Relations

Recap

Partial Order Relation

Website : https://www.goclasses.in/

Set is unordered collection of elements. But many times, we want Ordering among elements. We often use relations to order some or all of the elements of sets. For instance, we order words using the relation containing pairs of words (x, y), where x comes before y in the dictionary. We schedule projects using the relation consisting of pairs (x, y), where x and y are tasks in a project such that x must be completed before y begins.

We order the set of integers using the relation containing the pairs (x, y), where x is less than y. When we add all of the pairs of the form (x, x) to these relations, we obtain a relation that is reflexive, antisymmetric, and transitive. These are properties that characterize relations used to order the elements of sets.

Set : unordered

we want some ordering on elements of set :

$\underbrace{\text{Define a Relation}}_{\text{Partial order}}$ $\Rightarrow$ This Relation will put elements in some order.

In an $\boxed{\text{ordering}}$, ✗

OS → CO
CO → OS

Sym : Not Desirable

Antisym : ✓ ✓

$a_1 \quad a_2 \quad a_3 \quad - - - -$

In an $\boxed{\text{Ordering}}$

Transitive : Desirable

$$DL \rightarrow CO \rightarrow OS$$

$$a_1 \rightarrow a_2 \rightarrow a_3$$

Base set

A : unordered

when we Define POR R on A :

then Elements of A will have some ordering.

So, (A, R) is partially ordered set.

Partial Orders

- Many relations are equivalence relations:

$$x = y \qquad x \equiv_k y \qquad u \leftrightarrow v$$

- What about these sorts of relations?

$$x \leq y \qquad x \subseteq y$$

- These relations are called ***partial orders***, and we'll explore their properties next.

Antisymmetry

- A binary relation R over a set A is called ***antisymmetric*** if the following is true:

$$\forall a \in A.\ \forall b \in A.\ (a \neq b \land aRb \to \neg(bRa))$$

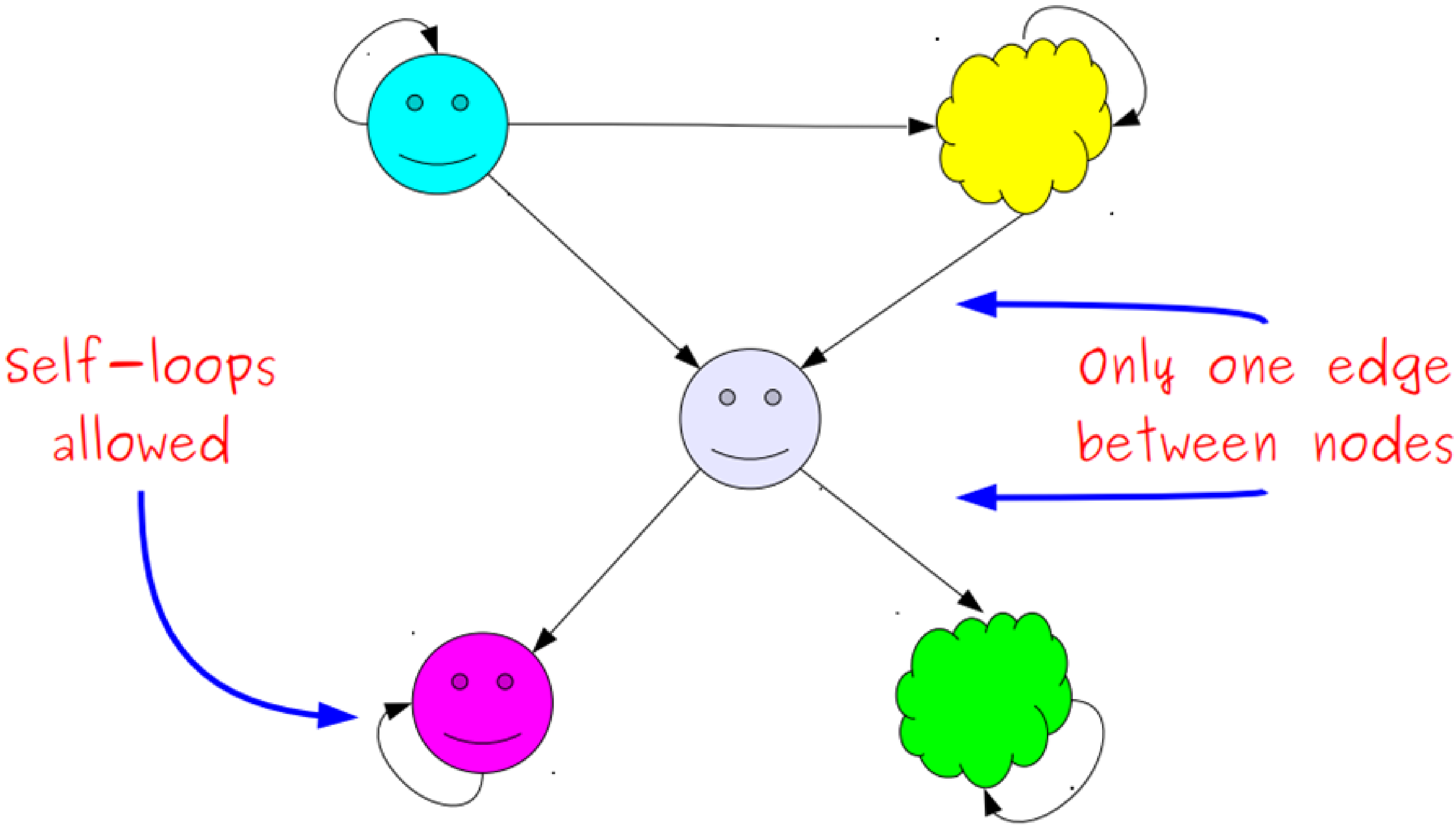
("If a is related to b and a ≠ b, then b is not related back to a.")

- Equivalently:

$$\forall a \in A.\ \forall b \in A.\ (aRb \land bRa \to a = b)$$

("If a is related to b and b is related back to a, then a = b.")

An Intuition for Antisymmetry



For any $x \in A$ and $y \in A$,
If xRy and $y \neq x$, then $y \not R x$.

Partial Orderings

Definitions

A relation R that is reflexive, antisymmetric, and transitive on a set S is called a **partial ordering** on S.

A set S together with a partial ordering R is called a **partially ordered set** or **poset**.

Partial Orders

Definition: A relation R on a set A is a ***partial order*** (or ***partial ordering***) for A if R is *reflexive, antisymmetric* and *transitive*.
A set A with a partial order is called a ***partially ordered set***, or ***poset.***

Examples:
The natural ordering " $\leq$ "on the set of real numbers $\mathbb{R}$.

For any set A, the subset relation $\subseteq$ defined on the power set P (A).

Integer division on the set of natural numbers $\mathbb{N}$.

Examples

1. $P = \{1, 2, \ldots, \}$ and $a \leq b$ has the usual meaning.
2. $P = \{1, 2, \ldots, \}$ and $a \preceq b$ if a divides b.
3. $P = \{A_1, A_2, \ldots, A_m\}$ where the A_i are sets and $\preceq = \subseteq$.

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution: Because $A \subseteq A$ whenever A is a subset of S, $\subseteq$ is reflexive. It is antisymmetric because $A \subseteq B$ and $B \subseteq A$ imply that $A = B$. Finally, $\subseteq$ is transitive, because $A \subseteq B$ and $B \subseteq C$ imply that $A \subseteq C$. Hence, $\subseteq$ is a partial ordering on $P(S)$, and $(P(S), \subseteq)$ is a poset. ◀

Example 3

Problem: Show that $\geq$ is a partial ordering on the set of integers.

Solution: Since $a \geq a$, this relation is reflexive. If $a \geq b$ and $b \geq a$, then $a = b$ which shows this relation is antisymmetric. If $a \geq b$ and $b \geq c$, then $a \geq c$ so this relation is transitive. Thus, $\geq$ is a partial ordering on the set of integers.

Examples

are the following partial orders?:

"$\leq$" on pairs of numbers? yes

$aRb \Leftrightarrow$ "a divides b" for nonzero integers? yes

"$<$" on pairs of numbers? no

$\geq$ on pairs of numbers yes

$\subseteq$ on pairs of subsets of a given universe? yes

A binary relation on a set S is called an order relation if it satisfies the following three conditions and then it is usually written $x \preceq y$ instead of $x \; R \; y$.

(i) *(Reflexive) For all $s \in S$ we have $s \preceq s$.*

(ii) *(Antisymmetric) For all $s, t \in S$ such that $s \neq t$, if $s \preceq t$ then $t \npreceq s$.*

(iii) *(Transitive) For all $r, s, t \in S$ such that $r \preceq s$ and $s \preceq t$ we have $r \preceq t$.*

A set S together with an order relation $\preceq$ is called a partially ordered set or poset. Formally, a poset is a pair $(S, \preceq)$. We shall, once the binary relation is defined, refer to the poset by the set S alone, not the pair.

If we use the alternative notation R for the relation, then the three conditions for an order relation are written as follows.

(i) For all $s \in S$ we have $(s, s) \in R$.

(ii) For all $s, t \in S$ such that $s \neq t$, if $(s, t) \in R$ then $(t, s) \notin R$.

(iii) For all $r, s, t \in S$ such that $(r, s) \in R$ and $(s, t) \in R$ we have $(r, t) \in R$.

$\underbrace{\text{POR}}$ "$\leq$" is generally uses for POR.

$\sim$ " " " " Eq. Rel.

IISc $\rightarrow$ PAV

Prog Analysis

Verification

$\leq$ symbol is usually placeholder of POR R.

$(\{1,2,3,4\}, R)$

xRy iff $x \mid y$

POR

$(\{1,2,3,4\}, \mid)$

$2 \mid 4$ ✓

$2 \nmid 3$ ✓

$(\{1,2,3,4\}, \leq)$

$x \leq y$ iff $x \mid y$.

Actual thing

x is related to y

Don't call it x less than equal to y

$2 \leq 3$ ✗

$2 \nmid 3$

$2 \leq 4$ ✓

$2 \mid 4$ ✓

4.8.4 Partial Order: GATE CSE 1996 | Question: 1.2 top

Let $X = \{2, 3, 6, 12, 24\}$, Let $\leq$ be the partial order defined by $X \leq Y$ if x divides y.

[redacted]

means x is related to y.

Actual thing

$2 \leq 3$ ✗

$2 \leq 6$ ✓

$2 \leq 12$ ✓

$3 \leq 2$ ✗

$3 \leq 12$ ✓

$12 \leq 6$ ✗

$2 \leq 3$ means $2 | 3$ → false

Partial Orders

- A binary relation R is a **partial order** over a set A iff it is
  - **reflexive**,
  - **antisymmetric**, and
  - **transitive**.

Why "partial"?

- A pair (A, R), where R is a partial order over A, is called a **partially ordered set** or **poset**.

$S_1 = \{ 2^{nd}$ sem, 1^{st} sem, 3^{rd} sem, 4^{th} sem $\}$

$S_2 = \{$ BTech, MTech, MBA, UPSC $\}$

Relation. R : xRy iff x "must be" Done before y.

↓

Order Relation

R on S_1 is Total order.

$1 \leq 2 \leq 3 \leq 4$

$\forall x, y \in S_1$; we have order b/w x, y.

1 R 3 $\qquad 1 \leq 3$

Symbol of POR

4, 2 $\qquad$ 2 R 4 $\qquad 2 \leq 4$

1, 4 $\qquad 1 \leq 4$

R on S_2 is partial order but not Total order.

Btech R mtech

BTech $\leq$ metch

symbol for partial order

MBA $\not R$ mtech

mtech $\not R$ MBA

not Comparable

MBA, Mtech have no order.

When a and b are elements of the poset $(S, \preccurlyeq)$, it is not necessary that either $a \preccurlyeq b$ or $b \preccurlyeq a$. For instance, in $(P(\mathbf{Z}), \subseteq)$, $\{1, 2\}$ is not related to $\{1, 3\}$, and vice versa, because neither set is contained within the other. Similarly, in $(\mathbf{Z}^+, |)$, 2 is not related to 3 and 3 is not related to 2, because $2 \nmid 3$ and $3 \nmid 2$. This leads to Definition 2.

The elements a and b of a poset $(S, \preccurlyeq)$ are called *comparable* if either $a \preccurlyeq b$ or $b \preccurlyeq a$. When a and b are elements of S such that neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, a and b are called *incomparable*.

In the poset $(\mathbf{Z}^+, |)$, are the integers 3 and 9 comparable? Are 5 and 7 comparable?

Solution: The integers 3 and 9 are comparable, because $3 \mid 9$. The integers 5 and 7 are incomparable, because $5 \nmid 7$ and $7 \nmid 5$. ◀

If R is POR on S then we say that $\boxed{\text{a, b are Comparable}}$ iff "one of them is Related to another."

a, b are Comparable iff aRb OR bRa.

If $a \neq b$ then aRb, bRa

Cannot happen together. because of Antisym.

If $a = b$ then aRb, bRa ✓

If $a \neq b$ then
$$\begin{cases} aRb \text{ OR } bRa \to a, b \text{ Comparable} \\ a \not{R} b \text{ and } b \not{R} a \to a, b \text{ Incomparable.} \end{cases}$$

$(\{1, 2, 3, 4\}, |)$

Comparable elements: 1, 2 ✓ $1 \mid 2$

4, 2 ✓ $2 \mid 4$

2, 3 ✗ $2 \nmid 3$, $3 \nmid 2$

Not Comparable element: 2, 3 ; 3, 4

$(\{1, 2, 3, 4\}, |)$ — Partial order
— Not Total order

$1 \leq 2 \leq 4$

$1 \leq 3$

Partial order

Not Total order

$2 \leq 4$

2 Divides 4.

$1 \leq 2$ means 1 Divides 2.

$(\{1, 2, 4, 8\}, |)$ — POR ✓
— TOR ✓

$\boxed{1 \leq 2 \leq 4 \leq 8}$ Total order

→ b/w every pair of elements, we have order.

④ Total order / Total order Relation:

a Relation which is Partial order
(RAT properties)

AND Every two elements are Comparable.
(whatever two elements we have, one of them must be related to another)

POR

Ref

Antisym

Transitive

TOR

POR AND

every pair of elements

Comparable.

Every TOR is POR.

The adjective "partial" is used to describe partial orderings because pairs of elements may be incomparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

If $(S, \preccurlyeq)$ is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and $\preccurlyeq$ is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Note: It is not required that two things be related under a partial order. That's the *partial* part of it.

If two objects are always related in a poset, it is called a *total order* or *linear order* or *simple order*. In this case (A, R) is called a *chain*.

The poset $(\mathbf{Z}, \leq)$ is totally ordered, because $a \leq b$ or $b \leq a$ whenever a and b are integers.

The poset $(\mathbf{Z}^+, |)$ is not totally ordered because it contains elements that are incomparable, such as 5 and 7.

$(Z, \leq)$ — Refl ✓
Antisym ✓
Trans. ✓

less than equal to

if $a, b \in Z$ then $a \leq b$ or $b \leq a$ ✓

$(Z^+, |)$ — POR

$\rightarrow$ Divides Relation is POR.

Not Total order.

2, 3 are InComparable.

3, 5 " " .

4.11.28 Relations: GATE2006-4

A relation R is defined on ordered pairs of integers as follows:

$$(x, y)R(u, v) \text{ if } x < u \text{ and } y > v$$

Then R is:

A. Neither a Partial Order nor an Equivalence Relation
B. A Partial Order but not a Total Order
C. A total Order
D. An Equivalence Relation

4.11.28 Relations: GATE2006-4

A relation R is defined on ordered pairs of integers as follows:

$$(x, y)R(u, v) \text{ if } x < u \text{ and } y > v$$

Base set $Z \times Z$

Then R is:

A. Neither a Partial Order nor an Equivalence Relation
B. A Partial Order but not a Total Order
C. A total Order
D. An Equivalence Relation

$R : Z \times Z \rightarrow Z \times Z$

$(x,y)\ R\ (u,v)$ iff $x<u$ and $y>v$

$<$

$>$

Ref : ×

$(a,a)\ R\ (a,a)$ ×

$(a,b)\ \cancel{R}\ (a,b)$ ✓

Same or Diff

false
$a<a$

$b>b$ false

Sym: $(a,b) R (c,d)$ means $a < c, b > d$

$(c,d) R (a,b)$ means $c < a$

not sym

Transitive !

$(2,4) \cancel{R} (3,4)$

$(3,4) \cancel{R} (2,4)$

$(a, b) \longrightarrow (c, d) \longrightarrow (e, f)$

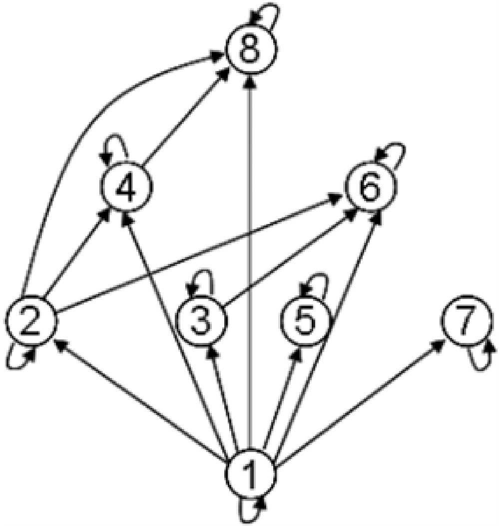
$a < c$ $\qquad$ $c < e$

$b > d$ $\qquad$ $d > f$

$$\left.\begin{array}{l} a < e \\ b > f \end{array}\right\} \quad \boxed{(a, b)\, R\, (e, f)}$$

So Trans.

As a small example, let S = {1, 2, 3, 4, 5, 6, 7, 8}, and let R be the binary relation "divides." So (2,4) ∈ R, (2, 6) ∈ R, etc. Using | as the symbol for "divides", we see that R is reflexive, since x | x; R is transitive since x | y and y | z implies x | z; and R is antisymmetric since we never have x | y and y | x where x ≠ y. Thus S and R form a poset. The following graph represents R:



The loops on each vertex show reflexivity; the arrows between nodes like 2, 4, and 8 show transitivity; and that fact that no two nodes have arrows to and from each other shows antisymmetry.

Another thing to notice about this relation is that only certain pairs of integers in $S \times S$ are related. We know that $4 \mid 8$, so (4, 8) is in the relation, and 4 comes "before" 8 in the ordering. Likewise 3 comes before 6, and 1 comes before all the others. This relation does not give an ordering of nodes like 3 and 7, however, since $3 \nmid 7$ and $7 \nmid 3$. That is why we say that the ordering is **partial**. The relation gives an ordering of some, but not all, pairs of vertices.

What if we wanted to list all the vertices in an order that does not violate the ordering provided by R? We know that 1, 2, 3, 4, 5, 6, 7, 8 would work. But so would 1, 5, 3, 2, 7, 4, 6, 8, since whenever $(x, y) \in R$, x comes before y in the list. We will return to this idea below.

Hasse Diagram; Property and Definition of POR.

POR R $\longrightarrow$ Hasse Diagram for R

R is POR $\longleftarrow$ Hasse Diagram for R

POR $\rightleftarrows$ HD

Any Relation R on set A has many Representations:

1. Graph Rep
2. Matrix "
3. Set "
4. Arrow Diagram Rep

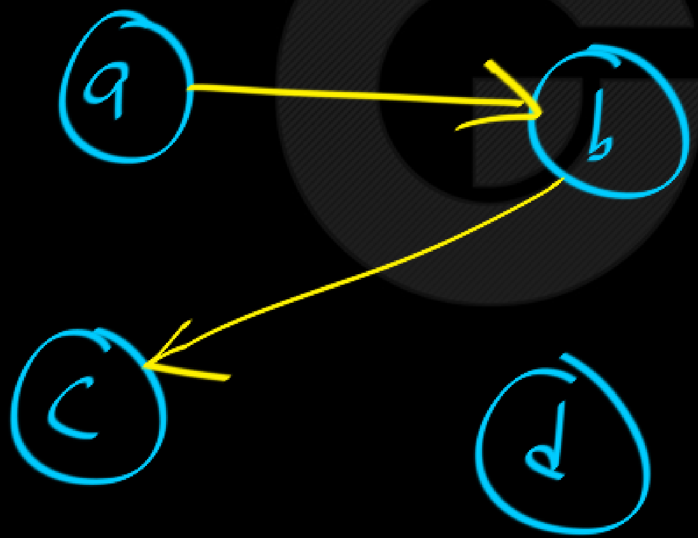
Partial order Relation R on set A has many Representations:

1. Graph Rep
2. Matrix "
3. Set "
4. Arrow Diagram Rep
5. Hasse Diagram

$A = \{a, b, c, d\}$ PoR R on A.

→ Given

Graph-like Rep of R:



aRc ✓

~~cRb~~

bRc ✓

aRa ✓

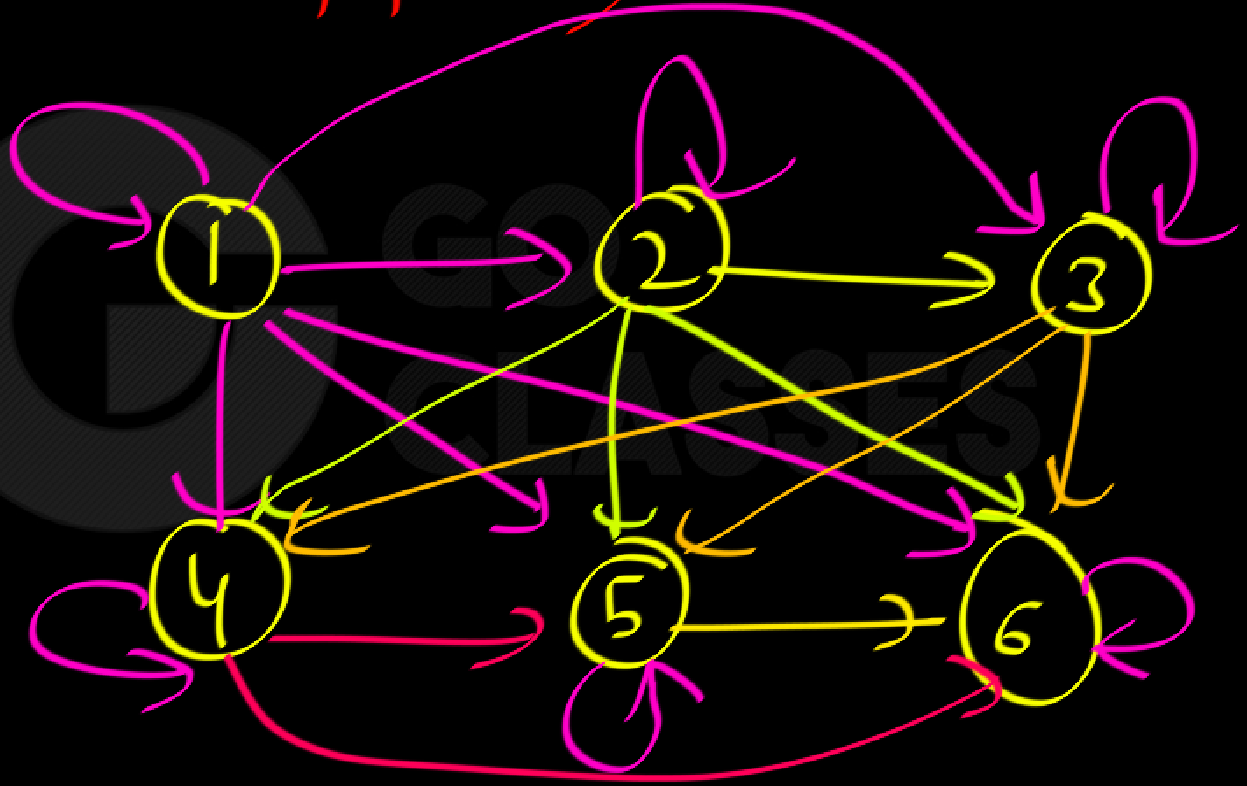
dRd

POR — Transitive ✓

Reflexive ✓

$(\{1,2,3,4,5,6\}, \leq)$

Graph :



$(\{1,2,3,4,5,6\}, \leq)$

POR

Ref

Trans

POR +

$\Rightarrow$ 1R6 ✓

2R2 ✓

2R5 ✓

Resume :

Run
walk

Don't show off.

Can be inferred

Hasse Diagram :

Don't show : self loops,
Transitive Edges

Hasse Diagram : — Graph-like Rep of POR.

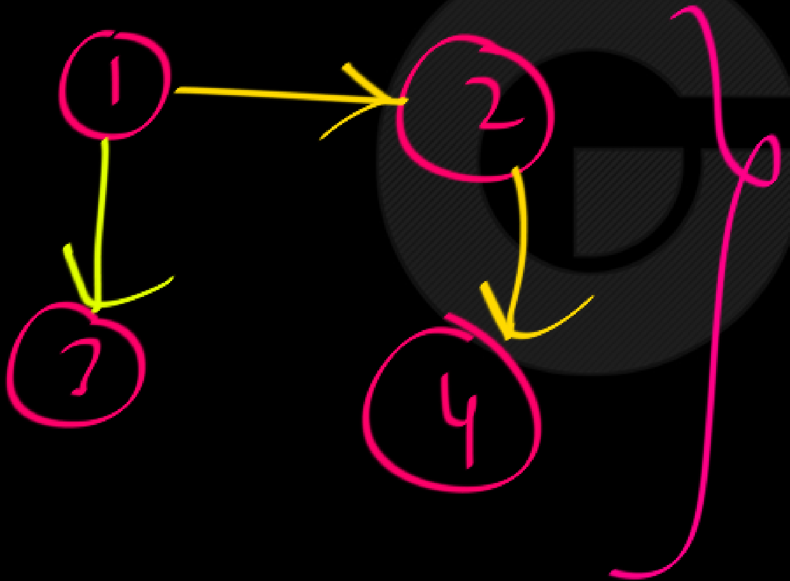
Do not show

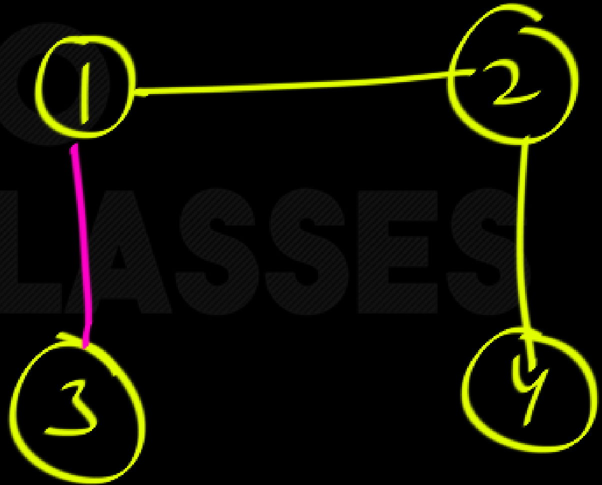
1. self loops
2. Trans. Edges
3. Remove "Arrows" AND Assume all Directions upward.

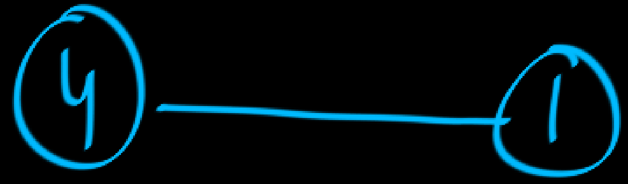
aRb

b
|
a

$$(\{1, 2, 3, 4\}, 1)$$



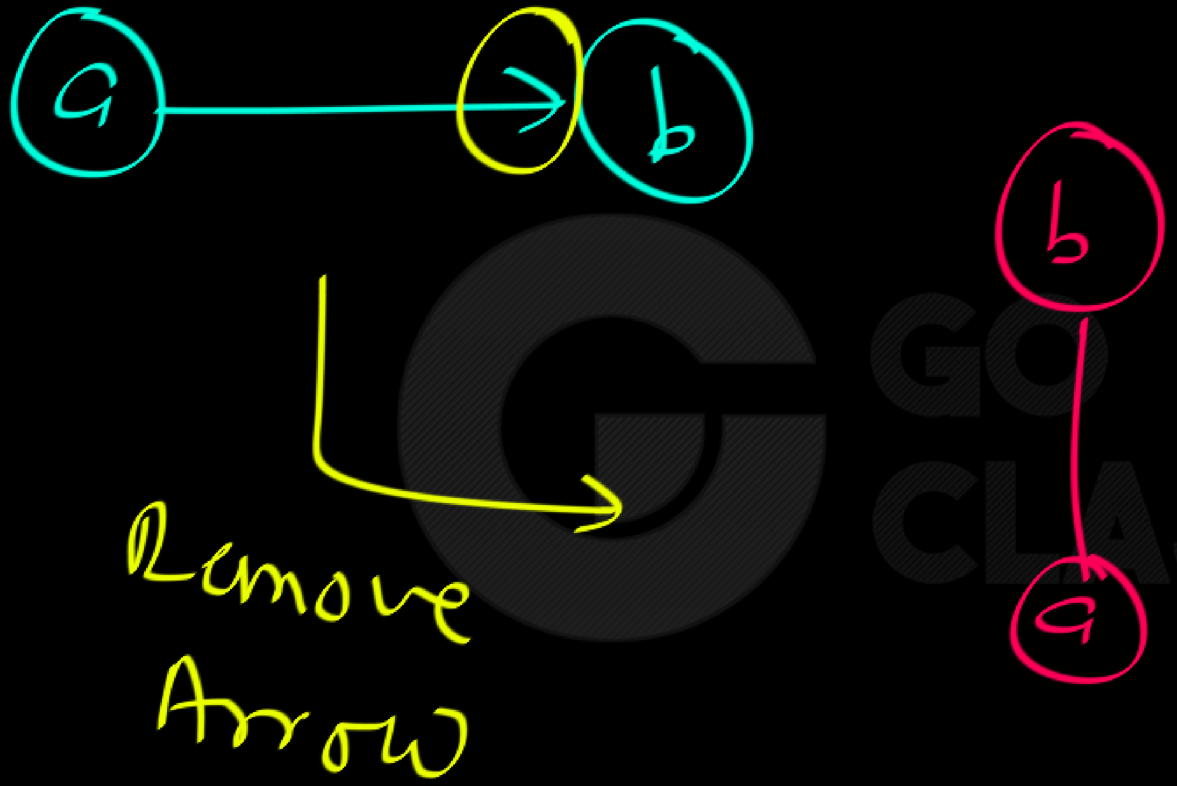




$4 \, R \, 1$, $1 \, R \, 4$ } can't say.

$(\{1,4\}, \leq)$ — $1 R 4$ ✓

$(\{1,4\}, \geq)$ — $4 R 1$ ✓



Assumption:

All Directions are upward,

Hasse Diagrams

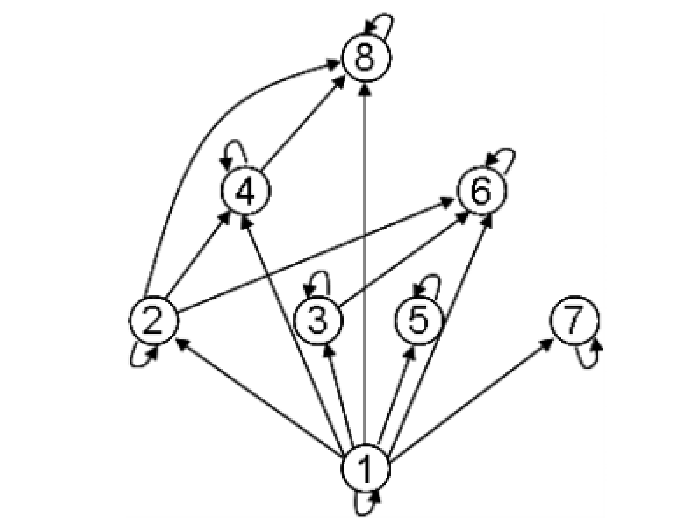
The graphs of partial orderings can be fairly complex. (Being both reflexive and transitive produces a lot of arcs.) One way to simplify these graphs is to use a special kind of representation known as a Hasse diagram.

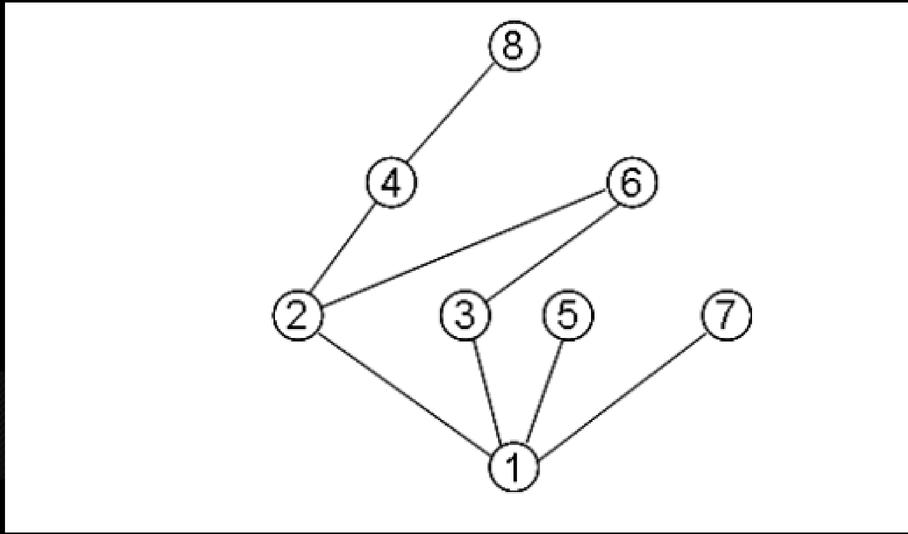
The general procedure for creating a Hasse diagram is as follows:

1. Draw the directed graph for the relation
2. Remove all the loops at each node, which must be there for reflexivity
3. Remove all edges that are there for transitivity; that is, wherever there are three nodes x, y, and z with edges from x to y and from y to z, remove the edge between x and z.
4. Arrange each edge so that its initial node is below its terminal node as indicated by the directed edges
5. Remove all arrows on the directed edges (since all edges point upward toward their terminal node)

Hasse Diagrams

- A ***Hasse diagram*** is a graphical representation of a partial order.
- No self-loops: by ***reflexivity***, we can always add them back in.
- Higher elements are bigger than lower elements: by ***antisymmetry***, the edges can only go in one direction.
- No redundant edges: by ***transitivity***, we can infer the missing edges.





Graph Rep

Hasse Diagram

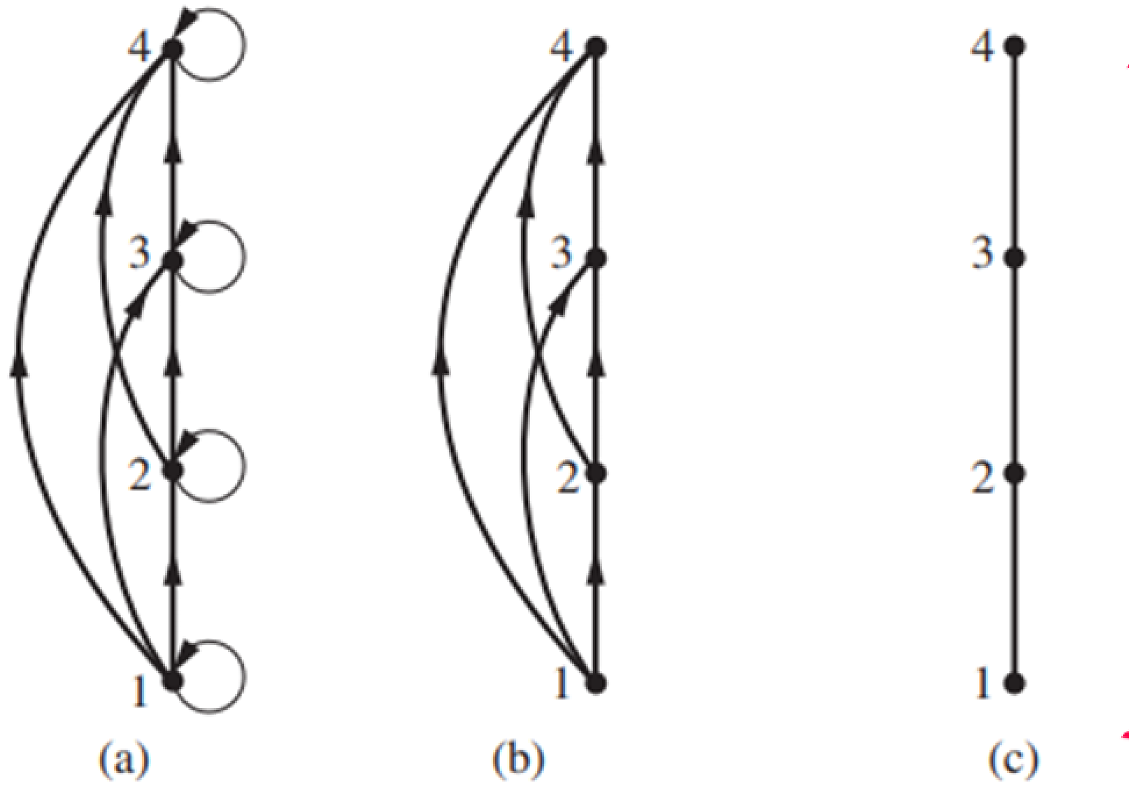


FIGURE 2 **Constructing the Hasse Diagram for ({1, 2, 3, 4}, ≤).**

1R4 ✓

4 ~~R~~ 3

3 ~~R~~ 2

3R4

2R4

Hasse Diagram: Example

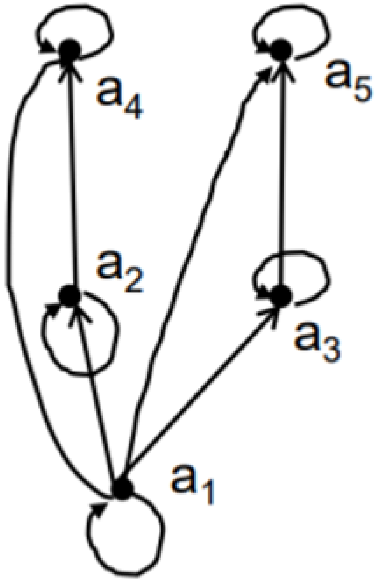
$a_4 \cancel{R} a_5$

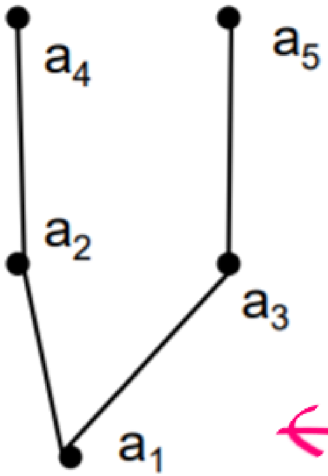
$a_1 R a_5$ ✓

$a_2 \cancel{R} a_5$

$a_2 R a_4$

Graph





HD

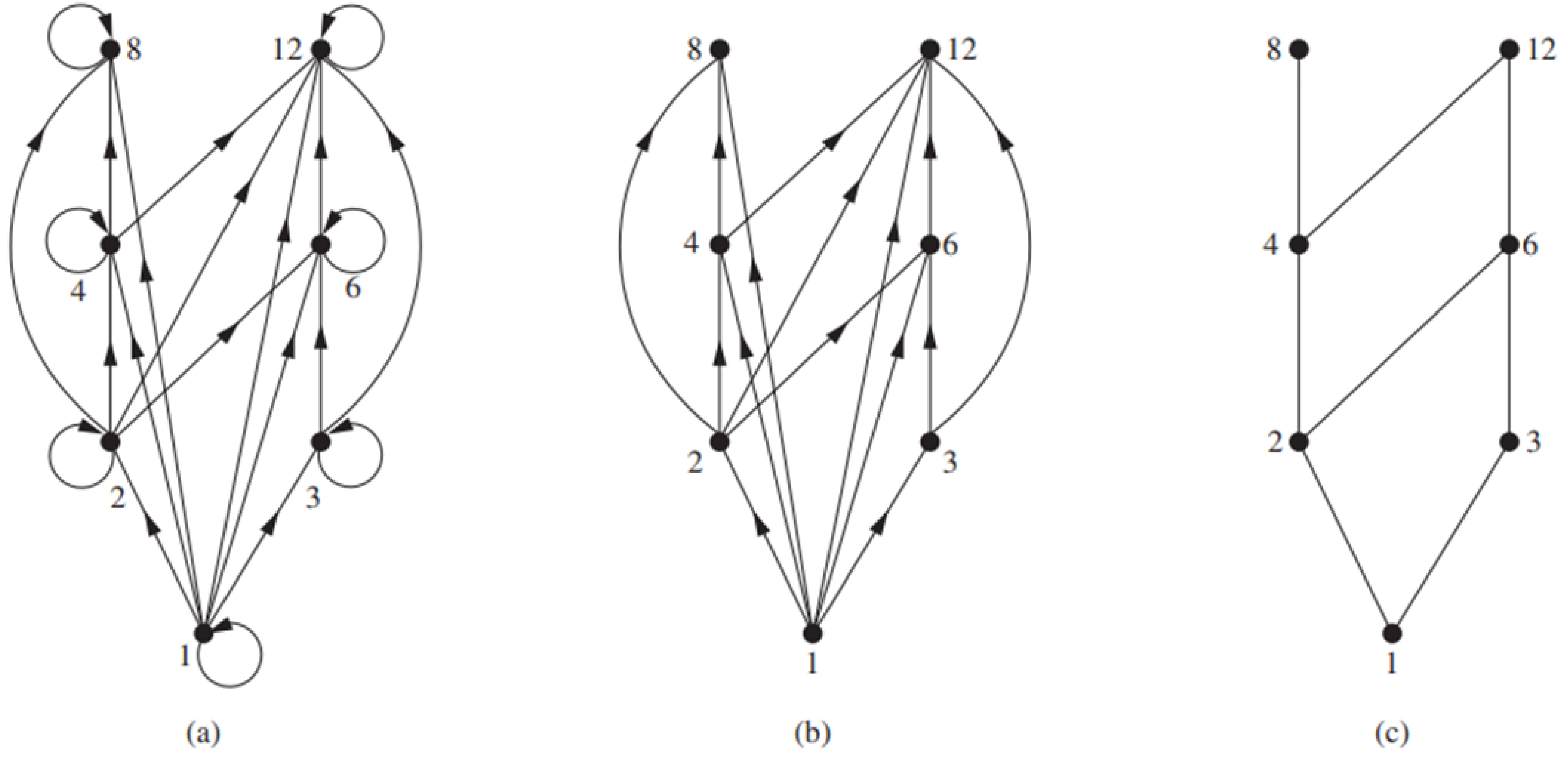


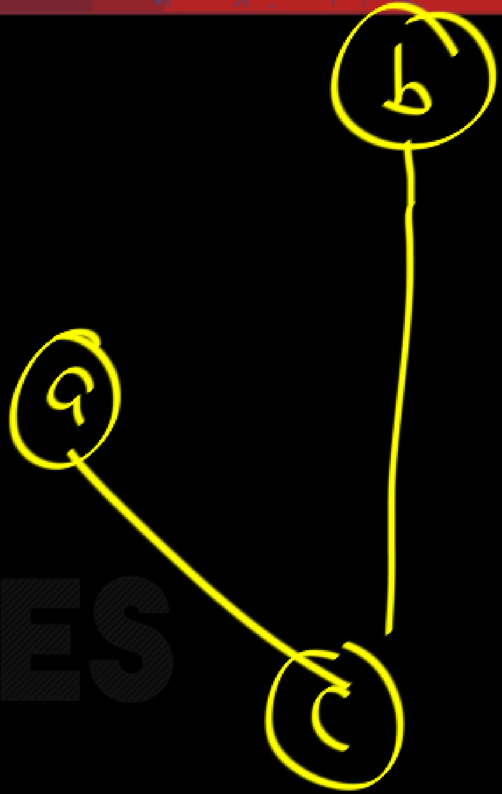
FIGURE 3 Constructing the Hasse Diagram of ({1, 2, 3, 4, 6, 8, 12}, |).

Note: In Hasse Diagram (HD) of a Relation R;

aRb iff there is "upward path" from a to b.

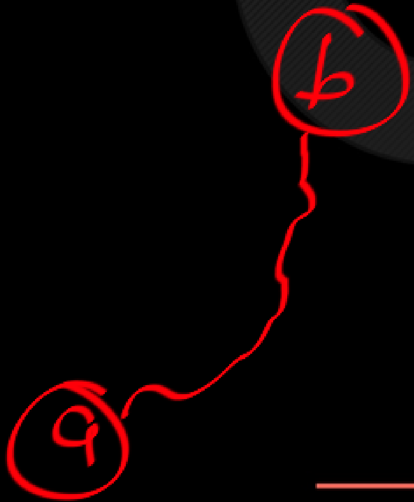
then

aRb



$a \not R b$

Note: In HD, whenever aRb then from a to b there will be upward path.



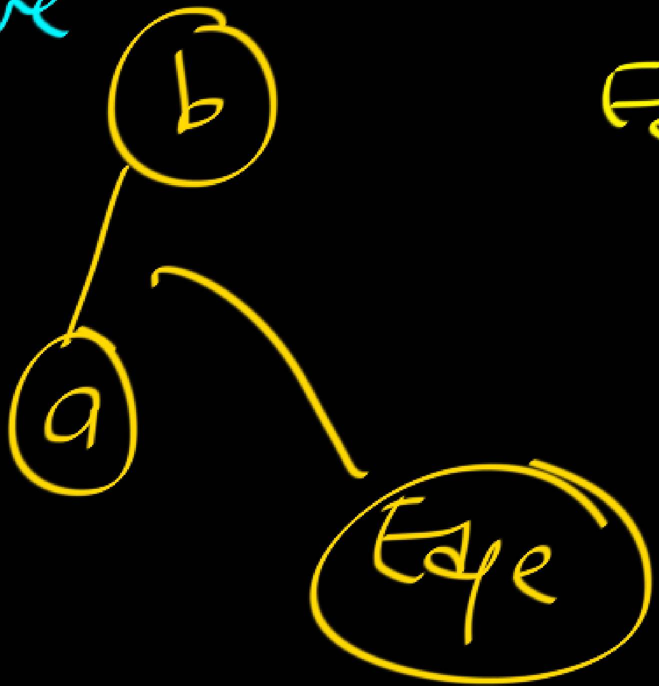
Q: In HD, whenever aRb then from a to b there will be upward edge ✗

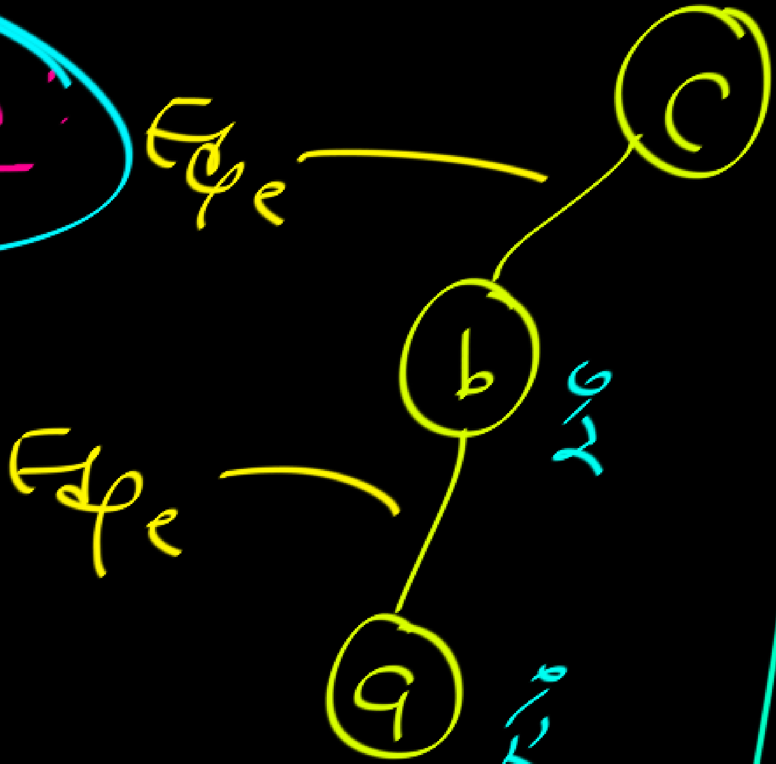
Eg: $(\{1,2,3\}, \leq)$

1R3 ✓

Edge vs Path:

Intuitive Concept





a—b—c

Path

Jaipur

Delhi

Chamba

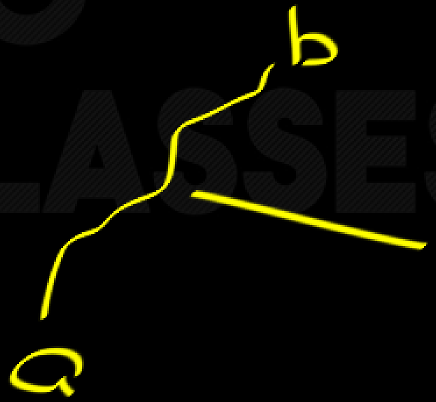
Every edge is also a path

Note: In HD, No Horizontal Edges,

Creation of HD:

Guidelines: (1) Start from bottom.

If aRb then



upward path

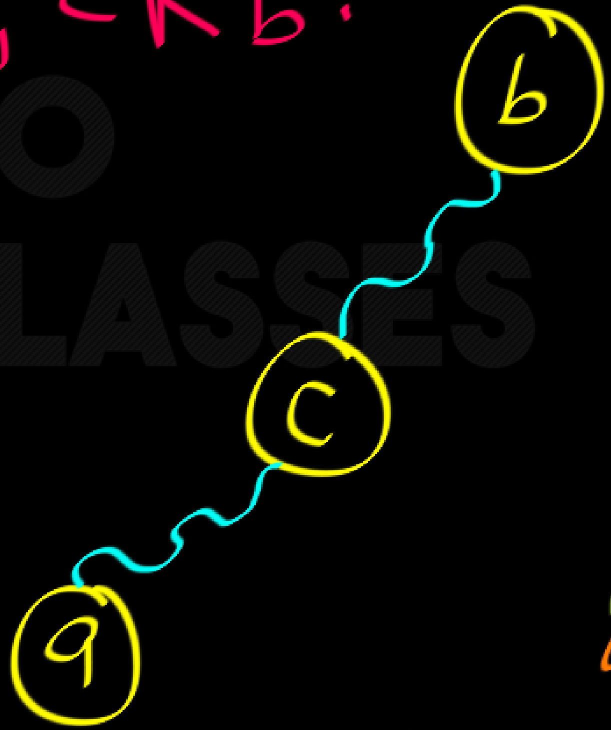
(2) aRb and there is a "c" such that aRc, cRb.

means

$a \to b$

is Transitive Edge.

Don't Show in HD.

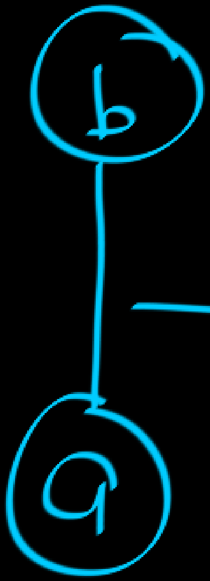


Trans Edge

$a \to c \to b$

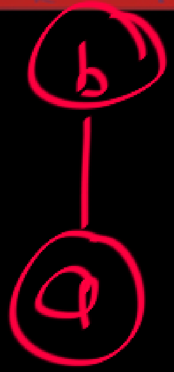
(3) aRb and there is no c such that aRc, cRb.

Means there is no intermediate person b/w a, b, So, $a \rightarrow b$ is not a Transitive Edge. So $\longrightarrow$



Edge in HD

Q: In HD, we have edge



then ?

Ans: There is no element "c" such that

$$a \rightarrow c \rightarrow b$$

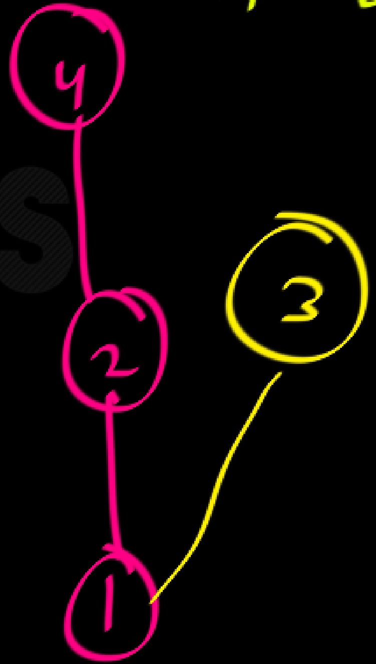
Eg: $(\{1, 2, 3, 4\}, \mid)$ — POR ✓

HD ✓

HD ; $1 R 2 R 4$; $1 R 3$

X 

2, 3 are Incomparable.

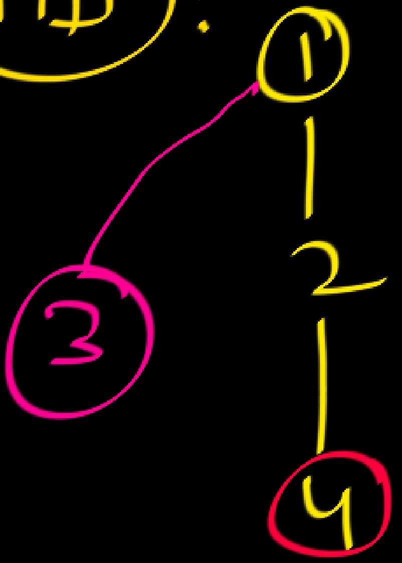


Eg: $(\{1, 2, 3, 4\}, \text{multiple of})$ — POR

xRy iff x is multiple of y.

HD

HD : $4 R 2 R 1$; $3 R 1$



$$4 \rightarrow 2 \rightarrow 1$$
$$3 \rightarrow 1$$

Ep: $(\{3, 4, 5\}, |)$ — R

3, 4 In Comparable

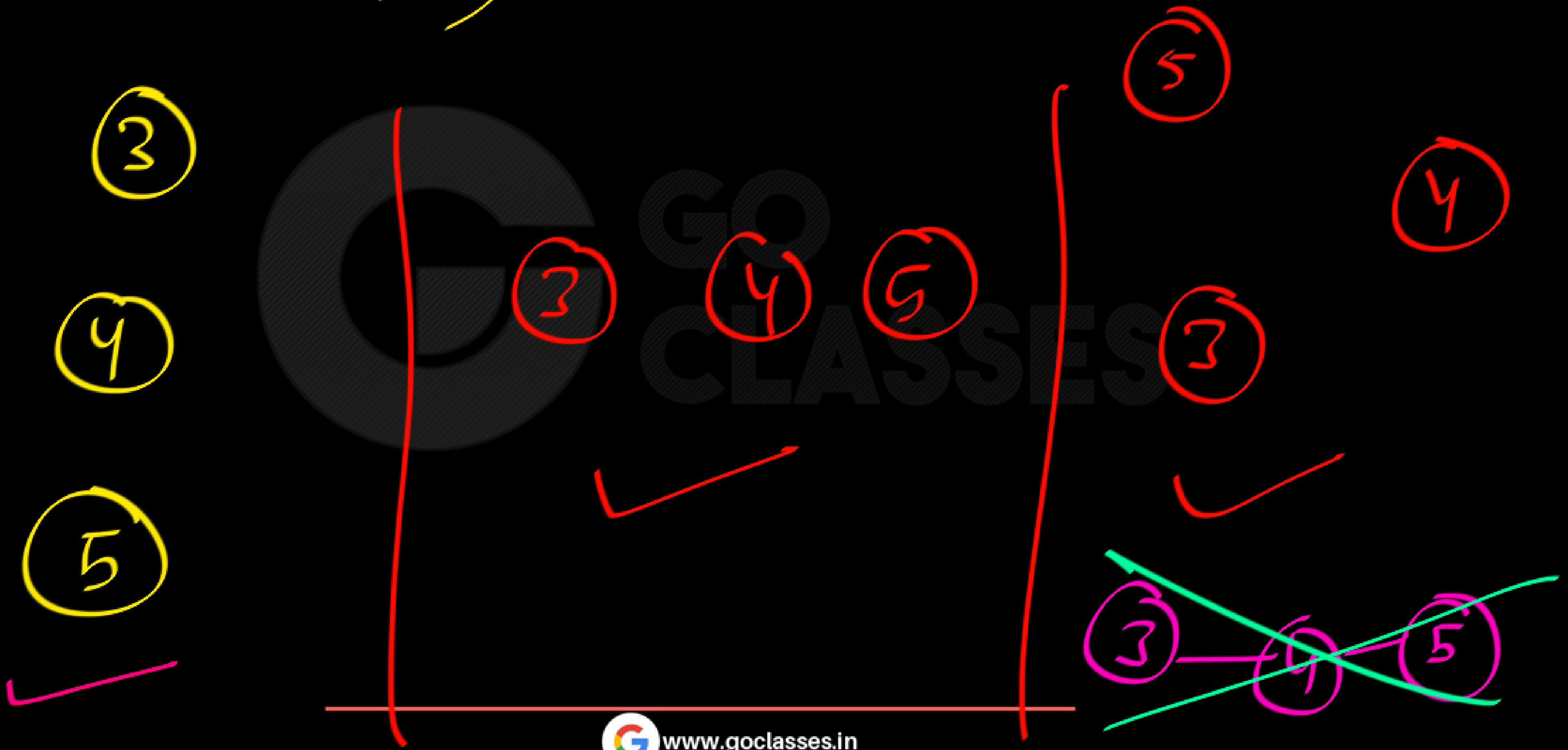
3, 5 " "

4, 5 " "

3 4 5

$R = \{(3,3), (4,4), (5,5)\}$

$(\{3,4,5\},1)$



Note: In HD, there is no concept of "levels".

There is only concepts of Upward paths.

Eg: $(P\{1,2\}, \subseteq)$

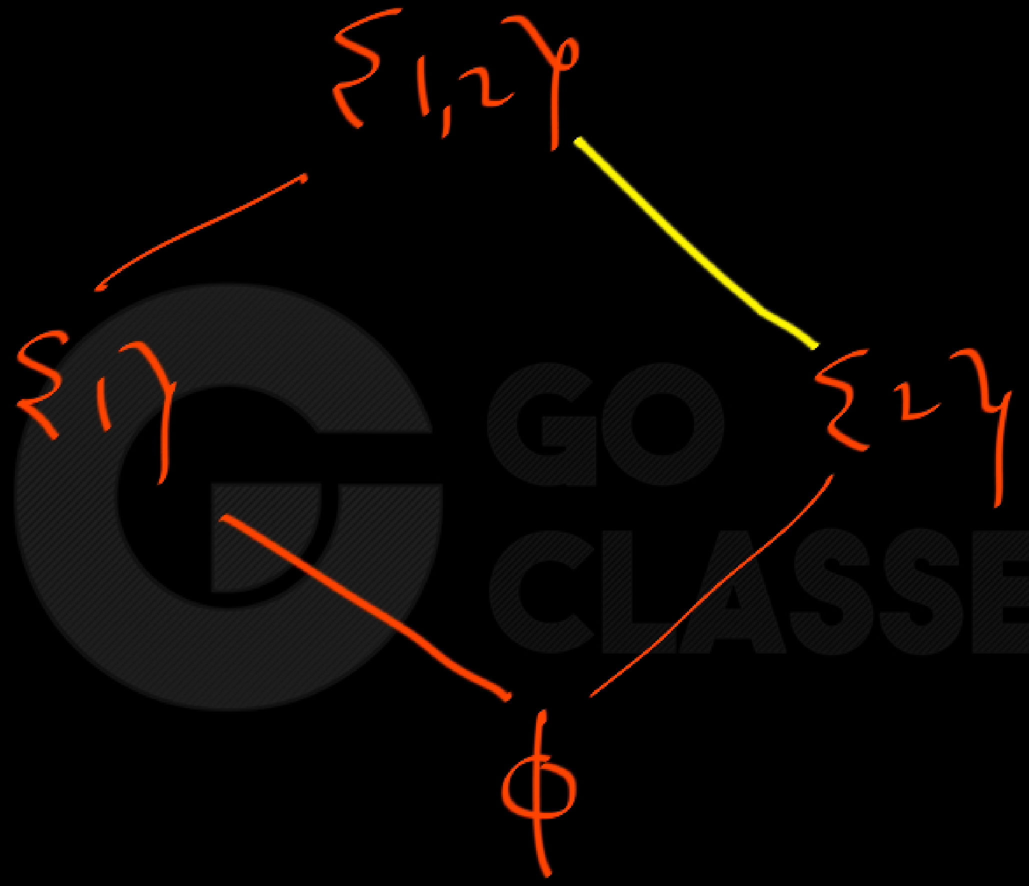
$(\{\phi, \{1\}, \{2\}, \{1,2\}\}, \subseteq)$

$\phi \subseteq \{1\} \subseteq \{1,2\}$

$\{2\}$

Incomparable

$\{1\}, \{2\}$

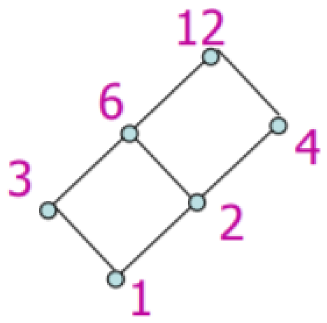

{1,2}
{1}
{2}
ϕ

Example

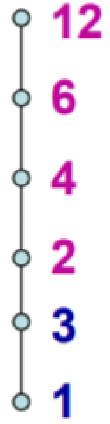
- On the same set E={1,2,3,4,6,12} we can define different partial orders:

12 ~~R~~ 6
6 R 12
2 R 12
12 R 12

2 R 12 ✓
4 ~~R~~ 6
6 ~~R~~ 4



3 R 12 ✓



3 R 2
2 ~~R~~ 3

$(\{a, b, c\}, \leq)$ — POR

Possible HDs :

a   b   c

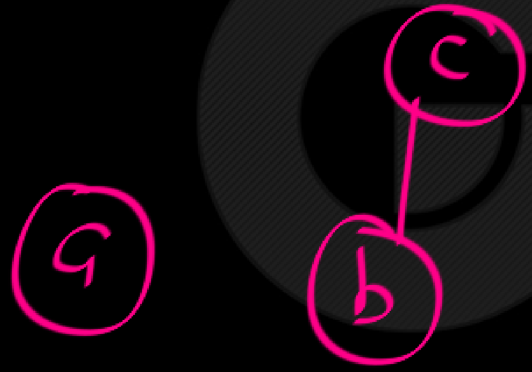
Identity Rel

$R = \{(a,a), (b,b), (c,c)\}$

$|R| = 3$

$(\{a, b, c\}, \leq)$ POR

Possible HDs:



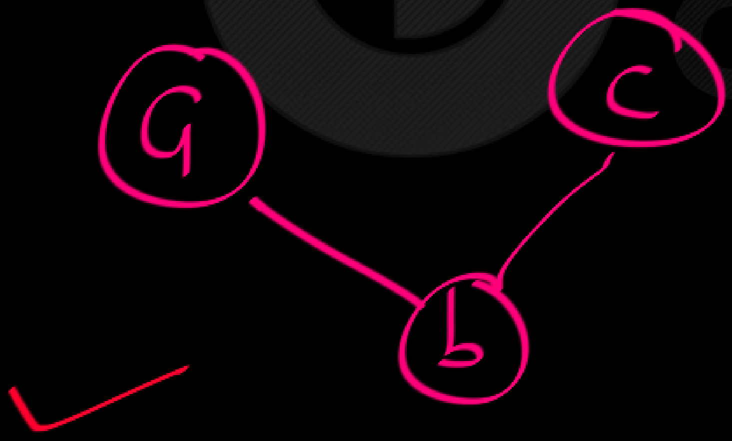
$R = \{(a,a), (b,b), (c,c), (b,c)\}$

$|R| = 4$

#Edges in HD = 1

$(\{a, b, c\}, \leq)$ — POR

Possible HDs:



$|R| = 3 + 1 + 1$

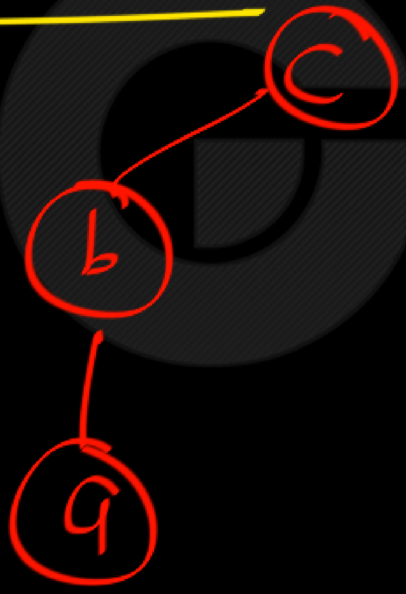
Ref Pairs

bRa

bRc

$(\{a, b, c\}, \leq)$ POR

Possible HDs :



$|R| = 3 + 2 + 1$

Ref Pairs

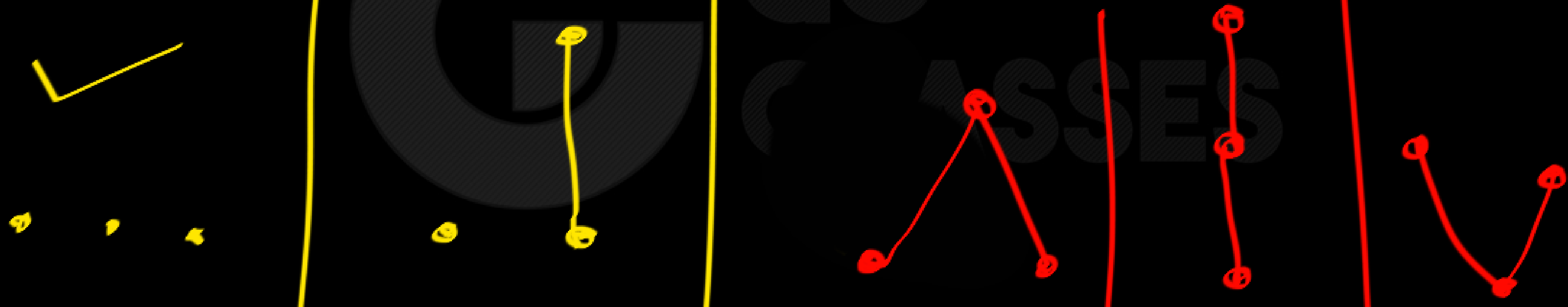
aRb
bRc

aRc

Trans Edge

With 3 elements: (Unlabelled elements)

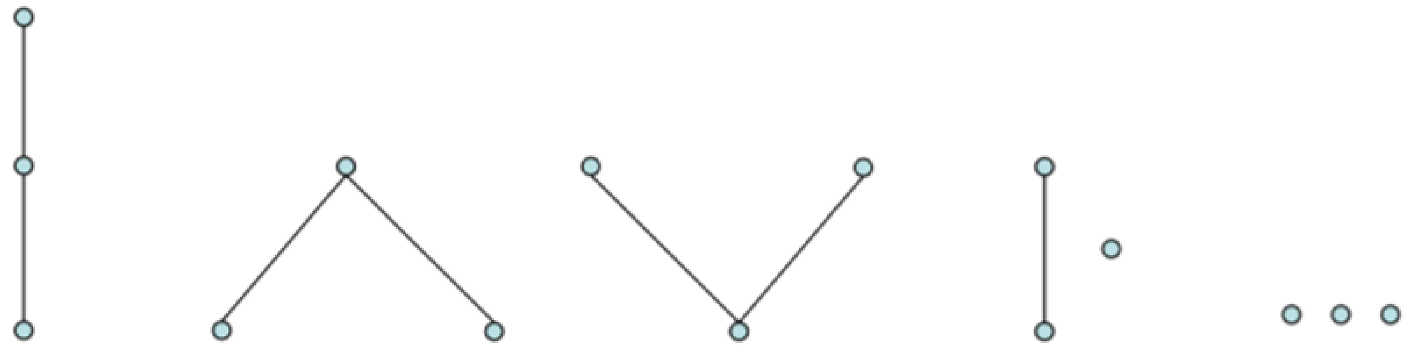
How many Unlabelled HD structures?



H_1 | H_2 | H_3 | H_4 | H_5

Example

- All possible partial orders on a set of three elements (modulo renaming)



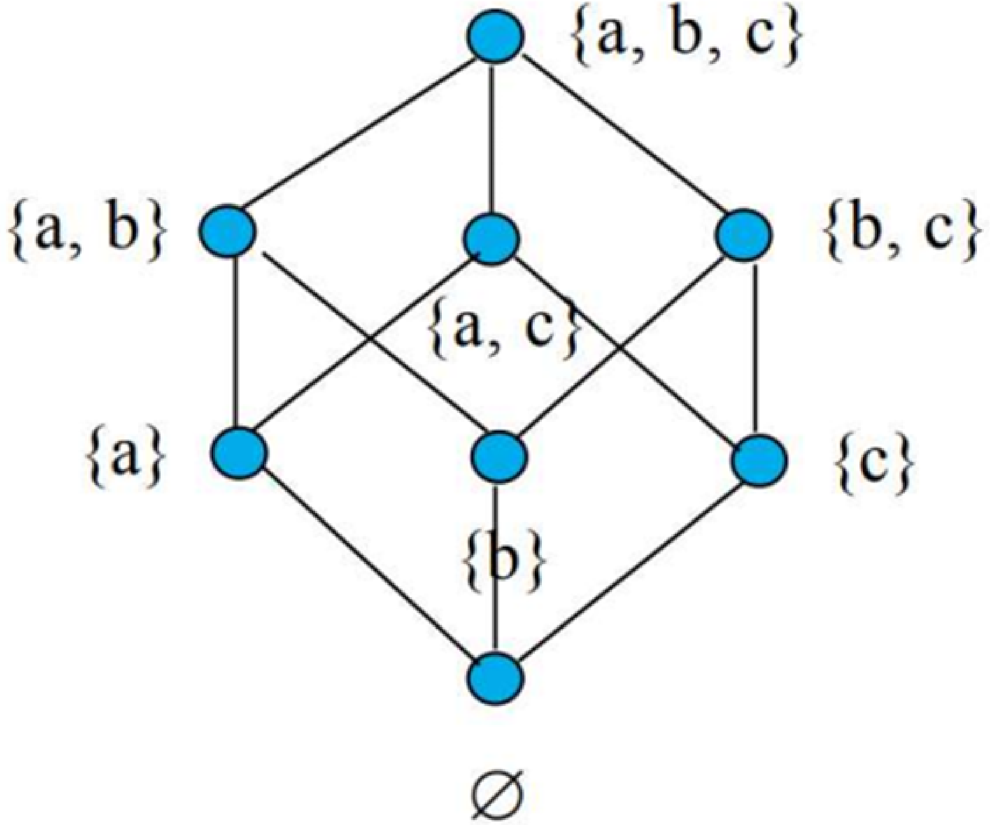
Example:

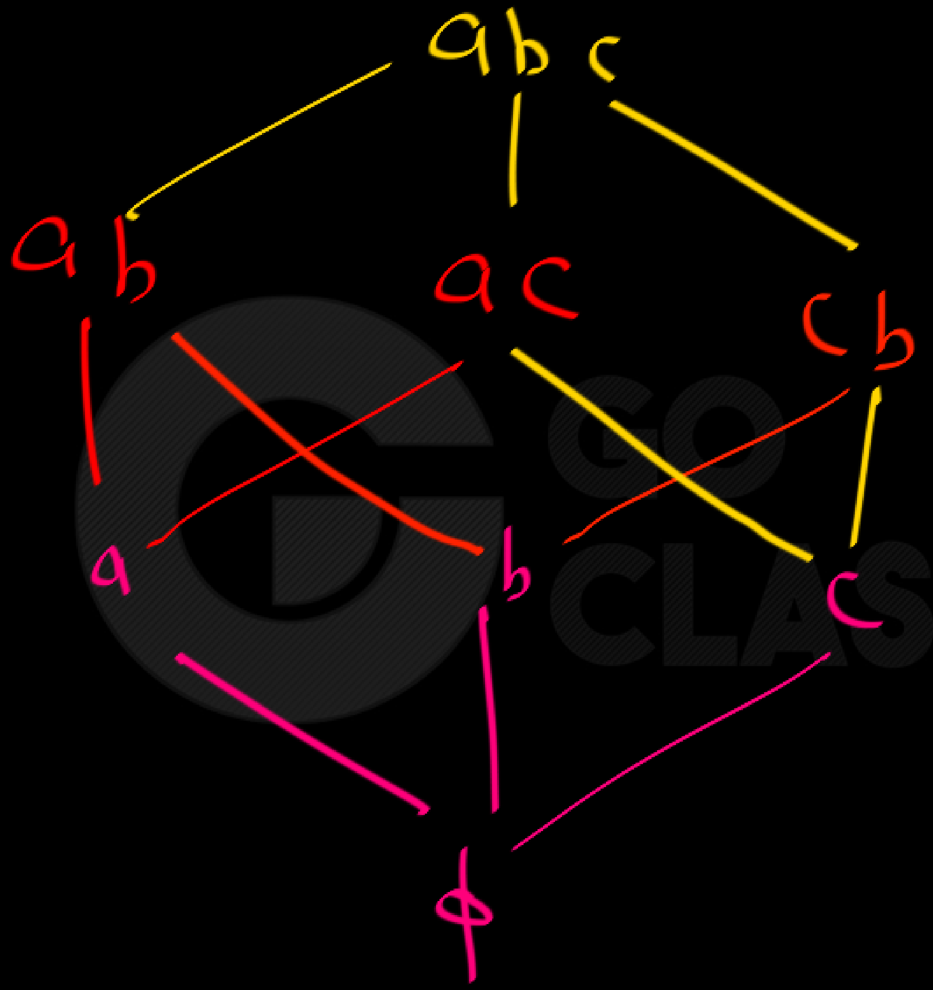
Construct the Hasse diagram of $(P(\{a, b, c\}), \subseteq)$.

$\phi \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a,b,c\}$

$\{b\}$ $\{b,c\}$

The digraph is





for Convenience

$\{a,b\} = ab$

$\{a\} = a$

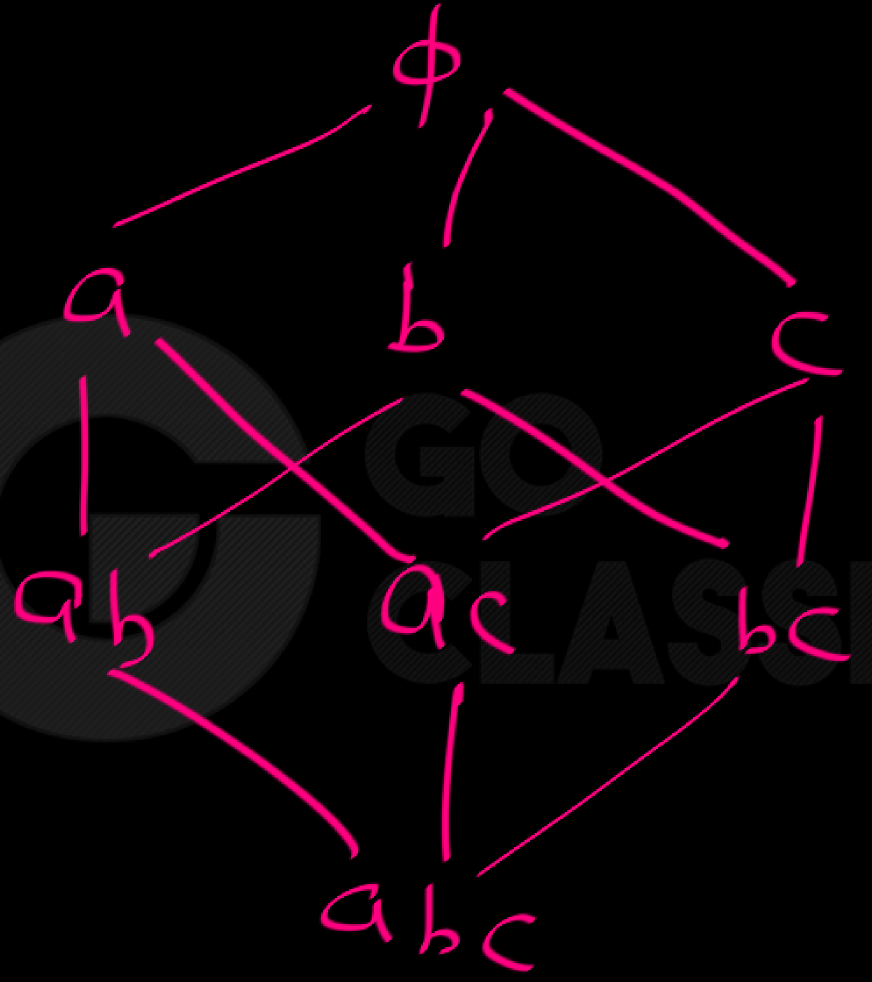
$\{b\} = b$

Example:

Construct the Hasse diagram of $(P(\{a, b, c\}),$ [redacted] $\supseteq)$

$\{a, b, c\} \; R \; \phi$

$\phi \; \cancel{R} \; \{a, b, c\}$



$ab \supseteq b$

$abc \supseteq ab$

$abc \supseteq a$

$abc \supseteq \phi$

Maximal and Minimal Elements

Definition: Let (A, R) be a poset. Then a in A is a *minimal element* if there does not exist an element b in A such that bRa.

Similarly for a *maximal element*.

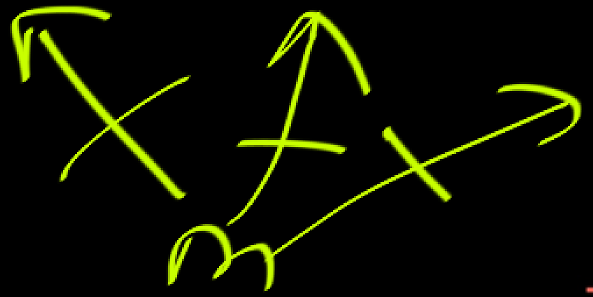
Maximal element "m";

$\forall a \neq m$; $m \not R a$

m not Relates to Anyone else

In HD, from m to anyone else

no edge



minimal element "m";

$\forall a \neq m$; $a \not{R}\, m$

No one else Relates to m

In HD, from anyone else to m, no edge

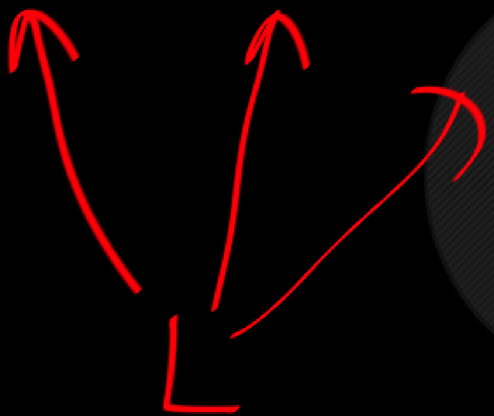
Greatest / maximum element: (G)

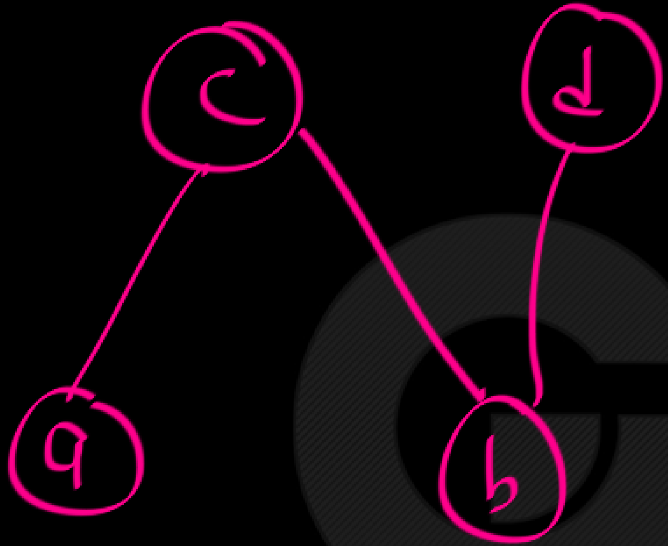
1. $\forall_a \; aRG$ ✓
2. Every element is Related to G.

Least element / minimum : (L)

$\forall a, \; L R a$

L is Related to everyone.



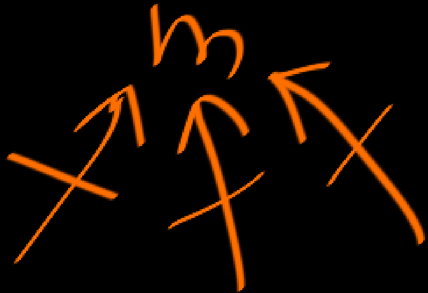


① maximal

c, d

② minimal :

a, b



③ Greatest G : DNE

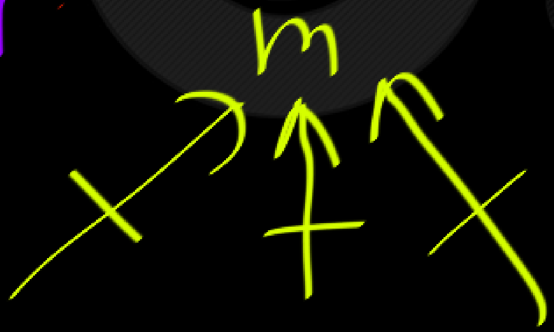
Doesnot exist

④ Least L : DNE

maximal

Noone is Above me

minimal

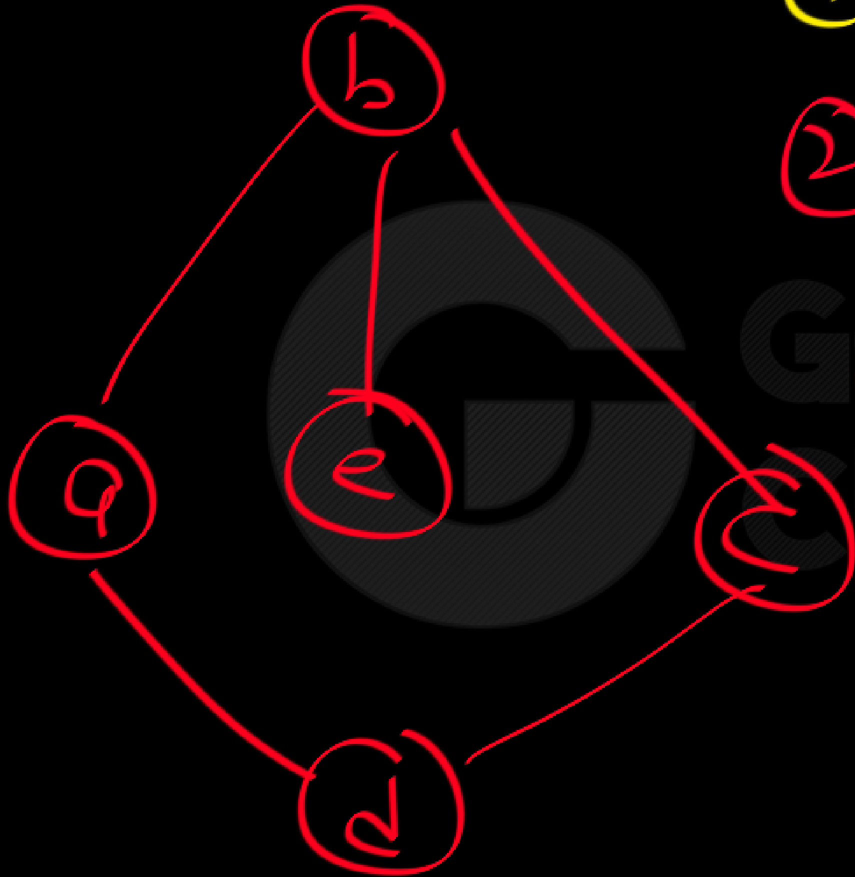


no one is below me.

Greatest : G

Everyone is below me.

Least L : → Everyone is Above me.



1. maximal : b
2. minimal : d, e
3. Greatest : b
4. least : DNE

$e \not R d,\ d \not R e$

Movie: Kong Vs Godzilla

Kong — Gorrila

Godzilla — Huge lizard

Kong: $\boxed{\text{Kong bows to no one.}}$

Kong — maximal

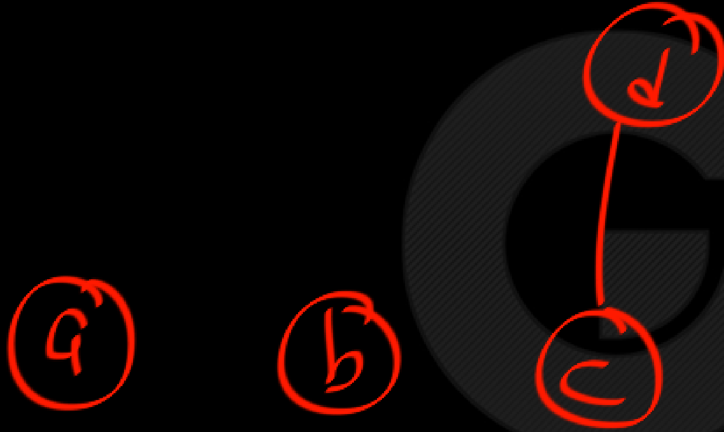
Everyone bows to me. — Greatest

— God

I bow to noone, — Trending Dialog of Almost everyone

main Jhukega nhi

Note: there can be more than one minimal and maximal element in a poset.



maximal : a, b, d

minimal: a, b, c

Q: In a poset, more than one greatest possible? — No.

Assume a, b are greatest.

$a \neq b$ | by def of Greatest

$\forall_x \, xRa \, ; \, \forall_x \, xRb$

$\forall_x \; xRa$

so bRa

$\forall_x \; xRb$

aRb

not poset

Contradiction

Greatest : "exist" or not exist

"Unique"

least : If exists, then unique

Q: In a poset, Can we have more than one Least element?

Ans: No.

If we have more than one least element then it is not a poset.

Assume a, b are least element.

$a \neq b$

by Definition of least element

$\forall_x \; aRx$ | $\forall_x \; bRx$

aRb

bRa

not Antisym

Proof, Complete Analysis

Simply Applications of Definitions

Apply them

Least and Greatest Elements

Definition: Let (A, R) be a poset. Then a in A is the *least element* if for every element b in A, aRb and b is the *greatest element* if for every element a in A, aRb.

Theorem: Least and greatest elements are unique.