Partial Order Relations

Next Sub-Topic:

Distributive Lattice

Website : https://www.goclasses.in/

Distributive Lattices :

If you were paying attention to the sort of properties that were being proved earlier, you may have been wondering what happened to the distributive property.

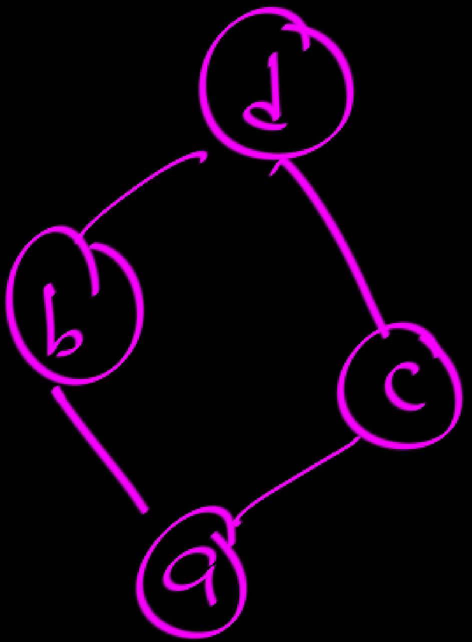
Why wasn't that included?

Well, there's a good reason. Not all lattices are distributive. Only the distributive ones are.

Distributive Property: $\forall a, b, c$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$



Distributive

$$b \wedge (c \vee d) \overset{??}{=} (b \wedge c) \vee (b \wedge d)$$

$$\underbrace{b \wedge d} \qquad\qquad \underbrace{a \vee b}$$

$$b \qquad\qquad\qquad b$$

DEFINITION 7.2-3: ***Distributive Lattice***

A lattice $\mathcal{A}$ is **distributive** *iff the following* **distributive laws** *hold for any x, y, and z in $\mathcal{A}$:*
$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; $\quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A lattice is Distributive iff $\forall_{x,y,z}$

$$\left\{ \begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array} \right\}$$

A non-distributive lattice

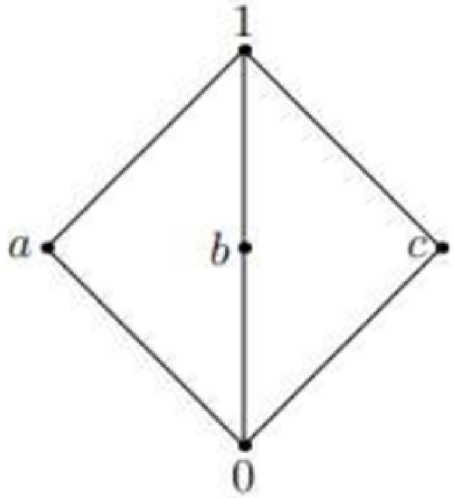
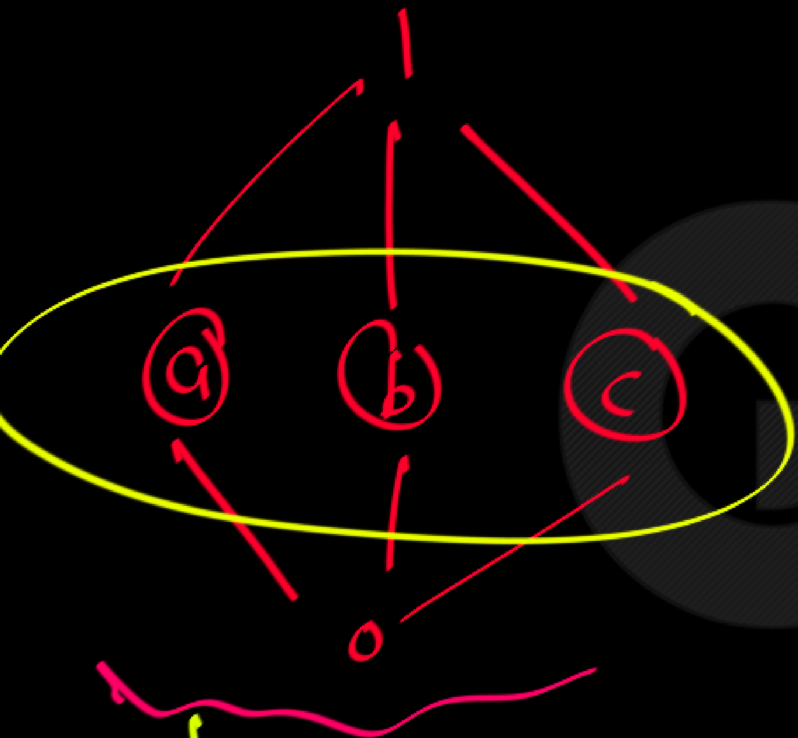


Figure: $a \cap (b \cup c) \neq (a \cap b) \cup (a \cap c)$



$$\underbrace{a \vee (b \wedge c)}_{a \vee 0} \overset{?}{=} \underbrace{(a \vee b)}_{1} \wedge \underbrace{(a \vee c)}_{1}$$

$$\underbrace{a \vee 0}_{a} \qquad \underbrace{1 \wedge 1}_{1}$$

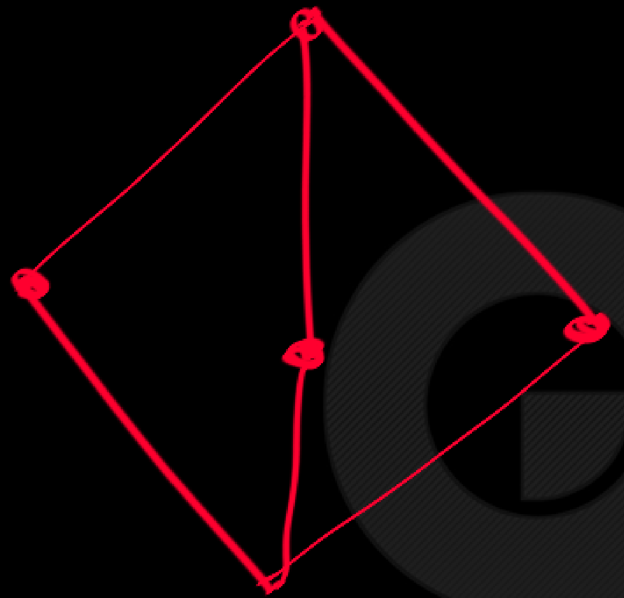
$$a \neq 1$$

not Dist. lattice

$$a \wedge \underbrace{(b \vee c)}_{1} \stackrel{?}{=} \underbrace{(a \wedge b)}_{0} \vee \underbrace{(a \wedge c)}_{0}$$

$$\underbrace{a \wedge 1}_{a} \qquad \qquad \underbrace{0 \vee 0}_{0}$$

$$a \neq 0$$

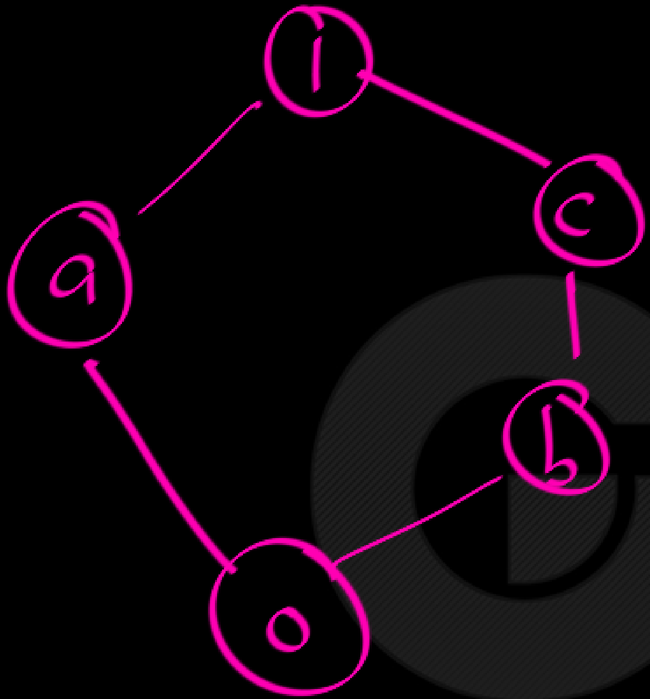


Kite Lattice

Diamond lattice

M_3 lattice

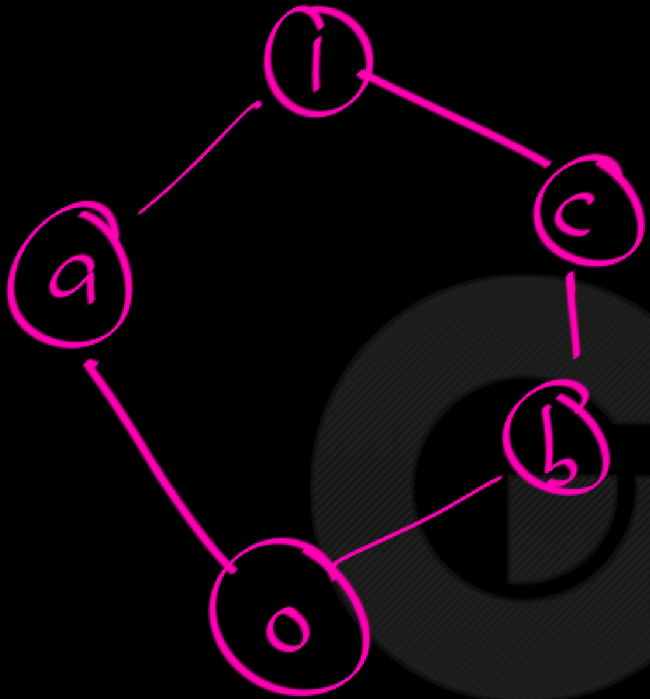
Not Distributive lattice



$$c \vee (b \wedge a) \overset{?}{=} (c \vee b) \wedge (c \vee a)$$

$$\underbrace{c \vee 0}_{c} = \underbrace{c \wedge 1}_{c}$$

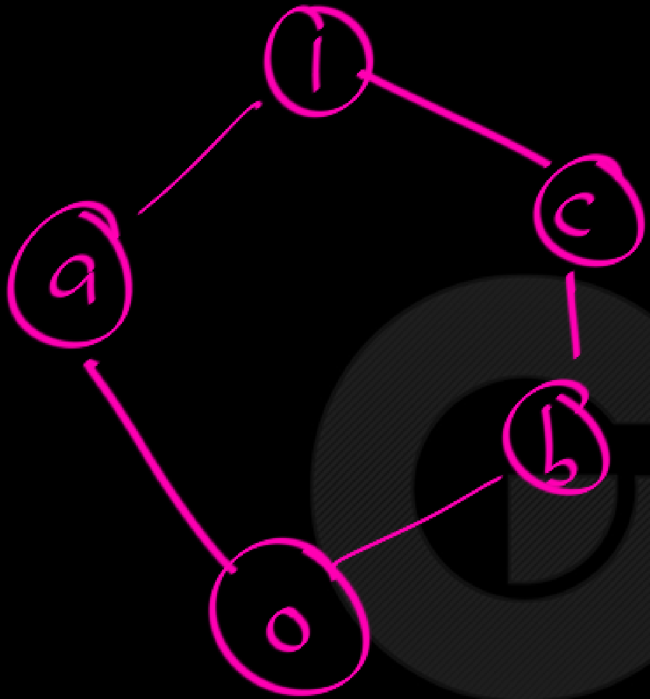
<u>Not Distributive</u>



$$a \vee (\underbrace{b \wedge c}_{c}) \overset{?}{=} (\underbrace{a \vee b}_{1}) \wedge (\underbrace{a \vee c}_{1})$$

$$\underbrace{a \vee c}_{1} = \underbrace{1 \wedge 1}_{1}$$

Not Distributive



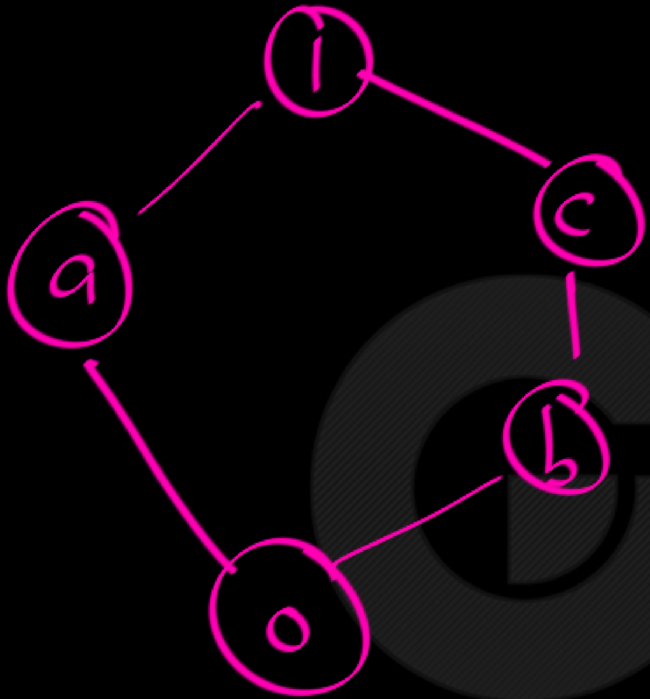
$$b \vee (a \wedge c) \overset{??}{=} (b \vee a) \wedge (b \vee c)$$

$$b \vee 0 \qquad\qquad 1 \wedge c$$

$$b \neq c$$

So, not Distributive

Not Distributive

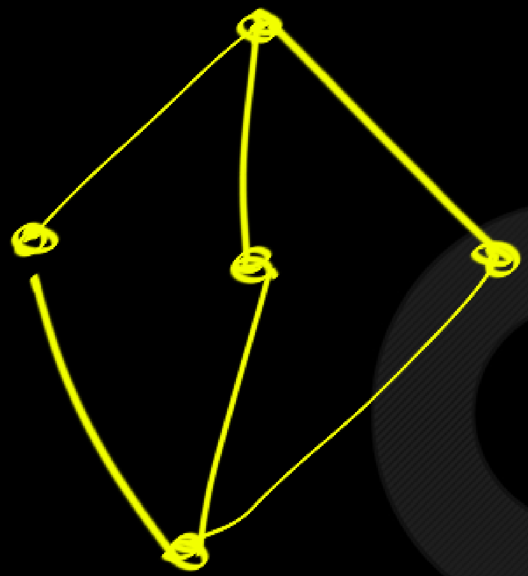


Pentagon Lattice

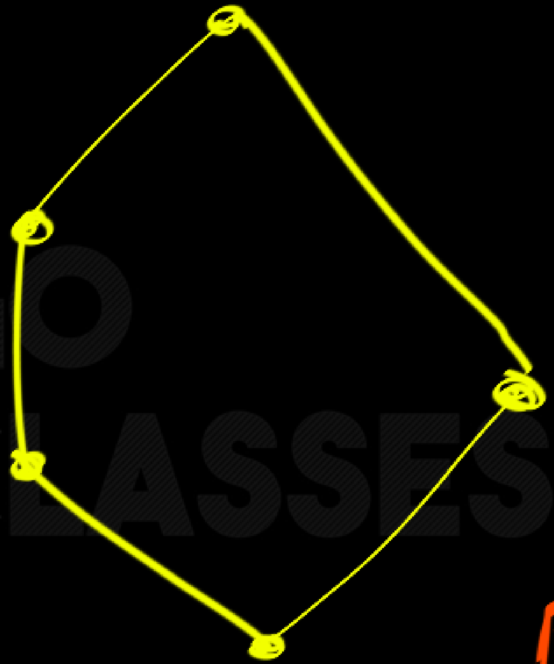
N_5 Lattice

Not Distributive

Not Distributive :



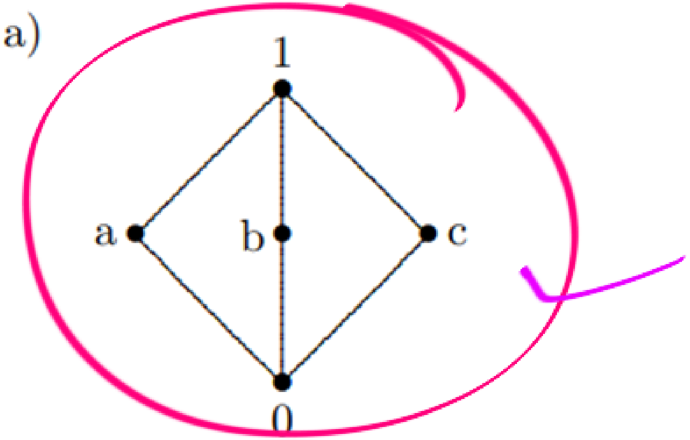
Kite / Diamond
(M_3)



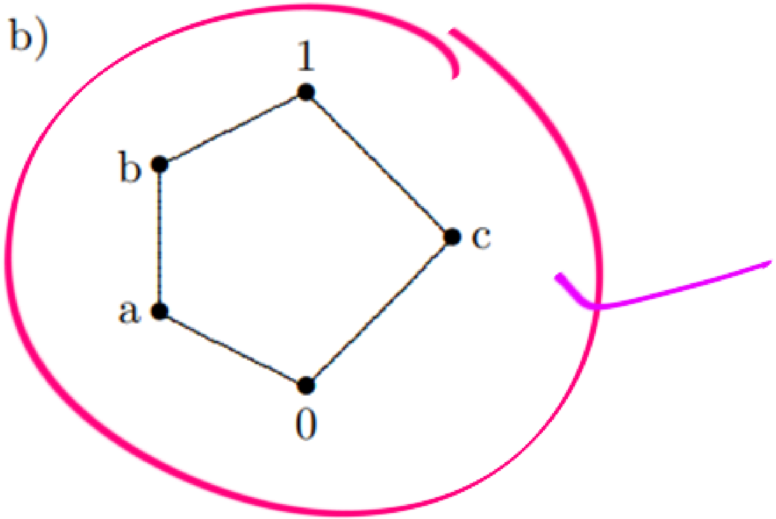
Pentagon
(N_5)

Show that the following simple but significant lattices are not distributive. (In fact, it has been proved that every non-distributive lattice contains a copy of one of these lattices.)

a)



b)



Solution

a) To see that the diamond lattice is not distributive, use the middle elements of the lattice: $a \wedge (b \vee c) = a \wedge 1 = a$, but $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$, and $a \neq 0$. Similarly, the other distributive law fails for these three elements (see Exercise 22a).

b) The pentagon lattice is also not distributive. See Exercise 22b.

The simplest *non-distributive* lattices are M_3, the "diamond lattice", and N_5, the "pentagon lattice". A lattice is distributive if and only if none of its sublattices is isomorphic to M_3 or N_5; a sublattice is a subset that is closed under the meet and join operations of the original lattice. Note that this is not the same as being a subset that is a lattice under the original order (but possibly with different join and meet operations).

Theorem:

kite, pentagon $\Rightarrow$ Not Distributive

L is Distributive iff there is no sublattice of L which is kite or pentagon.

for lattice L;

If $\exists$ sublattice S = kite structure

then L is not Distributive.

for lattice L ;

If $\exists$ sublattice S = Pentagon Structure

then L is not Distributive.

If L is not Distributive then

$\exists$ sublattice which has

structure of kite or pentagon.

Q: If lattice L has ≤ 4 elements

then L is Definitely Distributive.

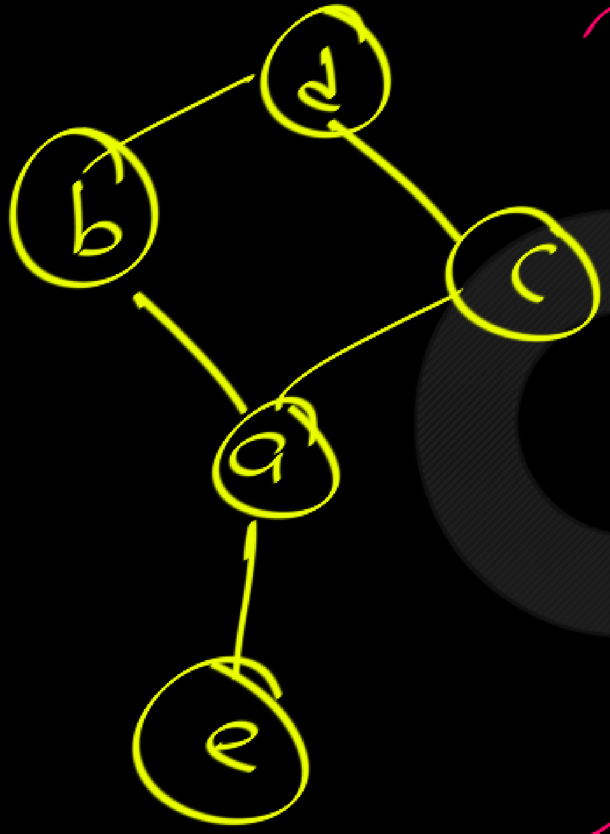
because you cannot M_3, N_5 as

Sublattice

need 5 element

Q: Which lattices of 5 elements are Distributive?

Ans: Every lattice other than exactly kite, Pentagon.



L

→ 5 elements

→ not kite

→ " pentagon

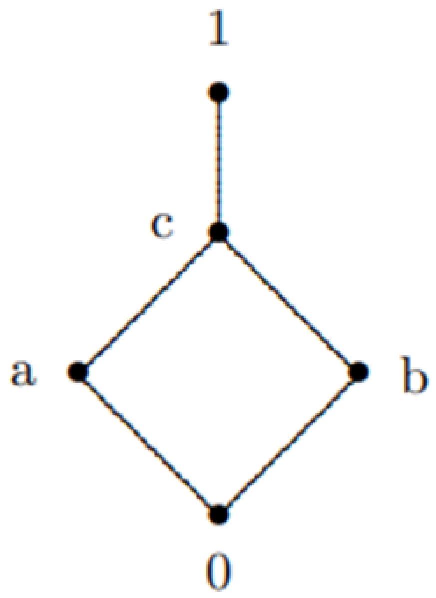
→ Sublattices of L

L , every other sublattice has ≤ 4 elements.

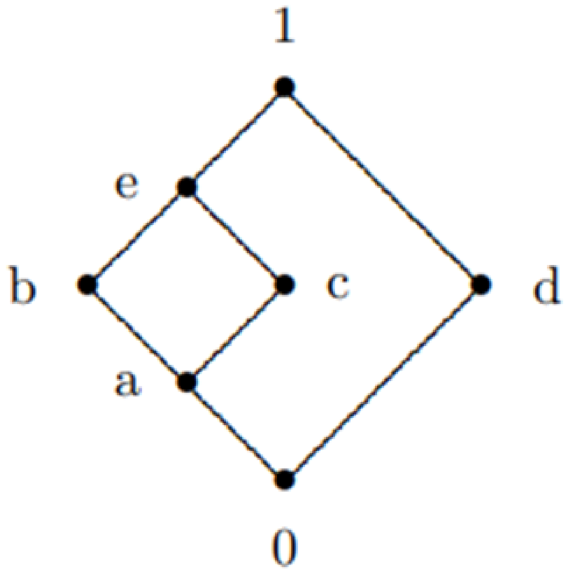
The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

EC a.



*b.

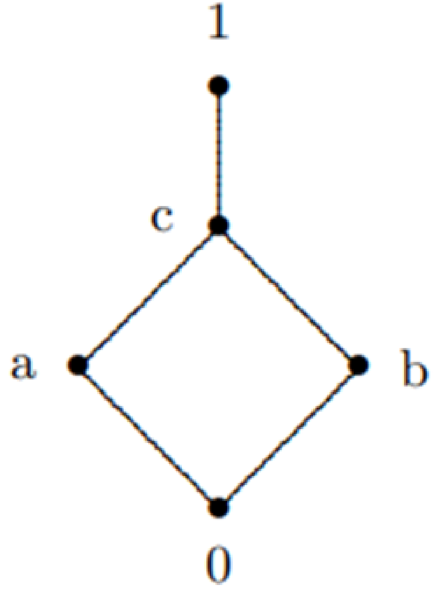


The following problems relate to distributive lattices.

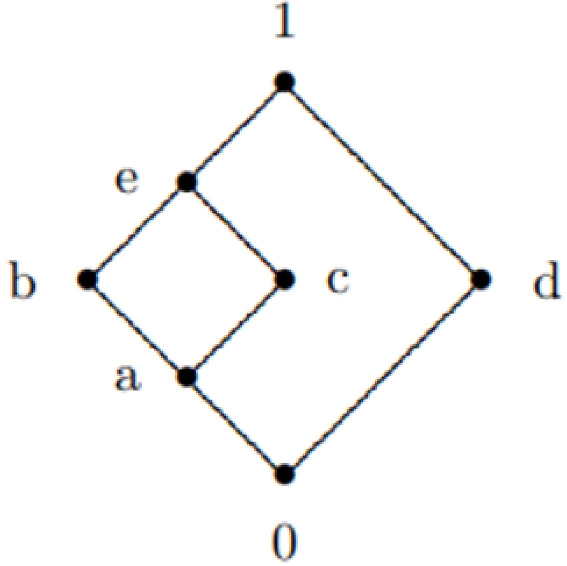
$\bar{d}' = \underline{b, c}$

*21. Are the following lattices distributive or not? Explain.

EC a.



*b.



$\bar{b}' = d$

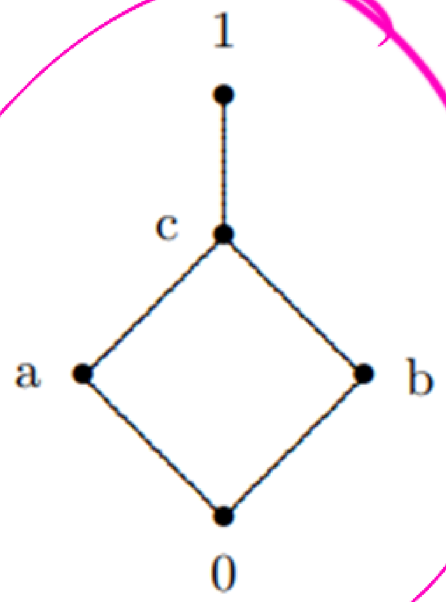
$\bar{c}' = d$

$\boxed{\bar{d}' = b, c} \Rightarrow$ So, Not Distributive.

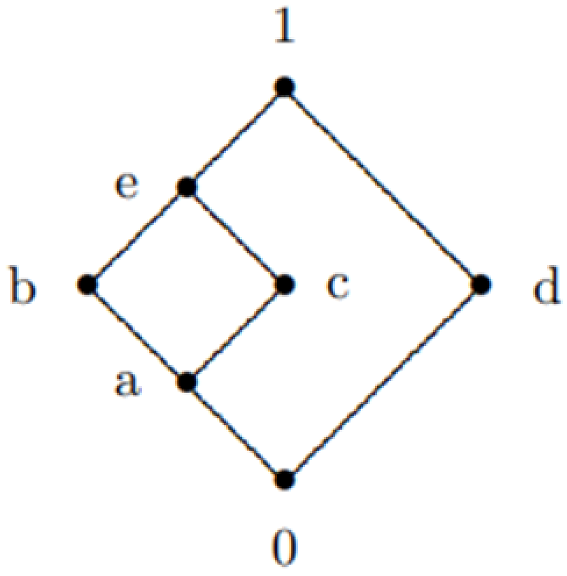
The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

EC a.



*b.



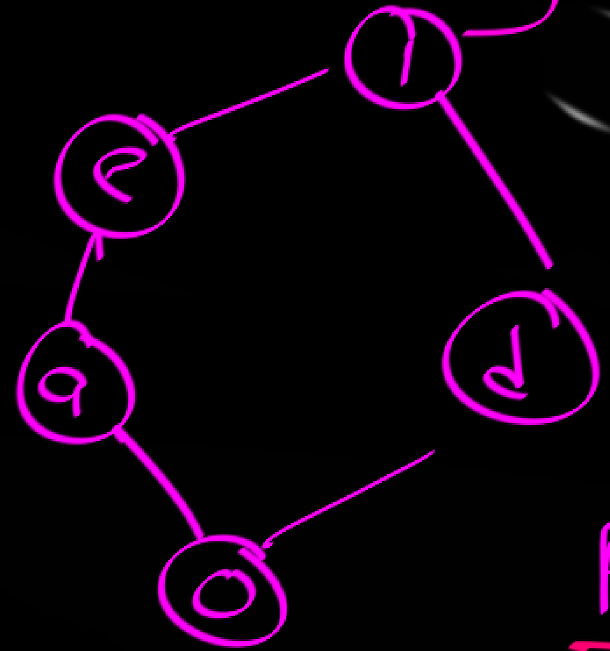
Distributive

The following problems relate to distributive lattices.

*21. Are the following lattices distributive or not? Explain.

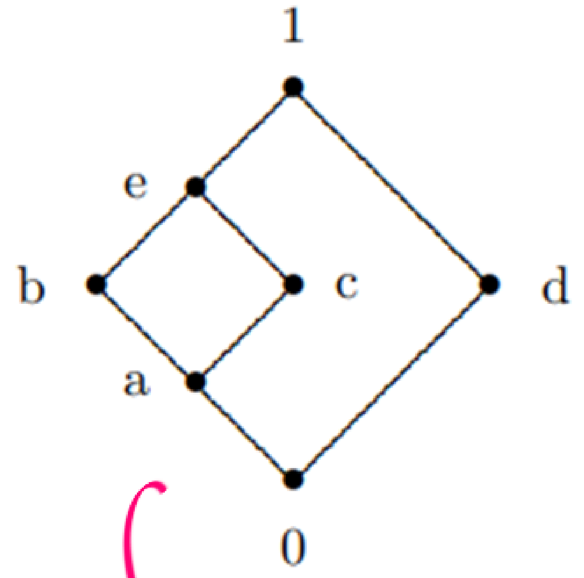
[redacted]

1 {0, a, e, d, 1} sublattice



Pentagon

*b.



Not Distributive.

The following problems relate to distributive lattices.

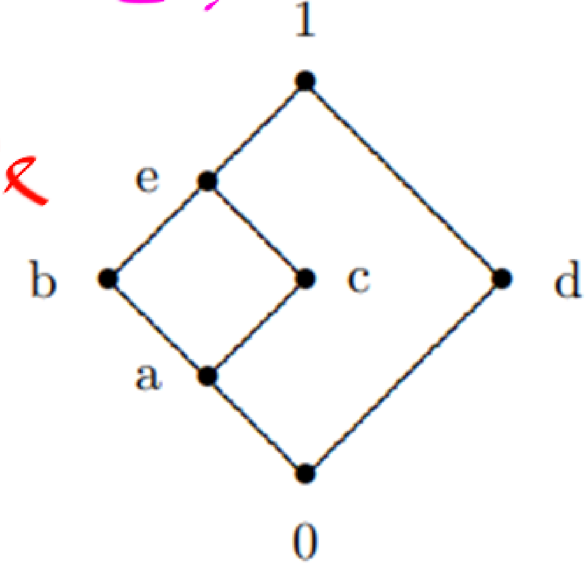
*21. Are the following lattices distributive or not? Explain.

[redacted]

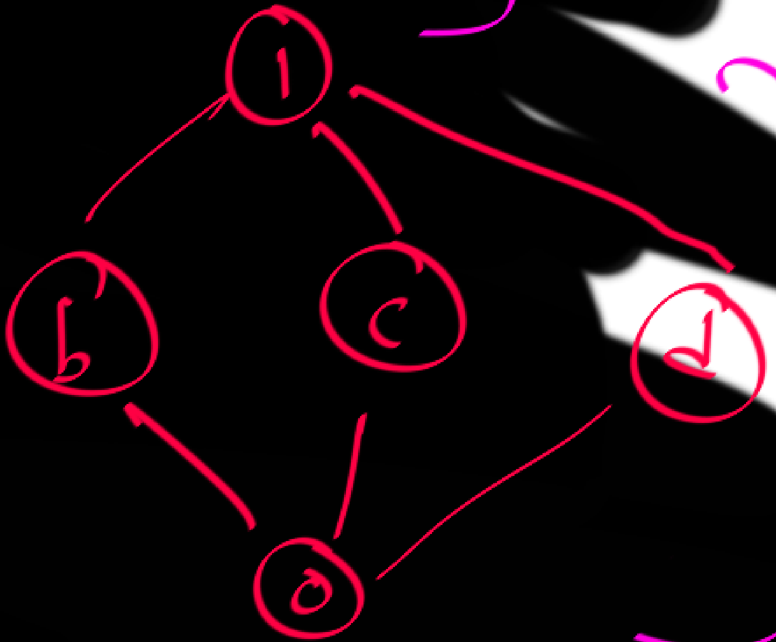
*b.

$b \vee c = e$

$L:$



① $\{b, c, d, 0, 1\}$ ⇒ wrong counter example



$b \vee c = 1$

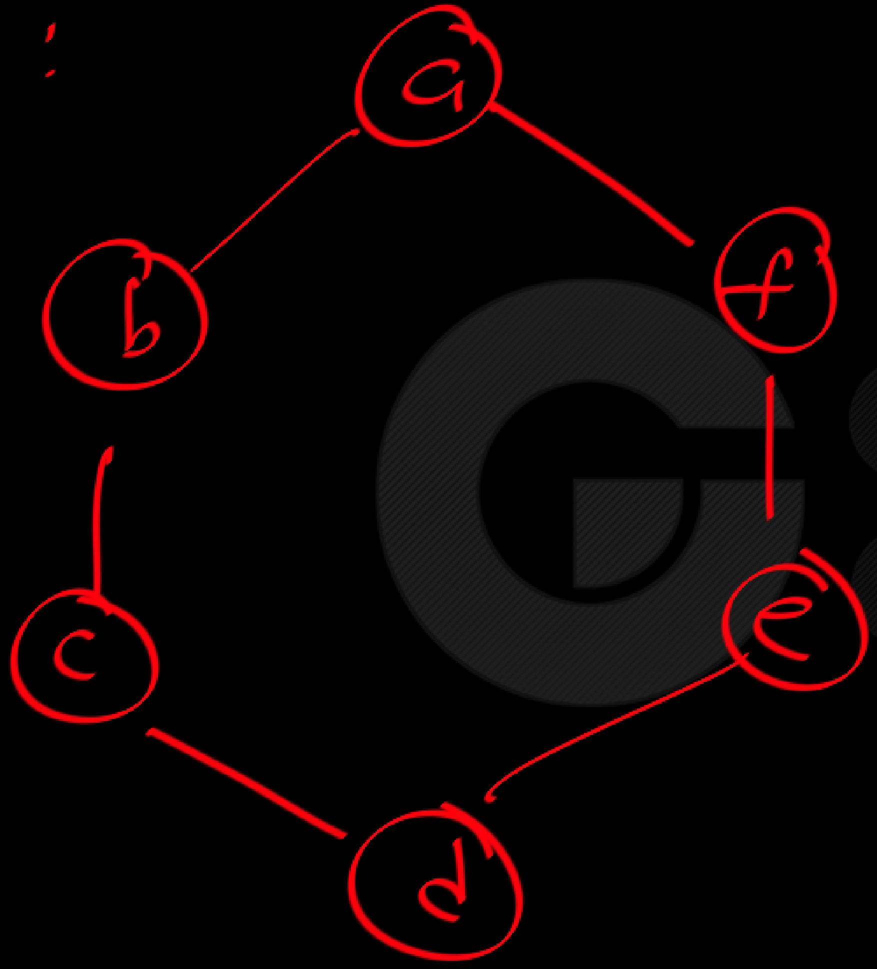
⇒ not sublattice of L.

Note: To make Sublattice,

take elements with their

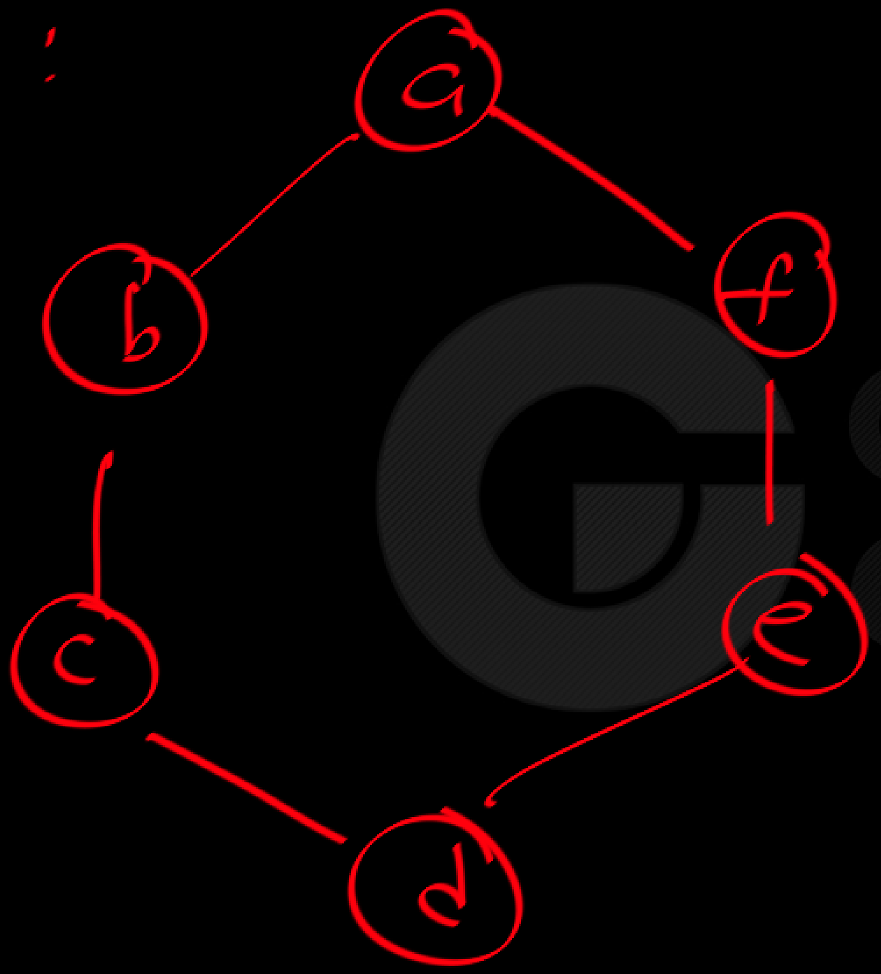
GLB, LUB.

Q :



— Distributive?

Q :



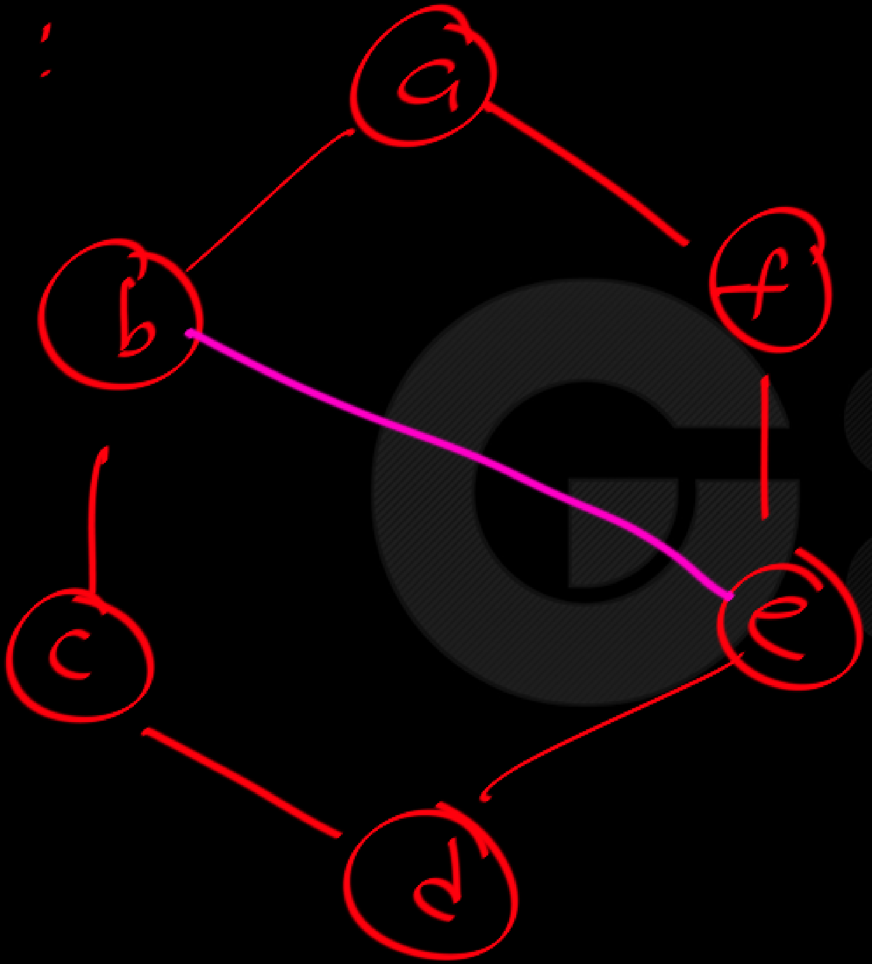
NOT Distributive

Sublattice

$\{d, c, b, a, f\}$

Pentagon

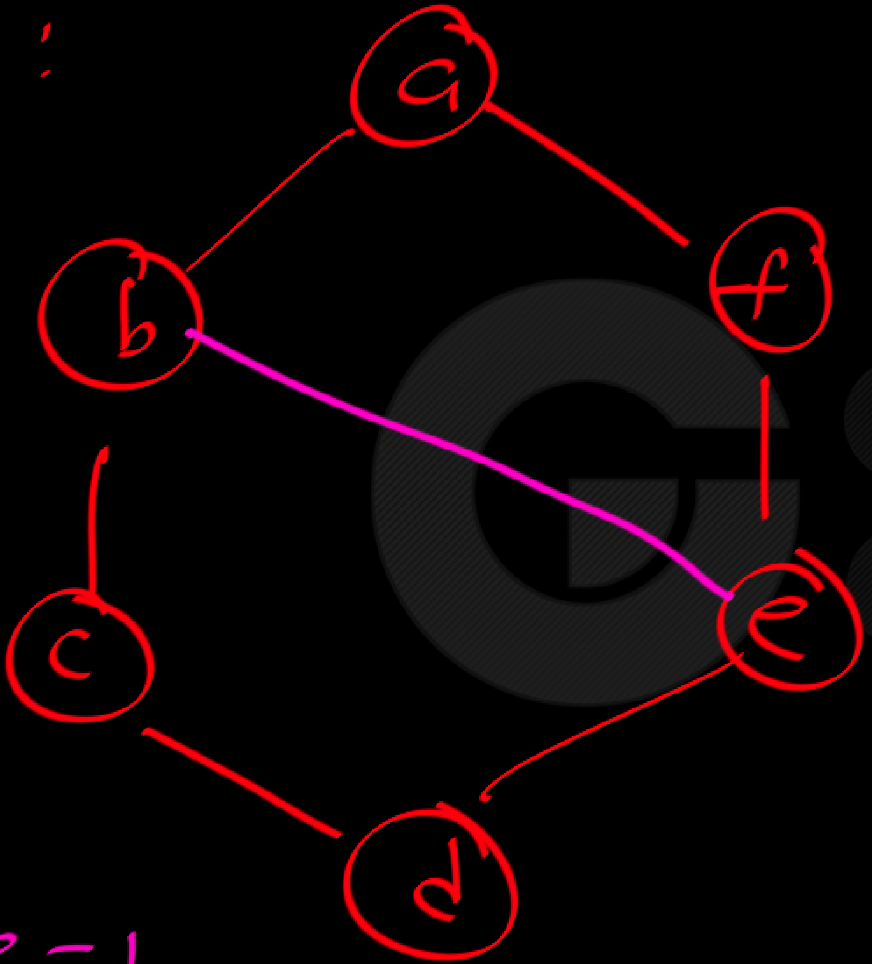
Q :



Distributive ??

Lattice ✓

Q :



$\{d, c, e, a, f\}$

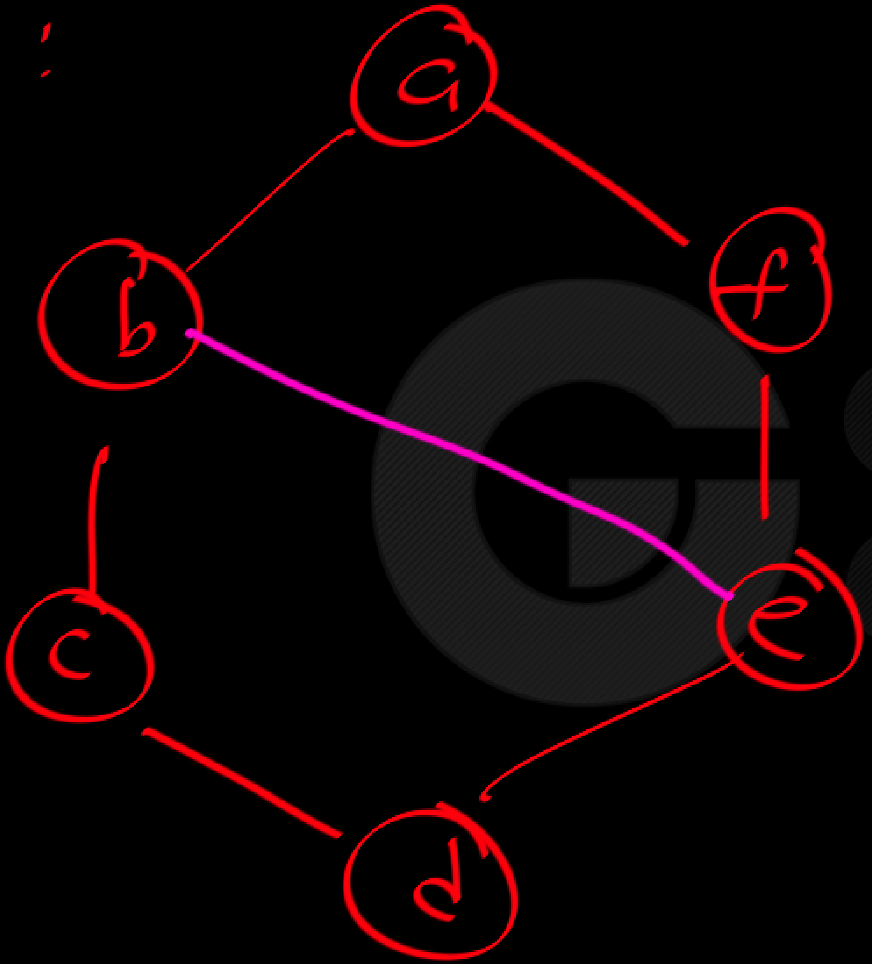
not a sublattice

$c \vee e = a$

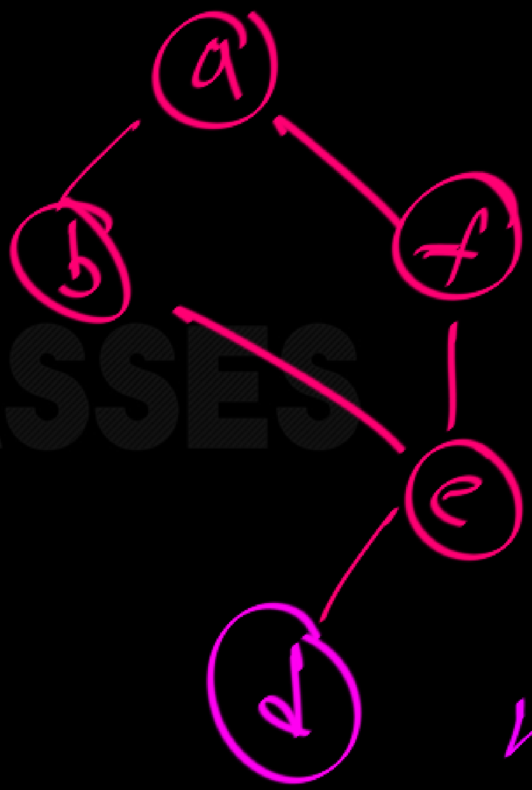
wrong counter Example

$c \vee e = b$

Q :

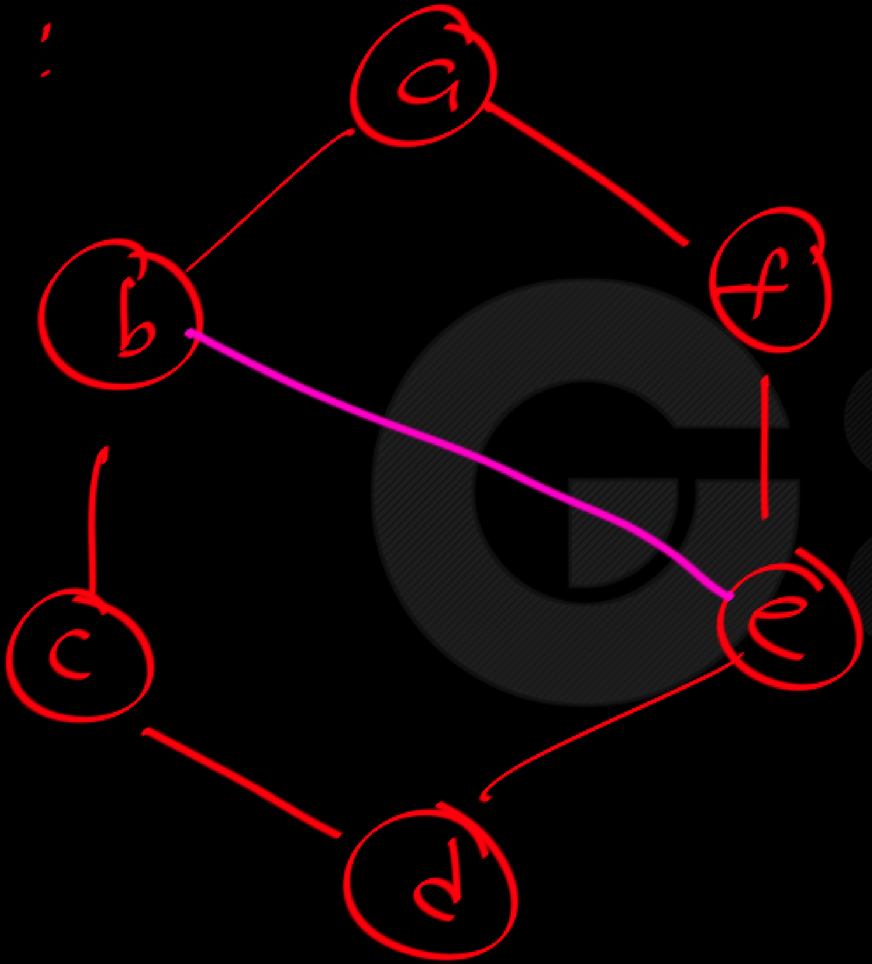


$\{d, b, e, f, a\} \Rightarrow$ Sublattice



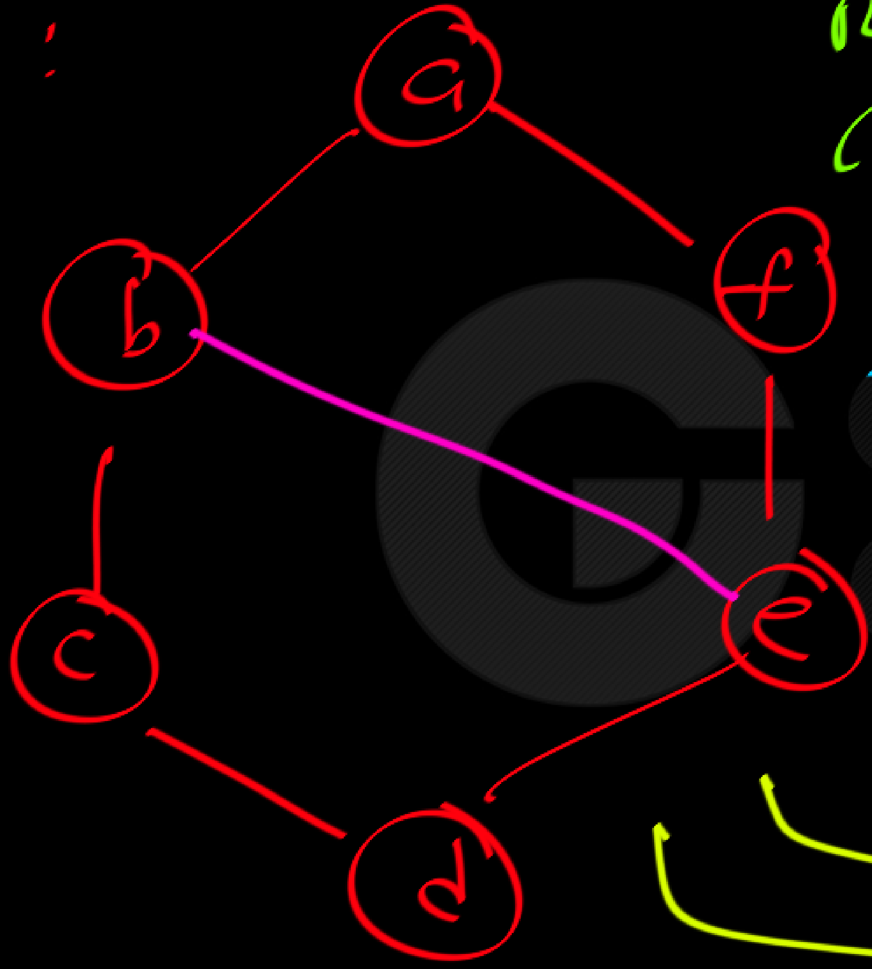
Neither kite, nor pentagon

Q :



Distributive

Q :



Remove ≥ 2 elements

kite ✗, Pentagon ✗

a

b

d

e

Not Sublattice

f

c

Sublattice but not kite, not N_5

Theorem: (one way theorem)

Distributive lattice $\Longrightarrow$ at most one complement for every element.

Theorem: (one way)

If some element has >1 complements then Not Distributive lattice.

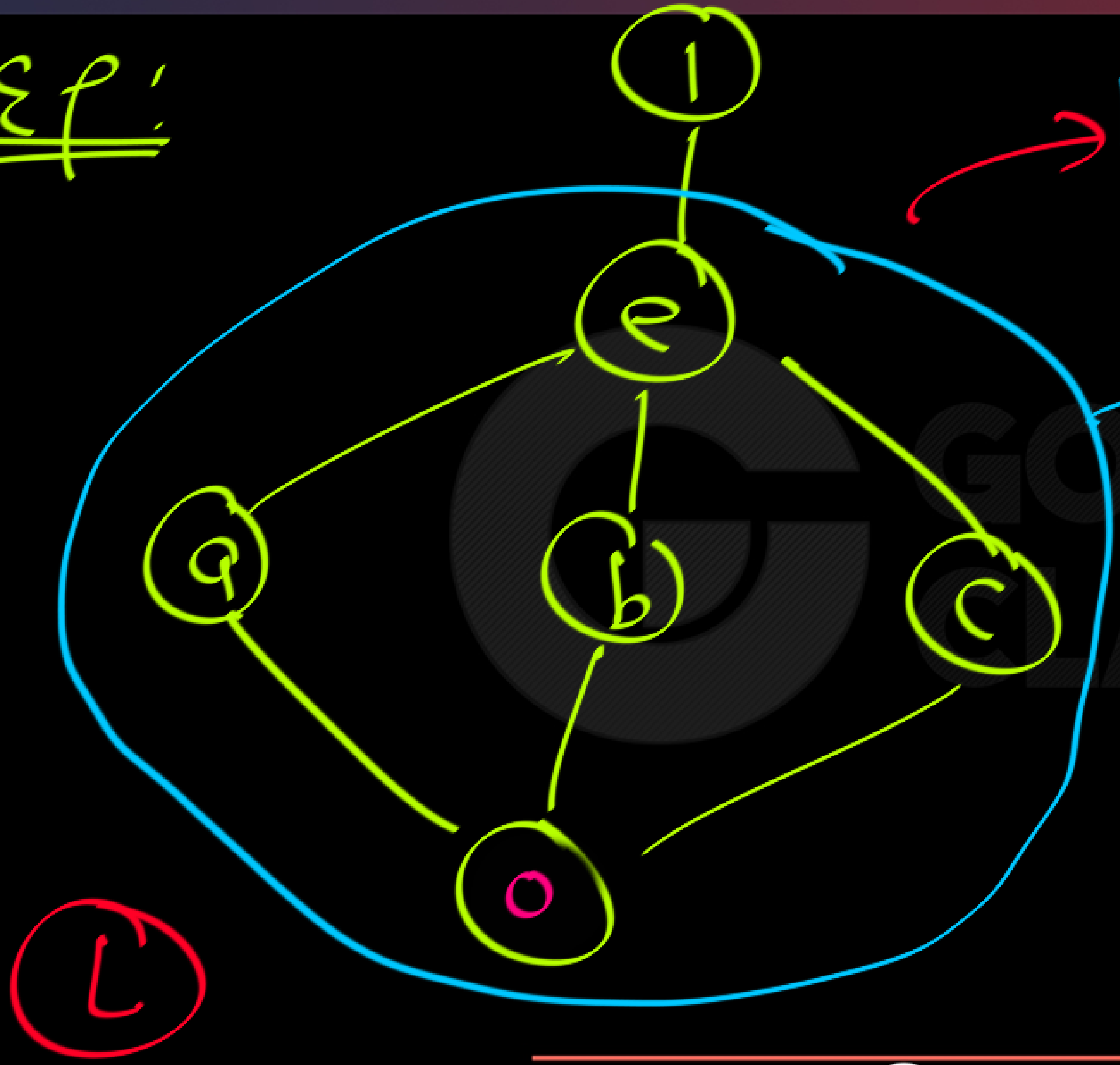
Note:

Every element has ≤ 1 Complement

$$\not\Downarrow$$

Distributive

Eg:



→ L Not Distributive

Kite Sublattice

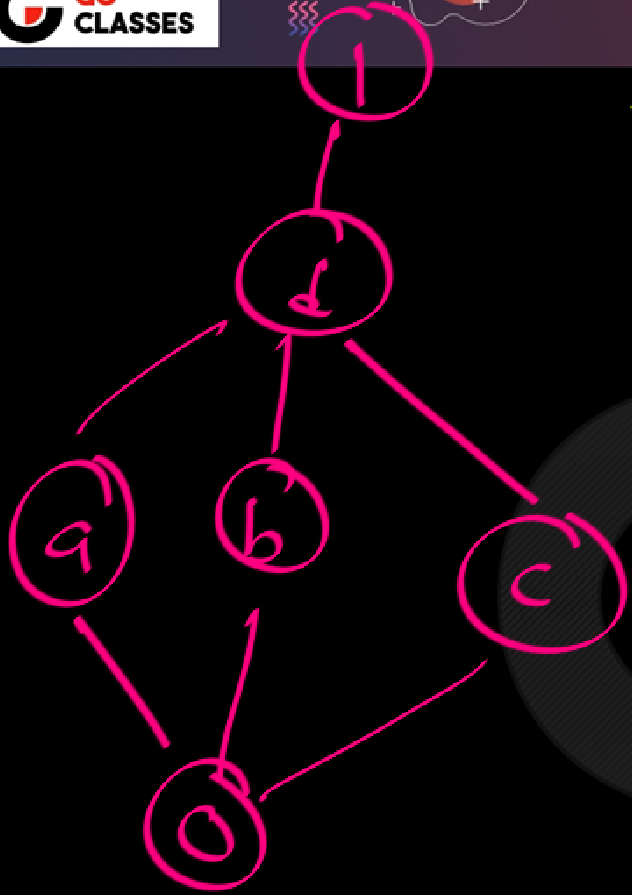
$1' = 0$

$0' = 1$

$c' = DNE$

$e' = DNE$

$a' = DNE = b'$



Every elements has ≤ 1 Complement but still it is not Distributive.

Q: Every element has exactly one Complements then Distributive ?

Q: Every element has exactly one Complements then Distributive?

No.

Proven by Dilworth

Right now, no-one knows the Counter ex.

Analogy : Right now, "n" is the greatest prime number that we know right now But we know there is a prime number $> n$.

But we don't know which (Right now).

Imp:

Dist. lattice $\longrightarrow$ ≤ 1 complement for every element

Note: To check if Given lattice L is Distributive :

(1) If $|L| \le 4 \implies$ Distributive

(2) If $|L| = 5 \implies$ Dist iff not kite or Pentagon

③ If L is <u>Total order</u> $\Longrightarrow$ Distribut-
-ive

④ find Complements of elements:
If some element has > 1
Complement then Not Distributive

But If every element has ≤ 1 Complement then

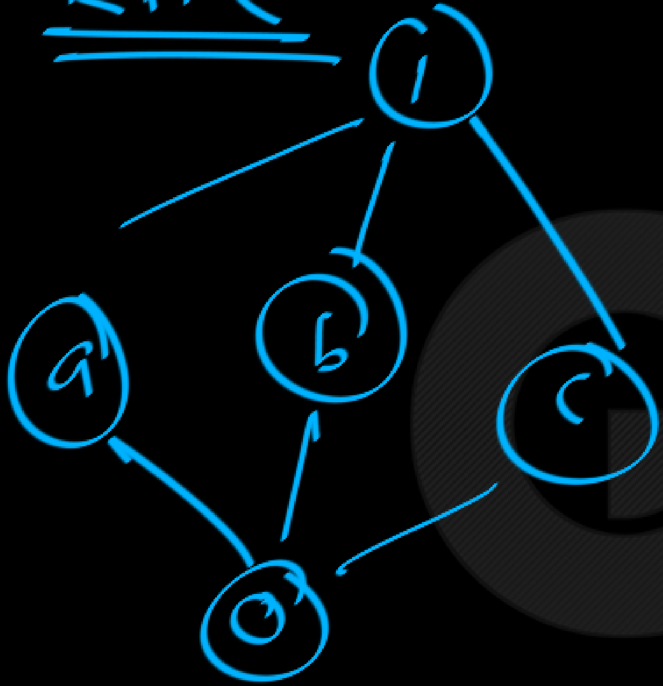
$$\boxed{\text{find kite or pentagon "Sublattice"}}$$

Every Total order is Distributive.

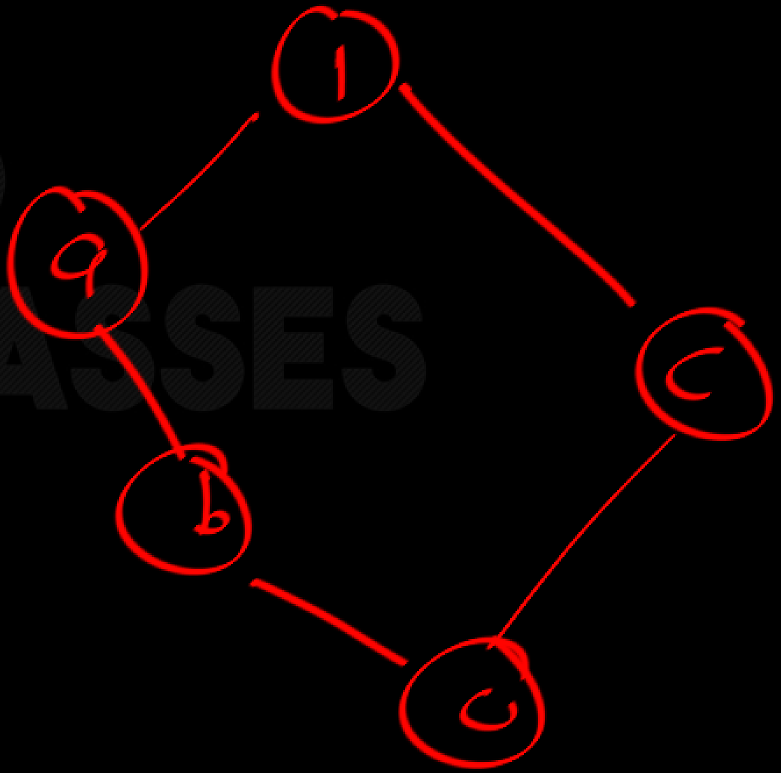
chain

$\Rightarrow$ Can not make kite or Pentagon sublattice ✓

Kite



Pentagon



Note: In Dist lattice:

Every element has ≤ 1 Comp.

In Comp. lattice:

Every element has ≥ 1 Comp.

In Dist Comp lattice :

Every element has exactly one Complement.

(only one way theorem)

Note:

If every element has exactly one Complement

Complemented

~~Distributive~~

PROPOSITION 7.2-4: ***Uniqueness of Complements in Distributive Lattices***

If $\langle \mathcal{A}, \leq \rangle$ is a bounded distributive lattice with minimum 0 and maximum 1, then complements are unique, provided they exist.

COROLLARY 1: ***Complemented Distributive Lattices Have Unique Complements***

Every element in a complemented distributive lattice has a unique complement. The complement of x will be denoted by $\overline{x}$.

Partial Order Relations

Next Sub-Topic:

Boolean Lattice/ Boolean Algebra

Website : https://www.goclasses.in/

Definition

A Boolean algebra is a lattice with 0 and 1 that is distributive and complemented.

Example

Let A be a set. Then $\langle \text{pow}(A), \subseteq \rangle$ is a Boolean algebra.

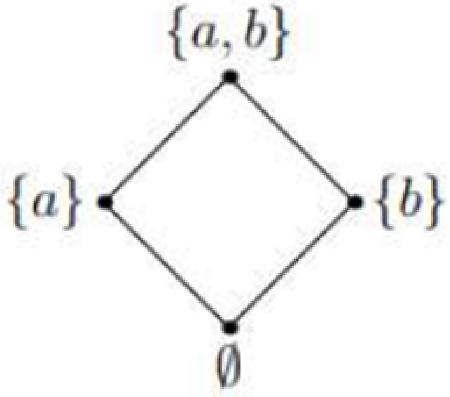


Figure: $A = \{a, b\}$.

Boolean Lattices: Definition and Properties

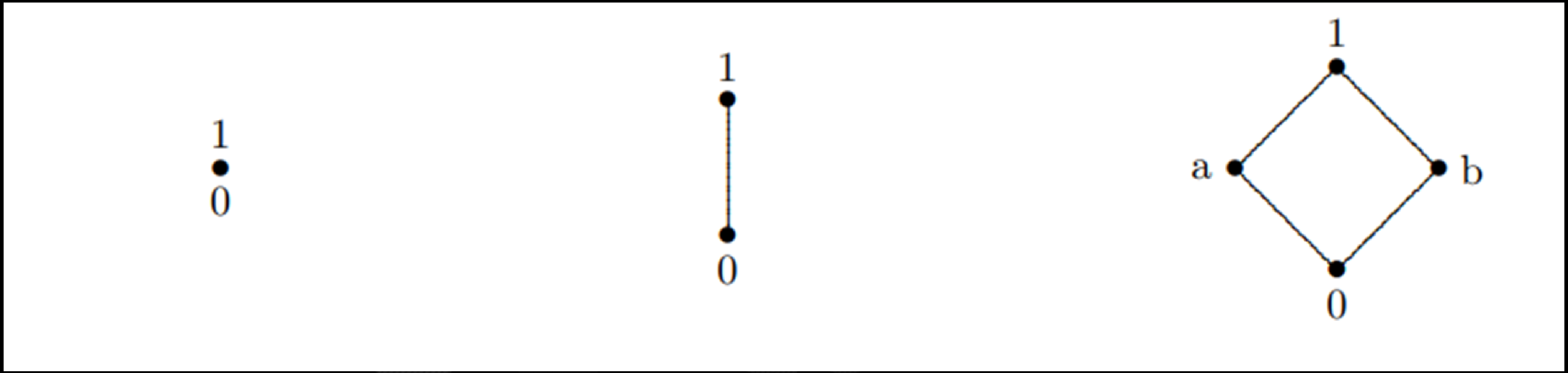
Complemented distributive lattices are an important type of lattice. Rather than call them by this mouthful, they are given a special name: they are called *Boolean lattices*.

DEFINITION 7.2-7: *Boolean Lattices*

A lattice $\langle \mathcal{A}, \leq, ^{-}, 0, 1 \rangle$ *is a* ***Boolean lattice*** *iff it is a complemented distributive lattice.*

Example

$\langle \{1, 2, 3, 6\}, | \rangle$ is a Boolean algebra.


1
0
1
0
1
a
b
0

Theorem (M. H. Stone, 1936)

Every finite Boolean algebra is isomorphic to the Boolean algebra $\langle \text{pow}(S), \subseteq \rangle$ of a finite set S.

Corollary

Every finite Boolean algebra has 2^n elements for some $n \in \mathbb{N}$.