Protograph-based LDPC Convolutional Codes for Correlated Erasure Channels

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***Abstract*—We consider terminated LDPC convolutional codes (LDPC-CC) constructed from protographs and explore the performance of these codes on correlated erasure channels including a single-burst channel (SBC) and Gilbert-Elliott channel (GEC). We consider code performance with a latency-constrained message passing decoder and the belief propagation decoder. We give theoretical bounds on the code efficiency over the SBC and describe a construction that achieves this bound. We show that the designed codes with belief propagation (BP) decoding perform as well as the regular LDPC-CCs presented in the literature on the binary erasure channel (BEC) and the GEC, while achieving significant gains on the SBC. In the case of windowed decoding, our codes perform much better than the best known regular LDPC-CCs over the BEC and the GEC, with very low decoding latencies.**

I. Introduction

LDPC Convolutional Codes (LDPC-CC) were first introduced in [1]. In [2] protograph-based terminated LDPC-CCs which can approach capacity over the binary erasure channel (BEC) were introduced. In this paper we are interested in evaluating the performance of this family of codes over correlated erasure channels. The analysis is carried out considering both the classical belief propagation (BP) decoder and the windowed decoder (WD) introduced in [3]. The motivation behind the adoption of a windowed decoder is the possibility to trade-off erasure correcting capability for reduced decoding latency.

We devote our attention to a subclass of asymptotically rate-$1/2$, regular LDPC-CCs and make some key observations related to the protograph structure. We analyze the performance on erasure channels with memory, specifically the single-burst channel (SBC) and the Gilbert-Elliott channel (GEC). Our aim is to achieve the best possible performance on the SBC and the GEC, while maintaining good performance on the BEC. We give bounds on the maximal size of a single burst of erasures – the maximum tolerable burst length (MTBL) – that the BP decoder can recover on the SBC, and we propose a construction rule that allows us to design codes with MTBL value guaranteed to be linearly increasing as a function of the memory m_s of the convolutional code. We show that an (n, k) terminated LDPC-CC belonging to the class of asymptotically regular $(J, 2J)$ codes [2] can never reach the MTBL performance of an (n, k) maximum distance separable (MDS) code. Finally we analyze the performance of the proposed codes over the GEC through numerical simulations.

This paper is organized as follows. In Section II we introduce the LDPC-CC terminology. We will introduce the two ensembles and codes used as continuing examples throughout the paper. In Section III, we describe the two decoding algorithms considered in the analysis. Section IV presents the erasure channels under consideration and the relevant figures of merit for each channel. We then present the results obtained through numerical simulation in Section V. We finally summarize our findings in Section VI.

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II. Protograph-based terminated LDPC-CC

A protograph [4] is a relatively small bipartite graph from which a larger graph can be obtained by a copy-and-permute procedure – the protograph is copied M times, and then the edges of the individual replicas are permuted among the M replicas to obtain a single, large bipartite graph. Suppose the protograph possesses N_P variable nodes (VNs) and M_P check nodes (CNs), with degrees $J_j, j = 1, ..., N_p$, and $K_i, i = 1, ..., M_p$, respectively. Then the derived graph will consist of $n = N_P M$ VNs and $m = M_P M$ CNs. The nodes of the protograph are labeled, so that if the VN V_j is connected to the CN C_i in the protograph, then V_j in a replica can only connect to one of the M replicated C_i's.

Protographs can be represented by means of an $M_P \times N_P$ bi-adjacency matrix $\mathbf{B}$, called the *base matrix* where the entry $\mathbf{B}_{m,n}$ represents the number of edges between CN C_m and VN V_n (a non-negative integer, i.e., multiple parallel edges – multiedges – are permitted.). The copy-and-permute operation is realized by replacing each edge (multiedge) in the base matrix $\mathbf{B}$ with a size-M permutation matrix (the sum of as many distinct size-M permutation matrices as the number of multiedges, respectively). For different values of M, different blocklengths of the derived Tanner graph can be achieved keeping the original graph structure imposed by the protograph. This means that the density evolution analysis can be performed within the protograph instead of the unstructured ensemble.Furthermore, protographs impose a structure on the derived graph, which facilitates the design of fast decoders and efficient encoders, as well as a refined control on the derived graph edge connections.

Analogous to LDPC block codes, LDPC-CC can also be derived by a protograph expansion. The parity-check matrices of these codes are composed of blocks of size-M square matrices. Let a be the greatest common divisor (gcd) of J and K, the VN and CN degrees respectively. Then there exist positive integers J' and K' such that $J = aJ'$, $K = aK'$, and $\gcd(J', K') = 1$. Starting from these parameters, it is possible to define the ensemble $\mathcal{C}_P(J, K)$ of LDPC-CC. If we start the convolutional code at time instant $t = 1$ and terminate it after L instants, we obtain a block code, described by the base matrix[1] $\mathbf{B}_{[1,L]}$ as in (1). The protograph of the terminated code has $N_p = LK'$ VNs and $M_p = (L + m_s)J'$ CNs,

$$\mathbf{B}_{[1,L]} = \begin{bmatrix} \mathbf{B}_0(1) & & & \\ \mathbf{B}_1(1) & \mathbf{B}_0(2) & & \\ \vdots & \mathbf{B}_1(2) & \ddots & \\ \mathbf{B}_{m_s}(1) & \vdots & \ddots & \mathbf{B}_0(L) \\ & \mathbf{B}_{m_s}(2) & \ddots & \mathbf{B}_1(L) \\ & & \ddots & \vdots \\ & & & \mathbf{B}_{m_s}(L) \end{bmatrix} \quad (1)$$

where $m_s = a - 1$ is the *memory* of the LDPC-CC and $\mathbf{B}_i(t), i = 0, .., m_s$ are $J' \times K'$ identical component base matrices with all entries equal to one. The rate of the LDPC-CC is therefore

$$R_L = 1 - \left(\frac{L + m_s}{L}\right)\frac{J'}{K'} = 1 - \left(1 + \frac{m_s}{L}\right)(1 - R) \quad (2)$$

where $R = 1 - \frac{J'}{K'}$ is the rate of the non-terminated code. Note that $R_L \to R$ and the LDPC-CC has a regular degree distribution [2] when $L \to \infty$. The construction above was proposed in [5] and allows the construction of some (J, K) regular LDPC-CCs. However, not all (J, K) regular LDPC-CC can be constructed, e.g. with this construction m_s becomes zero if J and K are relatively prime and consequently the resulting code is not convolutional. In [2], the authors addressed this problem by proposing a construction rule based on *edge spreading*.

In the following we analyze the case of $(J, 2J)$ regular LDPC-CC. Our aim is to analyze this subclass of asymptotically rate-$1/2$ codes and find protograph structures achieving good performances in both memoryless and correlated erasure channels. Since in this case we always have $J' = 1$ and $K' = 2$, the component base matrices $\mathbf{B}_i, i = 0, ..., m_s$ are 1×2 matrices. As a consequence, we can use two polynomials to represent the entries of the terminated protograph $\mathbf{B}$. The first column of the protograph $\mathbf{B}$ would comprise the first elements of the $\mathbf{B}_i$ matrices $p_0, p_1, \cdots, p_{m_s}$, with which we associate a polynomial $p(x) = p_0+p_1x+\cdots+p_{m_s}x^{m_s}$ of degree at most m_s. Similarly, the second column of $\mathbf{B}$ comprises the second elements of the $\mathbf{B}_i$ matrices, $q_0, q_1, \cdots, q_{m_s}$ and with these we associate a polynomial $q(x) = q_0+q_1+\cdots+q_{m_s}x^{m_s}$, also of degree at most m_s. Then, the $(2i+1)^{\text{th}}$ column of $\mathbf{B}$ can be associated with the polynomial $x^i p(x)$, and the $(2i + 2)^{\text{th}}$ column the polynomial $x^i q(x)$. We will use the polynomial of a column and its index interchangeably, e.g. when we say "choosing the polynomial $x^i p(x)$," we mean that we choose the $(2i + 1)^{\text{th}}$ column of $\mathbf{B}$. Similarly, by "summations of polynomials $p(x)$ and $q(x)$," we mean the collection of the corresponding columns of $\mathbf{B}$. Note that for the exnsembles to be $(J, 2J)$ regular, we will further have the constraint $p(1) = q(1) = J$.

The terminated ensemble can now be defined by specifying the parameters L and $\mathbf{B}_i,\ \forall\ i = 0, ..., m_s$, or equivalently, by giving L, $p(x)$ and $q(x)$. Note that our construction is similar to the one in [5] except that we do not require that $m_s = a-1$. We further disregard the requirement that the $\mathbf{B}_i$ matrices are identical and have only ones.

Ensembles and Codes

Throughout the paper, we consider the performance of two code ensembles on a multitude of channels. The first code ensemble is a regular LDPC-CC ensemble constructed in [2], and the second is constructed to perform well with windowed decoding [3]. For fair comparison between the two ensembles, i.e., for comparing two ensembles of the same rate and same asymptotic degree distribution, we keep the parameters m_s, J and L the same. Here we choose $m_s = 2$, $J = 3$, $K = 2J = 6$ and $L = 20$, so that the rate $R_L = 0.45$. Note that both the code ensembles are $(3, 6)$ regular LDPC-CC ensembles, and we refer to them henceforth as ensembles $\mathcal{A}$ and $\mathcal{B}$. Ensemble $\mathcal{A}$ is defined by

$$\mathbf{B}_0 = [1 \quad 1],\ \mathbf{B}_1 = [1 \quad 1],\ \mathbf{B}_2 = [1 \quad 1] \quad (3)$$

and ensemble $\mathcal{B}$ by

$$\mathbf{B}_0 = [2 \quad 2],\ \mathbf{B}_1 = [0 \quad 1],\ \mathbf{B}_2 = [1 \quad 0]. \quad (4)$$

Equivalently, the ensembles $\mathcal{A}$ and $\mathcal{B}$ can be defined using the polynomials $p_A(x) = q_A(x) = 1 + x + x^2$ and $p_B(x) = 2 + x^2, q_B(x) = 2 + x$ respectively. Let us denote the protographs defined by these submatrices (or polynomials) as $\mathbf{B}_\mathcal{A}$ and $\mathbf{B}_\mathcal{B}$ respectively. While dealing with finite-length codes, we keep the parameter M (and consequently, the blocklength n) also the same for the two codes. In such finite length comparisons, the code protographs have been expanded by means of circulant permutation matrices[2]. For this purpose, a girth optimization technique based on [6], [7] has been adopted [8]. The codes constructed from ensembles $\mathcal{A}$ and $\mathcal{B}$ are denoted A and B respectively.

III. Decoders

We consider two particular iterative decoders described below.

A. Belief Propagation (BP) Decoder

Since we are interested in terminated LDPC-CC, decoding can be performed in a manner similar to the case of an LDPC block code, i.e. each frame carrying a codeword obtained through the termination can be decoded with the sum-product algorithm (SPA). Note that the BP decoder has to wait for the entire codeword to be received to start decoding, and this implies a large latency and buffer requirement at the decoder.

[1]For a time-invariant terminated LDPC-CC, $\mathbf{B}_i(t) = \mathbf{B}_i\ \forall\ t = 1, 2, \cdots, L$, and $i = 0, 1, \cdots, m_s$.

[2]It follows that the resulting terminated LDPC-CCs can be seen also as quasi-cyclic block codes.

B. Windowed Decoder (WD)

It is desirable to avoid the large latency of the BP decoder described above in many applications, e.g. [9]. The convolutional structure of the code imposes a maximum distance constraint on the VNs connected to the same parity-check equations – two VNs that are at least $K'(m_s+1)$ columns apart (in the protograph) cannot be involved in the same equation. This characteristic can be exploited in order to perform continuous decoding of the received stream through a *window* that slides along the bit sequence.

Consider a terminated parity-check matrix $\mathbf{H}$ built from a base protograph $\mathbf{B}$ and a sliding window that covers[3] WJ' rows and WK' columns of the protograph $\mathbf{B}$. At the first time instant, $t=1$, the decoder performs message passing iterations over the equations within the sliding window with the aim of decoding all of the first MK' symbols in the window, called the *targeted symbols*. The sliding window shifts down J' rows and right K' columns in $\mathbf{B}$ after a maximum number of message passing iterations have been performed, or when all targeted symbols within the window have been recovered, and continues decoding at the new position at the second time instant, $t=2$. This is summarized in Figure 1. The latency of the WD is

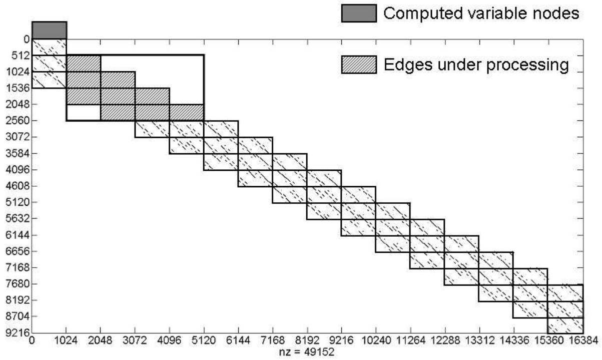

Fig. 1. Windowed decoder for sliding window size $W=4$, for a $(3,6)$ regular LDPC-CC with $L=16$ and $M=512$ at time instant $t=2$.

therefore a fraction $w=\frac{W}{L}$ of the latency in comparison with the BP decoder. It is clear that the performance achievable by the WD improves with increase in window size (latency) as more equations are considered. In [3], this windowed decoder was analyzed for the BEC.

IV. Channels and Figures of Merit

A. Binary erasure channel (BEC)

The BEC is one over which each symbol of the transmitted codeword is erased independently with a probability ε. The performance of LDPC-CCs on the BEC has been extensively studied, e.g. [1], [2], [3]. In the asymptotic setting, the figures of merit for the two decoders discussed in Section III are the *BP threshold* ε^*_{BP} and the *windowed threshold* ε^*_W, both of which were studied in [3].

For the finite length analysis, we will compare the performance with the Singleton bound, which represents the performance achievable by an (n,k) MDS code. This bound for the codeword error rate (CER) $\mathbb{P}_{CW}$ for the BEC can be expressed as

$$\mathbb{P}_{CW}=\sum_{j=n-k+1}^{n}\binom{n}{j}\varepsilon^j(1-\varepsilon)^{n-j} \tag{5}$$

For ensemble[4] $\mathcal{B}$, the trade-off between latency and performance in terms of ε^*_W is shown in Figure 2. Also shown is

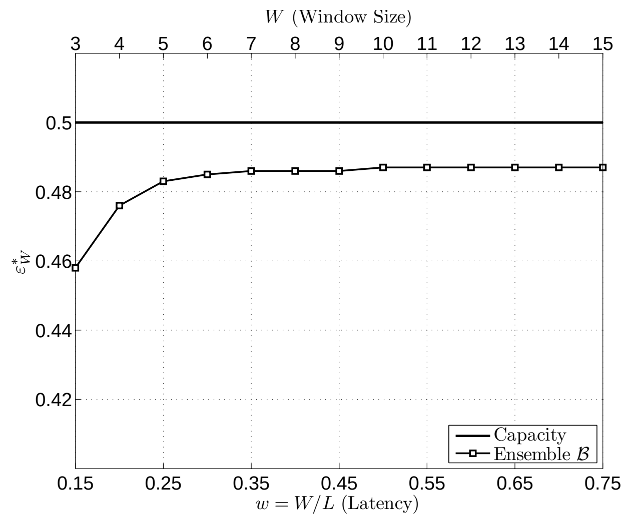

Fig. 2. Trade-off between latency and performance for ensemble $\mathcal{B}$.

the Shannon limit for windowed decoding – we give a limit for windowed decoding, rather than for the ensemble because for window sizes $m_s < W \le L$, the rate of the code within the window is 0.5, although the rate for the entire code is $R_L=0.45$. The reduced limit achievable with WD ($\varepsilon^*=0.5$ rather than $\varepsilon^*=0.55$) is perhaps the cost for reduced latency. Finite length performance of codes constructed from ensembles $\mathcal{A}$ and $\mathcal{B}$ over the BEC is evaluated in Section V-A.

B. Single-burst channel (SBC)

We now consider the performance of the ensembles over a channel that inserts a single burst of erasures within a codeword. In this case, we are interested in the MTBL [10], defined as the maximal length of a single burst of erasures that can be recovered by the BP decoder, and denoted Δ_{max}. Let us define a *protograph stopping set* to be a subset $S^{\mathbf{B}}$ of the VNs of the protograph $\mathbf{B}$ whose neighbouring CNs are connected at least twice to $S^{\mathbf{B}}$. Let us denote the minimum number of consecutive columns of the protograph that contains a protograph stopping set by $s^{\mathbf{B}}_{min}$. This parameter is of interest because we can give simple bounds for the MTBL based on $s^{\mathbf{B}}_{min}$, as will be shown later.

For ensemble $\mathcal{A}$, the first two columns of $\mathbf{B}_{\mathcal{A}}$ form a stopping set. This is clear from the highlighted columns below

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & \cdots \\ \mathbf{1} & \mathbf{1} & 1 & 1 & 0 & 0 & \cdots \\ \mathbf{1} & \mathbf{1} & 1 & 1 & 1 & 1 & \cdots \\ \mathbf{0} & \mathbf{0} & 1 & 1 & 1 & 1 & \cdots \\ \mathbf{0} & \mathbf{0} & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

[3]When the sliding window goes beyond the protograph, it is assumed to be bounded by the boundaries of the protograph.

[4]Ensemble $\mathcal{A}$ has zero windowed threshold [3].

Therefore, $s_{min}^{\mathbf{B}_\mathcal{A}} \leq 2$. Since no single column forms a protograph stopping set, $s_{min}^{\mathbf{B}_\mathcal{A}} \geq 2$, implying $s_{min}^{\mathbf{B}_\mathcal{A}} = 2$.

For ensemble $\mathcal{B}$, the highlighted columns of $\mathbf{B}_\mathcal{B}$ in the following matrix form a stopping set.

$$\begin{bmatrix} \mathbf{2} & 2 & 0 & \mathbf{0} & 0 & 0 & \cdots \\ \mathbf{0} & 1 & 2 & \mathbf{2} & 0 & 0 & \cdots \\ \mathbf{1} & 0 & 0 & \mathbf{1} & 2 & 2 & \cdots \\ \mathbf{0} & 0 & 1 & \mathbf{0} & 0 & 1 & \cdots \\ \mathbf{0} & 0 & 0 & \mathbf{0} & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, $s_{min}^{\mathbf{B}_\mathcal{B}} \leq 4$. As no three consecutive columns of $\mathbf{B}_\mathcal{B}$ constitute a protograph stopping set, it is clear that $s_{min}^{\mathbf{B}_\mathcal{B}} > 3$, so that $s_{min}^{\mathbf{B}_\mathcal{B}} = 4$. In fact, we can give a general construction of a protograph that achieves an $s_{min}^{\mathbf{B}}$ that increases linearly with m_s.

Proposition 1 (Achievability of $s_{min}^{\mathbf{B}} = 2m_s$): For a $(J, 2J)$ regular LDPC-CC with memory m_s and $J > 2$, we can achieve $s_{min}^{\mathbf{B}} = 2m_s$.

Proof: Consider the ensemble given by $p(x) = (J-1) + x^{m_s}$ and $q(x) = (J-1) + x$, and the polynomial $r(x) = p(x)a(x) + q(x)b(x)$ that represents all $(2^{2m_s-1} - 1)$ non-trivial subsets of the first $(2m_s - 1)$ columns of $\mathbf{B}$, for all choices of polynomials $a(x)$ and $b(x)$ with coefficients in $\{0, 1\}$ and maximal degrees $(m_s - 1)$ and $(m_s - 2)$ respectively, except the choice of all zero coefficients. It is not hard to see that $r(x)$ is a monic polynomial of degree $(m_s + i_1)$ where i_1 is the degree of $a(x)$ when $a(x) \neq 0$, and a monic polynomial of degree $(1 + i_2)$ where i_2 is the degree of $b(x)$ when $a(x) = 0$. Thus, $s_{min}^{\mathbf{B}} > 2m_s - 1$. Finally, notice that

$$p(x) + x^{m_s-1}q(x) = (J-1) + (J-1)x^{m_s-1} + 2x^{m_s},$$

with all coefficients strictly larger than 1. $p(x)$ corresponds to the first column of the protograph and $x^{m_s-1}q(x)$ to the $(2m_s)^{\text{th}}$ column. Thus, we have $s_{min}^{\mathbf{B}} = 2m_s$. ■

Note that ensemble $\mathcal{B}$ in (4) is an example of the above construction. We bring to the reader's attention here that other constructions that achieve $s_{min}^{\mathbf{B}} = 2m_s$ are also possible, i.e. the construction in Proposition 1 is not unique in achieving the mentioned $s_{min}^{\mathbf{B}}$. A drawback in this protograph construction, however, is that if m_s is increased to get larger $s_{min}^{\mathbf{B}}$, the code rate in (2) decreases linearly. We prove in the following that for any protograph, we cannot have $s_{min}^{\mathbf{B}} > 2m_s$ so that the above construction achieves the maximal $s_{min}^{\mathbf{B}}$.

Proposition 2 (Converse): For a $(J, 2J)$ regular LDPC-CC with memory m_s and $J > 2$, $s_{min}^{\mathbf{B}} \leq 2m_s$.

Proof: We will prove this by showing that in any protograph with $K = 2J$ and memory m_s, we can find $2m_s$ consecutive columns that will always contain a protograph stopping set. We know that since the code has a memory m_s, at least one of the polynomials defining the protograph of the ensemble – $p(x)$ and $q(x)$ – is of degree m_s. Without loss of generality, let us assume the degree of $p(x)$ to be m_s, and that of $q(x)$ be j, $1 \leq j \leq m_s$. Also, let the first non-zero coefficient[5] in $p(x)$ be the coefficient of x^{i_p}, $0 \leq i_p \leq m_s - 1$ and that in $q(x)$ be the coefficient of x^{i_q}, $0 \leq i_q \leq j - 1$. Then consider $2m_s$ consecutive columns of the protograph $\mathbf{B}$ starting with a column corresponding to the polynomial $p(x)$. Consider the subset of columns of $\mathbf{B}$ corresponding to the polynomial $r(x) = p(x)(x^{i_q} + x^{i_q+1} + \cdots + x^{j-1}) + q(x)(x^{i_p} + x^{i_p+1} + \cdots + x^{m_s-1})$. We claim that this is a protograph stopping set. To see this, consider $\hat{p}(x) = x^{i_p} + x^{m_s}$ and $\hat{q}(x) = x^{i_q} + x^j$. Note that $p(x)$ and $\hat{p}(x)$ (resp. $q(x)$ and $\hat{q}(x)$) have the same minimum and maximum degrees. Thus, $p(x)$ differs from $\hat{p}(x)$ in that its coefficients of x^{i_p} through x^{m_s} are possibly larger than the corresponding coefficients of $\hat{p}(x)$. Similarly for $q(x)$ and $\hat{q}(x)$. With $\hat{p}(x)$ and $\hat{q}(x)$ as the column polynomials defining $\mathbf{B}$, the columns corresponding to the above subset can be associated with the polynomial $\hat{r}(x) = \hat{p}(x)(x^{i_q} + x^{i_q+1} + \cdots + x^{j-1}) + \hat{q}(x)(x^{i_p} + x^{i_p+1} + \cdots + x^{m_s-1})$, which can be written as

$$\begin{aligned} \hat{r}(x) = (x^{i_p} + x^{m_s})(x^{i_q} + x^{i_q+1} + \cdots + x^{j-1}) + \\ (x^{i_q} + x^j)(x^{i_p} + x^{i_p+1} + \cdots + x^{m_s-1}) \\ = 2(x^{i_p+i_q} + \cdots + x^{i_p+j-1} + x^{i_p+j} + \cdots + x^{m_s+j-1}). \end{aligned}$$

Note that $\hat{r}(x)$ has all coefficients more than 1 and that the minimum and maximum degrees of $\hat{r}(x)$ are $(i_p + i_q)$ and $(m_s + j - 1)$ respectively. These degrees are the same as those of $r(x)$, and thus $r(x)$ can only differ from $\hat{r}(x)$ in having larger coefficients. Therefore, $r(x)$ also has all coefficients greater than 1. This shows that the chosen subset of columns form a protograph stopping set. Since all these columns are contained in a maximum of $2m_s$ consecutive columns of the protograph $\mathbf{B}$, $s_{min}^{\mathbf{B}} \leq 2m_s$. ■

We now show the relation between the parameters Δ_{max} and $s_{min}^{\mathbf{B}}$, assuming that $s_{min}^{\mathbf{B}} \geq 2$.

Lemma 1: $\Delta_{max} \leq M s_{min}^{\mathbf{B}} - 1$.

Proof: Clearly, all of the $M s_{min}^{\mathbf{B}}$ columns of the parity-check matrix corresponding to the $s_{min}^{\mathbf{B}}$ consecutive columns of $\mathbf{B}$ that contain a protograph-stopping set must contain a stopping set of the parity-check matrix. Therefore, if all symbols corresponding to these columns are erased, they cannot be retrieved, implying $\Delta_{max} \leq M s_{min}^{\mathbf{B}} - 1$. ■

Corollary 1: A $(J, 2J)$ regular LDPC-CC can never achieve the MTBL of an MDS code.

Proof: From the Singleton bound, we have $\Delta_{max} \leq n - k = (L + m_s)M$, assuming that the parity-check matrix is full rank. From Lemma 1 we have $\Delta_{max} \leq M s_{min}^{\mathbf{B}} - 1 \overset{\text{Prop. 2}}{\leq} 2m_s M - 1$. Since we require $m_s \leq L$ for a non-negative code rate in (2), $\Delta_{max} \leq 2m_s M - 1 \leq (L + m_s)M - 1 < (L + m_s)M$, which shows that the MTBL of an MDS code can never be achieved. ■

Despite the above discouraging result, we can guarantee an MTBL that linearly increases with m_s.

Lemma 2: $\Delta_{max} \geq M(s_{min}^{\mathbf{B}} - 2) + 1$.

Proof: From the definition of $s_{min}^{\mathbf{B}}$, it is clear that if one of the two extreme columns is completely known, all other symbols can be recovered, for otherwise the remaining columns will have to contain a protograph stopping set, violating the minimality of the stopping set distance $s_{min}^{\mathbf{B}}$ (The two extreme columns are *pivots* of the stopping set [11]). The largest solid burst that is guaranteed to have at least one of the extreme columns completely known is of length $M(s_{min}^{\mathbf{B}} - 2) + 1$. Therefore, $\Delta_{max} \geq M(s_{min}^{\mathbf{B}} - 2) + 1$. ■

[5] We disallow $i_p = m_s$ or $i_q = j$ because in this case a single column of $\mathbf{B}$ is a protograph stopping set, i.e. $s_{min}^{\mathbf{B}} = 1 < 2m_s$.

The actual values of the MTBL for the codes A and B are given in Section V-B.

C. Gilbert-Elliott Channel (GEC)

Finally, we evaluate the performances of codes A and B on the GEC [12], [13]. In this model, the channel is either in a "good" state G, where we assume the erasure probability is 0, or in a "bad" state B, in which we will assume the erasure probability is 1. The state process of the channel is a first-order Markov process with the transition probabilities $\mathbb{P}\{B \rightarrow G\} = g$ and $\mathbb{P}\{G \rightarrow B\} = b$. With these parameters, we can easily deduce [14] that the *average erasure rate* $\varepsilon = \mathbb{P}\{B\} = \frac{b}{b+g}$ and the *average burst length* $\Delta = \frac{1}{g}$. We will consider the GEC to be parameterized[6] by the pair (ε, Δ) and evaluate the performance of codes A and B in Section V-C.

The figure of merit in this case is the CER in comparison with that of an (n, k) MDS code, which is given by the Singleton bound [13].

V. Numerical Results

In this section we analyze the performance achievable by codes A and B over the channels described in Section IV. The terminated protographs describing the two ensembles have been expanded by a factor $M = 512$, resulting in codes with blocklength $n = LMK' = 20480$ and rate 0.45. The codes are constructed such that the girth is 12 for both codes A and B.

A. BEC

In Figures 3 and 4, the symbol error rate (SER) and the CER performance are depicted for codes A and B. Also shown in Figure 3 (as ticks on the ε-axis) are the ε^*_{BP} values for the ensembles, which are 0.4882 and 0.4881 respectively for ensembles $\mathcal{A}$ and $\mathcal{B}$, and the ε^*_W values for ensemble $\mathcal{B}$ (see Figure 2). As can be observed, code B clearly outperforms code A for small window sizes ($W = 3, 5$), confirming the effectiveness of the proposed design rule for windowed decoding. For larger window sizes ($W = 10$), the two codes show almost the same performance, with code A performing slightly better. Also shown for comparison in Figure 4 is the CER for an MDS code (marked SB for Singleton bound).

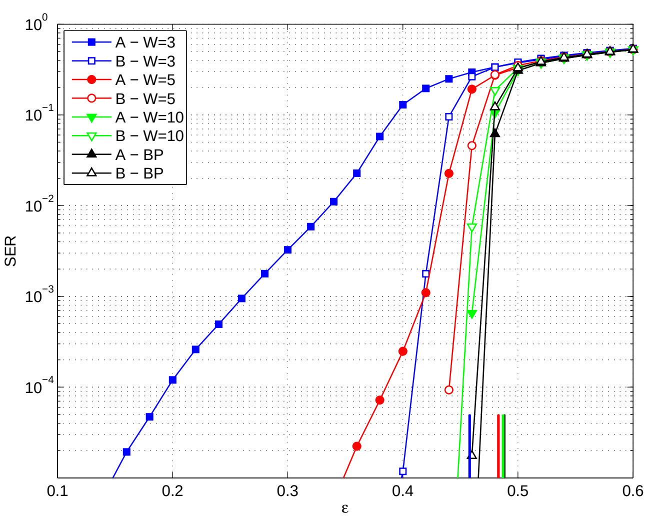

Fig. 3. SER performance for BP and Windowed Decoding over BEC.

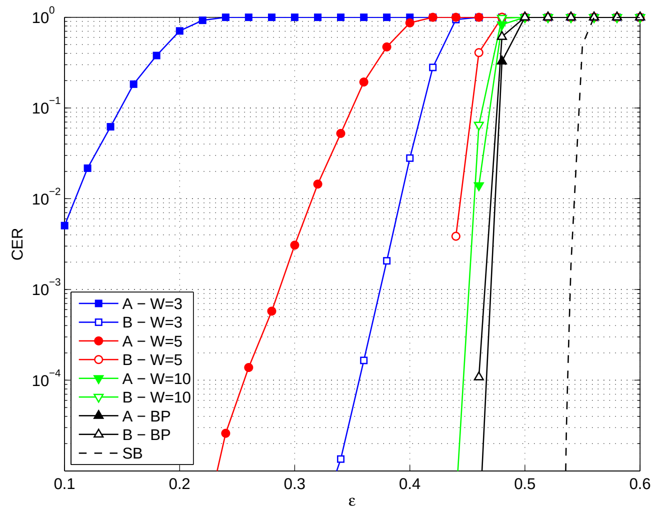

Fig. 4. CER performance for BP and Windowed Decoding over BEC with Singleton bound (SB).

B. SBC

The MTBL for codes A and B was computed using an exhaustive search algorithm, by feeding the decoder with a solid burst of erasures and testing all the possible locations of the burst. The MTBL for the codes we considered were 1023 and 1751 for codes A and B, respectively. Note that for code A, the MTBL $\Delta_{max} = 1023 = 2M-1$, i.e., code A achieves the upper bound from Lemma 1. More importantly, the maximum possible Δ_{max} was achievable while maintaining good performance over the BEC with the BP decoder. However, the MTBL for code B, $\Delta_{max} = 1751 < 2047 = 4M - 1$, is much smaller than the corresponding bound from Lemma 1. In this case, whereas other code constructions with Δ_{max} up to 2045 were possible, a trade-off between the BEC performance and MTBL was observed, i.e. the code that achieved $\Delta_{max} = 2045$ was found to be much worse over the BEC than both codes A and B considered here. Such a trade-off has also been observed by others, e.g. [15]. Nevertheless, our code design does give a large increase in MTBL ($> 70\%$) without increasing the memory m_s, i.e. without any sacrifice in terms of code rate. The MTBL achieved as a fraction of the maximum possible MTBL (achieved by MDS codes) $\Delta_{max}/(n-k)$ was roughly 9.1% and 15.5% for codes A and B respectively.

C. GEC

In Figures 5, 6 and 7 we show the CER performance obtained for codes A and B over GEC channels with $\Delta = 10, 50, 100$ respectively, and $\varepsilon \in [0.1, 0.6]$. As can be seen from the figures, for $W = 3$, code B always outperforms code A, while for $W = 5$ there is no such gain when $\Delta = 100$. However, for $W = 10$ and for BP decoding, code A slightly outperforms code B, similar to the trend observed for the BEC.

Note that the code B outperforms A for small ε when the average burst length $\Delta = 100$ for large window sizes and for

[6]Note that there is a one-to-one correspondence between the two pairs (b, g) and (ε, Δ).

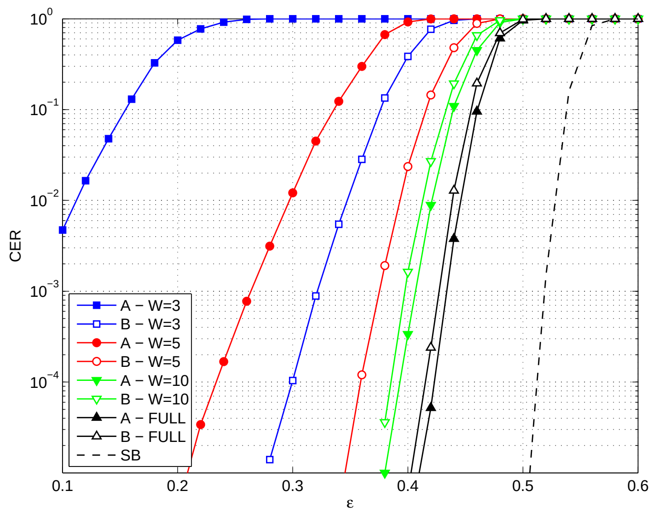

Fig. 5. CER Performance on GEC with $\Delta = 10$ with Singleton bound (SB).

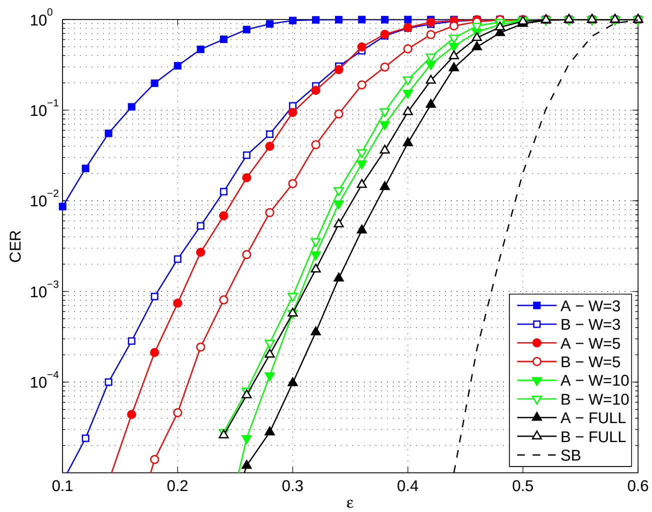

Fig. 6. CER Performance on GEC with $\Delta = 50$ with Singleton bound (SB).

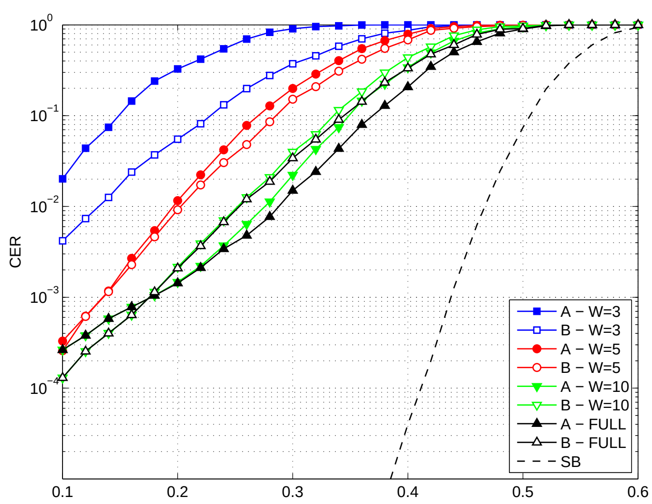

Fig. 7. CER Performance on GEC with $\Delta = 100$ with Singleton bound (SB).

BP decoding. This can be explained because in this regime, the probability of a burst is small but the average burst length is large. Therefore, when a burst occurs, it is likely to resemble a single burst in a codeword, i.e. the channel in this regime is similar to the SBC in which case we know that the code B is stronger than A. Also note the significant gap between the BP decoder performance and the Singleton bound, suggesting that unlike some moderate length LDPC block codes [16], LDPC-CCs are far from achieving MDS performance.

VI. Conclusion

In this paper we have analyzed the performance of $(J, 2J)$ protograph-based terminated LDPC convolutional codes (LDPC-CC) on correlated erasure channels. We have derived lower and upper bounds for the maximum tolerable burst length (MTBL) and proposed a construction rule that allows us to design protographs whose MTBL increases linearly with the memory m_s. These protographs also have good performance under BP and windowed decoding over the BEC. We proved that $(J, 2J)$ regular LDPC-CC cannot achieve the MTBL of MDS codes. Finally, we evaluated the performance of the proposed LDPC-CC on the GEC through numerical simulations, showing that the convolutional structure leads to far-from-optimal performance with respect to the Singleton bound.

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