Propositional Logic

Next Chapter:

Tautology in Propositional Logic

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GATE CSE AIR 53; AIR 67; AIR 206; AIR 256;

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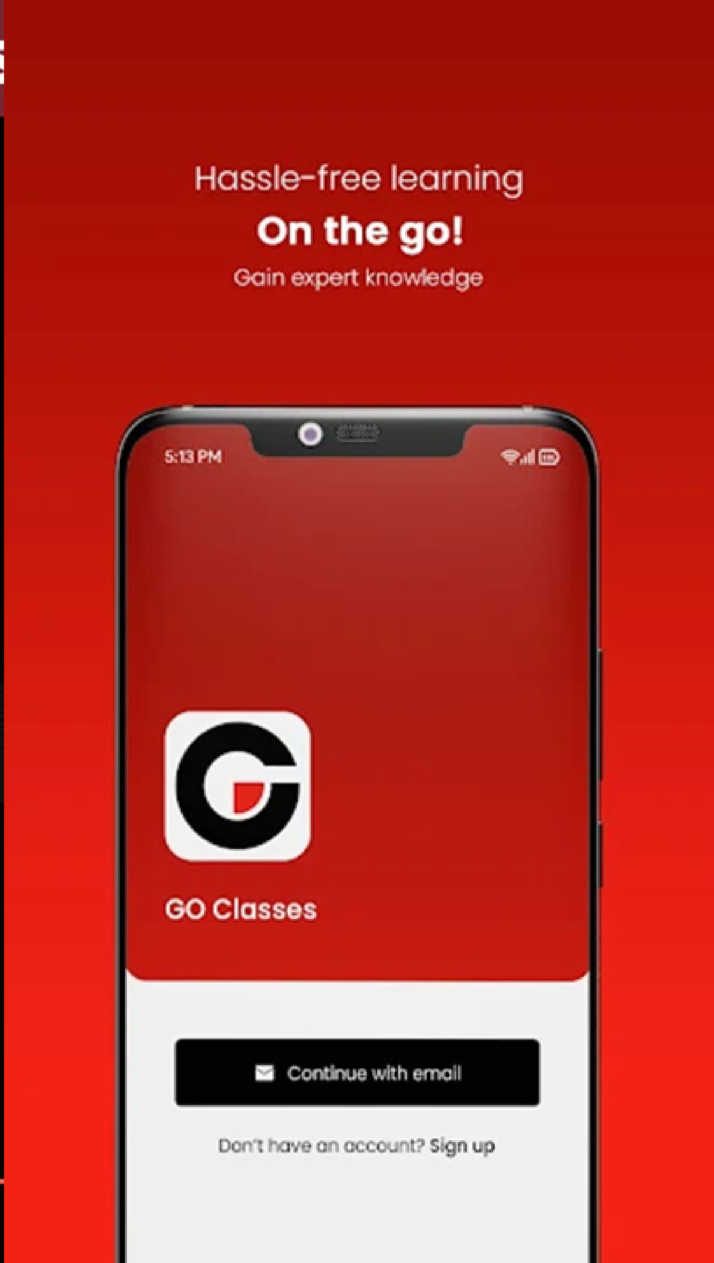

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Next Topic:

Tautology, Contradiction, Contingency in Propositional Logic

Compound Propositions :

$p \vee \bar{p}$ α_1

$p \wedge \bar{p}$ α_2

$p \rightarrow q$ α_3

propositions

Tautology α_1 α_2 Contradiction α_3

P	Q	$P \vee \overline{P}$	$P \wedge \overline{P}$	$P \to Q$
F	F	T	F	T
F	T	T	F	T
T	F	T	F	F
T	T	T	F	T

Always True

Always false

Contingency

Sometimes True and Sometimes false

$\boxed{\text{Some} \equiv \text{At least one}}$

Contingency = Sometimes True (NOT a Contradiction)

and

Sometimes false (NOT a Tautology)

$p \vee \overline{p} \equiv \text{True}$

$T \vee F = \text{True}$

$F \vee T = \text{True}$

$p \vee \overline{p} = \text{True}$

$p \wedge \overline{p} = \text{false}$

$p \wedge \overline{p} \equiv \text{false}$

Eg:

		Always True	Always false	Sometimes True / Sometime false	
p	q	$p \vee p'$	$p \wedge p'$	$p \to q$	$\neg p \to q$
F	F	T	F	T	F
F	T	T	F	T	T
T	F	T	F	F	T
T	T	T	F	T	T
		Tautology	Contradiction	Contingency	

Compound proposition / prop. formula

iitg.ac.in/dgoswami/Logic2.pdf

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Three Classes

A *Tautology* is a wff for which all truth table values are T.
A *Contradiction* is a wff for which all truth table values are F.
A *Contingency* is a wff that is neither a tautology nor a contradiction.

Examples. $P \vee \neg P$ is a tautology. $P \wedge \neg P$ is a contradiction. $P \rightarrow Q$ is a contingency.

Always True

Always false

Some True and Some false

Tautologies are a key concept in propositional logic, where a tautology is defined as a propositional formula that is true under any possible Boolean valuation of its propositional variables.

Tautology/Contradiction/Contingency:

(a) A compound proposition is called a tautology if that is always true, no matter what the truth values of the propositions that occur in it.

(b) A compound proposition that is always false is called a contradiction.

(c) A proposition that is neither a tautology nor a contradiction is called a contingency.

Let us look at the classic example of a tautology, $p \vee \neg p$. The truth table

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

shows that $p \vee \neg p$ is true no matter the truth value of p.

The proposition $p \vee \neg(p \wedge q)$ is also a tautology as the following the truth table illustrates.

Compound Statement

variables

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$p \vee \neg(p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Always True

Tautology

Examples

The following are contradictions:

(a) $p \wedge \neg p$ α_1

(b) $(p \vee q) \wedge (\neg p) \wedge (\neg q)$ → Truth Table → All false → (b) Contradiction

(a) $p \wedge \overline{p}$ α_1

$p = T \longrightarrow \alpha_1 = F$

$p = F \longrightarrow \alpha_1 = F$

α_1 Always false → α_1 Contradiction

Which of the following propositional logic expressions are Tautology ?

P : prop. Variable.

1. P
2. $\sim P$
3. $T \rightarrow P$
4. $P \rightarrow P$
5. $P \rightarrow \sim P$
6. $\sim P \rightarrow P$
7. T
8. F
9. $P \rightarrow F$

Which of the following propositional logic expressions are Tautology ?

P : prop. Variable

1. P → Contingency
2. ~P → Contingency
3. T→ P → Contingency
4. P→ P → Tautology
5. P→~P → Contingency
6. ~P→P → Contingency
7. T → Tautology
8. F → Contradiction
9. P→F → Contingency

P

{T, F} → ~~Contingency~~

P	$\bar{P}$
T	F
F	T

→ Contingency

$T \rightarrow p \equiv p$ → Contingency

p	$T \rightarrow p$
T	T
F	F

Contingency

$p \rightarrow p$ → Tautology

p	$p \rightarrow p$
T	T
F	T

Always True

$P \to F$

Constant (fixed)

P	$P \to F$
T	F
F	T

→ Contingency

→ Sometimes True and Sometimes false

Compound proposition

Definition 1. Given propositional formula p, then p is a

1. **tautology** iff p always evaluates to `true`, regardless of how its variables are assigned.

2. **fallacy** iff p always evaluates to `false`, regardless of how its variables are assigned.

3. **contingency** iff p's variables may be assigned in a way that makes p evaluate to `true`, and may be assigned in another way that makes p evaluate to `false`.

Note:

In propositional logic:

1. Tautology ≡ Valid ≡ Always True
2. Contradiction ≡ fallacy ≡ Unsatisfiable
3. Satisfiable ≡ (Tautology OR Contingency)

Satisfiable : Can satisfy i.e.

Can make it True

for at least one combination of Truth values.

Unsatisfiable : Always False ≡ Contradiction

α_2: $P \wedge (P \rightarrow Q)$ $\rightarrow$ Satisfiable ✓

$\rightarrow$ $P = T, Q = T$ $\Rightarrow$ $\alpha_2 = \text{True}$

α_1: $P \vee (P \rightarrow Q)$ — Satisfiable ✓

$\rightarrow$ $P = T$ $\Rightarrow$ $\alpha_1 = \text{True}$

$p \wedge \bar{p}$

Can't satisfy

Can't make it True

Always False $\equiv$ Contradiction

$\equiv$ Unsatisfiable

α_3

$p \lor \bar{p}$ $\longrightarrow$ Satisfiable

$p = F$ $\Longrightarrow$ $\alpha_3 = \text{True}$

2.1. Tautology/Contradiction/Contingency.

DEFINITION 2.1.1. *A* **tautology** *is a proposition that is always true.*

EXAMPLE 2.1.1. $p \vee \neg p$

DEFINITION 2.1.2. *A* **contradiction** *is a proposition that is always false.*

EXAMPLE 2.1.2. $p \wedge \neg p$

DEFINITION 2.1.3. *A* **contingency** *is a proposition that is neither a tautology nor a contradiction.*

EXAMPLE 2.1.3. $p \vee q \to \neg r$

Propositional formula

$\equiv$ Compound Proposition

$\equiv$ well formed formula (wff)

$\equiv$ Proposition

$\equiv$ Prop. expression

Proposition formula α → Compound Proposition

prop. variables in α

p_1	p_2	$---$	p_n	α
F	F	$---$	F	T
⋮	⋮		⋮	T
⋮	⋮		⋮	T
⋮	⋮		⋮	T
T	T	$---$	T	T

→ Tautology

Always True

Proposition formula α → Compound Proposition

prop. variables in α

P_1	P_2	---	P_n	α
F	F	---	F	F
⋮	⋮		⋮	F
⋮	⋮		⋮	F
⋮	⋮		⋮	F
T	T	---	T	F

→ Contradiction

Always false

Proposition formula α → Compound proposition

prop. variables in α

p_1	p_2	- - -	p_n	α
F	F	- - -	F	⋮
⋮		⋮	⋮	⋮
-	-	- - -	-	T
⋮		⋮	⋮	⋮
				F
T	T	- - -	T	⋮

Contingency

Sometimes True & Sometimes False

Proposition formula α

Satisfiable : at least one time True

P	Q	R	- - - -	Z	α
F	F	F	- - - -	F	
					T
T	T	T	- - - -	F	

$\alpha \rightarrow$ Satisfiable

Sometimes True

Proposition formula α

Satisfiable : at least one time True

Tautology OR Contingency

Proposition formula α

Satisfiable $\equiv$ Consistent $\equiv$ Sometimes True

Proposition formula α

Contradiction $\equiv$ fallacy $\equiv$ Always false

$\equiv$ Unsatisfiable

Proposition formula α

Valid ≡ Tautology ≡ Always True

Invalid ≡ Sometimes false

≡ falsifiable ≡ (Contingency or Contradiction)

Proposition formula α

falsifiable $\equiv$ Sometimes false

NOT falsifiable $\equiv$ Always True

$\equiv$ Tautology $\equiv$ Valid

Proposition formula α

Satisfiable $\equiv$ Consistent

Unsatisfiable $\equiv$ InConsistent $\equiv$ Contradiction

Always false

Proposition formula α

Satisfiable : Can satisfy $\equiv$ Sometimes True

falsifiable : Can make it false $\equiv$ Invalid

$\equiv$ NOT Tautology

EXERCISE 2.1.1. *Build a truth table to verify that the proposition* $(p \leftrightarrow q) \wedge (\neg p \wedge q)$ *is a contradiction.*

α

Can not make True

p	q	$\alpha = (p \leftrightarrow q) \wedge (\bar{p} \wedge q)$
F	F	F
F	T	F
T	F	F
T	T	F

Always false

Contradiction
$\equiv$ Unsatisfiable
$\equiv$ Inconsistent

Unfalsifiable $\equiv$ $\boxed{\text{Can NOT make false}}$

$\equiv$ Always True

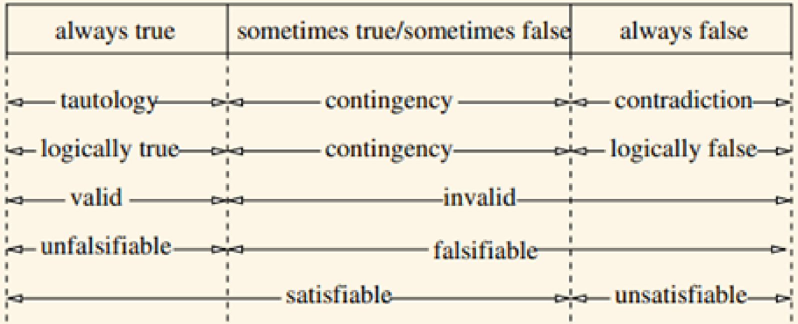
$\equiv$ Tautology $\equiv$ Valid

Validity, Unsatisfiability

- The truth value of a propositional formula depends on truth assignments to variables
- Example: $\neg p$ evaluates to true under under the assignment $p = F$ and to false under $p = T$
- Some formulas evaluate to true for every assignment, e.g., $p \lor \neg p$
- Such formulas are called tautologies or valid formulas
- Some formulas evaluate to false for every assignment, e.g., $p \land \neg p$
- Such formulas are called unsatisfiable formulas or contradictions

13.1. Valid, Invalid, Satisfiable and Unsatisfiable

- The truth-tables we saw in the previous lecture show that some wffs evaluate to true no matter how their atomic wffs are interpreted (every interpretation satisfies them), some evaluate to false for every interpretation of their atomic wffs (no interpretation satisfies them), and others are true for some interpretations and false for others (some interpretations satisfy them, others don't.) There is some terminology for all of this. (There would be, wouldn't there?)



Some books also use 'consistent' for 'satisfiable' and 'inconsistent' for 'unsatisfiable' but this is to be avoided.

We will use just the following: valid/invalid and satisfiable/unsatisfiable. I ignore all the others to avoid terminology overload:

  - A wff is *valid* if all interpretations satisfy it. Otherwise (i.e. if at least one does not satisfy it), it is *invalid*.
  - A wff is *satisfiable* if there is at least one interpretation that satisfies it. Otherwise (i.e. if no interpretation satisfies it), it is *unsatisfiable*.

Note how valid and satisfiable overlap: every valid wff is satisfiable, but not every satisfiable wff is valid. Similarly, every unsatisfiable wff is invalid, but not every invalid wff is unsatisfiable.

always true	sometimes true/sometimes false	always false
tautology	contingency	contradiction
logically true	contingency	logically false
valid	invalid	
unfalsifiable	falsifiable	
satisfiable		unsatisfiable

always true	sometimes true/sometimes false	always false
tautology	contingency	contradiction
logically true	contingency	logically false
valid	invalid	
unfalsifiable	falsifiable	
satisfiable		unsatisfiable

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a *tautology*. A compound proposition that is always false is called a *contradiction*. A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true. When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is **unsatisfiable**.

Note that a compound proposition is unsatisfiable if and only if its negation is true for all assignments of truth values to the variables, that is, if and only if its negation is a tautology.

Satisfiable:

α is Satisfiable if it possible to make α True in at least one case.

Satisfiable $\equiv$ either Tautology or Contingency.

Satisfiable $\equiv$ At least one time True.

Unsatisfiable $\equiv$ Contradiction $\equiv$ Always false

In Propositional logic ;

Tautology ≡ Valid ≡ Always True ≡ Equivalent to True

Invalid ≡ Not Tautology ≡ Contradiction or Contingency

Contradiction ≡ Unsatisfiable ≡ Always false ≡ Equival-
-ent to false.

Satisfiable ≡ Atleast one time True ≡ Tautology or Contingency.

At least one time false $\equiv$ Contradiction or Contingency

Contingency $\equiv$ At least one time True AND At least one time false

Contingency $\equiv$ Neither Tautology NOR Contradiction

Example 4. Let a and b be Boolean variables. Then

1. a is a contingency since $a = 0$ makes a false, and $a = 1$ makes a true.

2. $a \to b$ is a contingency since $(a = 1, b = 0)$ makes $a \to b$ false, while $(a = 1, b = 1)$ makes $a \to b$ true.

3. $a \vee \neg a$ is a tautology since, no matter how a is assigned, either a or $\neg a$ will evalute to `true`.

4. $a \wedge \neg a$ is a fallacy since, no matter how a is assigned, either a or $\neg a$ will evalute to `false`, and so a and $\neg a$ cannot both be true.

5. $a \to (b \to a)$ is a tautology. One way to see this is by constructing a truth table and verifying that the $a \to (b \to a)$-column consists of all 1's (see Table 3). Alternatively, notice that the only way this formula could be false is if a is true and $b \to a$ is false. But $b \to a$ is false only when a is false, which conflicts with the requirement that a be true. Therefore, the formula must be tautology.

6. $(a \to b) \to (\neg a \to \neg b)$ is a contingency, since it's true for $(a = 0, b = 0)$, but false for $(a = 0, b = 1)$.

Next Topic:

By Case Method

to Solve questions Quickly

Truth Table : Inefficient

Efficient method : "By Case" method

Analytical

P → T

OR

P → F

Q — T

OR

Q — F

R — T

OR

R — F

p	q	$p \land (p \to q)$
F	F	
F	T	
T	F	
T	T	

P=F

P=T

Two Cases

① P=F Case 1

② P=T Case 2

Q = T

Case 1 Q = T

Case 2 Q = F

P	Q	$(P \vee Q)$	$(P \wedge Q)$	$\sim (P \wedge Q)$	$(P \vee Q) \wedge \sim (P \wedge Q)$
T	T	T	T	F	**F**
T	F	T	F	T	**T**
F	T	T	F	T	**T**
F	F	F	F	T	**F**

Q = F

$\bar{P} \vee Q$

Truth Table :

P — T or F

Q — T or F

P	Q	$\bar{P} \vee Q$
T	F	F
T	T	T
F	T	T
F	F	T

P	Q
F	F
T	T
F	T
T	F

4 Combinations

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

Case1:
P=T

Case2:
P=F

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	T	**T**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	T	**F**
F	F	T	T	**F**
F	F	F	F	**T**

from P's POV:

Case1: P=F

Case2: P=T

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	T	**T**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	T	**F**
F	F	T	T	**F**
F	F	F	F	**T**

from Q's pov:

Case 1: $Q = T$

Case 2: $Q = F$

P	Q	R	$Q \vee R$	$P \Leftrightarrow (Q \vee R)$
T	T	T	T	**T**
T	T	F	T	**T**
T	F	T	T	**T**
T	F	F	F	**F**
F	T	T	T	**F**
F	T	F	T	**F**
F	F	T	T	**F**
F	F	F	F	**T**

R=T

R=F

Two Cases:

Case 1: R = T

Case 2: R=F

By Case method:

Q: Prop. logic Question → Prop variable

P, Q, R ...

Ans: By Case method:

Case 1 — P = T

Case 2 — P = F

Q:

c) In propositional logic, a *contradiction* is a compound proposition that is always *false*, no matter what the truth values of the propositional variables that occur in it are. Show by means of a truth table that $p \wedge \neg (q \vee p)$ is a contradiction.

Q: c) In propositional logic, a *contradiction* is a compound proposition that is always *false*, no matter what the truth values of the propositional variables that occur in it are. Show by means of a truth table that $p \wedge \neg (q \vee p)$ is a contradiction.

boring

Inefficient

time consuming

HW

p q

Q: c) In propositional logic, a *contradiction* is a compound proposition that is always *false*, no matter what the truth values of the propositional variables that occur in it are. Show by means of a truth table that $p \wedge \neg (q \vee p)$ is a contradiction.

$\alpha : \quad p \wedge \neg(p \vee q)$

Always false

In every case, false

Q: c) In propositional logic, a *contradiction* is a compound proposition that is always *false*, no matter what the truth values of the propositional variables that occur in it are. Show by means of a truth table that $p \wedge \neg (q \vee p)$ is a contradiction.

$\alpha : p \wedge \neg(p \vee q)$

Case 1: $p = F$

α : False

Case 2: $p = T$

$\alpha : T \wedge \neg(T \vee q)$

T

$\alpha : T \wedge F =$ false

α : $p \wedge \neg(p \vee q)$

Always false

Whether $p = T$ OR $p = F$

All possibilities

α : Contradiction

Examples

The following are all tautologies:

Always True

In every case True

(a) $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

(b) $p \vee \neg p$

(c) $(p \wedge q) \rightarrow p$

(d) $q \rightarrow (p \vee q)$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

Examples

The following are all tautologies:

(a) $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

(b) $p \vee \neg p$

(c) $(p \wedge q) \rightarrow p$

(d) $q \rightarrow (p \vee q)$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

Case 1: $p = T$

(a): $q' \leftrightarrow q'$

Same

True

Case 2: $p = F$

(a) $T \leftrightarrow T$

True

$F \vee \bar{q} = \bar{q}$

$T \wedge q = q$

Examples

The following are all tautologies:

(a) $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

(b) $p \vee \neg p$ → Always True

(c) $(p \wedge q) \rightarrow p$

(d) $q \rightarrow (p \vee q)$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

p	$p \vee \bar{p}$
T	T
F	T

$p \vee \bar{p}$ → Tautology

Always True

Examples

The following are all tautologies:

(a) $(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$

(b) $p \vee \neg p$ Always

(c) $(p \wedge q) \rightarrow p$ True

(d) $q \rightarrow (p \vee q)$

(e) $(p \vee q) \leftrightarrow (q \vee p)$

Case1: $p = T$

(c): True

Case2 $p = F$

(c) $F \longrightarrow F$

$\equiv$ True

$\boxed{\text{Anything} \longrightarrow \text{True}}$ $\equiv$ True

HW

3. (**9 points**) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$

b. $(p \vee \neg q) \wedge (\neg p \vee q)$

c. $(\neg p \vee q) \vee (p \wedge q)$

3. **(9 points)** Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$ Truth Table

[redacted]

Satisfiable $\equiv$ Consistent

p	q	$\bar{q} \vee (p \wedge q)$
F	F	T
F	T	—
T	F	—
T	T	—

NOT Contradiction

3. (**9 points**) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$

Truth Table

[redacted]

p	q	$\bar{q} \vee (p \wedge q)$
F	F	T
F	T	F
T	F	–
T	T	–

Contingency

3. (**9 points**) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$

Truth Table

[redacted]

p	q	$\overline{q} \vee (p \wedge q)$
F	F	T
F	T	F
T	F	T
T	T	T

3. (**9 points**) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

a. $\neg q \vee (p \wedge q)$

By Case method:

Case 1: $p = F$

$a : \bar{q} \vee F$

$\bar{q}$

$q = T \Rightarrow a = F$

$q = F \Rightarrow a = T$

$a :$ Contingency

Case 2: $p = T$

$a : \bar{q} \vee q = \text{True}$

3. (9 points) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

[redacted]

b. $(p \vee \neg q) \wedge (\neg p \vee q)$

By Case method:

Case 1: $p = T$

b: $T \wedge q = q$

$q = T \Rightarrow b = T$

$q = F \Rightarrow b = F$

(b) Contingency

Case 2: $p = F$

b: $\bar{q} \wedge T = \bar{q}$

Not needed

3. (**9 points**) Which of the following compound propositions are **tautologies**, which are **contradictions**, and which are **neither** (contingencies). Justify your answer for each.

[redacted]

c. $(\neg p \vee q) \vee (p \wedge q)$

$P = T$ Case 1:

$P = F$

$c: \; q \vee q$

$= q$

$q = T \rightarrow c = T$

$q = F \rightarrow c = F$

(c) Contingency

Satisfiable $\equiv$ Not Contradiction

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

(a) $(p \to r) \land (r \to p)$

(b) $(p \to q) \land (p \land \neg q)$

(c) $(p \lor q) \land (\neg p \land r) \land (r \lor q)$

(d) $(p \land q) \land (\neg p \lor r) \land (\neg q \lor \neg r)$

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

(a) $(p \to r) \land (r \to p)$

Case 1: $r = T$

Case 2: $r = F$

[redacted]

(a)

(a) : $T \land p = p$

$a = \text{True}$ when $p = T$, $r = T$

$T \to p \equiv p$

$\Rightarrow$ (a) satisfiable

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

[redacted]

(b) $(p \to q) \wedge (p \wedge \neg q)$

[redacted]

α

(b)

Case 1: $p = T$

(b): $q \wedge q' = \text{false}$

Case 2: $p = F$:

$T \wedge F = \text{false}$

$T \wedge \alpha = \alpha$

$T \to q \equiv q$

(b): Unsatisfiable $\equiv$ Contradiction

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

[redacted]

(c) $(p \vee q) \wedge (\neg p \wedge r) \wedge (r \vee q)$

[redacted]

Case 1:

$p = T$

(c): $T \wedge (F) \wedge (r \vee q)$

(c) = False

Case 2:

$p = F$

E

$\boxed{q \wedge r \wedge (r \vee q)}$

Case 1 $q = T$

Case 2 $q = F$

E: r

(c)

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

[redacted]

(c) $(p \lor q) \land (\neg p \land r) \land (r \lor q)$

[redacted]

when $p = F$, $q = T$
then
$(c) = r$

(c) True when
$p = F$, $q = T$, $r = T$

Satisfiable

Which of the following compound propositions are satisfiable?

If they are satisfiable, give a truth assignment. If they are not satisfiable, write the truth table.

(a) $(p \to r) \land (r \to p)$

(b) $(p \to q) \land (p \land \neg q)$

(c) $(p \lor q) \land (\neg p \land r) \land (r \lor q)$

(d) $(p \land q) \land (\neg p \lor r) \land (\neg q \lor \neg r)$

HW

Public Telegram Group Link

$\boxed{\text{By Case}}$ → time Saving

Next Topic:

Logical Equivalence

Ex:

It is Not true that John is both intelligent & Hardworking $\equiv \neg(I \wedge H)$

I H

$\equiv$ John is Not intel OR Not hardworking.

$\neg I \vee \neg H$

equivalent

$$\boxed{\neg(I \land H) \equiv \neg I \lor \neg H}$$

Equivalence

Using Intuition, Understand

Symbol for Equivalence :

$$\equiv$$

$$\neg(I \wedge H) \equiv \neg I \vee \neg H$$

I	H	$\neg(I \wedge H)$	$\neg I \vee \neg H$
F	F	T	T
F	T	T	T
T	F	T	T
T	T	F	F

Same Truth Table

Let P, Q be two propositions.

It is Not the case that both P, Q are true.

!!!

$\equiv$ Either P NOT true OR Q NOT True

Let p, q be two propositions.

It is Not the case that both p, q are true. $\equiv \neg(p \wedge q)$

$\equiv$ Either p Not true OR q Not True

$\equiv \neg p \vee \neg q$

equivalent

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

$P = F, Q = F$: T , T

$P = F, Q = T$: T , T

$P = T, Q = F$: T , T

$P = T, Q = T$: F , F

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

Same behaviour

for every truth value combination

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

p	q	$\neg(p \wedge q)$	$\equiv$	$\neg p \vee \neg q$
F	F	T		T
F	T	T		T
T	F	T		T
T	T	F		F

$P \to Q \not\equiv Q \to P$

$P = T, Q = T$: T, T

$P = F, Q = F$: T, T

$P = T, Q = F$: F, T

Different behaviour

$$P \to Q \not\equiv Q \to P$$

P	Q	$P \to Q$	$\not\equiv$ $Q \to P$
F	F	T	T
F	T	T	F
T	F	F	T
T	T	T	T

NOT Same

Note: Propositions: α, β

$\alpha \equiv \beta$ iff α, β have Equal truth value in every case.

Let P, Q be two propositions.

It is Not the case that at least one of P, Q is true.

||| intuition

$\equiv$ P false and Q false.

Let P, Q be two propositions.

It is Not the case that at least one of P, Q is true. $\equiv \neg(P \vee Q)$

$\equiv$ P false and Q false. $\equiv \neg P \wedge \neg Q$

$\neg P \quad \wedge \quad \neg Q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

p	q	$\neg(p \vee q)$	$\equiv$	$\neg p \wedge \neg q$
F	F	T		T
F	T	F		F
T	F	F		F
T	T	F		F

Same Truth Table

Suppose we have two propositions, **p** and **q**. The propositions are *equal* or **LOGICALLY EQUIVALENT** if they always have the same truth value. That is, **p** and **q** are logically equivalent if **p** is true whenever **q** is true, and vice versa, and if **p** is false whenever **q** is false, and vice versa. If **p** and **q** are logically equivalent, we write **p = q**.

Compound propositions: α, β

$\alpha \equiv \beta$ iff α, β have same Truth Table.

Compound propositions: α, β

$\alpha \equiv \beta$ iff α and β Always have equal truth value.

Compound propositions: α, β

$\alpha \equiv \beta$ iff

α True $\longrightarrow$ β True

and

α false $\longrightarrow$ β false

Compound propositions: α, β

$\alpha \equiv \beta$ iff

p_1 p_2 --- p_n	α	$\equiv$	β	$\alpha \leftrightarrow \beta$
	T		T	T
	T		T	T
	F		F	T
	T		T	T
	F		F	T

Always True

Tautology

Compound propositions: α, β

$\alpha \equiv \beta$ iff $\boxed{\alpha \leftrightarrow \beta \text{ is a Tautology}}$

Logical Equivalence

The compound propositions p and q are called *logically equivalent* if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.

symbol for Equivalence

$\alpha \equiv \beta$ iff α, β have same truth Table.

p_1	p_2	...	p_n	α	$\equiv$	β	$\alpha \leftrightarrow \beta$
–	–		–	T		T	T
–	–		.	T		T	T
–	–			T		T	T
–	–			F		F	T
–	–			F		F	T
–	–		–	T		T	T

Tautology

Always True

$\alpha \equiv \beta$ iff α, β have same truth Table.

p_1	p_2 ...	p_n	α	$\equiv$	β	$\alpha \to \beta$	$\beta \to \alpha$
–	–	–	T		T	T	T
–	–	.	T		T	T	T
-	–		T		T	T	T
.	–		F		F	T	T
-	–		F		F	T	T
–	–	–	T		T	T	T
							T

Tautology

Propositional Expressions ; α, β

$\alpha \equiv \beta$ iff

1. Same Truth table.
2. Always equal truth value.
3. $\alpha \leftrightarrow \beta$ is Tautology.
4. $\alpha \rightarrow \beta$ is Tautology and $\beta \rightarrow \alpha$ is Tautology.

Remark: The symbol $\equiv$ is not a logical connective, and $p \equiv q$ is not a compound proposition but rather is the statement that $p \leftrightarrow q$ is a tautology. The symbol $\Leftrightarrow$ is sometimes used instead of $\equiv$ to denote logical equivalence.

2.2. Logically Equivalent.

DEFINITION 2.2.1. *Propositions r and s are* **logically equivalent** *if the statement $r \leftrightarrow s$ is a tautology.*

Notation: If r and s are logically equivalent, we write

$$r \Leftrightarrow s.$$

Equivalence Symbol :

$\equiv$ → more popular

$\Longleftrightarrow$ → some Authors

You can determine whether compound propositions r and s are logically equivalent by building a single truth table for both propositions and checking to see that they have exactly the same truth values.

EXAMPLE 2.3.1. *Show that* $(p \to q) \wedge (q \to p)$ *is logically equivalent to* $p \leftrightarrow q$.

Solution 1. *Show the truth values of both propositions are identical.*

Truth Table:

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Intuitive

Means: $(p \to q)$ and $(q \to p)$

EXAMPLE 2.3.2. *Show* $\neg(p \to q)$ *is equivalent to* $p \wedge \neg q$.

Solution 1. Build a truth table containing each of the statements.

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Since the truth values for $\neg(p \to q)$ and $p \wedge \neg q$ are exactly the same for all possible combinations of truth values of p and q, the two propositions are equivalent.

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

p	q	$\neg(p \rightarrow q)$	$p \wedge \overline{q}$
F	F	F	F
F	T	F	F
T	F	T	T
T	T	F	F

Same

$\neg(p \to q) \overset{?}{\equiv} p \wedge \neg q$

equivalent

Using By Case method:

In All Cases, Equal

$\alpha \quad \neg(p \to q) \overset{?}{\equiv} p \wedge \neg q \quad \beta$

By Case method:

two cases;

Case 1: $p = T$

$\alpha : \neg q$
$\beta : \neg q$ } equal

Case 2: $p = F$

$\alpha : F$
$\beta : F$ } equal

$$\boxed{T \to q \equiv q}$$

So, in every case; $\alpha = \beta$

<u>Hence</u>; $\alpha \equiv \beta$ ✓

Next Topic:

Logical Equivalence Practice

Using “By Case” Method

$\alpha \equiv \beta$ $\longleftrightarrow$ in every case $\alpha = \beta$.

Part II: Proving logical equivalence using laws of propositional logic (50 pt.)

4. (30 pt.) Use the laws of propositional logic to prove that the following compound propositions are logically equivalent.

   a. (5 pt.) $(p \vee q) \wedge \neg(p \wedge \neg q)$ and q

   b. (5 pt.) $\neg p \rightarrow \neg(q \vee r)$ and $(q \rightarrow p) \wedge (r \rightarrow p)$

   c. (10 pt.) $\neg(p \vee (q \wedge (\neg p \rightarrow \neg r)))$ and $\neg p \wedge (q \rightarrow r)$

   d. (10 pt.) $p \leftrightarrow (p \wedge q)$ and $\neg p \vee q$

Part II: Proving logical equivalence using laws of propositional logic (50 pt.)

4. (30 pt.) Use the laws of propositional logic to prove that the following compound propositions are logically equivalent.

   a. (5 pt.) $(p \vee q) \wedge \neg(p \wedge \neg q)$ and q

   b. (5 pt.) $\neg p \rightarrow \neg(q \vee r)$ and $(q \rightarrow p) \wedge (r \rightarrow p)$

   c. (10 pt.) $\neg(p \vee (q \wedge (\neg p \rightarrow \neg r)))$ and $\neg p \wedge (q \rightarrow r)$

   d. (10 pt.) $p \leftrightarrow (p \wedge q)$ and $\neg p \vee q$

a) $\alpha : (p \vee q) \wedge \neg(p \wedge \overline{q})$ $\beta : q$

Case1 : $q = T$

$\beta : T$

$\alpha : T \wedge T = T$

equal

Case2 : $q = F$

$\alpha : p \wedge \neg(p) = F$

$\beta :$ false

equal

(b) $\alpha : \neg p \rightarrow \neg(q \vee r)$ $\beta : (q \rightarrow p) \wedge (r \rightarrow p)$

Case 1 : $p = T$

α : True

β : True

} equal

Case 2 : $p = F$

$\alpha : \neg(q \vee r)$

$\beta : \bar{q} \wedge \bar{r}$

} equivalent

$$\boxed{T \to Y \equiv Y} \qquad \boxed{Y \to F \equiv \overline{Y}}$$

$$T \to \underbrace{\neg(q \vee r)}_{Y} \equiv \neg(q \vee r)$$

Constant (fixed)

$Y \rightarrow F \equiv \boxed{\quad} \rightarrow \bar{Y}$

$T \;—\; \boxed{F}$

$F \;—\; \boxed{T}$

Part II: Proving logical equivalence using laws of propositional logic (50 pt.)

4. (30 pt.) Use the laws of propositional logic to prove that the following compound propositions are logically equivalent.

   a. (5 pt.) $(p \lor q) \land \neg(p \land \neg q)$ and q

   b. (5 pt.) $\neg p \rightarrow \neg(q \lor r)$ and $(q \rightarrow p) \land (r \rightarrow p)$

   c. (10 pt.) $\neg(p \lor (q \land (\neg p \rightarrow \neg r)))$ and $\neg p \land (q \rightarrow r)$

   d. (10 pt.) $p \leftrightarrow (p \land q)$ and $\neg p \lor q$

(c)

$\alpha : \neg(p \vee (q \wedge (\bar{p} \rightarrow \bar{r})))$

$\beta : \neg p \wedge (q \rightarrow r)$

Case 1: $p = T$

$\alpha : \neg(T) = F$

$\beta : F$

equal

Case 2: $p = F$

$\alpha : \neg(q \wedge \bar{r})$

$\beta : q \rightarrow r$

equivalent

$\neg(q \wedge \bar{r}) \equiv q \to r$

Case 1: $q = T$

$r \equiv r$

Case 2: $q = F$

$\text{True} \equiv \text{True}$

(D)

$\alpha : p \leftrightarrow (p \wedge q)$ $\beta : \neg p \vee q$

Case 1 : $p = \text{True}$

$\alpha : T \leftrightarrow q \equiv q$

$\beta : q$

Same

Case 2 : $p = F$

$\alpha : F \leftrightarrow F \equiv T$

$\beta : T$

Same

Constant

$T \leftrightarrow \varphi = \boxed{\quad}$

φ

T —— T

F —— F

Problem 3: Logical Equivalence *(30 pts)*

Prove that the following expressions are **logically equivalent**. You can use **either** a truth table **or** laws of propositional logic! If you choose to use a truth table, please begin the enumeration of your inputs as **F**, **F**, **F**, instead of **T**, **T**, **T**, as we have discussed in class and discussion. If you choose to use propositional logic laws, you **must** document which law you are using for every derivation.

$$p \Rightarrow (q \vee r),\ \big(p \wedge ({\sim}q)\big) \Rightarrow r,\ \big(p \wedge ({\sim}r)\big) \Rightarrow q$$

BEGIN YOUR ANSWER BELOW THIS LINE

Problem 3: Logical Equivalence *(30 pts)*

Prove that the following expressions are **logically equivalent**. You can use **either** a truth table **or** laws of propositional logic! If you choose to use a truth table, please begin the enumeration of your inputs as **F**, **F**, **F**, instead of **T**, **T**, **T**, as we have discussed in class and discussion. If you choose to use propositional logic laws, you **must** document which law you are using for every derivation.

$$p \Rightarrow (q \vee r), \quad (p \wedge (\sim q)) \Rightarrow r, \quad (p \wedge (\sim r)) \Rightarrow q$$

α β γ

BEGIN YOUR ANSWER BELOW THIS LINE

$\alpha \equiv \beta$; $\alpha \equiv \gamma$ $\Longrightarrow$ $\beta \equiv \gamma$

Problem 3: Logical Equivalence *(30 pts)*

Prove that the following expressions are **logically equivalent**. You can use **either** a truth table **or** laws of propositional logic! If you choose to use a truth table, please begin the enumeration of your inputs as **F**, **F**, **F**, instead of **T**, **T**, **T**, as we have discussed in class and discussion. If you choose to use propositional logic laws, you **must** document which law you are using for every derivation.

$$p \Rightarrow (q \vee r),\ (p \wedge (\sim q)) \Rightarrow r,\ (p \wedge (\sim r)) \Rightarrow q$$

BEGIN YOUR ANSWER BELOW THIS LINE

??

$\equiv$

$\alpha : \boxed{p \to (q \vee r)}$ $\beta : \boxed{(p \wedge \overline{q}) \to r}$

Case 1: $p = F$

$\alpha : T$
$\beta : T$
equal

Case 2: $p = T$

$\alpha : q \vee r$
$\beta : \overline{q} \to r$
equivalent

E_1

$q \vee r$

E_2

$\bar{q} \to r$

Case 1 : $q = T$

$E_1 : T$

$E_2 : T$

Same ✓

$q = F$

$E_1 : r$

$E_2 : r$

Same ✓

Problem 3: Logical Equivalence *(30 pts)*

Prove that the following expressions are **logically equivalent**. You can use **either** a truth table **or** laws of propositional logic! If you choose to use a truth table, please begin the enumeration of your inputs as **F**, **F**, **F**, instead of **T**, **T**, **T**, as we have discussed in class and discussion. If you choose to use propositional logic laws, you **must** document which law you are using for every derivation.

$$p \Rightarrow (q \vee r), \quad (p \wedge (\sim q)) \Rightarrow r, \quad (p \wedge (\sim r)) \Rightarrow q$$

BEGIN YOUR ANSWER BELOW THIS LINE

$\equiv$

$$p \rightarrow (q \vee r) \equiv p \wedge \bar{r} \rightarrow q$$

HW

Note :

$\alpha \equiv \beta$; and $\beta \equiv \gamma$

then $\boxed{\alpha \equiv \gamma}$??

p_1	p_2	----	p_n	α	$\equiv$	β	$\equiv$	γ
-	-		-	T		T		T
-	-			T		T		T
-	-			F		F		F
-	-			T		T		T
				⋮		⋮		⋮
-	-		-					
-	-		-					

$\alpha \equiv \gamma$

Note :

$\alpha \equiv \beta$; and $\beta \equiv \gamma$

then $\boxed{\alpha \equiv \gamma}$?? ✓

Note:

$\boxed{\text{If } \alpha \equiv \beta \text{ and } \beta \equiv \gamma \text{ then } \alpha \equiv \gamma.}$

2 Logical equivalence

HW

a) [redacted] demonstrate that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. (1 mark)

b) [redacted] demonstrate that $p \wedge q \equiv \neg(\neg p \vee \neg q)$. (1 mark)

c) [redacted] show that

$$p \to q \equiv \neg(p \wedge \neg q).$$

[redacted]

by case

d) [redacted] show that

$$p \leftrightarrow q \equiv (\neg p \vee q) \wedge (\neg q \vee p).$$

[redacted]