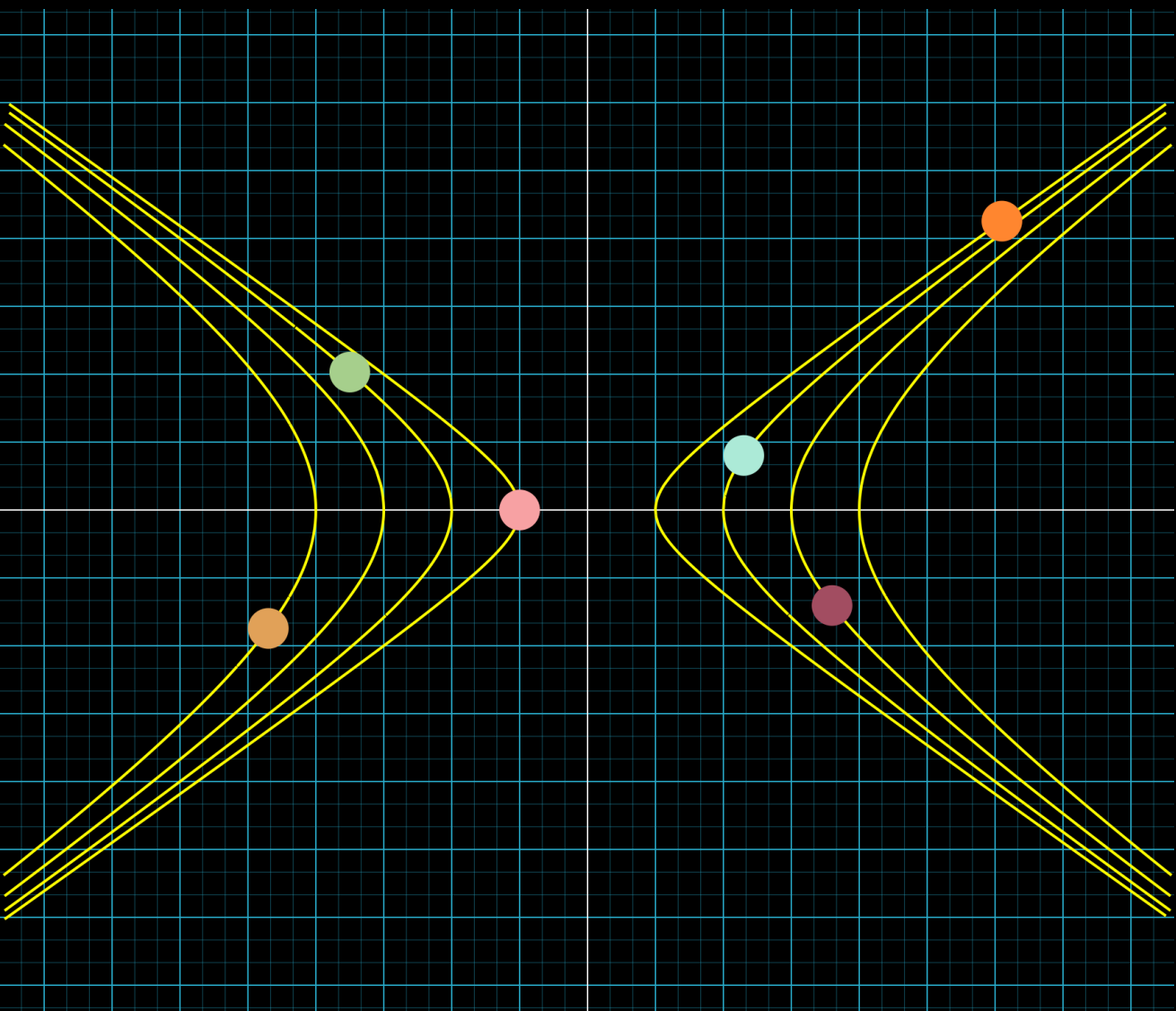


# Strongly correlated electrons in Sachdev-Ye-Kitaev models and Twisted bilayer graphene



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Shankar



A. S. Shankar

# Strongly interacting electrons in Sachdev-Ye-Kitaev models and Twisted Bilayer Graphene

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The cover represents two ways in which Hyperbolic geometries play a role in the low energy properties of electrons. The yellow curve represents the fermi surface near a van-Hove singularity and the random colored dots represent the SYK model. The graphics were made using the manim python package.



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# Samenvatting

Vaste stoffen bestaan uit elektronen en ionen. De meeste vaste stoffen kunnen effectief worden gemodelleerd met een theorie waarin de beweging van de elektronen allemaal als onafhankelijk van elkaar kan worden beschouwd. Dit proefschrift houdt zich bezig met enkele gevallen waarin deze beschrijving onvermijdelijk faalt, en meerdere-deeltjes-interacties de lage-energiefysica domineren. Dergelijke systemen zijn veel moeilijker te bestuderen. Soms kan vooruitgang worden geboekt met behulp van geavanceerde numerieke technieken, of door creatief geconstrueerde, precies oplosbare speelgoedmodellen.

Dit laatste is de benadering van het Sachdev-Ye-Kitaev (SYK)-model, dat een kwantumdot beschrijft met een groot aantal fermionen erop, die allemaal met elkaar interacteren door middel van een gerandomiseerde interactie met vier fermionen. De eenvoud van het model verhult een overvloed aan exotische verschijnselen die voortkomen uit een emergente conforme symmetrie in het infrarood, met eclectische toepassingen van vreemde metalen tot zwarte gaten. Door zogenaamde holografische dualiteit kan de lage-energietheorie van het model, die voortkomt uit het verbreken van de bovengenoemde conforme symmetrie, in kaart worden gebracht via verstoringen van een bijna  $\text{AdS}_2$  ruimtetijd.

Eén van die verschijnselen staat bekend als maximale chaos. Het SYK-model verzadigt een grens aan de snelheid waarmee in een kwantumsysteem verstoringen kunnen dissiperen in zijn vrijheidsgraden. Dit kenmerk blijft bestaan, zelfs als men bepaalde gewijzigde versies van het SYK-model construeert, zoals het bipartiete SYK-model. De mate van chaos verandert niet, zelfs niet als de schaaldimensies van de samenstellende fermionen worden veranderd, zoals beschreven in hoofdstuk 2. De eerste subleidende correctie van de chaos exponent blijft ook onafhankelijk te zijn van de schaaldimensie.

Wanneer twee identieke kopieën van het SYK-model aan elkaar worden gekoppeld door een tunnelinteractie, bij lage temperatuur beschrijft het holografische dual de ruimtetijd van een wormgat. Een op een dergelijke manier gemaakt Josephson-contact kan worden bestudeerd met behulp van de SYK-versie van het elektron-fonon-gekoppelde systeem, dat bekend staat als het Yukawa SYK-model. Supergeleiding in het Yukawa SYK-model vindt plaats bij lage temperaturen als het systeem tijdinversie niet schendt. In hoofdstuk 3 wordt aangetoond dat de niet-supergeleidende

toestand de correcte niet-analytische afhankelijkheid van de "gap" vertoont van de sterkte van de tunnelinteractie, kenmerkend voor de wormgattoestand. Er wordt aangetoond dat de supergeleidende toestand effectief kan worden beschreven door een twee-vloeistoffenmodel, waarbij de Cooperparen onafhankelijk aan beide zijden leven, terwijl excitaties van ongebonden elektronen nog steeds de wormgattoestand vormen.

Als tegenpool van de hyperbolische geometrie van de  $\text{AdS}_2$  ruimte, hebben we in het tweede deel van dit proefschrift een systeem beschouwd waarvan het niet-interagerende fermi-oppervlak is samengesteld uit een familie van hyperbolen. In gedraaid dubbellaags grafeen voor kleine draaihoeken is dit te zien in de vorm van van Hove-singulariteiten, die precies onder de magische hoek een hogere orde krijgen. In hoofdstuk 4 wordt aangetoond dat een kenmerk voor het waarnemen van het effect ervan in een realistische fysieke omgeving de reactie is op een magnetische onzuiverheid. De onzuiverheidsentropie vertoont duidelijke kenmerken nabij de magische hoek, en toont stromingen naar de verschillende soorten vaste punten die beschikbaar zijn in de faseruimte van het systeem. Totdat de magische hoek is bereikt, werkt het bestaan van de Dirac-kegel bij de laagste energieën als een afschermingsmiddel voor Kondo-screening, wat zich manifesteert als een entropie van  $\log 2$  per onzuiverheid. Bij de magische hoek domineert de verschijning van de hogere orde Van Hove-singulariteit de bandstructuur, en dit leidt tot een versterkt Kondo-effect tot de laagst mogelijke energieschalen.





# Summary

Solids are composed of electrons and ions. Most solids can be effectively modeled by a theory in which the motion of the electrons can all be considered independent to each other. This thesis is concerned with some cases where that description fails, and many body interactions dominate the low energy physics. Such systems are far more difficult to study, and sometimes progress can be made either with sophisticated numerical tools, or by creatively constructed exactly solvable toy models.

The latter is the approach taken by the Sachdev-Ye-Kitaev (SYK) model, which describes a quantum dot with a large number of fermions on it, all interacting with each other by means of a randomized four-fermion interaction. The simplicity of the model belies a plethora of exotic phenomena that arise from an emergent conformal symmetry in the infrared, with eclectic applications from strange metals to black holes. Through the holographic duality, the low energy theory of the model arising from the breaking of the aforementioned conformal symmetry can be mapped to perturbations of a nearly  $\text{AdS}_2$  spacetime.

One such phenomenon is known as maximal chaos. The SYK model saturates a bound on the rate at which a quantum system can dissipate perturbations into its degrees of freedom. This feature persists even when one constructs certain modified versions of the SYK model, such as the bipartite SYK model, and the rate of chaos doesn't change even when the scaling dimensions of the constituent fermions are changed as is described in chapter 2. The first subleading correction to the chaos exponent is also found to be independent of the scaling dimension.

When two identical copies of the SYK model are coupled together by a tunneling interaction, its holographic dual describes a wormhole spacetime at low temperature. A Josephson contact made in such a way can be studied using the SYK version of the electron-phonon coupled system, which is known as the Yukawa SYK model. Superconductivity in the Yukawa SYK model occurs at low temperatures if time reversal symmetry is present. In chapter 3, it was shown that the non-superconducting state shows the correct non-analytic dependence of the gap on the strength of the tunneling interaction, characteristic of the wormhole state. The superconducting state was shown to be effectively described by a two-fluid model, where the Cooper pairs live independently on the two sides, while single electron excitations still form the wormhole state.

In juxtaposition to the hyperbolic geometry of the  $\text{AdS}_2$  space, we consider a system whose non-interacting fermi surface was composed of a family of hyperbolae in the second part of this thesis. In twisted bilayer graphene for small twists, this is seen in the form of van Hove singularities, which turn higher order at precisely the magic angle. In chapter 4, a probe for sensing its effect in a realistic physical setting was shown to be the response of a magnetic impurity. The impurity entropy shows distinct signatures near the magic angle, showing flows to the different kinds of fixed points available in the phase space of the system. Until the magic angle is reached, the existence of the Dirac cone at the lowest energies acts as a deterrent for Kondo screening, which manifests itself as a  $\log 2$  entropy per impurity. At the magic angle, the appearance of the higher order van Hove singularity dominates the band structure, and it leads to an enhanced Kondo effect until the lowest energy scales possible.



# List of Publications

1. “ Lyapunov exponents in a Sachdev-Ye-Kitaev-type model with population imbalance in the conformal limit and beyond ”,

**A. S. Shankar**, M.Fremling, S. Plugge and L. Fritz  
*Phys. Rev. D*, 108, 094039 (2023).

2. “ Kondo effect in twisted bilayer graphene ”,

**A. S. Shankar**, D. O. Oriekhov, Andrew K. Mitchell, and L. Fritz  
*Phys. Rev. B*, 107, 245102 (2023).



# Curriculum Vitae

I was born on September 26<sup>th</sup>, 1996 in Chennai, a big city in the south of India. After finishing high school, I went to West Bengal for a five-year integrated Bachelor-Master program in Physics at the Indian Institute of Technology, Kharagpur.

During this period, I took courses in several areas of physics and engineering, including high energy physics and gravitational physics, with my favorite being condensed matter and statistical physics. I was also fortunate to get an early start at doing my own research by being selected for several internships throughout my bachelor and master studies. With Prof. Pinaki Sengupta at NTU Singapore, I worked on writing my first quantum Monte Carlo code to investigate the Heisenberg XXZ model on ladder-like systems. I then worked briefly with Prof. Barbara Terhal and Prof. Fabian Hassler on DMRG simulations at RTWH aachen, and then on numerical and analytical studies on the Anderson model on the Bethe lattice with Prof. Antonello Scardicchio at the ICTP, Trieste, looking for signatures of anomalous diffusion. I concluded my master studies with a thesis titled "Disordered Superconductors", working with Prof. Sudhansu Sekhar Mandal.

I started my PhD at Leiden university in the Quantum matter group of Prof. Jan Zaanen and Prof. Koenraad Schalm in August 2019. After a short stint there, I moved to the group of Prof. Vadim Cheianov in April 2021. It was also at this time that I started working very closely with Prof. Lars Fritz and his group at Utrecht University.

As the early years of my PhD programme were marred by the coronavirus pandemic, I attended several condensed matter schools online, including editions of the Princeton PSSC summer school and the Maglab winter school. After the pandemic ended, I was able to present my work at several schools and conferences, including poster presentations in Trieste, Italy, at the ICTP conference "From quantum criticality to flat bands", at the school "Emergence of quantum phases in novel materials", at CSIC, Madrid, Spain, at the Veldhoven Physics conference in the Netherlands, and an online talk at the APS March meeting in 2023, among others. I also had the opportunity to TA several undergraduate and graduate courses, in particular to write a set of extensive lecture notes on Information Geometry with Prof. Subodh Patil.

After the defense of my PhD, I will move on to begin a Postdoctoral position at the ICTP in Trieste.

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I'm also very grateful to the PhD students and postdocs that I worked with in Leiden and in Utrecht - Dima Oriekhov, Mikael Fremling and Stephan Plugge. I am also particularly grateful to the several intense discussions I had with Nicolas Chagnet over physics and on numerics in general. I would also like to acknowledge master students Alina Melissakis and Nina Doerfler whose theses I helped co-supervise- I learned quite a lot from this experience.

I extend thanks next to the several friends I made along the way. I will always have a special place for my covid-housemate Alicja Dutkiewicz, and my post-covid housemate Dimitrios Krommydas, with whom I felt that family was always close by.

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physics and otherwise. I hope I didn't miss anyone out!

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# Stellingen

Behorende bij het proefschrift

## Strongly correlated electrons in Sachdev-Ye-Kitaev models and Twisted bilayer graphene

1. Many crucial aspects of the SYK model persist upon reducing the connectivity of the interactions from all-to-all.  
[Chapter 2]
2. Coupled Yukawa SYK models allow one to compute the Josephson current between two superconductors non-perturbatively in the coupling interaction.  
[Chapter 3]
3. The wormhole solutions obtained in the Yukawa SYK model only allow transmission of electronic excitations, but not Cooper pairs.  
[Chapter 3]
4. The Kondo effect can be used as a probe to identify crucial qualitative features of the band structure of twisted bilayer graphene.  
[Chapter 4]
5. Although the mechanism of superconductivity in the Yukawa SYK model is **not** of the conventional electron-phonon type, it still shows an isotope effect. Conversely, an observation of the isotope effect is not an immediate guarantee of a BCS like superconducting mechanism.  
I. Esterlis, J. Schmalian *Phys. Rev. B* 100, 115132 (2019)
6. A Moire pattern in the real space lattice of a material is a strong indicator of the possible existence of flat bands in the said material.
7. The existence of a true quantum critical point has not been confirmed beyond reasonable doubt. This is because any function can be made to look like a power law on a log-log scale for less than a decade.
8. Arrays of SYK dots in the weak inter-dot tunneling limit can be used to model the phase diagram near the Mott phase of cuprates, but such a description fails in the strange metallic phase.  
A. A. Patel, H. Guo, I. Esterlis, S. Sachdev *Science* 381,790-793 (2023)
9. Although aggressively marketed as such, it is unclear whether quantum computers will be useful for understanding much about quantum many body problems.

Aravindh Swaminathan Shankar  
Leiden, 7<sup>th</sup> January 2025

