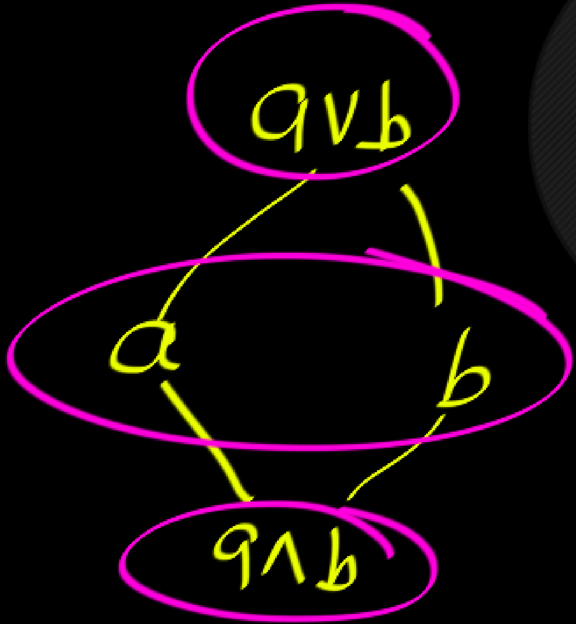
Set Theory : Lattices

Some Questions related to Maximal, Minimal, Greatest, Least Elements in Lattices.

Website : https://www.goclasses.in/

Lattice: a poset in which $\forall a, b$

$\exists (a \vee b)$, $\exists (a \wedge b)$.



$a \, R \, (a \vee b)$; $b \, R \, (a \vee b)$

$a \wedge b \, R \, a$; $(a \wedge b) \, R \, b$.

In a Lattice;

$$\forall x \;, \quad \left.\begin{array}{l} x \, R \, (x \vee y) \\ x \wedge y \, R \, x \end{array}\right\}$$

GO Classes

Test Series

On "GATE Overflow" Website

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let $a, b \in L$;

If $a \neq b$ then $a \neq a \vee b$. True Or False?

If $a \neq b$ then $a \neq a \wedge b$. True Or False?

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.
Let $a, b \in L$;

If $a \neq b$ then $a \neq a \vee b$. True Or False?

$(\mathbb{N}, \leq)$

TOR $\rightarrow$ Lattice

$a = 5$

$b = 2$

$a \neq b$

$\boxed{a \vee b = 5} = a$

$2 \leq 5$; $2 R 5$

5 — $2 \vee 5$

2 — $2 \wedge 5$

3

2

1

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

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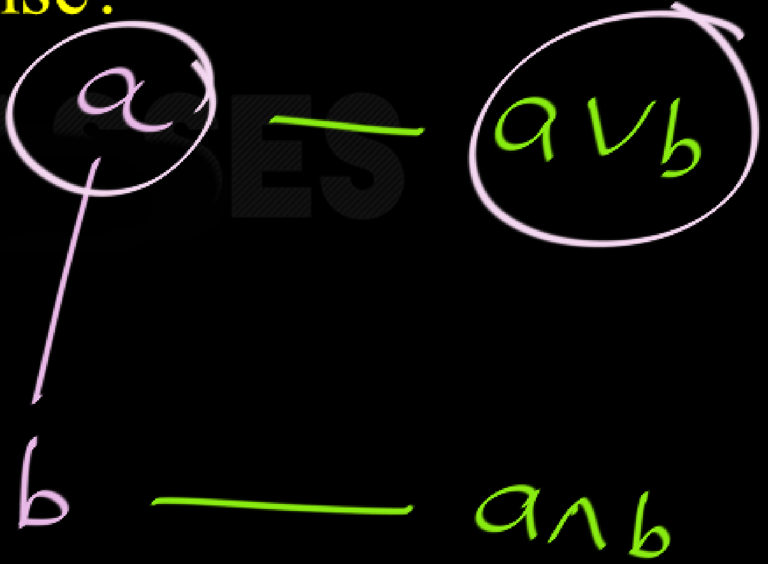
$a \neq b$; bRa



Quick Question:

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If $a \neq b$ then $a \neq a \wedge b$. True Or False?

$(N, \leq)$ Lattice

$a = 2$
$b = 5$

$b = 5$

$a = 2$

$a \wedge b = 2 = a$

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let $a, b \in L$;

If $a \neq b$ then $a \neq a \vee b$. True Or False?

If $a \neq b$ then $a \neq a \wedge b$. True Or False?

$\underbrace{a \neq b}$

aRb

$b = a \vee b$

$a = a \wedge b$

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let $a, b \in L$;

If a, b are incomparable then $a \neq a \vee b$. True Or False?

If a, b are incomparable then $a \neq a \wedge b$. True Or False?

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let a, b ∈ L ;

If a, b are incomparable then $a \neq a \vee b$. True Or False?

$a \not R b$

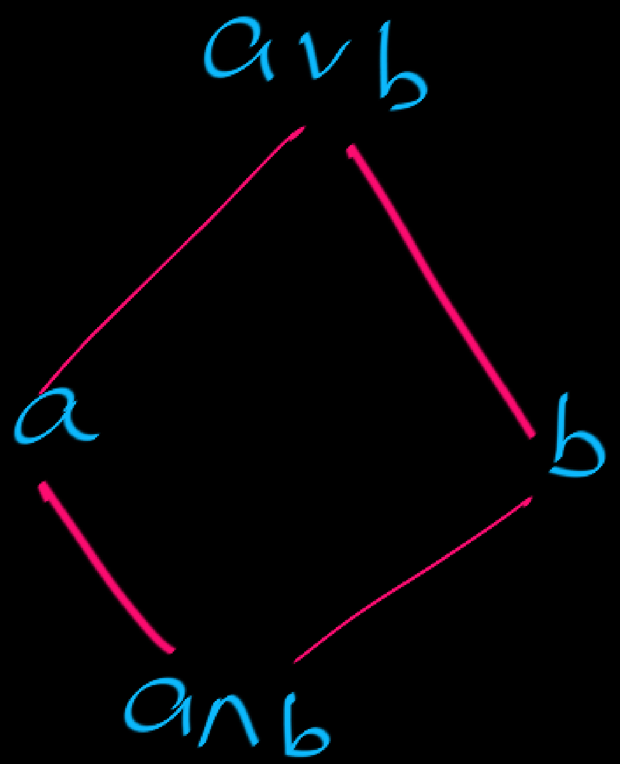
& $b \not R a$

Assume $a = a \vee b$

$b R (a \vee b)$

$b R a$ Contradiction

Lattice:



$(a \wedge b)\, R\, b$

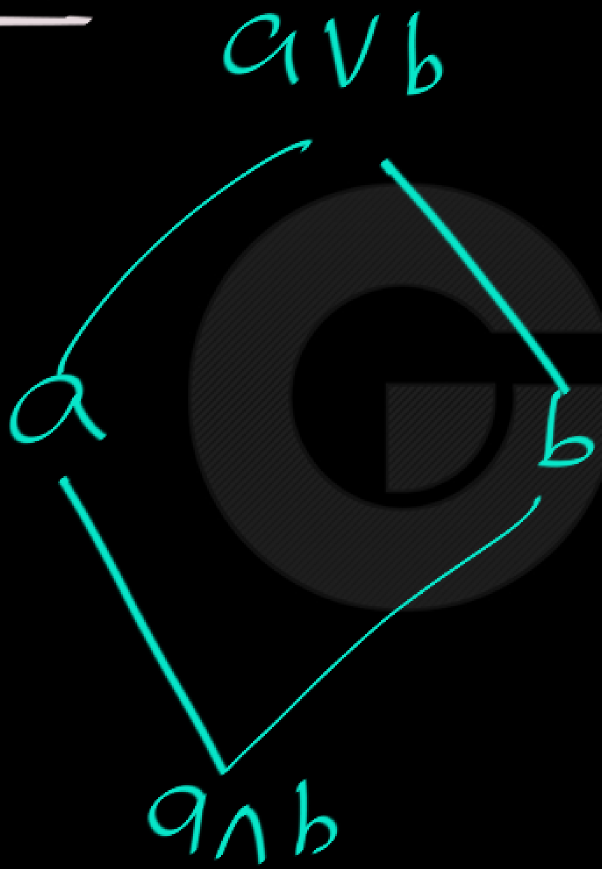
$a\, R\, (a \vee b)$

$(a \wedge b)\, R\, a$

$(a \wedge b)\, R\, (a \vee b)$

$\boxed{b\, R\, (a \vee b)}$

Lattice:



$$\left.\begin{array}{l} b \; R \; (a \vee b) \\ (a \wedge b) \; R \; b \end{array}\right\}$$

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let a, b ∈ L ;

If a, b are incomparable then $a \neq a \vee b$. True Or False?

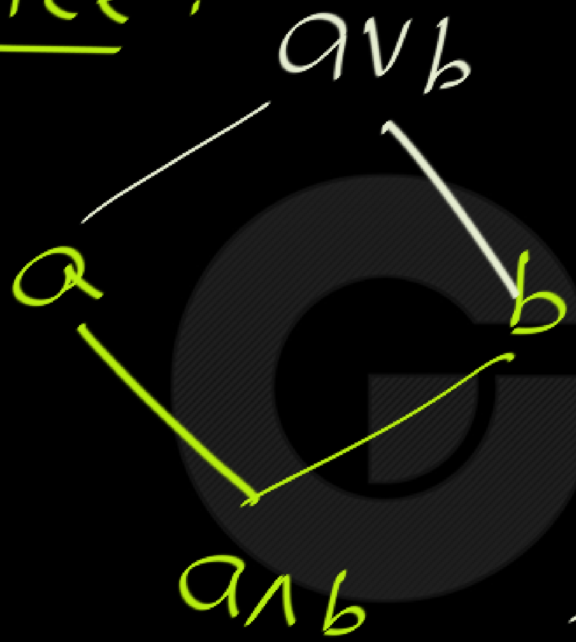
$a \cancel{R} b$ & $b \cancel{R} a$

Assume $\boxed{a = a \vee b}$

$b \; R \; (a \vee b)$ — In Every lattice

$b \; R \; a$ Contradiction

Lattice :



$(a \wedge b) \; R \; b$

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let $a, b \in L$;

If a, b are incomparable then $a \neq a \vee b$. True Or False?

If a, b are incomparable then $a \neq a \wedge b$. True Or False?

$a \not R b$

$b \not R a$

Assume : $\boxed{a = a \wedge b}$

$(a \wedge b) \, R \, b \rightarrow a R b$

Contradiction

Quick Question:

Assume L is a Lattice w.r.t. partial order relation R.

Let $a, b \in L$;

If a, b are incomparable then $a \neq a \vee b$. True Or False?

If a, b are incomparable then $a \neq a \wedge b$. True Or False?

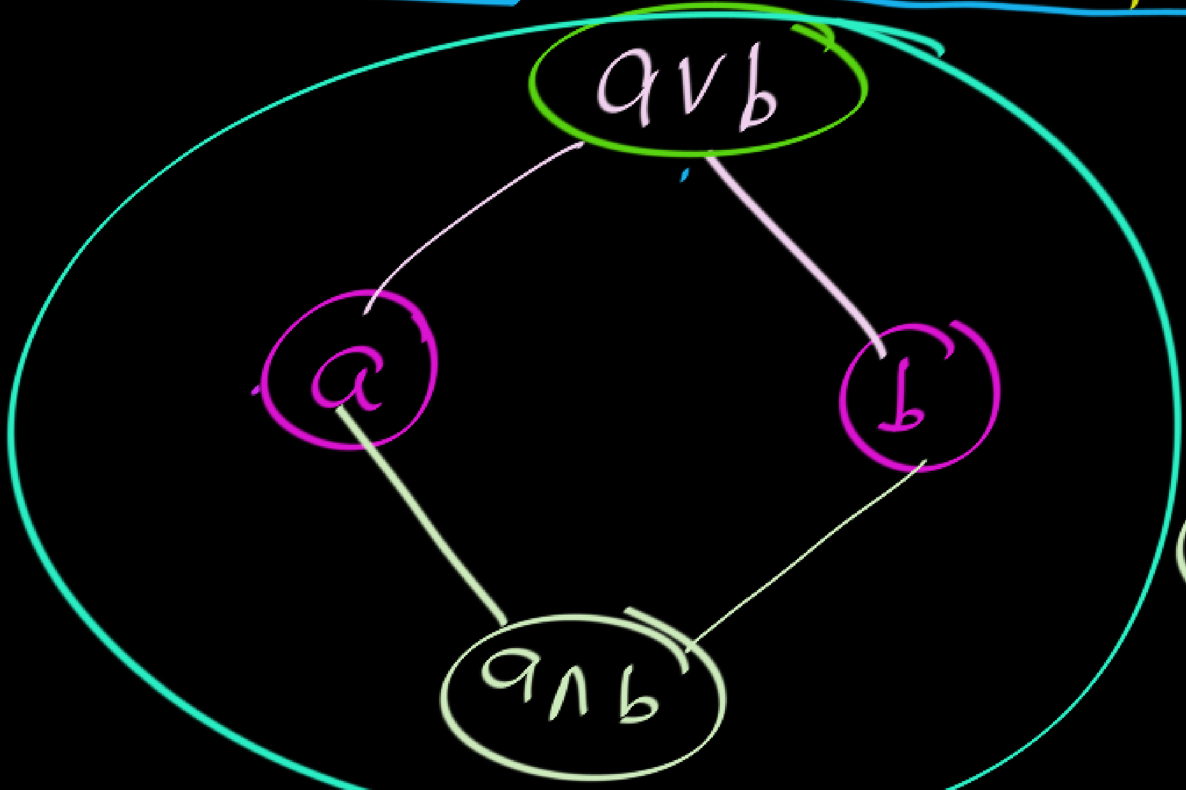
Note: In Every lattice;

If a, b are Incomparable then

$$\left.\begin{array}{l} a \neq a \vee b \\ a \neq a \wedge b \end{array}\right\} \checkmark \qquad \left.\begin{array}{l} b \neq a \vee b \\ b \neq a \wedge b \end{array}\right\} \checkmark$$

Lattice :

If a, b $\boxed{\text{NOT Comparable}}$



$$\left.\begin{array}{l} (a \vee b) \neq a \\ (a \vee b) \neq b \\ (a \wedge b) \neq a \\ (a \wedge b) \neq b \end{array}\right\}$$

Note :

Lattice : If $\boxed{a, b \text{ Comparable}}$

Assume : aRb

$(a \wedge b) = a$

$(a \vee b) = b$

$b = a \vee b$

$a = a \wedge b$

Q 1:

A lattice can have more than one maximal element. True/False?

A lattice can have more than one minimal element. True/False?

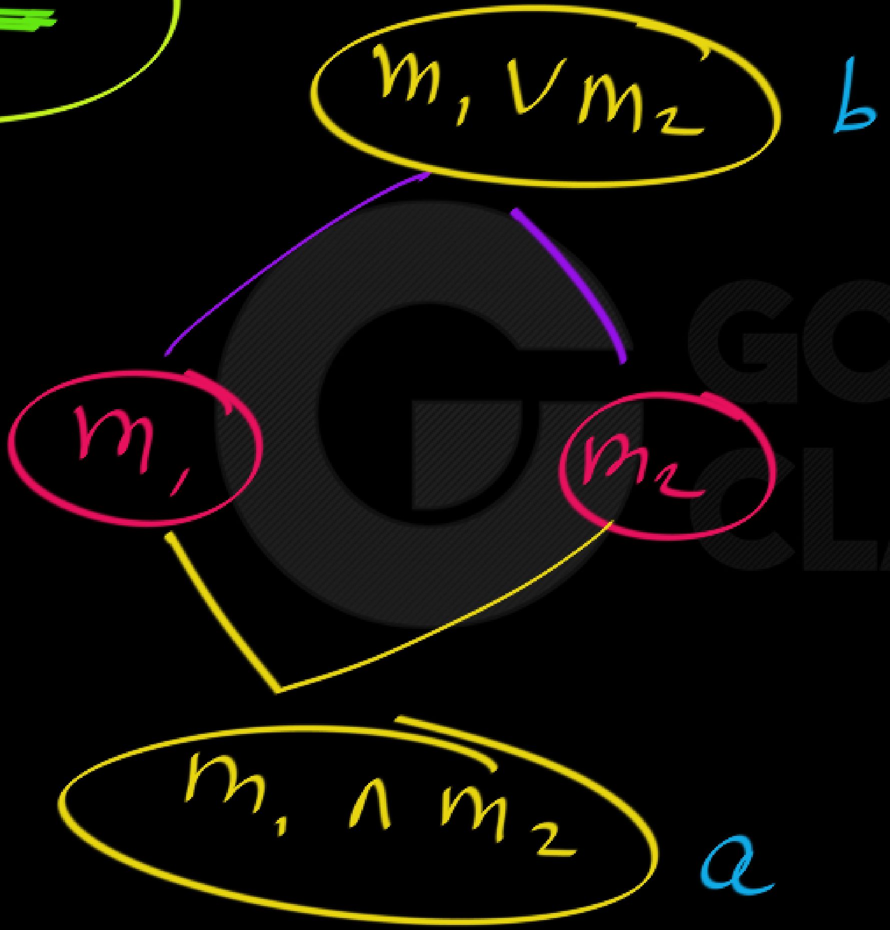
Q 1:

A lattice can have more than one maximal element. True/False?

Assume two different $\boxed{\text{maximal}}$ elements; $m_1 \neq m_2$

So; $m_1 \not R\, m_2$ & $m_2 \not R\, m_1$ $\longrightarrow$ m_1, m_2 Incomparable

Lattice



$m_1 \neq b$

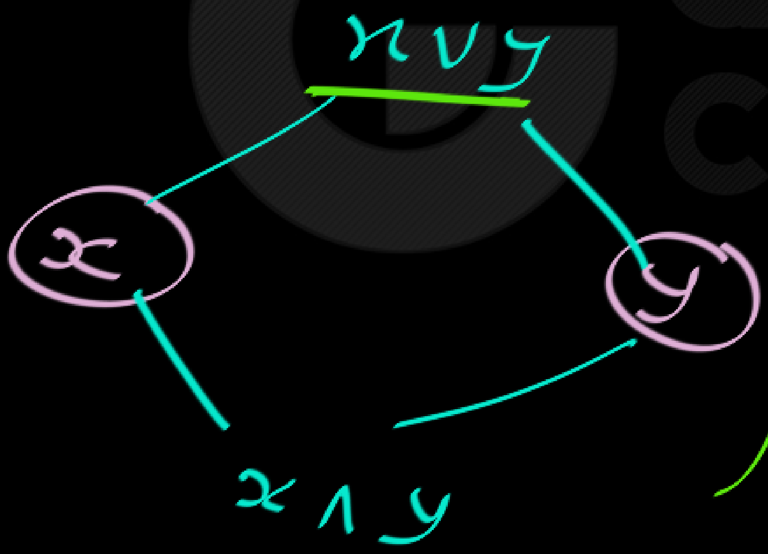
But

$m_1 \, R \, b$ → m_1 NOT maximal

Contradiction

Note: In **Lattice**,

If x, y are $\boxed{\text{NOT Comparable}}$,



$$\left.\begin{array}{l} x \neq x \vee y \\ x \neq x \wedge y \end{array}\right\}$$

Q 1:

A lattice can have more than one maximal element. True/False?

A lattice can have more than one minimal element. True/False?

NOTE

A lattice can have more than one maximal element.

not

A lattice can have more than one minimal element.

not

Q 1:

A lattice can have more than one maximal element. True/False?

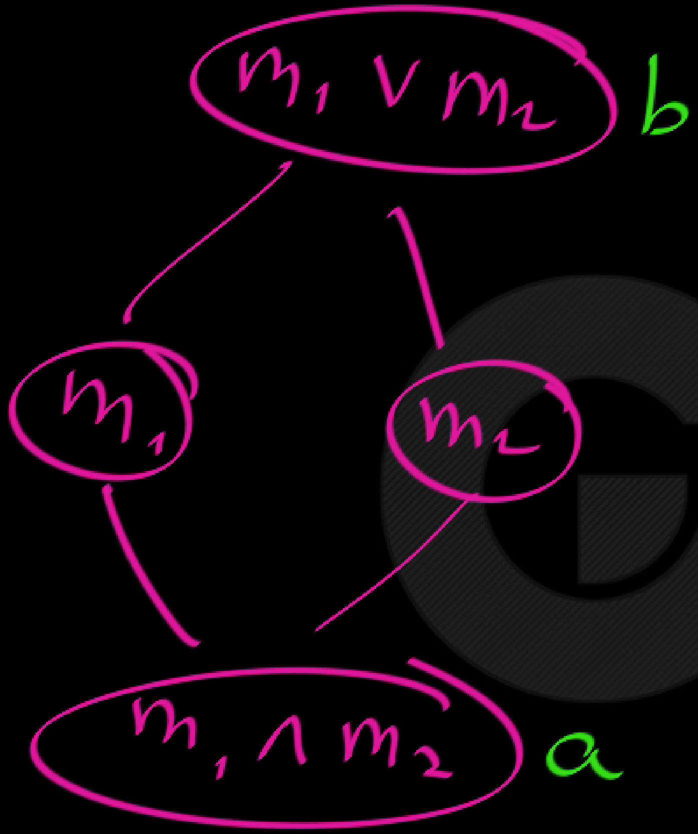
A lattice can have more than one minimal element. True/False?

Proof: Lattice; Assume $m_1 \neq m_2$ minimal

$m_2 \not R m_1$

$m_1 \not R m_2$

→ m_1, m_2 incomparable



$m_1 \neq a$

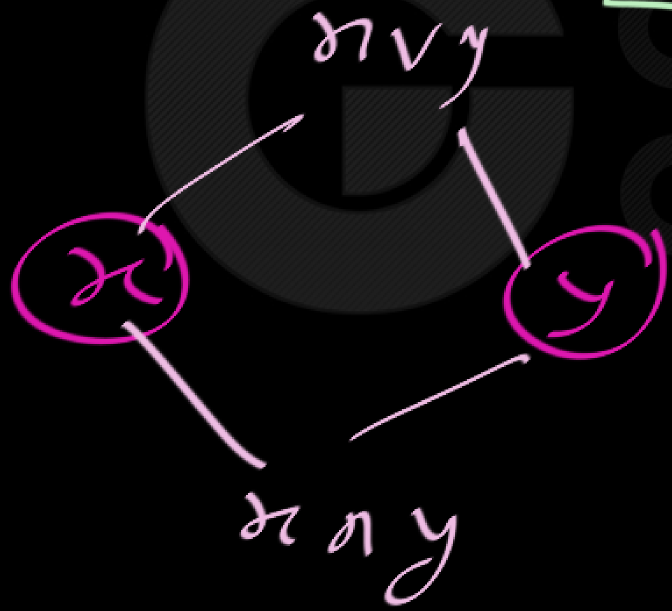
$a \, R \, m_1$

$\rightarrow$ m_1 NOT minimal

Contradiction

Note: Lattice:

If x, y $\boxed{\text{NOT Comparable:}}$



$$\left.\begin{aligned} x &\neq x \wedge y \\ x &\neq x \vee y \end{aligned}\right\}$$

Q 2:

Every lattice must have at least one maximal element.

True/False?

Every lattice must have at least one minimal element.

True/False?

Q 2:

Every lattice must have at least one maximal element.

True/False?

Every lattice must have at least one minimal element.

True/False?

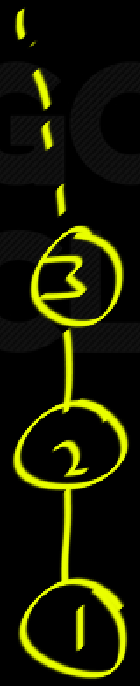
$(N, \leq)$ — maximal = DNE

minimal = 1

Toset → Lattice

$2 \vee 7 = 7$

$2 \wedge 7 = 2$



— minimal

$(\mathbb{Z}, \leq)$ — maximal = $\underline{DNE}$, minimal = $\underline{DNE}$

$\underbrace{(\mathbb{Z}, \leq)}$

Lattice

$-5 \vee 10 = 10$

$-5 \wedge 10 = -5$

$a \vee b = \max(a, b)$

$a \wedge b = \min(a, b)$

NOTE:

Every lattice has at most one maximal element.

Every lattice has at most one minimal element.

Q 3:

A finite lattice must have at least one maximal element.
True/False?

A finite lattice must have at least one minimal element.
True/False?

Q 3:

A finite lattice must have at least one maximal element.

True/False?

A finite lattice must have at least one minimal element.

True/False?

lattice $\longrightarrow$ poset with GLB, LUB $\forall_{x,y}$

finite poset $\Longrightarrow$ ≥ 1 maximal, ≥ 1 minimal

finite lattice $\longrightarrow$ finite poset $\longrightarrow$ ≥ 1 maximal

finite lattice $\Longrightarrow$ ≥ 1 minimal finite lattice $\longrightarrow$ ≥ 1 maximal

lattice $\longrightarrow$ ≤ 1 maximal

finite lattice $\longrightarrow$ 1 maximal

$\longrightarrow$ 1 minimal

NOTE:

A finite lattice has exactly one maximal element.

A finite lattice has exactly one minimal element.

Infinite Poset :

If Unique maximal $\longrightarrow$ Greatest

If $\underbrace{\text{'' minimal}}$ $\longrightarrow$ least finite lattice → 1 maximal

unique maximal

Greatest

finite lattice → 1 minimal

unique minimal

least

NOTE:

Every finite lattice has greatest element.

Every finite lattice has least element.

Q 4:

In a lattice, if we have unique maximal element M then M is the greatest element. True/False?

In a lattice, if we have unique minimal element m then m is the least element. True/False?

Q 4:

In a lattice, if we have unique maximal element M then M is the greatest element. True/False?

In a lattice, if we have unique minimal element m then m is the least element. True/False?

Lattice

If m — unique maximal

then m — Greatest.

Proof: Lattice; m — unique maximal

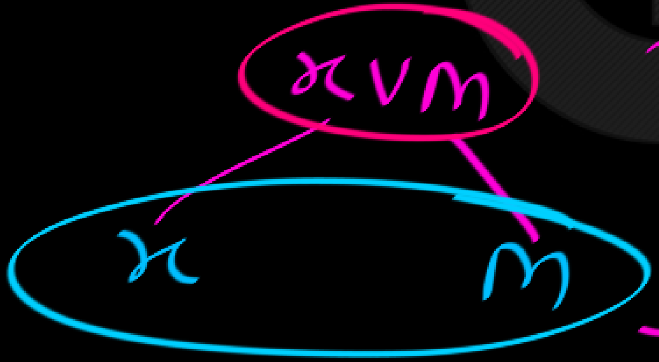
Assume that m — Not Greatest

then Some $x \neq m$; $x \not R m$.

m – maximal ; $x \neq m$

$x \not{R}\, m$; $m \not{R}\, x$

x, m are Not Comparable.



$m \, R \, (x \vee m)$

So m is not maximal.

Contradiction

Conclusion:

1. Every lattice has at most one maximal element.

2. Every lattice has at most one minimal element.

3. Every finite lattice has exactly one maximal element.

4. Every finite lattice has exactly one minimal element.

Conclusion:

5. Every finite lattice has greatest element.

6. Every finite lattice has least element.

7. In any lattice, if we have unique maximal element M then M is the greatest element.

8. In a lattice, if we have unique minimal element m then m is the least element.