Aggregating Algorithm

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Declaration

This report has been prepared on the basis of my own work. Where other published and unpublished source materials have been used, these have been acknowledged.

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Date of Submission: 29 August 2024

Signature:

Abstract

Your abstract goes here.

Contents

Acknowledgements

While the contents of this report are on the basis of my own work, none of this would have been possible without the patience and mentorship of my supervisor Dr. Yuri Kalnishkan to whom I am extremely grateful. It was your advice, clear explanations, and expertise that made this project what it is now and something that I am incredibly proud of. I would also like to express my gratitude to the group of friends who made this academic year possible, namely Cougar Tasker, Einstein Ebereonwu, Hayden Amarr, Mohammadreza Yazdian, Niraj Jain, and Ray Mahbub, without whom I would have struggled to maintain my discipline and motivation.

1 Introduction *(1,000)* (728)

1.1 Project Scope and Objectives (205)

The aim of this project is to implement methods of Prediction with Expert Advice, such as the Aggregating Algorithm, and to evaluate their performance in different scenarios, specifically targeting real-world applications.

As an introduction to the concepts explored in the chapters to come, these methods allow for the pooling of different prediction algorithms (known as 'experts') with the goal of improving prediction accuracy—allowing the final prediction to be nearly as accurate as the best-performing expert.

This project will encompass several key areas, including:

- **Explaining the Theory of Prediction with Expert Advice.** To effectively experiment with different methods of Prediction with Expert Advice, the underlying theory must first be understood by conducting a review of the relevant literature.
- **Implementing the Aggregating Algorithm.** This project will primarily investigate the Aggregating Algorithm introduced by Vovk (see [1], [2]).
- **Handling Specialist Experts.** Introduced by Freund [3], *Specialist Experts* may refrain from making predictions at certain points, meaning that the Aggregating Algorithm has to be modified slightly [4].
- **Applying Prediction with Expert Advice to Real-World Data.** The methods described in this report will be applied to real-world datasets in order to evaluate their practicality outside of theoretical models, including an investigation into the perception of randomness by utilising specialist experts.

1.2 Motivation and Interest in the Subject Area (261)

The motivation for selecting a project in this subject area is rooted in both my personal and professional interests, as well as the discussions I had with my now-supervisor, Dr. Yuri Kalnishkan, before finalising my selection.

During this academic year, the module that most piqued my interest was CS5200 – On-line Machine Learning because I was interested in the techniques that allowed machine learning models to gradually improve over time as more data became available to them without the need to retrain the model on the entire newly-updated dataset; something that had not been

covered previously by other modules. Due to the module's small size and frequent absentees, I was able to gain a deeper understanding of the module, in large part due to Dr. Kalnishkan's willingness to explain portions of the syllabus in extreme detail. Alongside the lectures, I felt like I was strongly suited to the contents of the module because it has strong ties to the field of statistics – another area that I thoroughly enjoyed throughout my education.

Regarding my professional aspirations, I am set to begin my career later this year and I am of the firm belief that the work that I have done in this subject area is highly relevant, not only to the job I am to start in September, but also for my career plan due to its relevance across a variety of industries – including finance, energy, and insurance.

Ultimately, the combination of all of these factors led me to pursue a project investigating on-line prediction, and prediction with expert advice.

1.3 Structure of the Dissertation (262)

The dissertation is split into distinct chapters, each dedicated to exploring a specific aspect of the work. The following outline guides the reader through the report by providing a brief overview of the contents of each chapter.

Chapters 2 through 5 contain a literature review organised to explain the concepts that the practical portion of the dissertation aims to explore. Chapter 2 defines the problem of On-line Prediction, outlining the scenarios that it applies to, and the protocols that such problems follow. Additionally, it explores how on-line learning differs from traditional batch learning and defines concepts that will be critical to understanding the following sections. Chapter 3 introduces the problem of Prediction with Expert Advice, explaining its significance and applications in the real world, as well as exploring some algorithms that are used to solve such problems – including their theoretical bounds. Chapter 4 introduces the Aggregating Algorithm that this report is centred around, exploring how it differs from other methods of Prediction with Expert Advice. Chapter 5 focuses on Specialist Experts, defining what they are and how the base Aggregating Algorithm must be modified to accommodate them.

Chapter 6 contains the practical portion of the dissertation, explaining how the research problem was handled based on the concepts explored in the literature review, the findings from conducting the requirements analysis and design processes, and the results found when comparing an individual's idea of "random" to that of a random number generator.

Finally, Chapter 7 contains a conclusion which discusses the findings of the investigation as well as a self-evaluation of the project.

2 Literature Review

2.1 Introduction

Purpose: Overview of the goals of the literature review.
Scope: Outline of the topics covered and their relevance to the dissertation.

2.2 Perceived Randomness

2.2.1 Are Humans Good Randomisers?

The definition of the term "random" is a contentious area for debate. In essence, randomness is an unobservable characteristic of a generating process therefore the act of trying to define it is somewhat contradictory. To determine if a process or sequence is random, it has to be put through statistical tests for specific properties which are deemed to be "random". However, because these tests are statistical, are the conclusions drawn *objectively* random or purely *subjective*?

Research into the human perception and generation of random sequences is a common topic within psychological papers yet the contradictory nature of findings results in a less-than-satisfactory answer to the question of "Are humans good randomisers?" – much like when defining the term itself.

The origins of such a question can be traced back to an observation made by Hans Reichenbach in The Theory of Probability [5]; he suggested that when asked to produce a series that seemed random to them, people untrained in the theory of probability would be unable to generate such a series and, instead, generate one that would contain patterns and biases, e.g. too many alternations than what was expected. This ultimately suggests that humans are not good randomisers which is the prevalent opinion to date. This behaviour is attributed to the fact that human-generated sequences often reflect the underlying psychological tendencies of subjects, rather than the unpredictability of true randomness.

While Reichenbach assumes the stance that humans are not good randomisers, the alternative to this conclusion was put forward by the work of Bruce Ross [6]. Ross explores the processes involved in randomising binary sequences and analyses the methods that people used, as well as the typical mistakes that they made, when attempting to create random sequences. In his study, Ross got 60 subjects to stamp cards with either an '$\mathcal{O}$' or an '$\mathcal{X}$' and place them singly in a 100-item sequence in the middle of a table that they thought to be random, with item frequencies of either $50 - 50$,

$60 - 40$, or $70 - 30$. These sequences were then scored against the expected properties of a random sequence and, based on the analysis conducted, resulted in the conclusion that "the prevalent *a priori* assumption that the human being is a systematically biased randomi[s]er was not borne out" [6] and that "[subjects] who are instructed to construct a random series give a fairly good approximation of the expected number of alternations" [7]. This, however, is not sufficient to deem that humans are not systematically biased randomisers.

2.2.2 Judgement vs. Production of Random Binary Sequences

As alluded to by the question posed in the previous subsection, the human perception of randomness is a *subjective*, rather than *objective*, quality. Because of this subjectivity, it will vary from person to person and two natural conclusions can be drawn from this—either that people have an incorrect idea of what randomness is and what it should look like, or that people intuitively know what true randomness should look like, but there is some internal functional limitation that prevents the judgement and production of such sequences [8], being so powerful that individuals may choose to forego an available statistical analysis in favour of this 'gut feeling' [9].

Being that the topic of this dissertation is Prediction with Expert Advice, specifically in the scenario of η-mixable Games—a subject to be introduced in the following section—the primary focus of this literature review will be centred around experiments conducted to explore the judgement and production of random binary sequences. These two categories both make interesting observations about the internal mechanism responsible for the human perception of randomness, namely "that [humans] see clumps or streaks in truly random series and expect more alternation, or shorter runs, than are there", and that "[humans] produce series with higher than expected alternation rates" [9].

2.2.2.1 Judgement of Random Binary Sequences

We will first explore *judgement*. In Willem Wagenaar's study titled "Appreciation of conditional probabilities in binary sequences", Wagenaar examines how people perceive and interpret the likelihood of certain events occurring given previous outcomes revealing a disparity between what was perceived to be random and what was truly random, as well a systematic recency bias that affected subject's judgements of conditional probabilities [8]. The study controlled the conditional probability of a 0 after 0 (1 after 1) as the exper-

imental variable, testing it between the range $0.2 - 0.8$ with 0.1 increments, i.e. 7 values, for first-, second-, and third-orders of dependency. To test this, subjects were shown 16 sets of 7 binary sequences (each generated with one of the conditional probabilities in the range) for each order of dependency and were asked to select and record the sequence in each set that looked the most random to them—explained as the sequence that looked the most likely to be produced when flipping a fair coin.

For reference, in a truly random binary sequence, the conditional probability of 0 after 0 ($\Pr(0|0)$) or 1 after 1 ($\Pr(1|1)$) for the first order of dependency is 0.5. However, Wagenaar identified sequences with conditional probabilities close to 0.4 were the ones perceived as the most random across all orders of dependency, affirming the position that humans aren't good randomisers. This study also highlights the bias in favour of 'negative recency', more commonly known as the gambler's fallacy wherein gamblers will tend to bet on red after a run of blacks (and vice versa) on a roulette wheel. This observation ultimately caused subjects to favour series with slightly more alternations than is expected of true randomness causing Wagenaar to postulate that this is because subjects "cannot process such a mathematical quantity as 'conditional probability'... Rather, they will look at some other characteristics like, for instance, the run-structure of the sequence" [8].

2.2.2.2 Production of Random Binary Sequences

Having introduced the topic and a systematic bias that affects how humans judge sequences to be random, we can now delve into generation—the category that this project will explore.

Examining Paul Bakan's work titled "Response-Tendencies in Attempts to Generate Random Binary Series", Bakan aimed to "allow for another test of the hypothesis that [a subject] will generate more runs than chance predicts under conditions somewhat different from those reported by Ross" in that biases in motor operations (e.g., favouring to use their dominant hand) was avoided [7]. As stated by Bakan, the main findings of this study are that subjects "exhibit consistent patterns of responses" and "deviate from randomness by having too many alternations in the series" when trying to generate a random binary sequence—a conclusion supported by [10]: "humans-produced sequences have too few symmetries and long runs, too many alternations among events, and too much balancing of event frequencies over relatively short regions" which may be explained by the fact that a human's short-term memory roughly spans 7 (+/- 2) items that constitutes the "window" that people try to achieve representative randomness in [11].

Lastly, we explore the work of Raymond Nickerson and Susan Butler titled "On producing random binary sequences" which forms the basis of the experiment that this project will carry out. Nickerson and Butler's experiment varies from previous ones carried out in that, instead of getting subjects to produce a single sequence that would later be aggregated into a larger collective, they got subjects to produce a number of sequences while attempting to be random since they noted that "randomness does not reveal itself in any single short sequence; it reveals itself in sets of such sequences. Or at least it has a better opportunity to reveal itself in a set of sequences rather than in a single member of such a set." [12]. In their methodology, subjects were tasked with producing 100 10-item random sequences—explained as the sequences likely to be recorded if 100 individuals were asked to flip a fair coin 10 times each—that would be statistically indistinguishable from if an actual coin were to have been flipped. The notion behind this experiment was that, if subjects' perceptions of randomness were good, then subjects would be able to produce sets of sequences with properties (i.e., number of heads per sequence, number of runs per sequence, run lengths, frequency of alternations and repetitions) that fell within expected percentages. Because each subject was made to produce several sequences, their results provided a stronger justification for a human's ability as a randomiser since it allows subjects time to prove that they can act randomly. What Nickerson and Butler found was that, while subjects weren't any good at producing truly random sequences since "in the aggregate, the sets of sequences produced by our participants differed quantitatively from those expected of a random process, so our results can be seen as supporting the prevailing view that people are not very good randomi[s]ers", the distribution shape produced by the aggregate of participants' sequences were qualitatively similar—"not indistinguishable, but close" [12]—to what was expected (shown in the figures below), suggesting that humans can be effective randomisers when part of a group.

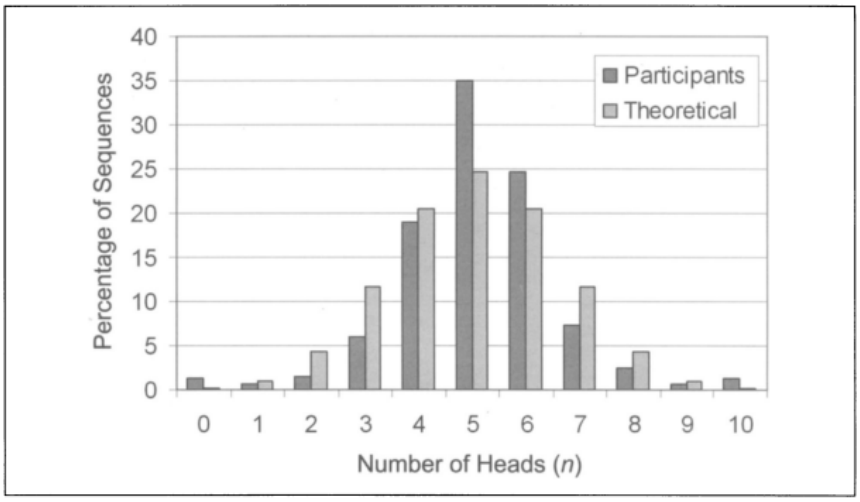

Figure 1: Percentage of 10-toss sequences with n heads, including the theoretical distribution, $X \sim B(10, 0.5)$, for comparison [12].

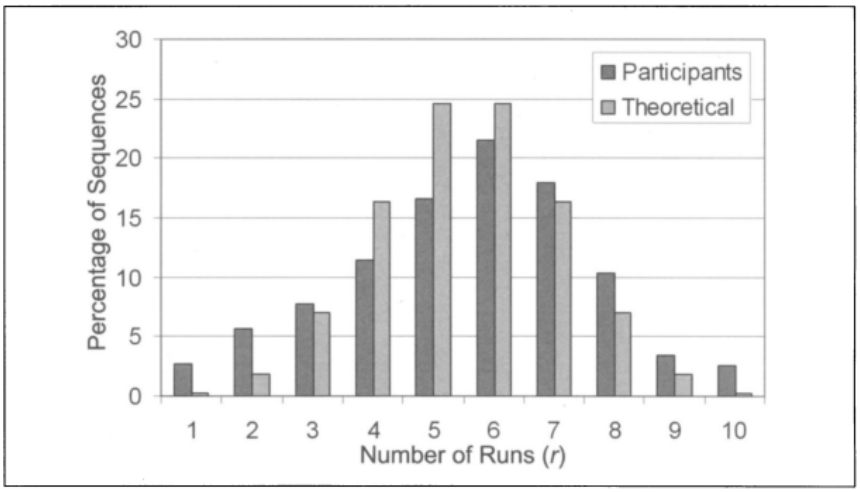

Figure 2: Percentage of all 10-toss sequences with r runs, including the theoretical distribution for comparison [12].

2.3 Prediction with Expert Advice

2.3.1 Introduction to On-line Prediction

Within the areas of Machine Learning and Statistics, there lies the problem of accurately "predicting future events based on past observations" [13] known as *on-line prediction*. This problem refers to methods where a model makes predictions sequentially and updates its parameters in real-time as new data points become available. There is a particular class of algorithm that is designed to tackle this, with one of the most notable being the "Strong" Aggregating Algorithm proposed by Volodymyr Vovk [1] which forms the basis of this study. The adjective "Strong" is emphasised with inverted commas to help distinguish the algorithm from the "Weak" Aggregating Algorithm proposed by Yuri Kalnishkan and Michael Vyugin [14] that will be touched upon but not explored in detail in this dissertation.

Given that the foundations of this dissertation lie firmly in this subject area, this section aims to lay a comprehensive foundation, exploring the key concepts and frameworks that will set the stage for the discussions in Chapter **TODO**.

On-line Prediction, Batch Learning and Timeseries Analysis

Herein the first distinction between on-line prediction and the traditional batch learning framework. With batch learning, a whole training set of labelled examples (x_i, y_i) is given to the learner at once in order to train a model. In contrast, on-line learning involves gradually feeding the learner information over time, requiring the model to continuously adapt to the new data it is given while requiring the learner to take actions on the basis of the information it already possesses instead of waiting for a complete picture. [4] This forced adaptability ensures that the predictions outputted by the algorithm remain accurate based on the information that the model deems as relevant as it gains additional knowledge, making these models particularly valuable in applications that require immediate responses and fluidity such as financial market analysis and weather forecasting.

Another distinction that needs to be made is between on-line prediction and timeseries analysis as, while these are both ways of handling sequential data in machine learning and statistics, they are unique. On-line learning is based on processing data points sequentially and updating predictive models in real-time whereas timeseries analysis is based on modelling and forecasting data that is collected over successive time intervals. The prior approach does not impose any strict assumptions about the underlying data-

generating process, even going so far as to not assume the existence of such a process [15], while the latter assumes a structured approach where observations are dependent on previous observations. These are typically modelled using stochastic processes such as *autoregressive integrated moving average (ARIMA)* or *state-space* models [16]. The majority of the literature on On-line Prediction takes a similar stance that no assumptions can be made about the sequence of outcomes that are observed. Because of this, the analyses are done over the worst-case and may be better in reality [13].

Notation

In on-line prediction, we consider a scenario where the elements of a sequence, known as ***outcomes***, ω_t occur at discrete times $\omega_1, \omega_2, \ldots$ which we assume to be drawn from a known ***outcome space*** Ω. In this problem, a learner is tasked with making ***predictions*** γ_t about these *outcomes* one at a time before they occur. Similarly, we assume that the learner's predictions are drawn from a known ***prediction space*** Γ which may or may not be the same as the *outcome space* Ω.

Once the learner has made their *prediction*, the true *outcome* is then revealed and the quality of the learner's prediction is assessed by a ***loss function*** $\lambda(\gamma_t, \omega_t)$. This function measures the discrepancy between the *prediction* and *outcome* or, more generally, quantifies the effect of when the *prediction* γ_t is confronted with the *outcome* ω_t [17] by mapping the input space $\Gamma \times \Omega$ to a subset of the real-number line $\mathbb{R}$, typically $[0, +\infty)$ [18].

Across several time steps T, the learner will suffer multiple losses which can be referred to as their cumulative loss up to time T. Their performance is measured by this cumulative loss, so their natural objective is to suffer as low a cumulative loss as they can.

Protocol 1 On-line Prediction Framework

1: FOR $t = 1, 2, \ldots$
2: learner L outputs $\gamma_t \in \Gamma$
3: nature outputs $\omega_t \in \Omega$
4: learner L suffers loss $\lambda(\gamma_t, \omega_t)$
5: END FOR

2.3.1.1 Games and Mixability

The combination of a *prediction space*, *outcome space*, and *loss function* can be referred to with a triple $< \Gamma, \Omega, \lambda >$, known as a ***Game*** G. *TODO: Explain mixability and touch on (Kalnishkan & Vyugin, 2008)* [14]

2.3.2 Prediction with Expert Advice

Framework: Description of the prediction with expert advice framework.
Mechanisms: Detailed explanation of how this framework operates.

Protocol 2 Prediction with Expert Advice Framework

FOR $t = 1, 2, \ldots$
    experts $E_1, \ldots, E_N$ output predictions$\gamma_t^1, \ldots, \gamma_t^N \in \Gamma$
    learner L outputs $\gamma_t \in \Gamma$
    nature outputs $\omega_t \in \Omega$
    experts $E_1, \ldots, E_N$ suffer losses $\lambda(\gamma_t^1, \omega_t), \ldots, \lambda(\gamma_t^N, \omega_t)$
    learner L suffers loss $\lambda(\gamma_t, \omega_t)$
END FOR

2.3.3 Aggregating Algorithm (AA)

Algorithm Description: Introduction to the Aggregating Algorithm.
Functionality: How the Aggregating Algorithm works in practice.

Algorithm 1 Aggregating Algorithm (AA)

initialise weights $w_0^i = q_i, i = 1, 2, \ldots, N$
FOR $t = 1, 2, \ldots$
    read the experts' predictions $\gamma_t^i, i = 1, 2, \ldots, N$
    normalise the experts' weights $p_{t-1}^i = w_{t-1}^i / \sum_{j=1}^N w_{t-1}^j$
    output $\gamma_t \in \Gamma$ that satisfies the inequality for all $\omega \in \Omega$:
        $\lambda(\gamma_t, \omega) \leq -\frac{C}{\eta} \ln \sum_{i=1}^N p_{t-1}^i e^{-\eta \lambda(\gamma_t^i, \omega)}$
    observe the outcome ω_t
    update the experts' weights $w_t^i = w_{t-1}^i e^{-\eta \lambda(\gamma_t^i, \omega_t)}, i = 1, 2, \ldots, N$
END FOR

$$\mathrm{Loss}_T(L) \leq C \cdot \mathrm{Loss}_T(\mathcal{E}_i) + \frac{C}{\eta} \ln \frac{1}{q_i} \tag{1}$$

2.3.4 Aggregating Algorithm for Specialist Experts (AASE)

Having introduced the Aggregating Algorithm in its base form, we can now discuss the modification that this paper's experiment will be centred around:

the Aggregating Algorithm for Specialist Experts.

The use of the term 'specialist' was first introduced by the work of Avrim Blum [19] for the Winnow and Weighted-Majority algorithms, and can be thought of as a natural extension to traditional experts insofar as it enables these 'specialists' to abstain from making a prediction "when the current expert does not fall into their 'specialty'". While the criteria for an expert to abstain from making a prediction is sufficient in our context, it can also be extended to allow for other scenarios like those suggested in [20], namely if "a prediction algorithm [sees] that its internal confidence is low and [decides] to skip a turn in order to re-train" or if a prediction algorithm breaks down, as would be the case if a regression algorithm "[has] its matrix very close to singular."

In order to accommodate these specialist experts, the Prediction with Expert Advice Framework given in (1) has to be modified as follows:

Protocol 3 Modified Prediction with Expert Advice Framework

FOR $t = 1, 2, \ldots$
    nature chooses a subset of experts $\mathcal{E}_i \in \mathcal{E}$ that are awake
    awake experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ output predictions $\gamma_t^1, \ldots, \gamma_t^N \in \Gamma$
    learner L outputs $\gamma_t \in \Gamma$
    nature outputs $\omega_t \in \Omega$
    awake experts $\mathcal{E}_1, \ldots, \mathcal{E}_N \in \mathcal{E}_i$ suffer losses $\lambda(\gamma_t^1, \omega_t), \ldots, \lambda(\gamma_t^N, \omega_t)$
    learner L and sleeping experts $\mathcal{E}_j \notin \mathcal{E}_i$ suffers loss $\lambda(\gamma_t, \omega_t)$
END FOR

As referenced above, another colloquial way of referring to 'specialist experts' is 'sleeping experts'; Freund postulated that "a specialist is awake when it makes a prediction and that it is asleep otherwise", going so far as to refer to the traditional on-line prediction framework as "the insomniac framework since it is a special case in which all specialists are awake all the time." [3] This colloquialism is useful when adapting the bounds of the base Aggregating Algorithm because a natural interpretation of what happens when an expert is sleeping is that it simply "joins the crowd" [20], meaning that it mimics the learner's prediction on the time steps that it is asleep because the learner's prediction is formed based on the weighted majority of experts' predictions. Given this definition, it can be seen that on some time steps t, the learner's prediction and the expert $\mathcal{E}_i$'s predictions are the same; $\gamma_t = \gamma_t^i$. Recall that, in the mixable case, the Aggregating Algorithm

guarantees that the following inequality is satisfied:

$$\sum_{t=1}^{T}\lambda(\gamma_t,\omega_t) \le \sum_{t=1}^{T}\lambda(\gamma_t^i,\omega_t) + \frac{1}{\eta}\ln\frac{1}{q_i}$$

Typically, the Aggregating Algorithm's performance is measured in terms of the learner's cumulative loss compared to the best expert's cumulative loss but given that, on certain time steps t, $gamma_t = \gamma_t^i$, it is clear that the corresponding terms in both sums cancel out and what is left are the sums over the time steps where the learner's and the expert's predictions are different, i.e. where expert $\mathcal{E}_i$ is awake. What follows from this is that, instead of wanting the learner's loss to be nearly as good as the best expert's loss over a period of time T, we judge the Aggregating Algorithm for Specialist Experts' performance based on the learner's loss compared to the best expert's $\mathcal{E}_i$ loss over the steps in which it was awake. A learner following the algorithm achieves a cumulative loss that satisfies the following inequality:

$$\sum_{\substack{t=1,2,\ldots,T:\\ \mathcal{E}_i \text{ is awake}\\ \text{on step } t}}^{T} \lambda(\gamma_t,\omega_t) \le C\cdot \sum_{\substack{t=1,2,\ldots,T:\\ \mathcal{E}_i \text{ is awake}\\ \text{on step } t}}^{T} \lambda(\gamma_t^i,\omega_t) + \frac{C}{\eta}\ln\frac{1}{q_i} \tag{2}$$

As is the case for the traditional Aggregating Algorithm, we make no assumptions about the outcome-generating mechanism (including the existence of such a mechanism) and this bound holds for *any* adversarial strategy, meaning that the adversary cannot inflict a large loss on the learner without inflicting a large loss on the specialists and ensuring that the performance will be good whenever there is a good mixture of specialists.

Algorithm 2 Aggregating Algorithm for Specialist Experts (AASE)

initialise weights $w_0^i = q_i, i = 1,2,\ldots,N$
FOR $t = 1,2,\ldots$
    read the awake experts' predictions $\gamma_t^i, i = 1,2,\ldots,N$
    normalise the awake experts' weights
        $p_{t-1}^i = w_{t-1}^i / \sum_{j:\mathcal{E}_j \text{ is awake}} w_{t-1}^j$
    output $\gamma_t \in \Gamma$ that satisfies the inequality for all $\omega \in \Omega$:
        $\lambda(\gamma_t,\omega) \le -\frac{C}{\eta}\ln\sum_{i:E_i \text{ is awake}} p_{t-1}^i e^{-\eta\lambda(\gamma_t^i,\omega)}$
    observe the outcome ω_t
    update the awake experts' weights $w_t^i = w_{t-1}^i e^{-\eta\lambda(\gamma_t^i,\omega_t)}$
    update the sleeping experts' weights $w_t^i = w_{t-1}^i e^{-\eta\lambda(\gamma_t,\omega_t)/C(\eta)}$
END FOR

2.4 Conclusions

Summary: Recap of key points covered in the literature review.
Implications: Implications of the reviewed literature for the current study.

3 Experiment Design and Methodology

3.1 Introduction

As discussed in Chapter 2.2.2.2, the experimental design used in this study closely aligns with the methodology established by Nickerson and Butler [12], serving as the foundational basis for this experiment. By adapting their established methodology to the current study and its novel application, this project not only compares the sequences generated by subjects with those expected from a random process according to the statistical definition, but also evaluates the predictability of each subject's responses using Prediction with Expert Advice and Vovk's Aggregating Algorithm [1]. In theory, the more random a subject's sequence, the less predictable their responses, leading to greater losses for both the Learner and each of the relevant Experts.

3.2 Experimental Design

This chapter outlines the experimental design, which aims to assess how well individuals can generate random binary sequences when compared to theoretical randomness and their previous inputs. The study employs a within-subject design, where each participant is exposed to all conditions of the independent variable which, in his case, means they are required to repeat the generation process several times. This approach provides a comprehensive view of each individual's performance in generating random sequences.

The independent variable in this experiment is the method by which the participants generate their binary sequences, as each participant is allowed to enter their sequences independently of one another. The dependent variables include the frequencies of 0s and 1s in each sequence, the number and length of runs within each sequence, and the predictions generated by the Experts and the Learner, which will be discussed in further detail in the following sections. The control variables of this study include the instructions given to the subject before beginning the experiment, the length of each binary sequence inputted, and the total number of sequences entered. These control variables were devised with the intention of facilitating a more accurate comparison and analysis of the collected data between participants.

This study's subjects primarily consisted of postgraduate students from the Computer Science Department at Royal Holloway, University of London, with additional participants from the Psychology Department to create a more representative sample. Each sample was tasked with generating

several 10-item sequences intended to mimic the results expected from a random process. These sequences were entered into a web application hosted on GitHub Pages. Each 10-item sequence consisted of 0s and 1s (representing Heads and Tails) arranged in any order that the subject chose, and participants were allowed to enter the sequences at their own pace. Prior to beginning the experiment, subjects were presented with the following instructions on a modal screen shown upon loading the web page:

> *Your task is to create a table of sequences each consisting of 10 items, either 0 or 1.*
>
> *Imagine that several people have each tossed a fair coin 10 times and the results of their tosses are recorded in a table, with each row recording the outcomes of the 10 tosses by one person.*
>
> *Your goal is to produce this table in such a way that if compared with a table of the results of actual coin tosses, it would not be possible to distinguish which table represented the actual coin tosses with statistical tests and which didn't.*

Herein lies the first divergence from Nickerson and Butler's original design because the sequences entered by the subjects are always displayed and were concatenated into a single, continuous sequence (as shown in Figure 3), which is then passed to the Aggregating Algorithm as ωs. In the original experiment, the sequence would only remain visible to the subject until they had entered 10 items, at which point it would disappear. The modification in this study allows the Aggregating Algorithm to better identify patterns in the user's inputs as an interval length of 10-bits would be insufficient in allowing the algorithm to determine which Experts should be given higher weighting in forming the Learner's predictions, thereby improving the learner's prediction accuracy. While the underlying algorithm treats the sequence differently to [12], the sequences are still presented to the subject in 10-item chunks (as shown in Figure 4), consistent with the original method, to better align with the human short-term memory span of approximately 7 $\pm$ 2 items cited in 2.2.2.2. This chunking allows subjects to quickly review their previous inputs and continue generating sequences that they perceive as random.

Given this foundation, we can now discuss how the Aggregating Algorithm for Specialist Experts (AASE) was applied to this experiment.

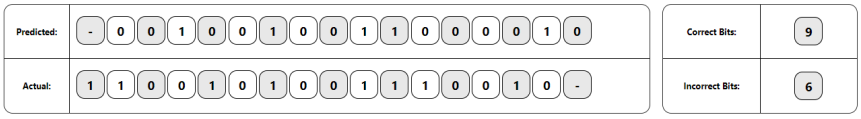

Figure 3: Inputted Sequence Displayed in the Web Application

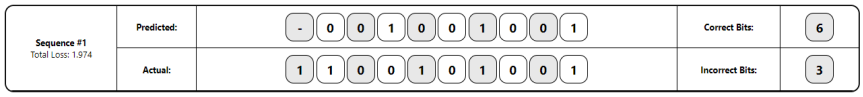

Figure 4: Past Sequences Displayed in the Web Application.

3.3 Applying the Aggregating Algorithm

The Aggregating Algorithm, as well as the broader framework of Prediction with Expert Advice, are integral to this study. To apply these methods to this scenario, we must first define the η-mixable game $G = \langle \Gamma, \Omega, \lambda \rangle$, as defined in Protocol 2 – Prediction with Expert Advice Framework. As suggested by Chapter 2, the primary focus of this project is Prediction with Expert Advice for the Discrete Binary Game which is formally defined by the triple consisting of the ***outcome space*** $\Omega = \{0, 1\}$, the ***prediction space*** $\Gamma = [0, 1]$, and the ***loss function*** given by Brier's (or Square) Loss, $\lambda_{\text{SQ}}(\gamma_t, \omega_t) = (\gamma_t - \omega_t)^2$. In practical terms, this means that Nature generates binary outcomes, either 0 or 1, while both the Learner and the Experts predict values within the range $[0, 1]$.

With the game formally established, it is essential to define the roles of Nature, the Learner and the Experts in our experimental context. In this project, which assesses a subject's ability to generate random binary sequences, one of the key metrics is the predictability of their inputs prior to pressing a key. Therefore, each subject naturally assumes the role of Nature, while the Aggregating Algorithm functions as the Learner attempting to pre-empt Nature. As the experiment involves multiple subjects, this supports the assertion made previously that "we make no assumptions about the outcome-generating mechanism (including the existence of such a mechanism)." Each subject possesses a unique internal concept of randomness, and the Aggregating Algorithm must perform adequately across all participants and all sequences.

Finally, the concept of an Expert, as well as the rationale for making use of Specialist Experts and the Aggregating Algorithm for Specialist Experts, must also be defined. For our purposes, an Expert can be thought of as a

function designed to predict the next outcome in a sequence based on the presence of a specific scenario. Since the experiment evaluates each subject's concept of randomness by statistically analysing conditional probabilities for different orders of dependency, it is natural to conceptualise the group of Experts as functions that search for specific prefixes within the sequence. As the subject inputs their sequence to the application, the last n bits are passed to the group of Experts, who then assess whether the sequence currently falls within their "area of expertise", i.e., whether the last n bits match the prefix that they are searching for.

Given that not every prefix will be relevant to each new subject input, the use of Specialist Experts is justified. If the current sequence does not match a Specialist's prefix, and thus "area of expertise", that Specialist is considered "asleep" and abstains from making a prediction, "joining the crowd". Conversely, Specialists whose prefixes match the sequence's ending are considered "awake" and make their predictions accordingly. It is important to note that there will never be a scenario where all Specialists are asleep, with at least one being awake at all timestamps, as there is always an Expert searching for the last $1 - n$ bits of the sequence.

Finally, the details of how an Expert functions within this experiment must be explained. In order to generate the Experts, binary sequences up to length 5 were generated and assigned to individual experts. The decision to limit the prefix length to 5-bits balances both computational efficiency with predictive accuracy, reflecting the established constraints of the human short-term memory typically spanning 7 ± 2 items. An Expert tracks the frequency of 0s and 1s following its specific prefix and makes predictions based on the ratio $\#1/(\#0 + \#1)$. As subjects type their sequences, each Expert checks the last x bits (corresponding to the length of their prefix) to determine whether they are awake. If so, the Specialist makes a prediction, which is then fed into the Aggregating Algorithm to inform the Learner's prediction for the next time stamp. For the sake of transparency, each Expert's last prediction, current prediction and status are displayed at the bottom of the application for user's reference if the subject is interested.

3.4 Data Analysis

After the experiment was conducted, the data was aggregated and analysed using various methods. The primary method was a chi-square goodness-of-fit test, comparing each subject's generated sequences to the distribution expected from a random process. To evaluate the Learner's performance, a secondary analysis was conducted, comparing both the cast (rounded) and

uncast (unrounded) predictions against the actual outcomes.

Given the experiment involves playing Brier's Game with $\Omega = \{0, 1\}$ and $\Gamma = [0, 1]$, the simplest strategy for the Learner would be to predict $1/2$ for every timestamp, which is the minimax prediction, resulting in a loss of $1/4$ each time. Over 10 steps, the maximum loss for the Learner would be $10/4 = 2.5$, which serves as a benchmark for assessing the Aggregating Algorithm's performance – loss less than 2.5 indicates that a Learner using the Aggregating Algorithm predicts better than this naive strategy.

These methods of analysis are, in fact, somewhat adversarial, as the more statistically random a subject's input is, the more challenging it should become for the Experts, and thus the Learner, to make accurate predictions. Consequently, a better fit to statistical randomness should result in poorer Learner performance as the subject's sequences should exhibit no discernable patterns.

3.5 Procedure

This chapter outlines the step-by-step procedure followed to conduct the experiment based on what was outlined in the previous chapters.

1. Participants were selected from the student community at Royal Holloway, University of London. Those recruited included postgraduate students from the Computer Science Department, as well as students from the Psychology Department, in order to create a diverse sample.

2. Participants were directed to the application hosted on GitHub Pages designed to collect and analyse generated binary sequences. Its development and the issues faced will be outlined in a following chapter.

3. During the experiment, subjects were tasked with generating several 10-item binary sequences by entering 0s and 1s into the application such that the results would appear random if subjected to statistical tests. For transparency, the application displayed the internal workings of each Expert at the bottom of the screen, though participants could disable this feature if they desired.

4. After finishing the experiment, the subjects were asked to email their results to Andrew.Barraclough.2018@live.rhul.ac.uk to be subjected to the methods outlined in the previous section, namely chi-square goodness-of-fit and comparison to the loss inflicted by following the naive strategy.

4 Analysis of Perceived Randomness

Data Presentation: Presentation of collected data in an organised manner.
Analytical Techniques: Methods used to analyse the data.
Results: Detailed presentation of findings.
Discussion: Interpretation of results in the context of perceived randomness.
Comparison with Literature: How the findings align or differ from existing research.

5 Conclusions *(1,500)*

Summary of Findings: Recap of the main findings of the study.
Contributions: Discussion on the contributions of the study to the field.
Limitations: Identification of any limitations encountered during the research.
Future Work: Suggestions for future research based on the findings and limitations of this study.

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