Set Theory

Next Topic:

Subsets, Power Sets

Website : https://www.goclasses.in/

Subsets

- A set S is called a ***subset*** of a set T (denoted $\boldsymbol{S \subseteq T}$) if all elements of S are also elements of T.
- Examples:
  - $\{ 1, 2, 3 \} \subseteq \{ 1, 2, 3, 4 \}$
  - $\{ b, c \} \subseteq \{ a, b, c, d \}$
  - $\{ \text{H}, \text{He}, \text{Li} \} \subseteq \{ \text{H}, \text{He}, \text{Li} \}$
  - $\mathbb{N} \subseteq \mathbb{Z}$ (*every natural number is an integer*)
  - $\mathbb{Z} \subseteq \mathbb{R}$ (*every integer is a real number*)

Subset

Definition:

A set A is a subset of B if every element of A is also in B.

- We write $A \subseteq B$ if A is a subset of B.
- Clearly, for any set A, the empty set $\emptyset$ (which does not contain any element) and A itself are both subsets of A.

Definition:

If $A \subseteq B$ but $A \neq B$, then A is a proper subset of B, and we write $A \subset B$.

Subsets and Elements

- We say that $\boldsymbol{S \in T}$ if, among the elements of T, one of them is *exactly* the object S.
- We say that $\boldsymbol{S \subseteq T}$ if S is a set and every element of S is also an element of T. (S has to be a set for the statement $S \subseteq T$ to be true.)
  - Although these concepts are similar, ***they are not the same!*** Not all elements of a set are subsets of that set and vice-versa.

Proper Subsets

- By definition, any set is a subset of itself. *(Why?)*
- A **proper subset** of a set S is a set T such that
  - $T \subseteq S$
  - $T \neq S$
- There are multiple notations for this; they all mean the same thing:
  - $T \subsetneq S$
  - $T \subset S$

Definition 2.12 *For two sets S and T we say that S* **is a subset of** *T if each element of S is also an element of T. In formal notation $S \subseteq T$ if for all $x \in S$ we have $x \in T$.*

If $S \subseteq T$ then we also say T contains S which can be written $T \supseteq S$. If $S \subseteq T$ and $S \neq T$ then we write $S \subset T$ and we say S is a *proper* subset of T.

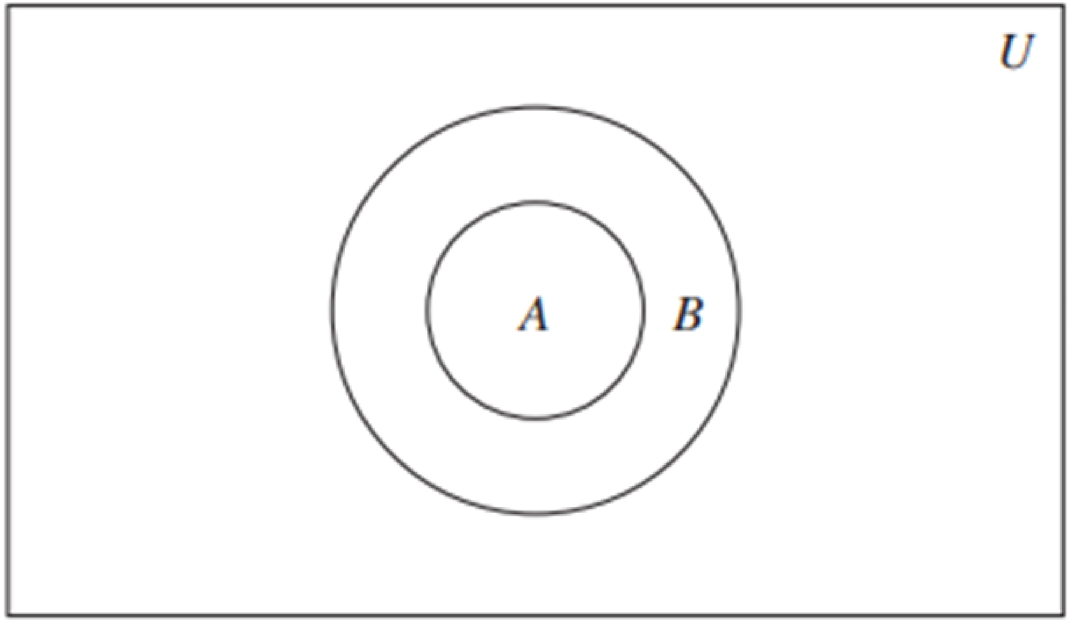


FIGURE 2 Venn Diagram Showing that A Is a Subset of B.

let $A = \{a_1, a_2, a_3, ---, a_n\}$

and $\boxed{A \subseteq B}$ means every element of A is in B.

then

$B = \{a_1, a_2, ---, a_n, b_1, b_2, ---, b_m\}$

Example 2.9 Subsets

If $A = \{a, b, c\}$ then A has eight different subsets:

$$\emptyset \qquad \{a\} \qquad \{b\} \qquad \{c\}$$

$$\{a, b\} \quad \{a, c\} \quad \{b, c\} \quad \{a, b, c\}$$

Notice that $A \subseteq A$ and in fact each set is a subset of itself. The empty set $\emptyset$ is a subset of every set.

Example 2.9 Subsets

If $A = \{a, b, c\}$ *then* [redacted] Powerset of A

$$P(A) = \left\{ \begin{matrix} \emptyset, & \{a\}, & \{b\}, & \{c\}, \\ \{a, b\}, & \{a, c\}, & \{b, c\}, & \{a, b, c\} \end{matrix} \right\}$$

Notice that $A \subseteq A$ *and in fact each set is a subset of itself. The empty set* $\emptyset$ *is a subset of every set.*

Set $S = \{1, 2\}$

Powerset of S = $P(S)$ = set of all subsets of S.

$P(S) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Definition 2.13 *The set of all subsets of a set S is called the* **powerset** *of S. The notation for the powerset of S is $\mathcal{P}(S)$.*

Given a set S, the *power set* of S is the set of all subsets of the set S. The power set of S is denoted by $\mathcal{P}(S)$.

Set S

$$P(S) = \{ A \mid A \subseteq S \}$$

$S = \{$ ,  $\}$

$|S| = 2$

$\wp(S) = \{ \emptyset, \{$  $\}, \{$  $\}, \{$ ,  $\} \}$

This is the ***power set*** of S, the set of all subsets of S. We write the power set of S as $\wp(S)$.

Formally, $\wp(S) = \{ T \mid T \subseteq S \}$.

(Do you see why?)

$|P(S)| = 4$

What is the power set of the set $\{0, 1, 2\}$?

Solution: The power set $\mathcal{P}(\{0, 1, 2\})$ is the set of all subsets of $\{0, 1, 2\}$. Hence,

$$\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}.$$

Note that the empty set and the set itself are members of this set of subsets. ◀

What is $\wp(\emptyset)$?

Set $\phi = \{ \}$ $\quad |\phi| = 0$

Subsets of ϕ : ϕ

$P(\phi) = \{ \phi \}$ $\quad |P(\phi)| = 1$

What is $\wp(\emptyset)$?

Answer: $\{\emptyset\}$

Remember that $\emptyset \neq \{\emptyset\}$!

Set $S = \{\phi\}$ → Subsets :

→ $|S| = 1$

$\{\ \}$ ✓

$\{\phi\}$ ✓

$P(S) = P(\{\phi\})$

$P(S) = \{\ \phi\ ,\ \{\phi\}\ \}$

$|P(S)| = 2$

What is the power set of the empty set? What is the power set of the set $\{\emptyset\}$?

Solution: The empty set has exactly one subset, namely, itself. Consequently,

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

The set $\{\emptyset\}$ has exactly two subsets, namely, $\emptyset$ and the set $\{\emptyset\}$ itself. Therefore,

$$\mathcal{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}.$$

If a set has n elements, then its power set has 2^n elements. We will demonstrate this fact in several ways in subsequent sections of the text.

Power set

Definition

The power set of set A is the set of all subsets of A. We denote it by $P(A)$.

Example:

- $P(\{1,2,3\}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
- $P(\emptyset) = \{\emptyset\}$.
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

Number of Subsets Possible :

Q : $S = \{a, b\}$

T I T I

Subsets of S = $\{\ \}$, $\{a\}$

$\{b\}$, $\{a, b\}$

subsets = 4

Take NOT Take

$S = \{a\} \longrightarrow$ #subset = 2

To create subset of S, Every element of S has two choices : Take it or NOT take it.

Subset of S : $\{a\}, \{\ \}$

$S = \{a, b, c\}$

T I T I T I

2 2 2

To create subset of S, Every element of S has two choices : Take it or NOT take it.

#subsets : $2 \times 2 \times 2 = 2^3 = 8$

Set S, $|S| = n$

$S = \{a_1, a_2, a_3, --, a_n\}$

$\phantom{S = \{} 2 \quad 2 \quad 2 \qquad\quad 2$

Number of subsets $= \underbrace{2 \times 2 \times -- \times 2}_{n \text{ times}} = 2^n$

$|P(S)| = \text{Number of subsets} = 2^n$

Note: for any set S,

$$|P(S)| = 2^{|S|}$$

Set $S \longrightarrow |P(S)| = 2^n$

$\downarrow$

n elements

Note: for any set S,

$$|P(S)| = 2^{|S|}$$

$$|P(P(S))| = 2^{\left(2^{|S|}\right)}$$

$S = \{a\}$ —— $|S| = n$

$P(S) = \{ \{a\}, \phi \}$ — $|P(S)| = 2^n$

$P(P(S)) = \{ \phi, \{\phi\}, \{\{a\}\}, \{\{a\}, \phi\} \}$

$|P(P(S))| = 2^{(2^n)}$

$$\underbrace{2^{2^n} = 2^{(2^n)}}_{\text{Same}} \neq \left(2^2\right)^n$$

$$\left(2^2\right)^n = \underbrace{\left(2^2\right)\left(2^2\right)\cdots\left(2^2\right)}_{n \text{ times}} = 2^{2+2+\cdots+2} = 2^{2n} = 4^n$$

$$\boxed{a^{b^c} = a^{(b^c)}}$$

Set Theory

Next Topic:

Set Operations

Universal Set, Venn Diagrams

Website : https://www.goclasses.in/

When performing set theoretic computations, you should declare the domain in which you are working. In set theory this is done by declaring a universal set.

Definition 2.8 *The* **universal set**, *at least for a given collection of set theoretic computations, is the set of all possible objects.*

If we declare our universal set to be the integers then $\{\frac{1}{2}, \frac{2}{3}\}$ is not a well defined set because the objects used to define it are not members of the universal set. The symbols $\{\frac{1}{2}, \frac{2}{3}\}$ do define a set if a universal set that includes $\frac{1}{2}$ and $\frac{2}{3}$ is chosen. The problem arises from the fact that neither of these numbers are integers. The universal set is commonly written $\mathcal{U}$. Now that we have the idea of declaring a universal set we can define another operation on sets.

Universal Set and Empty Set

- The universal set U is the set containing everything currently under consideration.
  - Content depends on the context.
  - Sometimes explicitly stated, sometimes implicit.
- The empty set is the set with no elements. Symbolized by $\emptyset$ or $\{\}$.

U — Contains EVERYTHING Under Consideration

ϕ — Contains Nothing.

for Discussions of Vowels :

Possible universal sets :

1. $\{a, e, i, o, u\}$
2. English Alphabet
3. All Alphabets of all languages
4. ~~$\{a, e, i, o\}$~~
5. ~~$\{a, e, i, o, u, 1, 2, \alpha\}$~~

Venn Diagrams

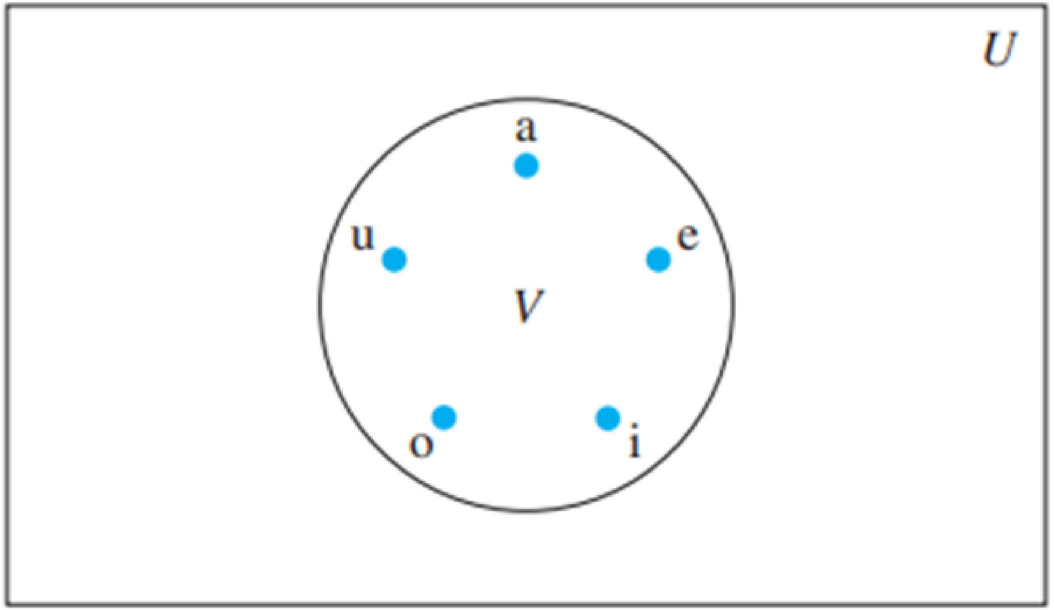
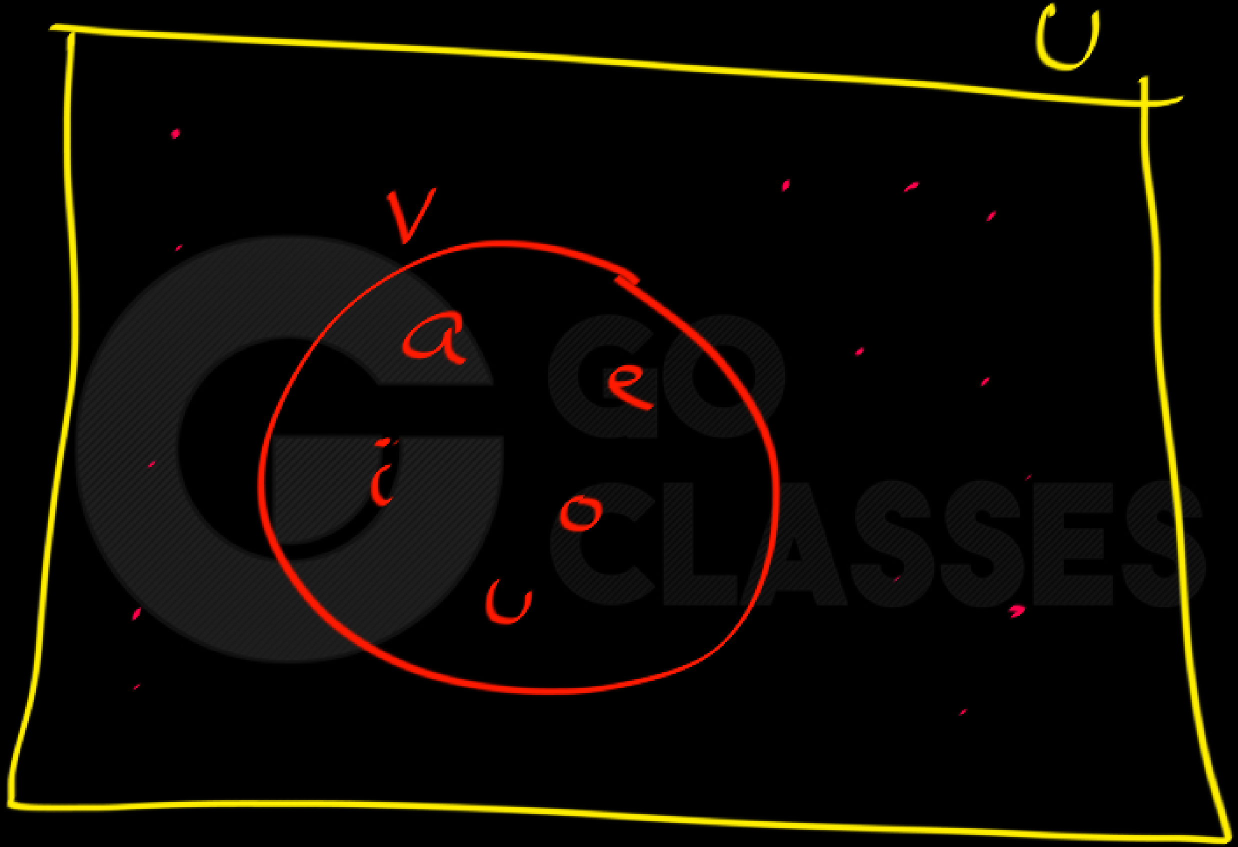


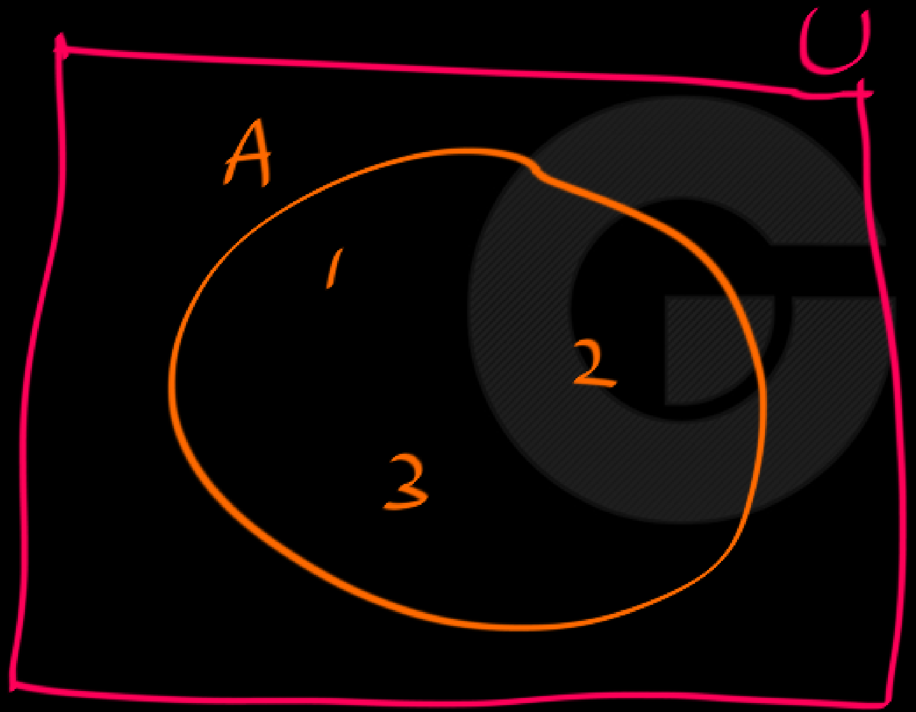
FIGURE 1 Venn Diagram for the Set of Vowels.


U
V
a
e
i
o
u

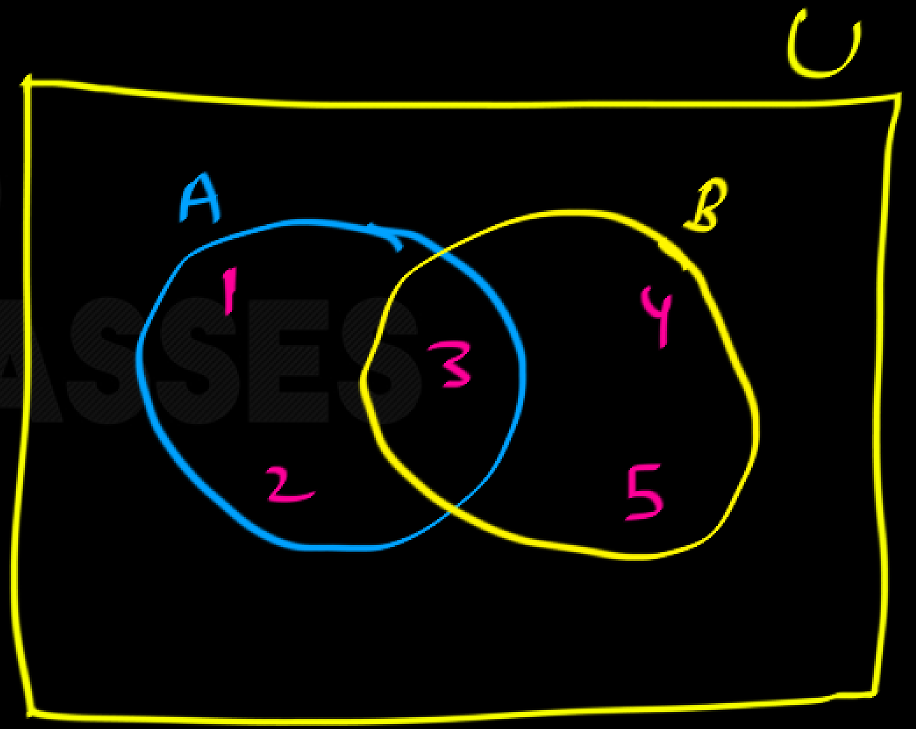
- A={Orange, Apple, Banana} is a set containing the names of three fruits.
- B={Red, Blue, Black, White, Grey} is a set containing five colors.
- $\{x \mid x$ takes CSE191 at UB in Spring 2014$\}$ is a set of 220 students.
- {N,Z,Q,R} is a set containing four sets.
- $\{x \mid x \in \{1, 2, 3\}$ and $x > 1\ \}$ is a set of two numbers.

Note: When discussing sets, there is a universal set U involved, which contains all objects under consideration. For example: for A, the universal set might be the set of names of all fruits. for B, the universal set might be the set of all colors.
In many cases, the universal set is implicit and omitted from discussion. In some cases, we have to make the universal set explicit.

$A = \{1, 2, 3\}$



$A = \{\underbrace{1, 2, 3}\}$, $B = \{\underbrace{3, 4, 5}\}$



$A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$

$D = \{1, 5, 8\}$

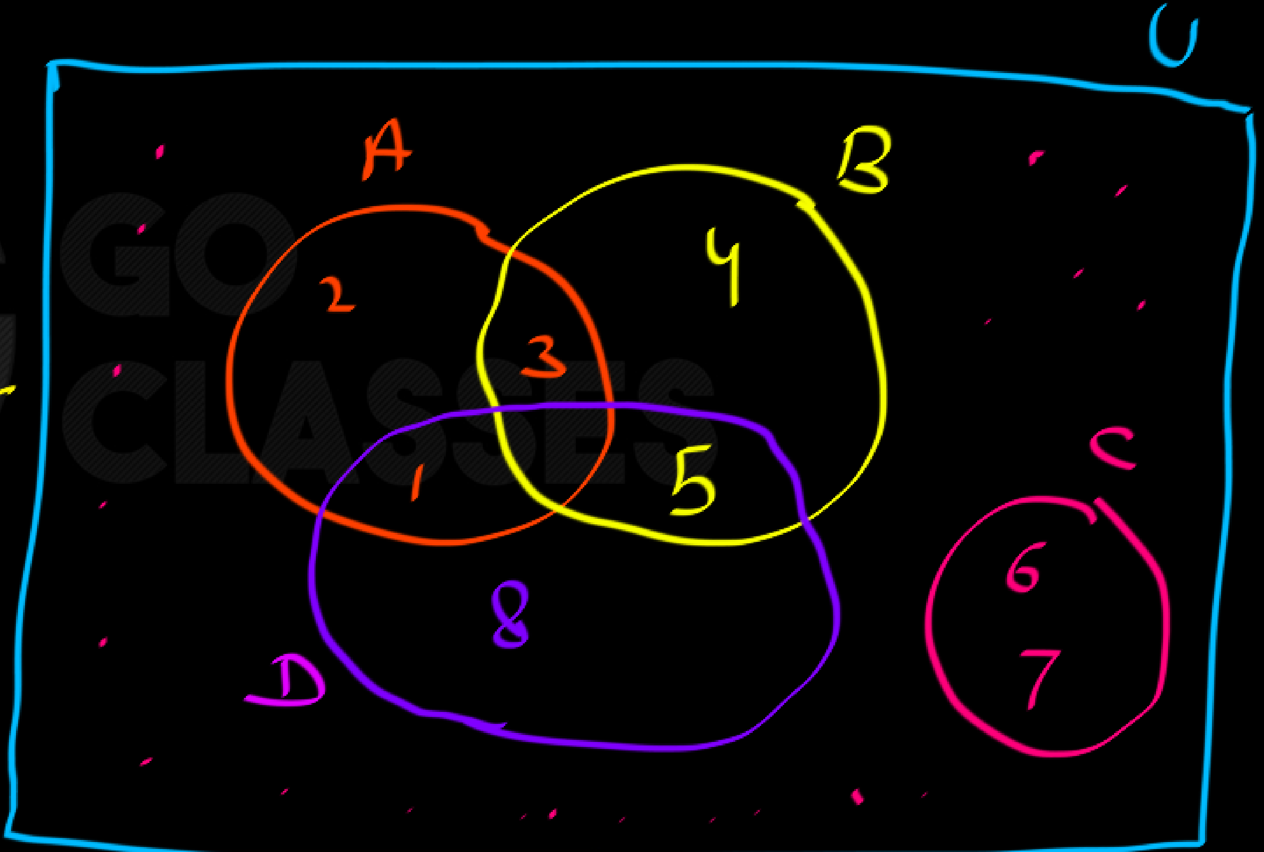
Possible universal set:

$\{1, 2, 3, 4, 5, 6, 7, 8\}$ ✓

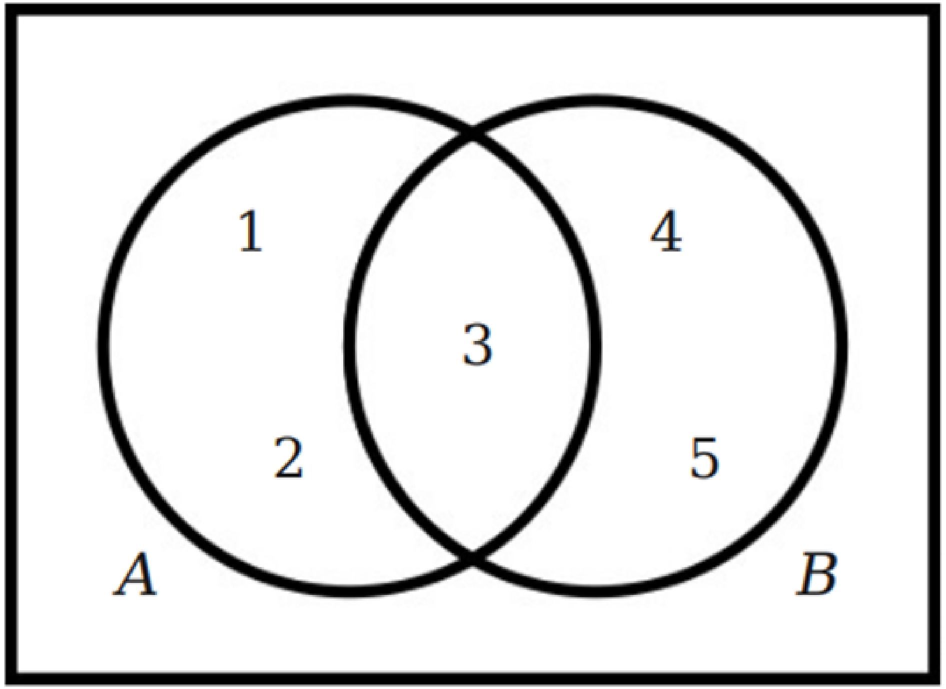
N ✓ Q ✓

Z ✓

R ✓ Even Numbers ✗

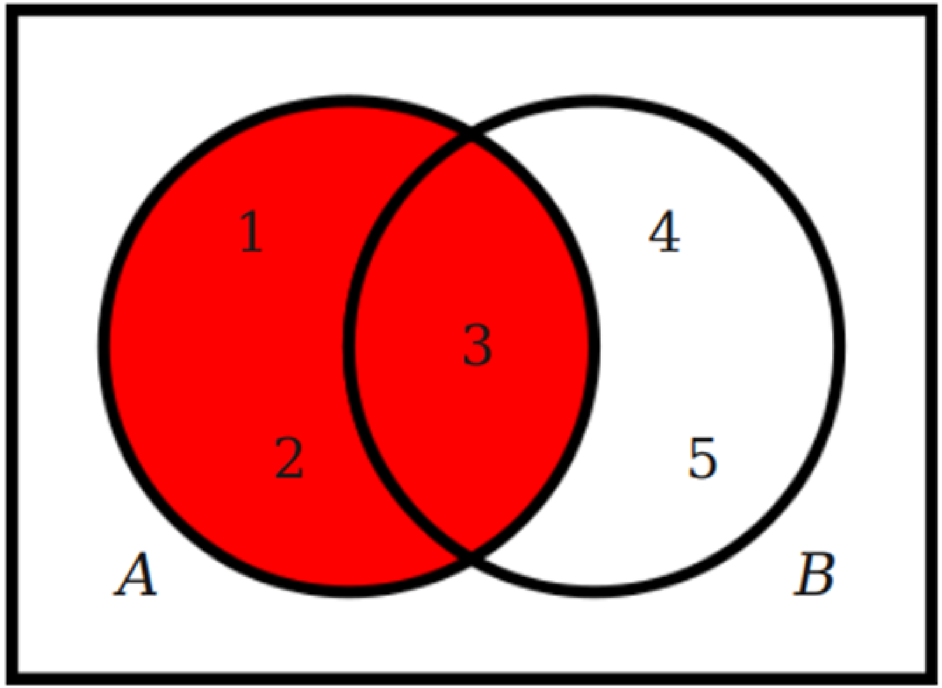


Venn Diagrams



$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

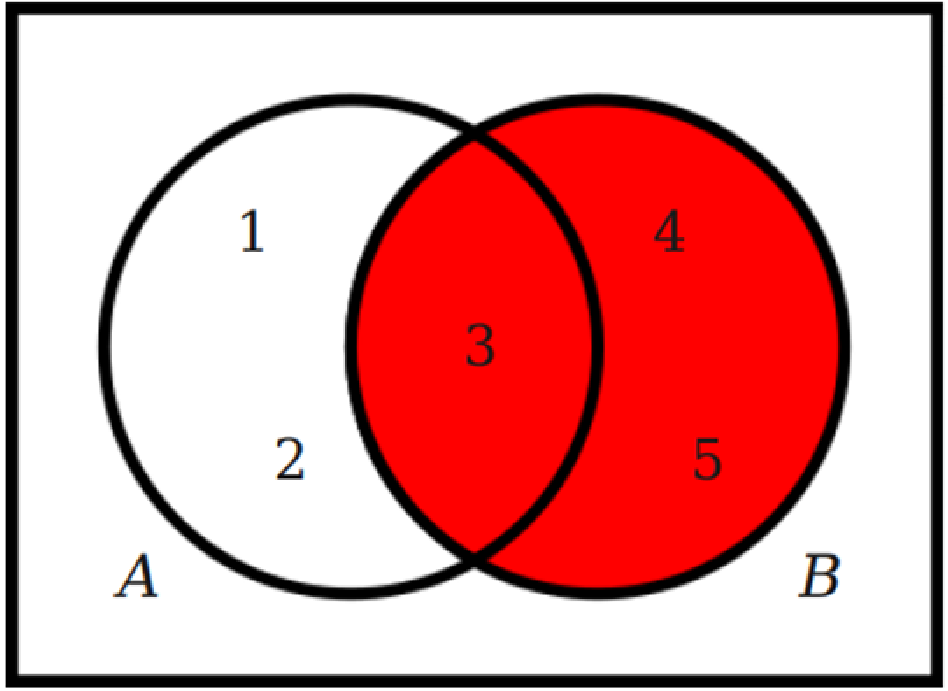
Venn Diagrams



A

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



B

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Set operations

Recall:

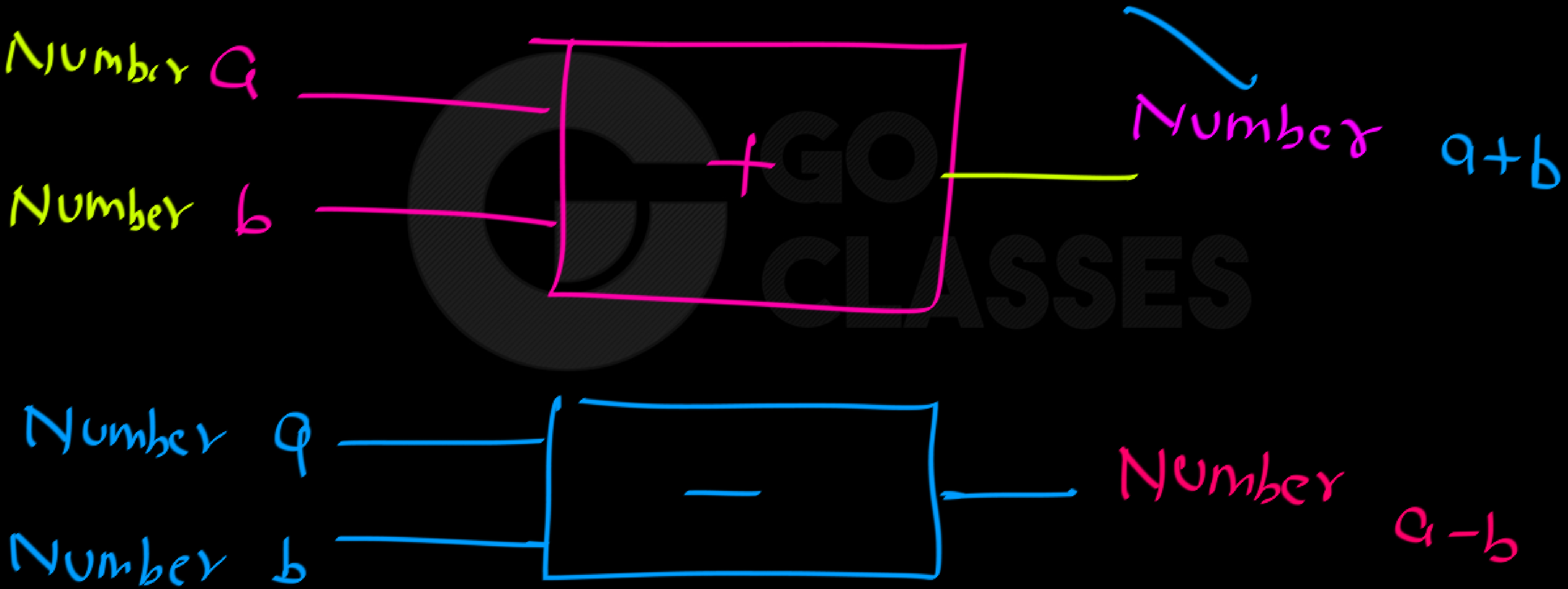
- We have $+, -, \times, \div, \ldots$ operators for numbers.
- We have $\vee, \wedge, \neg, \rightarrow \ldots$ operators for propositions.

Question:

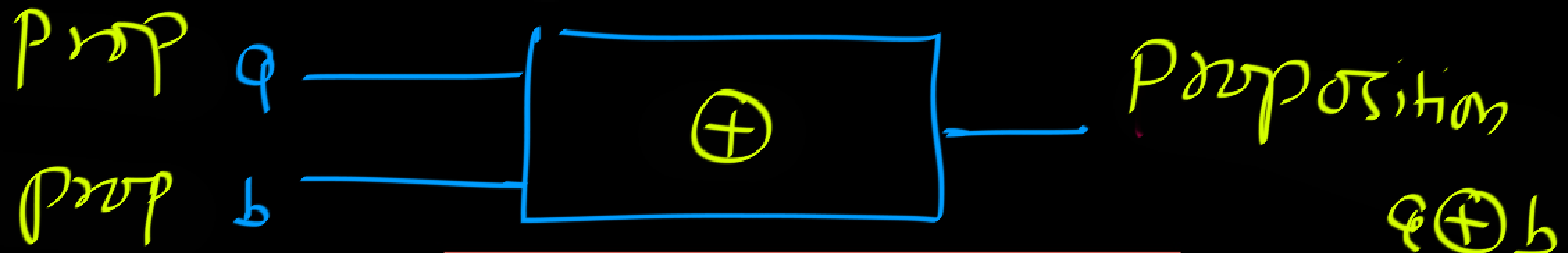
What kind of operations do we have for sets?

Answer: union, intersection, difference, complement, ...

On Numbers :



On Propositions:



Set operation :

Set A

Set B

Set

A # B

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

Intersection ≡ Common elements

$M \cap K$: the set of students who love Mangoes and Kiwis.

$$M \cap K = \{x \mid (x \in M) \land (x \in K)\}$$

intersection

and

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

union ≡ merge all elements

M U K: the set of students who love Mangoes or Kiwis (or both).

$$M \cup K = \{ x \mid (x \in M) \vee (x \in K) \}$$

OR

OR ≡ inclusive OR

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

sym. Difference ≡ Exclusively in one set.

$K \Delta M$: the set of students who love Kiwis Or Mangoes But Not Both.

$$K \Delta M = K \oplus M = \{ x \mid (x \in M) \oplus (x \in K) \}$$

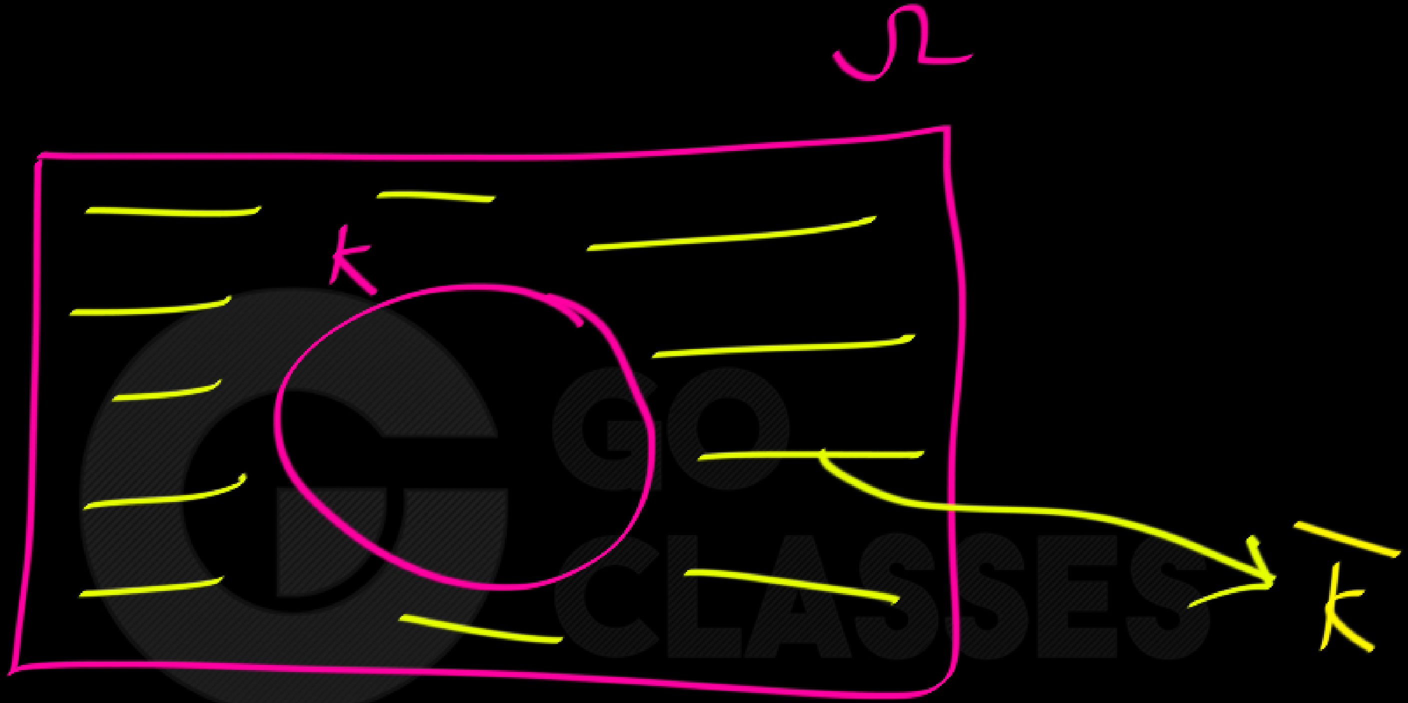
Symmetric Difference

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

$\overline{M}$: the set of students who don't love Mangoes

Complement of M

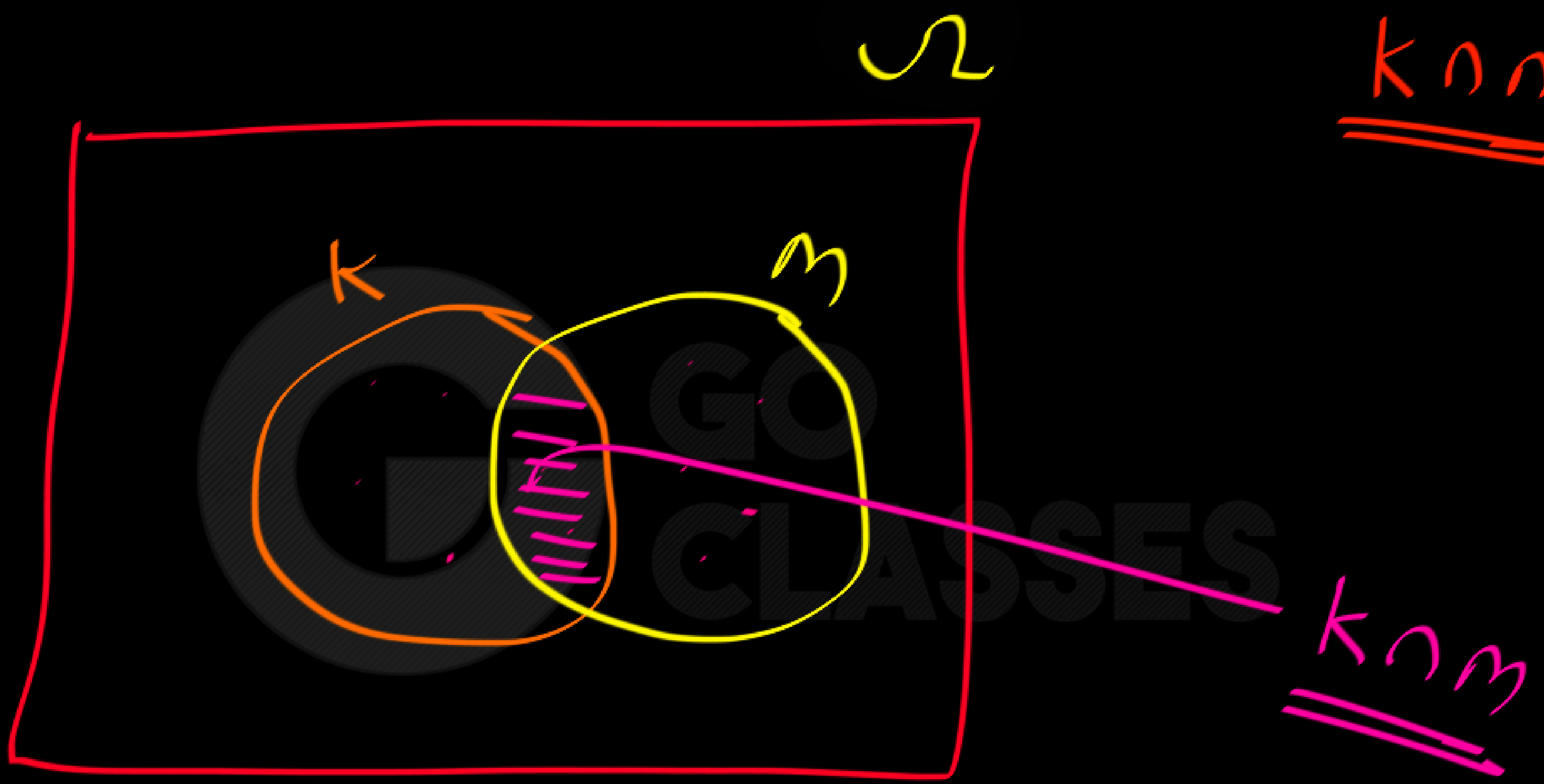
$\overline{M} = \{ x \mid x \notin M \}$


Ω
K
$\overline{K}$

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

$\overline{K}$: the set of students who don't love Kiwis.

$$\overline{K} = \{ x \mid x \notin K \}$$



$K \cap M$

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

Set Difference

M – K: the set of students who love Mangoes But Not Kiwis.

$$M - K = \{ x \mid x \in M \text{ but } x \notin K \}$$

Suppose M is the set of students who love Mangoes, and K is the set of students who love Kiwis.

K – M: the set of students who love Kiwis But Not Mangoes.

$$K - M = \{x \mid x \in K, x \notin M\}$$

- The *union* of sets A and B is the set $A \cup B = \{x : x \in A \vee x \in B\}$.
- The *intersection* of sets A and B is the set $A \cap B = \{x : x \in A \wedge x \in B\}$.
- The *set difference of A and B* is the set $A \setminus B = \{x : x \in A \wedge x \notin B\}$. Alternate notation: $A - B$.
- The *symmetric difference of A and B* is $A \oplus B = (A \setminus B) \cup (B \setminus A)$. Note: $A \oplus B = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$.

The *universe*, $\mathcal{U}$, is the collection of all objects that can occur as elements of the sets under consideration.

- The *complement of* A is $A^c = \mathcal{U} \setminus A = \{x : x \notin A\}$.

Definition 2.7 *The* **union** *of two sets S and T is the collection of all objects that are in either set. It is written $S \cup T$. Using curly brace notion*

$$S \cup T = \{x : (x \in S) \text{ or } (x \in T)\}$$

The symbol *or* is another Boolean operation, one that is true if either of the propositions it joins are true. Its symbolic equivalent is $\vee$ which lets us re-write the definition of union as:

$$S \cup T = \{x : (x \in S) \vee (x \in T)\}$$

Set Union

GO Classes

Definition:

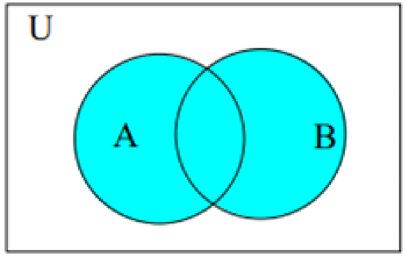
The union of two sets A and B is the set that contains exactly all the elements that are in either A or B (or in both).

- We write $A \cup B$.
- Formally, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Example:

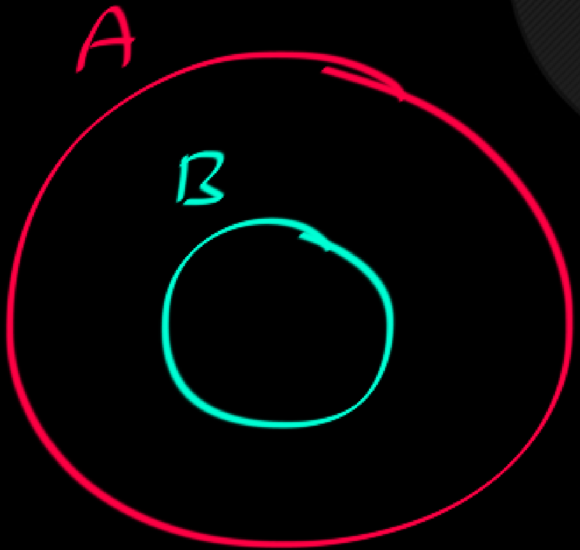
- $\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}$
- $\{x \mid x > 0\} \cup \{x \mid x > 1\} = \{x \mid x > 0\}$

Venn Diagram of Union Operation:



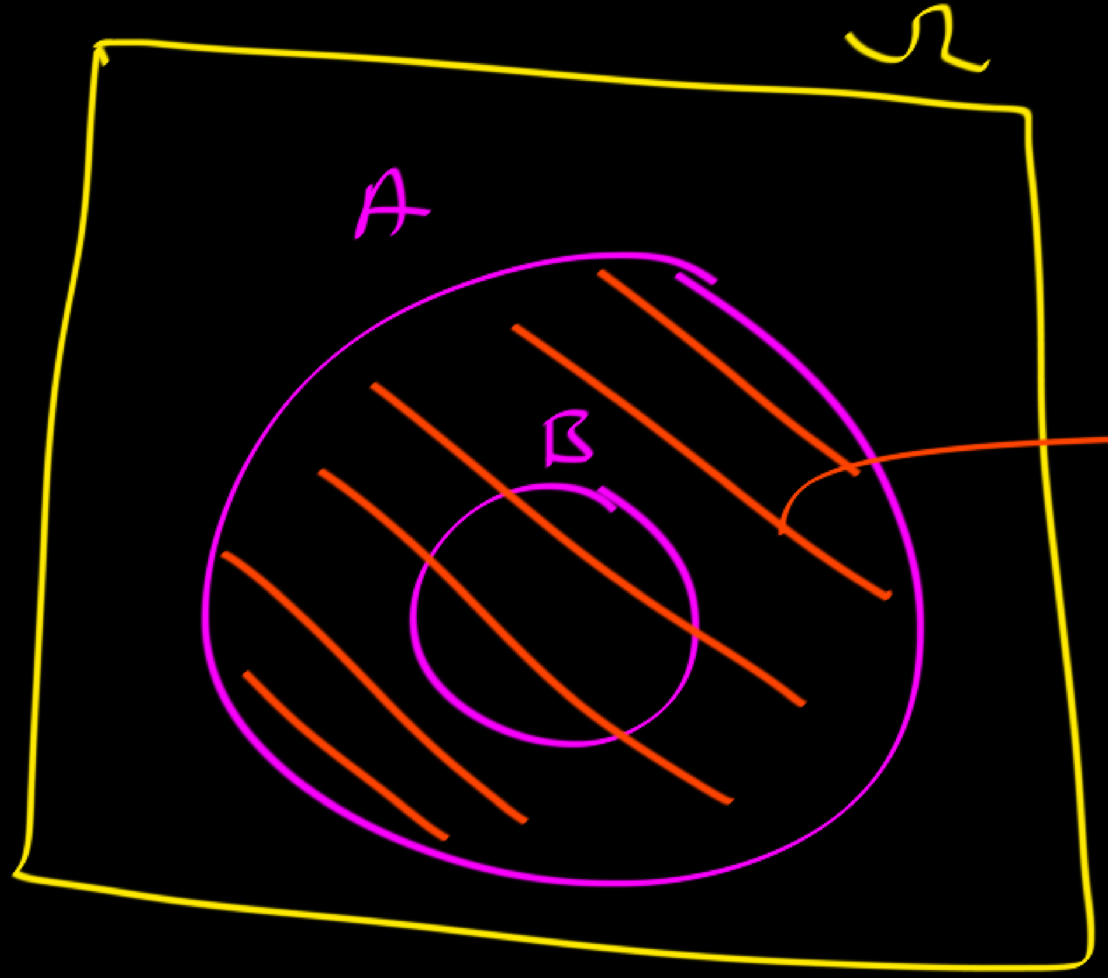
$A = \{x \mid x > 0\}$ $\quad$ $B = \{x \mid x > 1\}$

$A \cup B = \{x \mid x > 0\} = A$



Every element of B is in A

$B \subseteq A$



when $B \subseteq A$
then

$\underline{A \cup B} = A$

Example 2.4 Unions of sets.

Suppose $S = \{1,2,3\}$, $T = \{1,3,5\}$, *and* $U = \{2,3,4,5\}$.

Then:

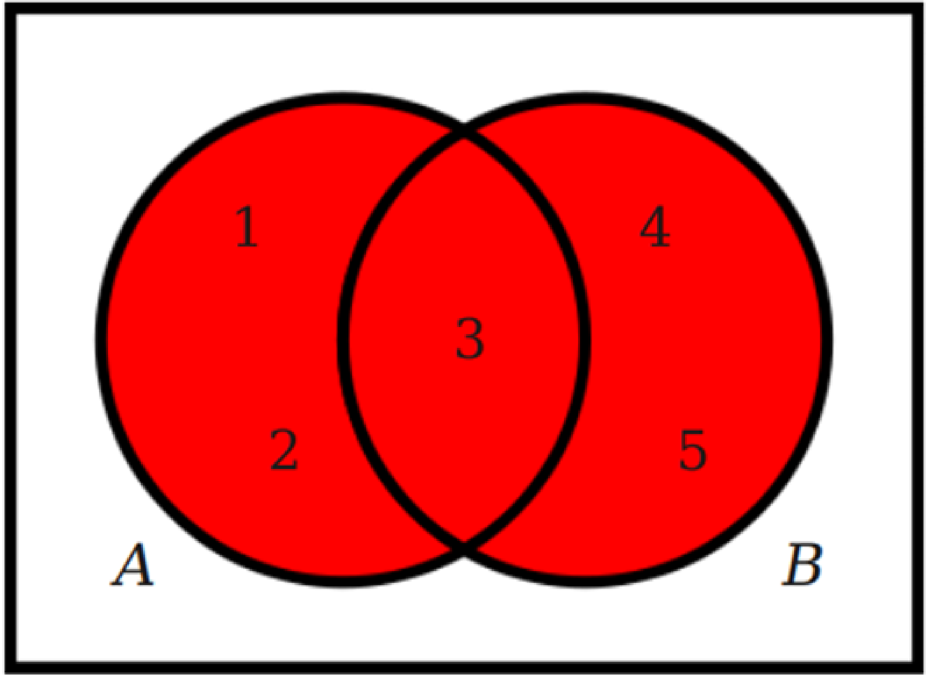
$S \cup T = \{1,2,3,5\}$,

$S \cup U = \{1,2,3,4,5\}$, *and*

$T \cup U = \{1,2,3,4,5\}$

$S \cup T \cup U =$
$\{1, 2, 3, 5, 4\}$

Venn Diagrams



Union

$A \cup B$

$\{ 1, 2, 3, 4, 5 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

$$A \cup B \cup C \cup D = \left\{ x \;\middle|\; \begin{array}{l} (x \in A) \text{ OR } (x \in B) \\ \text{OR } (x \in C) \text{ OR } (x \in D) \end{array} \right\}$$

Definition 2.5 *The* **intersection** *of two sets S and T is the collection of all objects that are in both sets. It is written $S \cap T$. Using curly brace notation*

$$S \cap T = \{x : (x \in S) \text{ and } (x \in T)\}$$

The symbol *and* in the above definition is an example of a Boolean or logical operation. It is only true when both the propositions it joins are also true. It has a symbolic equivalent $\wedge$. This lets us write the formal definition of intersection more compactly:

$$S \cap T = \{x : (x \in S) \wedge (x \in T)\}$$

Set intersection

Definition:

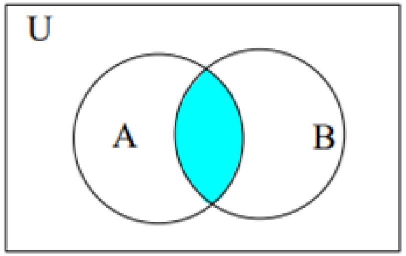
The intersection of two sets A and B is the set that contains exactly all the elements that are in both A and B.

- We write $A \cap B$.
- Formally, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Example:

- $\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$
- $\{x \mid x > 0\} \cap \{x \mid x > 1\} = \{x \mid x > 1\}$

Venn Diagram of Intersection Operation:



Example 2.3 Intersections of sets

Suppose $S = \{1, 2, 3, 5\}$,
$T = \{1, 3, 4, 5\}$, *and* $U = \{2, 3, 4, 5\}$.
Then:

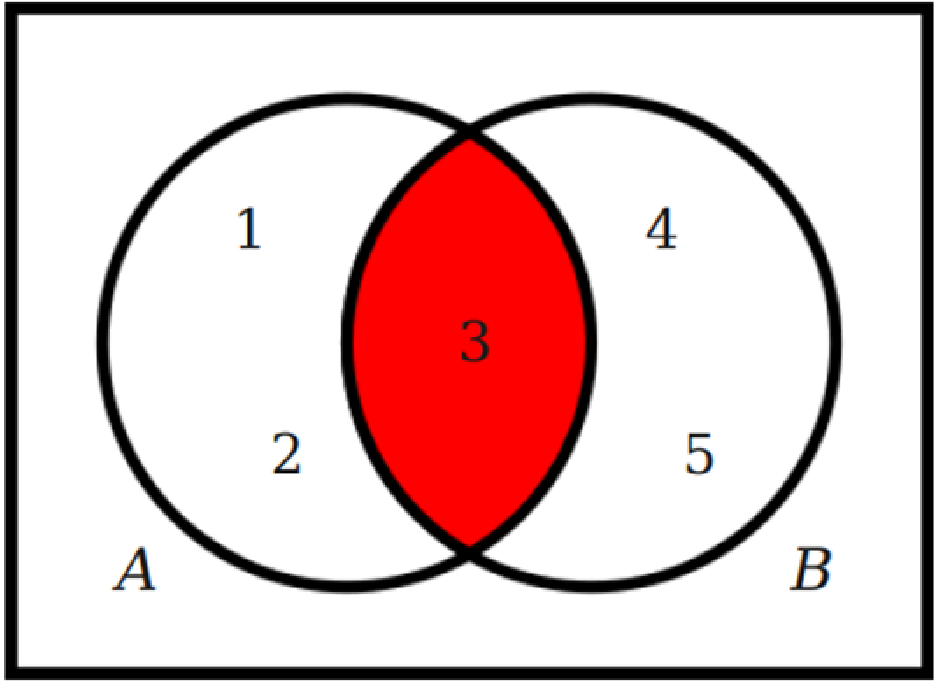
$$S \cap T = \{1, 3, 5\},$$

$$S \cap U = \{2, 3, 5\}, \text{ and}$$

$$T \cap U = \{3, 4, 5\}$$

$S \cap T \cap U = \{3, 5\}$

Venn Diagrams



Intersection
$A \cap B$
$\{ 3 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

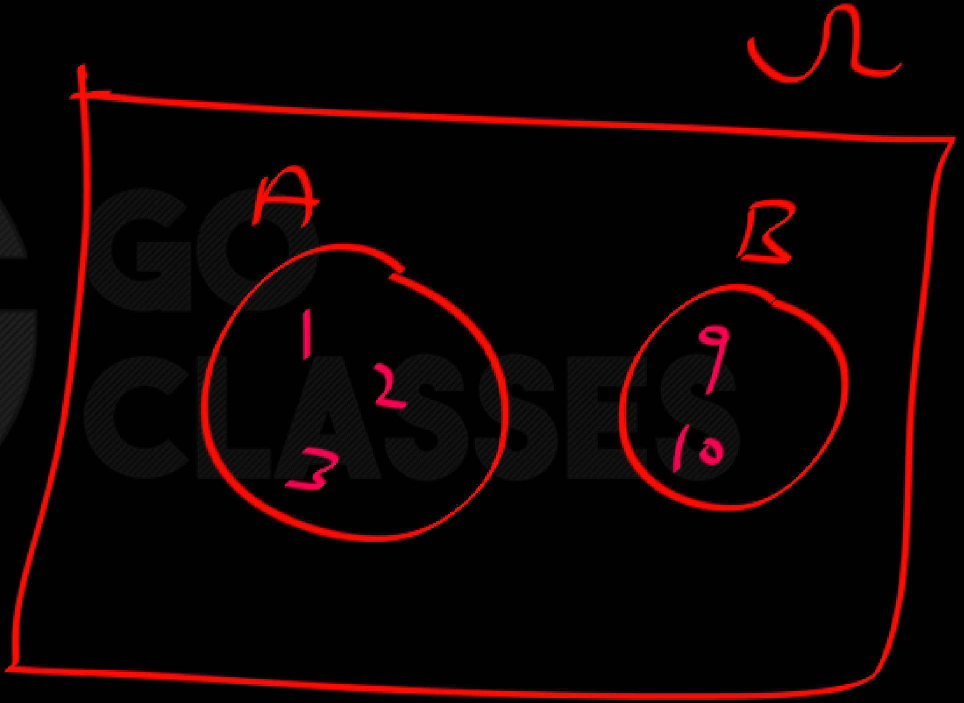
$$\underbrace{A \cap B \cap C}_{\text{Common in all of them.}} = \left\{ x \;\middle|\; \begin{array}{l} (x \in A) \text{ AND } (x \in B) \\ \text{AND } (x \in C) \end{array} \right\}$$

$A = \{1, 2, 3\}$

$B = \{9, 10\}$

nothing Common

Disjoint Sets



Definition 2.6 *If A and B are sets and $A \cap B = \emptyset$ then we say that A and B are* disjoint, *or* disjoint sets.

If $A \cap B = \phi$ then

A, B are Disjoint sets.

Disjoint set

Definition:

Two sets A and B are disjoint if $A \cap B = \emptyset$.

Example:

- $\{1, 2, 3\} \cap \{4, 5\} = \emptyset$, so they are disjoint.
- $\{1, 2, 3\} \cap \{3, 4, 5\} \neq \emptyset$, so they are not disjoint.
- $Q \cap R^{+} \neq \emptyset$, so they are not disjoint.
- $\{x \mid x < -2\} \cap R^{+} = \emptyset$, so they are disjoint.

$A = \{a\}$

$B = \phi$

$\Big\}$ Disjoint $\rightarrow$ Definition

$A \cap B = \phi$

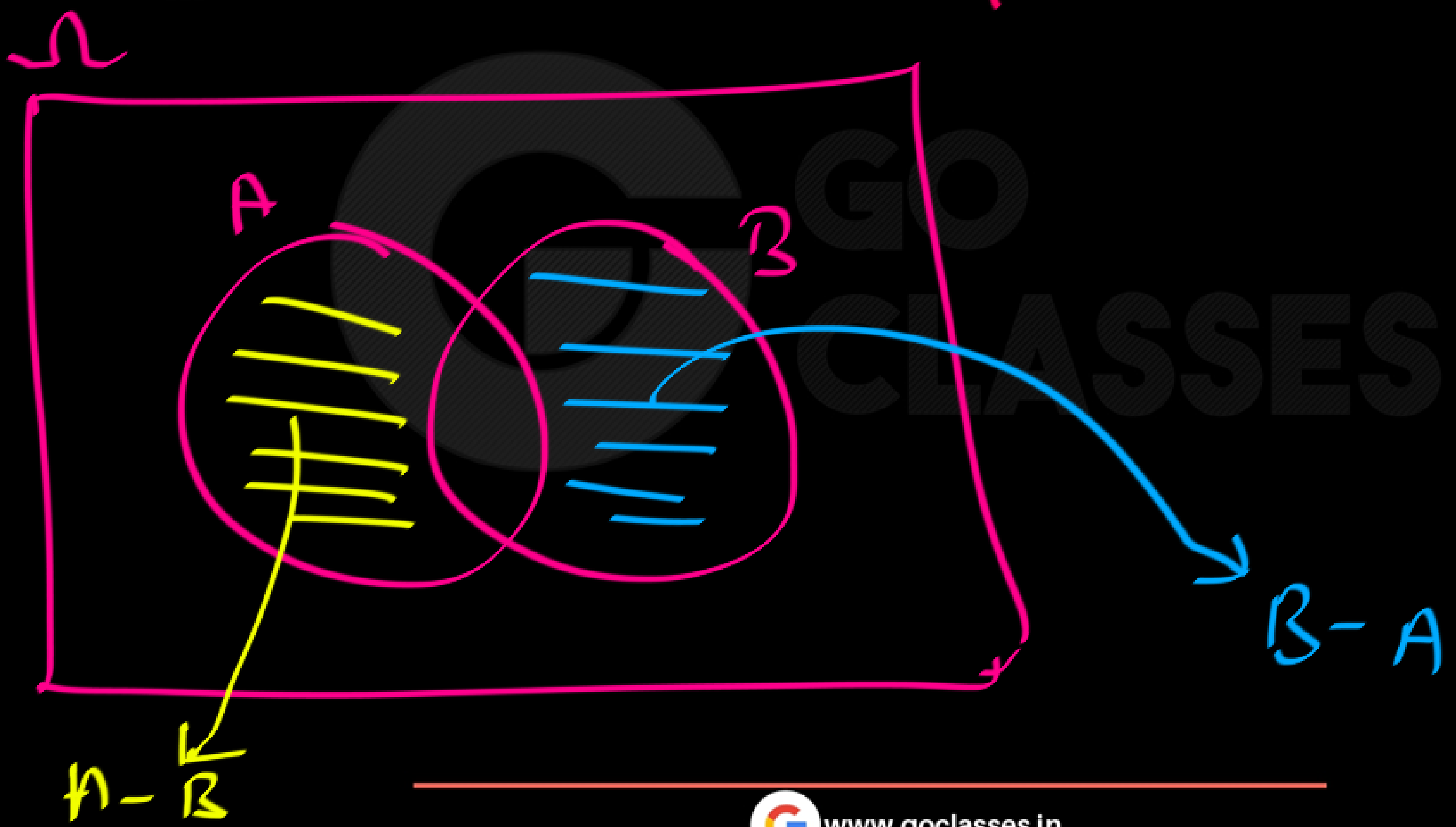
$A = \phi$

$B = \phi$

$\Big\}$ Disjoint $\rightarrow$ $A \cap B = \phi$

So Disjoint

$$A - B = A \setminus B = \{ x \mid x \in A, x \notin B \}$$



Set difference

Definition:

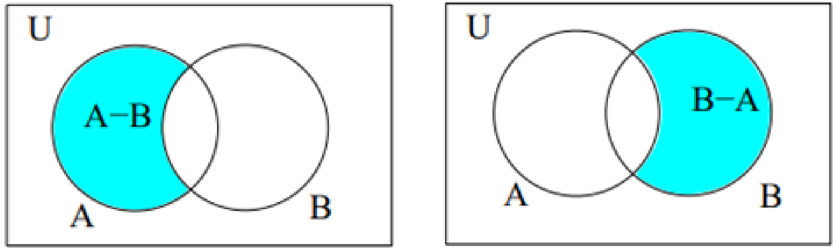
The difference of set A and set B, denoted by $A - B$, is the set that contains exactly all elements in A but not in B.

- Formally, $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$.

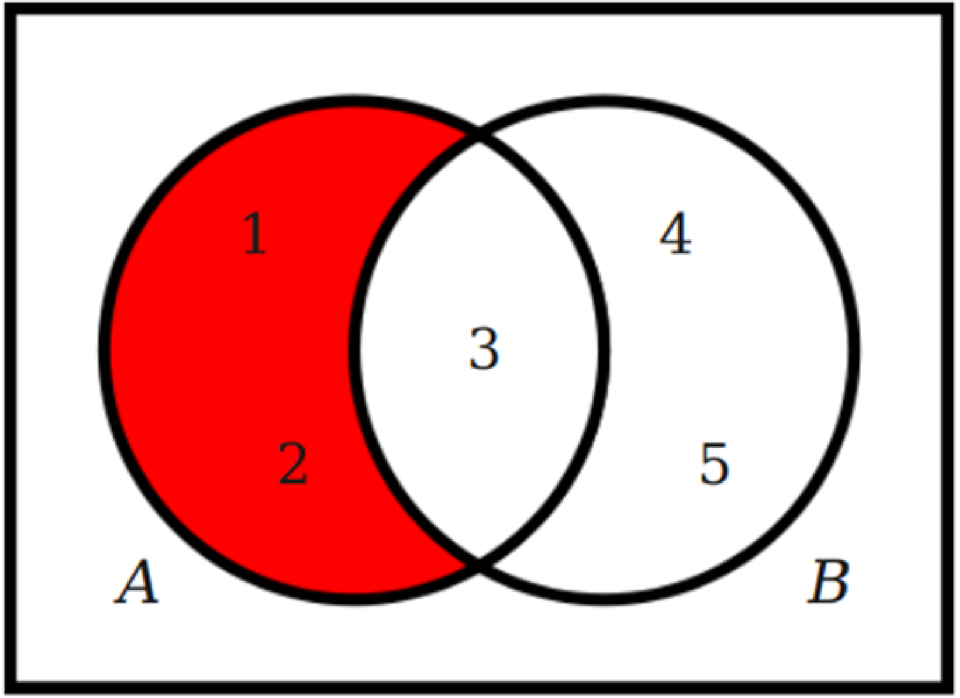
Example:

- $\{1,2,3\} - \{3,4,5\} = \{1,2\}$
- $R - \{0\} = \{x \mid x \in R \wedge x \neq 0\}$
- $Z - \{2/3, 1/4, 5/8\} = Z$

Venn Diagram of Difference Operation:



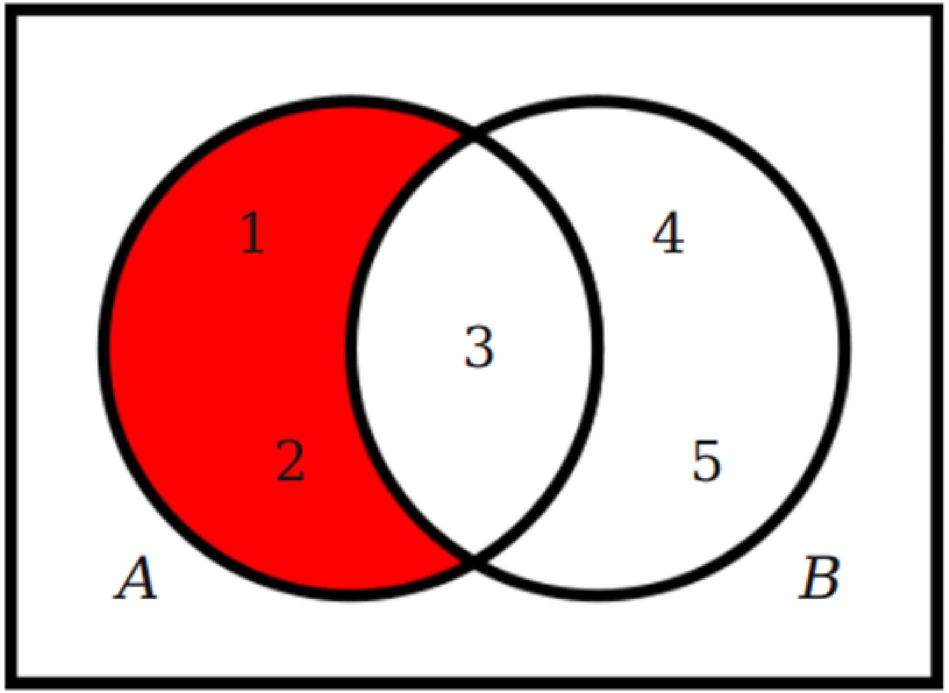
Venn Diagrams



Difference
$A - B$
$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

Venn Diagrams



Difference

$A \setminus B$

$\{ 1, 2 \}$

$A = \{ 1, 2, 3 \}$
$B = \{ 3, 4, 5 \}$

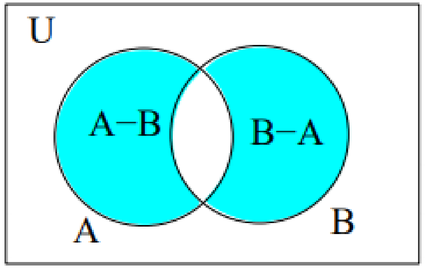
Symmetric Difference

Definition:

The symmetric difference of set A and set B, denoted by $A \oplus B$, is the set containing those elements in exactly one of A and B.

- Formally: $A \oplus B = (A - B) \cup (B - A)$.

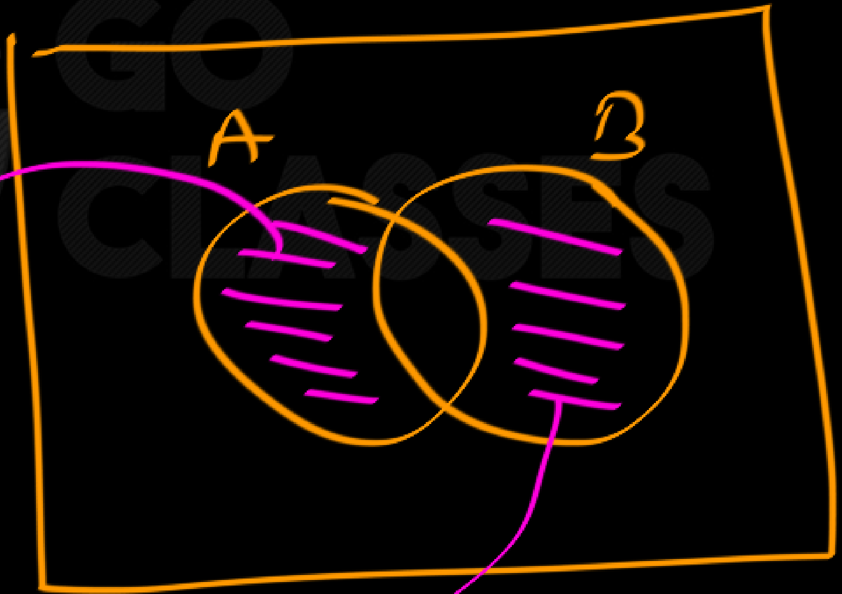
Venn Diagram of Symmetric Difference Operation:



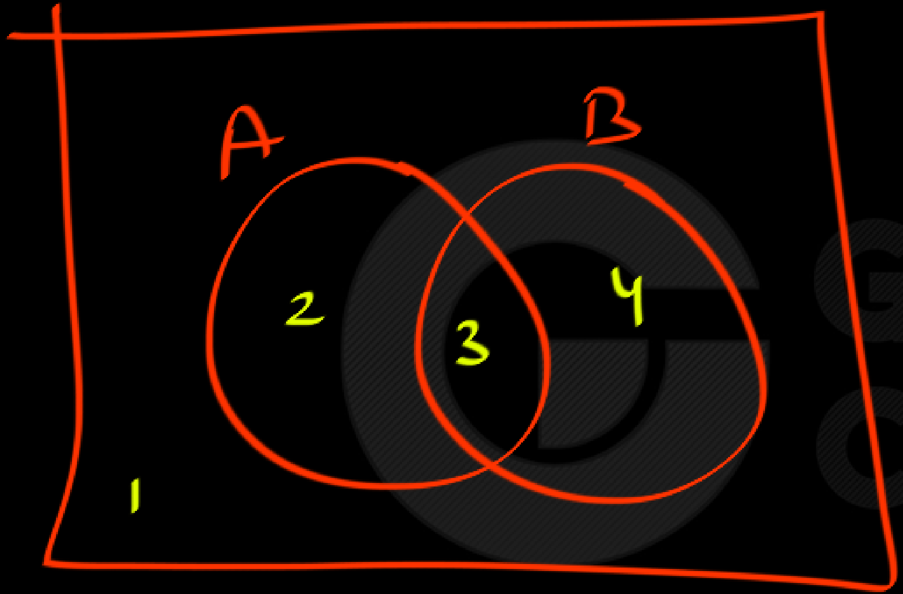
$\{1,2,3\} \,\Delta\, \{3,4,5\} = \{1,2,4,5\}$

$\oplus$

$A \,\Delta\, B$



$A \,\Delta\, B = (A-B) \cup (B-A)$



A — $2, 3$

B — $3, 4$

$A \cap B$ — 3

$A \cup B$ — $2, 3, 4$

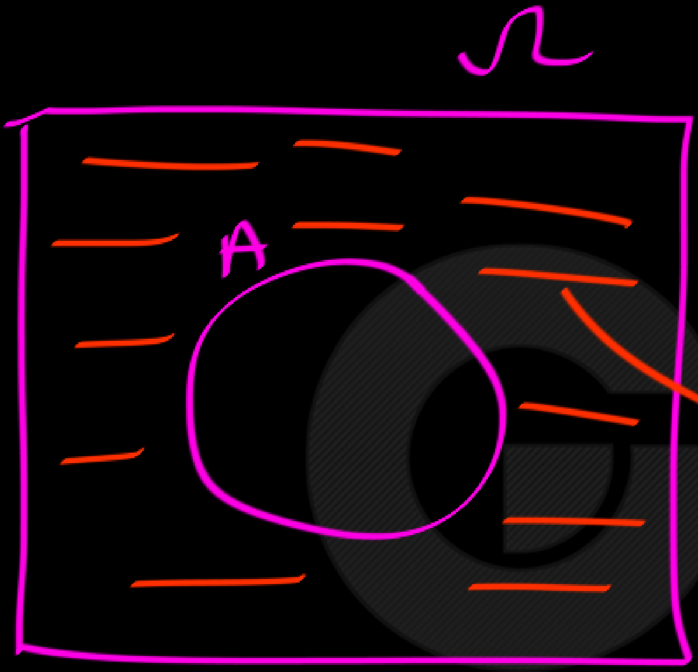
$A - B$ — 2

$B - A$ — 4

$A \Delta B$ — $2, 4$

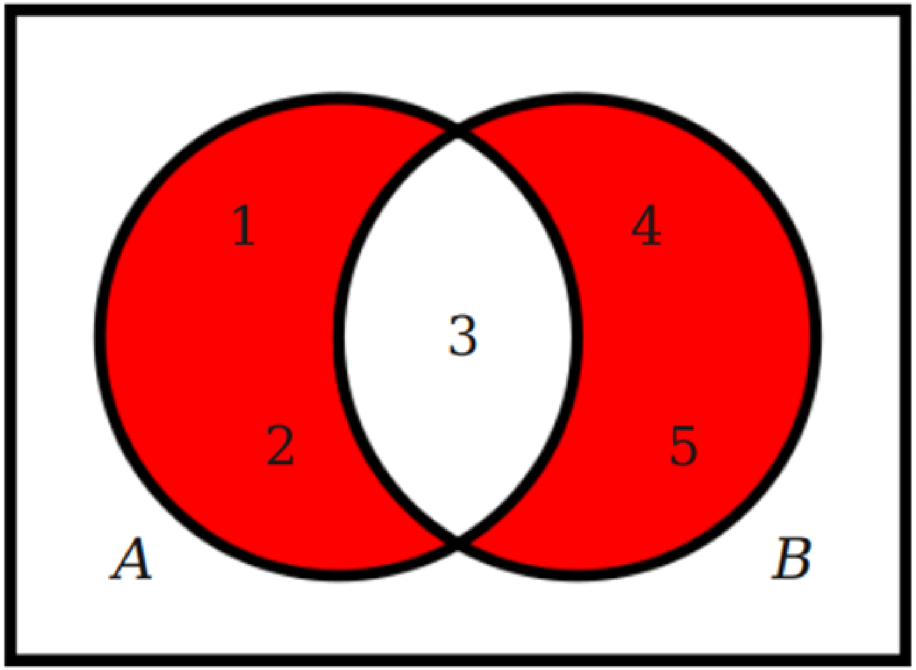
$\overline{A}$ — $1, 4$

$\overline{B}$ — $2, 1$



$$\overline{A} = \Omega - A = \Omega \setminus A$$

Venn Diagrams



Symmetric Difference

$A \,\Delta\, B$

$\{\ 1, 2, 4, 5\ \}$

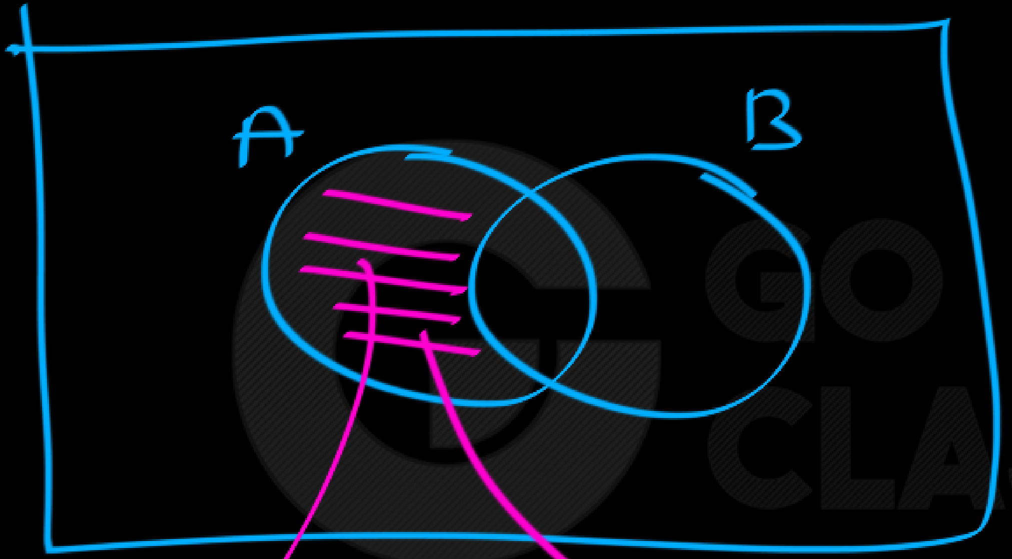
$A = \{\ 1, 2, 3\ \}$
$B = \{\ 3, 4, 5\ \}$

$$A \triangle B = (A - B) \cup (B - A)$$ ✓

$$A \triangle B = (A \cup B) - (A \cap B)$$ ✓

$$\boxed{A - B = A \cap \overline{B}}$$

In DBMS, Extremely Imp.



$$\underbrace{A \cap \overline{B}}$$

in A and not in B

$$A \cap \overline{B} = A - B$$

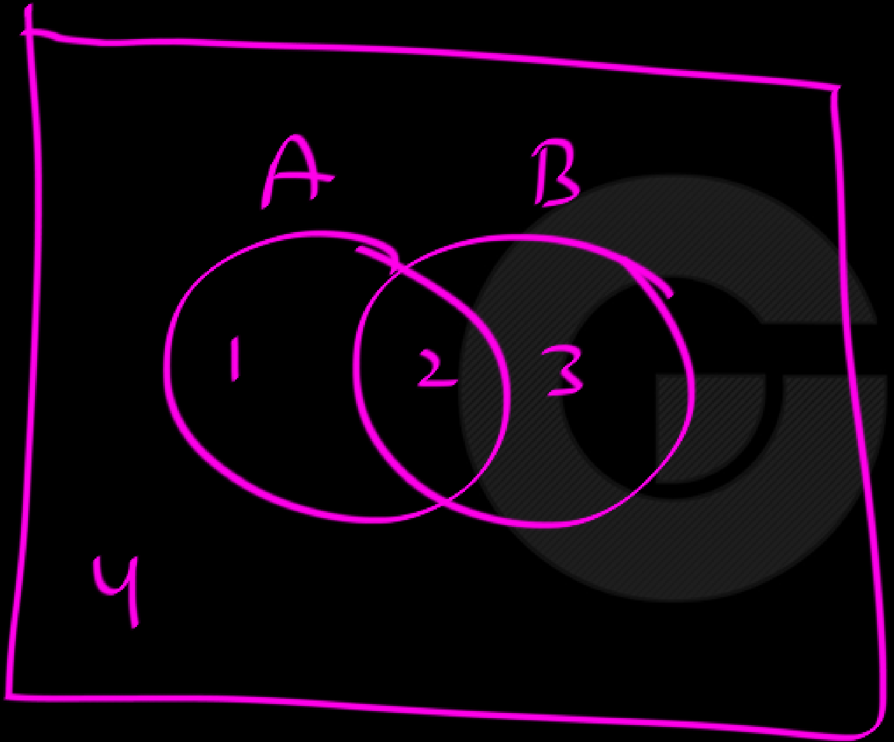
Quick;

① $x \in A \cap B$ means $x \in A$ AND $x \in B$

② $x \in A \cup B$ means $x \in A$ OR $x \in B$ OR both

③ $x \in \overline{A} \cap B$ means $x \in B$ AND $x \notin A$

④ $x \in \overline{A} \cup B$ means $x \in B$ OR $x \notin A$ OR both



$(\overline{A} \cap B)$ — 3 — $(B - A)$

$\overline{A} \cup B$ — $\underbrace{2, 3}_{B}$, 4

$\overline{(A - B)}$

$\overline{A} - B$ — 4

$\overline{A \cup B}$

Definition 2.9 *The* **compliment** *of a set S is the collection of objects in the universal set that are not in S. The compliment is written S^c. In curly brace notation*

$$S^c = \{x : (x \in \mathcal{U}) \wedge (x \notin S)\}$$

or more compactly as

$$S^c = \{x : x \notin S\}$$

however it should be apparent that the compliment of a set always depends on which universal set is chosen.

Set complement

GO Classes

Definition:

The complement of set A, denoted by $\overline{A}$, is the set that contains exactly all the elements that are not in A.

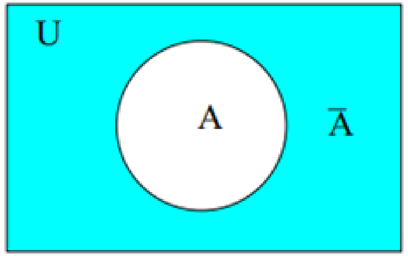
- Formally, $\overline{A} = \{x \mid x \notin A\}$.
- Suppose U is the universe. Then, $\overline{A} = U - A$.

Example:

Let the universe be R.

- $\overline{\{0\}} = \{x \mid x \neq 0 \wedge x \in R\}$ $\quad\quad$ $\overline{R^+} = \{x \mid x \leq 0 \wedge x \in R\}$.

Venn Diagram of Complement Operation:



$$\overline{A} = \{ x \mid x \notin A \}$$ ✓

$$\overline{A} = \{ x \mid x \notin A,\ x \in \Omega \}$$ ✓

$$\boxed{\overline{A} = \Omega - A}$$

Obvious (implicit)

universal set

Example 2.6 Set Compliments

$\overline{A} = \Omega - A = \Omega \setminus A$

(i) Let the universal set be the integers. Then the compliment of the even integers is the odd integers.

(ii) Let the universal set be $\{1, 2, 3, 4, 5\}$*, then the compliment of* $S = \{1, 2, 3\}$ *is* $S^c = \{4, 5\}$ *while the compliment of* $T = \{1, 3, 5\}$ *is* $T^c = \{2, 4\}$.

(iii) Let the universal set be the letters $\{a, e, i, o, u, y\}$. *Then* $\{y\}^c = \{a, e, i, o, u\}$.

Example for calculating set operations

Example:

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then:

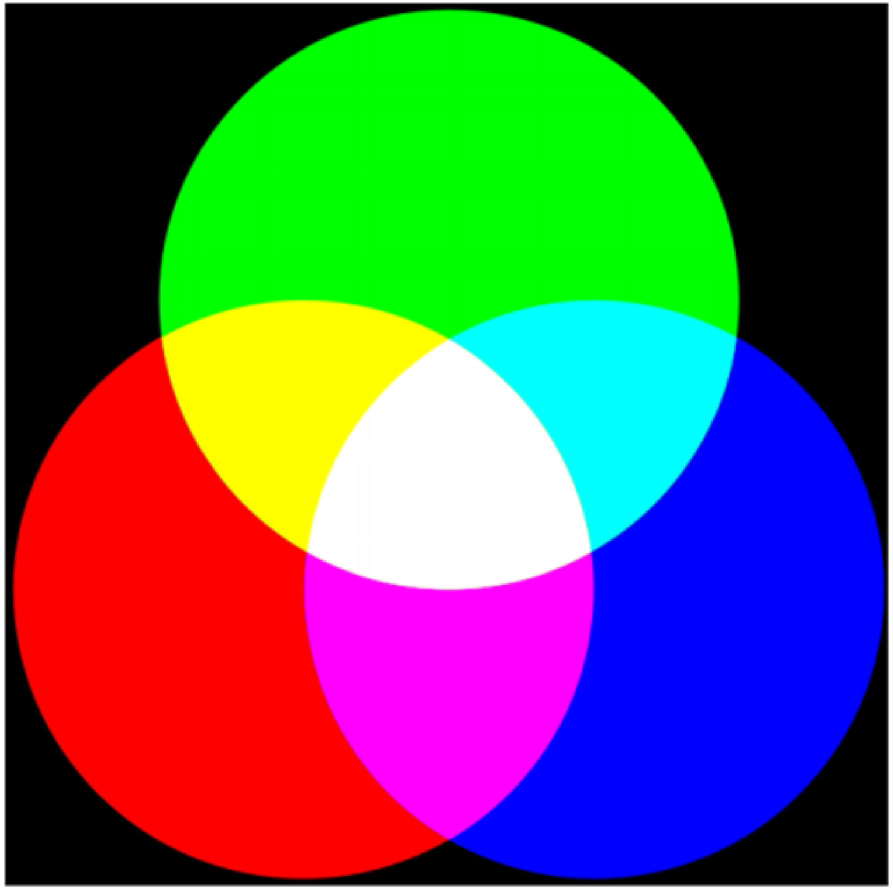
- $A \cap B = \{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $A - B = \emptyset$
- $B - A = \{6, 7, 8\}$

Example for set operations

Suppose A is the set of students who loves CSE 191, and B is the set of students who live in the university dorm.

- $A \cap B$: the set of students who love CSE 191 and live in the university dorm.
- $A \cup B$: the set of students who love CSE 191 or live in the university dorm.
- $A - B$: the set of students who love CSE 191 but do not live in the university dorm.
- $B - A$: the set of students who live in the university dorm but do not love CSE 191.

Venn Diagrams for Three Sets



Set Theory

Next Topic:

(Equal sets)Set Equality

Set Identities

Website : https://www.goclasses.in/

Definition:

Two sets are equal if and only if they have the same elements.

- Note that the order of elements is not a concern since sets do not specify orders of elements.
- We write $A = B$, if A and B are equal sets.

Example:

- $\{1,2,3\} = \{2,1,3\}$
- $\{1, 2, 3, 4\} = \{x \in Z$ and $1 \leq x < 5\}$

$\{1,2,3\} = \{2,1,3\}$

$\{1,2,3\} = \{3,3,3,1,2,2\}$

$\{1,2,3\} \neq \{1,2,4\}$

$\{a\} \neq a \rightarrow$ not even a set

$A = \{a_1, a_2, a_3\}$ $\qquad$ $B = \{a_1, a_2, a_3\}$

let $A = B$

$A = B$ **means** for all x

$A \subseteq B$

$\boxed{x \in A \longrightarrow x \in B}$

and

$\boxed{x \in B \longrightarrow x \in A}$

$B \subseteq A$

$$A = \{1, 2, 3\} \overset{??}{=} \{1, 2, 2, 3, 3, 3\} = B$$

$A = B$

$A \subseteq B$ ✓

$B \subseteq A$ ✓

for all x

$\boxed{x \in A \longrightarrow x \in B}$: $A \subseteq B$

$x \in B \longrightarrow x \in A$: $B \subseteq A$

Fact:

Suppose A and B are sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

This fact is often used to prove set identities.

To say $A = B$

$A \subseteq B$

and

$B \subseteq A$

Showing Two Sets are Equal To show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$.

Sets may have other sets as members. For instance, we have the sets

$$A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\} \qquad \text{and} \qquad B = \{x \mid x \text{ is a subset of the set } \{a, b\}\}.$$

Note that these two sets are equal, that is, $A = B$. Also note that $\{a\} \in A$, but $a \notin A$.

How to prove set identities

To prove a set identity:

$$X = Y$$

it is necessary and sufficient to show two things:

- $X \subseteq Y$ and
- $Y \subseteq X$

Equivalently, it is necessary and sufficient to show two things:

- $x \in X \rightarrow x \in Y$ and
- $x \in Y \rightarrow x \in X$

How to prove that $A \subseteq B$?

Assume Arbitrary (Random) $x \in A$

then Show/prove that

$x \in B$.

How to prove that $B \subseteq A$?

Assume Arbitrary (Random) $x \in B$

then Show/prove that

$x \in A$.

How to prove that $A = B$?

Prove that
$$\begin{cases} A \subseteq B \ \checkmark \\ \text{and} \\ B \subseteq A \ \checkmark \end{cases}$$

Note: $x \notin B$ and $x \notin C$

then $x \notin B-C$

When $\boxed{x \in B-C}$?

$x \in B$, $x \notin C$

$x \notin B$

$\Downarrow$

$x \notin B-S$

~~$x \in S$ then~~

$x \notin P-S$

Proving Equivalence

- You may be asked to show that a set is
  - a subset of,
  - proper subset of, or
  - equal to another set.
- To prove that A is a subset of B, use the equivalence discussed earlier $A \subseteq B \Leftrightarrow \forall x(x \in A \Rightarrow x \in B)$
  - To prove that $A \subseteq B$ it is enough to show that for an arbitrary (nonspecific) element x, $x \in A$ implies that x is also in B.
  - Any proof method can be used.
- To prove that A is a proper subset of B, you must prove
  - A is a subset of B **and**
  - $\exists x\ (x \in B) \wedge (x \notin A)$

Proving Equivalence

- Finally to show that two sets are equal, it is sufficient to show independently (much like a biconditional) that
  - $A \subseteq B$ and
  - $B \subseteq A$
- Logically speaking, you must show the following quantified statements:

$$(\forall x\ (x \in A \Rightarrow x \in B)) \wedge (\forall x\ (x \in B \Rightarrow x \in A))$$

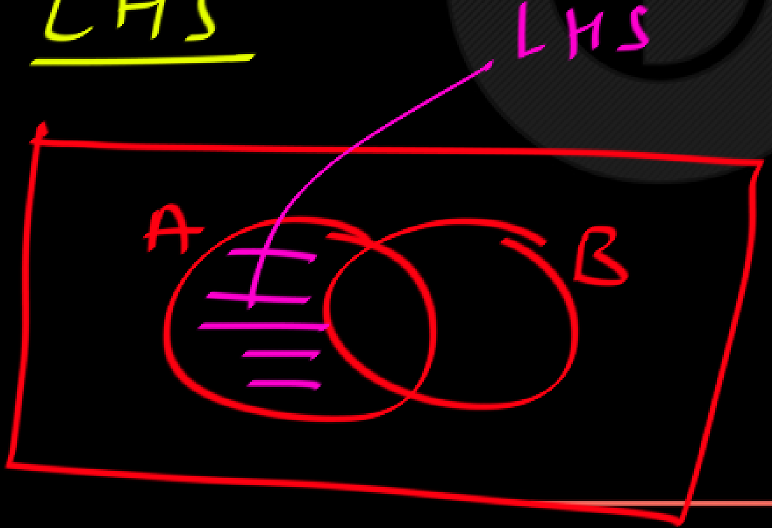
Q: let A, B be sets.

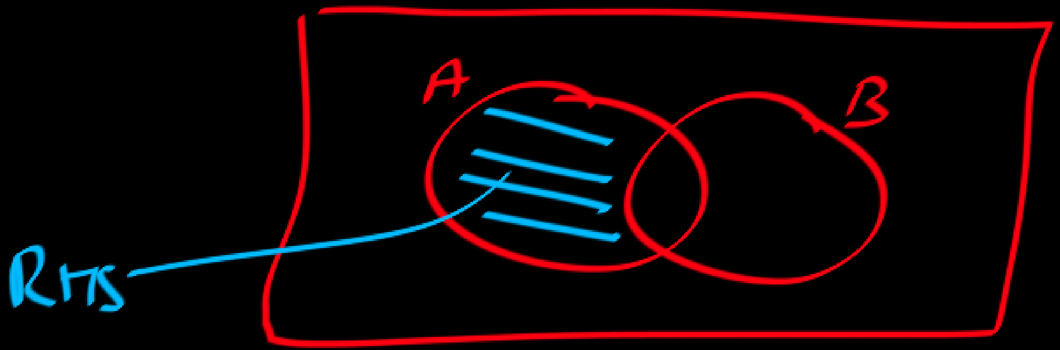
$\boxed{A - B} = \boxed{A \cap \overline{B}}$ means in A AND not in B

Proof 1: Venn Diagram

LHS

LHS = RHS





Q: let A, B be sets.

$\boxed{A - B} = \boxed{A \cap \overline{B}}$??

Proof: Complete Analysis:

① LHS $\subseteq$ RHS ?

② RHS $\subseteq$ LHS ?

if True then LHS = RHS

Check $A - B \subseteq A \cap \overline{B}$??

Proof: $\boxed{\text{Assume } x \in A - B}$

means $x \in A$ and $x \notin B$

means $x \in A$ and $x \in \overline{B}$

means $\boxed{x \in A \cap \overline{B}}$

So $A - B \subseteq A \cap \overline{B}$

1. $\left(\underbrace{x \in S}_{} \text{ and } \underbrace{x \in P}_{}\right) \longleftrightarrow \left(\underbrace{x \in S \cap P}_{}\right)$

2. $\left(\underbrace{x \in A}_{} \text{ and } \underbrace{x \in \overline{B}}_{}\right) \longleftrightarrow \left(x \in A \cap \overline{B}\right)$

3. $\left(x \in A - B\right) \longleftrightarrow \left(x \in A \text{ and } x \notin B\right)$

4. $x \notin B$ then $x \in \overline{B}$

5. $x \in \overline{B}$ then $x \notin B$

Check: $A \cap \overline{B} \subseteq A - B$??

Proof: Assume $\boxed{x \in A \cap \overline{B}}$

means $x \in A$ and $x \in \overline{B}$

means $x \in A$ and $x \notin B$

means $\boxed{x \in A - B}$

So $A \cap \overline{B} \subseteq A - B$

So we have proven

$$\left.\begin{aligned} A - B &\subseteq A \cap \overline{B} \\ A \cap \overline{B} &\subseteq A - B \end{aligned}\right\}$$

So

$$\boxed{A - B = A \cap \overline{B}}$$

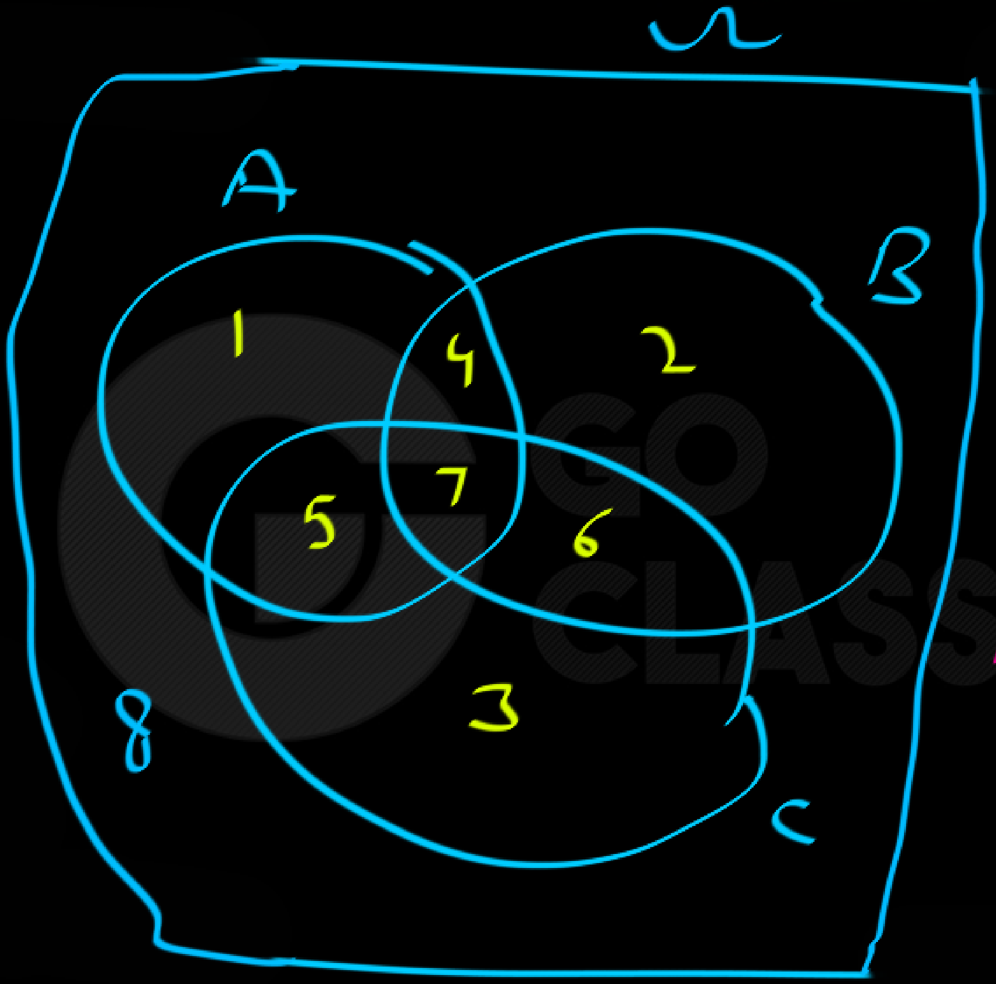
$B = 2467$

$C = 3567$

$B - C = 24$

$A \cap (\bar{B} \cap C)$

$= 5$



$A = 1457$

$A - B = 1, 5$

$A - (B \cup C) = 1$

$A - (B \cap C) = 1, 4, 5$

$A - (B - C) =$

$1457 - (24)$

$= 157$

Example for proving set identities

GO Classes

Example:

Show that if A, B, C are sets, then:

$$(\overline{A} - B) - C = (\overline{A} - C) - (B - C)$$

Proof: First we show $x \in \text{LHS} \rightarrow x \in \text{RHS}$:

$x \in (\overline{A} - B) - C$ (by the definition of “set difference”)

$\Rightarrow x \in (\overline{A} - B)$ but $x \notin C$. Hence $x \in \overline{A}, x \notin B$ and $x \notin C$.

So $x \in \overline{A} - C$ and $x \notin B - C$. This means $x \in (\overline{A} - C) - (B - C) = \text{RHS}$.

Next we show $x \in \text{RHS} \rightarrow x \in \text{LHS}$:

$x \in (\overline{A} - C) - (B - C)$ (by definition of “set difference”)

$\Rightarrow x \in (\overline{A} - C)$ but $x \notin (B - C)$. Hence: $x \in \overline{A},\ x \notin C$ and $x \notin (B - C)$.

Here: $x \notin B - C$ means either $x \notin B$ or $x \in C$.

Since the latter contradicts $x \notin C$, we must have $x \notin B$.

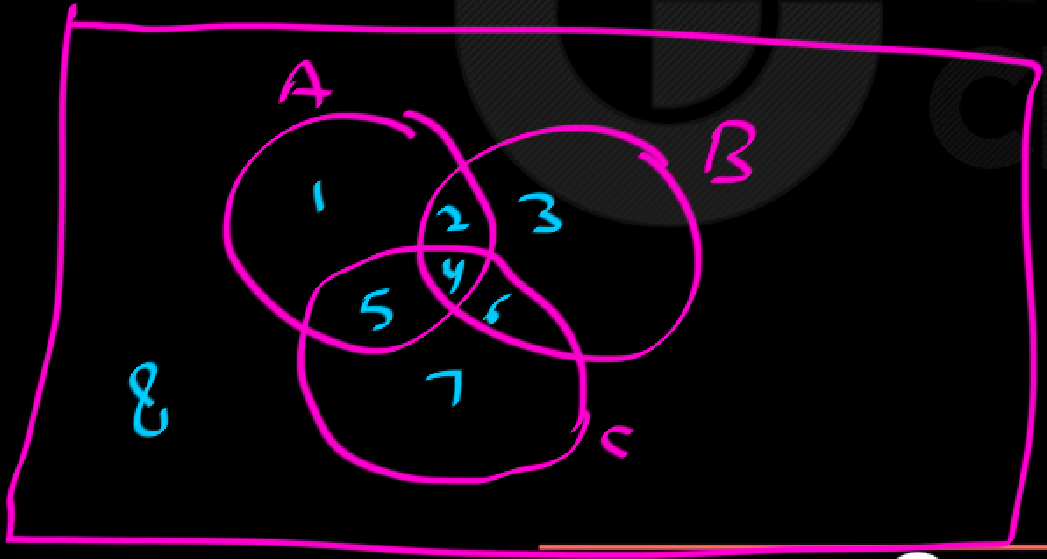
This implies $x \in (\overline{A} - B) - C = \text{LHS}$.

Q: Prove that

$$(\bar{A}-B)-C = \underbrace{(\bar{A}-C)}_{83} - \underbrace{(B-C)}_{23}$$

83 — 23

Proof1: Venn;



$\bar{A}-B$ = 87

$$\underbrace{(\bar{A}-B)}_{87} - \underbrace{C}_{4567} = \boxed{8 = \text{LHS}}$$

$$\boxed{8 = \text{RHS}}$$

Q: Prove that

$$(\bar{A} - B) - C = (\bar{A} - C) - (B - C)$$

Proof: Analysis

$$(\bar{A} - B) - C \subseteq (\bar{A} - C) - (B - C) \checkmark$$

$$(\bar{A} - C) - (B - C) \subseteq (\bar{A} - B) - C$$

Check: $(\bar{A} - B) - C \subseteq (\bar{A} - C) - (B - C)$

Assume $\boxed{x \in (\bar{A} - B) - C}$

means $x \in (\bar{A} - B)$ and $x \notin C$

means $\boxed{x \in \bar{A}}$ and $x \notin B$ and $\boxed{x \notin C}$

means $\boxed{x \in \bar{A} - C}$ and $x \notin B$ and $x \notin C$

$x \in \bar{A} - C$ and $\boxed{x \notin B}$

$\boxed{x \in \bar{A} - C}$ and $\boxed{x \notin B - C}$

$$\boxed{x \in (\bar{A} - C) - (B - C)}$$

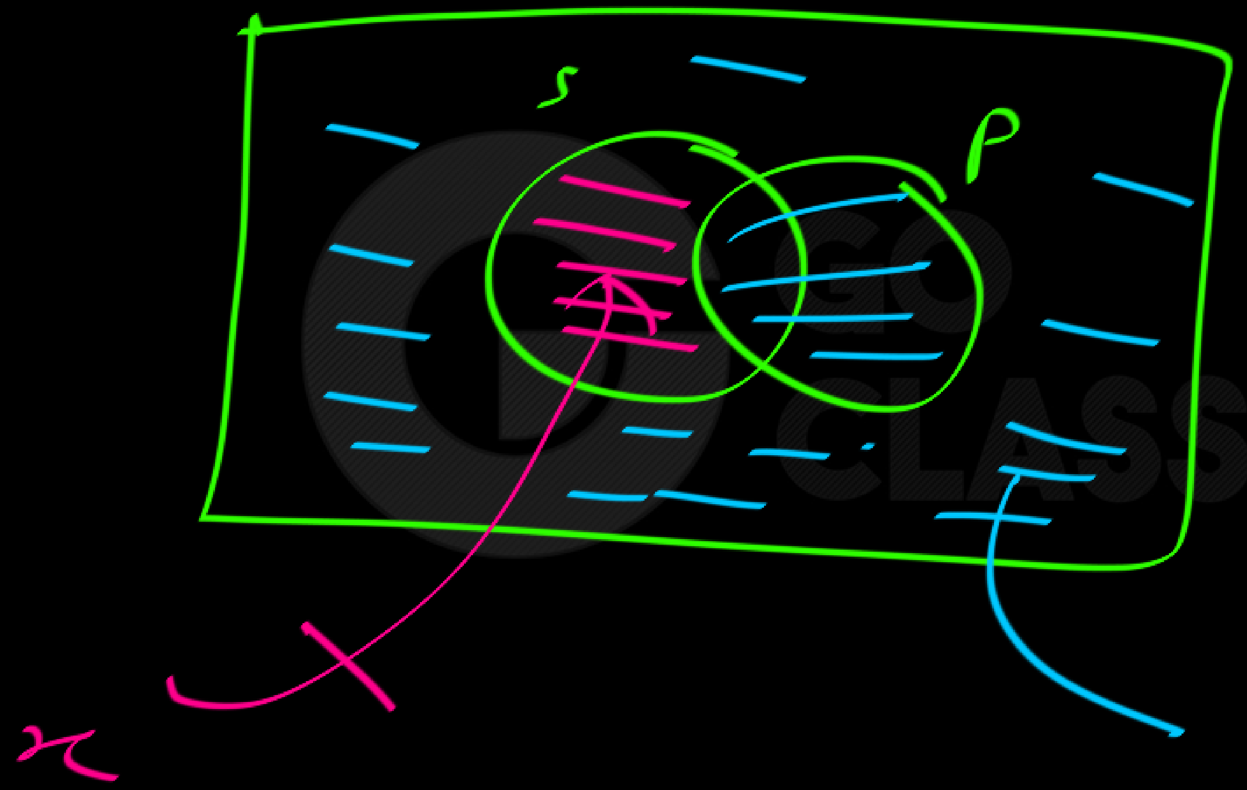
Hence : $(\bar{A} - B) - C \subseteq (\bar{A} - C) - (B - C)$

Note;

$x \notin S$ then $\boxed{x \notin S - P}$ ✓

$\boxed{x \notin S - P \text{ then } x \notin S}$ ✗

$x \notin S - P$ then $x \notin S$ OR $x \in P$ ✓



$x \notin S - P$

means

$x \notin S$ OR $x \in P$

$\bar{S} \cup P$