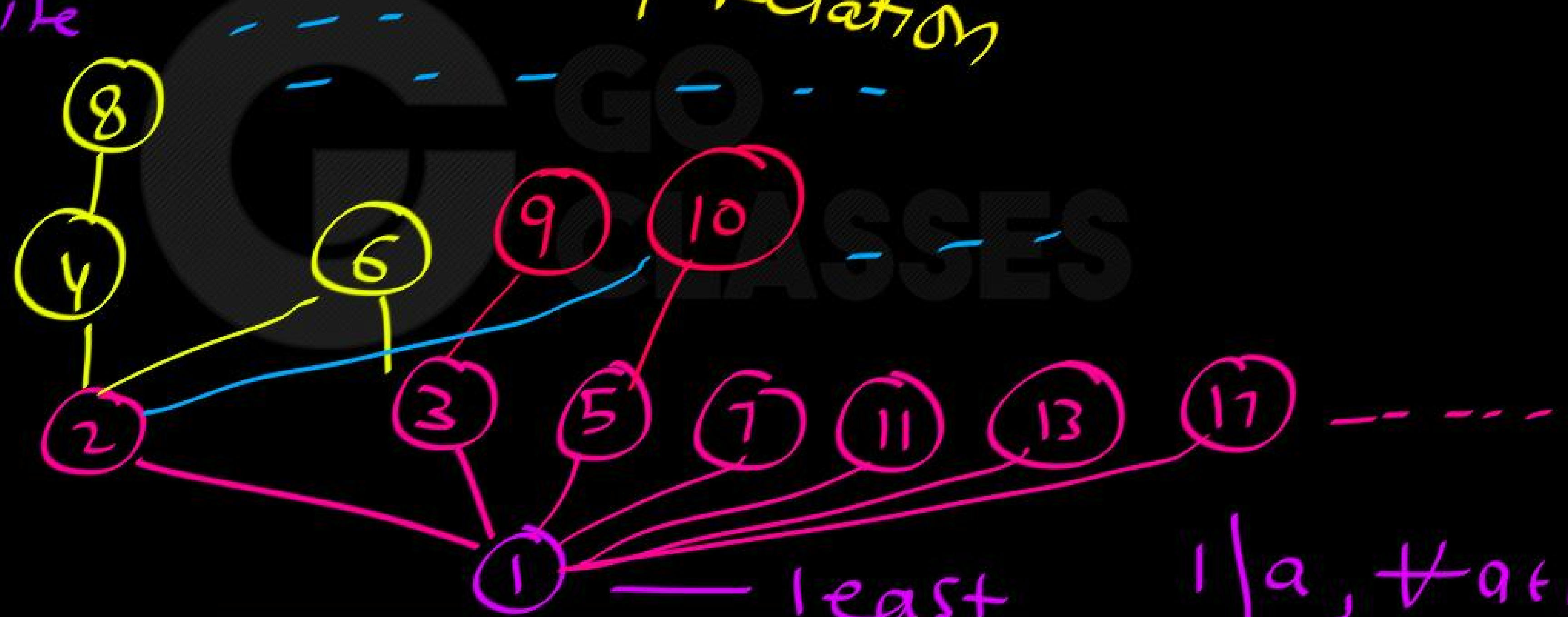
$\frac{0}{0}$ → value → indeterminate

$\frac{8}{4}$ = value → 2

Q: $(N, |)$ $\quad N = \{1, 2, 3, 4, ---\}$

infinite

Divisibility Relation



least

$1 | a, \forall a \in N$

$(N, |)$

least : 1 $\quad \boxed{1 | a, \forall a \in N}$

Greatest : DNE

Q : $(W, |)$ $\quad W = \{\underbrace{0, 1, 2, 3, 4, --}_{\text{Base set}}\}$

Ref : ✓ $\quad \forall a \in W \quad a|a$

$1|1$ ✓ ; $2|2$ ✓ ; $0|0$ ✓

$0 = 0(-5)$

Antisym ✓

Trans ✓

$(W, |)$ — <u>least</u>: 1 ✓ $\quad 1 \mid a, \ \forall_{a \in W}$

<u>Greatest</u>: 0 ✓ $\quad a \mid 0, \ \forall_{a \in W}$

$0 \mid 0$ ✓

$1 \mid 0$ ✓

$2 \mid 0$ ✓

$\vdots$

Divisibility Relation:

- Ref ✓
- Antisym ✓
- Transitive ✓

for whole numbers

$(Z, |)$ — Not Antisym

$1 | -1$; $-1 | 1$
$2 | -2$; $-2 | 2$
} Not Antisym

$(\mathbb{N}, |) \rightarrow$ Poset

$(\mathbb{Z}, |) \rightarrow$ Not Sym ; Not Antisym

$2 | 4$ $\quad$ $1 | -1$

$4 \nmid 2$ $\quad$ $-1 | 1$

$(\omega, |)$

$(\{1,2,3\}, |) \rightarrow$ Poset

$(\{1\}, |) -$ Ref ; Sym ; AntiSym, Trans

Divisibility Relation

$(\{2,3\}, |) -$ Ref ✓ ; Sym ✓ ; AntiSym ; Trans.

$\{(2,2), (3,3)\}$

$(\underbrace{\{1, 2, 3\}}, |)$ — Refl ✓

Antisym ✓

Symm ✗

Trans ✓

$1 | 2$ ✓

$2 \nmid 1$

$(D_n, |)$ — Lattice

D_n → Set of Divisors of n

$\boxed{(D_n, |)}$ — Lattice

Complete Analysis of

$$(D_n, |)$$

$n \in \mathbb{N}$

Positive

D_n = set of all Divisors of n.

$n = 4$

$D_n = \{1, 2, 4\}$

$n = 5$

$D_5 = \{1, 5\}$

$n = 6$

$D_6 = \{1, 2, 3, 6\}$

$D_{20} = \{1, 2, 4, 5, 10, 20\}$

$20 = 2^2 \times 5$

$D_{32} = \{1, 2, 4, 8, 16, 32\}$

$32 = 2^5$

$(D_n, |)$ — poset $\begin{cases} \text{Ref} \checkmark \; a|a \\ \text{Antisym} \checkmark \\ \text{Trans} \checkmark \end{cases}$

$D_n \to$ Base Set

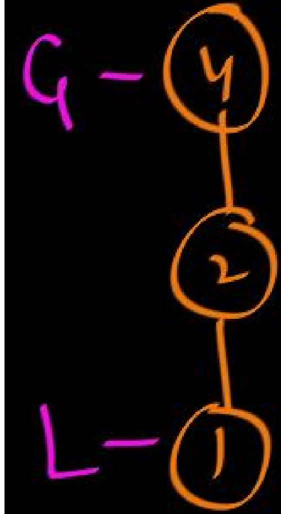
$n \to \in N$

$| \to$ Divisibility Relation

$(D_n, |)$ — poset, for $n \in N$

$(D_4 = \{1, 2, 4\}, |)$ — Toset → Lattice ✓

Distributive



Lattice ✓

Complemented ✗

$\bar{2}^{1} = \text{DNE}$

$2 \vee 4 = 4$

$2 \wedge 4 = 2$

$1 \vee 4 = 4$

$1 \wedge 4 = 4$

$1 \vee 2 = 2$

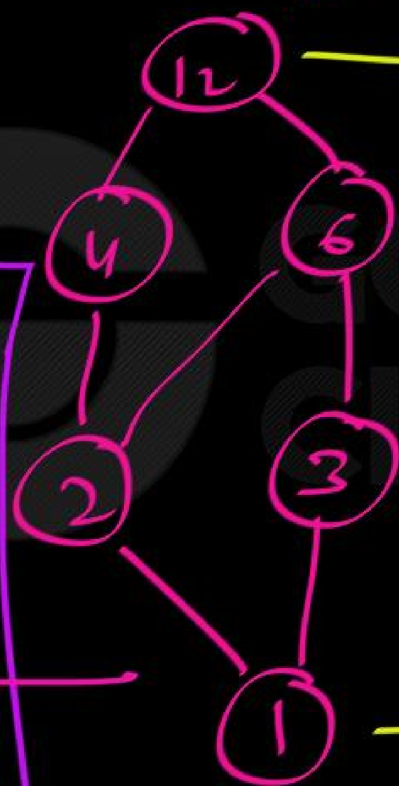
$1 \wedge 2 = 1$

$(D_{12}, |)$

$D_{12} = \{1, 2, 3, 4, 6, 12\}$

$12 = 2^2 \times 3$

Divisors of 12



Greatest

Lattice

least

$4 \vee 6 = \text{LCM}(4,6) = 12$

$4 \wedge 6 = \text{GCD}(4,6) = 2$

$2 \vee 3 = 6$

$2 \wedge 3 = 1$

$3 \vee 4 = 12$

$3 \wedge 4 = 1$

$2 \vee 6 = 6$

$2 \wedge 6 = 2$

Note: $(D_n, |)$ — poset ✓

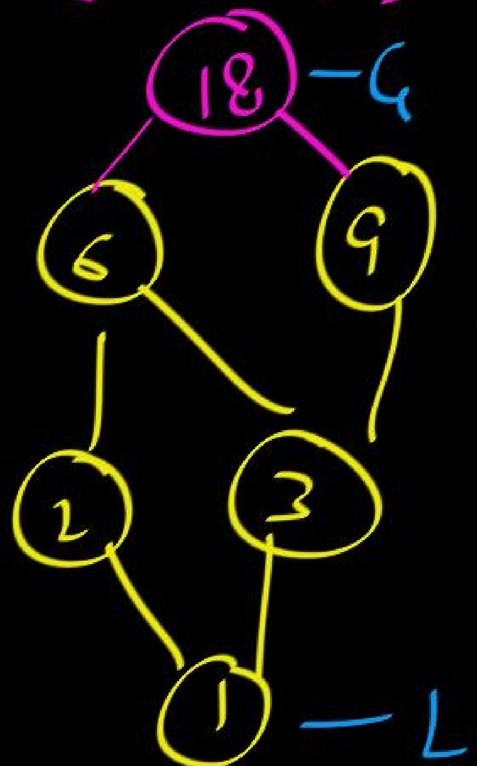
① Poset

② Least: 1 ; $1 | a$, $\forall a \in D_n$

③ Greatest: n ; $\forall_{a \in D_n}$ $a | n$

④ Lattice: LUB = LCM ✓ ; GLB = GCD ✓

$(D_{18}, |)$



$18 = 3^2 \times 2$

$D_{18} = \{\underbrace{1, 2, 3, 6, 9, 18}\}$

$\forall a \in D_{18}, \; a | 18$ So Greatest $= 18$

Least: 1

$2 \wedge 9 = GCD(2,9) = 1$

$2 \vee 9 = LCM(2,9) = 18$

$(D_{18}, |)$

$\left\{ \begin{array}{l} 2 \vee 6 = \mathrm{LCM}(2,6) = 6 \\ 2 \wedge 6 = \mathrm{GCD}(2,6) = 2 \end{array} \right.$

$2 R 6 \Rightarrow \left. \begin{array}{l} 2 \vee 6 = 6 \\ 2 \wedge 6 = 2 \end{array} \right\}$

$(D_n, |)$ — $LUB\{a,b\} = a \vee b = LCM(a,b)$

$GLB\{a,b\} = a \wedge b = GCD(a,b)$

Lattice

$(D_n, |)$ — Bounded Lattice $\begin{cases} \text{Greatest} = n \\ \text{Least} = 1 \end{cases}$

Q: $(D_n, |)$ is Distributive ?

Q: $(D_n, |)$ is Distributive ?

Yes.

$\Rightarrow$ GLB = GCD

$\Rightarrow$ LUB = LCM

$(D_n, |)$

$\wedge = \text{GCD}\ ;\ \vee = \text{LCM}$

Distributive Property

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

GCD LCM

GCD LCM

Note (Fact from Number Theory):

GCD and LCM Distribute Over Each Other.

Proof NOT required for us. Skip it.
But the proof can be found below. (Don't read, Not required)

https://proofwiki.org/wiki/GCD_and_LCM_Distribute_Over_Each_Other

So, $(D_n, |)$ — Distributive Lattice

Q: Is $(D_n, |)$ Complemented Lattice?

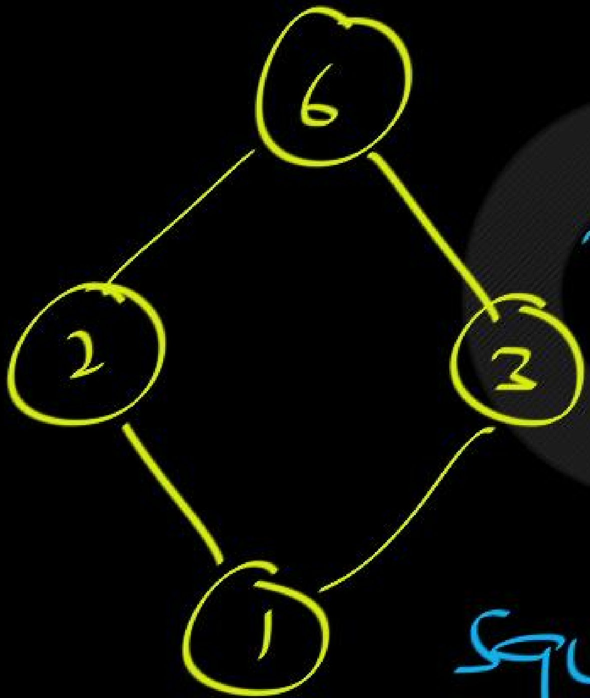
$(D_4 = \{1, 2, 4\}, |)$ — total order

Not Comp

$\bar{2}' = \text{DNE}$

D_4 = Not Comp Lattice

$(D_6 = \{1, 2, 3, 6\}, |)$



Boolean Lattice

Complemented

Square Structure

$$\left\{ \begin{array}{ll} \bar{1}' = 6 \; ; & \bar{2}' = 3 \\ \bar{6}' = 1 \; ; & \bar{3}' = 2 \end{array} \right\}$$

$(D_5 = \{1, 5\}, |)$

D_4 — Not Comp

D_7, D_5, D_6 — Comp

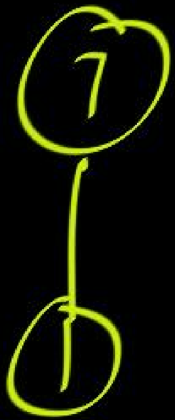
Boolean Lattice

Complementes lattice

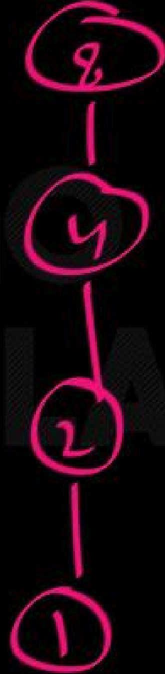
D_8 = Not Comp.

$1^{-1} = 5 \; ; \; 5^{-1} = 1$

$D_7 = \{1, 7\}$



$D_8 = \{1, 2, 4, 8\}$ $8 = 2^3$



Not Complemented

$\bar{2}' = \text{DNE}$

$\bar{4}' = \text{DNE}$

$(D_n, |)$ — may or may not be Complemented.

(Depend on n)

Q: for which n, $(D_n, |)$ is Complemented Lattice?

Ans: n = square-free

$8 = 2^3 = 2^2 \times 2$

not square free → $(D_8, |)$ Not Comp.

$\boxed{\text{Square free}}$: n is square free iff

n is not Divisible by p^2

where p = prime.

$6 = 2 \times 3$

square free → No prime number p such that $p^2 \mid 6$.

$(D_6, |)$ Complemented Lattice.

$10 = 2 \times 5$

Prime factorisation of Square free number:

$150 = 2 \times 3 \times 5^2$

not Sq. free

$5^2 \mid 150$

n = Sq. free

$n = p_1^1 \times p_2^1 \times \cdots \times p_m^1$

$$154 = 2^1 \times 7^1 \times 11^1$$

sq. free

$$n = 2^4 \times 3^5 = \text{not sq. free}$$

not sq. free

$$3^2 \mid n$$

Square free number (n):

Def1: No prime p such that $p^2 \mid n$.

Def2: Prime factorization of n

$$n = p_1^{1} \times p_2^{1} \times \cdots \times p_m^{1}$$

Note:

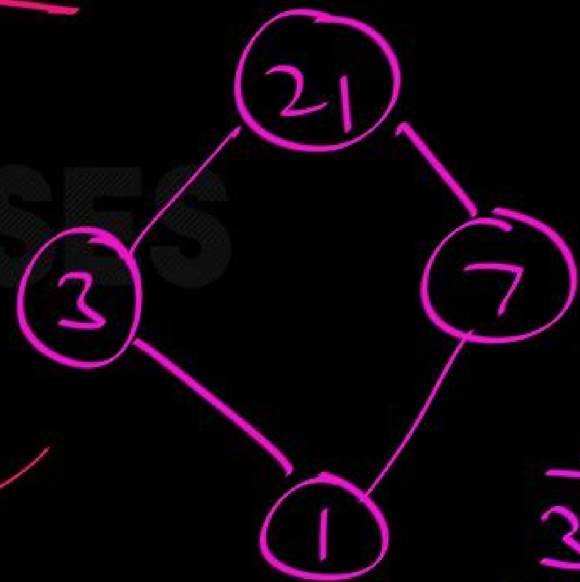
$(D_n, |)$ is Complemented iff

n is square free.

$(D_{21}, |)$ — Complemented

$21 = 7^1 \times 3^1$ — sq. free

$D_{21} = \{1, 3, 7, 21\}$



Boolean Lattice.

$21^{-1} = 1$

$1^{-1} = 21$

$3^{-1} = 7$

$7^{-1} = 3$

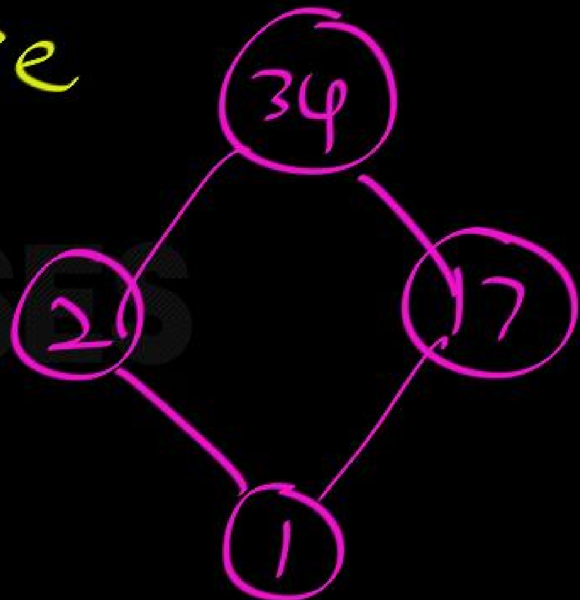
$(D_{34}, |)$

Complemented

$34 = 17^1 \times 2^1$

sq. free

$D_{34} = \{1, 2, 17, 34\}$



$(D_{50}, |)$

not Complemented

$50 = 2 \times 5^2$

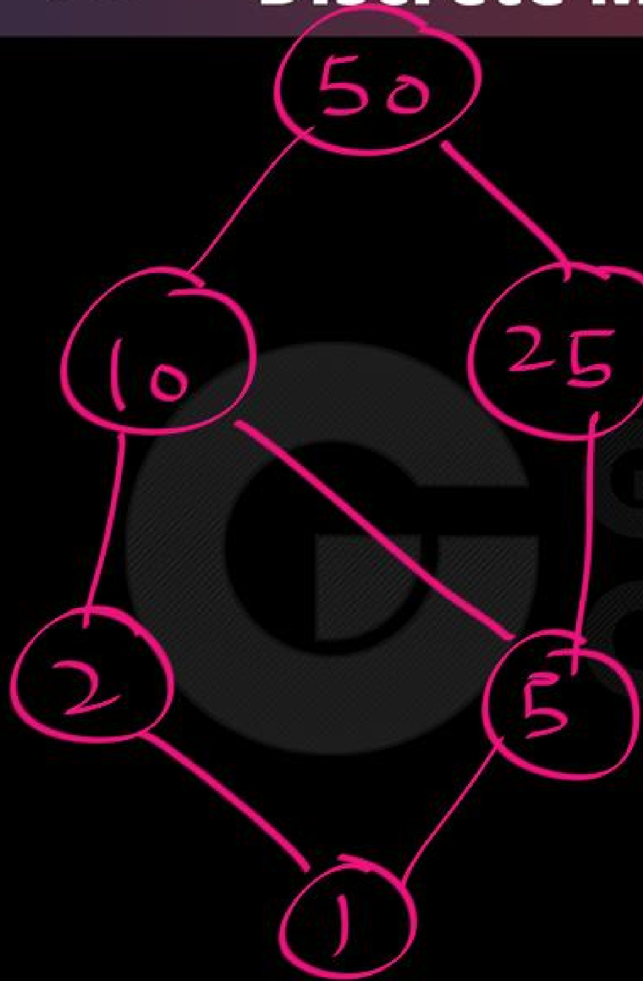
not sq. free

Prime $p = 5$

$5^2 / 50$

$D_{50} = \{1, 2, 5, 10, 25, 50\}$

D_{50}



$2^{-1} = 25$

$25^{-1} = 2$

$1^{-1} = 50$

$50^{-1} = 1$

Not Complemented

$10^{-1} = \text{DNE}$

$10^{-1} \neq 25$

$\boxed{5^{-1} = \text{DNE}}$

$10 \wedge 25 = GCD(10, 25)$

$= 5 \neq 1$

Square free number :

10

$10 = 2^1 \times 5^1$

2 ——— b = 5

$D_{10} = \{1, 2, 5, 10\}$

$lcm(2, 5) = 10$

$2^{-1} = 5$
$5^{-1} = 2$

least

Greatest

$\overline{(\text{least})} = \text{Greatest}$

$(\overline{G}) = L$

Square free number :

Greatest n

$$n = 2^{1} \times 3^{1} \times 7^{1} \times 11^{1}$$

$\overline{2}^{1} = 3 \times 7 \times 11$

$\overline{3}^{1} = 2 \times 7 \times 11$

$\overline{7}^{1} = 2 \times 3 \times 7$

$\overline{11}^{1} = 2 \times 3 \times 7$

$\overline{6}^{1} = 7 \times 11$

$\overline{2 \times 3 \times 7} = 11$

$30 = 2 \times 3 \times 5$ → sq. free

$(D_{30}, |)$

$$\left\{\begin{array}{l} \bar{1} = 30 \\ \bar{2} = 3 \times 5 \\ \bar{3} = 2 \times 5 \\ \bar{5} = 2 \times 3 \end{array}\right.$$

$\overline{30} = 1$

$\overline{(2 \times 3)} = \bar{6} = 5$

$\overline{(2 \times 5)} = \overline{10} = 3$

$\overline{(3 \times 5)} = \overline{15} = 2$

$n = 2^2 \times \underbrace{3^3 \times 5^2}$

$2^{-1} = DNE$

$(2^2)^{-1} = 3^3 \times 5^2$

$(3^3)^{-1} = 2^2 \times 5^2$

$(3)^{-1} = DNE$

$(3^2)^{-1} = DNE$

$(2^2 \times 3^3)^{-1} = 5^2$

Conclusion:

$(D_n, |)$ $n \in N$

D_n = set of all Divisors of n

$D_1 = \{1\}$ $D_3 = \{1, 3\}$ $D_{14} = \{1, 2, 7, 14\}$

$D_2 = \{1, 2\}$ $D_4 = \{1, 2, 4\}$

$(D_n, |)$

1. Poset ✓
2. Lattice ✓
3. GLB = GCD, LUB = LCM

$$\boxed{\begin{aligned} a \wedge b &= GLB(a,b) \\ &= GCD(a,b) \end{aligned}}$$

$$\boxed{\begin{aligned} a \vee b &= LUB(a,b) \\ &= LCM(a,b) \end{aligned}}$$

$(D_n, |)$

(4) Distributive Lattice

because LCM, GCD Distribute over each other.

(5) Least = 1
Greatest = n

$\left.\begin{array}{l}\text{Least} = 1 \\ \text{Greatest} = n\end{array}\right\}$ Bounded lattice

$(D_n, |)$

(6) Complemented iff n is Square free

(7) Boolean lattice/Boolean Algebra iff n is Square free.

$(D_n, |)$

8 Complements of elements of D_n:

$n = p_1^1 \times p_2^1 \times \cdots \times p_m^1$

sq. free

$$n = p_1 \times p_2 \times p_3 \times p_4$$

$$(p_1)^{-1} = p_2 \times p_3 \times p_4 = \frac{n}{p_1}$$

$$(p_2)^{-1} = \frac{n}{p_2}$$

$$(p_2 \times p_3)^{-1} = \frac{n}{p_2 \times p_3}$$

$$(p_1 p_2 p_3)^{-1} = \frac{n}{p_1 p_2 p_3}$$

Eg: $n = 5^1 \times 7^1 \times 19^1 \times 23^1$

sq. free

$n^{-1} = \frac{n}{n}$

$1^{-1} = \frac{n}{1}$

$(D_n, 1)$ – Complemented Lattice.

$(5)^{-1} = \frac{n}{5}$

$5 \times 19 = \frac{n}{5 \times 19}$

$(5 \times 7 \times 23)^{-1}$

$= \frac{n}{5 \times 7 \times 23}$

Result: when n is square free

then $(D_n, |)$ is Complemented Lattice

and $\forall a \in D_n$

$$\boxed{n^{-1} = \frac{n}{n}}$$

$$\boxed{a^{-1} = \frac{n}{a}}$$

$$\boxed{1^{-1} = \frac{n}{1}}$$

When n is not square free then in $(D_n, |)$ lattice, some elements do not have complement.

Those elements which have complements for them

$$a^{-1} = \frac{n}{a}$$

$108 = 2^2 \times 3^3$

not sq. free

$(2^2)^{-1} = \frac{108}{2^2} = 3^3$

$LCM(4, 27) = 108$
$GCD(4, 27) = 1$

$2^{-1} = DNE$

$2^{-1} = \frac{108}{2} = 54$ wrong

$2 \vee 54 = 54 \neq 108$

$(2^2)^{-1} = \frac{n}{2^2} = 3^3$

$$108 = 2^2 \times 3^3$$

$$3^{-1} = DNE$$

$$(3^{-1}) = DNE$$

$$(3^2)^{-1} = DNE$$

$$(3^3)^{-1} = \frac{108}{3^3} = 2^2$$

$$3^{-1} = \frac{108}{3} = 36$$

$$3 \vee 36 = 36 \neq 108$$

So $3^{-1} \neq 36$ So $3^{-1} = DNE$

$$n = p_1^4 \times p_2^2 \times p_3^6$$

$$p_2^{-1} = DNE$$

$$(p_1^4)^{-1} = \frac{n}{p_1^4}$$

$$(p_2^2)^{-1} = \frac{n}{p_2^2}$$

$$(p_1)^{-1} = DNE \; ; \; (p_1^2)^{-1} = DNE \; ; \; (p_1^3)^{-1} = DNE$$

$$8 = 2^3$$

$$(2^3)^{-1} = 8^{-1} = \frac{8}{8} = 1$$

$$\left.\begin{array}{l} 2^{-1} = DNE \\ 4^{-1} = DNE \end{array}\right\}$$

Show that the following divides-relations are partial orders on $\mathcal{A}$. Is either a total order?

a) $m \mid n$ iff m divides n, $\mathcal{A} = \mathbb{N}$.

b) $m \mid n$ iff m divides n, $\mathcal{A} =$ the set of all positive-integer divisors of 36.

Solution

a) Since $n \mid n$ for all $n \in \mathbb{N}$, the relation is reflexive. It is also antisymmetric: if $m \mid n$ and $n \mid m$, then $m = n$. And it is transitive: if $k \mid m$ and $m \mid n$, then $k \mid n$.
It is not connected, however. For example, neither 2 nor 3 divides the other. Thus, the relation is not a total order.

b) Here the divisibility relation is restricted to $\mathcal{A} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.
The relation here remains reflexive, antisymmetric, transitive, and not connected. Thus, it too is not a total order on $\mathcal{A}$.

Diagram the following posets:

a) The poset of Example 3b: the divisors of 36 ordered by $m \mid n$.

b) The poset $\mathcal{P}(S)$ for $S = \{0, 1, 2\}$, ordered by $R \subseteq T$.

Diagram the following posets:

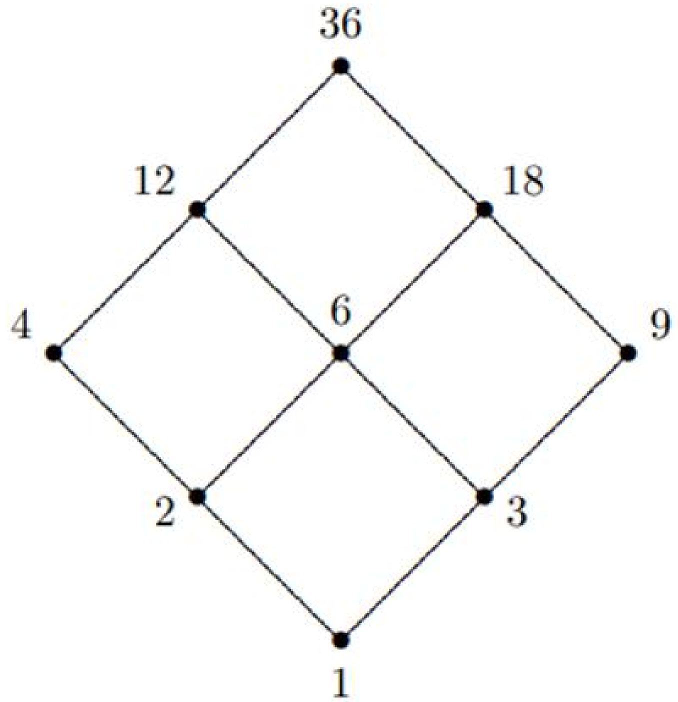
a) The poset of Example 3b: the divisors of 36 ordered by $m \mid n$.

b) The poset $\mathcal{P}(S)$ for $S = \{0, 1, 2\}$, ordered by $R \subseteq T$.

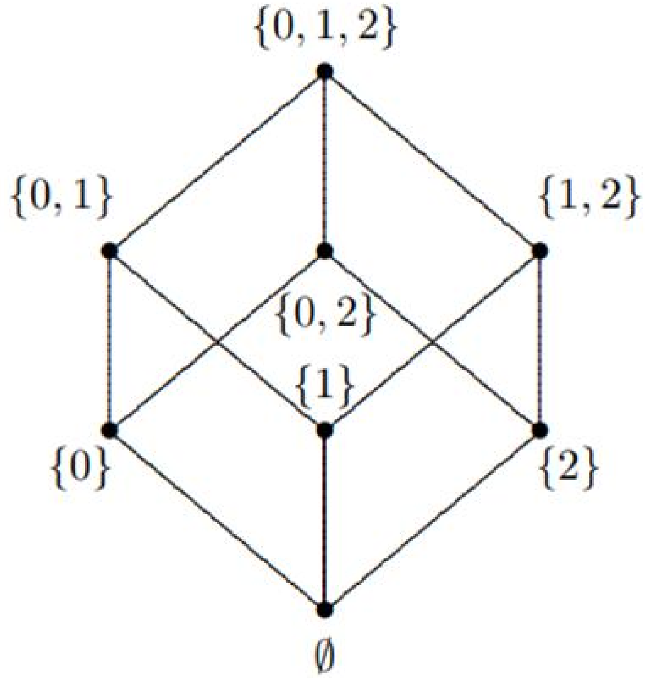
Solution

These are graphed by the following Hasse diagrams.

a)



b)



Problems 2-4: Divisor Lattices

The following problems have to do with lattices in which the partial order is the divisibility relation.

*2. Let $\langle \mathcal{D}_{12}, \mid \rangle$ denote the poset of all divisors of 12.

*a. Show that $\mathcal{D}_{12}$ is a lattice by drawing out the Hasse diagram for the poset and then verifying that each pair of divisors has both a meet and a join. How do meet and join relate to the numbers in terms of divisibility?

*b. Is $\mathcal{D}_{12}$ a complemented lattice? Explain.

EC c. Is $\mathcal{D}_{12}$ a distributive lattice? To check whether $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, how many different equations must be checked? Explain.

d. If the bottom point 1 and the top point 12 are deleted from $\mathcal{D}_{12}$, is the result still a lattice? Explain.

e. Is $\mathcal{D}_{12}$ a Boolean lattice? Explain.

3. Let $\langle \mathcal{D}_{30}, \mid \rangle$ denote the poset of all divisors of 30.
   a. Show that $\mathcal{D}_{30}$ is a lattice. Explain.
   b. Is $\mathcal{D}_{30}$ a complemented lattice? Explain.
   c. Is $\mathcal{D}_{30}$ a distributive lattice? Explain.
   d. Is $\mathcal{D}_{30}$ a Boolean lattice? Explain.

4. Let $\langle \mathcal{D}_n, \mid \rangle$ denote the poset of all divisors of n, where n is a positive integer.
   a. Prove that $\mathcal{D}_n$ is a lattice. Carefully explain why $x \wedge y = \gcd(x, y)$ and $x \vee y = \operatorname{lcm}(x, y)$.
   b. When will $\mathcal{D}_n$ be a complemented lattice? Explain.
   c. When will $\mathcal{D}_n$ be a distributive lattice? Explain.
   Hint: to check $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, express x, y, and z via their prime factorizations and relate meet and join to how many factors of each prime (possibly 0) must be used. You may assume that $\langle \mathbb{N}, \leq \rangle$ is a distributive lattice (see Exercise 19).
   d. For which n will $\mathcal{D}_n$ be a Boolean lattice?