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Article

Novel Modeling and Control Strategies for a HVAC System Including Carbon Dioxide Control

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Abstract: Conventional heating, ventilating, and air conditioning (HVAC) systems have traditionally used the temperature and the humidity ratio as the quantitative indices of comfort in a room. Recently, the carbon dioxide (CO_2) concentration has also been recognized as having an important contribution to room comfort. This paper presents the modeling of an augmented HVAC system including CO_2 concentration, and its control strategies. Because the proposed augmented HVAC system is multi-input multi-output (MIMO) and has no relative degree problem, the dynamic extension algorithm can be employed; then, a feedback linearization technique is applied. A linear-quadratic regulator (LQR) is designed to optimize control performance and to stabilize the proposed HVAC system. Simulation results are provided to validate the proposed system model, as well as its linearized control system.

Keywords: HVAC system; MIMO system; dynamic extension algorithm; feedback linearization

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1. Introduction

HVAC systems are automatic systems that control temperature and humidity in buildings, providing people with a comfortable environment. The use of HVAC systems represents more than 50% of the world energy consumption [1–4]. Thus, balancing occupant comfort and energy efficiency is a main goal of HVAC control strategies.

In most previous studies, HVAC systems have been modeled considering only the temperature and the humidity ratio [5–8]. A nonlinear HVAC model that includes dynamics of temperature and humidity ratio is proposed in [5], which includes the design of an observer to estimate the thermal and moisture loads. In [6], an adaptive fuzzy output feedback controller is proposed, based on an observer for the HVAC system. In [7,8], a back-stepping controller and a decentralized nonlinear adaptive controller are respectively applied to the same model.

Recently, the CO_2 concentration has been recognized as having an important contribution to room comfort [9,10]. Some researchers have proposed hybrid HVAC systems that represent the temperature and humidity ratio as continuous states and CO_2 concentration as a discrete state [11,12]. However, because these states are strongly interrelated, it is more appropriate to integrate these continuous and discrete dynamics into a single model that includes temperature, humidity ratio, and CO_2 concentration as states.

This paper presents a modeling and control strategy for a novel HVAC system that considers temperature, humidity ratio, and CO_2 concentration. In the process of modeling, the dynamic extension algorithm of [13] is employed to deal with non-interacting control problem and no relative degree problem. After the dynamic extension process, a feedback linearization method can be applied to the proposed HVAC system to convert a bilinear system into a linear system. Linear controllers, pole placement and LQR can be designed for the linearized novel HVAC system to stabilize it and improve its control performance.

This paper is organized as follows: in Section 2, we present the bilinear model for the conventional HVAC system, including valve dynamics. Section 3 presents a novel HVAC system including CO_2 concentration and its applicability in the feedback linearization method. Also, dynamic extension algorithm is applied for solving the no relative degree and interacting control problems in the MIMO system. In Section 4, we describe the design of linear controllers for the linearized HVAC system, such as pole placement and LQR controllers, to improve the system's control performance and to verify the effectiveness of the proposed model.

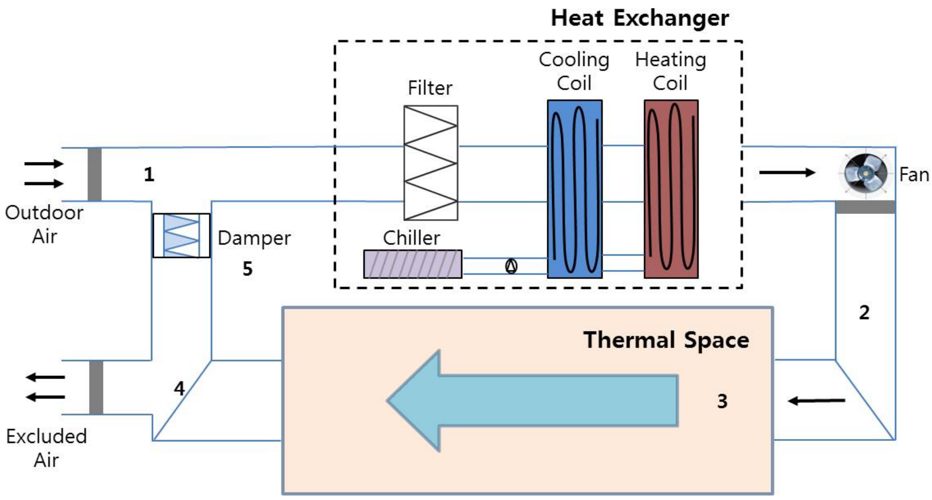
2. Conventional HVAC System with Temperature and Humidity Ratio

As mentioned, conventional HVAC systems control only temperature and humidity. In this paper, we consider the single-zone system shown in Figure 1 as a representative conventional HVAC system. It consists of the following components: a heat exchanger; a chiller, which provides chilled water to the heat exchanger; a circulating air fan; the thermal space; connecting ductwork; dampers; and mixing air components [5]. The conventional HVAC system controls the temperature and humidity ratio as follows [5]:

- Fresh air is introduced into the system and is mixed in a 25:75 ratio with recirculated air (position 5) at the flow mixer.
- Second, air mixed at the flow mixer (position 1) enters the heat exchanger, where it is conditioned.

- Third, the conditioned air is moved out of the heat exchanger; this air is ready to enter the thermal space, and is called supply air (position 2).
- Fourth, the supply air enters the thermal space (position 3), where it offsets the sensible (actual heat) and latent (humidity) heat loads acting upon the system.
- Finally, the air in the thermal space is drawn through a fan (position 4); 75% of this air is recirculated and the rest is exhausted from the system.

Figure 1. Model of the representative conventional HVAC system.



The control inputs for a conventional HVAC system are the flow rate of air, which is varied using a variable-speed fan (position 2), and the flow rate of water from the chiller to the heat exchanger. However, in our proposed HVAC system, the air recirculation rate (position 4) is added as a new control input, and some modifications are made to the basic operation rules listed above. That is to say, the 75% recirculation rate listed above in the first and last steps becomes a variable quantity, and is used as the third control input.

2.1. Mathematical Modeling of Conventional HVAC System

The conventional HVAC system is a model considering the temperature and the humidity ratio as states. The differential equations describing the dynamic behavior of the HVAC system in Figure 1 can be derived from energy conservation principles and are given by [5]:

$$\begin{cases} \dot{T}_3 = \dfrac{60F}{V_s}(T_2 - T_3) - \dfrac{60h_{fg}F}{C_pV_s}(W_s - W_3) + \dfrac{1}{(1-\mu)C_pV_s}\left(Q_o - h_{fg}M_o\right) \\ \dot{W}_3 = \dfrac{60F}{V_s}(W_s - W_3) + \dfrac{M_o}{\rho V_s} \\ \dot{T}_2 = \dfrac{60F}{V_{he}}(T_3 - T_2) + \dfrac{60F(1-\mu)}{V_{he}}(T_o - T_3) - \dfrac{60Fh_w}{C_pV_{he}}\{(1-\mu)W_o + \mu W_3 - W_s\} - \dfrac{6000}{\rho C_p V_{he}} gpm \end{cases} \tag{1}$$

The dynamic system given by Equation (1) can be converted into a state variable form for the purposes of control. Let $u_1 = F$, $u_2 = gpm$, $z_1 = T_3$, $z_2 = W_3$, $z_3 = T_2$ and define the following parameters: $\alpha_1 = \frac{60}{V_s}$, $\alpha_2 = \frac{60h_{fg}}{C_pV_s}$, $\alpha_3 = \frac{1}{\rho_a C_p V_s}$, $\alpha_4 = \frac{1}{\rho_a V_s}$, $\beta_1 = \frac{60}{V_{he}}$, $\beta_2 = \frac{1}{\rho_a C_p V_{he}}$, and $\beta_3 = \frac{60h_w}{C_pV_{he}}$. Then, the dynamic equations given in Equation (1) can be written in the following state variable form.

$$\begin{cases} \dot{z}_1 = u_1\alpha_1(z_3 - z_1) - u_1\alpha_2(W_s - z_2) + \alpha_3\left(Q_o - h_{fg}M_o\right) \\ \dot{z}_2 = u_1\alpha_1(W_s - z_2) + \alpha_4 M_o \\ \dot{z}_3 = u_1\beta_1(z_1 - z_3) + (1-\mu)u_1\beta_1(T_o - z_1) - u_1\beta_3\{(1-\mu)W_o + \mu x_2 - W_s\} - 6000u_2\beta_2 \\ y_1 = z_1 \\ y_2 = z_2 \end{cases} \quad (2)$$

The conventional HVAC system of Equation (2) is a 2-input, 2-output MIMO system: its inputs are the volumetric air flow rate and the chilled water flow rate, and its outputs are the temperature and the humidity ratio of the thermal space.

2.2. Adding Valve Dynamics to the Conventional HVAC Model

The Control input signals $\mathbf{u} = [u_1 u_2]^T$ in the system described in Equation (2) are implemented using liquid valves. The valve dynamics can be modeled as follows in which $\psi(s)$ is the valve inherent characteristic and $u(s)$ is the flow rate of the liquid which enters the valve [14,15]:

$$u(s) = \frac{1}{(1+\tau s)}\psi(s) \quad (3)$$

By considering the characteristic of a linear valve as $\psi(s) = kv(s)$, the valve transfer function can be written as:

$$u_1 = \frac{k_1}{1+\tau_1 s}v_1, u_2 = \frac{k_2}{1+\tau_2 s}v_2 \quad (4)$$

where k_1, k_2, τ_1, and τ_2 are the constant gains and the time constants, respectively; $\mathbf{v} = [v_1 v_2]^T$ is the control signal applied to the actuator; and $\mathbf{u} = [u_1 u_2]^T$ is the signal that is input to the HVAC system.

An augmented state space model with the new state vector, $\mathbf{z} = [z_1 z_2 z_3 u_1 u_2]^T = [z_1 z_2 z_3 z_4 z_5]^T$ can be derived as:

$$\begin{cases} \dot{\boldsymbol{z}} = \boldsymbol{f}(\boldsymbol{z}) + \boldsymbol{g}(\boldsymbol{z})\boldsymbol{v} = \begin{bmatrix} a_1(\boldsymbol{z}) \\ a_2(\boldsymbol{z}) \\ a_3(\boldsymbol{z}) \\ a_4(\boldsymbol{z}) \\ a_5(\boldsymbol{z}) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{k_1}{\tau_1} & 0 \\ 0 & \frac{k_2}{\tau_2} \end{bmatrix}\boldsymbol{v} \\ \boldsymbol{y} = [z_1 \;\; z_2]^T \end{cases} \quad (5)$$

where:

$$\begin{cases} a_1(\boldsymbol{z}) = \{\alpha_1(z_3 - z_1) - \alpha_2(W_s - z_2)\}u_1 + \alpha_3\left(Q_o - h_{fg}M_o\right) \\ a_2(\boldsymbol{z}) = \alpha_1(W_s - z_2)u_1 + \alpha_4 M_o \\ a_3(\boldsymbol{z}) = [\{\beta_1(z_1 - z_3) + (1-\mu)\beta_1(T_0 - z_1) - \beta_3\{(1-\mu)W_o - \mu z_2 - W_s\}]u_1 - 6000\beta_2 u_2 \\ a_4(\boldsymbol{z}) = -\frac{u_1}{\tau_1} \\ a_5(\boldsymbol{z}) = -\frac{u_2}{\tau_2} \end{cases}$$

Thus, the system in Equation (5) represents a conventional HVAC system to which valve dynamics have been applied to implement the control signals. This conventional system can regulate only the temperature and the humidity ratio; other factors such as CO_2 concentration, which affects the health of occupants or workers indoors, cannot be considered.

3. Novel Modeling of HVAC System including CO_2 Concentration

If the recirculated air contains too much CO_2, it can affect the health and work efficiency of the building's occupants. Therefore, CO_2 concentration should be one of the quantitative indices of room comfort, along with temperature and humidity ratio. In this paper, we propose an HVAC system that continuously controls all three of these indices.

From the mass balance equation, the average CO_2 concentration C_s in the room can be represented as [14]:

$$V_s\dot{C}_s = C_g + (1-\mu)(C_i - C_o) \tag{6}$$

where $C_s = C_i - C_o$, C_g is the amount of CO_2 generated in the room; C_i is the CO_2 concentration in the inlet air; C_o is the CO_2 concentration of air leaving the room, and $(1-\mu), 0 \leq \mu \leq 1$, is the air exchange rate.

3.1. Proposed HVAC System Model

The proposed HVAC system model includes CO_2 concentration as a state. The differential Equation (6) can be integrated into the dynamic equations in (1). The valve dynamics of u_3 are added to the control input vector $\mathbf{u} = [u_1 u_2]^T$, and the control signal applied to the actuator of v_3 also can be added to the actuator input vector $\mathbf{v}$.

Let $u_3 = 1-\mu$, $\mathrm{x}_1 = z_1$, $\mathrm{x}_2 = z_2$, $\mathrm{x}_3 = z_3$, $\mathrm{x}_4 = C_s$, and let an augmented state vector $\mathbf{x} = [\mathrm{x}_1 \mathrm{x}_2 \mathrm{x}_3 \mathrm{x}_4 u_1 u_2 u_3]^{\boldsymbol{T}}$. Then, the whole dynamics can be written in the state variable form as:

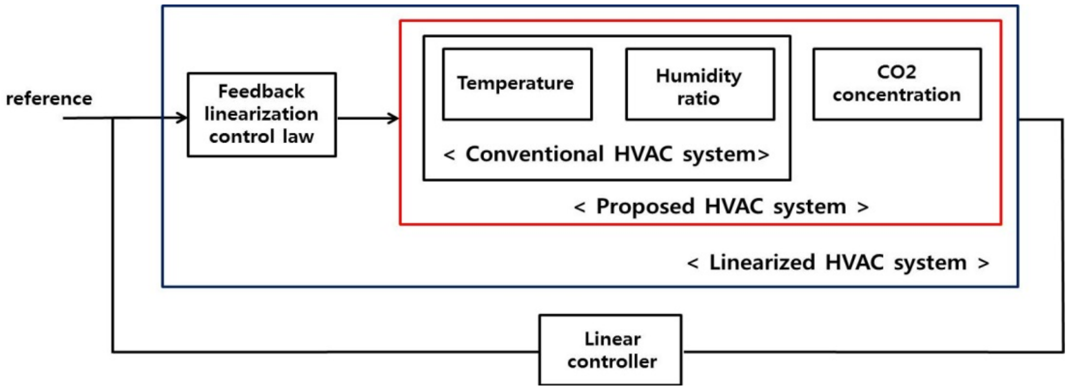
$$\begin{cases} \dot{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}) + \sum_{i=1}^{3} \boldsymbol{g}_i(\mathbf{x})\boldsymbol{v}_i = \begin{bmatrix} a_1(\mathbf{x}) \\ a_2(\mathbf{x}) \\ a_3(\mathbf{x}) \\ a_4(\mathbf{x}) \\ a_5(\mathbf{x}) \\ a_6(\mathbf{x}) \\ a_7(\mathbf{x}) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{k_1}{\tau_1} & 0 & 0 \\ 0 & \frac{k_2}{\tau_2} & 0 \\ 0 & 0 & \frac{k_3}{\tau_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \\ \boldsymbol{y} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^T \end{cases} \tag{7}$$

where:

$$
\begin{cases}
a_1(\mathbf{x}) = \{\alpha_1(\mathrm{x}_3 - \mathrm{x}_1) - \alpha_2(W_s - \mathrm{x}_2)\}u_1 + \alpha_3(Q_o - h_{fg}M_o) = \gamma_1 u_1 + \alpha_3(Q_o - h_{fg}M_o) \\
a_2(\mathbf{x}) = \alpha_1(W_s - \mathrm{x}_2)u_1 + \alpha_4 M_o = \gamma_2 u_1 + \alpha_4 M_o \\
a_3(\mathbf{x}) = [\{\beta_1(\mathrm{x}_1 - \mathrm{x}_3) + (1-\mu)\beta_1(T_0 - \mathrm{x}_1) - \beta_3\{(1-\mu)W_o - \mu\mathrm{x}_2 - W_s\}]u_1 - 6000\beta_2 u_2 \\
\qquad\quad = \gamma_3 u_1 + \gamma_4 u_2 \\
a_4(\mathbf{x}) = \dfrac{1}{V_s}(u_3 C_s + C_g) = \gamma_5 u_3 + \dfrac{C_g}{V_s} \\
a_5(\mathbf{x}) = -\dfrac{u_1}{\tau_1} \\
a_6(\mathbf{x}) = -\dfrac{u_2}{\tau_2} \\
a_7(\mathbf{x}) = -\dfrac{u_3}{\tau_3}
\end{cases}
$$

The novel HVAC system of (7) is a 3-input, 3-output MIMO system: its inputs are the volumetric air flow rate, the chilled water flow rate, and the outdoor air flow rate, and its outputs are the temperature, humidity ratio, and CO_2 concentration of the thermal space. This proposed HVAC system can be linearized using a feedback linearization control method, as shown in Figure 2. Thus, we can finally obtain a linearized HVAC system that can be controlled using linear controllers.

Figure 2. Overall block diagram for controlling the proposed HVAC system.



3.2. Conditions for Input-Output Feedback Linearization

An input-output feedback linearization method can be applied to the state space model given in (7) to track the desired temperature, humidity ratio, and CO_2 concentration only when the decoupling matrix is non-singular. However, the decoupling matrix (refer to Appendix A2):

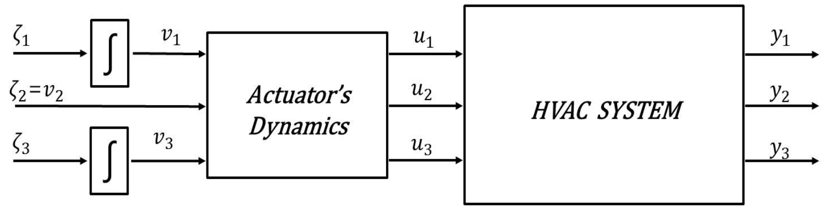
$$
D(\mathbf{x}) = \begin{bmatrix} L_{g_1}L_f^1\mathrm{x}_1 & L_{g_2}L_f^1\mathrm{x}_1 & L_{g_3}L_f^1\mathrm{x}_1 \\ L_{g_1}L_f^1\mathrm{x}_2 & L_{g_2}L_f^1\mathrm{x}_2 & L_{g_3}L_f^1\mathrm{x}_2 \\ L_{g_1}L_f^1\mathrm{x}_4 & L_{g_2}L_f^1\mathrm{x}_4 & L_{g_3}L_f^1\mathrm{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{\gamma_1 k_1}{\tau_1} & 0 & 0 \\ \frac{\gamma_2 k_1}{\tau_1} & 0 & 0 \\ 0 & 0 & \frac{\gamma_5 k_3}{\tau_3} \end{bmatrix}
$$

is singular; here, $L(\cdot)$ represents the Lie derivative and the total relative degree is $6 \neq$ n, where n is the system order [16]. When the invertibility condition is violated, some method is needed to carry out an input-output linearization; the dynamic extension algorithm used herein is such a method.

Dynamic Extension Algorithm

Because input-output linearization can be achieved only when the decoupling matrix $D(\mathbf{x})$ is non-singular, employing the dynamic extension algorithm involves choosing some new inputs that are the derivatives of some of the original system inputs, in such a way that the decoupling matrix $D(\mathbf{x})$ becomes non-singular, as shown in Figure 3 [16].

Figure 3. Concept of the dynamic extension algorithm [13].



Let state vector $\bar{\mathbf{x}} = [\mathbf{x}\ v_1\ v_3]^T = [\mathrm{x}_1\ \mathrm{x}_2\ \mathrm{x}_3\ \mathrm{x}_4\ u_1\ u_2\ u_3\ v_1\ v_3]^T$. Then, the state space can be written as:

$$\begin{cases} \dot{\bar{\mathbf{x}}} = \bar{\boldsymbol{f}}(\bar{\mathbf{x}}) + \displaystyle\sum_{i=1}^{3} \bar{\boldsymbol{g}}_i(\bar{\mathbf{x}})\boldsymbol{\zeta}_i = \begin{bmatrix} \bar{a}_1(\bar{\mathbf{x}}) \\ \bar{a}_2(\bar{\mathbf{x}}) \\ \bar{a}_3(\bar{\mathbf{x}}) \\ \bar{a}_4(\bar{\mathbf{x}}) \\ \bar{a}_5(\bar{\mathbf{x}}) \\ \bar{a}_6(\bar{\mathbf{x}}) \\ \bar{a}_7(\bar{\mathbf{x}}) \\ \bar{a}_8(\bar{\mathbf{x}}) \\ \bar{a}_9(\bar{\mathbf{x}}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \zeta_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dfrac{k_2}{\tau_2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \zeta_2 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \zeta_3 \\ \boldsymbol{y} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^T \end{cases} \tag{8}$$

where:

$$\begin{cases} a_1(\mathbf{x}) = \{\alpha_1(\mathrm{x}_3 - \mathrm{x}_1) - \alpha_2(W_s - \mathrm{x}_2)\}u_1 + \alpha_3(Q_o - h_{fg}M_o) = \gamma_1 u_1 + \alpha_3\left(Q_o - h_{fg}M_o\right) \\ a_2(\mathbf{x}) = \alpha_1(W_s - \mathrm{x}_2)u_1 + \alpha_4 M_o = \gamma_2 u_1 + \alpha_4 M_o \\ a_3(\mathbf{x}) = [\{\beta_1(\mathrm{x}_1 - \mathrm{x}_3) + (1-\mu)\beta_1(T_0 - \mathrm{x}_1) - \beta_3\{u_3 W_o - (1-u_3)\mathrm{x}_2 - W_s\}]u_1 - 6000\beta_2 u_2 \\ \quad = \gamma_3 u_1 + \gamma_4 u_2 \\ a_4(\mathbf{x}) = \dfrac{1}{V_s}(u_3 C_s + C_g) = \gamma_5 u_3 + \dfrac{C_g}{V_s} \\ \bar{a}_5(\bar{\mathbf{x}}) = -\dfrac{u_1}{\tau_1} + \dfrac{k_1}{\tau_1} v_1 \\ \bar{a}_6(\bar{\mathbf{x}}) = -\dfrac{u_2}{\tau_2} \\ \bar{a}_7(\bar{\mathbf{x}}) = -\dfrac{u_3}{\tau_3} + \dfrac{k_3}{\tau_3} v_3 \\ \bar{a}_8(\bar{\mathbf{x}}) = 0 \\ \bar{a}_9(\bar{\mathbf{x}}) = 0. \end{cases}$$

The decoupling matrix as changed by the dynamic extension (refer to Appendix A3):

$$\overline{\boldsymbol{D}}(\bar{\mathrm{x}}) = \begin{bmatrix} L_{\bar{g}_1}L_{\bar{f}}^2\mathrm{x}_1 & L_{\bar{g}_2}L_{\bar{f}}^2\mathrm{x}_1 & L_{\bar{g}_3}L_{\bar{f}}^2\mathrm{x}_1 \\ L_{\bar{g}_1}L_{\bar{f}}^2\mathrm{x}_2 & L_{\bar{g}_2}L_{\bar{f}}^2\mathrm{x}_2 & L_{\bar{g}_3}L_{\bar{f}}^2\mathrm{x}_2 \\ L_{\bar{g}_1}L_{\bar{f}}^2\mathrm{x}_4 & L_{\bar{g}_2}L_{\bar{f}}^2\mathrm{x}_4 & L_{\bar{g}_3}L_{\bar{f}}^2\mathrm{x}_4 \end{bmatrix} = \begin{bmatrix} \dfrac{\gamma_1 k_1}{\tau_1} & \dfrac{\alpha_1\gamma_4 k_2 u_1}{\tau_2} & 0 \\ \dfrac{\gamma_2 k_1}{\tau_1} & 0 & 0 \\ 0 & 0 & \dfrac{\gamma_5 k_3}{\tau_3} \end{bmatrix}$$

is non-singular. The vector relative degree is $\{3 \quad 3 \quad 3\}$ and the total relative degree is equal to the system order n, which means that there are no internal dynamics [16]. Therefore, we can achieve relative degree and non-interacting control.

3.3. Equivalent Linearization HVAC System

To be able to apply the feedback linearization method conveniently, we can change system (8) into the equivalent system (9) by using Equation (A3):

$$\begin{cases} \dot{\mathbf{x}}' = \boldsymbol{A}\mathbf{x}' + \boldsymbol{B}[\boldsymbol{F}(\mathbf{x}') + \boldsymbol{G}(\mathbf{x}')\boldsymbol{\zeta}] \\ \mathbf{y} = [y_1 \quad y_2 \quad y_3]^T = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^T \end{cases} \tag{9}$$

where:

$\mathbf{x}' = \left[\mathrm{x}_1 \mathrm{x}_1^{(1)} \mathrm{x}_1^{(2)} \ \mathrm{x}_2 \ \mathrm{x}_2^{(1)} \ \mathrm{x}_2^{(2)} \ \mathrm{x}_4 \ \mathrm{x}_4^{(1)} \ \mathrm{x}_4^{(2)}\right]^T$;

$\boldsymbol{F}(\mathbf{x}') = [f_1 f_2 f_3]^T$;

$\boldsymbol{G}(\mathbf{x}') = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$;

$\boldsymbol{A} = diag\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right]$;

$\boldsymbol{B} = diag\left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right]$;

and $\mathbf{C} = \mathrm{diag}[[1 \quad 0 \quad 0] \quad [1 \quad 0 \quad 0] \quad [1 \quad 0 \quad 0]]$.

This equivalent system can apply the feedback linearization law to linearize the HVAC system. By putting proper control gain, the linearized HVAC system can be regulated to maintain the set points of temperature, humidity ratio, and CO_2 concentration.

4. Control Strategies Using Feedback Linearization Control

The proposed HVAC system is linearized by a feedback linearization method. The linearized HVAC system shown can be controlled by linear controllers such as pole placement and LQR controllers.

Table 1 shows the numerical values of system parameters used in the simulations. The initial state and reference values are given in Table 2.

Table 1. Numerical values for system parameters.

Parameter	Value	Unit
ρ	0.074	$\mathrm{lb/ft^3}$
V_{he}	60.75	$\mathrm{ft^3}$
V_s	58464	$\mathrm{ft^3}$
C_p	0.24	Btu/lb °F
W_s	0.0070	lb/lb
M_o	166.06	lb/hour
Q_o	289,897.52	Btu/hour
T_o	85	°F
W_o	0.018	lb/lb
C_o	400	ppm
τ_1, τ_2, τ_3	0.008	hour
k_1, k_2, k_3	5	-

Table 2. Initial state and reference values.

$T_{initial}$	76 °F	$W_{initial}$	0.021 lb/lb	$C_{initial}$	1300 ppm
$T_{reference}$	71 °F	$W_{reference}$	0.0092 lb/lb	$C_{reference}$	1200 ppm

4.1. Design of Pole Placement Control for Linearized HVAC System

Let the reference signal $\boldsymbol{y_d} = [y_{1d} \quad y_{2d} \quad y_{3d}]^T$, $\boldsymbol{y_d^{(3)}} = \left[y_{1d}^{(3)} \quad y_{2d}^{(3)} \quad y_{3d}^{(3)}\right]^T$, and matrix $\boldsymbol{Y_d} = \left[y_{1d} \quad y_{1d}^{(1)} \quad y_{1d}^{(2)} y_{2d} \quad y_{2d}^{(1)} \quad y_{2d}^{(2)} y_{3d} \quad y_{3d}^{(1)} \quad y_{3d}^{(2)}\right]^T$. Define the tracking errors as $\mathrm{e}_1 = y_1 - y_{1d}$, $\mathrm{e}_2 = y_2 - y_{2d}$ and $\mathrm{e}_3 = y_3 - y_{3d}$, and let the error matrix $\boldsymbol{e} = \mathbf{x}' - \boldsymbol{Y_d} = \left[e_1\, e_1^{(1)} e_1^{(2)}\, e_2\, e_2^{(1)}\, e_2^{(2)}\, e_3\, e_3^{(1)}\, e_3^{(2)}\right]^T$.

The feedback linearization control law for the proposed HVAC system (9) is designed as:

$$\boldsymbol{\zeta} = \boldsymbol{G}^{-1}(-\boldsymbol{F} + \boldsymbol{y_d^{(3)}} - \boldsymbol{Ke}) \tag{10}$$

choosing $\boldsymbol{K} = \mathrm{diag}([k_{11} \quad k_{12} \quad k_{13}], [k_{21} \quad k_{22} \quad k_{23}], [k_{31} \quad k_{32} \quad k_{33}])$ so that the polynomial $s^3 + k_{i1}s^2 + k_{i2}s^1 + k_{i3} = 0, i = 1, 2, 3$ has all its roots strictly in the left-half complex plane, thereby meeting the desired performance specifications such as those for the transient response of the steady-state error. By substituting Equation (10) into Equation (9), the linearized HVAC system can finally be obtained as follows:

$$\dot{\mathbf{x}}' = \boldsymbol{A}\mathbf{x}' + \boldsymbol{B\eta} \tag{11}$$

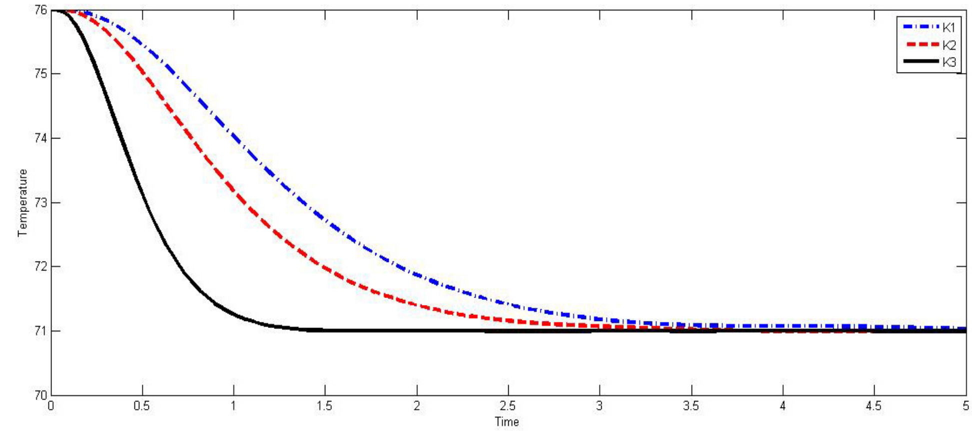
where $\boldsymbol{\eta} = \boldsymbol{y_d^{(3)}} - \boldsymbol{Ke}$.

The gain K_1 corresponds to the case in which the pole is located at −2, −2, and −4, whereas the gain K_2 corresponds to the case in which the pole is located at −2, −3, and −5, and the gain K_3 corresponds to the case in which the pole is located at −5, −6, and −7. According to the pole placement, the control performance is varied. Figure 4 show the system responses in terms of temperature, humidity ratio, and CO_2 concentration, respectively. Table 3 shows the control performance metrics of settling time, rising time, settling max value, and settling min value for each value of gain.

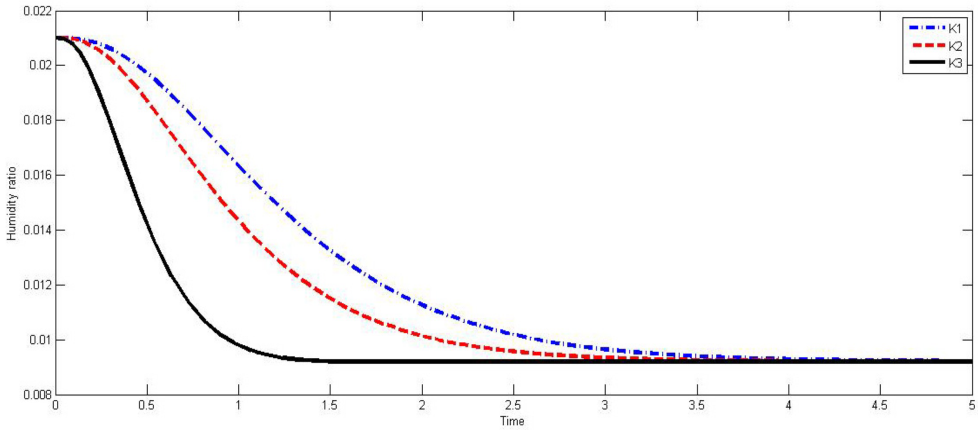
Table 3. Control performance metrics of the pole placement controller.

	Temperature			Humidity ratio			CO_2 concentration		
	K_1	K_2	K_3	K_1	K_2	K_3	K_1	K_2	K_3
T_s	3.2695	2.8431	1.1618	3.3889	2.7312	1.1595	3.3857	2.7270	1.1581
T_r	1.8919	1.5184	0.6804	1.8971	1.5124	0.6795	1.8970	1.5119	0.6794
Settling min	70.9962	71.0172	70.9927	0.0092	0.0092	0.0092	1200.1	1200	1199.9
Settling max	71.4929	71.4741	71.4297	0.0104	0.0104	0.0102	1209.9	1210	1208.5

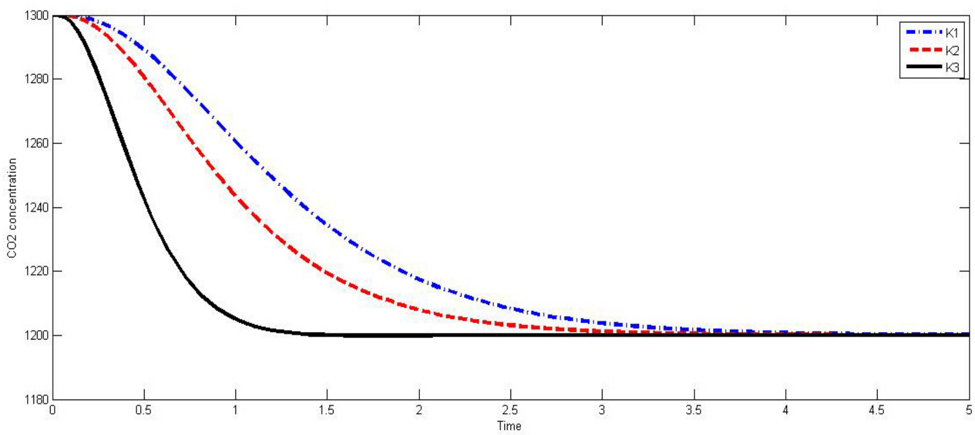
Figure 4. (**a**) Temperature response for each pole placement; (**b**) Humidity ratio response for each pole placement; (**c**) CO_2 concentration response for each pole placement.



(**a**)



(**b**)



(**c**)

4.2. Design of Linear Quadratic Regulator for Linearized HVAC System

The linearized HVAC system of (11) can be controlled by a linear controller. A linear-quadratic regulator (LQR) aims at designing stable controller which can minimize the cost function J represents the performance characteristic requirement as well as the controller input limitation [17]. The cost function is:

$$\boldsymbol{J} = \frac{\mathbf{1}}{\mathbf{2}} \int_{\mathbf{0}}^{\infty} \left[\mathbf{x}'^{\boldsymbol{T}} \boldsymbol{Q} \mathbf{x}' + \boldsymbol{\eta}^{\boldsymbol{T}} \boldsymbol{R} \boldsymbol{\eta} \right] \boldsymbol{dt} \tag{12}$$

where $\boldsymbol{Q}$ is a positive semi-definite weight matrix and $\boldsymbol{R}$ is a positive definite weight matrix. The weighting matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ are chosen by the Bryson's rule (refer to Appendix A5) [18].

The feedback control law that minimizes the values of cost is:

$$\boldsymbol{\eta} = -\boldsymbol{k}'\boldsymbol{e} + \boldsymbol{y}_{\boldsymbol{d}}^{(\mathbf{3})} \tag{13}$$

where $\boldsymbol{e} = \mathbf{x}' - \boldsymbol{Y_d} = \left[e_1\, e_1^{(1)} e_1^{(2)}\, e_2\, e_2^{(1)}\, e_2^{(2)}\, e_3\, e_3^{(1)}\, e_3^{(2)} \right]^T$; $\boldsymbol{K}'$ is given by $\boldsymbol{K}' = \boldsymbol{R}^{-\mathbf{1}}\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{P}$; and P is found by solving the continuous time algebraic Riccati equation:

$$\boldsymbol{A^T P} + \boldsymbol{PA} - \boldsymbol{PBR^{-1}B^T P} + \boldsymbol{Q} = \mathbf{0} \tag{14}$$

Table 4 shows the gain values resulting from the LQR process solving algebraic Riccati equation.

Table 4. LQR gain values.

	9 × 9 Gain matrix
$\boldsymbol{K}_1'$	diag([100 44.7850 9.9785], [100 44.7850 9.9785], [100 44.7850 9.9785])
$\boldsymbol{K}_2'$	diag([100 54.7214 14.4721], [100 54.7214 14.4721], [100 54.7214 14.4721])
$\boldsymbol{K}_3'$	diag([100 88.6706 34.3124], [100 88.6706 34.3124], [100 88.6706 34.3124])

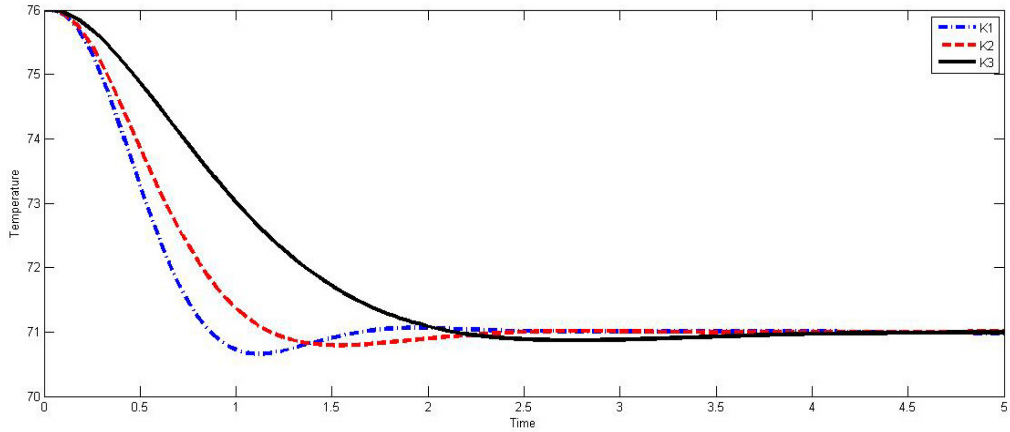
The LQR controller was applied to the proposed HVAC system model (11) and the simulation results are shown in Figure 5. The temperature response of the proposed HVAC system is shown in Figure 5(a), and its humidity ratio response and CO_2 concentration response are shown respectively in Figure 5(b,c).

From these simulation results, we can see that the proposed HVAC system is effective, and that linear controllers are suitable for application to the proposed HVAC system model. Table 5 shows the control performance metrics of settling time, rising time, settling max value, and settling min value for each value of gain.

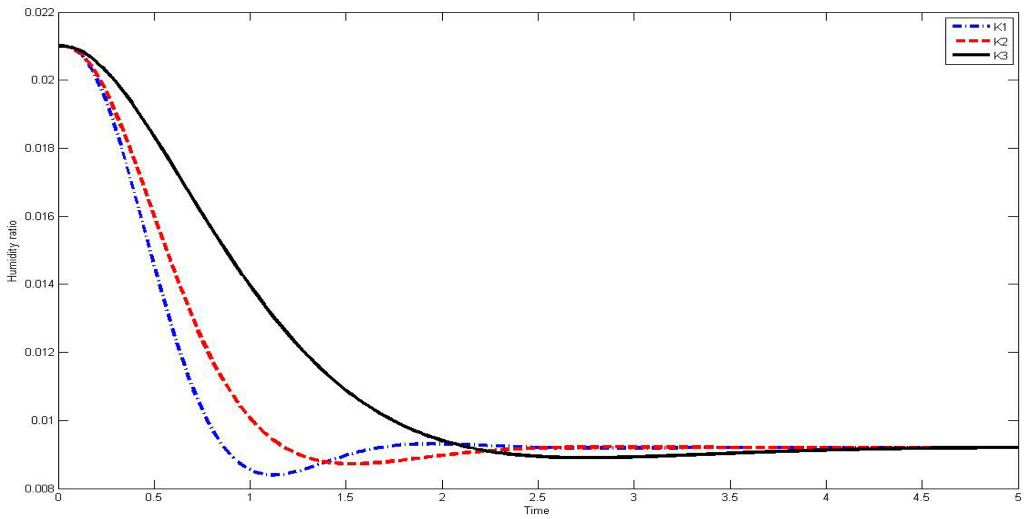
Table 5. Control performance metrics of the LQR controller.

	Temperature			**Humidity ratio**			**CO_2 concentration**		
	$\boldsymbol{K_1}$	$\boldsymbol{K_2}$	$\boldsymbol{K_3}$	$\boldsymbol{K_1}$	$\boldsymbol{K_2}$	$\boldsymbol{K_3}$	$\boldsymbol{K_1}$	$\boldsymbol{K_2}$	$\boldsymbol{K_3}$
$\boldsymbol{T_s}$	1.4886	2.0107	3.2315	1.4982	1.9880	3.2028	1.5049	1.9873	3.1964
$\boldsymbol{T_r}$	0.5279	0.7187	1.3093	0.5277	0.7189	1.3098	0.5277	0.7189	1.3098
Settling min	70.6601	70.7922	70.8702	0.0084	0.0087	0.0089	1193.2	1196	1197.5
Settling max	71.4249	71.3586	71.4777	0.0102	0.0100	0.0103	1208.5	1207.2	1209.6

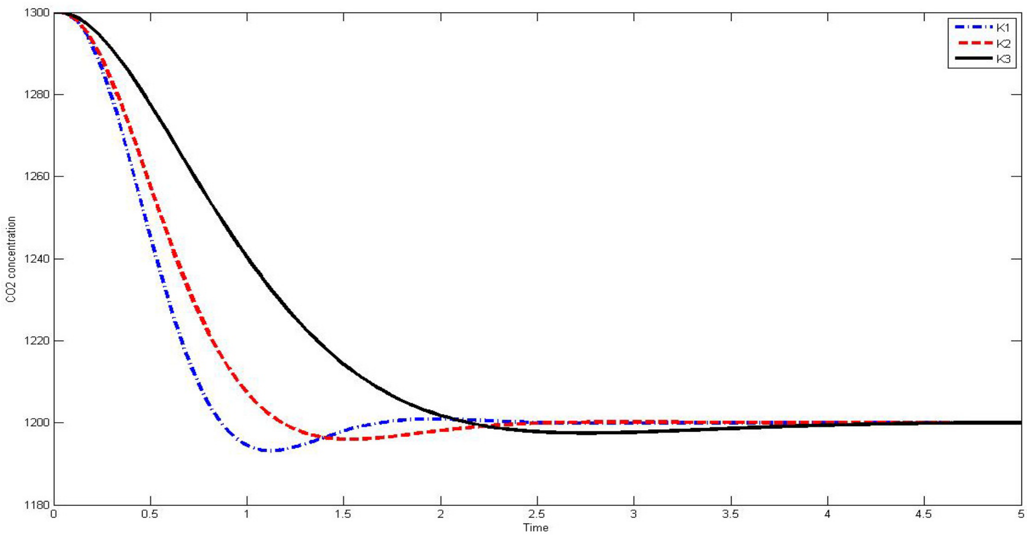
Figure 5. (**a**) Temperature response by LQR; (**b**) Humidity ratio response by LQR; (**c**) CO_2 concentration response by LQR.



(**a**)



(**b**)



(**c**)

5. Conclusions

Herein we have presented a novel HVAC system model that considers not only temperature and humidity ratio, but also CO_2 concentration as the quantitative indices of comfort in a room. In applying an input-output feedback linearization method to linearize the HVAC system, problems of singularity, no relative degree, and interacting controls were encountered and a dynamic extension algorithm was used to solve these problems. The key contribution of this report is the addition of a continuous CO_2 concentration state and corresponding valve dynamics to a conventional HVAC system to allow continuous control of CO_2 concentration. Two types of linear controllers, pole placement and LQR controllers, were able to regulate the linearized HVAC system at the desired set point. Simulation results validated the proposed HVAC model, demonstrating its effectiveness in maintaining comfortable conditions. In future work, we will conduct further study on developing disturbance observer based controllers or intelligent controllers using fuzzy logic or artificial neural networks for a HVAC system considering parameter uncertainty and disturbance effect.

Author Contributions

A co-author, Joil-Il Park, cooperates on dynamic extension algorithm for achieving relative degree and applying linearization method to the HVAC system in Section 3.2.1. A co-author, Mignon Park, cooperates on deriving modeling of HVAC system including CO_2 concentration in Section 3.1. And Jaeho Baek cooperates on the equivalent linearization HVAC system in Section 3.2.2.

Nomeclature

h_w	Enthalpy of liquid water
W_o	Humidity ratio of outdoor air
h_{fg}	Enthalpy of water vapor
V_{he}	Volume of heat exchanger
W_s	Humidity ratio of supply air
W_3	Humidity ratio of thermal space
C_p	Specific heat of air
T_o	Temperature of outdoor air
M_o	Moisture load
Q_o	Sensible heat load
T_2	Temperature of supply air
T_3	Temperature of thermal space
V_s	Volume of thermal space
ρ	Air mass density
F	Volumetric air flow rate ($\mathrm{ft}^3/\mathrm{min}$)
gpm	Chilled water flow rate ($\mathrm{gal/min}$)
μ	Air recirculation rate

Appendix A1

Consider the MIMO system of the form:

$$\dot{\mathbf{x}} = \boldsymbol{f}(\mathbf{x}) + \boldsymbol{G}(\mathbf{x})\boldsymbol{u} \tag{A1}$$

where $\mathbf{X}$ is the $\mathrm{n} \times 1$ state vector; $\boldsymbol{u}$ is the $\mathrm{m} \times 1$ control input vector (of components u_i); $\mathbf{y}$ is the $\mathrm{m} \times 1$ vector of system outputs (of components y_i); $\mathbf{f}$ and $\mathbf{h}$ are smooth vector fields; and $\mathbf{G}$ is a $\mathrm{n} \times \mathrm{m}$ matrix whose columns are smooth vector fields $\boldsymbol{g_i}$.

Assume that r_i is the smallest integer for which at least one of the inputs appears in $y_i^{(r_i)}$; then:

$$y_i^{(r_i)} = L_{\boldsymbol{f}}^{r_i} h_i + \sum_{j=1}^{m} L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^{r_i - 1} h_i u_j \tag{A2}$$

with $L_{\boldsymbol{g}_j} L_{\boldsymbol{f}}^{r_i-1} h_i(\mathbf{x}) \neq 0$ for at least one j. Applying Equation (A2) for each output y_i yields:

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_m^{(r_m)} \end{bmatrix} = \begin{bmatrix} L_{\boldsymbol{f}}^{r_1} h_1(\mathbf{x}) \\ \vdots \\ L_{\boldsymbol{f}}^{r_m} h_m(\mathbf{x}) \end{bmatrix} + \boldsymbol{D}(\mathbf{x})\boldsymbol{u}$$

Therefore, the decoupling matrix $\boldsymbol{D}(\mathbf{x})$ is defined as:

$$\boldsymbol{D}(\mathbf{x}) = \begin{bmatrix} L_{g_1} L_f^{r_i-1} h_1 & \cdots & L_{g_m} L_f^{r_i-1} h_1 \\ L_{g_1} L_f^{r_i-1} h_2 & \cdots & L_{g_m} L_f^{r_i-1} h_2 \\ \vdots & \cdots & \vdots \\ L_{g_1} L_f^{r_i-1} h_m & \cdots & L_{g_m} L_f^{r_i-1} h_m \end{bmatrix}$$

where $1 \leq \mathrm{i}, \mathrm{j} \leq \mathrm{m}$.

For example, in our proposed HVAC system case, $\mathrm{m} = 3$ and $\mathrm{r} = 3$:

$$\boldsymbol{D}(\mathbf{x}) = \begin{bmatrix} L_{g_1} L_f^2 h_1 & L_{g_2} L_f^2 h_1 & L_{g_3} L_f^2 h_1 \\ L_{g_1} L_f^2 h_2 & L_{g_2} L_f^2 h_2 & L_{g_3} L_f^2 h_2 \\ L_{g_1} L_f^2 h_3 & L_{g_2} L_f^2 h_3 & L_{g_3} L_f^2 h_3 \end{bmatrix} = \begin{bmatrix} L_{g_1} L_f^2 \mathrm{x}_1 & L_{g_2} L_f^2 \mathrm{x}_1 & L_{g_3} L_f^2 \mathrm{x}_1 \\ L_{g_1} L_f^2 \mathrm{x}_2 & L_{g_2} L_f^2 \mathrm{x}_2 & L_{g_3} L_f^2 \mathrm{x}_2 \\ L_{g_1} L_f^2 \mathrm{x}_4 & L_{g_2} L_f^2 \mathrm{x}_4 & L_{g_3} L_f^2 \mathrm{x}_4 \end{bmatrix}.$$

Appendix A2

From the system given in (6):

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}) \quad h_2(\mathbf{x}) \quad h_3(\mathbf{x})]^{\boldsymbol{T}} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^{\boldsymbol{T}}$$

and the output y_1:

$$\begin{cases} y_1 = h_1(\mathbf{x}) = \mathrm{x}_1 \\ \dot{y}_1 = \dot{\mathrm{x}}_1 = \gamma_1 u_1 + \alpha_3 \big(Q_o - h_{fg} M_o\big) \\ \ddot{y}_1 = \ddot{\mathrm{x}}_1 = \dot{\gamma}_1 u_1 + \gamma_1 \dot{u}_1 = \dot{\gamma}_1 u_1 + \gamma_1 a_5(\mathbf{x}) + \dfrac{\gamma_1 k_1}{\tau_1} v_1 \ (\because \dot{u}_1 = a_5(\mathbf{x}) + \dfrac{k_1}{\tau_1} v_1) \end{cases}$$

Therefore, the relative degree $r_1 = 2$ with respect to output y_1. According to Equation (A2):

$$y_1^{(2)} = L_{\boldsymbol{f}}^2 h_1 + L_{\boldsymbol{g}_1} L_{\boldsymbol{f}}^1 h_1 v_1 + L_{\boldsymbol{g}_2} L_{\boldsymbol{f}}^1 h_1 v_2 + L_{\boldsymbol{g}_3} L_{\boldsymbol{f}}^1 h_1 v_3$$

- $L_{g_1}L_f^1h_1$: $\frac{\gamma_1 k_1}{\tau_1}$;
- $L_{g_2}L_f^1h_1$: 0;
- $L_{g_3}L_f^1h_1$: 0.

The above procedure is similar to that applied for output y_2:

$$\begin{cases} y_2 = h_2(\mathbf{x}) = \mathrm{x}_2 \\ \dot{y}_2 = \dot{\mathrm{x}}_2 = \gamma_2 u_1 + \alpha_4 M_o \\ \ddot{y}_2 = \ddot{\mathrm{x}}_2 = \dot{\gamma}_2 u_1 + \gamma_2 \dot{u}_1 = \dot{\gamma}_2 u_1 + \gamma_2 a_5(\mathbf{x}) + \dfrac{\gamma_2 k_1}{\tau_1} v_1 \ (\because \dot{u}_1 = a_5(\mathbf{x}) + \dfrac{k_1}{\tau_1} v_1) \end{cases}$$

Thus, the relative degree $r_2 = 2$ with respect to output y_2. By Equation (A2):

$$y_2^{(2)} = L_f^2 h_2 + L_{g_1}L_f^1 h_2 v_1 + L_{g_2}L_f^1 h_2 v_2 + L_{g_3}L_f^1 h_2 v_3$$

- $L_{g_1}L_f^1h_2$: $\frac{\gamma_2 k_1}{\tau_1}$;
- $L_{g_2}L_f^1h_2$: 0;
- $L_{g_3}L_f^1h_2$: 0.

In the case of output y_3:

$$\begin{cases} y_3 = h_3(\mathbf{x}) = \mathrm{x}_4 \\ \dot{y}_3 = \dot{\mathrm{x}}_4 = \gamma_5 u_3 + \dfrac{C_g}{V_s} \\ \ddot{y}_3 = \ddot{\mathrm{x}}_4 = \dot{\gamma}_5 u_3 + \gamma_5 \dot{u}_3 = \dot{\gamma}_5 u_3 + \gamma_5 a_7(\mathbf{x}) + \dfrac{\gamma_5 k_3}{\tau_3} v_3 \ (\because \dot{u}_3 = a_7(\mathbf{x}) + \dfrac{k_3}{\tau_3} v_3). \end{cases}$$

Therefore, the relative degree $r_3 = 2$ with respect to output y_3. By Equation (A2):

$$y_3^{(2)} = L_f^2 h_3 + L_{g_1}L_f^1 h_3 v_1 + L_{g_2}L_f^1 h_3 v_2 + L_{g_3}L_f^1 h_3 v_3$$

- $L_{g_1}L_f^1h_3$: 0;
- $L_{g_2}L_f^1h_3$: 0;
- $L_{g_3}L_f^1h_3$: $\frac{\gamma_5 k_3}{\tau_3}$.

From the above results, the vector relative degree is $\{2 \quad 2 \quad 2\}$, the total relative degree is 6, and the decoupling matrix is:

$$D(\mathbf{x}) = \begin{bmatrix} L_{g_1}L_f^1\mathrm{h}_1 & L_{g_2}L_f^1\mathrm{h}_1 & L_{g_3}L_f^1\mathrm{h}_1 \\ L_{g_1}L_f^1\mathrm{h}_2 & L_{g_2}L_f^1\mathrm{h}_2 & L_{g_3}L_f^1\mathrm{h}_2 \\ L_{g_1}L_f^1\mathrm{h}_3 & L_{g_2}L_f^1\mathrm{h}_3 & L_{g_3}L_f^1\mathrm{h}_3 \end{bmatrix} = \begin{bmatrix} \frac{\gamma_1 k_1}{\tau_1} & 0 & 0 \\ \frac{\gamma_2 k_1}{\tau_1} & 0 & 0 \\ 0 & 0 & \frac{\gamma_5 k_3}{\tau_3} \end{bmatrix}.$$

Appendix A3

From the system given in (7):

$$\mathbf{y} = \mathbf{h}(\bar{\mathbf{x}}) = [h_1(\bar{\mathbf{x}}) \quad h_2(\bar{\mathbf{x}}) \quad h_3(\bar{\mathbf{x}})]^{\boldsymbol{T}} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^{\boldsymbol{T}}$$

and the output y_1:

$$\begin{cases} y_1 = h_1(\bar{\mathbf{x}}) = \mathrm{x}_1 \\ \dot{y}_1 = \dot{\mathrm{x}}_1 = \gamma_1 u_1 + \alpha_3\left(Q_o - h_{fg}M_o\right) \\ \ddot{y}_1 = \ddot{\mathrm{x}}_1 = \dot{\gamma}_1 u_1 + \gamma_1 \dot{u}_1 \\ \dddot{y}_1 = \dddot{\mathrm{x}}_1 = \left\{\alpha_1(\dot{\gamma}_3 u_1 + \gamma_3 \dot{u}_1) + \alpha_1 \dot{\gamma}_4 u_2 - \dfrac{\alpha_1\gamma_4 u_2}{\tau_2} + \dfrac{\alpha_1\gamma_4 k_2}{\tau_2}\zeta_2 - \alpha_1 \ddot{\mathrm{x}}_1 + \alpha_2 \ddot{\mathrm{x}}_2\right\} u_1 + \dot{\gamma}_1 \dot{u}_1 + \dot{\gamma}_1 u_1 \\ -\dfrac{\gamma_1 \dot{u}_1}{\tau_1} + \dfrac{\gamma_1 k_1}{\tau_1}\zeta_1 \\ = \left\{\alpha_1(\dot{\gamma}_3 u_1 + \gamma_3 \dot{u}_1) + \alpha_1 \dot{\gamma}_4 u_2 - \dfrac{\alpha_1\gamma_4 u_2}{\tau_2} - \alpha_1 \ddot{\mathrm{x}}_1 + \alpha_2 \ddot{\mathrm{x}}_2\right\} u_1 + \dot{\gamma}_1 \dot{u}_1 + \dot{\gamma}_1 u_1 + \dfrac{\gamma_1 k_1}{\tau_1}\zeta_1 \\ + \dfrac{\alpha_1 \gamma_4 k_2 u_1}{\tau_2}\zeta_2. \end{cases}$$

Therefore, the relative degree $\bar{r}_1 = 3$ with respect to output y_1. According to Equation (A2):

$$y_1^{(3)} = L_{\bar{f}}^3 h_1 + L_{\bar{g}_1} L_{\bar{f}}^2 h_1 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_1 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_1 \zeta_3$$

- $L_{\bar{g}_1} L_{\bar{f}}^2 h_1 : \frac{\gamma_1 k_1}{\tau_1}$
- $L_{\bar{g}_2} L_{\bar{f}}^2 h_1 : \frac{\alpha_1 \gamma_4 k_2 u_1}{\tau_2}$
- $L_{\bar{g}_3} L_{\bar{f}}^2 h_1 : 0$.

The above procedure is similar to that applied for output y_2:

$$\begin{cases} y_2 = h_2(\bar{\mathbf{x}}) = \mathrm{x}_2 \\ \dot{y}_2 = \dot{\mathrm{x}}_2 = \gamma_2 u_1 + \alpha_4 M_o \\ \ddot{y}_2 = \ddot{\mathrm{x}}_2 = \dot{\gamma}_2 u_1 + \gamma_2 \dot{u}_1 \\ \dddot{y}_2 = \dddot{\mathrm{x}}_2 = \ddot{\gamma}_2 u_1 + \dot{\gamma}_2 \dot{u}_1 + \dot{\gamma}_2 \dot{u}_1 + \gamma_2 \ddot{u}_1 = \ddot{\gamma}_2 u_1 + 2\dot{\gamma}_2 \dot{u}_1 - \dfrac{\gamma_2 \dot{u}_1}{\tau_1} + \dfrac{\gamma_2 k_1}{\tau_1}\zeta_1. \end{cases}$$

Thus, the relative degree $\bar{r}_2 = 3$ with respect to output y_2. By Equation (A2):

$$y_2^{(3)} = L_{\bar{f}}^3 h_2 + L_{\bar{g}_1} L_{\bar{f}}^2 h_2 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_2 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_2 \zeta_3$$

- $L_{\bar{g}_1} L_{\bar{f}}^2 h_2 : \frac{\gamma_2 k_1}{\tau_1}$
- $L_{\bar{g}_2} L_{\bar{f}}^2 h_2 : 0$
- $L_{\bar{g}_3} L_{\bar{f}}^2 h_2 : 0$.

In the case of output y_3:

$$\begin{cases} y_3 = h_3(\bar{\mathbf{x}}) = \mathrm{x}_4 \\ \dot{y}_3 = \dot{\mathrm{x}}_4 = \gamma_5 u_3 + \dfrac{C_g}{V_s} \\ \ddot{y}_3 = \ddot{\mathrm{x}}_4 = \dot{\gamma}_5 u_3 + \gamma_5 \dot{u}_3 \\ \dddot{y}_3 = \dddot{\mathrm{x}}_3 = \ddot{\gamma}_5 u_3 + \dot{\gamma}_5 \dot{u}_3 + \dot{\gamma}_5 \dot{u}_3 + \gamma_5 \ddot{u}_3 = \ddot{\gamma}_5 u_3 + 2\dot{\gamma}_5 \dot{u}_3 - \dfrac{\gamma_5 \dot{u}_3}{\tau_3} + \dfrac{\gamma_5 k_3}{\tau_3}\zeta_3. \end{cases}$$

Therefore, the relative degree $r_3 = 3$ with respect to output y_3. By Equation (A.2):

$$y_3^{(3)} = L_{\bar{f}}^3 h_3 + L_{\bar{g}_1} L_{\bar{f}}^2 h_3 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_3 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_3 \zeta_3$$

- $L_{\bar{g}_1} L_{\bar{f}}^2 h_3 : 0$;

- $L_{\bar{g}_2} L_{\bar{f}}^2 h_3 : 0$;
- $L_{\bar{g}_3} L_{\bar{f}}^2 h_3 : \frac{\gamma_5 k_3}{\tau_3}$.

From the above results, the vector relative degree is $\{3 \quad 3 \quad 3\}$, the total relative degree is 9, and the decoupling matrix is:

$$\overline{\boldsymbol{D}}(\bar{\mathbf{x}}) = \begin{bmatrix} L_{\bar{g}_1} L_{\bar{f}}^2 \mathrm{x}_1 & L_{\bar{g}_2} L_{\bar{f}}^2 \mathrm{x}_1 & L_{\bar{g}_3} L_{\bar{f}}^2 \mathrm{x}_1 \\ L_{\bar{g}_1} L_{\bar{f}}^2 \mathrm{x}_2 & L_{\bar{g}_2} L_{\bar{f}}^2 \mathrm{x}_2 & L_{\bar{g}_3} L_{\bar{f}}^2 \mathrm{x}_2 \\ L_{\bar{g}_1} L_{\bar{f}}^2 \mathrm{x}_4 & L_{\bar{g}_2} L_{\bar{f}}^2 \mathrm{x}_4 & L_{\bar{g}_3} L_{\bar{f}}^2 \mathrm{x}_4 \end{bmatrix} = \begin{bmatrix} \frac{\gamma_1 k_1}{\tau_1} & \frac{\alpha_1 \gamma_4 k_2 u_1}{\tau_2} & 0 \\ \frac{\gamma_2 k_1}{\tau_1} & 0 & 0 \\ 0 & 0 & \frac{\gamma_5 k_3}{\tau_3} \end{bmatrix}.$$

Appendix A4

From Equation (A2) and our proposed 3-input (ζ_i) and 3-output (y_i) MIMO HVAC system given in (7):

$$\mathbf{y} = \mathbf{h}(\mathbf{x}) = [h_1(\mathbf{x}) \quad h_2(\mathbf{x}) \quad h_3(\mathbf{x})]^{\boldsymbol{T}} = [\mathrm{x}_1 \quad \mathrm{x}_2 \quad \mathrm{x}_4]^{\boldsymbol{T}}$$

$$\begin{cases} y_1^{(3)} = L_{\bar{f}}^3 h_1 + \sum_{j=1}^{3} L_{\bar{g}_j} L_{\bar{f}}^{r_i - 1} h_i \zeta_j = L_{\bar{f}}^3 h_1 + L_{\bar{g}_1} L_{\bar{f}}^2 h_1 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_1 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_1 \zeta_3 \\ = f_1 + g_{11}\zeta_1 + g_{12}\zeta_2 + g_{13}\zeta_3 \\ y_2^{(3)} = L_{\bar{f}}^3 h_2 + \sum_{j=1}^{3} L_{\bar{g}_j} L_{\bar{f}}^{r_i - 1} h_i \zeta_j = L_{\bar{f}}^3 h_2 + L_{\bar{g}_1} L_{\bar{f}}^2 h_2 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_2 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_2 \zeta_3 \\ = f_2 + g_{21}\zeta_1 + g_{22}\zeta_2 + g_{23}\zeta_3 \\ y_3^{(3)} = L_{\bar{f}}^3 h_3 + \sum_{j=1}^{3} L_{\bar{g}_j} L_{\bar{f}}^{r_i - 1} h_i \zeta_j = L_{\bar{f}}^3 h_3 + L_{\bar{g}_1} L_{\bar{f}}^2 h_3 \zeta_1 + L_{\bar{g}_2} L_{\bar{f}}^2 h_3 \zeta_2 + L_{\bar{g}_3} L_{\bar{f}}^2 h_3 \zeta_3 \\ = f_3 + g_{31}\zeta_1 + g_{32}\zeta_2 + g_{33}\zeta_3. \end{cases} \tag{A3}$$

Appendix A5

Bryson's rule specifies the weighting of the Q and R matrices where they are selected to be diagonal matrices:

$$Q_{ii} = \frac{1}{\text{maximum acceptance values of } {x'}_i^{2}}, i \in \{1,2, \dots, l\}$$

$$R_{jj} = \frac{1}{\text{maximum acceptance values of } \eta_j^2}, j \in \{1,2, \dots, m\}$$

This corresponds to the following criteria:

$$\boldsymbol{J} = \frac{\mathbf{1}}{\mathbf{2}} \int_{\mathbf{0}}^{\infty} \left[\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{l}} \boldsymbol{Q}_{\boldsymbol{ii}} \boldsymbol{x'}^{\mathbf{2}} + \rho \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{m}} \boldsymbol{R}_{\boldsymbol{jj}} \boldsymbol{\eta}^{\mathbf{2}} \right] \boldsymbol{dt} \tag{A4}$$

where $\boldsymbol{x}'$ is state vector; $\boldsymbol{\eta}$ is control input, and $\rho = \left(\frac{\text{max state error}}{\text{max control input}}\right)^2$.

Conflicts of Interest

The authors declare no conflict of interest.

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