PROOF TECHNIQUES

Proofs

GATE OVERFLOW

GO CLASSES

www.goclasses.in

Basic Proof Techniques

Homework 1

“Direct Proofs”

Complete Discrete Mathematics Course Link:

https://www.goclasses.in/courses/Discrete-Mathematics-Course

Proofs : Anyone who doesn't believe there is creativity in mathematics clearly has not tried to write proofs. Finding a way to convince the world that a particular statement is necessarily true is a mighty undertaking and can often be quite challenging. There is not a guaranteed path to success in the search for proofs.

For example, in the summer of 1742, a German mathematician by the name of Christian Goldbach wondered whether every even integer greater than 2 could be written as the sum of two primes. Centuries later, we still don't have a proof of this apparent fact (computers have checked that "Goldbach's Conjecture" holds for all numbers less than 4×10^{18}, which leaves only infinitely many more numbers to check).

We study Proof Techniques to improve our Analytical Skills.

Aim to study proof techniques is to understand the proofs which keep occurring in all the subjects.

Don't get demotivated if you couldn't come up with a proof, if you couldn't write a proof, if you couldn't think in a particular way of proving.... Be happy when you understand a proof which is taught to you.

There is NO need to practice "proof" related questions from books. Whatever we have studied(or will study throughout the course) in the our lectures, in our homeworks, in practice sets; is **MORE than ENOUGH**.

Direct proof of $P \Rightarrow Q$

This is the simplest, and most natural, method of proof. *Try this first before considering other methods.*

Assume P.
...
[Logical deductions]
...
Therefore Q.

Subscribe GO Classes YouTube Channel :

https://www.youtube.com/channel/UCyvbBJhLJ-5YqwPA884vjoQ

Elementary methods of proving $P \Rightarrow Q$.

Basic method to prove "If P then Q" type of statements :

Direct Proof: Assume P, follow logical deductions, conclude Q.

Direct Proof

Assume P. Explain, explain, . . . , explain. Therefore Q.

Q1.(Sums and products of even/odd numbers.)

Prove the following statements:

(a) If n and m are both odd, then n + m is even.

(b) If n is odd and m is even, then n + m is odd.

(c) If n and m are both even, then n + m is even.

(d) If n and m are both odd, then nm is odd; otherwise, nm is even.

Solution 1 : We give a proof for the first statement, (a); the other statements, (b)–(d), can be proved in an analogous/similar way.

(a) Proof of "n and m odd $\Rightarrow$ n + m even":

- Suppose(Given that) n and m are odd integers.
- Then $n = 2k + 1$ and $m = 2l + 1$ for some $k, l \in Z$, by the definition of an odd integer.
- Therefore $n + m = (2k + 1) + (2l + 1) = 2(k + l + 1)$.
- Since k and l are integers, so is $k + l + 1$.
- Hence $n + m = 2p$ with $p = k + l + 1 \in Z$.
- By the definition of an even integer, this shows that n + m is even.

Q 2. (Even/odd squares:)

Prove the following:

(a) Let n be an integer. If n is odd, then n^2 is odd.

(b) Let n be an integer. If n is even, then n^2 is even.

Q3.

Prove that:

For all integers a, b, and c, if $a|b$ and $b|c$ then $a|c$.

(Here $x|y$, read "x divides y" means that y is a multiple of x, i.e., that x will divide into y with remainder zero).

Solution 3.

Even before we know what the divides symbol means, we can set up a direct proof for this statement. It will go something like this: Let a, b, and c be arbitrary integers. Assume that $a|b$ and $b|c$. Dot dot dot dot. Therefore $a|c$.

How do we connect the dots? We say what our hypothesis ($a|b$ and $b|c$) really means and why this gives us what the conclusion ($a|c$) really means. Another way to say that $a|b$ is to say that $b = ka$ for some integer k (that is, that b is a multiple of a). What are we going for? That $c = la$, for some integer l (because we want c to be a multiple of a). Here is the complete proof.

Proof: Let a, b, and c be integers. Assume that $a|b$ and $b|c$. In other words, b is a multiple of a and c is a multiple of b. So there are integers k and j such that $b = ka$ and $c = jb$. Combining these (through substitution) we get that $c = jka$. But jk is an integer, so this says that c is a multiple of a. Therefore $a|c$.

Q4.

Prove that if n and m are positive, even integers, then nm is divisible by 4.

Q5.

A perfect number is a positive integer n such that the sum of the factors of n is equal to 2n (1 and n are considered factors of n). So 6 is a perfect number since $1 + 2 + 3 + 6 = 12 = 2 * 6$. Similarly, the divisors of 28 are 1, 2, 4, 7, and 14, 28 and $1 + 2 + 4 + 7 + 14+28 = 2^{*}28$.

Prove that a prime number cannot be a perfect number.

Hint : What are the divisors of a prime number ?

Solution 5:

Prove that a prime number cannot be a perfect number.

Answer :

Prime numbers cannot be perfect. By definition, a number N is perfect if the sum of its divisors is 2N.

For any prime number P, it's divisors are P and 1. The sum of these divisors is (P+1), which is always less than 2P.

Q 6.

Prove that there does not exist an integer $n > 3$ such that $n, n+2, n+4$ are each prime.

i.e. For $n>3$, show that the integers $n, n+2$, and $n+4$ cannot all be prime (i.e. at least one of them must be Non-prime)

Hint : Every integer n can be written in one and only one of the following forms: $n=3k$ OR $n=3k+1$ OR $n=3k+2$, where k is some integer.

Solution 6:

When we divide any integer n by 3, we get remained 0 or 1 or 2. So,

Every integer n can be written in one and only one of the following forms:

$n=3k$ OR $n=3k+1$ OR $n=3k+2$, where k is some integer.

Assume $n>3$ is prime. Then n cannot have the form $n=3k$ (else 3 would be a factor). Thus you have either $n=3k+1$ or $n=3k+2$. In the first case you have that $n+2=3k+3=3(k+1)$, in the second one you have that $n+4=3k+6=3(k+2)$.

Thus you can never have n, n+1 and n+2 primes at the same time.

Q7.

Prove that if p, q are positive integers such that $p|q$ and $q|p$, then $p = q$.

Q 8.

Prove that for any integer x, the integer $x(x + 1)$ is even.

Q9.

(An integer a is a **perfect square** if there is an integer b such that $a = b^2$.)

If m and n are perfect square, then m+n+2√(mn) is a perfect square.

Video Solution:

https://youtu.be/_nDAsHxErAk

Solution 9:

If m and n are perfect square, then $m+n+2\sqrt{(mn)}$ is a perfect square.

Proof

$m = a^2$ and $n = b^2$ for some integers a and b

Then $m + n + 2\sqrt{(mn)} = a^2 + b^2 + 2ab$

$= (a + b)^2$

So $m + n + 2\sqrt{(mn)}$ is a perfect square.

Q10. Assume "a" is some integer.

Prove

$$7 \mid 4a \rightarrow 7 \mid a$$

Video Solution: https://youtu.be/n-deKicFaH8

NOTE :

Complete Discrete Mathematics & Complete Enginnering Mathematics

Courses, by GO Classes, are FREE for ALL learners.

Click here to watch : https://www.goclasses.in/s/store/

SignUp/Login (for free) on Goclasses website and start learning.

Join GO Classes Doubt Discussion Telegram Group :

https://t.me/GATECSE_Goclasses

GO Classes Channel: https://t.me/goclasses_cse

Join GATEOverflow Doubt Discussion Telegram Group :

https://t.me/gateoverflow_CSE

DEEPAK POONIA

IISc Bangalore

GATE AIR 53 ; 67 ; 206

www.goclasses.in +91 6302536274

SACHIN MITTAL

IISc Bangalore

Ex Amazon scientist ; GATE AIR 33

www.goclasses.in +91 6302536274