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Scribe 4 - CMA

- **Differentiation**

  Consider $f(x) = \sin(e^{x^2})$

  $f'(x) = \cos(e^{x^2}) \times e^{x^2} \times 2x$

Computers can perform numerical methods to differentiate

Numerical diff.

$f : [a,b] \to \mathbb{R}$ → continuously differentiable in (a,b)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

For small $h > 0$,

$$\delta_h^+(x) = \frac{f(x+h) - f(x)}{h}, \quad \delta_h^-(x) = \frac{f(x) - f(x-h)}{h}$$

Error of approximation

f is twice differentiable in (a,b).

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\zeta)$$

where $\zeta \in [x, x+h]$

$$|\delta_h^+(x) - f'(x)| = \frac{h}{2} f''(\zeta)$$

If x is small, 'h' can be somewhat large compared to the case where x is large, then h has to be small enough for good approximation

Centred finite diff

For small $h > 0$,

$$\delta_h^c(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Assuming f is thrice differentiable.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(\xi_1) \rightarrow ①$$

$$f(x-h) = f(x) - h'f(x) + \frac{h^2}{2} f''(\xi_2) \rightarrow ②$$

① − ②

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{h^2}{2}\left(f''(\xi_1) - f''(\xi_2)\right)$$

$$\frac{f(x+h) - f(x-h)}{2h} = f'(x) + \frac{h}{4}\left(f''(\xi_1) - f''(\xi_2)\right)$$

$$\Rightarrow \left|\delta_h^c(x) - f'(x)\right| = \frac{h}{4}\left(\left|f''(\xi_1) - f''(\xi_2)\right|\right)$$

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Numerical integration

→ Trapezoidal formula:

Divide $[a,b]$ into m intervals of equal length. Let $x_k = a + kh$ for $k \in \{0,1,\dots m\}$ & $h = \frac{(b-a)}{m}$.

Then

$$I_m(f) = \frac{b-a}{2m} \sum_{k=1}^{m} \left[f(x_k) + f(x_{k+1}) \right]$$

For twice differentiable $f(x)$ on the interval $[a,b]$, we have

$$E_n^T(f) := \int_a^b f(x)\,dx - T_n(f) = \frac{-h^2(b-a)}{12} f''(c_n)$$

Error for one sub-interval:

$$E := \int_\alpha^{\alpha+h} f(x)\,dx - \frac{h}{2}\left[f(\alpha) + f(\alpha+h)\right].$$

$$= \int_\alpha^{\alpha+h} \left[f(\alpha) + f'(\alpha)(x-\alpha) + \frac{f''(c_\xi)}{2}(x-\alpha)^2 \right] dx - \frac{h}{2}\left[f(\alpha) + f(\alpha) + hf'(\alpha) + \frac{h^2 f''(c_\xi)}{2} \right]$$

After cancelling

$$E = \frac{-h^3}{12} f''(c_\xi)$$

$$I(f) = \int_{x_0}^{x_n} f(x)\,dx \quad ; \quad T_n(f) = \sum_{i=0}^{n-1} \frac{h}{2}\left(f(x_i) + f(x_{i+1})\right)$$

$$E_n^T(f) = \int_a^b f(x)\,dx - T_n(f)$$

Combing with above result,

$$E_n^T(f) = \frac{-h^3}{12} f''(r_1) + \dots + \frac{-h^3}{12} f''(r_n)$$

$$= \frac{-h^3 n}{12} \underbrace{\left[\frac{f''(r_1) + \dots f''(r_n)}{n} \right]}_{c_n}$$

Hence

$$E_n^T(f) = \frac{-h^2(b-a)}{12} f''(\phi_n) \quad \text{where } \phi_n \in [a,b]$$