First Order Logic

Next Chapter:

Arguments in FOL

- By Deepak Poonia (IISc Bangalore)

Instructor:

Deepak Poonia

IISc Bangalore

GATE CSE AIR 53; AIR 67;
AIR 107; AIR 206; AIR 256

Discrete Mathematics Complete Course:

https://www.goclasses.in/courses/Discrete-Mathematics-Course

GO CLASSES

Test Series

GO Classes

On "GATE Overflow" Website

Join **GO+ GO Classes Combined Test Series** for BEST quality tests, matching GATE CSE Level:

Visit www.gateoverflow.in website to join Test Series.

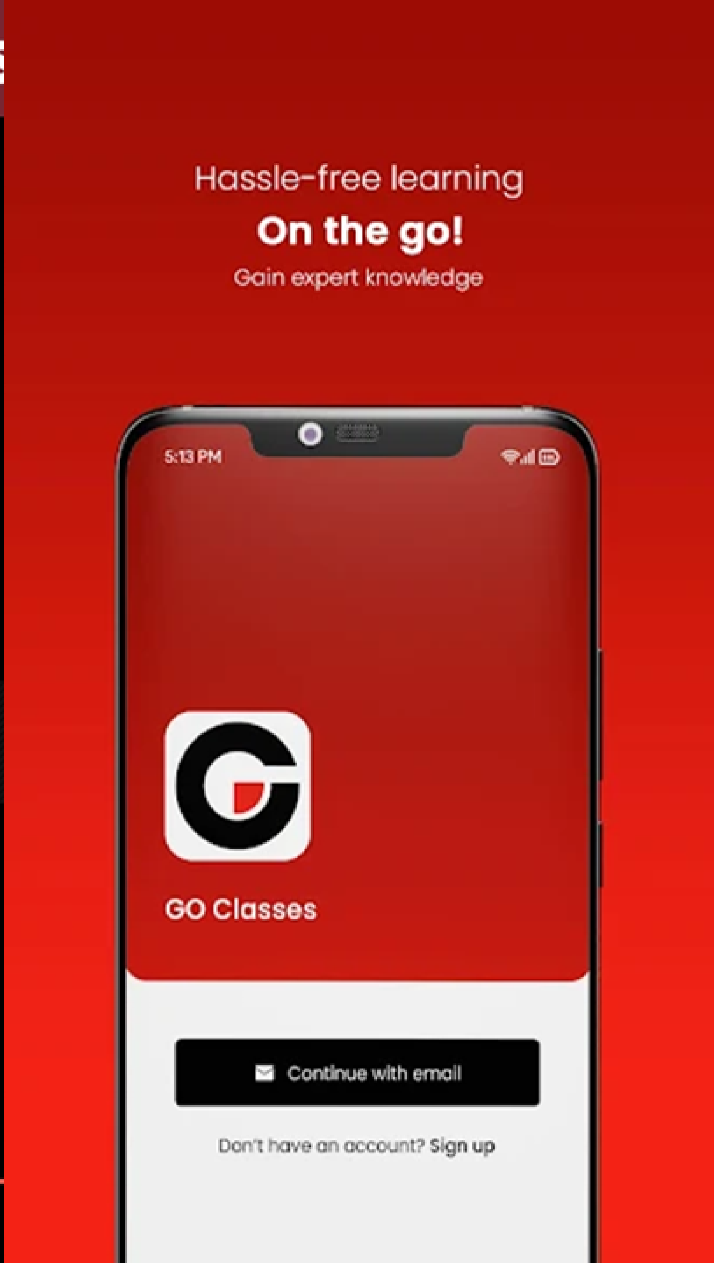
1. **Quality Questions:** No Ambiguity in Questions, All Well-framed questions.
2. Correct, **Detailed Explanation**, Covering Variations of questions.
3. **Video Solutions.**

https://gateoverflow.in/blog/14987/gate-overflow-and-go-classes-test-series-gate-cse-2024

Download the GO Classes Android App:

https://play.google.com/store/apps/details?id=com.goclasses.courses

Search “GO Classes” on Play Store.



NOTE :

Complete Discrete Mathematics & Complete Engineering Mathematics Courses, by GO Classes, are **FREE** for ALL learners.

Visit here to watch : https://www.goclasses.in/s/store/

SignUp/Login on Goclasses website for free and start learning.

We are on Telegram. **Contact us** for any help.

Link in the Description!!

Join GO Classes **Doubt Discussion** Telegram Group :

Username:

@GATECSE_Goclasses

We are on Telegram. **Contact us** for any help.

Join GO Classes Telegram Channel, Username: @GOCLASSES_CSE

Join GO Classes **Doubt Discussion** Telegram Group :

Username: @GATECSE_Goclasses

(Any doubt related to Goclasses Courses can also be asked here.)

Join GATEOverflow Doubt Discussion Telegram Group :

Username: @GateOverflow_CSE

Next Topic:

Arguments in FOL

Arguments in prop. logic:

Arg. $\left[\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \hline \therefore C \end{array}\right.$

P_1, P_2, P_3 } Premises

$\therefore C$ } Conclusion

Arg.

Valid

Invalid

Arguments in first order logic:

Arg. $\left[\begin{array}{l} P_1 \\ P_2 \\ P_3 \\ \hline \therefore C \end{array}\right.$ $\left.\begin{array}{l} \\ \\ \\ \end{array}\right\}$ Premises, $\}$ Conclusion

Arg.

Valid

Invalid

In FOL:

Argument

P_1

P_2

P_3

$\therefore c$

$\boxed{\text{FOL expression}}$

Note:

Arg. is Valid iff

[Whenever ALL Premises are True then Conclusion must be True.]

Arg.

$$\left.\begin{array}{c} P_1 \\ P_2 \\ P_3 \\ \hline \therefore\ c \\ \hline \end{array}\right\} \rightarrow \text{Arg is valid iff}$$

$(P_1 \wedge P_2 \wedge P_3 \longrightarrow c)$ is Valid → Always True.

Arg 1:

$\forall_x P(x)$

$\therefore P(b)$

Valid

Domain

a, b, c

Arg 1:

$$\forall_x P(x)$$

$$\therefore P(c)$$

Valid

Domain

a, b, c
...

Arg 1:

$\forall_x P(x)$

$P(b)$

valid

Universal instantiation

Domain

a, b, c

Arg 2 :

$$\left.\begin{array}{l} P(b) \\ \hline \therefore \forall_x P(x) \\ \hline \end{array}\right\} \underline{\text{Valid}}\,?$$

Domain

a, b, c

Arg 2:

$$\begin{array}{l} P(b) \\ \hline \therefore\ \forall x\, P(x) \\ \hline \end{array}$$

Invalid

Domain

a, b, c, ...

Arg 3:

for arbitrary m, $P(m)$

$\therefore \forall x\, P(x)$

Valid?

Domain

a, b, c

$\vdots$

Arg 3:

$\boxed{\text{whatever } m \in \text{Domain}}$

for arbitrary m, $P(m)$

$\therefore \quad \forall x\, P(x)$

Valid ✓

Universal Generalization

Domain

a, b, c

Specific Vs Arbitrary

↓

Particular

Eg: a

b

↓

Any / whatever

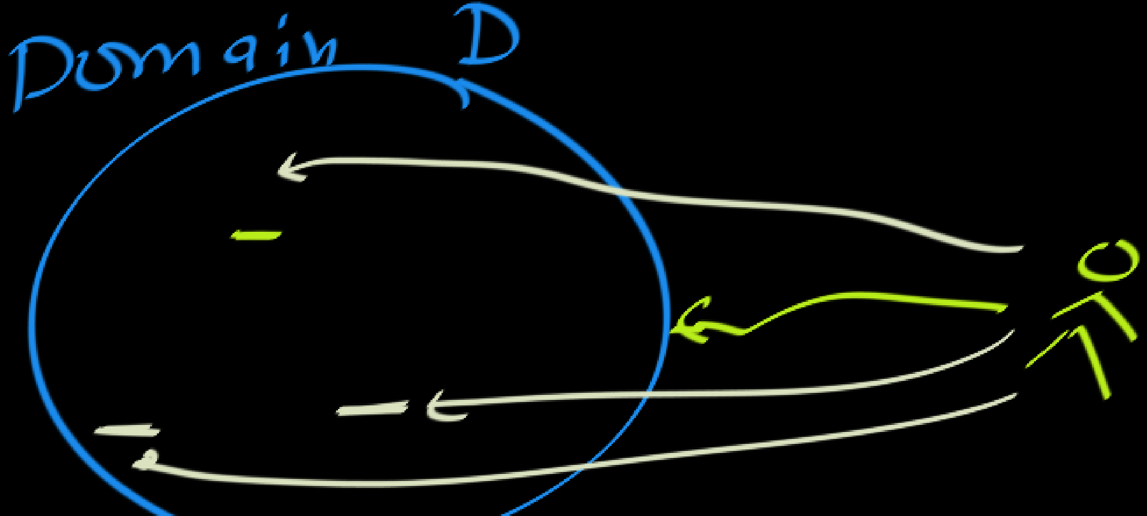
Domain

a, b, c

$$\begin{array}{l} \text{for arbitrary } x \in D,\ P(x) \\ \hline \therefore\ \forall x\, P(x) \\ \hline \end{array}$$

Valid

$\boxed{\text{Universal Generalization}}$



(3)

$$\begin{array}{l} P(a) \\ \hline \therefore \exists x\, P(x) \\ \hline \end{array}$$

Valid

Existential Generalization

Domain

a, b, c

(4)

$$\exists x\, P(x)$$

$$\therefore\; P(a)$$

Invalid

Domain

a, b, c ...

5

$\exists x\, P(x)$

---

for "some" particular $m \in$ Domain, $P(m)$

Domain

a, b, c

Valid

Existential Instantiation

Q :

$\exists x\, P(x)$

$\therefore$ for arbitrary m , $P(m)$

Invalid

Domain

a, b, c

Q :

$\forall x\, P(x)$

$\therefore$ for arbitrary m, $P(m)$

---

valid

Domain

a, b, c

GO CLASSES

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$	Existential generalization

TABLE 2 Rules of Inference for Quantified Statements.

Rule of Inference	*Name*
$\dfrac{\forall x P(x)}{\therefore\ P(c)}$	Universal instantiation
$\dfrac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$	Universal generalization
$\dfrac{\exists x P(x)}{\therefore\ P(c) \text{ for some element } c}$	Existential instantiation
$\dfrac{P(c) \text{ for some element } c}{\therefore\ \exists x P(x)}$	Existential generalization

$\forall x\, P(x)$

$\therefore\ P(b)$

Valid

Universal instantiation

$P(b)$

$\therefore\ \forall x\, P(x)$

Invalid

Note:

these Names are Not important.

$$\forall x(P(x) \rightarrow Q(x))$$

$P(a)$, where a is a particular element in the domain

$\therefore Q(a)$

Valid?

$\boxed{\forall x(P(x) \to Q(x))}$

$P(a)$, where a is a particular element in the domain

$\therefore Q(a)$

Domain

a,

for every element x, if P(x) = T then Q(x) = T.

P(a) = True

∴ Q(a)

Valid

$P \to Q$

P

$\therefore Q$

Modus Ponens

$$\forall x (P(x) \rightarrow Q(x))$$

$\neg Q(a)$, where a is a particular element in the domain

$$\therefore \neg P(a)$$

Valid?

$$\forall x (P(x) \rightarrow Q(x))$$

$\neg Q(a)$, where a is a particular element in the domain

$$\therefore \neg P(a)$$

Domain

$a, \ldots$

for everyone x, if $P(x) = T$ then $Q(x) = T$ ✓

$Q(a) = F \equiv \neg Q(a)$

$P(a) = F \equiv \neg P(a)$

Valid

$\forall x(P(x) \to Q(x))$

$\neg Q(a)$, where a is a particular element in the domain

$\therefore \neg P(a)$

$\forall x(\neg Q(x) \to \neg P(x))$

$\neg Q(a)$

$\neg P(a)$

Valid

Domain

$a, \ldots$

$p \rightarrow q$

$\neg q$

$\therefore \neg p$

Modus Tollens

$\forall x\,(P(x) \rightarrow Q(x))$

$\neg Q(a)$

$\therefore \neg P(a)$

Q: $\exists x \left(P(x) \rightarrow Q(x) \right)$

$P(a)$; a is a particular element in Domain

$\therefore Q(a)$

Valid?

Q: $\exists x \left(P(x) \rightarrow Q(x) \right)$

Domain: a, b, c, ...

$P(a)$; a is a particular element in Domain

$\therefore Q(a)$

Invalid

$P(b) = T$
$Q(b) = T$
$P(a) = T$
$Q(a) = F$

(1) Every man is mortal

Aristotle is a man

Therefore: Aristotle is mortal

(2) Aristotle is a man

Therefore: Someone is a man

Valid

(1) Every man is mortal $\forall x\,(man(x) \rightarrow mortal(x))$

x

Domain: Aristotle, ...

Aristotle is a man $man(Aristotle)$

Therefore: Aristotle is mortal $mortal(Aristotle)$

(2) Aristotle is a man $man(Aristotle)$

Therefore: Someone is a man $\exists x\,(man(x))$

Valid

1

$\forall x \, (P(x) \rightarrow Q(x))$

$P(a)$

$\therefore \; Q(a)$

Valid

→ Similar to modus ponens

(2)

$$P(b)$$

$$\therefore \exists x\, P(x)$$

Existential Generalization

Universal generalization is the rule of inference that states that $\forall x P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the domain. Universal generalization is used when we show that $\forall x P(x)$ is true by taking an arbitrary element c from the domain and showing that $P(c)$ is true. The element c that we select must be an arbitrary, and not a specific, element of the domain. That is, when we assert from $\forall x P(x)$ the existence of an element c in the domain, we have no control over c and cannot make any other assumptions about c other than it comes from the domain. Universal generalization is used implicitly in many proofs in mathematics and is seldom mentioned explicitly. However, the error of adding unwarranted assumptions about the arbitrary element c when universal generalization is used is all too common in incorrect reasoning.

Existential instantiation is the rule that allows us to conclude that there is an element c in the domain for which $P(c)$ is true if we know that $\exists x P(x)$ is true. We cannot select an arbitrary value of c here, but rather it must be a c for which $P(c)$ is true. Usually we have no knowledge of what c is, only that it exists. Because it exists, we may give it a name (c) and continue our argument.

Existential generalization is the rule of inference that is used to conclude that $\exists x P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the domain for which $P(c)$ is true, then we know that $\exists x P(x)$ is true.

Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

Show that the premises “Everyone in this discrete mathematics class has taken a course in computer science” and “Marla is a student in this class” imply the conclusion “Marla has taken a course in computer science.”

$P_1 : \forall x \left(DM(x) \rightarrow CS(x) \right)$

$P_2 : DM(marla)$

$\therefore CS(marla)$

Valid

$\equiv \forall x \left(P(x) \rightarrow Q(x) \right)$

$P(a)$

$\therefore Q(a)$

Show that the premises "Everyone in this discrete mathematics class has taken a course in computer science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in computer science."

Solution: Let $D(x)$ denote "x is in this discrete mathematics class," and let $C(x)$ denote "x has taken a course in computer science." Then the premises are $\forall x(D(x) \rightarrow C(x))$ and $D(\text{Marla})$. The conclusion is $C(\text{Marla})$.

Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

Show that the premises “A student in this class has not read the book,” and “Everyone in this class passed the first exam” imply the conclusion “Someone who passed the first exam has not read the book.”

$P_1 : \boxed{\exists x \, \neg Book(x)}$

$P_2 : \forall x \; Exam(x)$

$\therefore \exists x \, (Exam(x) \land \neg Book(x))$

Valid

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

P_1: $\boxed{\exists x \neg Book(x)}$ $\neg Book(k)$

P_2: $\forall x \; Exam(x)$ $Exam(k)$

$\therefore \exists x \left(Exam(x) \wedge \neg Book(x) \right)$ — True

x → Domain

All students in class

witness: k

Show that the premises "A student in this class has not read the book," and "Everyone in this class passed the first exam" imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and $P(x)$ be "x passed the first exam." The premises are $\exists x(C(x) \wedge \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$. The conclusion is $\exists x(P(x) \wedge \neg B(x))$. These steps can be used to establish the conclusion from the premises.

27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ and $\forall x(P(x) \wedge R(x))$ are true, then $\forall x(R(x) \wedge S(x))$ is true.

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 27

27. Use rules of inference to show that if $\forall x(P(x) \rightarrow (Q(x) \land S(x)))$ and $\forall x(P(x) \land R(x))$ are true, then $\forall x(R(x) \land S(x))$ is true.

Conclusion

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 27

1: $\forall x \left(P(x) \rightarrow \left(Q(x) \wedge S(x) \right) \right)$

2: $\forall x \left(P(x) \wedge R(x) \right)$

$\therefore \forall x \left(R(x) \wedge S(x) \right)$

Valid

1: $\forall x \left(P(x) \rightarrow \left(Q(x) \land S(x) \right) \right)$

2: $\forall x \left(P(x) \land R(x) \right)$

$\therefore \forall x \left(R(x) \land S(x) \right)$ → Valid Conclusion

Domain:

	P	R	S	Q
a:	T	T	T	T
b	T	T	T	T
c	T	T	T	T
⋮				

28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 28

28. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$ and $\forall x((\neg P(x) \wedge Q(x)) \rightarrow R(x))$ are true, then $\forall x(\neg R(x) \rightarrow P(x))$ is also true, where the domains of all quantifiers are the same.

Conclusion

by Default

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 28

1: $\forall x \left(P(x) \vee Q(x) \right)$

2: $\forall x \left((\neg P(x) \wedge Q(x)) \rightarrow R(x) \right)$

$\therefore \forall x \left(\neg R(x) \rightarrow P(x) \right)$

1: $\forall x \left(P(x) \lor Q(x) \right)$

2: $\forall x \left((\neg P(x) \land Q(x)) \rightarrow R(x) \right)$

$\therefore \forall x \left(\neg R(x) \rightarrow P(x) \right)$

Valid Conclusion

$\boxed{\text{any} \rightarrow T \equiv T}$

Domain:

	P	Q	R
a	T	T	
b	T	F	
c	F	T	T
⋮			

To Check Validity of Argument:

method ①

All Premises True

↓

Is Conclusion True

method ②

Make Conclusion : false

↓

See if we can make All Premises True

To Check Validity of Argument:

Proposition logic: method 2 is better.

first order logic: method 1 is better.

1: $\forall x \left(P(x) \vee Q(x) \right)$: True

2: $\forall x \left((\neg P(x) \wedge Q(x)) \longrightarrow R(x) \right)$

$\therefore \forall x \left(\neg R(x) \longrightarrow P(x) \right)$

false

false

Domain:

	P	Q	R
a			
b:	F	T	F
c			
⋮			

1: $\forall x \left(P(x) \vee Q(x) \right)$

2: $\forall x \left((\neg P(x) \wedge Q(x)) \rightarrow R(x) \right)$

$\therefore \forall x \left(\neg R(x) \rightarrow P(x) \right)$

Valid Arg

After making Conclusion false; we can NOT make All premises True

Checking Validity of Arg ;

Premises $\Rightarrow$ Conclusion ✓

$\neg$ Conclusion $\Rightarrow$ $\neg$ Premises ✓

29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 29

29. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x \neg P(x)$ are true, then $\exists x \neg R(x)$ is true.

Conclusion

Source: Kenneth H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Exercise 1.6 Question 29

Method 1 :

All Premises True

Check Conclusion

1: $\forall x\,(P(x) \lor Q(x))$

2: $\forall x\,(S(x) \lor \neg Q(x))$

3. $\forall x\,(R(x) \rightarrow \neg S(x))$

4. $\exists x\,(\neg P(x))$

$\therefore\ \exists x\ \neg R(x)$

Valid Conclusion

Domain:

	P	Q	R	S
a	F	T	F	T
b				
c				
⋮				

method 2:

make
Conclusion : false

Try to make All premises True

1: $\forall x\,(P(x) \vee Q(x))$: T

2: $\forall x\,(S(x) \vee \neg Q(x))$: T

3: $\forall x\,(R(x) \rightarrow \neg S(x))$: T

4. $\exists x\,(\neg P(x))$: Can't make True

$\therefore \exists x\, \neg R(x)$: false

Domain:

	P	Q	R	S
a	T	F	T	F
b	T	F	T	F
c	T	F	T	F
⋮				

Valid Arg ✓

1. (20 points) Prove the validity of the following arguments using the rules of inference (for propositional or predicate logic, whenever appropriate).

(i.) (10 points)

$$\begin{array}{r} p \vee q \\ p \vee r \\ \neg p \\ \hline \therefore q \wedge r \end{array}$$

(ii.) (10 points)

$$\begin{array}{r} \forall x\,(P(x) \vee Q(x)) \\ \forall x\,(Q(x) \rightarrow R(x)) \\ \exists x\,\neg P(x) \\ \hline \therefore \exists x\,R(x) \end{array}$$

1. (20 points) Prove the validity of the following arguments using the rules of inference (for propositional or predicate logic, whenever appropriate).

(i.) (10 points)

method 1:

All Premises True

Check Conclusion

1: $p \vee q$ T : $q = T$

2: $p \vee r$ T : $r = T$

3: $\neg p$ T : $p = f$

$\therefore q \wedge r$ T

Valid

(ii.) (10 points)

$$\forall x\,(P(x) \vee Q(x))$$
$$\forall x\,(Q(x) \rightarrow R(x))$$
$$\exists x\,\neg P(x)$$

$$\therefore \exists x\,R(x)$$

1. (20 points) Prove the validity of the following arguments using the rules of inference (for propositional or predicate logic, whenever appropriate).

(i.) (10 points)

method 2:

make Conclusion false

[redacted]

Try to make All premises True

$$p \vee q$$
$$p \vee r$$
$$\neg p$$

$\therefore q \wedge r$: False

[redacted]

3 Cases

Valid

Case 1: q : F, r : F

Case 2: q : T, r : F

Case 3: q : F, r : T

Can Not make All premises True

Prop. logic: method 2 better most times.

first order logic: method 1 better most times

ADVICE: You must be Pro in both method.

1. (20 points) Prove the validity of the following arguments using the rules of inference (for propositional or predicate logic, whenever appropriate).

(i.) (10 points)

$$\begin{array}{r} p \vee q \\ p \vee r \\ \neg p \\ \hline \therefore q \wedge r \end{array}$$

(ii.) (10 points)

$$\begin{array}{r} \forall x\,(P(x) \vee Q(x)) \\ \forall x\,(Q(x) \to R(x)) \\ \exists x\,\neg P(x) \\ \hline \therefore \exists x\, R(x) \end{array}$$

Method 1

1: $\forall x\,(P(x) \vee Q(x))$: T

2: $\forall x\,(Q(x) \rightarrow R(x))$: T

3: $\exists x\,\neg P(x)$: T

$\therefore\ \exists x\, R(x)$: Valid Conclusion

Valid Argument

Domain:

	P	Q	R
a			
b	F	T	T ✓
c			
⋮			

method 2

1: $\forall x\,(P(x) \lor Q(x))$ → True

2: $\forall x\,(Q(x) \to R(x))$ → True

3: $\exists x\, \neg P(x)$: Can't make True

$\therefore\ \exists x\, R(x)$: false

Domain:

	P	Q	R
a	T	F	F
b	T	F	F
c	T	F	F
⋮			

Valid Arg

FOL : Nothing Remaining

Optional : L1

L2

L3

L4 : Tautology in FOL → 'GATE ques.'

In FOL :

Tautology $\neq$ Valid

FOL :

$\boxed{\text{Completely Complete}}$

Solve All GATE PYQs, $\boxed{\text{GoClasses Test Series DM Test -1}}$