Question-8.2.1

EE24BTECH11038 - MALAKALA BALA SUBRAHMANYA ARAVIND

Question: Find the area of the circle $4x^2 + 4y^2 = 9$, which is interior to the parabola $x^2 = 4y$

Solution:

First we need to calculate the point of intersections of the two given curves
Let $\mathbf{V_1}, \mathbf{u_1}, f_1$ be the parameters of the parabola, and let $\mathbf{v_2}, \mathbf{u_2}, f_2$ be the parameters of the circle.
for parabola

$$\mathbf{v_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{0.1}$$

$$\mathbf{u_1} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \tag{0.2}$$

$$f = 0 \tag{0.3}$$

for circle

$$\mathbf{v_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.4}$$

$$\mathbf{u_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{0.5}$$

$$f_2 = -\frac{9}{4} \tag{0.6}$$

The intersection of two conics with parameters $\mathbf{V}_i, \mathbf{u}_i, \mathbf{f}_i,\ i = 1, 2$ is defined as

$$x^T \left(V_1 + \mu V_2\right) x + 2 \left(u_1 + \mu u_2\right)^T x + \left(f_1 + \mu f_2\right) = 0 \tag{0.7}$$

$$x^T \begin{pmatrix} 1+\mu & 0 \\ 0 & \mu \end{pmatrix} x + 2 \begin{pmatrix} 0 \\ -2 \end{pmatrix}^T x - \frac{9}{4}\mu = 0 \tag{0.8}$$

we can get μ by solving the below equation

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{0.9}$$

$$\begin{vmatrix} 1+\mu & 0 & 0 \\ 0 & \mu & -2 \\ 0 & -2 & -\frac{9}{4}\mu \end{vmatrix} = 0 \tag{0.10}$$

$$(1+\mu)\left(-\frac{9}{4}\mu^2 - 4\right) = 0 \tag{0.11}$$

$$\mu = -1 \tag{0.12}$$

now solving the equation 0.8 by placing the value of μ

$$x^T \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} x + 2 \begin{pmatrix} 0 \\ -2 \end{pmatrix}^T x + \frac{9}{4} = 0 \tag{0.13}$$

$$\left(x^T \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 \\ -4 \end{pmatrix}^T \right) x = -\frac{9}{4} \tag{0.14}$$

$$\tag{0.15}$$

on placing $x^2 = 4y$ and solving we get the point of intersections as

$$\begin{pmatrix} \sqrt{2} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ \frac{1}{2} \end{pmatrix} \tag{0.16}$$

The area bounded by the curves $x^2 + y^2 = \frac{9}{4}$ and $x^2 = 4y$ is

$$\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{\frac{9}{4} - x^2} - \frac{x^2}{4}\, dx = 2\left[\frac{9}{4} \sin^{-1}\left(\frac{x}{\frac{3}{2}} \right) + x\sqrt{\frac{9}{4} - x^2} + \frac{1}{4}\frac{x^3}{3} \right]_0^{\sqrt{2}} \tag{0.17}$$

$$\approx 3.0063 \tag{0.18}$$

Computational Solution:

Let $A(x_n)$ be the area enclosed by the curve $y(x)$ from $x = x_0$ to $x = x_n$, $(x_0, x_1, \ldots x_n)$ be equidistant points with step-size h.

$$A(x_n + h) = A(x_n) + \frac{1}{2} h (y(x_n + h) + y(x_n)) \tag{0.19}$$

We can repeat this till we get required area.
Discretizing the steps, making $A(x_n) = A_n, y(x_n) = y_n$ we get,

$$A_{n+1} = A_n + \frac{1}{2} h (y_{n+1} + y_n) \tag{0.20}$$

We can write y_{n+1} in terms of y_n using first principle of derivative. $y_{n+1} = y_n + hy'_n$

$$A_{n+1} = A_n + \frac{1}{2} h ((y_n + hy'_n) + y_n) \tag{0.21}$$

$$A_{n+1} = A_n + \frac{1}{2} h (2y_n + hy'_n) \tag{0.22}$$

$$A_{n+1} = A_n + hy_n + \frac{1}{2} h^2 y'_n \tag{0.23}$$

$$x_{n+1} = x_n + h \tag{0.24}$$

In the given question ,$y_n = \sqrt{\frac{9}{4} - x_n^2} - \frac{x_n^2}{4}$ and $y'_n = -\frac{-x_n}{\sqrt{\frac{9}{4} - x_n^2}} - \frac{x_n}{2}$

The general difference equation will be given by

$$A_{n+1} = A_n + hy_n + \frac{1}{2}h^2 y_n' \tag{0.25}$$

$$A_{n+1} = A_n + h\left(\sqrt{\frac{9}{4} - x_n^2} - \frac{x_n^2}{4}\right) + \frac{1}{2}h^2\left(-\frac{-x_n}{\sqrt{\frac{9}{4} - x_n^2}} - \frac{x_n}{2}\right) \tag{0.26}$$

$$x_{n+1} = x_n + h \tag{0.27}$$

Area computed by computational method is 3.0054
The area computed by the theoretical method is 3.063