Aggregating Algorithm

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Declaration

This report has been prepared on the basis of my own work. Where other published and unpublished source materials have been used, these have been acknowledged.

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Signature:

Abstract

Your abstract goes here.

Contents

Acknowledgements

While the contents of this report are on the basis of my own work, none of this would have been possible without the patience and mentorship of my supervisor Dr. Yuri Kalnishkan to whom I am extremely grateful. It was your advice, clear explanations, and expertise that made this project what it is now and something that I am incredibly proud of. I would also like to express my gratitude to the group of friends who made this academic year possible, namely Cougar Tasker, Einstein Ebereonwu, Hayden Amarr, Mohammadreza Yazdian, Niraj Jain, and Ray Mahbub, without whom I would have struggled to maintain my discipline and motivation.

1 Introduction

1.1 Project Scope and Objectives

The aim of this project is to implement the Aggregating Algorithm for Specialist *(Sleeping)* Experts, a method of Prediction with Expert Advice, to scenarios involving human-generated sequences and evaluate its predictive performance in pre-empting what the human subject will input, effectively testing how good human subjects are at generating random inputs.

As an introduction to the concepts that will be explored in the sections to come, this algorithm allows for the effective pooling of different prediction algorithms, known as 'Experts', in order to improve the algorithm's prediction accuracy. By aggregating several predictions, this allows the final prediction outputted by the algorithm to be nearly as accurate as the best-performing Expert.

This project will encompass several key areas, including:

- **Explaining the Theory of Perceived Randomness.** The basis of this study revolves around the human perception of randomness, which is different to objective randomness, therefore the underlying psychological mechanisms for how humans judge and perceive randomness must be understood.

- **Explaining the Theory of Prediction with Expert Advice.** The other portion of this study is firmly based in the subject matter of Prediction with Expert Advice, primarily focussing on the Aggregating Algorithm and the Aggregating Algorithm for Speciallist Experts so the underlying theory must be explored with a thorough review of the current literature.

- **Implementing the Aggregating Algorithm.** This project will primarily investigate the Aggregating Algorithm introduced by Vovk (see [1], [2]).

- **Handling Specialist Experts.** Introduced by Freund [3], *Specialist Experts* may refrain from making predictions at certain points, meaning that the Aggregating Algorithm has to be modified slightly [4].

- **Evaluating the Performance of Human Subjects in Generating Statistically Random Seuqences.** Through conducting the experiment outlined in this paper, this study will present how well the subjects were able to generate a "random" sequence when compared to the statistical definition used by statisticians.

- **Evaluating the Performance of the Algorithm in Predicting Human-Generated Outcomes.** The experiment will also compare how well the Aggregating Algorithm for Specialist Experts was able to pre-empt each subject's sequences, in a somewhat adversarial comparison to statistical randomness.

1.2 Motivation and Interest in the Subject Area

The motivation for selecting a project in this subject area is rooted in both my personal and professional interests, as well as the discussions I had with my now-supervisor, Dr. Yuri Kalnishkan, before finalising my selection.

During this academic year, the module that most piqued my interest was CS5200 – On-line Machine Learning because I was interested in the techniques that allowed machine learning models to gradually improve over time as more data became available to them without the need to retrain the model on the entire newly-updated dataset; something that had not been covered previously by other modules. Due to the module's small size and frequent absentees, I was able to gain a deeper understanding of the module, in large part due to Dr. Kalnishkan's willingness to explain portions of the syllabus in extreme detail. Alongside the lectures, I felt like I was strongly suited to the contents of the module because it has strong ties to the field of statistics – another area that I thoroughly enjoyed throughout my education.

Regarding my professional aspirations, I am set to begin my career later this year and I am of the firm belief that the work that I have done in this subject area is highly relevant, not only to the job I am to start in September, but also for my career plan due to its relevance across a variety of industries – including finance, energy, and insurance.

Ultimately, the combination of all of these factors led me to pursue a project investigating on-line prediction, and prediction with expert advice.

1.3 Structure of the Dissertation

The following study is split into several distinct chapters and subsections, each dedicated to exploring a specific aspect behind the study being conducted. The following outline guides you, the reader, through the report by providing a brief overview of the contents of each chapter.

Chapter 2 contains the Literature Review that is organised to explain the underlying theory that the practical portion of this study aims to investigate.

Section 2.2 explores the human perception of randomness and how it is, in fact, different from the objective definition of randomness, and aims to provide context to answer the question “Are humans good randomisers?” in both judgement and generation scenarios.

Section 2.3 is diverse in its contents. Subsection 2.3.1 defines the problem of On-line Prediction, outlining the scenarios in which it is applicable, and the protocols that such problems follow. Additionally, it explores how On-line Prediction differs from the traditional Machine Learning frameworks of Batch Learning and Timeseries Analysis and defines concepts that will be critical to understanding the experiment being conducted. Subsection 2.3.2 expands on the information presented in the previous subsection by defining the modifications to the original framework to now incorporate the predictions made by a pool of Experts and how the Learner aggregates those to inform its own. It also gives two examples of merging strategies that could be used to accomplish that task. Subsection 2.3.3 introduces the algorithm that is the main focus of this study, delving into the rationale behind the algorithm and discussing the bounds guaranteed by utilising it. Lastly, Subsection 2.3.4 defines a modification to the original Aggregating Algorithm that allows it to be used in scenarios involving Specialist Experts, a term used to define any Expert that has the ability to abstain from making a prediction.

Chapter 3 is centred on the practical applications of the theory and how it was used to derive the experiment that is to be conducted, including how the experiment was designed, how the Aggregating Algorithm was applied to the proposed scenario and how the data gathered from the subjects will be analysed.

Chapter 4 will discuss the findings of the study in detail, ultimately attempting to answer the hypothesied question “are humans good randomisers?”.

Finally, Chapter 5 contains a conclusion that summarises the findings of the study, as well as a self-evaluation of the project.

2 Literature Review

2.1 Introduction

Through exploring the human capacity for judging and generating random binary sequences, this Literature Review will examine a variety of studies that delve into the complex, and somewhat subconscious, relationship between what a human perceives as random and what is truly random according to the statistical definition.

The term 'random" inherently means something that is happening by chance with no cause or reason, however, the way in which statisticians measure randomness is through statistical tests, ultimately meaning that there is some measurable quantity to determine whether something conforms to what we objectively consider random. That being said, humans have a subjective view on randomness which may deviate significantly from what is considered objective randomness. The discrepancy between objective and subjective randomness forms the basis of this study, particularly in the context of whether humans are capable of producing sequences that are statistically indistinguishable from those that are expected of a random process.

A fundamental debate that is found throughout the literature reviewed in this study is whether humans, when asked to perceive and generate random sequences, can truly do so without identifying "patterns" that happen occur by coincidence regarding judgement tasks, or without introducing subconscious patterns and biases regarding production tasks. Early research conducted by Reichenbach, with subsequent studies performed by Ross, Wagenaar, Nickerson (to name a few), highlight a complex and often contradictory knowledge base with some of their findings indicating that humans are inherently poor at identifying and generating random sequences, while others suggest the opposite in that humans might approximate randomness better than previously thought.

This review does not aim to provide a definitive answer to the question, however, it does synthesise the current understanding of how humans perceive randomness, as well as connects these findings to the broader context of prediction algorithms—by understanding the limitations of human-generated randomness, insights can be gained into how these subconscious biases might cause algorithms specifically designed for prediction in scenarios with incomplete information, such as the Aggregating Algorithm, to perform better than would be expected of a truly random process. This hypothesis forms the basis of the experimental work performed in this study.

2.2 Perceived Randomness

2.2.1 Are Humans Good Randomisers?

The definition of the term "random" is a contentious area for debate. In essence, randomness is an unobservable characteristic of a generating process therefore the act of trying to define it is somewhat contradictory. To determine if a process or sequence is random, it has to be put through statistical tests for specific properties which are deemed to be "random". However, because these tests are statistical, are the conclusions drawn *objectively* random or purely *subjective*?

Research into the human perception and generation of random sequences is a common topic within psychological papers yet the contradictory nature of findings results in a less-than-satisfactory answer to the question of "Are humans good randomisers?" – much like when defining the term itself.

The origins of such a question can be traced back to an observation made by Hans Reichenbach in The Theory of Probability [5]; he suggested that when asked to produce a series that seemed random to them, people untrained in the theory of probability would be unable to generate such a series and, instead, generate one that would contain patterns and biases, e.g. too many alternations than what was expected. This ultimately suggests that humans are not good randomisers which is the prevalent opinion to date. This behaviour is attributed to the fact that human-generated sequences often reflect the underlying psychological tendencies of subjects, rather than the unpredictability of true randomness.

While Reichenbach takes the stance that humans are not good randomisers, the alternative to this conclusion was put forward by the work of Bruce Ross [6]. Ross explores the processes involved in randomising binary sequences and analyses the methods that people used, as well as the typical mistakes that they made, when attempting to create random sequences. In his study, Ross got 60 subjects to stamp cards with either an '$\mathcal{O}$' or an '$\mathcal{X}$' and place them singly in a 100-item sequence that they thought to be random in the middle of a table, with item frequencies of either $50-50$, $60-40$, or $70-30$. These sequences were then scored against the expected properties of a random sequence and, based on the analysis conducted, resulting in "the prevalent *a priori* assumption that the human being is a systematically biased randomi[s]er [not being] borne out" [6] and that "[subjects] who are instructed to construct a random series give a fairly good approximation of the expected number of alternations" [7]. This, however, is not sufficient to deem that humans are not systematically biased randomisers.

2.2.2 Judgement vs. Production of Random Binary Sequences

As alluded to by the question posed in the previous subsection, the human perception of randomness is a *subjective*, rather than *objective*, quality. Because of this subjectivity, it will vary from person to person and two natural conclusions can be drawn from this—either that people have an incorrect idea of what randomness is and what it should look like, or that people intuitively know what true randomness should look like, but there is some internal functional limitation that prevents the judgement and production of such sequences [8], being so powerful that individuals may choose to forego an available statistical analysis in favour of this 'gut feeling' [9].

Being that the topic of this dissertation is Prediction with Expert Advice, specifically in the scenario of η-mixable Games—a subject to be introduced in the following section—the primary focus of this literature review will be centred around experiments conducted to explore the judgement and production of random binary sequences. These two categories both make interesting observations about the internal mechanism responsible for the human perception of randomness, namely "that [humans] see clumps or streaks in truly random series and expect more alternation, or shorter runs, than are there", and that "[humans] produce series with higher than expected alternation rates" [9].

2.2.2.1 Judgement of Random Binary Sequences

We will first explore *judgement*. In Willem Wagenaar's study titled "Appreciation of conditional probabilities in binary sequences", Wagenaar examines how people perceive and interpret the likelihood of certain events occurring given previous outcomes revealing a disparity between what was perceived to be random and what was truly random, as well a systematic recency bias that affected subject's judgements of conditional probabilities [8]. The study controlled the conditional probability of a 0 after 0 (1 after 1) as the experimental variable, testing it between the range $0.2 - 0.8$ with 0.1 increments, i.e. 7 values, for first-, second-, and third-orders of dependency. To test this, subjects were shown 16 sets of 7 binary sequences (each generated with one of the conditional probabilities in the range) for each order of dependency and were asked to select and record the sequence in each set that looked the most random to them—explained as the sequence that looked the most likely to be produced when flipping a fair coin.

For reference, in a truly random binary sequence, the conditional probability of 0 after 0 ($\Pr(0|0)$) or 1 after 1 ($\Pr(1|1)$) for the first order of dependency is 0.5. However, Wagenaar identified sequences with conditional probabilities close to 0.4 were the ones perceived as the most random across all orders of dependency, affirming the position that humans aren't good randomisers. This study also highlights the bias in favour of 'negative recency', more commonly known as the gambler's fallacy wherein gamblers will tend to bet on red after a run of blacks (and vice versa) on a roulette wheel. This observation ultimately caused subjects to favour series with slightly more alternations than is expected of true randomness causing Wagenaar to postulate that this is because subjects "cannot process such a mathematical quantity as 'conditional probability'... Rather, they will look at some other characteristics like, for instance, the run-structure of the sequence" [8].

2.2.2.2 Production of Random Binary Sequences

Having introduced the topic and a systematic bias that affects how humans judge sequences to be random, we can now delve into generation—the category that this project will explore.

Examining Paul Bakan's work titled "Response-Tendencies in Attempts to Generate Random Binary Series", Bakan aimed to "allow for another test of the hypothesis that [a subject] will generate more runs than chance predicts under conditions somewhat different from those reported by Ross" in that biases in motor operations (e.g., favouring to use their dominant hand) was avoided [7]. As stated by Bakan, the main findings of this study are that subjects "exhibit consistent patterns of responses" and "deviate from randomness by having too many alternations in the series" when trying to generate a random binary sequence—a conclusion supported by [10]: "humans-produced sequences have too few symmetries and long runs, too many alternations among events, and too much balancing of event frequencies over relatively short regions" which may be explained by the fact that a human's short-term memory roughly spans 7 (+/- 2) items that constitutes the "window" that people try to achieve representative randomness in [11].

Lastly, we explore the work of Raymond Nickerson and Susan Butler titled "On producing random binary sequences" which forms the basis of the experiment that this project will carry out. Nickerson and Butler's experiment varies from previous ones carried out in that, instead of getting subjects to produce a single sequence that would later be aggregated into a larger collective, they got subjects to produce a number of sequences while attempting to be random since they noted that "randomness does not reveal itself in any single short sequence; it reveals itself in sets of such sequences. Or at least it has a better opportunity to reveal itself in a set of sequences rather than in a single member of such a set." [12]. In their methodology, subjects

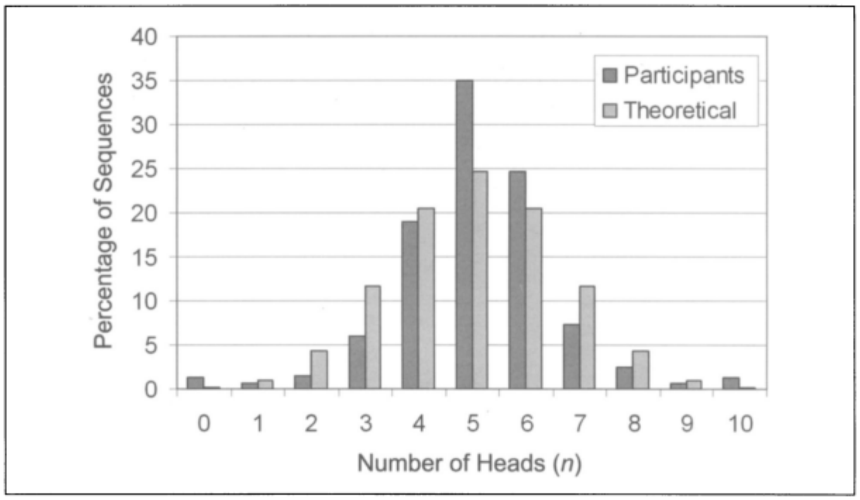

Figure 1: Percentage of 10-toss sequences with n heads, including the theoretical distribution, $X \sim B(10, 0.5)$, for comparison [12].

were tasked with producing 100 10-item random sequences—explained as the sequences likely to be recorded if 100 individuals were asked to flip a fair coin 10 times each—that would be statistically indistinguishable from if an actual coin were to have been flipped. The notion behind this experiment was that, if subjects' perceptions of randomness were good, then subjects would be able to produce sets of sequences with properties (i.e., number of heads per sequence, number of runs per sequence, run lengths, frequency of alternations and repetitions) that fell within expected percentages. Because each subject was made to produce several sequences, their results provided a stronger justification for a human's ability as a randomiser since it allows subjects time to prove that they can act randomly. What Nickerson and Butler found was that, while subjects weren't any good at producing truly random sequences since "in the aggregate, the sets of sequences produced by our participants differed quantitatively from those expected of a random process, so our results can be seen as supporting the prevailing view that people are not very good randomi[s]ers", the distribution shape produced by the aggregate of participants' sequences were qualitatively similar—"not indistinguishable, but close" [12]—to what was expected (shown in the figures below), suggesting that humans can be effective randomisers when part of a group.

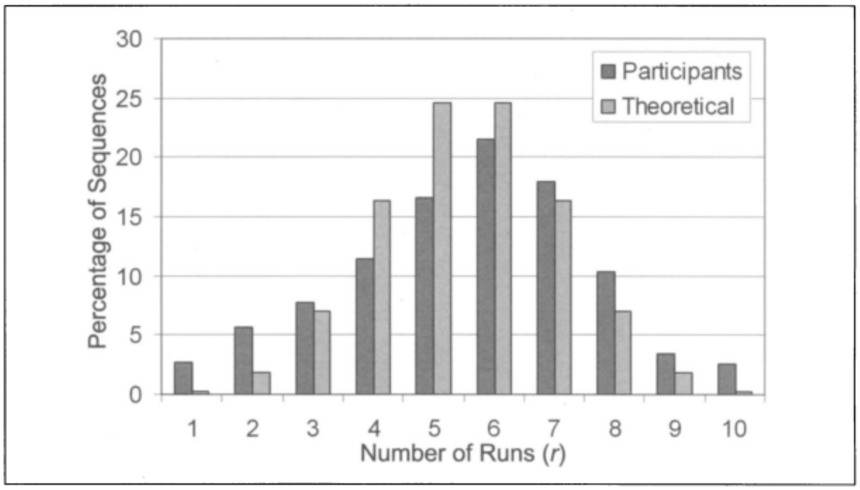

Figure 2: Percentage of all 10-toss sequences with r runs, including the theoretical distribution for comparison [12].

2.3 Prediction with Expert Advice

While understanding Perceived Randomness highlights the limitations of human intuition in recognising and generating random sequences, these insights naturally extend to the domain of prediction, particularly in scenarios where decisions must be made with some level of uncertainty. Just as individuals struggle to perceive randomness accurately, the issue may extend to how they generate "random" sequences, accidentally creating predictable patterns that may emerge as a result of subconscious biases they are unaware of. This is where the concept of Prediction with Expert Advice becomes relevant. The following sections will explore the mechanics of On-line Prediction and Prediction with Expert Advice, then introduce two methods used to combine the advice from a pool of Experts in order to improve prediction accuracy.

2.3.1 Introduction to On-line Prediction

Within the areas of Machine Learning and Statistics, there lies the problem of accurately "predicting future events based on past observations" [13] known as On-line Prediction—the framework for which is displayed below. This problem refers to methods where a model makes predictions sequentially and updates its parameters in real-time as new data points become available.

There is a particular class of algorithm that is designed to tackle this, with one of the most notable being the "Strong" Aggregating Algorithm proposed by Volodymyr Vovk [1] which forms the basis of this study. The adjective "Strong" is emphasised with speech marks to help distinguish it from the "Weak" Aggregating Algorithm proposed by Yuri Kalnishkan and Michael Vyugin [14] that will not be explored in detail in this paper.

Given that the foundations of this study lie firmly within On-line Prediction, this section aims to lay a comprehensive foundation, exploring the key concepts and frameworks that will set the stage for the discussions in Chapter 4.

Protocol 1 On-line Prediction Framework

FOR $t = 1, 2, \dots$
    learner L outputs $\gamma_t \in \Gamma$
    nature outputs $\omega_t \in \Omega$
    learner L suffers loss $\lambda(\gamma_t, \omega_t)$
END FOR

2.3.1.1 On-line Prediction, Batch Learning and Timeseries Analysis

In order to effectively explain On-line Prediction, we must first compare and contrast it with some alternative frameworks used in Machine Learning: Batch Learning and Timeseries Analysis.

First, we will explore the distinction between the On-line Prediction and Batch Learning frameworks. With Batch Learning, a whole training set of labelled examples (x_i, y_i) is given to the Learner at once and used to train a model. In contrast, On-line Prediction involves gradually feeding the Learner information over time, requiring any models made to continuously adapt to the new data it is given whilst also requiring the Learner to take actions based on the incomplete information it currently possess rather than waiting for a complete picture [4]. This forced adaptability ensures that the predictions outputted by an On-line Prediction algorithm remain accurate to the information that the model deems as relevant as it continues to gain additional information. This fact makes these models particularly valuable in applications that require immediate responses and flexibility in predictions, such as financial market analysis and weather forecasting.

The second distinction we will explore is between On-line Prediction and Timeseries Analysis since, while they are both ways of handling sequential data, they are unique. On-line Prediction is based on processing data points sequentially and updating a predictive model in real-time whereas Time-series Analysis is based on modelling and forecasting data that is collected over successive time intervals. The prior approach does not impose any strict assumptions about the underlying data-generating process, even going so far as to not assume the existence of such a process [15], while the latter assumes a structured approach in which the current observation is dependent on the previous observations. This assumption leads to the data-generating processes being modelled with stochastic processes, such as *autoregressive integrated moving average (ARIMA)* or *state-space models* [16].

The majority of literature on On-line Prediction takes a similar stance that no assumptions can be made about the sequence of outcomes that are observed and, because of this, the analyses are done over the worst-case scenario and may, in fact, be better in reality [13].

2.3.1.2 Notation

Having now clearly defined what On-line Prediction is and is not, we can delve into the notation Protocol 1 presents.

Consider a scenario where the elements of a sequence, known as ***outcomes*** ω_t, occur at discrete time steps $t \in T$ $\omega_1, \omega_2, \ldots, \omega_T$ which we assume to be drawn from a known ***outcome space*** Ω. In this scenario, the Learner is tasked with making ***predictions*** γ_t about these outcomes before they occur. Similarly to the outcomes themselves, we assume that the Learner's predictions are drawn from a known ***prediction space*** Γ which may or may not be the same as the outcome space Ω.

Once the Learner has made their prediction, the true outcome is revealed and the quality of the Learner's prediction is measured by ***loss function*** $\lambda(\gamma_t, \omega_t)$. In essence, this function measures the discrepancy between the prediction and the outcome—known as "regret", meaning it quantifies the effect of when the prediction γ_t is confronted with the outcome ω_t in hindsight. It does this by mapping the input space $\Gamma \times \Omega$ to a subset of the real-number like $mathbbR$, typically $[0, +\infty)$ [17].

$$\lambda(\gamma_t, \omega_t) : \Gamma \times \Omega \to [0, +\infty) \quad (1)$$

Across multiple time steps T, the Learner will suffer multiple individual losses which can be referred to collectively as the cumulative loss up until time T defined below. The Learner's performance is effectively measured by this cumulative loss, so their natural objective is to try to suffer as low a cumulative loss ass possible.

$$\text{Loss}_L(T) = \sum_{t=1}^{T} \lambda(\gamma_t, \omega_t) \tag{2}$$

2.3.1.3 Games

With the necessary notation established, we can now delve into the related concept of a ***Game*** which introduces a strategic perspective to On-line Prediction.

Formally, a Game G is denoted with the triple $\langle \Gamma, \Omega, \lambda \rangle$ which refers to a specific prediction space, outcome space and loss function. Informally, it makes sense to refer to this triple as a game because it encapsulates the interactive, yet adversarial, nature of the problem due to conflicting goals of the Learner and Nature which closely resembles the ***Repeated Game*** framework discussed in Game Theory [18].

For our purposes, the Learner must perform sequential decision-making and must output a prediction γ taken from Γ without knowing the true outcome ω from Ω in advance which highlights the imbalance in the game and the potential for adversarial games to be played. Given that the Learner's goal is to minimise their cumulative loss, it is natural to assume that Nature's goal is to select outcomes ωs that try to inflict as much cumulative loss on the Learner over time as possible which is similar to how two players in an adversarial game, such as chess, must develop strategies to optimise their performance based on the information available to them and the the actions they have seen from the other player(s).

¡TODO: INCLUDE EXAMPLES OF GAMES¿

2.3.2 Prediction with Expert Advice

With the foundations of On-line Prediction established, we can now focus on a more nuanced approach to prediction: Prediction with Expert Advice. The approach builds on the existing framework presented in Protocol 1 by introducing a critical element in the form of the aggregation of multiple Experts' opinions with the central idea being that, by harnessing the insights and predictions from a pool of Experts, the Learner can make more informed and accurate predictions, particularly in complex scenarios with high levels of uncertainty and/or incomplete information.

As shown above, a Learner in the Prediction with Expert Advice Framework has access to a pool of Experts, denoted $\mathcal{E}_1, \ldots, \mathcal{E}_N$, to help inform their predictions. At each time step t, each of the Experts provides the

Protocol 2 Prediction with Expert Advice Framework

FOR $t = 1, 2, \ldots$
  experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ output predictions$\gamma_t^1, \ldots, \gamma_t^N \in \Gamma$
  learner L outputs $\gamma_t \in \Gamma$
  nature outputs $\omega_t \in \Omega$
  experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ suffer losses $\lambda(\gamma_t^1, \omega_t), \ldots, \lambda(\gamma_t^N, \omega_t)$
  learner L suffers loss $\lambda(\gamma_t, \omega_t)$
END FOR

Learner with a prediction $\gamma_t^1, \ldots, \gamma_t^N$ which the Learner must combine to form a single prediction γ_t.

The theoretical basis for this framework lies in the theory that no single Expert, no matter how knowledgeable, can consistently outperform a well-constructed aggregate of advice from multiple Experts. While the Learner's objective in this scenario is still to minimise the amount of cumulative loss they suffer over time, the manner in which they achieve this is influenced by both the accuracy of each individual Expert, as well as how well each Expert's predictions are incorporated into their aggregated prediction.

In non-adversarial scenarios, it is safe to assume that the Experts' goal is to minimise the individual cumulative losses over time much like the Learner, however, we cannot safely assume that every Expert in the pool will behave in such a manner. Because of this, we must develop a framework to essentially treat each Expert in the pool as a black box, meaning that the Learner has no knowledge of the internal prediction mechanisms of each Expert nor what each individual Expert's goal is, to guarantee that the cumulative loss suffered by the Learner is almost as good as the cumulative loss suffered by the best Expert.

$$\forall n, \forall T : \mathrm{Loss}_L(T) \lesssim \mathrm{Loss}_{E_n}(T) \tag{3}$$

2.3.2.1 Mixability

Before we continue to delve into potential solutions for the problem of Prediction with Expert Advice, it is vital that we introduce another vital concept known as ***mixability*** which refers to a property of the loss function that allows for efficient aggregation of probabilistic predictions.

More precisely, a loss function $\lambda(\gamma_t, \omega_t) : \Gamma \times \Omega \to [0, +\infty)$ is called mixable if there exists a constant $\eta > 0$ such that there exists an aggregated ("mixed") prediction $\gamma \in \Gamma$ that satisfies the following inequality for any set of probabilistic predictions $\{\gamma^i\}_{i=1}^N$ over the outcomes in Ω:

$$\lambda(\gamma_t, \omega_t) \leq -\frac{1}{\eta}\sum_{n=1}^{N} p^n e^{-\eta\lambda(\gamma_t^n, \omega_t)} \tag{4}$$

The significance of this is that it allows the Learner to effectively combine the predictions from multiple sources, ensuring that the aggregated predictions suffer a loss that is close, if not equal, to the best possible weighted combination of individual predictions. Ultimately, this means that mixable prediction strategies can compete with, or outperform, any fixed prediction strategy with hindsight.

With that mentioned, we can now discuss the mathematical methods that can be used to combine the predictions from multiple Experts, namely the Halving, Weighted Majority, and ultimately Aggregating Algorithms. Each of these algorithms aims to leverage the strength of each Expert's predictions to make the Learner's own more accurate while also minimising the impact of inaccurate predictions by hedging so that, in the event that adversarial experts are present, the Learner wouldn't suffer great cumulative loss due to poor reliability.

2.3.2.2 Halving Algorithm

Given that we are now looking at problems involving a pool of Experts, consider the ***simple prediction game***, i.e. $\langle \Gamma = \Omega = \{0, 1\}, \lambda_{\mathrm{SQ}}(\gamma_t, \omega_t) = (\gamma_t - \omega_t)^2 \rangle$, where we know in advance that there is a perfect Expert that always gives the correct prediction. Despite the fact that the perfect Expert will not make mistakes, we as the Learner still might since we do not know which of the N Experts is perfect, thus a strategy must be devised and followed to locate them while suffering as little cumulative loss as possible.

Since we know in advance that there is a perfect Expert, we do not need to pay attention to any Experts that we have seen to make mistakes. In order to do this, we need to create and maintain two lists: a ***whitelist*** and a ***blocklist***.

Initially, all N Experts begin in the whitelist and make predictions on every time step t as before, however, as soon as an Expert makes a mistake, i.e., $\gamma_t^n \neq \omega_t$, it is moved to the blocklist with no way of returning to the whitelist. This means that on each time step t, we only consider the predictions of experts that are currently a member of the whitelist with the simplest strategy for the Learner being to listen to the majority of Experts' predictions.

The size of the whitelist on after step t is denoted by W_t and initially all Experts are in the whitelist, i.e. $W_0 = N$. By using this method, whenever the Learner makes a mistake, the size of the whitelist shrinks.

Say that there is an error on step t. Before this step, the size of the whitelist was W_{t-1} and because we made a mistake, this means that at least half of the experts on the whitelist also made a mistake and are therefore removed, meaning $W_t \leq \frac{W_{t-1}}{2}$. More generally, consider that by time T the Learner has made m mistakes, meaning that $W_T \leq \frac{W_0}{2^m}$ with $W_T \geq 1$ because of the presence of at least one perfect Expert.

$$1 \leq W_T \leq \frac{W_0}{2^m} = \frac{N}{2^m} \tag{5}$$

Meaning that a Learner following the Halving Algorithm in the Simple Prediction Game will achieve: By rearranging this formula, we get that a Learner following the Halving Algorithm in the Simple Prediction game will achieve the following for every sequence of outcomes provided that there is a perfect expert.

$$\mathrm{Loss}_L(T) \leq \lfloor \log_2 N \rfloor \tag{6}$$

2.3.2.3 Weighted Majority Algorithm

Consider now an extension to the Halving Algorithm known as the Weighted Majority Algorithm which can be played on the ***discrete binary prediction game***, i.e. $\langle \Gamma = [0,1], \Omega = \{0,1\}, \lambda_{\mathrm{SQ}}(\gamma_t, \omega_t) = (\gamma_t - \omega_t)^2 \langle$ and removes the need for a perfect Expert. To do so, we have to introduce a weight for each expert at each time step, denoted as w_t^n, that effectively represents how much the Learner trusts the expert and listens to their advice.

Let each Expert begin with an initial weight $W_0^n = 1$. Now, instead of being moved from the whitelist to the blocklist whenever an Expert makes a mistake, their weights are updated as follows:

$$w_t^n = w_{t-1}^n \beta^{\lambda(\gamma_t^n, \omega_t)} \tag{7}$$

where $\beta = e^{-\eta} < 1$, with $\eta > 0$ representing a "learning rate" which determines how severely the trust in an expert is eroded upon making a mistake. If an expert does not make a mistake, then $\lambda(\gamma_t^n, \omega_t) = 0$ meaning that $e^{-\eta\lambda(\gamma_t^n, \omega_t)} = 1$.

With this notation, it can be seen how the Halving Algorithm is actually a special case of the Weighted Majority Algorithm where $\beta = e^{-\eta} = 0$ as the trust in an Expert gets reduced to zero when they make a mistake.

Algorithm 1 Weighted Majority Algorithm

initialise weights $w_0^i = 1, i = 1, 2, \ldots, N$
FOR $t = 1, 2, \ldots$
    read the experts' predictions $\gamma_t^i, i = 1, 2, \ldots, N$
    calculate the sum of weights $v_t^0 = \sum_{n:\gamma_t^n=0} w_{t-1}^n$
        and $v_t^1 = \sum_{n:\gamma_t^n=1} w_{t-1}^n$
    if $v_t^0 > v_t^1$, predict $\gamma_t = 0$; otherwise predict $\gamma_t = 1$
    observe the outcome ω_t
    update the experts' weights $w_t^i = w_{t-1}^i \beta^{\lambda(\gamma_t^i, \omega_t)}, i = 1, 2, \ldots, N$
END FOR

The upper bound for a Weighted Majority Algorithm is as follows:

$$\mathrm{Loss}_L(T) \leq \frac{\ln(\frac{1}{\beta})}{\ln(\frac{2}{1+\beta})} \mathrm{Loss}_{E_n}(T) + \frac{\ln(N)}{\ln(\frac{2}{1+\beta})} \tag{8}$$

The proof for this bound is more complicated than the proof for the Halving Algorithm and, thus, will not be discussed in this paper but can be found at [19].

In the next subsection, we will explore the Aggregating Algorithm, the final method of merging Experts' predictions and a concrete implementation of the Prediction with Expert Advice that formalises how a Learner's predictions are made as accurate as possible.

2.3.3 Aggregating Algorithm (AA)

Having now established the value of Expert advice when attempting to make predictions, we must now explore how to effectively integrate it into our predictive models. The Aggregating Algorithm is one of the key algorithms that addresses this challenge, providing a way to combine various Expert opinions into a single, actionable prediction that is informed via weighting.

Much like the Weighted Majority Algorithm discussed previously, the Aggregating Algorithm works by maintaining a set of weights for each Expert, representing the Learner's confidence in the accuracy of their predictions. As predictions are made over time, these weights are adjusted based on the loss associated with each expert's predictions with Experts that output predictions closer to the actual outcomes being rewarded with higher weights and Experts that perform poorly being punished with their weights reduced.

Algorithm Description: Introduction to the Aggregating Algorithm.
Functionality: How the Aggregating Algorithm works in practice.

Algorithm 2 Aggregating Algorithm (AA)

initialise weights $w_0^i = q_i, i = 1, 2, \ldots, N$
FOR $t = 1, 2, \ldots$
  read the experts' predictions $\gamma_t^i, i = 1, 2, \ldots, N$
  normalise the experts' weights $p_{t-1}^i = w_{t-1}^i / \sum_{j=1}^N w_{t-1}^j$
  output $\gamma_t \in \Gamma$ that satisfies the inequality for all $\omega \in \Omega$:
    $\lambda(\gamma_t, \omega) \leq -\frac{C}{\eta} \ln \sum_{i=1}^N p_{t-1}^i e^{-\eta\lambda(\gamma_t^i, \omega)}$
  observe the outcome ω_t
  update the experts' weights $w_t^i = w_{t-1}^i e^{-\eta\lambda(\gamma_t^i, \omega_t)}, i = 1, 2, \ldots, N$
END FOR

$$\mathrm{Loss}_T(L) \leq C \cdot \mathrm{Loss}_T(\mathcal{E}_i) + \frac{C}{\eta} \ln \frac{1}{q_i} \tag{9}$$

2.3.4 Aggregating Algorithm for Specialist Experts (AASE)

While the Aggregating Algorithm provides a robust framework for incorporating Expert advice, there are certain scenarios that require a more nuanced approach. This is where the Aggregating Algorithm for Specialist Expert comes into play, which will be the approach that this study's experiment is centred.

The use of the term 'Specialist' was first introduced by the work of Avrim Blum [20] for the Winnow and Weighted-Majority algorithms, and can be thought of as a natural extension to traditional Experts insofar as it enables these 'Specialists' to abstain from making a prediction "when the current Expert does not fall into their '[speciality]"'. While the criteria for an Expert to abstain from making a prediction is sufficient in our context, it can also be extended to allow for other scenarios like those suggested in [21], namely if "a prediction algorithm [sees] that its internal confidence is low and [decides] to skip a turn in order to re-train" or if a prediction algorithm breaks down, as would be the case if a regression algorithm "[has] its matrix very close to singular."

In order to accommodate these Specialist Experts, the Prediction with Expert Advice Framework given in (1) has to be modified as follows:

Protocol 3 Modified Prediction with Expert Advice Framework

FOR $t = 1, 2, \ldots$
  nature chooses a subset of experts $\mathcal{E}_i \in \mathcal{E}$ that are awake
  awake experts $\mathcal{E}_1, \ldots, \mathcal{E}_N$ output predictions $\gamma_t^1, \ldots, \gamma_t^N \in \Gamma$
  learner L outputs $\gamma_t \in \Gamma$
  nature outputs $\omega_t \in \Omega$
  awake experts $\mathcal{E}_1, \ldots, \mathcal{E}_N \in \mathcal{E}_i$ suffer losses $\lambda(\gamma_t^1, \omega_t), \ldots, \lambda(\gamma_t^N, \omega_t)$
  learner L and sleeping experts $\mathcal{E}_j \notin \mathcal{E}_i$ suffers loss $\lambda(\gamma_t, \omega_t)$
END FOR

As referenced above, another colloquial way of referring to 'Specialist Experts' is 'sleeping Experts'; Freund postulated that "a Specialist is awake when it makes a prediction and that it is asleep otherwise", going so far as to refer to the traditional On-line Prediction framework as "the insomniac framework since it is a special case in which all Specialists are awake all the time." [3] This colloquialism is useful when adapting the bounds of the base Aggregating Algorithm because a natural interpretation of what happens when an Expert is sleeping is that it simply "joins the crowd" [21], meaning that it mimics the learner's prediction on the time steps that it is asleep because the learner's prediction is formed based on the weighted majority of Experts' predictions. Given this definition, it can be seen that on some time steps t, the learner's prediction and the Expert $\mathcal{E}_i$'s predictions are the same; $\gamma_t = \gamma_t^i$. Recall that, in the mixable case, the Aggregating Algorithm guarantees that the following inequality is satisfied:

$$\sum_{t=1}^{T} \lambda(\gamma_t, \omega_t) \leq \sum_{t=1}^{T} \lambda(\gamma_t^i, \omega_t) + \frac{1}{\eta} \ln \frac{1}{q_i}$$

Typically, the Aggregating Algorithm's performance is measured in terms of the learner's cumulative loss compared to the best Expert's cumulative loss but given that, on certain time steps t, $gamma_t = \gamma_t^i$, it is clear that the corresponding terms in both sums cancel out and what is left are the sums over the time steps where the learner's and the Expert's predictions are different, i.e. where Expert $\mathcal{E}_i$ is awake. What follows from this is that, instead of wanting the learner's loss to be nearly as good as the best Expert's loss over a period of time T, we judge the Aggregating Algorithm for Specialist Experts' performance based on the learner's loss compared to the best Expert's $\mathcal{E}_i$ loss over the steps in which it was awake. A learner following the algorithm achieves a cumulative loss that satisfies the following inequality:

$$\sum_{\substack{t=1,2,\ldots,T: \\ \mathcal{E}_i \text{ is awake} \\ \text{on step } t}}^{T} \lambda(\gamma_t, \omega_t) \leq C \cdot \sum_{\substack{t=1,2,\ldots,T: \\ \mathcal{E}_i \text{ is awake} \\ \text{on step } t}}^{T} \lambda(\gamma_t^i, \omega_t) + \frac{C}{\eta} \ln \frac{1}{q_i} \tag{10}$$

As is the case for the traditional Aggregating Algorithm, we make no assumptions about the outcome-generating mechanism (including the existence of such a mechanism) and this bound holds for *any* adversarial strategy, meaning that the adversary cannot inflict a large loss on the learner without inflicting a large loss on the Specialists and ensuring that the performance will be good whenever there is a good mixture of Specialists.

Algorithm 3 Aggregating Algorithm for Specialist Experts (AASE)

```
initialise weights $w_0^i = q_i, i = 1, 2, \ldots, N$
FOR $t = 1, 2, \ldots$
    read the awake experts' predictions $\gamma_t^i, i = 1, 2, \ldots, N$
    normalise the awake experts' weights
        $p_{t-1}^i = w_{t-1}^i / \sum_{j:\mathcal{E}_j \text{ is awake}} w_{t-1}^j$
    output $\gamma_t \in \Gamma$ that satisfies the inequality for all $\omega \in \Omega$:
        $\lambda(\gamma_t, \omega) \leq -\frac{C}{\eta} \ln \sum_{i:\mathcal{E}_i \text{ is awake}} p_{t-1}^i e^{-\eta\lambda(\gamma_t^i, \omega)}$
    observe the outcome $\omega_t$
    update the awake experts' weights $w_t^i = w_{t-1}^i e^{-\eta\lambda(\gamma_t^i, \omega_t)}$
    update the sleeping experts' weights $w_t^i = w_{t-1}^i e^{-\eta\lambda(\gamma_t, \omega_t)/C(\eta)}$
END FOR
```

2.4 Conclusions

Having gone through the established knowledge base on these topics, this Literature Review has now traced how the perception of randomness in both judging and generating random sequences has changed over time, highlighting key studies that have informed the current understanding of these cognitive processes with the conflicting findings highlighting the complexity of the task. Ultimately, the disagreement suggests that while humans may not be inherently good randomisers individually when compared to the statistical definition of random, their performance can vary significantly depending on the conditions in which they are tested with an interesting observation being made in that when sequences generated by multiple individuals are aggregated, the distribution becomes more like what is expected of a truly random process.

In the context of Prediction with Expert Advice, these insights are particularly interesting because the imperfections of human-generated "random" sequences, primarily marked by the tendency to favour sequences with a greater number of alternations, or that contain longer runs than what is expected, might actually be more predictable than realised by making use of On-line Prediction algorithms, providing evidence for the fact that humans are, in fact, bad randomisers.

By contextualising the experiment conducted in this study with the frameworks established by various papers cited within this Chapter, this Literature Review sets the stage for the subsequent analysis of human-generated "random" sequences with the findings from the experiment assisting in providing a deeper understanding of the human perception of randomness and the potential implications of that on predictive computational models.

3 Experiment Design and Methodology

3.1 Introduction

As discussed in Chapter 2.2.2.2, the experimental design used in this study closely aligns with the methodology established by Nickerson and Butler [12], serving as the foundational basis for this experiment. By adapting their established methodology to the current study and its novel application, this project not only compares the sequences generated by subjects with those expected from a random process according to the statistical definition, but also evaluates the predictability of each subject's responses using Prediction with Expert Advice and Vovk's Aggregating Algorithm [1]. In theory, the more random a subject's sequence, the less predictable their responses, leading to greater losses for both the Learner and each of the relevant Experts.

3.2 Experimental Design

This chapter outlines the experimental design, which aims to assess how well individuals can generate random binary sequences when compared to theoretical randomness and their previous inputs. The study employs a within-subject design, where each participant is exposed to all conditions of the independent variable which, in his case, means they are required to repeat the generation process several times. This approach provides a comprehensive view of each individual's performance in generating random sequences.

The independent variable in this experiment is the method by which the participants generate their binary sequences, as each participant is allowed to enter their sequences independently of one another. The dependent variables include the frequencies of 0s and 1s in each sequence, the number and length of runs within each sequence, and the predictions generated by the Experts and the Learner, which will be discussed in further detail in the following sections. The control variables of this study include the instructions given to the subject before beginning the experiment, the length of each binary sequence inputted, and the total number of sequences entered. These control variables were devised with the intention of facilitating a more accurate comparison and analysis of the collected data between participants.

This study's subjects primarily consisted of postgraduate students from the Computer Science Department at Royal Holloway, University of London, with additional participants from the Psychology Department to create a more representative sample. Each sample was tasked with generating several 10-item sequences intended to mimic the results expected from a random process. These sequences were entered into a web application hosted on GitHub Pages. Each 10-item sequence consisted of 0s and 1s (representing Heads and Tails) arranged in any order that the subject chose, and participants were allowed to enter the sequences at their own pace. Prior to beginning the experiment, subjects were presented with the following instructions on a modal screen shown upon loading the web page:

> *Your task is to create a table of sequences each consisting of 10 items, either 0 or 1.*
>
> *Imagine that several people have each tossed a fair coin 10 times and the results of their tosses are recorded in a table, with each row recording the outcomes of the 10 tosses by one person.*
>
> *Your goal is to produce this table in such a way that if compared with a table of the results of actual coin tosses, it would not be possible to distinguish which table represented the actual coin tosses with statistical tests and which didn't.*

Herein lies the first divergence from Nickerson and Butler's original design because the sequences entered by the subjects are always displayed and were concatenated into a single, continuous sequence (as shown in Figure 3), which is then passed to the Aggregating Algorithm as ωs. In the original experiment, the sequence would only remain visible to the subject until they had entered 10 items, at which point it would disappear. The modification in this study allows the Aggregating Algorithm to better identify patterns in the user's inputs as an interval length of 10-bits would be insufficient in allowing the algorithm to determine which Experts should be given higher weighting in forming the Learner's predictions, thereby improving the learner's prediction accuracy. While the underlying algorithm treats the sequence differently to [12], the sequences are still presented to the subject in 10-item chunks (as shown in Figure 4), consistent with the original method, to better align with the human short-term memory span of approximately 7 $\pm$ 2 items cited in 2.2.2.2. This chunking allows subjects to quickly review their previous inputs and continue generating sequences that they perceive as random.

Given this foundation, we can now discuss how the Aggregating Algorithm for Specialist Experts (AASE) was applied to this experiment.

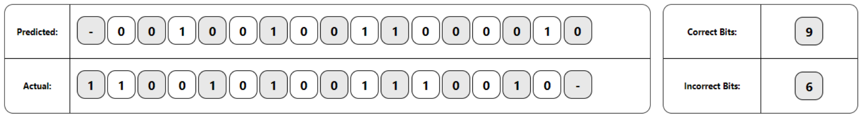

Figure 3: Inputted Sequence Displayed in the Web Application

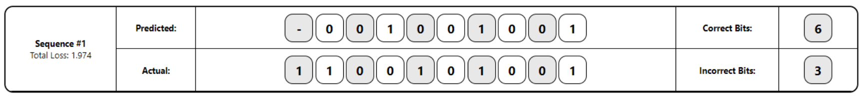

Figure 4: Past Sequences Displayed in the Web Application.

3.3 Applying the Aggregating Algorithm

The Aggregating Algorithm, as well as the broader framework of Prediction with Expert Advice, are integral to this study. To apply these methods to this scenario, we must first define the η-mixable game $G = \langle \Gamma, \Omega, \lambda \rangle$, as defined in Protocol 2 – Prediction with Expert Advice Framework. As suggested by Chapter 2, the primary focus of this project is Prediction with Expert Advice for the Discrete Binary Game which is formally defined by the triple consisting of the ***outcome space*** $\Omega = \{0, 1\}$, the ***prediction space*** $\Gamma = [0, 1]$, and the ***loss function*** given by Brier's (or Square) Loss, $\lambda_{\mathrm{SQ}}(\gamma_t, \omega_t) = (\gamma_t - \omega_t)^2$. In practical terms, this means that Nature generates binary outcomes, either 0 or 1, while both the Learner and the Experts predict values within the range $[0, 1]$.

With the game formally established, it is essential to define the roles of Nature, the Learner and the Experts in our experimental context. In this project, which assesses a subject's ability to generate random binary sequences, one of the key metrics is the predictability of their inputs prior to pressing a key. Therefore, each subject naturally assumes the role of Nature, while the Aggregating Algorithm functions as the Learner attempting to pre-empt Nature. As the experiment involves multiple subjects, this supports the assertion made previously that "we make no assumptions about the outcome-generating mechanism (including the existence of such a mechanism)." Each subject possesses a unique internal concept of randomness, and the Aggregating Algorithm must perform adequately across all participants and all sequences.

Finally, the concept of an Expert, as well as the rationale for making use of Specialist Experts and the Aggregating Algorithm for Specialist Experts, must also be defined. For our purposes, an Expert can be thought of as a function designed to predict the next outcome in a sequence based on the presence of a specific scenario. Since the experiment evaluates each subject's concept of randomness by statistically analysing conditional probabilities for different orders of dependency, it is natural to conceptualise the group of Experts as functions that search for specific prefixes within the sequence. As the subject inputs their sequence to the application, the last n bits are passed to the group of Experts, who then assess whether the sequence currently falls within their "area of expertise", i.e., whether the last n bits match the prefix that they are searching for.

Given that not every prefix will be relevant to each new subject input, the use of Specialist Experts is justified. If the current sequence does not match a Specialist's prefix, and thus "area of expertise", that Specialist is considered "asleep" and abstains from making a prediction, "joining the crowd". Conversely, Specialists whose prefixes match the sequence's ending are considered "awake" and make their predictions accordingly. It is important to note that there will never be a scenario where all Specialists are asleep, with at least one being awake at all timestamps, as there is always an Expert searching for the last $1 - n$ bits of the sequence.

Finally, the details of how an Expert functions within this experiment must be explained. In order to generate the Experts, binary sequences up to length 4 were generated and assigned to individual experts. The decision to limit the prefix length to 4-bits balances both computational efficiency with predictive accuracy, reflecting the established constraints of the human short-term memory typically spanning 7 ± 2 items—with more powerful equipment, a higher prefix length could be considered. An Expert tracks the frequency of 0s and 1s following its specific prefix and makes predictions based on the ratio $\#1/(\#0 + \#1)$. As subjects type their sequences, each Expert checks the last x bits (corresponding to the length of their prefix) to determine whether they are awake. If so, the Specialist makes a prediction, which is then fed into the Aggregating Algorithm to inform the Learner's prediction for the next time stamp. For the sake of transparency, each Expert's last prediction, current prediction and status are displayed at the bottom of the application for user's reference if the subject is interested.

3.4 Data Analysis

After the experiment was conducted, the data was aggregated and analysed using various methods. The primary method was a chi-square goodness-of-fit test, comparing each subject's generated sequences to the distribution expected from a random process. To evaluate the Learner's performance, a secondary analysis was conducted, comparing both the cast (rounded) and uncast (unrounded) predictions against the actual outcomes.

Given the experiment involves playing Brier's Game with $\Omega = \{0, 1\}$ and $\Gamma = [0, 1]$, the simplest strategy for the Learner would be to predict $1/2$ for every timestamp, which is the minimax prediction, resulting in a loss of $1/4$ each time. Over 10 steps, the maximum loss for the Learner would be $10/4 = 2.5$, which serves as a benchmark for assessing the Aggregating Algorithm's performance – loss less than 2.5 indicates that a Learner using the Aggregating Algorithm predicts better than this naive strategy.

These methods of analysis are, in fact, somewhat adversarial, as the more statistically random a subject's input is, the more challenging it should become for the Experts, and thus the Learner, to make accurate predictions. Consequently, a better fit to statistical randomness should result in poorer Learner performance as the subject's sequences should exhibit no discernable patterns.

3.5 Procedure

This chapter outlines the step-by-step procedure followed to conduct the experiment based on what was outlined in the previous chapters.

1. Participants were selected from the student community at Royal Holloway, University of London. Those recruited included postgraduate students from the Computer Science Department, as well as students from the Psychology Department, in order to create a diverse sample.

2. Participants were directed to the application hosted on GitHub Pages designed to collect and analyse generated binary sequences. Its development and the issues faced will be outlined in a following chapter.

3. During the experiment, subjects were tasked with generating several 10-item binary sequences by entering 0s and 1s into the application such that the results would appear random if subjected to statistical tests. For transparency, the application displayed the internal workings of each Expert at the bottom of the screen, though participants could disable this feature if they desired.

4. After finishing the experiment, the subjects were asked to email their results to Andrew.Barraclough.2018@live.rhul.ac.uk to be subjected to the methods outlined in the previous section, namely chi-square goodness-of-fit and comparison to the loss inflicted by following the naive strategy.

4 Analysis of Perceived Randomness

Data Presentation: Presentation of collected data in an organised manner.
Analytical Techniques: Methods used to analyse the data.
Results: Detailed presentation of findings.
Discussion: Interpretation of results in the context of perceived randomness.
Comparison with Literature: How the findings align or differ from existing research.

As discussed throughout the paper, a human's perception of randomness often deviates significantly from true mathematical randomness due to their inherent cognitive biases. This chapter delves into the findings from the experiment conducted, aiming to uncover patterns in how individuals judge and interpret random sequences.

- Histograms of Aggregated Results
- Chi-Square Goodness of Fit Test
- Loss Line Charts
- Differences in Human- and AI-Generated Sequences

4.1 Chi-Square Goodness of Fit Tests

4.1.1 Distribution of the Number of Heads

4.1.2 Distribution of the Number of Runs

4.1.3 Distribution of Run Lengths

4.2 Regret Analysis

As previously discussed, the Learner uses the predictions of various Experts in order to form its own prediction of the next bit that will be entered by the subject (acting as Nature in this context). The subject's task is to input bits in a way that is random, and thereby unpredictable to the Learner.

In the line chart, when the line is above the x-axis, it means that the difference between the individual expert's loss and the learner's is positive. This means that the expert is performing worse than the learner's method of aggregating predictions.

When the difference is negative, it indicates that the learner is performing worse than the expert, implying that the expert's predictions were close to the actual bits entered by the subject.

If a plot frequently shows the learner's loss being less than that of the experts, it suggests the learner is effectively combining the experts' predicitons and could imply that the subject's bit sequence has some underlying patterns or predictability that the learner is able to exploit, despite the subject's attempt to be random.

Conversely, if the line is frequently above the x-axis, it could indicate that the learner is not effectively aggregating the experts' predicitons, possibly because the subject's sequence is truly random and highly unpredictable. This could suggest that the subject's notion of randomness is strong, and that the learner struggles to predict the next bit accurately.

If the subject's input is truly random, no expert nor the learner should consistently outperform random guessing. The plot might show a mix of positive and negative values, with no clear trend. This randomness would make it difficult for the learner to maintain a consistent advantage.

If the subject's input is only perceived as random, then the learner might perform better over time by effectively leveraging the experts. A plot with the learner frequently outperforming the experts suggests that the subject's sequence is not truly random and that the learner is able to exploit subtle patterns.

5 Conclusions *(1,500)*

Summary of Findings: Recap of the main findings of the study.
Contributions: Discussion on the contributions of the study to the field.
Limitations: Identification of any limitations encountered during the research.
Future Work: Suggestions for future research based on the findings and limitations of this study.

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