# CLASSIFICATION OF GROUPS OF ORDER 40 UP TO ISOMORPHISM MATH H113 SPRING 2010

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## 1. First Steps

Let G be a group of order 40. $|G| = 40 = 8 \cdot 5 = 2^3 \cdot 5$.

By the third Sylow Theorem $s_5(G)|8$, so $s_5(G) =$ 1,2,4, or 8. However, also by the third Sylow Theorem, $s_5(G) = 1 \mod 5$. Thus, $s_5(G) = 1$ and $G$ has a normal subgroup of order 5, which must be cyclic.

Similarly, $s_2(G)|5$, so $s_2(G) = 1$ or 5. $5 = 1 \mod 2$, so this does not narrow down the number of Sylow-2 subgroups.

Because $|P_2| = 8$, every element in $P_2$ has order dividing 8. In $C_5$, every element has order 1 (identity) or 5 (not identity). Thus $P_2 \cap C_5 = 1$. Since $P_2C_5 \subset G$ and $|P_2C_5| = 5 * 8/1 = 40$ (exercise 41), $P_2C_5 = G$

Therefore, by exercise 66, $G = C_5 \rtimes P_2$ (where the semidirect product may be trivial).

## 2. Case when $s_2(G) = 1$

In this case, $P_2$ is normal, so $G = C_5 \times P_2$. Remember that $P_2$ can be one of 5 groups:

3 of them abelian: $C_8$ $\quad C_4 \times C_2$ $\quad C_2^3$

2 of them not: $D_8$ (group of symmetries of the square) $\quad Q$ (the quaternions)

This immediately gives us 5 non-isomorphic groups of order 40 that G can be:

(1) $C_5 \times C_8 = C_{40}$

(2) $C_5 \times C_4 \times C_2 = C_{20} \times C_2$

(3) $C_5 \times C_2^3 = C_{10} \times C_2^2$

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(4) $C_5 \times D_8$

(5) $C_5 \times Q$

Note that the first 3 are abelian and the last two are not.

## 3. Case when $s_2(G) = 5$

Now we look to find nontrivial semidirect products of $C_5$ and $P_2$.

Let us start with an observation. Observation: $C_n \rtimes C_2 \cong D_{2n}$

$D_{2n} =< \sigma, \tau | \sigma^n = \tau^2 = (1,1), \tau\sigma\tau = \sigma^{-1} >$ [1]

Let the cyclic subgroup of order n be generated by a and the cyclic subgroup of order 2 be generated by b. Let $\sigma = (a,1), \tau = (1,b)$. Then $\sigma^n = (a^n, 1) = (1,1)$ and $\tau^2 = (1, b^2) = (1,1)$. $\tau\sigma\tau = (1,b)(a,1)(1,b) = (1,b)(a,b) = (\varphi_b(a), b^2) = (\varphi_b(a), 1)$. Note that $\varphi_b(a)$ is a generator of $C_n$ and $\varphi_b(\varphi_b(a)) =$ a. So $\varphi_b(a) = a^{-1}$. Thus $(\varphi_b(a), 1) = (a^{-1}, 1)$. Therefore, $C_n \rtimes C_2 = D_{2n}$.

### 3.1. Subcase when $P_2 = C_8$.

Denote $C_8$ as $< b >$. In this subcase, we have two nontrivial homomorphisms from $C_8$ into $Aut(C_5) = C_4 =< \alpha >$ $(\alpha(a) = a^2, \alpha^2(a) = a^{-1}, \alpha^3(a) = a^3, \alpha^4(a) = a)$, $\varphi_1$ and $\varphi_2$, where $\varphi_1(b) = \alpha$ and $\varphi_2(b) = \alpha^2$. Let K denote the kernel of a homomorphism. $K(\varphi_1) = \{b^4, 1\}$ and $K(\varphi_2) = \{b^2, b^4, b^6, 1\}$. Note that we can also form more nontrivial homomorphisms by sending b to $\alpha^3$, for example. However, since $\alpha^3$ is another generator of $C_4$, these semidirect products will be isomorphic. Since the kernel are of different sizes, the two semidirect products constructed by these homomorphisms are not isomorphic. Therefore, we have two more groups of order 40:

(6) $C_5 \rtimes_{\varphi_1} C_8 = < a, b | a^5 = b^8 = 1, bab^{-1} = a^2 >$

(7) $C_5 \rtimes_{\varphi_2} C_8 = < a, b | a^5 = b^8 = 1, bab^{-1} = a^{-1} >$

### 3.2. Subcase when $P_2 = C_4 \times C_2$.

There are five nontrivial homomorphisms from $C_4 \times C_2$ to $C_4$. Consider $C_4 \times C_2 = < b, c | b^4 = c^2 = 1, bcb^{-1} = c >$ and $C_4 =< a >$. Then we have:

1. $\varphi_1$: $b \to \alpha, c \to 1$, $K(\varphi_1) =< c >\cong C_2$
2. $\varphi_2$: $b \to \alpha, c \to \alpha^2$, $K(\varphi_2) =< b^2c >\cong C_2$
3. $\varphi_3$: $b \to \alpha^2, c \to 1$, $K(\varphi_3) =< b^2, c >\cong C_2 \times C_2$

(4) $\varphi_4$: $b \to \alpha^2, c \to \alpha^2$, $K(\varphi_4) =< b^2, b^2c >\cong C_2 \times C_2$
(5) $\varphi_5$: $b \to 1, c \to \alpha^2$, $K(\varphi_5) =< b >\cong C_4$

We can form automorphisms between the semidirect products constructed from $\varphi_1$ and $\varphi_2$ by using an automorphism of $C_4 \times C_2$ and using exercise 68. Create this automorphism by mapping $c$ to $b^2$ and $b$ to $b$. Similarly, we can form an automorphism between the semidirect products constructed by $\varphi_3$ and $\varphi_4$. Send $b^2$ to $b^2$, $b^2c$ to $c$, and $b$ (in light of the kernel of the two homomorphisms from $C_4 \times C_2$ to $C_4$). In the end, with relabeling, we are left with 3 more groups of order 40:

(8) $C_5 \rtimes_{\varphi_1} (C_4 \times C_2) = < a, b, c | a^5 = b^4 = c^2 = 1, bab^{-1} = a^2, cac^{-1} = a, cbc^{-1} = b >$

(9) $C_5 \rtimes_{\varphi_2} (C_4 \times C_2) = < a, b, c | a^5 = b^4 = c^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a, cbc^{-1} = b >$

(10) $C_5 \rtimes_{\varphi_3} (C_4 \times C_2) = < a, b, c | a^5 = b^4 = c^2 = 1, bab^{-1} = a, cac^{-1} = a^{-1}, cbc^{-1} = b >$

Note that (8) and (9) correspond to taking a semidirect product of $C_5$ and $C_4$ and then the cross product with $C_2$ (this corresponds to the two nontrivial homomorphisms $C_4$ to $C_4$.) Then (10) corresponds to the semidirect product of $C_5$ and $C_2$ crossed with $C_4$ (this corresponds to the one nontrivial homomorphism $C_2$ to $C_4$). We can re-write them as follows:

(8) $(C_5 \rtimes_{\varphi_1} C_4) \times C_2$

(9) $(C_5 \rtimes_{\varphi_2} C_4) \times C_2$

(10) $(C_5 \rtimes C_2) \times C_4 = D_{10} \times C_4 = < \sigma, \tau, x | \sigma^5 = \tau^2 = x^4 = 1, x\sigma x^{-1} = \sigma, \tau\sigma\tau = \sigma^{-1}, x\tau x^{-1} = \tau >$ where $D_{10}$ is the group of symmetries of a regular pentagon.

### 3.3. Subcase when $P_2 = C_2^3$.

Since the kernel of any homomorphism is a subgroup, we have that any kernel of a nontrivial homomorphism is isomorphic to either $C_2$ or $C_2 \times C_2$. However, if only one component is sent to $\alpha^2$, then the kernel has 4 elements. Similarly, if 2 or 3 components are sent to $\alpha^2$, there are 4 elements in the kernel. Noting this and symmetry, we get one more subgroup:

(11) $C_5 \rtimes C_2 \times C_2^2 = D_{10} \times C_2^2 =< \sigma, \tau, a, b | \sigma^5 = \tau^2 = a^2 = b^2 = 1, \tau\sigma\tau = \sigma^{-1}, aba^{-1} = b, a\sigma a^{-1} = \sigma, a\tau a^{-1} = \tau >$

### 3.4. Subcase when $P_2 = D_8$.

If $\varphi(\sigma) = \alpha$ and $\varphi(\tau) = 1$ or $\varphi(\sigma) = \alpha$ and $\varphi(\tau) = \alpha^2$, then $\varphi(\tau\sigma\tau) = \alpha \neq \alpha^3 = \varphi(\sigma^3)$.

If $\varphi(\sigma) = \alpha^2$ and $\varphi(\tau) = 1$, then $\varphi(\sigma^4) = 1 = \varphi(\tau^2)$ and $\varphi(\tau\sigma\tau) = \alpha^2 = \alpha^6 = \varphi(\sigma^3)$. Thus we have a nontrivial homomorphism, denote it $\varphi_1$

In a similar fashion, we find that there are two more nontrivial homomorphisms: $\varphi_2$ sends $\sigma$ to 1 and and $\tau$ to $\alpha^2$, while $\varphi_3$ sends $\sigma$ and $\tau$ to $\alpha^2$.

$K(\varphi_1) = < \sigma^2, \tau > \cong C_2 \times C_2$, $K(\varphi_2) = < \sigma > \cong C_4$, $K(\varphi_3) = < \sigma^2, \sigma\tau > \cong C_2 \times C_2$. We can form an automorphism of $D_8$ by sending $\sigma^2$ to $\sigma^2$, $\sigma\tau$ to $\tau$, and $\tau$ to $\sigma\tau$. Thus the semidirect products constructed from the first and third homomorphisms are isomorphic, so we get two more groups:

(12) $C_5 \rtimes_{\varphi_1} D_8 = < a, \sigma, \tau | a^5 = \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^3, \tau a \tau = a, \sigma a \sigma^{-1} = a^{-1} >$

(13) $C_5 \rtimes_{\varphi_2} D_8 = < a, \sigma, \tau | a^5 = \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^3, \tau a \tau = a^{-1}, \sigma a \sigma^{-1} = a >$

Note that in (13), we have that, since a and $\sigma$ commute, $|a\sigma|$ = lcm(5,4) = 20. Also, $\tau\sigma a\tau = \sigma^{-1}\tau a\tau = \sigma^{-1}a^{-1} = (a\sigma)^{-1}$ and $\tau^2 = 1$. Thus, this group is $D_{40}$, the group of symmetries of a regular 20-gon.

(13) $D_{40} =< \sigma, \tau | \sigma^{20} = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} >$

3.5. **Subcase when $P_2 = Q$.**

We want nontrivial homomorphisms from $Q$ to $C_4$. We can represent Q as $< i, j | ij = k, i^2 = j^2 = k^2 = 1, ij = i^2 ji >$ [1]
Suppose $\varphi \colon Q \to C_4$ and let $\varphi(i) = \alpha, \varphi(j) = \alpha^3$. Then $\varphi(ij) = 1 \neq \alpha^2 = \varphi(i^2 ji)$.

If $\varphi(i) = \alpha, \varphi(j) = \alpha^2$, then $\varphi(i^2) \neq \varphi(j^2)$.

If $\varphi(i) = \alpha, \varphi(j) = \alpha$, then $\varphi(k^2) = 1 \neq \alpha^2 = \varphi(i^2)$.

If $\varphi(i) = \alpha, \varphi(j) = 1$, then $\varphi(i^2) \neq \varphi(j^2)$. So $\varphi(i) \neq \alpha$. Similarly, $\varphi(i) \neq \alpha^3$ as $\alpha^3$ is also a generator of $C_4$ and thus a and $\alpha^3$ are essentially the same in $C_4$ (they can be exchanged via automorphism).

If $\varphi(i) = \alpha^2, \varphi(j) = \alpha^2$, then $\varphi(ij) = \varphi(i^2 ji)$ and $\varphi(i^2) = \varphi(j^2) = \varphi(k^2)$. Thus, we have a nontrivial homomorphism. Similarly, we can work out nontrivial homomorphisms by picking 2 elements of $\{i, j, k\}$ and sending them to $\alpha^2$ and sending the third element to 1. We therefore have a total of 3 nontrivial homomorphisms. However, by exercise 72, automorphisms of Q consist of sending i to any element of $\{\pm i, \pm j, \pm k\}$ and then j to any of the other four elements in $\{\pm i, \pm j, \pm k\} \setminus < \alpha(i) >$. Thus, each of the semidirect products constructed by these homomorphisms will be isomorphic.

Thus, have one more subgroup of order 40:

(14) $C_5 \rtimes Q = < a, i, j | a^5 = 1, ij = k, i^2 = j^2 = k^2 = 1, ij = i^2 ji, iai^{-1} = a^{-1}, jaj^{-1} = a^{-1} >$

## 4. Center, Commutator, and Frattini Subgroups

Proposition: For $C_5 \rtimes_\varphi P_2$, $\varphi$ non-trivial, $Z(G) = K(\varphi) \cap Z(P_2)$. Since $C_5$ is acted upon nontrivially, at least one element in a does not commute with some element in $P_2$. Let a be this element. Then no element in $< a >$ is in the center as $< a >$ is cyclic and of prime order, so any element generates the whole subgroup. If $b \in P_2$ acts trivially on a, i.e. $\varphi_b(a) = a$, then b commutes with every element in $< a >$. If b also commutes with every element in $P_2$, then b $\in Z(G)$. We now only need to look at elements in $(< a > \cdot P_2) \backslash (< a > \cup P_2)$. Consider $a \cdot b$, and take any element that acts upon a nontrivially, say c. Since $c, b \in P_2, cb \in P_2$. We have $cab = (1, c)(a, 1)(1, b) = (1, c)(a, b) = (\varphi_c(a), cb)$. While $abc = (a, 1)(1, b)(1, c) = (a, bc)$. Thus, ab is not in the center as $\varphi_c(a) \neq a$. So we have exhausted all possibilities for the center, and we have $Z(G) = K(\varphi) \cap Z(P_2)$.

**4.1. ($C_5 \times C_8 = C_{40}$).** Abelian so Z(G) = G, [G,G] = 1. The maximal subgroups are $< a^2 >$ and $< a^5 >$. FratG = $< a^{10} >$.

**4.2. ($C_5 \times C_4 \times C_2 = C_{20} \times C_2$).** Abelian so Z(G) = G, [G,G] = 1. $C_5 \times C_4 \times C_2 = < a, b, c >$. Maximal subgroups: $< a, b >$, $< a, b^2, c >$, $< b, c >$. FratG = $< b^2 >$.

**4.3. ($C_5 \times C_2^3 = C_{10} \times C_2$).** Abelian so Z(G) = G, [G,G] = 1. $C_5 \times C_2 \times C_2 \times C_2 = < a, b, c, d >$ Maximal subgroups include: $< a, b, c >$, $< a, b, d >$, $< a, c, d >$, $< b, c, d >$. Thus, FratG = 1.

**4.4. ($C_5 \times D_8$).** First component ($C_5$) is abelian. In $D_8$, only 1, $\sigma^2 \in$ Z(G). Thus, Z(G) = $< a, \sigma^2 >$. In [G,G], note that the first component will always cancel. Consider the second component in the sense of $[D_8, D_8]$. $[\sigma, \tau] = \sigma\tau\sigma^{-1}\tau = \sigma^2$. Similarly, $[\sigma^3, \tau] = \sigma^2$. Since $\sigma^2$ commutes, we have $[D_8, D_8] = < \sigma^2 >$. Thus, [G,G] = $< \sigma^2 >$. Maximal subgroups: $D_8$, $< a, \sigma >$, $< a, \tau, \sigma^2 >$. Thus, FratG = $< \sigma^2 >$.

**4.5. ($C_5 \times Q$).** First component ($C_5$) is abelian. In $Q$, only 1,-1 $\in$ Z(Q). Thus, Z(G) = $< a, -1 >$. Similar to (4), in [G,G], the first component will always cancel. In [Q,Q], take any 2 elements of $\{i, j, k\}$, e.g. i,j. Then [i,j] = ij(-i)(-j) = kk = -1. Eventually, we find that [Q,Q] = $< -1 >$. Thus, [G,G] = $< -1 >$. Maximal subgroups: $Q$, $< a, i >$, $< a, j >$, $< a, k >$. FratG = $< -1 >$.

4.6. **($C_5 \rtimes_{\varphi_1} C_8 = < a, b|a^5 = b^8 = 1, bab^{-1} = a^2 >$).** By the proposition, $Z(G) =< b^4 >$. For the commutator, $[b, a] = bab^{-1}a^{-1} = a^2a^{-1} = a$. Thus $< a >\subset [G, G]$. Since a is normal, taking the commutator of anything in $< a >$ with anything in $< b >$ will result in something in a. Thus [G,G] = $< a >$. Maximal subgroups: $< b >, < a, b^2 >, a < b > a^{-1}, a^2 < b > a^{-2}, a^3 < b > a^{-3}, a^4 < b > a^{-4}$. The intersection of the first two subgroups is $< b^2 >$ In the third subgroup, $ab^2a^4 = aa^{-4}b^2 = a^2b^2$. Thus, $b^2 \notin a < b > a^{-1}$. However, $b^4$ is in the kernel of the homomorphism. So $FratG =< b^4 >$.

4.7. **($C_5 \rtimes_{\varphi_2} C_8 = < a, b|a^5 = b^8 = 1, bab^{-1} = a^{-1} >$).** By the proposition, $Z(G) =< b^2 >$. For the commutator, $[b, a] = bab^{-1}a^{-1} = a^{-1}a^{-1} = a^3$. Thus $< a >\subset [G, G]$. Similar to (6), $< a >$ is normal so $[G, G] =< a >$. Maximal subgroups: $< b >, < a, b^2 >, a < b > a^{-1}, a^2 < b > a^{-2}, a^3 < b > a^{-3}, a^4 < b > a^{-4}$. Since $b^2 \in Z(G)$, we have that $b^2$ is in all of the maximal subgroups. Thus, $FratG =< b^2 >$.

4.8. **($(C_5 \rtimes_{\varphi_1} C_4) \times C_2 = < a, b, c|a^5 = b^4 = c^2 = 1, bab^{-1} = a^2, cac^{-1} = a, cbc^{-1} = b >$).** By the proposition, $Z(G) =< c >$. Since $< c >$ is in the center, we look at commutators formed by elements from a and b. $[b, a] = bab^{-1}a^{-1} = a^2a^{-1} = a$, so $< a >\subset [G, G]$. Since a is normal, the commutators will only be in $< a >$. Therefore we have $[G, G] =< a >$. Maximal subgroups: $< a, b >, < a, b^2, c >, x < b, c > x^{-1}, x \in< a >$. The intersection of the first three subgroups (when x = 1 for the third subgroup) is $< b^2 >$. However, since $b^2$ is not in the kernel of the homomorphisms, $ab^2a^{-1} = aab^2 \neq b^2$. So $b^2 \notin a < b, c > a^{-1}$. c is in the center, so it is in all of the subgroups and $FratG = 1$.

4.9. **($(C_5 \rtimes_{\varphi_2} C_4) \times C_2 = < a, b, c|a^5 = b^4 = c^2 = 1, bab^{-1} = a^{-1}, cac^{-1} = a, cbc^{-1} = b >$).** By the proposition, $Z(G) =< c >$. Similar to (8), we look at commutators formed from elements of a and b. $[b, a] = bab^{-1}a^{-1}a^{-1}a^{-1} = a^3$, so $< a^3 >=< a >\subset [G, G]$. Again, similar to (8), $< a >$ is normal, so any other commutator will be in $< a >$. Therefore we have $[G, G] =< a >$. Maximal subgroups: $< a, b >, < a, b^2, c >, < b, c >, x < b, c > x^{-1}, x \in< a >$. $b^2$ is in the kernel of the homomorphism, so in this case, $FratG =< b^2 >$.

4.10. **($D_{10} \times C_4 = < \sigma, \tau, x|\sigma^5 = \tau^2 = x^4 = 1, x\sigma x^{-1} = \sigma, \tau\sigma\tau = \sigma^{-1}, x\tau x^{-1} = \tau >$).** $D_{10}$ has trivial center since $\sigma^5 = 1$ and 5 is odd. Everything in $C_4$ is abelian within that subgroup, but with the cross product, these elements are in the center of the entire group. Thus, Z(G) = $< x >$. $[\tau, \sigma] = \tau\sigma\tau\sigma^{-1} = \sigma^{-1}\sigma^{-1} = \sigma^3$. So $< \sigma^3 >=< \sigma >\subset [G, G]$. Since $< x >$ is in the center, it doesn't contribute to the commutator. With $< \sigma >$ normal in $D_{10}$, we cannot get any other elements in the commutator. Thus, $[G, G] = < \sigma >$. Maximal subgroups include: $< \sigma, x >$, $< \sigma, \tau, x^2 >$, $y < \tau, x > y^{-1}$ for $y \in< \sigma >$ The intersection of the first three subgroups is $< x^2 >$. This is contained in the other subgroups as well, so $FratG =< x^2 >$.

4.11. **($D_{10} \times C_2^2 =< \sigma, \tau, a, b|\sigma^5 = \tau^2 = a^2 = b^2 = 1, \tau\sigma\tau = \sigma^{-1}, aba^{-1} = b, a\sigma a^{-1} = \sigma, a\tau a^{-1} = \tau >$).** Similar to (10), the first component ($D_{10}$) has trivial center, while the second component is abelian and thus is in the center of the group. Therefore, $Z(G) =< a, b >$. Since elements from the second component are in the center, we only need to look at elements from $D_{10}$. $[\tau, \sigma] = \tau\sigma\tau\sigma^{-1} = \sigma^{-1}\sigma^{-1} = \sigma^3$. So $< \sigma^3 > = < \sigma >\subset [G, G]$.

Since $<\sigma>$ is normal in in $D_{10}$, we have that $[G,G] =<\sigma>$. Maximal subgroups: $<\sigma, a, b>, x<\tau, a, b>x^{-1}$ for $x \in <\sigma>$, $<\sigma, \tau, a>, <\sigma, \tau, b>$. Thus, $FratG = 1$.

4.12. **($C_5 \rtimes_{\varphi_1} D_8 = <a, \sigma, \tau | a^5 = \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}, \tau a \tau = a, \sigma a \sigma^{-1} = a^{-1}>$).** By the proposition, $Z(G) =<\sigma^2>$. From (4), we have that the second component ($D_8$) contributes $<\sigma^2>$ to the commutator. Also, $[\sigma, a] = \sigma a \sigma^{-1} a^{-1} = a^{-1}a^{-1} = a^3$. So $<a^3>=<a>\subset [G,G]$. Since a is normal, any other commutator with a will produce another element in $<a>$. Therefore, $[G,G] =<\sigma^2, a>$. Maximal subgroups: $<a, \sigma^2, \tau>$, $<a, \sigma>, xD_8x^{-1}$ for $x \in <a>$. The intersection of the first two subgroups is $<\sigma^2, a>$. Since $\sigma^2$ is in the center, so it is also in the other subgroups. Thus, $FratG =<\sigma^2>$.

4.13. **($D_{40} =<\sigma, \tau | \sigma^{20} = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}>$).** The only nontrivial element in the center is $\sigma^{20/2} = \sigma^{10}$. So $Z(G) =<\sigma^{10}>$. $[\tau, \sigma] = \tau\sigma\tau\sigma^{-1} = \sigma^{-1}\sigma^{-1} = \sigma^{18}$. Similarly, the commutator of $\sigma^k$ falls within $<\sigma^2>$. So $[G,G] =<\sigma^2>$. Maximal subgroups include $<\sigma>, <\tau, \sigma^2>, x<\tau, \sigma^5>x^{-1}$ for $x \in <\sigma^4>$. So $FratG =<\sigma^{10}>$.

4.14. **($C_5 \rtimes Q = <a, i, j | a^5 = 1, ij = k, i^2 = j^2 = k^2 = 1, ij = i^2ji, iai^{-1} = a^{-1}>$).** By the proposition $Z(G) =<i^2>=<-1>$. Similar to (5), elements in the second component will only contribute -1 to the commutator. $[a, i] = (a^{-1})^2 = a^3$, so we have $<a^3>=<a>\subset Z(G)$. Since the first component ($C_5$) is normal, we have $[G,G] =<a, i^2>$. Maximal subgroups: $<i, j>$ (Q), $<a, i>, <a, j>, <a, i \cdot j>, x<i, j>x^{-1}$ for $x \in <a>$. The intersection of the first four subgroups is $<i^2>$. Since $<i^2>$ is in the kernel of the homomorphism, it is also in the other maximal subgroups. So $FratG =<i^2>$.

## 5. Confirmation that these groups are not isomorphic

We know that if the respective sylow-2 subgroups are not isomorphic, then the groups cannot be isomorphic. Also, if $s_2(G)$ is different for two groups, then they cannot be isomorphic. From, these observations, we immediately get that (1), (2), (3), (4), and (5) are not isomorphic to any other group listed.

$Z(6) \cong C_2$ and $Z(7) \cong C_4$, so these groups are not isomorphic.

(8) and (9) have different Frattini subgroups, so they are not isomorphic. $Z(10) \cong C_4$ while $Z(8) \cong Z(9) \cong C_2$, so (10) is not isomorphic to (8) or (9).

(11) is the only group with $s_2(G) = 5$ and $P_2 \cong C_2^3$, so it is not isomorphic to any other group.

Although (12) and (13) have isomorphic centers, commutators, frattini subgroups, they are still not isomorphic. Take any element of order 5 in (12) (in the sylow-5 subgroup). Denote it a. Then $C_G(a) =<\tau, \sigma^2, a>$, which has no element of order 20. Consider $\sigma^4$ of order 5 in $D_{40}$. $C_G(\sigma^4) =<\sigma>\cong C_{20}$. Thus, (12) and (13) are not isomorphic.

(14) is the only group with $s_2(G) = 5$ and $P_2 \cong Q$, so it is not isomorphic to any other group.

## 6. Conclusion

To conclude, there are 14 non-isomorphic groups of order 40:

(1) $C_{40}$
(2) $C_{20} \times C_2$
(3) $C_{10} \times C_2^2$
(4) $C_5 \times D_8$
(5) $C_5 \times Q$
(6) $C_5 \rtimes_{\varphi_1} C_8$
(7) $C_5 \rtimes_{\varphi_2} C_8$
(8) $(C_5 \rtimes_{\varphi_1} C_4) \times C_2$
(9) $(C_5 \rtimes_{\varphi_2} C_4) \times C_2$
(10) $D_{10} \times C_4$
(11) $D_{10} \times C_2^2$
(12) $C_5 \rtimes D_8$
(13) $D_{40}$
(14) $C_5 \rtimes Q$

The first 3 are abelian, and the last 11 are not.

## References

[1] Lang, S. *Algebra*, Springer-Verlag, 2002.