Midterm Adventure

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Rules of Engagement

You may discuss the problems below with anyone in the following set of people:

$$P = \{\text{Anyone currently enrolled in MATH 3310, Kaylee Bodily, Caroline Torman, Erin Pitts, Brent Thomas}\}.$$

The documentation of your adventure should be created by teams of at most 4 and is to be submitted by Friday March 6, 2026 as a PDF via Canvas (the way we've done all semester). One submission per team.

No outside resources are to be used on this experience. Try to get through this with your brain and the brains of your team (if you choose to be part of one).

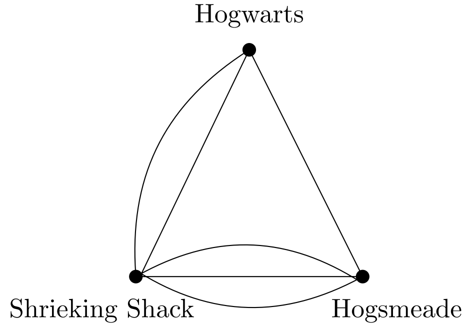
1 The Adventure Begins

Sneaking into Hogsmeade

Sub-Adventure 1.1

Argus Filch and Severus Snape are playing a game in which they try to destroy all secret passages between Hogwarts, Hogsmeade, and the Shrieking Shack. Argus and Severus take turns destroying passages. On their turn, they select a location and destroy any positive number of passages to the other two locations. The winner of the game is the player who is able to destroy all the remaining passages on their turn.

As an example, suppose there are 2 passages between the Shrieking Shack and Hogwarts, 1 passage between Hogwarts and Hogsmeade, and 3 passages between the Shrieking Shack and Hogsmeade.



Suppose, in general, there are x passages between the Shrieking Shack and Hogwarts, y passages between Hogwarts and Hogsmeade, and z passages between the Shrieking Shack and Hogsmeade.

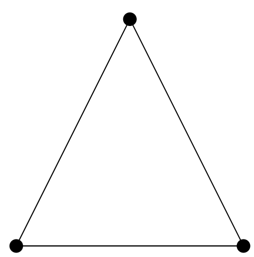
Determine, with proof, the values of x, y, z for which Severus Snape (the first player) can always win if a particular strategy is employed for each move.

Solution: Claim: Player 1 (Snape) has a winning strategy for every starting position in which the number of passages in each side is not equal.

Proof. We can prove this through induction.

Let x, y, z = N meaning that all of the routes have the same number of passages.

Base Case: N = 1



In this case the player going first will always lose as long as their opponent understands the game. Player 1 can either remove one passage or 2 in either case player 2 can remove the remaining passages.

Hypothesis

Assume that this is true for N = K + 1 meaning that x + 1, y + 1, z+1 all equal N.

Inductive step:

In any of these cases Player 1 can remove from 1 or 2 of the values meaning that Player 2 can match them to create the same position where x y z = N and eventually get to the base case x, y, z = 1

Therefore any position where x, y, z = N is an unwinable position for the player going first.

In any case where x, y, z $\neq$ the winning strategy for player one goes like this. The player finds the side or sides that have the least amount of passageways and removes passageways from the other sides until they match and all sides are equal. This forces player 2 into the same position where x, y, z = N which is unwinnable.

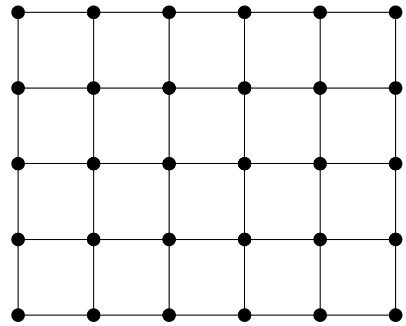
Therefore There is a winning strategy for player 1 for every starting position in which the number of passages in each side are not equal.

□

Dots and Boxes

Sub-Adventure 1.2

Two players, Link and Zelda, alternately select an *edge* (a line segment connecting two dots) on the grid graph shown below and color it red. The loser of the game is the player who is forced to select an edge that creates a red C_4 — a red square on 4 vertices.



Confirm or deny, with proof, whether Zelda (the first player) can always win if she employs a particular strategy for each move.

Solution:

2 Prime Time

Sub-Adventure 2.1

Prove or disprove whether you can order the years $2011, \ldots, 2026$ so that the resulting 64-digit number is prime.

Solution:

Claim

It is not possible to order the years 2011, 2012, ..., 2026 to form a 64-digit number that is prime.

Proof

We shall prove this by proving that any such 64-digit number formed by concatenating the years 2011 to 2026 is divisible by 11, and thus composite (Composite means not Prime).

First we will start by refering to a few things from our notes. According to Theorem 2, part 6, we know that an integer X is divisible by 11 if the alternating sum of its digits, $\sum_{i=0}^{n}(-1)^i d_i$, is divisible by 11.

So we will concatinate each year, which which will give us our 64-digit number because we have 16 years and each year has 4 digits. Because each years is just 4 digits, which is an even number of digits, the alternating sum of each year will always be the same, (For example 2011 will always be $1 - 1 + 0 - 2$ no matter where we place it in the 64 digit number).

We calculate the individual alternating sums $(d_0 - d_1 + d_2 - d_3)$ for each year in the set:

So here we will calculate the alternating sum for each year:

- 2011: $1 - 1 + 0 - 2 = -2$
- 2012: $2 - 1 + 0 - 2 = -1$
- 2013: $3 - 1 + 0 - 2 = 0$
- 2014: $4 - 1 + 0 - 2 = 1$

- 2015: $5 - 1 + 0 - 2 = 2$
- 2016: $6 - 1 + 0 - 2 = 3$
- 2017: $7 - 1 + 0 - 2 = 4$
- 2018: $8 - 1 + 0 - 2 = 5$
- 2019: $9 - 1 + 0 - 2 = 6$
- 2020: $0 - 2 + 0 - 2 = -4$
- 2021: $1 - 2 + 0 - 2 = -3$
- 2022: $2 - 2 + 0 - 2 = -2$
- 2023: $3 - 2 + 0 - 2 = -1$
- 2024: $4 - 2 + 0 - 2 = 0$
- 2025: $5 - 2 + 0 - 2 = 1$
- 2026: $6 - 2 + 0 - 2 = 2$

If we add all of those values together we get:

$$-2 - 1 + 0 + 1 + 2 + 3 + 4 + 5 + 6 - 4 - 3 - 2 - 1 + 0 + 1 + 2 = 11$$

Since 11 is divisible by 11 ($11 = 11 \cdot 1$), then our 64 digit number X is also divisible by 11 based on Theorem 2, part 6.

By Definition 4 from our notes, 11 is a divisor of X, meaning $X = 11k$ for some integer k. Since X is a 64-digit number, it is clear that $1 < 11, k < X$.

Using our definition of a composite number that we proved in HW 2 Question 1, an integer n is composite if it can be written as $n = st$ where $1 < s, t < n$.

If we set $n = X$, $s = 11$, and $t = k$, we have shown that X meets the criteria for being composite. Therefore, X is not prime by our definition of prime because if something is composite, it is not prime. So, no ordering of the years 2011 to 2026 can result in a prime number.

Conclusion

Thus, we have shown that any 64-digit number formed by concatenating the years 2011 to 2026 is divisible by 11 and therefore composite. Hence, it is not possible to order the years to form a prime number.

3 Counting Functions

Sub-Adventure 3

If A and B are sets, the notation B^A stands for the set of all functions that map A into B. Note that $f : A \xrightarrow{1-1} B$ denotes that f is a function from A into B that is injective. Note that $f : A \xrightarrow{\text{onto}} B$ denotes that f is a surjective function mapping A into B.

The goal is to count the number of functions with the properties indicated by the row and column headings in the table. For example, in entry number 8 should be the number of functions that are surjective (onto) that map n distinguishable objects to x indistinguishable objects. In mathematical notation, this is

$$\left|\left\{f \in B^A : A = \{a_1, a_2, \ldots, a_n\}, |B| = x, f : A \overset{\text{onto}}{\longrightarrow} B\right\}\right|.$$

While entry 6 should house the number

$$\left|\left\{f \in B^A : |A| = n, B = \{b_1, b_2, \ldots, b_x\}, f : A \overset{\text{1-1}}{\longrightarrow} B\right\}\right|.$$

A	B	unrestricted	onto	injective
distinguishable	distinguishable	1.	2.	3.
indistinguishable	distinguishable	4.	5.	6.
distinguishable	indistinguishable	7.	8.	9.
indistinguishable	indistinguishable	10.	11.	12.

Please determine with an argument for each, a formula enumerating the set of functions corresponding to entries 1-12.

Solution:

4 Tiling

Sub-Adventure 4.1

Let T_n be the number of ways to perfectly tile a $1 \times n$ floor with 1×1 and 1×2 tiles. We say $T_0 = 1$ because the tiling with no tiles perfectly tiles a 1×0 floor.

(a) Give a recursive formula for T_n, be sure to include initial values.

Solution:
Showing the initial values, leading up to the recursive formula for T_n:
$T_0 = 1$
$T_1 = 1$ (Using a single 1×1 tile)
$T_2 = 2$ (Using either two 1×1 tiles or a single 1×2 tile)
$T_3 = 3$ (Using either three 1×1 tiles, or a 1×2 tile and a 1×1 tile, in different orders)
$T_4 = 5$ (Using either four 1×1 tiles, or a combination of 1×2 tiles and 1×1 tiles, or two 1×2 tiles)
...
$T_n = T_{n-1} + T_{n-2}$ for $n \geq 2$

(b) Please give a combinatorial proof of the following theorem. Note that the notation $\lfloor n/2 \rfloor$ is a way to round down to the nearest integer.
Theorem 1. *For every integer $n \geq 0$,*

$$T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}$$

Solution:

(c) Please prove the following theorem.

Theorem 2. *For every integer $n \geq 0$,*

$$T_{n+1} = 1 + \sum_{i=1}^{n} T_{i-1}$$

Solution:

Sub-Adventure 4.2

Please prove the following theorem.

Theorem 3. *For every $n \in \mathbb{Z}^+$, if any one square is removed from a $2^n \times 2^n$ chessboard, the result can be perfectly tiled with*  *–shaped tiles.*

Solution: