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Seminar - uvod v raziskovalno delo

(Seminar - Introduction to Research Work)

Predstavitve grafov z induciranimi podgrafi Hammingovih grafov

(Representations of Graphs as Induced Subgraphs of Hamming Graphs)

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Izvleček: V tej seminarski nalogi obravnavamo inducirane podgrafe Hammingovih grafov. Inducirane vložitve grafov v večje grafe s posebno strukturo pogosto omogočajo boljše razumevanje lastnosti danih grafov. Z uporabo vložitev v kartezične produkte kvocientnih grafov sta Klavžar in Peterin leta 2005 opisala inducirane podgrafe Hammingovih grafov. V seminarski nalogi preučujemo kartezično dimenzijo grafa, invarianto, ki so jo leta 2017 vpeljali Milanič, Muršič in Mydlarz. Študija zajema tako strukturne kot računske vidike kartezične dimenzije grafov. Na kratko si ogledamo tudi Hammingovo dimenzijo grafov, ki so jo leta 2013 vpeljali Klavžar, Peterin in Zemljič.

Key words documentation

Abstract: In this term project paper we consider induced subgraphs of Hamming graphs. Representation of graphs as induced subgraphs of a larger graph with a special structure often leads to a better understanding of the properties of the given graphs. Using embeddings into Cartesian products of quotient graphs Klavžar and Peterin characterized induced subgraphs of Hamming graphs in 2005. We study the Cartesian dimension of graphs, an invariant introduced by Milanič, Muršič, and Mydlarz in 2017. The study consist of structural, algorithmic, and complexity aspects of the Cartesian dimension of graphs. We also take a quick look at Hamming dimension defined by Klavžar, Peterin and Zemljič in 2013.

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e.g.   for example

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1   Introduction

Representing graphs as induced subgraphs of a larger graph with a special structure often leads to a better understanding of the properties of a given graph. In this paper, we take a closer look at Hamming graphs. Hamming graphs are named after the mathematician Richard Hamming and are used in various branches of mathematics and computer science. By definition, they are the Cartesian products of complete graphs. In order to solve the problem of efficiently recognizing whether a graph is a Hamming graph, we can use the prime factorization algorithms with respect to the Cartesian product. Imrich and Klavžar present an algorithm for recognizing Hamming graphs which is linear in both time and space [12].

Hamming graphs have been characterized in various ways. In [19], Mulder has characterized the Hamming graphs as interval-regular graphs with some additional properties. This is analogous to his characterization of hypercubes as bipartite interval-regular graphs [19]. In [4], Egawa showed that if G is a connected graph with the same proper convex subgraphs as a Hamming graph where all r complete graphs have the same size n, then $|V(G)| \geq n^r$, with equality if and only if G is isomorphic to Hamming graph. In [18], Mollard gives two characterization of Hamming graphs. The first one is the analogue of the characterization of the hypercube as a $(0, 2)$-graph (a connected graph in which any two vertices have either 2 common neighbours or none at all) of maximal order; the second introduces the notion of quasi-interval.

Isometric subgraphs of Hamming graphs, which are called partial Hamming graphs, have also been extensively studied by now. Graham and Winkler [7] proved that every graph has a representation as an isometric subgraph of a Cartesian product graph. The key construction is based on embedding into Cartesian products of the so-called quotient graphs with respect to a certain relation defined on the edge set of a graph. Feder [6] followed with a similar approach in order to obtain such representations for stronger embeddability conditions: 2-isometric representation, weak retract representation, and Cartesian prime factorization.

A different direction was taken by Klavžar and Peterin in [13], treating isometry as the strongest property, and considering subgraphs, induced subgraphs, and isometric subgraphs of Hamming graphs. They showed, roughly speaking, that embeddings into Cartesian product of quotient graphs can be applied also to subgraphs and induced

subgraphs of Hamming graphs.

Several dimension-like graph invariants based on embeddings into product graphs have been studied by now. Milanič, Muršič, and Mydlarz study in [17] the following question: how difficult is it to determine if a given graph G can be realized in $\mathbb{R}^d$ so that vertices are mapped to distinct points and two vertices are adjacent if and only if the corresponding points are on a common line that is parallel to some axis? Any such mapping is referred as a d-realization of G and we say that a graph is d-realizable if it has a d-realization. A graph G is d-realizable if and only if it is an induced subgraph of a Cartesian product of d complete graphs, and the minimum non-negative integer d such that G is d-realizable (if such a d exists) is defined in [17] as the Cartesian dimension of G. In the literature, d-realizable graphs were studied under different names such as arrow graphs in [2], $(d-1)$-plane graphs and $(d-1)$-line graphs of d-partite d-uniform hypergraphs in [1], d-dimensional cellular graphs in [8], d-dimensional chess board graphs in [24], and d-dimensional gridline graphs in [20].

Recently, Sangha and Zito studied d-realizable graphs in the more general context of the so-called Line-of-Sight (LoS) networks in [22]. Line of Sight (LoS) networks offer a model of wireless communication that incorporates visibility constraints. Vertices of such networks can be embedded in finite d-dimensional grids of size n, and two vertices are adjacent if they share a line of sight and are at a distance of less than ω, where two points $x = (x_1, \ldots, x_d)$ and $x' = (x'_1, \ldots, x'_d)$ share a line of sight if there exists some $j \in \{1, \ldots, d\}$ such that $x_i = x'_i$ for all $i \in \{1, \ldots, d\} \setminus \{j\}$. They studied large independent sets in LoS networks. They proved that the computational problem of finding a largest independent set can be solved in polynomial time for one-dimensional LoS networks. However, for $d \geq 2$, the (decision version of) the problem becomes NP-hard for any fixed $\omega \geq 3$, even if restricted to $\omega \in O(n^{1-\epsilon})$ for any fixed $\epsilon > 0$. In addition, they showed that the problem is also NP-hard when $\omega = n$ for $d \geq 3$. This result extends earlier work, which demonstrated that the problem is solvable in polynomial time for gridline graphs when $d = 2$. For the small-dimensional cases, $d \in \{2, 3\}$, Peterson suggested an application of d-realizable graphs to robotics in [20]: if the movement of a robot is restricted to be along axis-parallel directions only and turns are allowable only at certain points, then a shortest path in a d-realized graph gives the number of turns required.

The dimension of a Hamming graph is the number of its factors, that is, the number of coordinates of its vertices. Another graph dimension defined in [14], the Hamming dimension of a graph G, is introduced as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. This invariant was introduced in order to significantly increase the number of graphs with a non-trivial dimension that come from the Cartesian product of graphs, as most of the graph dimensions do

not attain large values.

In this project paper we will take a closer look at representations of graphs as induced subgraphs of Hamming graphs. After summarizing the basic definitions and preliminaries in Chapter 2, we present in Chapter 3 the work done by Klavžar and Peterin, including a characterization of induced subgraphs of Hamming graphs. The main result in that section is a necessary and sufficient condition for a graph to be an induced subgraph of a (d-dimensional) Hamming graph. In Chapter 4 we discuss the properties of the Cartesian dimension, an invariant introduced by Milanič, Muršič, and Mydlarz, and examine how it is related with what is presented in Chapter 3. Graphs having dimension 0, 1, or 2 are well understood, while determining whether a graph has Cartesian dimension at most 3 is NP-complete. We also analyze the Cartesian dimension of some special graphs. In Chapter 5 we shortly introduce the definition of Hamming dimension, following [14], and present (without proofs) some bounds for the Hamming dimension of Sierpiński graphs.

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2 Preliminaries

Before we start with the results, we will fix the terminology used in the project paper and summarize some basic result about graphs. We only consider finite, undirected, and simple graphs without isolated vertices.

A graph consists of vertices which are connected by edges, and we write a graph G as $G = (V, E)$, where V is the vertex set and E is the edge set of G. If $G = (V, E)$, we write $V(G) = V$ and $E(G) = E$. A *induced subgraph* of a graph G is graph consisting of a subset of the vertices of G together with all edges of G whose endpoints are both in this subset. For $S \subseteq V(G)$, we denote by $G - S$ the subgraph of G induced by $V(G) \setminus S$. For $F \subseteq E(G)$, we denote by $G - F$ the graph $(V(G), E(G) \setminus F)$. The *degree* of a vertex is the number of edges incident to the vertex. A graph is *d-regular* if all its vertices are of degree d, and *cubic* if it is 3-regular. The *neighborhood* of a vertex v in a graph G is the set of all vertices adjacent to v. The neighbourhood of v together with vertex v is called the *closed neighborhood* of v. A *cut-edge* is an edge whose removal increases the number of components. The *distance* between two vertices u and v in a graph G is the number of edges in a shortest path connecting them; we denote it by $d_G(u, v)$. A subgraph H of a graph G is an *isometric subgraph* of G if $d_H(u, v) = d_G(u, v)$ for each pair of vertices u, v of H. The fact that two graphs G and H are isomorphic to each other will be denoted by $G \cong H$.

Let us now go through some special families of graphs. A *path* P_n is a graph whose vertices can be listed in order $v_1, v_2, \ldots, v_n$ such that the edges are exactly the pairs $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n - 1$. Similarly, a cycle C_n is a graph whose vertices can be listed in order $v_1, v_2, \ldots, v_n$ such that the edges are exactly the pairs $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n$ (where $v_{n+1} = v_1$). A cycle with exactly 3 edges is called a *triangle*. Given a graph G, a path or a cycle *in G* is a subgraph of G isomorphic to a P_n or a C_n. A *tree* is a connected acyclic graph. A graph is *bipartite* if the vertices can be partitioned into two sets V_1 and V_2 such that all edges go only from V_1 to V_2 (that is, no edges go from V_1 to V_1 or from V_2 to V_2). A *complete graph* is a graph with an edge between every two vertices; a complete graph with n vertices is denoted by K_n. Figure 1 shows the diamond which is a K_4 minus one edge and the claw graph, which is the complete bipartite graph $K_{1,3}$.

Let $\mathcal{F}$ be a set of graphs. A graph G is said to be $\mathcal{F}$-*free* if no induced subgraph
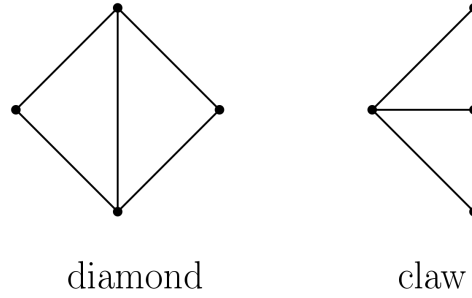
diamond   claw

Figure 1: The diamond and the claw.

of G is isomorphic to a graph from $\mathcal{F}$. Given two graphs G and H, we say that G is H-*free* if it is $\{H\}$-free. A *clique* in a graph is a set of pairwise adjacent vertices. A *maximal clique* is a clique that is not a subset of a larger clique. For a graph G, $L(G)$ denotes the *line graph* of G. The vertices of $L(G)$ are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent, see Figure 2 for an example. The *clique graph* $K(G)$ of a graph G has as its vertex set the maximal cliques of G, with two vertices of $K(G)$ adjacent whenever their corresponding cliques in G have some vertex in common.

G $L(G)$
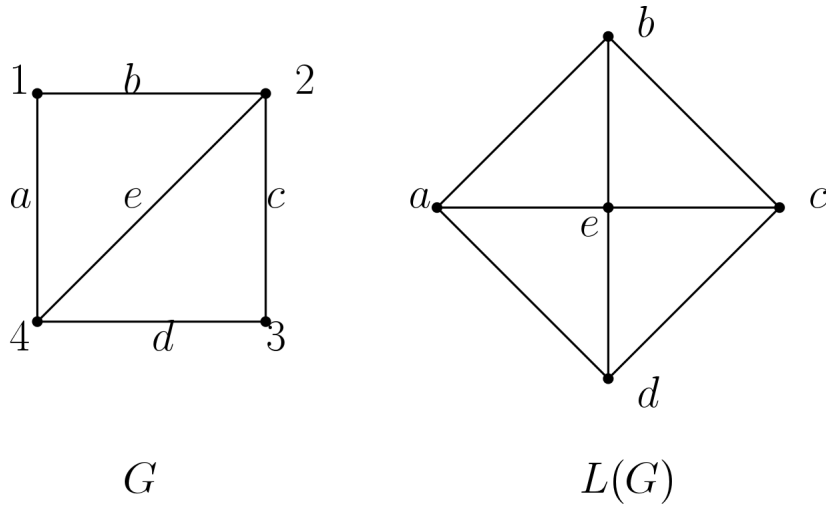
Figure 2: A line graph.

Given a graph G, a *d-edge-labeling* of G is a mapping from $E(G)$ to some set L of labels, where $|L| = d$. Given a d-edge-labeling of G and a set $F \subseteq E(G)$, we say that F is *monochromatic* if the labeling is the same for any two edges in F. An *edge coloring* of a graph G is a mapping from $E(G)$ to some set L of labels such that no two incident edges have the same label. In this case, labels may also be referred to as *colors*. The *chromatic index* of a graph G is the smallest non-negative integer k such

that G has an edge coloring with k colors. Unless stated otherwise, a *labeling* (or more precisely a *k-labeling*) of a graph G will always mean an edge-labeling (with k labels).

The following lemma proved, e.g., in [26], describes when an edge in a graph is cut-edge.

Lemma 2.1. *An edge of a graph G is a cut-edge if and only if it does not belong to any cycle.*

Proof. Take any edge $e = \{u, v\}$. We may assume without loss of generality that G is connected, since otherwise we can replace G with the connected component of G containing e. This will not affect any of the two properties of e consider in the statement of the lemma. Remove this edge from our graph. If the resulting graph has the same number of component as G, then there is some path from u to v not involving e; consequently, if we add e to the end of this path, we get a cycle. Thus, if e is not a cut-edge, then e belongs to a cycle.

Conversely, suppose that $e = \{u, v\}$ lies in a cycle C. Then $P = C - e$ is a path from u to v that does not use e. Pick any two vertices x, y in G; because G is connected, there is a path from x to y in G. Take this path, and edit it as follows: whenever the edge e shows up, replace this with the path P (or P traced backwards, if needed). This then creates a walk from x to y; by deleting cycles if necessary, this walk can be turned into a path from x to y, and thus the graph $G - e$ is connected. So if e is contained in a cycle, it is not a cut-edge. $\square$

A *decision problem* is a problem for which the set of instances divides into two sets depending on whether the answer is YES or NO. In computational complexity theory, the following three classes of decision problems are of interest:

- P – the set of decision problems that can be solved by a polynomial-time algorithm. Intuitively: P is the set of problems that can be solved efficiently.

- NP – the set of decision problems with the following property: If the answer is YES, then there exists a certificate that enables us to verify this fact in polynomial time. Intuitively: NP is the set of problems for which we can quickly verify a positive answer if we are given a solution.

- *co-NP* – the set of decision problems with the following property: If the answer is NO, then there exists a certificate that enables us to verify this fact in polynomial time.

3   Characterizing Induced Subgraphs of Hamming Graphs

In this chapter we will present necessary and sufficient conditions for a graph to be an induced subgraph of a Hamming graph and some results related to those conditions.

3.1   The definition of a KP-labeling

Before presenting to the result we need to go through some definitions.

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which a vertex (a, x) is adjacent to a vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. See Figure 3 for an example. For a fixed vertex a of G, the vertices $\{(a, x) \mid x \in V(H)\}$ induce a subgraph of $G \square H$ isomorphic to H, called an *H-layer* of $G \square H$. Analogously we define *G-layers*.
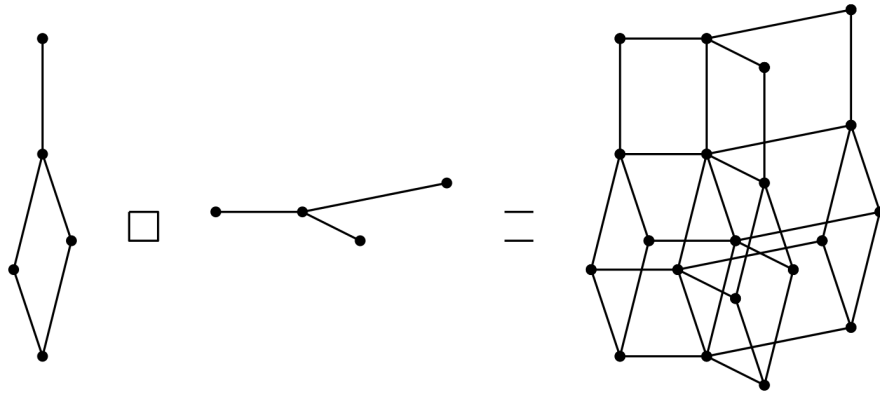
Figure 3: The Cartesian product of graphs.

More generally, the Cartesian product $G_1 \square G_2 \square \ldots \square G_n$ of graphs $G_1, G_2, \ldots, G_n$ is the the graph with vertex set $V(G_1) \times V(G_2) \times \ldots \times V(G_n)$ in which a vertex $(u_1, u_2, \ldots, u_n)$ is adjacent to a vertex $(v_1, v_2, \ldots, v_n)$ if and only if there exist an $i \in \{1, 2, \ldots, n\}$ such that $u_i v_i \in E(G_i)$ and $u_j = v_j$ for every $j \in \{1, 2, \ldots, n\} \setminus \{i\}$.

The map $p_G : V(G \square H) \to G$ defined by $p_G(a, x) = a$, is called a *projection*. The image of an edge $(a, x)(b, y)$ under projection p_G is an edge when $x = y$, and a vertex

when $a = b$. Extending the definition, a projection $p_G : V(G \square H_1 \square H_2 \square \ldots \square H_n) \to G$ is defined by $p_G(a, h_1, h_2, \ldots, h_n) = a$.

Cartesian products of complete graphs are known as *Hamming graphs*. They can also be described as follows. For $i = 1, 2, \ldots, n$ let $r_i \geq 2$ be given integers. Let G be the graph whose vertices are the n-tuples $b_1 b_2 \cdots b_n$ with $b_i \in \{0, 1, \ldots, r_i - 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one coordinate. The vertex set of G is the same as that of $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ and edges in $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ between two vertices are exactly those that have same vertex in every graph K_{r_i} but one. Since in complete graphs we have all possible edges, it is easy to see that G is isomorphic to $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$. Given a Hamming graph $H = K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$, we define the *color map* $c : E(H) \to \{1, 2, \ldots, n\}$ by setting, for all edges $uv \in E(H)$, $c(uv) = i$ where u and v differ in coordinate i.

Let G be a connected graph and let $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ be a partition of $E(G)$. The *quotient graph* G/F_i has connected components of $G - F_i$ as vertices, two components C and C' being adjacent whenever there exists an edge of F_i connecting a vertex of C with a vertex of C'. For each i, define a map $f_i : V(G) \to V(G/F_i)$ by $f_i(v) = C$, where C is the component of $G - F_i$ containing v. Then let

$$f : V(G) \to V((G/F_1) \square (G/F_2) \square \ldots \square (G/F_k))$$

be the natural coordinate-wise mapping, that is,

$$f(v) = (f_1(v), f_2(v), \ldots, f_k(v)).$$

We call f the *quotient map* of G with respect to $\mathcal{F}$. Note that f need not be one-to-one in general and that it is possible that some quotient graphs are the one vertex graph, an example is shown in Figure 4.
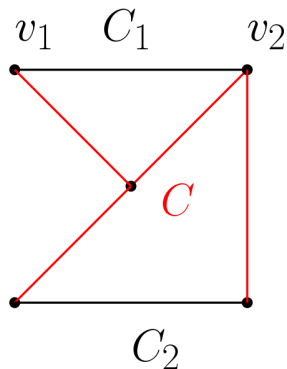
Figure 4: An example of a graph such that the quotient map is not one-to-one.

However, all the partitions $\mathcal{F}$ introduced later will lead to one-to-one mappings with non-trivial quotient graphs.

A partition $\{F_1, F_2, \ldots, F_k\}$ of $E(G)$ naturally leads to an edge-labeling $\ell : E(G) \to \{1, 2, \ldots, k\}$ by setting $\ell(e) = i$, where $e \in F_i$.

Since the concept of the quotient graph is a central concept of this chapter, we also visualize it to get a better understanding.
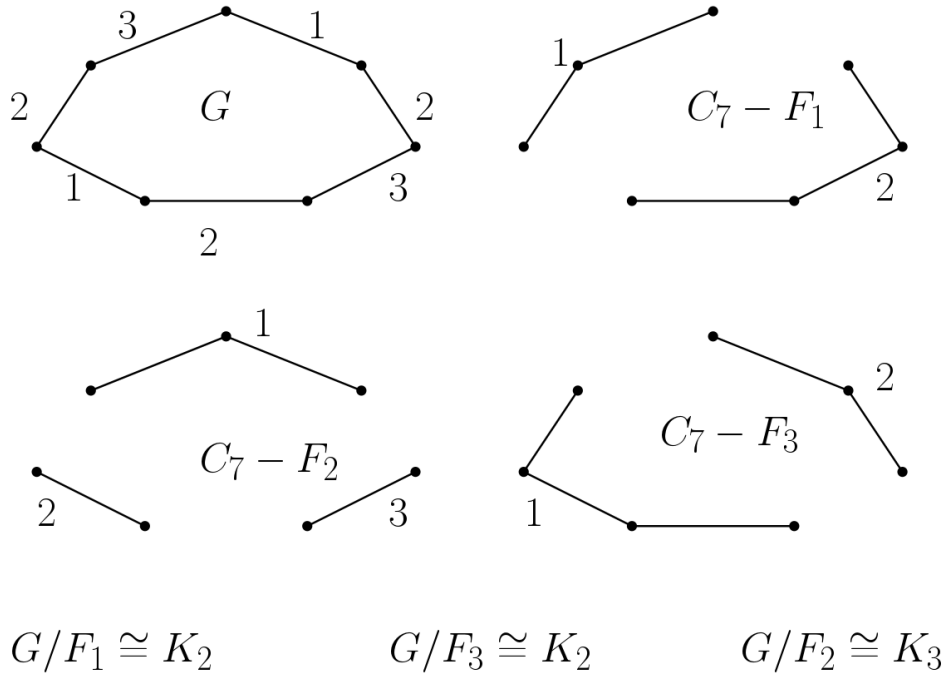
Figure 5: A quotient graph of C_7.

Consider the labeling of the 7-cycle, C_7, given in Figure 5, and the partition of $E(C_7)$ obtained by defining F_i to be the set of edges labeled i, for $i \in \{1, 2, 3\}$. As we can see, C_7/F_1 and C_7/F_3 have only two vertices and since there is an edge in the corresponding F_i connecting those two components, each of C_7/F_1 and C_7/F_3 is simply a K_2, while C_7/F_2 has three vertices but since any two components of $C_7 - F_2$ are connected with an edge from F_2, the graph C_7/F_2 is simply a K_3.

Klavžar and Peterin defined two conditions in their characterization of induced subgraphs of Hamming graphs. They actually named them by letters B and C, and also defined condition A, which they needed for characterizing subgraphs of Hamming graphs, but since we will not need condition A, we will name the two conditions simply by numbers 1 and 2 (following [17]).

We say that a d-edge-labeling of a graph G is a *(d-)KP-labeling* if it satisfies the following conditions:

Condition 1 The edges of any triangle have the same label.

Condition 2 For any two non-adjacent vertices u and v in G, there exist different labels i and j which both appear on any induced u, v-path.

Example 3.1. Consider the two edge labelings of the same graph G as in Figure 6.
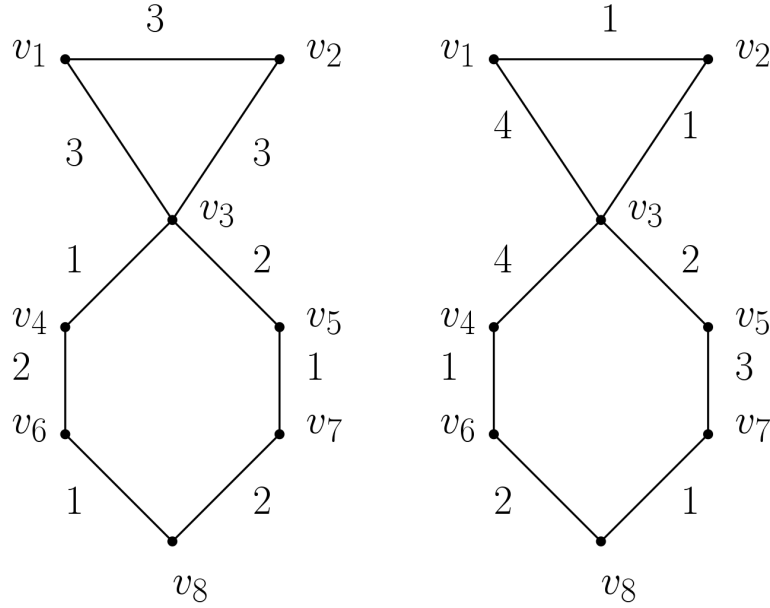
Figure 6: The labeling on the left is a KP-labeling, while the one on the right is not.

The labeling in the left side of the figure satisfies condition 1, as the edges of the triangle have the same label. It also satisfies condition 2, as it can be checked that for arbitrary vertices u and v of G with $d_G(u, v) \geq 2$ there exist different labels i and j which both appear on any induced u, v-path. Therefore the labeling in the left side of the figure is a KP-labeling.

The labeling in the right of the figure satisfies neither condition 1 nor 2, as the edges of the triangle do not have the same label, and also if we take vertices v_5 and v_8 then the two paths $(v_5, v_3, v_4, v_6, v_8)$ and (v_5, v_7, v_8) have only one label in common. Moreover, if we consider vertices v_1 and v_4, then only one label appears on the path (v_1, v_3, v_4). ▲

3.2   KP-labeling: basic results

Examples of graphs with no KP-labeling are the C_5 and the diamond. The fact that these two graphs, as well as some others, have no KP-labeling, will be proved in Chapter 4, see Theorem 4.7. For the diamond, the argument is particularly simple: the two triangles of the diamond share an edge, and by Condition 1 edges in any triangle must be all labeled with the same label; but then Condition 2 will fail.

Next we try to understand Conditions 1 and 2 on cycle graphs. In this respect, we have the following result.

Lemma 3.2. *Let G be a labeled graph satisfying Condition 2 and let C be an induced cycle in G of length at least 4. Then every label of C is present more than once on C.*

Proof. The labels of a u, v-path of length 2 on a C must be different, otherwise we cannot find two labels appearing in every u, v-path, hence by Condition 2 those two labels should appear also along the other u, v-path in C. So for any label take two vertices u and v that have distance two and such that a u, v-path of length two contains the edge with that label. Then, the label is present on both u, v-paths in C, therefore the label is present more than once on C. □

In order to prove the main result (Theorem 3.5), we will need also the following lemma.

Lemma 3.3. *Let G be a labeled graph satisfying Conditions 1 and 2 and let u, v be vertices of G with $d_G(u, v) \geq 2$. Then, if labels i and j appear on every induced u, v-path, they appear on every u, v-path.*

Proof. Suppose that labels i and j appear on every induced u, v-path but not on every u, v-path. Let $P = x_1 x_2 \ldots x_r$, with $x_1 = u$, $x_r = v$, be a u, v-path of minimal length that does not contain both labels i and j. Then P is not induced since every induced path contains both labels, hence we have an edge between some non-consecutive vertices, say $e = x_k x_l$ with $l - k > 1$. We may assume that e is chosen so that $l - k$ is as small as possible. By the minimality of P, the path $x_1 x_2 \ldots x_k x_l x_{l+1} \ldots x_r$ (which is shorter than P) contains both labels i and j. Hence, the label of the edge $x_k x_l$ is either i or j, otherwise P would contain both labels i and j. Assume without loss of generality that the label is i. Then, label j appears either on the path $x_1 x_2 \ldots x_k$ or on the path $x_l x_{l+1} \ldots x_r$ (note that it cannot happen that $k = 1$ and $l = r$ as $d_G(u, v) \geq 2$). It follows that i does not appear on the path $x_k x_{k+1} \ldots x_l$. But then label i appears only once on the induced cycle $C = x_k x_{k+1} \ldots x_l x_k$. If C is a triangle, we have a contradiction with Condition 1, otherwise with Lemma 3.2. □

If G is a tree, then assigning a different label to each edge results in a labeling satisfying Conditions 1 and 2, hence in a KP-labeling. Note that every edge in a tree is a cut-edge. More generally, we can ask: which edges in a graph that has a KP-labeling can receive a unique label in some KP-labeling? This question is answered by the following result.

Proposition 3.4. *Let G be a graph having a KP-labeling and let e be an edge in G. Then G has a KP-labeling in which e receives a unique label if and only if e is a cut-edge.*

Proof. First we will show that if e is a cut-edge then G has a KP-labeling in which e receives a unique label. Fix an arbitrary KP-labeling of G.

Relabel edge e with a new label, say i, not used before. We will show that the new labeling is also a KP-labeling. We need to verify the following two conditions:

Condition 1: Every triangle is monochromatic.

Condition 2: for every pair of distinct non-adjacent vertices u, v, there exist different labels j and k which both appear on every induced u, v-path.

Condition 1 holds, since e is not part of any cycle (see Lemma 2.1) and therefore every triangle is still monochromatic. To verify Condition 2, note that any two u, v-paths either both contain edge e or none (otherwise e would be part of a cycle, contradicting Lemma 2.1).

If none of the induced u, v-paths contains edge e then the new KP-labeling still satisfies Condition 2 for the pair u, v because of the original KP-labeling satisfied the condition for this pair.

Otherwise all induced u, v-paths contain e. Let the original label of e be i_0. Also let j, k be the labels that appear on every u, v-path. We consider two cases. If $i_0 \in \{j, k\}$ then we take the new label of e and one of the two labels that appeared on every u, v-path different from the original label of e, more formally $(\{j, k\} \setminus \{i_0\}) \cup \{i\}$. If $i_0 \notin \{j, k\}$ then take the two labels that appeared on every u, v-path, more formally $\{j, k\}$.

For the converse direction, suppose G has a KP-labeling in which e receives a unique label. Then by Lemma 3.2 we have that e does not belong to a cycle and from Lemma 2.1 we have that e is a cut-edge. $\square$

3.3   Induced subgraphs of Hamming Graphs

The following characterization of induced subgraphs of Hamming graphs is due to Klavžar and Peterin [13].

Theorem 3.5. *Let G be a connected graph. Then G is an induced subgraph of a Hamming graph of dimension d if and only if G has a d-KP-labeling.*

Proof. Let G be an induced subgraph of $H = K_{n_1} \square K_{n_2} \square \ldots \square K_{n_d}$. To make things easier denote $p_i = p_{K_{n_i}}$, which is the projection to the complete graph K_{n_i} of H and also consider the labeling $E(G)$ induced by the color map c of H. We will show that this labeling satisfies Conditions 1 and 2.

Condition 1 is clear. Indeed, if u, v, and w induce a triangle, then they all lie in the same layer of H and by the alternative description of Hamming graphs we have that each pair is adjacent so they differ in exactly in one coordinate. Therefore color map

c will map those edges to the coordinate in which they differ and so the edges uv, uw, and vw receive the same label.

We next show that Condition 2 is satisfied, too. Let u and v be two vertices of G with $d_G(u, v) \geq 2$. Suppose that there is no label that appears on all induced u, v-paths. Now if we see u and v as d-tuples and we understand walking along paths as changing coordinates. Therefore $p_i(u) = p_i(v)$ for all i because if we look at one induced u, v-path that does not have label i, that means than we never change the ith coordinate as we traverse this path and therefore the ith coordinates are the same. This implies $u = v$, contrary to $d_G(u, v) \geq 2$. Suppose now that all induced u, v-paths have exactly one label in common, say i. We have $p_j(u) = p_j(v)$ for all $j \neq i$ and consequently $p_i(u) \neq p_i(v)$ (since $u \neq v$). Vertices $p_i(u)$ and $p_i(v)$ are adjacent in K_{n_i}, since K_{n_i} is a complete graph. Hence, u and v are adjacent in H and therefore also in G, which is impossible, since $d_G(u, v) \geq 2$.

Conversely, let ℓ be a d-edge-labeling of G that fulfills Conditions 1 and 2. Let $\mathcal{F} = \{F_1, F_2, \ldots F_d\}$ be the partition of $E(G)$ induced by ℓ and let f be the quotient map of G with respect to $\mathcal{F}$. We claim that f embeds G as an induced subgraph into $(G/F_1) \square (G/F_2) \square \ldots \square (G/F_d)$.

We show first that f is one-to-one. Suppose that vertices x and y are not adjacent in G. Then by Condition 2 and Lemma 3.3, there exist labels i and j such that on every x, y-path in G we find labels i and j. So x and y are in different components in both $G \setminus F_i$ and $G \setminus F_j$. Already the first fact assures that $f(x) \neq f(y)$. Let next x and y be adjacent vertices of G and let $\ell(xy) = i$. We can see that $f_j(x) = f_j(y)$ for every $j \neq i$, vertices x and y belong to the same component $G - F_j$ as they are adjacent.

Now let us take a look at f_i. Suppose that there exists an x, y-path $P = x_1 x_2 \ldots x_r$ in $G \setminus F_i$, where $x_1 = x$ and $x_r = y$. We can assume that P is shortest among all x, y-paths in $G \setminus F_i$. If P is induced in $G - xy$ we have a contradiction with Condition 1 when $r = 3$ since P does not contain label i (recall that P is in $G \setminus F_i$) but a triangle will be formed with xy and two edges in P. When $r > 3$, we have a contradiction with Lemma 3.2 if P is an induced path in $G - xy$ because $P + xy$ would form a cycle in which label i is present only once. Thus P is not induced in $G - xy$, $r > 3$, and there are adjacent vertices x_j and x_k with $k > j + 1$ and $\{j, k\} \neq \{1, r\}$. By the minimality of P we have $\ell(x_j x_k) = i$. We can select j and k such that $k - j$ is minimal among all such vertices x_j and x_k. Then the cycle $C = x_j x_{j+1} \ldots x_{k-1} x_k$ is induced. If C is a triangle we have a contradiction with Condition 1, otherwise we have a contradiction with Lemma 3.2. Hence, $f_i(x) \neq f_i(y)$ and we have shown that f is one-to-one.

Let xy be an edge of G with $\ell(xy) = i$. Then, by the above, x and y are in different components of $G - F_i$. Moreover, they belong to the same component in any of the graphs $G - F_j$, for $j \neq i$. It follows that f maps edges to edges.

Let x and y be non-adjacent vertices of G. Then x and y are in different components of at least two graphs $G - F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates. Hence f maps non-edges to non-edges, and G is isomorphic to the subgraph $(G/F_1) \Box (G/F_2) \Box \ldots \Box (G/F_d)$ induced by $f(V(G))$, as claimed.

To complete the proof we show that G is also an induced subgraph of the Hamming graph $K_{|V(G/F_1)|} \Box K_{|V(G/F_2)|} \Box \ldots \Box K_{|V(G/F_k)|}$. Let x and y be non-adjacent vertices of G. By the same reasoning as above, x and y are in different components of at least two graphs $G - F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates and this fact remains valid after adding edges to the factor graphs. $\square$

Example 3.6. Let us see that C_7 is an induced subgraph of a Hamming graph by showing that there exists a labeling of C_7 that fulfills Conditions 1 and 2. The cycle C_7 is an induced subgraph of $K_2 \Box K_3 \Box K_2$.

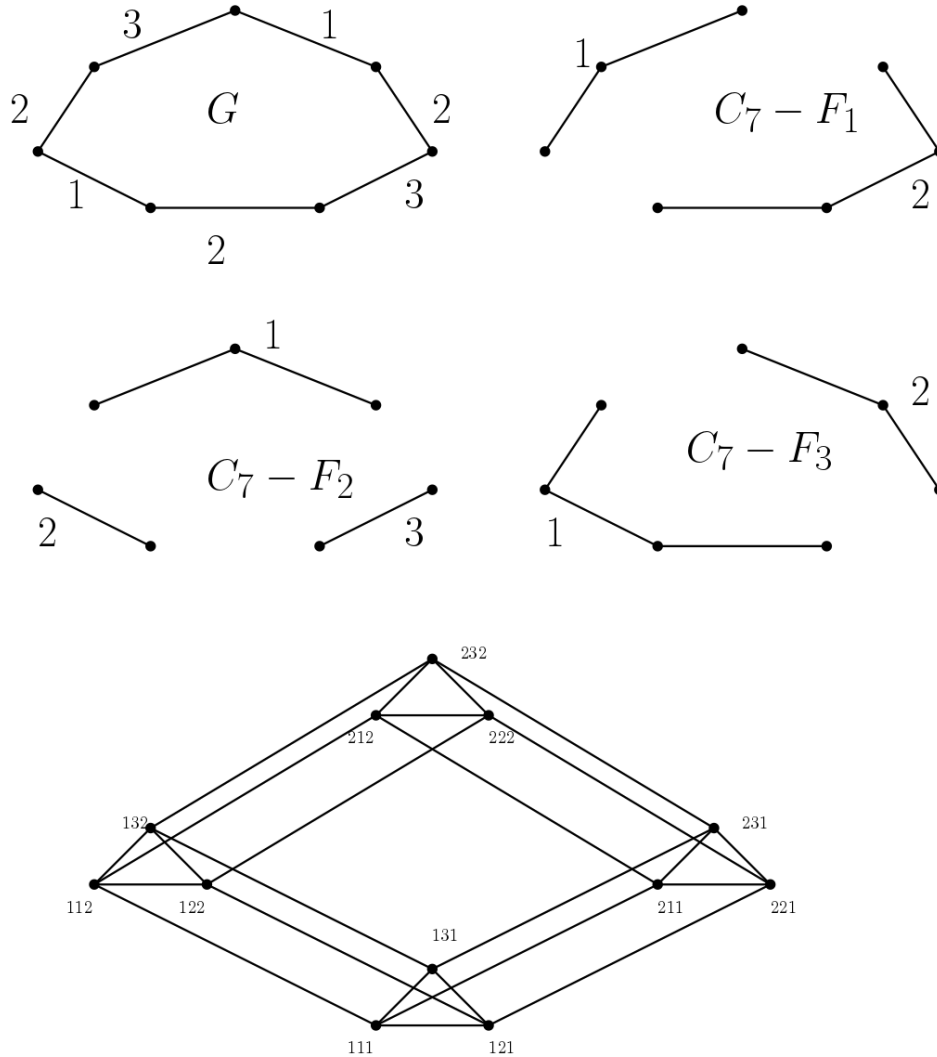
Figure 7: C_7 as an induced subgraph of $K_2 \Box K_3 \Box K_2$.

In Figure 7 take the cycle $(112, 212, 232, 231, 221, 121, 111)$ and using the color map c, edge $\{111, 112\}$ gets label 3, edge $\{112, 212\}$ gets label 1, and edge $\{212, 232\}$ gets label 2, edge $\{232, 231\}$ gets label 3, edge $\{231, 221\}$ gets label 2, edge $\{221, 121\}$ gets label 1, and edge $\{121, 111\}$ gets label 2. We can easily check that edges of each triangle have the same label (in fact, here there is no triangle, so the condition is satisfied trivially) and for any two vertices u and v of C_7 with $d_{C_7}(u, v) \geq 2$, there exist different labels i and j which both appear on any induced u, v-path.

The cycle C_7 has a labeling in which Conditions 1 and 2 hold, as shown in the first graph in Figure 7. Now let us take a look at where does the map f send vertices of C_7. We follow the order clockwise from top-most vertex f maps the first vertex to 112, the second vertex to 212, the third vertex to 232, the fourth vertex to 231, the fifth vertex to 221, sixth vertex to 121, and the seventh vertex to 111. As we can see $G \cong f(G)$ and an induced subgraph of $K_2 \square K_3 \square K_2$.   ▲

Note that the quotient graphs obtained in the proof of Theorem 3.5 need not be complete. For instance, consider the path P_4 together with the labeling 1, 2, 1 along the edges of the path (in order). Then get $G/F_1 \cong P_3$.

4 Cartesian Dimension

Given a non-negative integer d, a *d-realization* of a graph $G = (V, E)$ is an injective mapping $\varphi_G : V \to \mathbb{R}^d$ such that two vertices $u, v \in V$ are adjacent if and only if $\varphi_G(u)$ and $\varphi_G(v)$ differ in exactly one coordinate. A graph G is said to be *d-realizable* if it has a d-realization. Note that G is d-realizable if and only if G has a d-realization $\varphi_G : V \to \mathbb{N}^d$ or a d-realization $\varphi_G : V \to S^d$ where $S \subseteq \mathbb{N}$ is sufficiently large, since $V(G)$ is finite.

Any graph that has a d-realization also has a $(d + 1)$-realization (simply add one coordinate constantly equal to 0). This naturally leads to a definition: the Cartesian dimension of a graph $G = (V, E)$, denoted $\mathrm{Cdim}(G)$, is defined as the minimum non-negative integer d such that G is d-realizable, if such an integer exists, and ∞, otherwise. Note that K_1 is the only graph of Cartesian dimension 0.

For a graph G to have Cartesian dimension 1 it must have a 1-realization, or, in other words, we can map all the vertices of the graph to $\mathbb{R}$. Because the map has to be injective, every vertex is mapped to different a real number. But then every two vertices differ in exactly one coordinate, which makes them adjacent. Therefore the graphs of Cartesian dimension 1 are exactly the complete graphs of order at least 2.

Example 4.1. The cycle C_6 has Cartesian dimension 2. Indeed, since it is not a complete graph it cannot have dimension 1, and we find a 2-realization of C_6 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ by the map $\varphi = \varphi_{C_6} : V \to \mathbb{R}^2$ given by $\varphi(v_1) = (1, 1)$, $\varphi(v_2) = (3, 1)$, $\varphi(v_3) = (3, 3)$, $\varphi(v_4) = (2, 3)$, $\varphi(v_5) = (2, 2)$, and $\varphi(v_6) = (1, 2)$ (see Figure 8).

An easy check shows that indeed for any edge the corresponding values of φ differ in exactly one coordinate, while for any non-edge they differ in both coordinates. ▲

We will see that certain diamond-free graphs may have a 2-realization. In order to show that we first need the following lemma and two of its corollaries.

Lemma 4.2. *If C is a maximal clique in a diamond-free graph G and $v \in V(G) \setminus C$, then v is adjacent to at most one vertex in C.*

Proof. Suppose for a contradiction that v is adjacent to two distinct vertices in C, say x and y. Since C is a maximal clique, there exist a vertex in C, say z, such that is not adjacent to v (otherwise $C \cup \{v\}$ would be a clique properly containing C,

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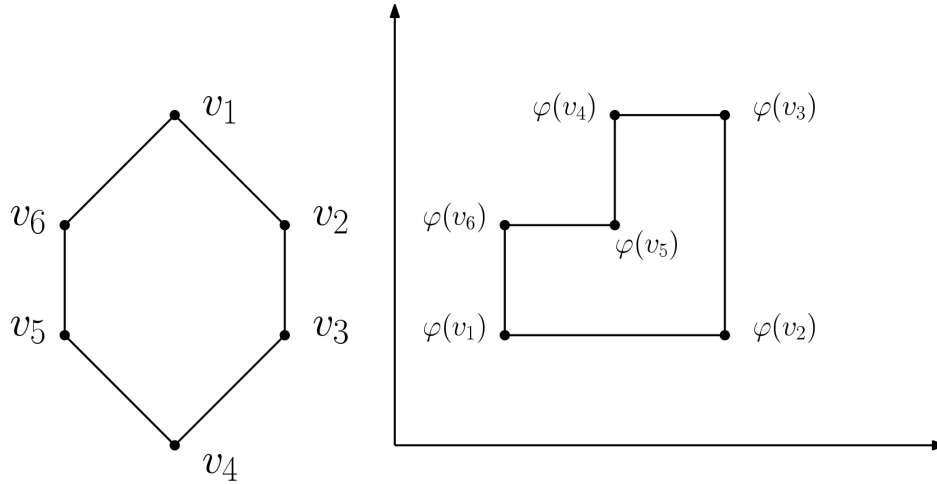
Figure 8: A 2-realization of C_6.

contradicting the maximality of C). But now, vertices v, x, y, z induce a diamond in G, a contradiction. □

Corollary 4.3. *If C and C' are maximal cliques in a diamond-free graph G such that $C \neq C'$ and $C \cap C' \neq \emptyset$, then $|C \cap C'| = 1$ and there are no edges between vertices of $C \setminus C'$ and vertices of $C' \setminus C$.*

Proof. If $|C \cap C'| \geq 2$ then any vertex in $C \setminus C'$ would be adjacent to at least two vertices in C', contradicting Lemma 4.2. Thus, $|C \cap C'| = 1$. Let $\{v\} = C \cap C'$. If a vertex $x \in C \setminus C'$ is adjacent to a vertex $y \in C' \setminus C$, then x is a vertex not in C' adjacent to at least two vertices in C' (namely, v and y), again contradicting Lemma 4.2. □

Corollary 4.4. *If G is a $\{diamond, claw\}$-free graph, then every vertex $v \in V(G)$ is contained in at most two maximal cliques.*

Proof. Suppose $v \in V(G)$ is contained in three distinct maximal cliques of G, say C_1, C_2, C_3. For $i \in \{1, 2, 3\}$, let $v_i \in C_i \setminus \{v\}$. Then v, v_1, v_2, v_3 induce a claw by Corollary 4.3, a contradiction. □

Theorem 4.5. *For every graph G, the following conditions are equivalent:*

1. *$\mathrm{Cdim}(G) \leq 2$.*

2. *G is the line graph of a bipartite graph.*

3. *G is diamond-free and its clique graph $K(G)$ is bipartite.*

4. *G is $\{claw, diamond, C_5, C_7, \ldots\}$-free.*

Proof. First, we prove that condition 1 implies condition 4. We show that none of the graphs in condition 4 can be an induced subgraph of a graph with $\text{Cdim}(G) \leq 2$. Suppose that G is not {claw, diamond, C_5, C_7, ... }-free. Clearly, since G is not complete, $\text{Cdim}(G)$ cannot be equal to 1. Now suppose that $\text{Cdim}(G) = 2$, and fix a 2-realization φ_G of G.

First let us consider the diamond. Suppose G has a diamond graph, H, as induced subgraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(H) = \{v_1 v_2, v_1 v_3, v_2 v_3, v_2 v_4, v_3 v_4\}$. Now because v_1, v_2, and v_3 induce a complete graph on 3 vertices, all three together differ in the same coordinate, similarly for v_2, v_3 and v_4. It follows that also v_1 and v_4 differ only in one coordinate but they are not adjacent, a contradiction.

Next suppose G has a claw graph, H, as induced subgraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(H) = \{v_1 v_2, v_1 v_3, v_1 v_4\}$. Without loss of generality assume that $\varphi_G(v_1)$ differs only in the first coordinate with $\varphi_G(v_2)$. Because v_2 and v_3 are not adjacent, $\varphi_G(v_1)$ and $\varphi_G(v_3)$ must differ only in the second coordinate. Because v_1 and v_4 are adjacent, they must differ in only one coordinate, but either coordinate is not good as that would imply adjacency between v_4 and v_2, or between v_4 and v_3, but there is no edge among them.

Finally, consider an induced subgraph H of G such that H is isomorphic to an odd cycle of length more than 3. Let the vertex set of H be $V(H) = \{v_1, v_2, \ldots, v_k\}$ with edge set $E(H) = \{v_1 v_2, v_2 v_3, \ldots, v_k v_1\}$ and notice that since there is no triangle, no three vertices lie in the same line. So if we consider the color map defined in Section 3.1, then by following the cycle we must alternate by colors 1 and 2, but the cycle is odd, so this is not possible.

Now, we prove that condition 4 implies condition 3. Suppose that G is {claw, diamond, C_5, C_7,...}-free. We only need to show that $K(G)$ is bipartite, which is equivalent to showing that $K(G)$ has no odd cycle. Suppose $K(G)$ has an odd cycle. First note that by Corollary 4.4 no three maximal cliques (that is, vertices of $K(G)$) share a vertex. Thus we form an odd cycle in $K(G)$ by following the common vertices of cliques. But G is {$C_5, C_7, \ldots$}-free, therefore any odd cycle in $K(G)$ has length 3. Let those cliques that form a triangle be A_1, A_2, A_3 and let v_1 be the common vertex of A_1 and A_2, let v_2 be the common vertex of A_2 and A_3, and let v_3 be the common vertex of A_3 and A_1. Because v_1 and v_2 are in A_2 they are adjacent, and similarly for the other pairs. Vertex v_3 is not in A_2 but is adjacent to two vertices in A_2 (namely to v_1 and v_2), a contradiction with Corollary 4.3. Therefore $K(G)$ is bipartite.

Now let us show that condition 3 implies condition 1. Since $K(G)$ is bipartite, we can properly color the vertices of $K(G)$ with two colors. Map the vertices of G in $\mathbb{R}^2$ by

placing cliques of G having one color on distinct vertical lines and cliques having the other color on distinct horizontal lines. By Corollary 4.3, no two vertices are mapped to the same point. This gives a 2-realization of G, showing that $\mathrm{Cdim}(G) \leq 2$.

Next, we show that condition 2 implies condition 1. Suppose that condition 2 holds. Then G is the line graph of a bipartite graph, say H. Let (X, Y) be a bipartition of H. Fix an ordering of X, say $X = \{x_1, \ldots, x_k\}$, and an ordering of Y, say $Y = \{y_1, \ldots, y_l\}$. For every two vertices $x_i \in X$ and $y_j \in Y$ such that $\{x_i, y_j\} \in E(H)$, define $\varphi_G(\{x_i, y_j\}) = (i, j)$. Then φ_G is one-to-one mapping from $V(G)$ to $\mathbb{R}^2$. Moreover, the fact that G is the line graph of H implies that φ_G is a 2-realization of G. This shows that $\mathrm{Cdim}(G) \leq 2$.

Finally, we show that condition 1 implies condition 2. Suppose that $\mathrm{Cdim}(G) \leq 2$ and fix a 2-realization φ_G of G. Identify each vertex $v \in V(G)$ with the set $S_v = \{(x, 1), (y, 2)\}$ where $\varphi_G(v) = (x, y)$. Then, two distinct vertices $u = \{(x^u, 1), (y^u, 2)\}$ and $v = \{(x^v, 1), (y^v, 2)\}$ are adjacent in G if and only if either $(x^u, 1) = (x^v, 1)$ or $(y^u, 2) = (y^v, 2)$. Moreover, since φ_G is one-to-one, both conditions cannot happen simultaneously. Thus, $\{u, v\} \in E(G)$ if and only if $S_u \cap S_v \neq \emptyset$, which implies that G is the line graph of some graph H such that $E(H) = \{S_v \mid v \in V(G)\}$. By construction, if two vertices of H are adjacent, then one of them is of the form $(x, 1)$ and the other one of $(y, 2)$ where $\varphi_G(v) = (x, y)$ for some $v \in V(G)$. Thus, H is bipartite. □

We move on to understand when a graph is d-realizable. First let us answer the question when, in a d-realizable graph, if a vertex is adjacent to two other vertices, those two vertices are adjacent.

Lemma 4.6. *Let φ_G be a d-realization of a graph G and let a vertex $v \in V(G)$ be adjacent to vertices x and y. Then x and y are adjacent if and only if both of them differ from v in the same coordinate.*

Proof. First we prove the forward implication. Suppose that x and y are adjacent. Then $\varphi_G(x)$ and $\varphi_G(y)$ differ in exactly one coordinate, say in the kth coordinate. Also say $\varphi_G(v)$ differs from $\varphi_G(x)$ exactly in the ith coordinate and from $\varphi_G(y)$ in the jth coordinate. Now suppose $\varphi_G(x)$ and $\varphi_G(y)$ do not differ from $\varphi_G(v)$ in the same coordinate. This means that $i \neq j$, also note that k cannot be equal to both i and j as this would imply $i = j$. So w.l.o.g. $k \neq j$. Now because $i \neq j$, the jth coordinate of $\varphi_G(x)$ is the same as the jth coordinate of $\varphi_G(v)$ and therefore differs from jth coordinate of $\varphi_G(y)$, but also $k \neq j$ which means that $\varphi_G(x)$ and $\varphi_G(y)$ differ in more than one coordinate. This contradicts the fact that x and y are adjacent.

The proof of the converse implication is simpler. Suppose that $\varphi_G(x)$ and $\varphi_G(y)$ differ from $\varphi_G(v)$ in the same coordinate. Because x and y are both adjacent to v, they

images differ from that of v in exactly that coordinate in which they differ with $\varphi_G(v)$. In particular, they differ in exactly one coordinate. Therefore x and y are adjacent. $\square$

Theorem 4.7. *For every non-negative integer d, any d-realizable graph is $\{K_{1,d+1},$ diamond, $K_{2,3}$, $C_5\}$-free.*

Proof. Let G be a d-realizable graph, and fix a d-realization φ_G of G.

For $K_{1,d+1}$: Suppose v is the hub (the vertex with largest degree) of an induced $K_{1,d+1}$ in G. Then $\varphi_G(v)$ differs from the images of the $d+1$ neighbors of v in different coordinates by Lemma 4.6, but there are only d coordinates available, a contradiction.

For the diamond: If v_1, v_2, x and y are vertices of an induced diamond in G with v_1 and v_2 the vertices of degree 2, then, invoking Lemma 4.6 we see that $\varphi_G(v_1)$ differs from $\varphi_G(x)$ and $\varphi_G(y)$ in the same coordinate. Similarly $\varphi_G(v_2)$ difers from $\varphi_G(x)$ and $\varphi_G(y)$ in the same coordinate. Hence $\varphi_G(v_1)$ and $\varphi_G(v_2)$ differ from $\varphi_G(x)$ and $\varphi_G(y)$ in the same coordinate, and thus either $x = y$ or x is adjacent to y, a contradiction.

For $K_{2,3}$: Let x and y be adjacent to all of v_1, v_2, v_3 in an induced $K_{2,3}$ in G. Then by Lemma 4.6, $\varphi_G(v_1)$, $\varphi_G(v_2)$, $\varphi_G(v_3)$ differ from $\varphi_G(x)$ in 3 distinct coordinates.

We may assume that all coordinate values of $\varphi_G(x)$ are zeros. Then $\varphi_G(y)$ is at a distance of 2 from $\varphi_G(x)$, so $\varphi_G(y)$ has exactly 2 non-zero coordinates. Hence $\varphi_G(y)$ has a zero in coordinate where either $\varphi_G(v_1)$, $\varphi_G(v_2)$ or $\varphi_G(v_3)$ is non-zero, and it follows that $\varphi_G(y)$ differs from the images of one of these 3 vertices in 3 coordinates, and therefore y is not adjacent to that vertex, a contradiction.

For the C_5: Let $v_1, \ldots, v_5$ be the vertices of a 5-cycle in cyclic order. We may assume that the coordinates of $\varphi_G(v_1)$ are all zeros, that is,

$$\varphi_G(v_1) = (0, 0, 0, \ldots).$$

Without loss of generality we can say

$$\varphi_G(v_2) = (\alpha, 0, 0, \ldots), \text{ for some } \alpha \in \mathbb{R}$$

By Lemma 4.6 we have without loss of generality

$$\varphi_G(v_3) = (\alpha, \beta, 0, 0, \ldots), \text{ for some } \beta \in \mathbb{R}$$

Now if $\varphi_G(v_4)$ differed from $\varphi_G(v_3)$ in any coordinate other than the first two, then v_4 would be at a distance in G at least 3 from v_1 and it would be impossible to find v_5 to complete the 5-cycle. Hence $\varphi_G(v_4)$ differs from $\varphi_G(v_3)$ in one of the first 2 coordinates. If

$$\varphi_G(v_4) = (\alpha, \gamma, 0, 0, \ldots), \text{ for some } \gamma \in \mathbb{R}$$

then v_4 is adjacent to v_2. Hence

$$\varphi_G(v_4) = (\delta, \beta, 0, 0, \ldots), \text{ for some } \delta \in \mathbb{R}$$

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and the only choice for v_5 are

$$(\delta, 0, 0, 0, \ldots)$$

and

$$(0, \beta, 0, 0, \ldots)$$

In the first case v_5 is adjacent to v_2, and, in the second one, to v_3; a contradiction. $\square$

We now take a closer look at the 3-dimensional case and present two results. Both are related to the Klavžar-Peterin characterization which is needed for hardness proof for recognizing 3-realizable graphs developed in paper [17].

In the first result, we will see that the defining properties of a 3-KP-labeling are satisfied for a graph as soon as they are satisfied for the family of all its induced subgraphs isomorphic to a cycle or to a P_3.

Theorem 4.8. *Let G be a graph. A 3-edge-labeling of G is a KP-labeling if and only if it satisfies the following two conditions:*
Condition 3: *For every induced cycle C of G, the restriction of the labeling to $E(C)$ is a KP-labeling of C.*
Condition 4: *No induced P_3 is monochromatic.*

Proof. Let us start by showing the necessity of the two conditions. If G is 3-KP-labeling and H is an induced subgraph of G, then the restriction of the labeling to $E(H)$ is a 3-KP-labeling of H because Conditions 1 and 2 can be easily verified to still hold. Hence Condition 3 is necessary. Condition 4 follows from the fact that every induced P_3, say v_1, v_2, v_3, must contain two different labels since otherwise Condition 2 would be violated for v_1 and v_3 in G.

Now we prove sufficiency. For Condition 1 we need to show that every triangle is monochromatic but from Condition 3 we know that for every induced triangle of G, the restriction to edges, which is the same triangle is a KP-labeling, but a triangle is KP-labeled if and only if it is monochromatic.

Now let us show that Condition 2 also holds by contradiction.

Suppose that there is a 3-edge-labeling $\ell : E(G) \to \{1, 2, 3\}$ satisfying Conditions 3 and 4, but not Condition 2. Singe Condition 2 does not hold in G, it means that G contains two different induced paths of length at least two, say P and Q, intersecting at their endpoints u and v, such that no pair of different labels appears on both P and Q.

Due to Condition 4, on each of the paths P and Q at least two different labels appear. Since no pair of different labels appears on both P and Q, we may assume that P and Q take alternatingly labels 1, 2 and 1, 3, respectively. Moreover, assume that P and Q were chosen so as to minimize $|V(P)| + |V(Q)|$.

We continue with a definition to facilitate our way in the proof. Given a path R and two of its vertices x and y, denote by R_{xy} the subpath of R between x and y, and by V_R^{-xy} the set $V(R) \setminus \{x, y\}$. Also, say a path is k-labeled if exactly k different labels appear on its edges.

We claim that $V_P^{-uv} \cap V_Q^{-uv} = \emptyset$. Suppose that there is something in the intersection, say $w \in V_P^{-uv} \cap V_Q^{-uv}$. First we see that P_{uw} and Q_{uw} cannot be 1-labeled since otherwise because of Condition 4 none of the paths can have length more than 2, from which it must follow that u and w are adjacent, but this is in contradiction with minimality of $|V(P)| + |V(Q)|$.

So it must be that P_{uw} and Q_{uw} are 2-labeled. Since there are only 3 labels, it must be the case that $P_{uw} \cup Q_{uw}$ would be 3-labeled, which is again a contradiction with the minimality of $|V(P)| + |V(Q)|$.

We add also some further definitions. For $t \in \{u, v\}$ and $xy \in E(G)$ with $(x, y) \in V_P^{-uv} \times V_Q^{-uv}$, a cycle $C = P_{tx} - xy - Q_{yt}$ such that either $P_{tx} - xy$ or $xy - Q_{yt}$ is an induced path will be called a PQ-cycle.

Note that a PQ-cycle cannot be 3-labeled: if, without loss of generality say $P' = P_{tx} - xy$ was an induced path, then we can find a shorter pair of paths, namely P' and Q_{yt}, which would contradict the minimality of $|V(P)| + |V(Q)|$.

Let us investigate the cycle $C_0 = P \cup Q$. From our assumption that no two labels appear in both paths P, Q, from Condition 3 we have on induced cycles, the restriction of the labeling to $E(C_0)$ is a KP-labeling of C_0 it must follow that C_0 is not an induced cycle as the restriction of the labeling to $E(C_0)$ is obviously not a KP-labeling of C_0. Let xy be a chord in C_0 such that $\{x, y\} \cap \{u, v\} = \emptyset$ and such that $x \in V(P)$ is closest to u (where the distance is measured within P), and y is the neighbor of x in Q closest to v (where the distance is measured within Q).

Now we observe two cycles that are formed, namely $C_1 = P_{ux} - xy - Q_{yu}$ and $C_2 = P_{vx} - xy - Q_{yv}$. Because of how we selected x and y each of C_1, C_2 is either a PQ-cycle or a triangle, implying that neither of them is 3-labeled. If C_1 is monochromatic, then, as $E(C_0) \subset E(C_1) \cup E(C_2)$ while C_1 and C_2 share the label of xy, it would follow that C_2 was 3-labeled, symmetrically if C_2 was monochromatic. Thus, C_1 and C_2 are 2-labeled.

As C_1 and C_2 are 2-labeled, they share exactly one label. By definition, any PQ-cycle contains a P_3 from either P or Q, hence (recalling that P and Q alternate labels 1, 2 and 1, 3, respectively), C_1 and C_2 share label 1. Such is then the label of xy. However, one of the two edges incident to x in P is also labeled with 1, forming with xy a monochromatic induced P_3 (as part of either C_1 or C_2), which contradicts Condition 4. $\square$

Example 4.9. We will see an example showing that a statement analogous to The-

orem 4.8 does not work for 4-edge-labelings. We find a 4-labeled graph satisfying Conditions 3 and 4 that is not KP-labeled.

Take the graph G with 4 paths of length 6 sharing only the endpoint vertices, say u and v. (See Figure 9.) Label the paths in following way:

path 1: 1, 2, 3, 1, 2, 3

path 2: 2, 3, 4, 2, 3, 4

path 3: 3, 4, 1, 3, 4, 1

path 4: 4, 1, 2, 4, 1, 2

A quick check shows that every induced cycle is KP-labeled and also no P_3 is monochromatic, which means that Conditions 3 and 4 are satisfied.
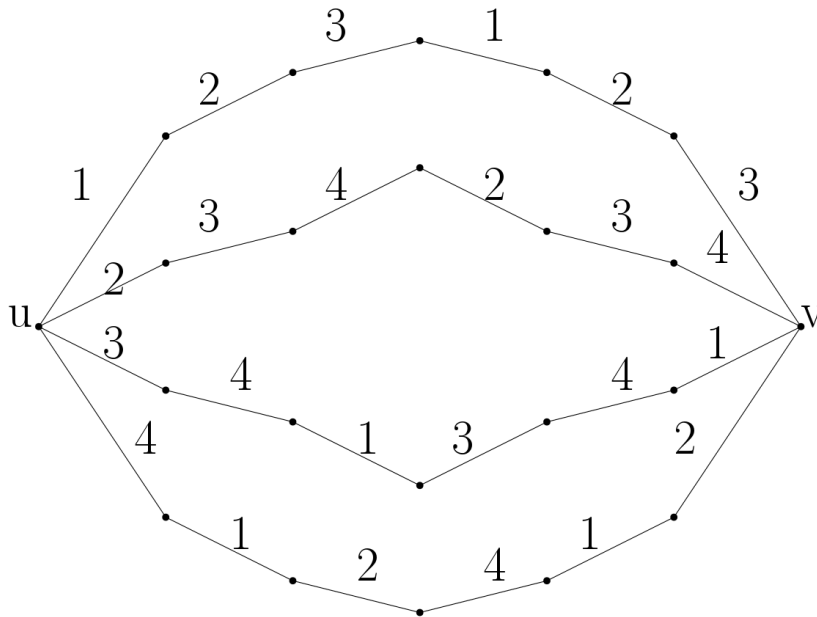

Figure 9: A counterexample to the analogue of Theorem 4.8 for 4-edge-labelings.

However, if we check whether this labeling is a KP-labeling of G, taking non-adjacent vertices u and v, we can see that each pair of the 4 paths share exactly 2 labels, but from the pigeonhole principle we cannot find labels i and j which both appear on any induced u, v-path. This example can be generalized to show an analogue of that Theorem 4.8 does not work for d-labeling for any $d > 3$. ▲

From Condition 1 we know that every 3-KP-labeling of a 3-cycle is constant (monochromatic). The following result analyzes longer cycles.

Lemma 4.10. *Let C be a cycle of length at least 4. A 3-edge-coloring of C with colors 1, 2, 3 is a KP-labeling if and only if*

- *either it is a 2-edge-coloring of C, or*

- *possibly after permuting the labels 1, 2, 3, cycle C contains a cyclically ordered sequence of 6 distinct (not necessarily consecutive) edges labeled 1, 2, 3, 1, 2, 3, respectively.*

Proof. Let us start by proving the forward direction. Suppose that C is KP-labeled with an edge-coloring using colors 1, 2 and 3 such that all three labels appear on $E(C)$. Fix a cyclic order σ of the edges of C. Without loss of generality, C has a pair of consecutive edges in σ, say e_1 and e_2, that are labeled 1 and 2 respectively.

By Lemma 3.2 each label must appear at least twice on C. Then, the sequence 1,2,3,3 must appear as a such sequence of the labels of edges in order σ, we stress it here from the lemma that this order is not necessarily consecutive (we could have something like 1,2,3,1,2,1,2,3). Let P be the maximal subpath of C containing e_1 and e_2 having no edge labeled 3, and let e and e' be the two edges of $E(C) \setminus E(P)$ incident to an edge in P. By the maximality of P, edges e and e' are labeled 3 and, since each each label appears twice, they must be distinct.

Let P' be the subpath of C formed by the edges not in $E(P) \cup \{e, e'\}$. Since P has only labels 1 and 2, in order not to violate Condition 2 both of these labels must appear in $E(C) \setminus E(P)$ and since $e, e' \in E(C) \setminus E(P)$ are labeled 3, it follows that labels 1,2 must appear in $E(P')$. If labels 1 and 2 appear in order σ on P' (not necessarily consecutively), we are done.

Suppose then that all occurrences of 2 appear in σ before all occurrences of 1 on $E(P')$. If on P there is an occurrence of label 2 appearing (in σ) before an occurrence of 1, we are done again. By way of contradiction, suppose that this is not the case, that is, suppose that all occurrences of 1 appear (in σ) before all occurrences of 2 on P. In this case, C can be divided into two parts so that the occurrences of 1 appear only in one part and the occurrences of 2 in the other part, which will violate Condition 2.

Now let us prove the other direction. Since there are no triangles, Condition 1 is satisfied. If C is 2-edge-labeled, then clearly Condition 2 holds. Suppose now, without loss of generality, that C contains a cyclically ordered sequence of 6 distinct (not necessarily consecutive) edges labeled 1, 2, 3, 1, 2, 3, respectively. Let F denote a fixed set of 6 edges with the above property. Consider now an arbitrary pair u, v of non-adjacent vertices of C. Let P and Q be the two u, v−subpaths of C. At least one of P and Q, say P, contains at least three consecutive edges from F. Hence, any two distinct labels appearing on Q will appear on every induced u, v-path. Since this is true for an arbitrary pair u, v of nonadjacent vertices, Condition 2 is satisfied.   □

4.1   NP-completeness of testing realizability in $d \geq 3$ dimension

In this section we will present a result due to Milanič, Muršič, and Mydlarz in [17]. We try to understand the question: How difficult is it to determine if a given graph G can be realized in $\mathbb{R}^d$? We show that for $d = 3$, determining whether $\mathrm{Cdim}(G) \leq 3$ is NP-complete. In fact, for all $d \geq 3$ it was shown in [17] that determining whether a given graph G satisfies $\mathrm{Cdim}(G) \leq d$ is NP-complete, even for connected bipartite graphs.

Theorem 4.11. *Given a graph G, determining whether $\mathrm{Cdim}(G) \leq 3$ is NP-complete, even for connected bipartite graphs of maximum degree at most 3.*

Proof. In order to show that determining whether $\mathrm{Cdim}(G) \leq 3$ is NP-complete, we need a polynomially checkable certificate of the fact that $\mathrm{Cdim}(G) \leq 3$ is any 3-realization of G of the form $\varphi_G : V(G) \to \{1, 2, \ldots, |V(G)|\}^3$. Therefore, the problem is in NP (on any class of input graphs).

In [11] it was showed that it is NP-complete to determine the chromatic index of an arbitrary graph. The problem remains NP-complete even for cubic graphs, so to show hardness, we make a reduction from the 3-edge-coloring problem in cubic graphs.

Let G be a cubic graph that is the input for the 3-edge-coloring problem. We may assume that G is connected. Construct a graph G' from G by replacing each edge xy of G with the structure shown in Figure 10. We will refer to such a structure as XVWY.
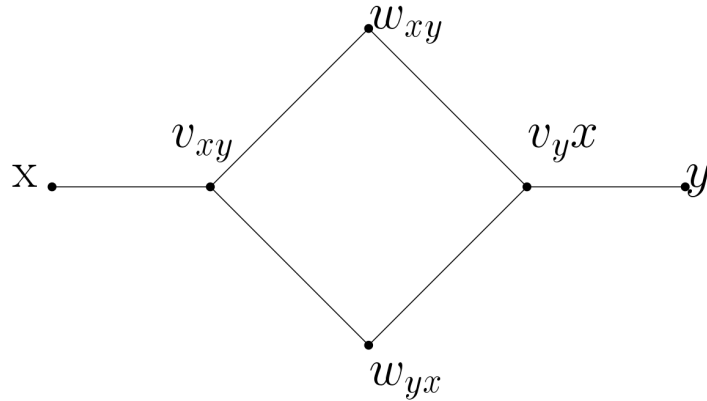

Figure 10: A gadget XVWY replacing each edge xy.

Formally,

$$V(G') = V(G) \cup \bigcup_{xy \in E(G)} \{v_{xy}, v_{yx}, w_{xy}, w_{yx}\},$$

$$E(G') = \bigcup_{xy \in E(G)} \{xv_{xy}, v_{xy}w_{xy}, v_{xy}w_{yx}, v_{yx}w_{xy}, v_{yx}w_{yx}, yv_{yx}\}.$$

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Now let us show that G' is bipartite. Letting $V_1 = V(G) \cup \bigcup_{xy \in E(G)} \{w_{xy}, w_{yx}\}$ and $V_2 = \bigcup_{xy \in E(G)} \{v_{xy}, v_{yx}\}$, we see that (V_1, V_2) is a bipartition of G' since both V_1 and V_2 are independent sets (that is, every possible edge in G' is between V_1 and V_2). Thus, G' is a bipartite graph with vertices of degrees 2 and 3 only. We will show that G is 3-edge-colorable if and only if $\mathrm{Cdim}(G') \le 3$.

Let us show the backward direction. Let $\mathrm{Cdim}(G') \le 3$. By Theorem 3.5, G' has a 3-KP-labeling. Then by Lemma 4.10 we infer that the cycle $C = v_{xy} - w_{xy} - v_{yx} - w_{yx} - v_{xy}$ in G' must be 2-KP-labled. Since we have a 3-KP-labeling, this implies that the edges xv_{xy} and yv_{yx} must have the same label ℓ_{xy}, different from labels used in C.

Since G' is triangle-free, any KP-labeling of G' is an edge-coloring (otherwise, Condition 4 would be violated). Therefore, by labeling each edge $xy \in E(G)$ with ℓ_{xy}, we get a 3-edge-coloring of G.

Now suppose that G has a 3-edge-coloring using colors 1, 2, 3. For each edge xy of G labeled $i \in \{1, 2, 3\}$, let $\{j, k\} = \{1, 2, 3\} \setminus \{i\}$ and label the associated edges of G' as follows: edges xv_{xy} and yv_{yx} with i, edges $v_{xy}w_{xy}$ and $v_{yx}w_{yx}$ with j, and edges $v_{xy}w_{yx}$ and $v_{yx}w_{xy}$ with k.

We claim that the so obtained labeling of G' is a KP-labeling. Since we have a 3-edge-labeling of G', by Thoerem 4.8, it suffices to check that Conditions 3 and 4 are satisfied. First we check that no induced P_3 is monochromatic. If the edges are stricly from one gadget XVWY then it follows from the labeling that no P_3 there is monochromatic.

If P_3 is formed from two gadgets XVWY having a vertex in common, the labels of this P_3 are taken from the incident edges in G, but the label in G is 3-edge-coloring and it has no monochromatic P_3, therefore Condition 4 holds. In order to verify that Condition 3 holds, note that G' has two types of induced cycles:

- 4-cycles. They only appear in the gadget XVWY; they are properly 2-edge-colored and hence KP-labeled by Lemma 4.10.

- Cycles of length greater than 4. Each such cycle C has length $4p$ for some $p \ge 3$ because the cycle must close in some vertex from set $V(G)$ and by a simple counting argument we get that it must be of length $4p$, and arises from a (not necessarily induced) p-cycle C' in G.

  We will show that such cycles satisfy the 123123-condition and apply Lemma 4.10. Let $x_1, x_2, \ldots, x_p$ be a cyclic order of vertices in C'. Without loss of generality, let 1, 2, 3, 1 be the labels (in this order) on some shortest path from x_1 to x_2 in C. We consider what are possible labels on the edges of any shortest path from x_2 to x_3 in C.

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Since some shortest path from x_1 to x_2 ends with label 1, no shortest path from x_2 to x_3 starts with 1, therefore it must start with 2 or 3. If it starts with 2, it must end with labels 1,3,2 or 3,1,2; else if it starts with 3, it must end with labels 1,2,3 or 2,1,3. In every case we can find labels 2,3 in this order and thus, along cycle C we find 6 distinct edges labeled 1, 2, 3, 1, 2, 3 in order. This shows that C satisfies the 123123-condition.

It follows that Condition 3 is satisfied, hence by Theorem 4.8 G' has a 3-KP-labeling. By Theorem 3.5, we conclude that $\mathrm{Cdim}(G') \leq 3$. $\qquad\square$

4.2   Cartesian dimension of grid graphs

A *(two-dimensional) grid graph* is the Cartesian product two paths. The results below will demonstrate that the Cartesian dimension is the same for any Cartesian product of two path graphs of length at least 3, namely 4. To prove these results, we will use Theorem 3.5 which can be understood as follows: for a connected graph G and positive integer d, $\mathrm{Cdim}(G) \leq d$ if and only if G has a d-KP-labeling.

Theorem 4.12.

$$Cdim(P_i \square P_j) = \begin{cases} 0, & \textit{if } i = j = 1; \\ 1, & \textit{if } (i,j) \in \{(1,2),(2,1)\}; \\ 2, & \textit{if } i = 1 \textit{ or } j = 1 \textit{ or } i = j = 2; \\ 3, & \textit{if } i = 2 \textit{ and } j \geq 3 \textit{ or vice versa;} \\ 4, & \textit{if } i \geq 3 \textit{ and } j \geq 3; \end{cases}$$

Proof. Without loss of generality, we may assume that $i \leq j$.

If $i = 1$, then $P_1 \square P_j \cong P_j$; if $j = 1$ then there is no edges, therefore $\mathrm{Cdim}(P_i \square P_j) = 0$; if $j = 2$, then there is only one edge, therefore $\mathrm{Cdim}(P_i \square P_j) = 1$; and if $j \geq 2$, then alternate by following the path horizontal and vertical in $\mathbb{R}^2$, therefore $\mathrm{Cdim}(P_i \square P_j) = 2$.
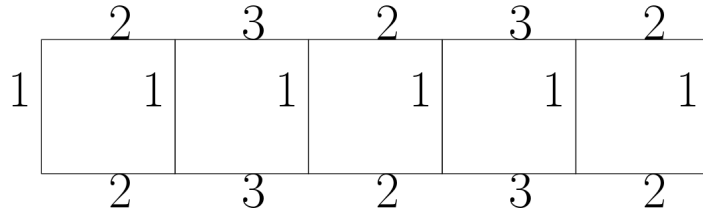
If $i = 2$ and $j = 2$, then $\mathrm{Cdim}(P_i \square P_j) = 2$.

If $j \geq 3$, from Theorem 4.7, we have that $\mathrm{Cdim}(P_2 \square P_j) \geq 3$, since $P_2 \square P_j$ has $K_{1,3}$ as induced subgraph.

Next, we find a KP-labeling of the graph with 3 labels.

We label the copies of the first factor (P_2) with label 1 and both copies of P_j the same by alternating between 2 and 3. An example of how we label $P_2 \square P_6$ is shown in Figure 11.

For any two non-adjacent vertices, u and v, if they belong to a C_4, we take label 1 and the other label from set $\{2,3\}$ to satisfy Condition 2. If they do not belong to some C_4, then we take labels 2 and 3. Note that u and v are in different components

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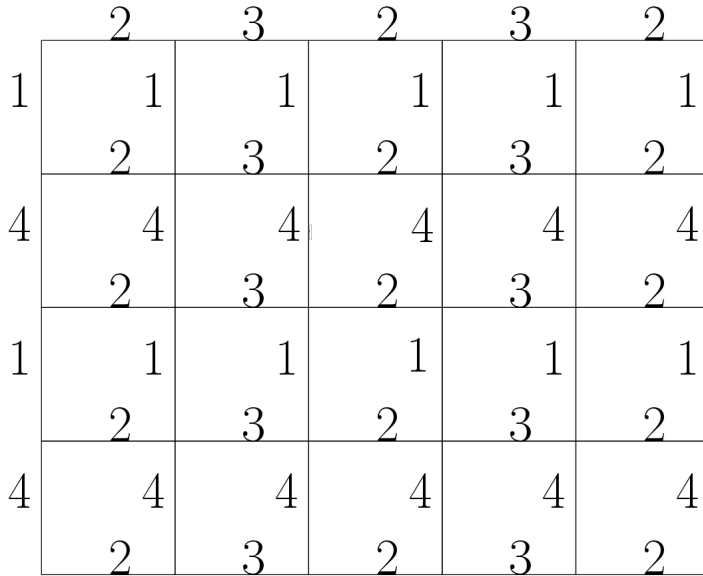
Figure 11: A KP-labeling of $P_2 \square P_6$.

when taking the partition of graph with respect to labels 2 or 3, meaning that every induced u, v-path has an edge labeled i, for i in $\{2, 3\}$. Therefore, this labeling is a KP-labeling and from Theorem 3.5, we have that $\text{Cdim}(P_2 \square P_j) \leq 3$.

Combining both inequalities, we have that $\text{Cdim}(P_2 \square P_j) = 3$ if $j \geq 3$.

Finally, we look at the case $i \geq 3$. In this case, from Theorem 4.7, we have that $\text{Cdim}(P_i \square P_j) \geq 4$, since $P_i \square P_j$ has $K_{1,4}$ as induced subgraph.

Next, we find a KP-labeling of the graph with 4 labels.

Figure 12: A KP-labeling of $P_5 \square P_6$.

We label the copies of P_i identically by alternating between 1 and 4; and the copies of P_j identically by alternating between 2 and 3. An example of how we label $P_5 \square P_6$ is shown in Figure 12.

For any two non-adjacent vertices, u and v, when they belong to a C_4, we take the two labels in this cycle. However, if any two nonadjacent vertices, u with coordinates (a, x) and v with coordinates (b, y), do not belong to a C_4, then either $d_{P_i}(a, b) \geq 2$ or $d_{P_j}(x, y) \geq 2$. If $d_{P_i}(a, b) \geq 2$, we take labels 2 and 3; otherwise, we take labels 1 and 4. Similarly to the previous proposition, vertices u and v would belong to different

components when removing either of the chosen labels. Therefore, this label is a KP-labeling and from Theorem 3.5, we have that $\text{Cdim}(P_i \square P_j) \leq 4$.

Combining both inequalities, we have that $\text{Cdim}(P_i \square P_j) = 4$, if $i \geq 3$. □

5 Hamming Dimension

Graph products offer a variety of possibilities to introduce the concept of a graph dimension. A classical result of Graham and Winkler [7] asserts that there is precisely one such embedding that is irredundant and has the largest number of factors. The embedding is is called the canonical isometric representation. Knowing that, this embedding is unique among all irredundant isometric embeddings with respect to the largest possible number of factors. The latter number is defined to be the *dimension* of a graph.

The strong product $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; distinct vertices (u, u') and (v, v) are adjacent in $G \boxtimes H$ if and only if $u = v$ and u' is adjacent to v', or $u' = v'$ and u is adjacent to v, or u is adjacent to v and u' is adjacent to v'.

Back in 1938 Schoenberg [23] proved that every connected graph admits an isometric embedding into the strong product of paths. It is hence natural to define the *strong isometric dimension*, $idim(G)$, of a graph G as the least number k such that G embeds isometrically into the strong product of k paths.

Eppstein considered the following representation problem in [5]: to which unweighted undirected graphs one can assigned integer coordinates in some d-dimensional space $\mathbb{Z}^d$ such that the distance between two vertices in the graph is equal to the L_1-distance (taxicab) between their coordinates? Call the minimum possible dimension d of such an embedding (if one exists) the *lattice dimension* of the graph.

The strong isometric dimension is universal in the sense that as soon as a graph is not the path graph, then its dimension is finite and bigger than 1. A similar conclusion can be stated for the so-called direct dimension of a graph defined in [3].

On the other hand, for one of the most important graph products, the Cartesian product, no such universal dimension is known. While the isometric dimension is useful as soon as the dimension of a graph is more than 1, it was proved in [21] that almost all graphs have isometric dimension 1. In other words, for almost all graphs G the isometric dimension yields no new insight about G. Also, only the so called partial cubes, a special (although important) subclass of bipartite graphs, have finite lattice dimension.

To significantly increase the number of graphs with a non-trivial dimension, Klavžar,

Peterin, and Zemljič defined in [14] the *Hamming dimension* $\mathrm{Hdim}(G)$ of a graph G as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. Here, we say that a graph G is an *irredundant* induced subgraph of the Cartesian product $G_1 \square G_2 \square \ldots \square G_k$ if each G_i has at least two vertices and for every i, every vertex of G_i appears as a coordinate of some vertex of G. If G is not an induced subgraph of any Hamming graph, we set $\mathrm{Hdim}(G) = \infty$.

In Figure 13 we show the 6-cycle as an induced subgraph of two different Hamming graphs, $K_3 \square K_3$ and $K_2 \square K_2 \square K_2$.
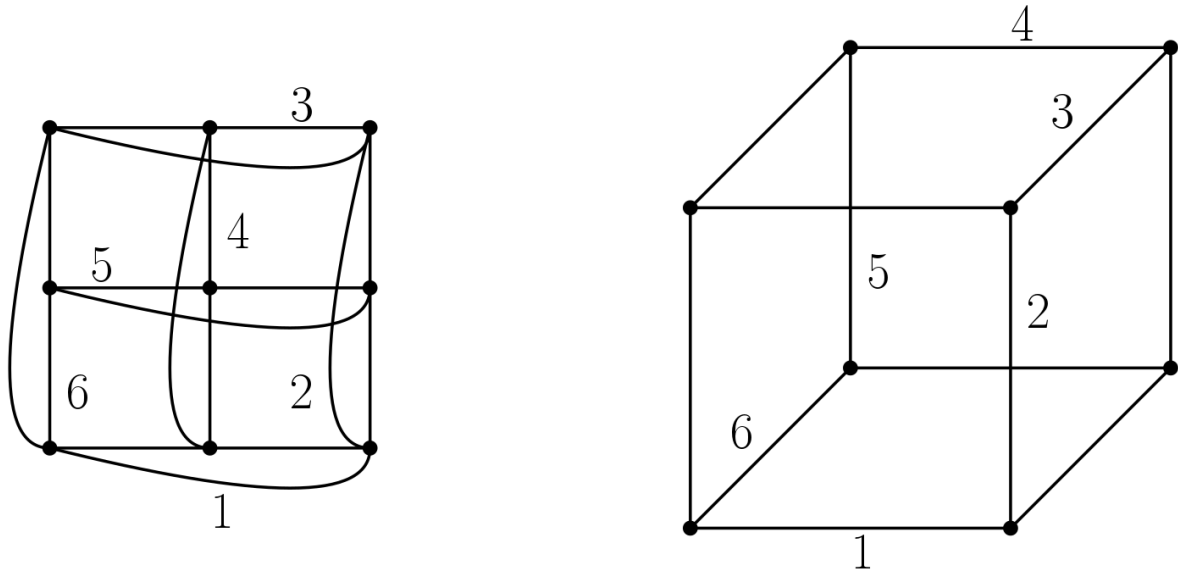

Figure 13: A C_6 induced in $K_3 \square K_3$ and in $K_2 \square K_2 \square K_2$.

Clearly, $\mathrm{Hdim}(G) = 1$ if and only if G is a complete graph. From Theorem 3.5, we infer that $\mathrm{Hdim}(G) < \infty$ if and only if G has a KP-labeling.

5.1   Hamming dimension of Sierpiński graphs

The general problem of determining the Hamming dimension of a graph seems very challenging. Here we will study this concept on Sierpiński graphs. Roughly speaking, as shown in [14], all Sierpiński graphs (except in the trivial cases) have Hamming dimension bigger than 1 and finite.

On the other hand, all but base 3 Sierpiński graphs have isometric dimension 1.

Graphs whose drawings can be viewed as approximations to the famous Sierpiński triangle have been studied intensely in the past 25 years. The interest for these graphs comes from many different sources such as games like the Chinese rings or the Tower of Hanoi, topology, physics, the study of interconnection networks, and elsewhere.

Sierpiński graphs S_n^k were studied for the first time by Klavžar and Milutinović in [15]. In computer science, a very similar class of graphs (known as WK-recursive networks) was introduced earlier by Vecchia and Sanges in [25].

The study in [15] was motivated in part by the fact that for $k = 3$ these graphs are isomorphic to the Tower of Hanoi graphs and in part by topological studies.

A 1-perfect code in a graph G is a vertex subset of G with the property that the closed neighborhoods of its elements form a partition of $V(G)$. The graphs S_n^k were investigated from numerous points of view, we recall some of them. These graphs contain unique 1-perfect codes, proven by Klavžar, Milutinović, and Petr in [16].

Metric properties of Sierpiński graphs were investigated in [10]. To determine the chromatic number of these graphs is easy, while in [9] it is proved that they are in edge- and total coloring class 1, except those isomorphic to a complete graph of odd or even order, respectively.

The Sierpiński graph S_n^k, where $k, n \geq 1$, is defined on the vertex set $\{1, \ldots, k\}^n$, two different vertices $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ being adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \ldots, h - 1$;

(ii) $u_h \neq v_h$; and

(iii) $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

An example of Sierpiński graph is shown in Figure 14. Below we will state without proof some results from [14].

Theorem 5.1. *For any $n \geq 2$*

$$\text{Hdim}(S_n^3) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2} \cdot n - \frac{9}{4}.$$

Proposition 5.2. *(i)* $\text{Hdim}(S_3^2) = 3$, $\text{Hdim}(S_3^3) = 6$.

(ii) For any $k \geq 4$, $\text{Hdim}(S_k^2) = 2$.

Theorem 5.3. *(i)* $\text{Hdim}(S_3^n) \leq 5 \cdot 3^{n-3} + 1$ *for all $n \geq 3$.*

(ii) $\text{Hdim}(S_k^n) \leq \frac{2}{k-1} k^{n-2} + \frac{2k-4}{k-1}$ *for all $k \geq 4$ and $n \geq 2$.*

Corollary 5.4. *For any $k \geq 4$,* $\text{Hdim}(S_k^3) = 4$.
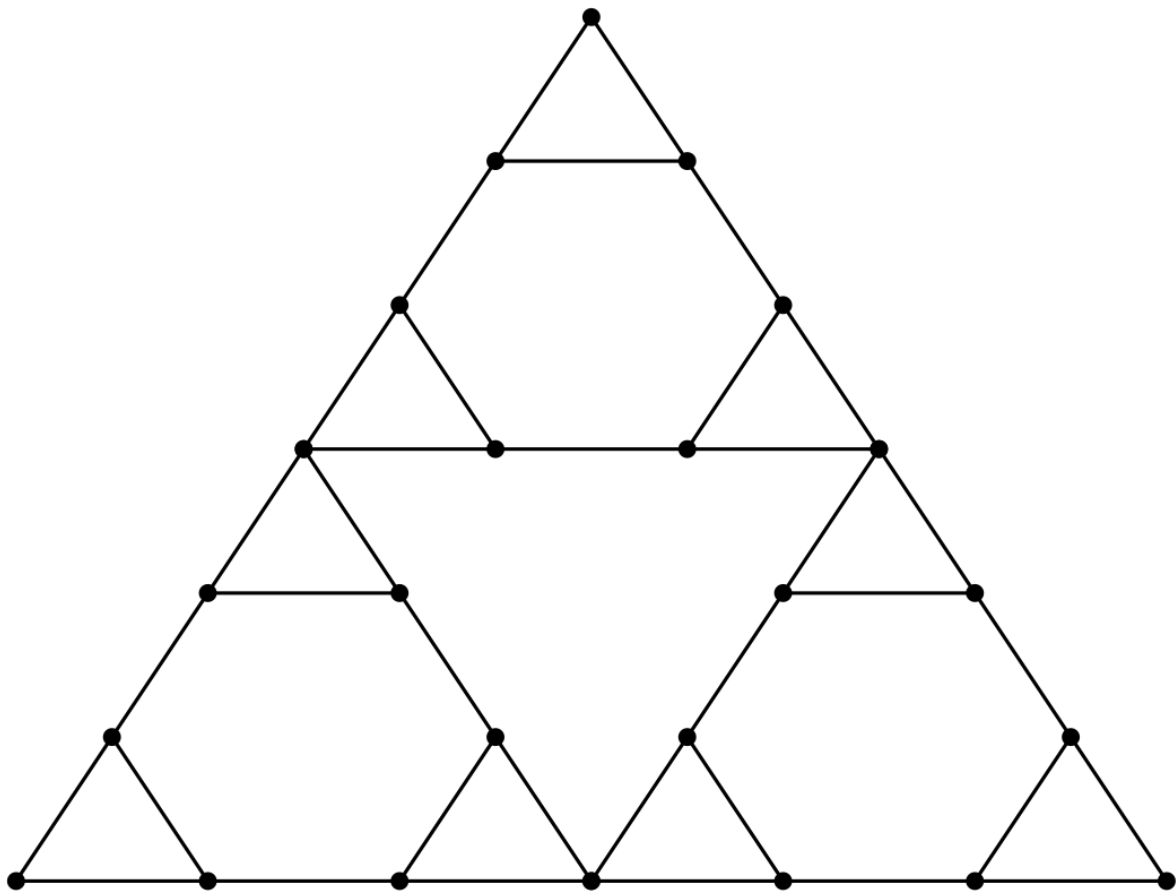
Figure 14: The Sierpiński graph S_3^3.

6 Conclusion

This project paper is focused on carefully examining the representations of graphs as induced subgraphs of Hamming graphs. We have seen that a graph G is an induced subgraph of a Hamming graph if and only if there exists a labeling of $E(G)$ fulfilling the two conditions:

1. edges of a triangle receive the same label,

2. for any vertices u and v at distance at least two, there exist two labels which both appear on any induced u; v-path.

We have noticed that for Cartesian dimension, the two-dimensional case corresponds to the class of line graphs of bipartite graphs and is well-understood, while for all $d \geq 3$ the problem of determining whether a given graph G is d-realizable is NP-complete.

We also briefly discussed the Hamming dimension of graphs and stated without proofs some results about the Hamming dimension of Sierpiński graphs.

It would be interesting to perform a further study and comparison of the Cartesian and the Hamming dimensions of graphs, both in general as well as for graphs from particular classes, such as planar graphs, graphs of small maximum degree, generalized Petersen graphs, Johnson graphs, etc.

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