

11.16.3.5.1

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Question:

A fair coin with 1 marked on one face and 6 on the other and a fair die are both tossed. Find the probability that the sum of numbers that turn up is 3.

Solution:

Textual solution:

The probability of a given event A (where A : Sum of the numbers is 3) is computed by considering the mutual independence of the coin and the die. Since the coin and die are independent events, we calculate the probability of each event separately and then multiply them.

The probability of the coin showing 1 is:

$$P(\text{Coin} = 1) = \frac{1}{2} \quad (0.1)$$

The probability of the die showing 2 is:

$$P(\text{Die} = 2) = \frac{1}{6} \quad (0.2)$$

Since the events are independent, the probability of both events happening together is:

$$P(A) = P(\text{Coin} = 1) \times P(\text{Die} = 2) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}. \quad (0.3)$$

Computational solution:

COMPUTATION OF PROBABILITIES FOR TOSSING A COIN AND ROLLING A DIE

To compute the probability of obtaining specific outcomes when tossing a fair coin and rolling a six-sided die, we rely on two key concepts: the **Probability Mass Function (PMF)** and the **Cumulative Distribution Function (CDF)**.

Definitions

Probability Mass Function (PMF): The PMF represents the probability of each individual outcome in the sample space S . For the coin:

$$C = \{1, 6\} \quad (0.4)$$

and for the die:

$$D = \{1, 2, 3, 4, 5, 6\}. \quad (0.5)$$

The combined sample space is:

$$S = \{(c, d) \mid c \in C, d \in D\}, \quad (0.6)$$

with the total number of outcomes:

$$|S| = |C| \times |D| = 2 \times 6 = 12. \quad (0.7)$$

The PMF is given as:

$$P((c, d)) = \frac{1}{12}, \quad \forall (c, d) \in S. \quad (0.8)$$

Cumulative Distribution Function (CDF): The CDF represents the cumulative probability of outcomes up to a given sum x , defined as:

$$F(x) = P(\text{Sum} \leq x) = \sum_{\text{Sum} \leq x} P((c, d)).$$

For this experiment:

$$F(x) = \begin{cases} 0, & x < 2, \\ \frac{\text{Number of outcomes where Sum} \leq x}{12}, & x \in [2, 12] \\ 1, & x > 12. \end{cases}$$

Simulation Process

We simulate the tossing of the coin and rolling of the die using the following steps:

- 1) A coin produces outcomes in the set:

$$C = \{1, 6\}.$$

- 2) A die produces outcomes in the set:

$$D = \{1, 2, 3, 4, 5, 6\}.$$

- 3) For each simulated trial, a random integer C is generated for the coin, such that $C \in \{1, 6\}$, using a random number generator function:

$$C = (\text{rand}() \bmod 2) + 1.$$

- 4) A random integer D is generated for the die, such that $D \in \{1, 2, 3, 4, 5, 6\}$, using another random number generator function:

$$D = (\text{rand}() \bmod 6) + 1.$$

- 5) The sum of the coin and die outcomes is calculated for each trial:

$$\text{Sum} = C + D.$$

- 6) The number of occurrences where the sum equals 12 is tracked over N trials, where N is the total number of simulations.
- 7) The probability of obtaining a sum of 12 is then computed by dividing the number of successful trials (where the sum equals 12) by the total number of trials N

Calculation of Probabilities

Probability of Each Outcome (PMF): The probability of obtaining each sum i ($i \in [2, 3, \dots, 12]$) is computed as:

$$P(i) = \frac{\text{Number of occurrences of Sum} = i}{N}. \quad (7.1)$$

Cumulative Probability (CDF): The cumulative probability up to sum i is:

$$F(i) = \sum_{k=2}^i P(k). \quad (7.2)$$

Probability of Sum = 3: The probability of obtaining a sum of 3 is determined by counting the favorable outcomes where the coin shows 1 and the die shows 2. Given that the coin has two possible outcomes $\{1, 6\}$ and the die has six possible outcomes $\{1, 2, 3, 4, 5, 6\}$, the only favorable combination is when the coin shows 1 and the die shows 2.

Therefore, the probability is:

$$P(\text{Sum} = 3) = \frac{1}{12}. \quad (7.3)$$

Output Representation

The computed probabilities are represented in two forms:

- ****PMF****: The probabilities of obtaining each sum $\{2, 3, \dots, 12\}$.
- ****CDF****: The cumulative probabilities up to each sum, showing the cumulative likelihood of outcomes.

Z-transform approach

Z-transform for the coin (X):

$$G_X(z) = \sum_{n=0}^{\infty} P_n(X)z^{-n} \quad (7.4)$$

Substituting the PMF:

$$G_X(z) = \frac{1}{2}z^{-1} + \frac{1}{2}z^{-6} \quad (7.5)$$

Z-transform for the die (Y):

$$G_Y(z) = \sum_{n=0}^{\infty} P_n(Y)z^{-n} \quad (7.6)$$

Substituting the PMF:

$$G_Y(z) = \frac{1}{6}z^{-1} + \frac{1}{6}z^{-2} + \frac{1}{6}z^{-3} + \frac{1}{6}z^{-4} + \frac{1}{6}z^{-5} + \frac{1}{6}z^{-6} \quad (7.7)$$

The Z-transform of the sum $S=X+Y$ is:

$$G_S(z) = G_X(z) \cdot G_Y(z) \quad (7.8)$$

$$G_S(z) = \left(\frac{1}{2}z^{-1} + \frac{1}{2}z^{-6} \right) \cdot \left(\frac{1}{6}z^{-1} + \frac{1}{6}z^{-2} + \frac{1}{6}z^{-3} + \frac{1}{6}z^{-4} + \frac{1}{6}z^{-5} + \frac{1}{6}z^{-6} \right) \quad (7.9)$$

The probability $p_3(S)$ is the coefficient of z^{-3} in $G_S(z)$

$$G_S(z) = \frac{1}{2}z^{-1} \cdot \frac{1}{6}(z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}) + \frac{1}{2}z^{-6} \cdot \frac{1}{6}(z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6}) \quad (7.10)$$

$$G_S(z) = \frac{1}{12}(z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7}) + \frac{1}{12}(z^{-7} + z^{-8} + z^{-9} + z^{-10} + z^{-11} + z^{-12}) \quad (7.11)$$

$$G_s(z) = \frac{1}{12}(z^{-2} + z^{-3} + z^{-4} + z^{-5} + z^{-6} + z^{-7} + z^{-7} + z^{-8} + z^{-9} + z^{-10} + z^{-11} + z^{-12}) \quad (7.12)$$

The coefficient of z^{-3} in the expansion is $\frac{1}{12}$

Therefore the probability is $\frac{1}{12}$

The Z-transform converges for $|Z| > 0$ since the PMFs are finite sequences.

