Question-10.3.2.1.1

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Question: 10 students of Class X took part in a Mathematics quiz. If the number of girls is 4 more than the number of boys, find the number of boys and girls who took part in the quiz.
Solution:
Let the number of boys be x and the number of girls be y. From the question, we can frame the following equations.

$$x + y = 10 \tag{0.1}$$

$$y = x + 4 \tag{0.2}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \tag{0.3}$$

$$\tag{0.4}$$

Any non-singular matrix can be represented as a product of a lower triangular matrix L and an upper triangular matrix U

$$A\mathbf{x} = LU\mathbf{x} = \mathbf{b} \tag{0.5}$$

The upper triangular matrix U is found by row reducing A,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \xrightarrow{R_2 -> R_2 + R_1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \tag{0.6}$$

Let

$$L = \begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \tag{0.7}$$

l_{21} is the multiplier used to zero a_{21}, so $l_{21} = -1$.
Now

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \tag{0.8}$$

Now we can get the solution to our problem by the two step process,

$$L\mathbf{y} = \mathbf{b} \tag{0.9}$$

$$U\mathbf{x} = \mathbf{y} \tag{0.10}$$

Using forward substitution to solve the first equation,

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 4 \end{pmatrix} \tag{0.11}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix} \tag{0.12}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix} \tag{0.13}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \tag{0.14}$$

Therefore the number of girls are 7 and boys are 3

Numerical Computation for LU Decomposition

We want to decompose A as the product of a lower triangular matrix L and an upper triangular matrix U

$$A = LU \tag{0.15}$$

L is a lower triangular matrix with ones on the diagonal

$$L = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ L_{21} & 1 & 0 & \cdots & 0 \\ L_{31} & L_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & L_{n3} & \cdots & 1 \end{pmatrix} \tag{0.16}$$

U is an upper triangular matrix

$$\begin{pmatrix} U_{11} & 0 & 0 & \cdots & 0 \\ U_{12} & U_{22} & 0 & \cdots & 0 \\ U_{13} & U_{23} & U_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{1n} & U_{2n} & U_{3n} & \cdots & U_{nn} \end{pmatrix} \tag{0.17}$$

The first row of U is simply the first row of A

$$U_{1j} = A_{1j} \tag{0.18}$$

The first column of L is computed as

$$L_{i1} = \frac{A_{i1}}{U_{11}}, \quad \text{for} \quad i = 2, 3, \ldots, n. \tag{0.19}$$

Subsequent columns of U are computed as

$$U_{kj} = A_{kj} - \sum_{m=1}^{k-1} L_{km} U_{mj} \tag{0.20}$$

for $j = k, k+1, \ldots, n$

Subsequent columns of L are computed as

$$L_{ik} = \frac{A_{ik} - \sum_{m=1}^{k-1} L_{im} U_{mk}}{U_{kk}} \tag{0.21}$$

for $i = k + 1, k + 2, \ldots, n$

This systematic approach ensures that the matrix A is decomposed into L and U