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(Representations of graphs as induced subgraphs of Hamming graphs)

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Abstract: In this final project paper we consider induced subgraphs of Hamming graphs. The representation of graphs as induced subgraphs of a larger graph with a special structure often allows a better understanding of the properties of the given graphs. Using embeddings into Cartesian products of quotient graphs Klavžar and Peterin in [12] characterize induced subgraphs of Hamming graphs. To get a better understanding we also study the Cartesian dimension defined in [2]. The study consist of structural, algorithmic, and complexity aspects of the Cartesian dimension of graphs.

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A Title of First Appendix
B Title of Second Appendix

List of Abbreviations

i.e. that is
e.g. for example

1 Introduction

The representation of graphs as induced subgraphs of a larger graph with a special structure often allows a better understanding of the properties of a given graph. A particularly convenient situation arises when the given graph has certain unique properties as a subgraph of the larger graph which is embedded in. In this paper, we will take a closer look at Hamming graphs. Hamming graphs are named after the mathematician Richard Hamming and they are used in various branches of mathematics and computer science. By definition, they are the Cartesian product of complete graphs. In order to solve the problem of effectively recognizing whether a graph is a Hamming graph, we can use the prime factorization algorithms with respect to the Cartesian product. In [1] present an algorithm for recognizing Hamming graphs which is linear in both time and space.
These graphs have been characterized differently by various people. Mulder in [4] has characterized the Hamming graphs as interval-regular graphs with some additional properties. This is analogous to his characterization of hypercubes as bipartite interval-regular graphs [4]. Whereas, in [5] Egawa showed that if G is a connected graph with the same proper convex subgraphs as Hamming graph where each complete graph has the same size, then $|V(G)| \geq n^r$ with equality if and only if G is isomorphic to Hamming graph. In [6], Mollard gives a characterization of Hamming graphs analogous to the characterization of hypercubes as (0,2) graphs of maximal order. Isometric subgraphs of Hamming graphs, which are called partial Hamming graphs, have been extensively studied by now.
Graham and Winkler [19] proved that every graph has the best representation as an isometric subgraph of a Cartesian product. The key construction is based on embedding into Cartesian products of the so-called quotient graphs with respect to a certain relation defined on the edge set of a graph. Feder [14] followed with a similar approach in order to obtain such representations for stronger embeddability conditions: 2-isometric representation, weak retract representation, and Cartesian prime factorization. Opposite direction took Klavžar and Peterin in [12], treating isometry as the strongest property, consider subgraphs, induced subgraphs, and isometric subgraphs of Hamming graphs. They showed, approximately, that embeddings into Cartesian product of quotient graphs can be applied also to subgraphs and induced subgraphs of

Hamming graphs.

Several graph dimensions based on embeddings into product graphs have been studied by now. In the work done by Martin Milanič, Peter Muršič, and Marcelo Mydlarz in [2], they studied the question on how difficult is it to determine if a given graph G can be realized in $\mathbb{R}^d$ so that vertices are mapped to distinct points and two vertices are adjacent if and only if the corresponding points are on a common line that is parallel to some axis? Any such mapping is referred as a d-realization of G and say that a graph is d-realizable if it has a d-realization. d-realizable graphs were studied under different names such as arrow graphs in [7], $(d-1)$-plane graphs and $(d-1)$-line graphs of d-partite d-uniform hypergraphs in [8], d-dimensional cellular graphs in [9], d-dimensional chess board graphs in [3], and d-dimensional gridline graphs in [10]. Recently, Sangha and Zito studied d-realizable graphs in the more general context of the so-called Line-of-Sight (LoS) networks in [18]. Line of Sight (LoS) networks offer a model of wireless communication that incorporates visibility constraints. Vertices of such networks can be embedded in finite d-dimensional grids of size n, and two vertices are adjacent if they share a line of sight and are at a distance of less than ω where two points $x = (x_1, \ldots, x_d)$ and $x' = (x'_1, \ldots, x'_d)$ share a line of sight if there exists some $j \in \{1, \ldots, d\}$ such that $x_i = x'_i$ for all $i \in \{1, \ldots, d\} \backslash \{j\}$. They studied large independent sets in LoS networks. They proved that the computational problem of finding a largest independent set can be solved optimally in polynomial time for one dimensional LoS networks. However, for $d \geq 2$, the (decision version of) the problem becomes NP-hard for any fixed $\omega \geq 3$ and even if is chosen to be a function of n that is $O(n^{1-\epsilon})$ for any fixed $\epsilon > 0$. In addition they showed that the problem is also NP-hard when $\omega = n$ for $d \geq 3$. This result extends earlier work which demonstrated that the problem is solvable in polynomial time for gridline graphs when $d = 2$. For the small-dimensional cases, $d \in \{2, 3\}$, Peterson suggested an application of d-realizable graphs to robotics in [10]: if the movement of a robot is restricted to be along axis-parallel directions only and turns are allowable only at certain points, then a shortest path in a d-realized graph gives the number of turns required.

Another graph dimension defined in [20], the Hamming dimension of a graph G is introduced as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph, which came as the need in order to significantly increase the number of graphs with a non-trivial dimension that comes from the Cartesian product of graphs.

In this project paper we will take a closer look at representations of graphs as induced subgraphs of Hamming graphs. In Chapter 3 we will present the work done by Sandi Klavžar and Iztok Peterin, their new definitions helps to give a characterization for induced subgraphs of Hamming graphs. The main result in that section is the necessary

and sufficient condition for a graph to be an induced subgraph of a Hamming graph.
In Chapter 4 we see the properties of Cartesian dimension defined by Martin Milanič, Peter Muršič, and Marcelo Mydlarz. We will see the properties of Cartesian dimension and how it is related with what was done in Chapter 3. We will see how for Cartesian dimension less than 3 is well studied and to know whether a graph has Cartesian dimension less than 4 is NP-complete. We also analyze Cartesian dimension of some special graphs.
In Chapter 5 we shortly introduce the definition of Hamming dimension. We will notice some bounderies of Hamming dimension of Sierpiński graphs without proofs.

2 Preliminaries

Before we start with the results, we will go through terminogoly used in the rest of the project paper, and basic result related to graphs. Unless stated otherwise, we talk about finite, undirected, and simple graphs without isolated vertices. A graph is made up of vertices which are connected by edges, and we write graph G by $G = (V, E)$, where V is the vertex set and E is the set of edges in G. The degree of a vertex is the number of edges that are incident to the vertex, the distance between two vertices u and v in a graph G is the number of edges in a shortest path connecting them and denote it by $d_G(u, v)$. A cut edge is an edge that when removed from a graph creates more components than previously in the graph. Let us now go through some special graphs, a path in a graph is a finite sequence of edges which joins a sequence of vertices. A cycle in a graph is a non-empty trail in which the only repeated vertices are the first and last vertices, a cycle which has 3 edges is called triangle. A tree is a connected acyclic graph. A graph is bipartite if the vertices can be partition into two sets V_1 and V_2 such that all edges go only from V_1 to V_2(no edges go from V_1 to V_1 or from V_2 to V_2). A complete graph is a graph with n vertices and an edge between every two vertices, denoted by K_n. A induced subgraph is a subset of the vertices of a graph G together with any edges whose endpoints are both in this subset. In Figure 1 is shown diamond and claw graph.

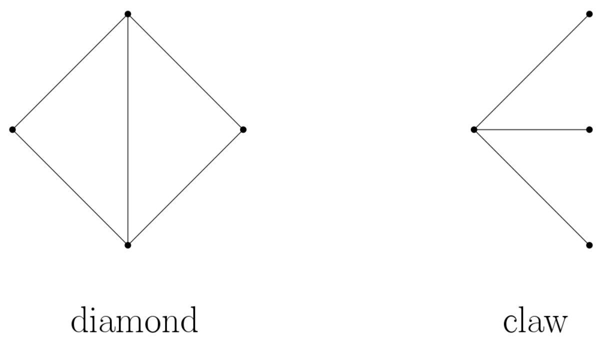

Figure 1: Diamond and claw.

A graph G is said to be $\mathfrak{F}$-free if no induced subgraph of G is isomorphic to a graph from $\mathfrak{F}$. A clique in a graph is a set of pairwise adjacent vertices, a maximal clique is such that it is not a subset of a larger clique. Let a graph G, $L(G)$ denotes the

line graph of G, the vertices of $L(G)$ are the edges of G and two vertices of $L(G)$ are adjacent if the corresponding edges of G are adjacent, in Figure 2 is shown an example. The adjacency matrix $A(G) = ((a_{ij}))$ of a graph G is a square-symmetric (0,1)-matrix whose rows and columns correspond to the vertices of G and $a_{ij} = 1$ if the vertices i and j are adjacent. The clique graph $K(G)$ of a graph G has as its vertex set the cliques of G, with two vertices adjacent whenever they have some vertex of G in common.
Given a graph G, a d-edge-labeling of G is a mapping from $E(G)$ to some set L of labels, where $|L| = d$. Given a d-edge-labeling of G and a set $F \subset E(G)$, we say that F is monochromatic if the labeling is the same for any edge in F. An edge coloring of a graph G is a mapping from $E(G)$ to some set L of labels so that no two incident edges have the same label, labels may also be referred to as colors.

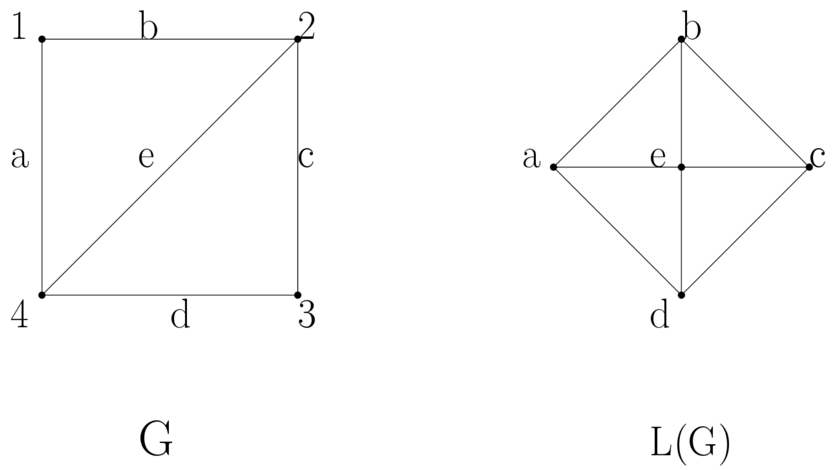

Figure 2: Diamond and claw.

A *decision problem* is a problem in which the set of instances divides into two sets depending on whether the answer is YES or NO. In computational complexity theory, the following three classes of decision problem are of interest:

- *P* is the set of decision problems that can be solved by a polynomial algorithm. Intuitively: *P* is the set of problems that can be solved efficiently.

- *NP* is the set of decision problems with the following property: If the answer is YES then there exists a certificate that enables us to verify this fact in polynomial time. Intuitively: NP is the set of problems for which we can quickly verify a positive answer if we are given a solution.

- *co-NP* is the set of decision problems with the following property: If the answer is NO then there exists a certificate that enables us to verify this fact in polynomial time.

The following lemma describes when an edge is cut-edge

Lemma 2.1. *An edge $e = \{u, v\}$ of a graph G is a cut-edge if and only if it does not belong to any cycle.*

Proof. Take any edge $e = \{u, v\}$. Remove this edge from our graph: if the graph is still connected, then there is some path from u to v not involving e; consequently, if we add e to the end of this path, we get a cycle. Thus, if e is not a cut-edge, then e belongs to a cycle.

Conversely, suppose that $e = \{u, v\}$ lies in a cycle. Let P be the path from u to v that does not use e (i.e. goes the other way around the cycle.) Pick any x, y in G; because G is connected, there is a path from x to y in G. Take this path, and edit it as follows: whenever the edge e shows up, replace this with the path P (or P traced backwards, if needed.) This then creates a walk from x to y; by deleting cycles if necessary, this walk can be turned into a path from x to y, and thus $G - e$ is connected. So if e is contained in a cycle, its not a cut-edge. $\square$

3 Characterizing Subgraphs of Hamming Graphs

In this chapter we will talk about necessary and sufficent conditions for a graph to be an induced subgraph of a Hamming graph and some results regarding those conditions.

3.1 KP-labeling

We take a closer look at the characterization of induced subgraphs of Hamming graphs due to Klavžar and Peterin.
Before going to the result we need to go through couple of definitions and some results.

The Cartesian product $G\Box H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which a vertex (a, x) is adjacent to a vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For a fixed vertex a of G, the vertices $\{(a, x)|x \in V(H)\}$ induce a subgraph of $G\Box H$ isomorphic to H, called an H*-layer* of $G\Box H$. Analogously we define G*-layers*. A subgraph of $G\Box H$ is called non-trivial if it intersects at least two G-layers and at least two H-layers.
The map $p_G : V(G\Box H) \to G$ defined by $p_G(a, x) = a$, is called a projection. The image of an edge $(a, x)(b, y)$ under a projection is an edge when $x = y$ or a vertex in when $a = b$.
Cartesian products of complete graphs are known as Hamming graphs. But they can

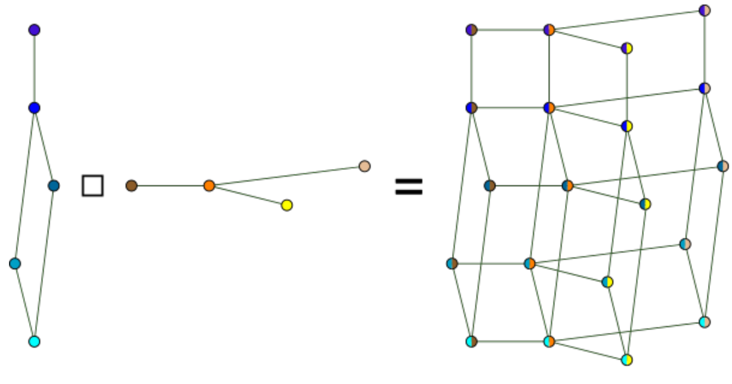

Figure 3: The Cartesian product of graphs.

be described also as follows. For $i = 1, 2, \ldots, n$ let $r_i \geq 2$ be given integers. Let G be the graph whose vertices are the n-tuples $b_1 b_2 \cdots b_n$ with $b_i \in \{0, 1, \ldots r_i - 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one coordinate. The vertex set of G is the same as that of $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ and edges in $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ between two vertices are exactly those that have same vertex in every graph K_{r_i} but one and since in complete graphs we have every possible edge, then it is easy to see that G and $K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ are isomorphic. For an edge uv of $H = K_{r_1} \square K_{r_2} \square \ldots \square K_{r_n}$ we define the color map $c : E(H) \to \{1, 2, \ldots n\}$ with $c(uv) = i$, where u and v differ in coordinate i.

Let G be a connected graph and let $\mathcal{F} = \{F_1, F_2, \ldots F_k\}$ be a partition of $E(G)$. The *quotient graph* G/F_i has connected components of $G \backslash F_i$ as vertices, two components C and C' being adjacent whenever there exists an edge of F_i connecting a vertex of C with a vertex of C'. For each i, define a map $f_i : V(G) \to V(G/F_i)$ by $f_i(v) = C$, where C is the component of $G \backslash F_i$ containing v. Then let

$$f : V(G) \to V(G/F_1 \square G/F_2 \square \ldots \square G/F_k)$$

be the natural coordinate-wise mapping, that is

$$f(v) = (f_1(v), f_2(v), \ldots, f_k(v))$$

We call f the *quotient map* of G with respect to $\mathfrak{F}$. Note that f need not be one-to-one in general and that it is possible that some quotient graphs are the one vertex graph, an example is shown in Figure 3.1. However, all the partitions $\mathfrak{F}$ introduced later will lead to one-to-one mappings with non-trivial quotient graphs.

A partition $\{F_1, F_2, \ldots, F_k\}$ of $E(G)$ naturally leads to an edge-labeling $l : E(G) \to \{1, 2, \ldots, k\}$ by setting $l(e) = i$, where $e \in F_i$. Unless stated otherwise, a labeling (or more precisely a k-labeling) of G will mean an edge-labeling (with k labels).

Quotient graph is the central concept in this chapter, so to get a better understanding we also visualize it. We define the partition of $E(C_7)$ by defining F_i having edges with label i, as we can see C_7/F_1 and C_7/F_3 have only two vertices and since there is an edge in the correspoding F_i connecting those two components, both of $C_7 \backslash F_1$ and C_7/F_3 are simply K_2, while $C_7 \backslash F_2$ has three vertices but each two of compontents are connected with an edge from F_2 so $C_7 \backslash F_2$ is simply K_3.

Sandi Klavžar and Iztok Peterin defined two conditions to get a better understanding. They actually named them by letters B and C, and also defined condition A which is needed for subgraphs oh Hamming graphs but since we will not need condition A will name just by numbers 1 and 2.

We say that a d-edge-labeling of G is a (d-)KP-labeling if it satisfies the following conditions:

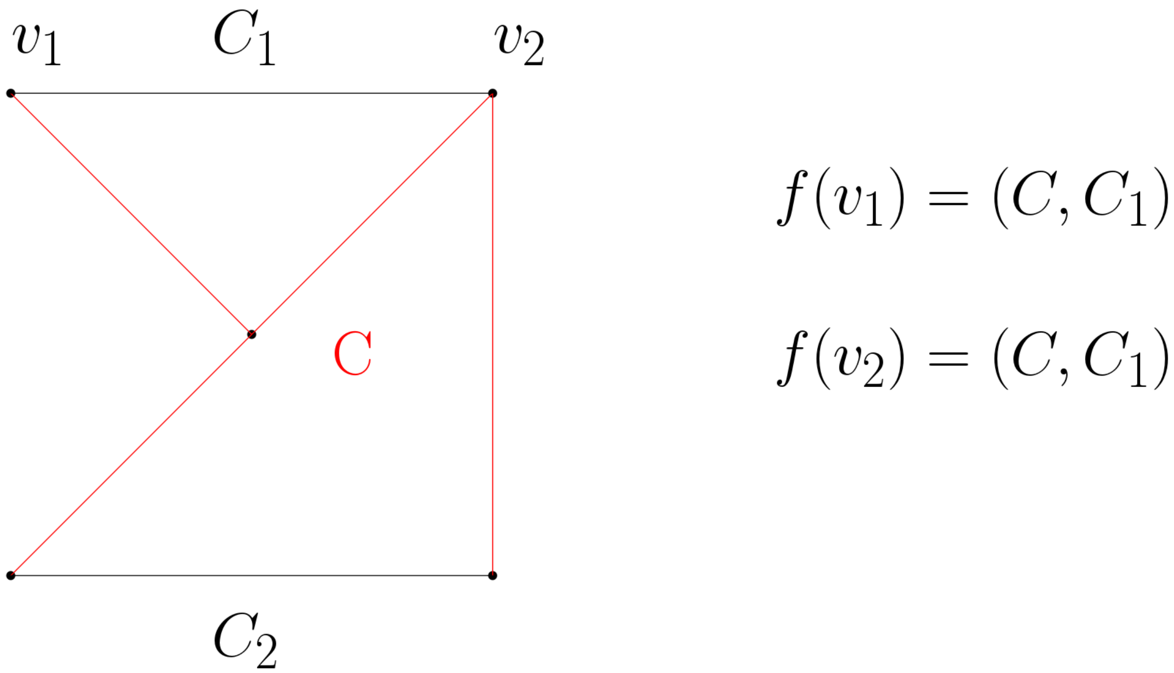

Figure 4: An example of a graph that the quotient map is not one-to-one

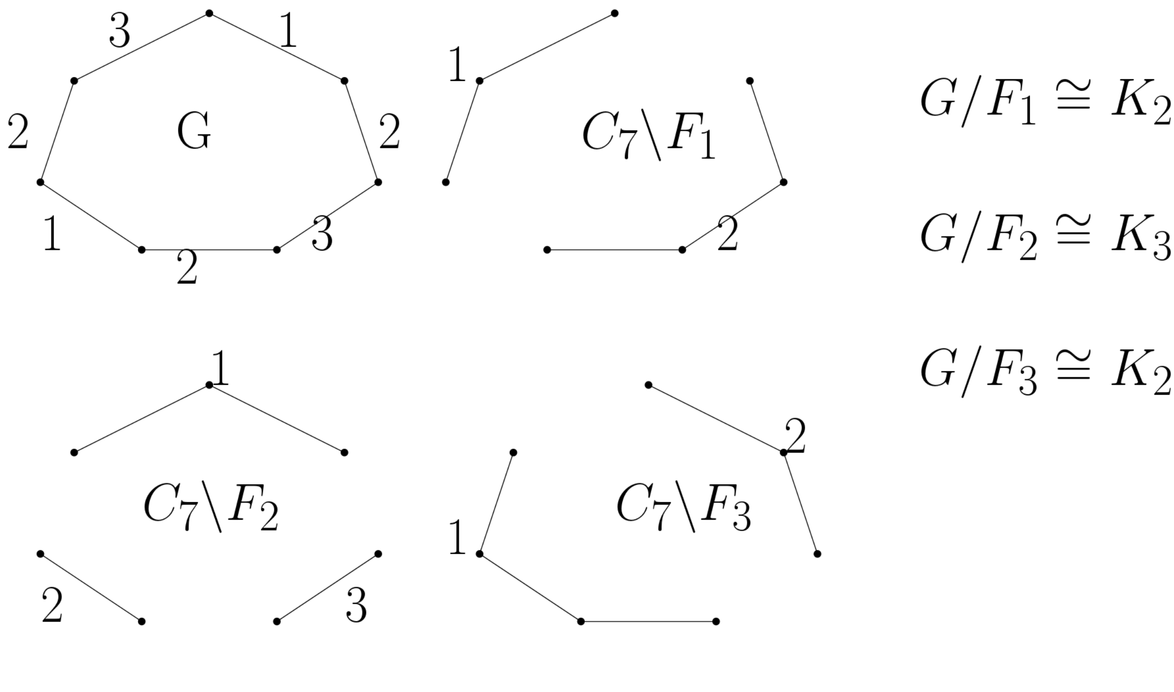

Figure 5: Quotient graph of C_7

Condition 1 The edges of any triangle have the same label.

Condition 2 Let u and v be arbitrary vertices of G with $d_G(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced u, v-path.

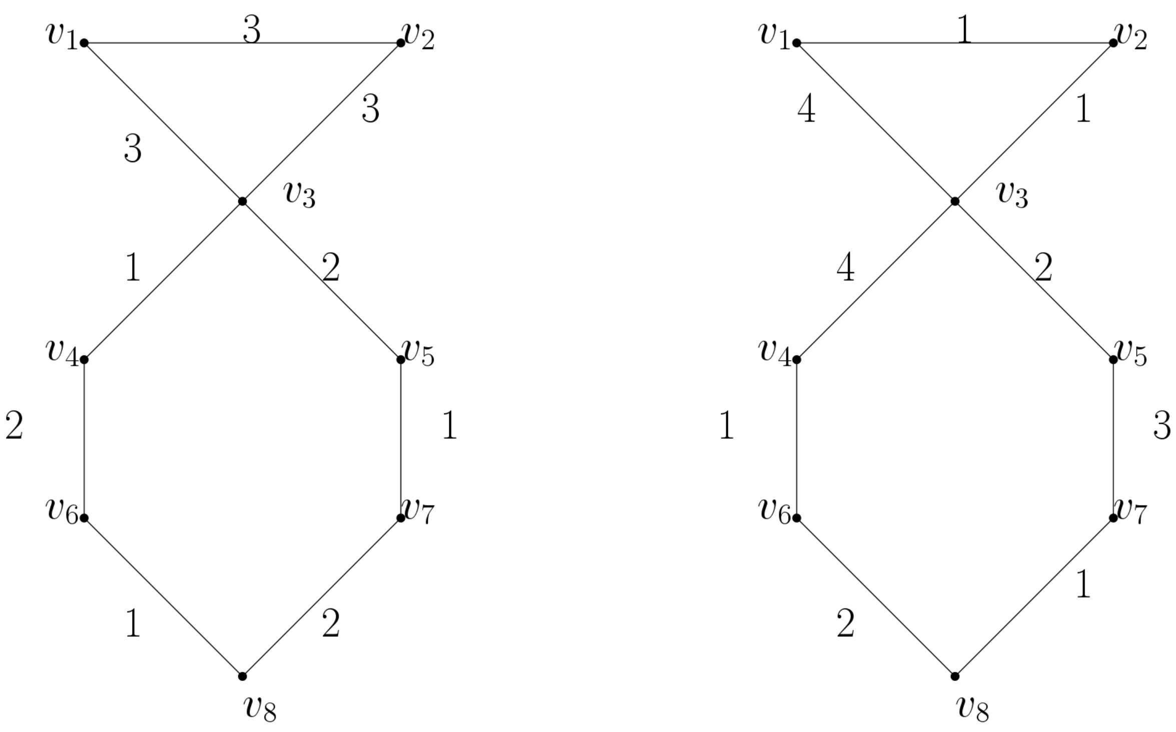

Figure 6: The label in left is a KP-labeling, while the one in right is not.

Example 3.1. Let us see two examples of edge labeling of the same graph G as in Figure 3.1.
The label in the left side of figure satisfies condition 1 as the edges of the triangle have the same label, also it satisfies condition 2 as it can be checked that for arbitrary vertices u and v of G with $d_G(u, v) \geq 2$ there exist different labels i and j which both appear on any induced u, v-path. Therefore the label in left is a KP-labeling.
The label in right does not satisfy neither condition 1 nor 2, as the edges of the triangle do not have the same label, and also if we take vertices v_5 and v_8 then paths $P_1 = v_5v_3v_4v_6v_8$ and $P_2 = v_5v_7v_8$ have only one label in their intersecion.

3.2 Induced subgraphs of Hamming Graphs

Next we try to understand those Conditions 1 and 2 on cycle graphs and we infer the following result

Lemma 3.2. *Let G be a labeled graph fulfilling Condition 2 and let C_k, $k \geq 4$, be an induced cycle of G. Then every label of C_k is present more than once on C_k.*

Proof. The labels of a u, v-path of length 2 on a C_k are different, hence by Condition 2 those two labels should appear also along the other path. So for any label take two vertices u and v that have distance two and such that a u, v-path of length two contains the edge with that label, meaning that the label is present on both disjoin paths therefore the label is present more than once on C_k □

Before going to the main result we will need also the following lemma that tells us that we only need to check the induced paths.

Lemma 3.3. *Let G be a labeled graph fulfilling Conditions 1 and 2 and let u, v be vertices of G with $d_G(u, v) \geq 2$. Then, if labels i and j appear on every induced u, v-path, they appear on every u, v-path.*

Proof. Suppose that labels i and j appear on every induced u, v-PATH but not on every u, v-path. Let $P = x_1x_2...x_r, x_1 = u$, with $x_r = v$, be a u, v-path of minimal length that does not contain both labels i and j. Then P is not induced since every induced path contains both labels, hence we have an edge between some non-consecutive vertices say $e = x_kx_l$ with $l - k > 1$. We may assume that e is chosen so that $l - k$ is as small as possible. By the minimality of P, the path $x_1x_2...x_kx_lx_{l+1}...x_r$ (which is shorter than P) contains both labels i and j. Hence, the label of the edge x_kx_l is either i or j, otherwise P would contain both labels i and j. Assume without loss of generality that the label is i. Then, using minimality again, label j appears on the path $x_1x_2...x_k$ or on $x_lx_{l+1}...x_r$ (note that it cannot happen that $k = 1$ and $l = r$ as $d_G(u, v) \geq 2$). It follows that i does not appear on the path $x_kx_{k+1}...x_l$. But then label i appears only once on the cycle $C = x_kx_{k+1}...x_lx_k$. If C is a triangle, we have a contradiction with Condition 1, otherwise with Lemma 3.2. □

If G is a tree, then assigning a different label to each edge result in a labeling satisfying Conditions 1 and 2, hence in a KP-labeling. Note that every edge in a tree is a cut-edge. More generally, we can ask: which edges in a graph that has a KP-labeling can receive a unique label in some KP-labeling? This question is answered by the following result.

Proposition 3.4. *Let G be a graph having a KP-labeling and let e be an edge in G. Then G has a KP-labeling in which e receives a unique label if and only if e is a cut-edge.*

Proof. First we will show that if e is a cut-edge then G has a KP-labeling in which e receives a unique label. Fix an arbitrary KP-labeling of G.
Relabel edge e with a new label, say i, not used before. We will show that the new labeling is also a KP-labeling. We need to verify the following two conditions:
Condition 1: Every triangle is monochromatic.
Condition 1 holds, since e is not part of any cycle (see Lemma 2.1) and therefore every triangle is still monochromatic.

Condition 2: for every pair of distinct non-adjacent vertices u, v, there exist different labels j and k which both appear on every iduced u, v-path.
To verify condition 2, note that any two u, v-paths either both contain edge e or none (otherwise e would be part of a cycle, but e is a cut-edge)
If none of the induced u, v-paths contains edge e then the new KP-labeling still satisfies Condition 2 for the pair u, v because of the original KP-labeling satisfied the condition for this pair.
Otherwise all induced u, v-paths contain e, original label of e be i_0. We consider/distinguish two cases:

Case 1 If $i_0 \in \{j, k\}$ then simply take the set $(\{j, k\} \setminus \{i_0\}) \cup \{i\}$.

Case 2 If $i_0 \notin \{j, k\}$ then simply take the set $\{j, k\}$.

For the converse direction, suppose G has a KP-labeling in which e receives a unique label (by Corollary). Suppose for contradiction that e is not a cut-edge. Then by Lemma 2.1 e is part of a cycle C. If C is a triangle by Condition 1 and every edge of C must have the same label as e, but then e would not be labeled uniquely. Therefore C cannot be triangle. Let $V(C) = \{v_1, v_2, ..., v_p\}$ in cyclic order and without loss of generality $e = v_1v_2$. Let $u = v_1$ and $v = v_3$. Then since e is labeled uniquely, the two induced paths forming a cycle from u to v. These two paths cannot share any two labels so Condition 2 cannot hold, a contradiction. □

Theorem 3.5. *Let G be a connected graph. Then G is an induced subgraph of a Hamming graph of dimension d if and only if G has a $d - KP$-labeling.*

Proof. Let G be an induced subgraph of $H = K_{n_1} \square K_{n_2} \square \cdots \square K_{n_d}$. To make things easier denote $p_i = p_{K_{n_i}}$, which is the projection of the complete graph K_{n_i} of H and also consider the labeling $E(G)$ induced by the color map c of H. We will show that this labeling satisfies Conditions 1 and 2.
Condition 1 is clear. Indeed, if u, v, and w induce a triangle, then they all lie in the same layer of H and by the alternative description of Hamming graphs we have that each pair is adjacent so they differ in exactly in one coordinate. Therefore color map c will map those edges to the coordinate in which they differ and so the edges uv, uw, and vw receive the same label.
We next show that Condition 2 is satisfied, too. Let u and v be two vertices of G with $d_G(u, v) \geq 2$. Suppose that there is no label that appears on all induced u, v-paths. Now if we see u and v as $d - tuples$ and we understand paths as changing coordinates and therefore $p_i(u) = p_i(v)$ for all i because if we look at one induced path that does not have label i that means than we never change the i^{th} coordinate as we traverse this path and therefore the i^{th} coordinates are the same, which implies $u = v$ contrary to $d_G(u, v) \geq 2$. Suppose now that all induced u, v-paths have exactly one label in

common, say i. We have $p_j(u) = p_j(v)$ for all $j \neq i$ and consequently $p_i(u) \neq p_i(v)$ (since $u \neq v$). Vertices $p_i(u)$ and $p_i(v)$ are adjacent in K_{n_i}, since K_{n_i} is a complete graph. Hence, u and v are adjacent in H and therefore also in G which is impossible.
Conversely, let ℓ be a d-edge-labeling of G that fulfills Conditions 1 and 2. Let $\mathcal{F} = \{F_1, F_2, \ldots F_d\}$ be the partition of $E(G)$ induced by l and let f be the quotient map of G with respect to $\mathcal{F}$. We claim that f embeds G as an induced subgraph into $G = G/F_1 \Box G/F_2 \Box \ldots \Box G/F_d$.
We show first that f is one-to-one. Suppose that vertices x and y are not adjacent in G. Then by Condition 2 and Lemma 3.3, there exist labels i and j such that on every x, y-path in G we find labels i and j. So x and y are in different components in both $G \backslash F_i$ and $G \backslash F_j$. Already the first fact assures that $f(x) \neq f(y)$. Let next x and y be adjacent vertices of G and let $l(xy) = i$. We can see that $f_j(x) = f_j(y)$ for every $j \neq i$, they belong to same component they are adjacent, now let us take a look at f_i. Suppose that there exists an x, y-path $P = x_1 x_2 \ldots x_r$ in $G \backslash F_i$, where $x_1 = x$ and $x_r = y$. We can assume that P is shortest among all x, y-paths in $G \backslash F_i$. If P is induced in $G - xy$ we have a contradiction with Condition 1 when $r = 3$ since P does not take the label i (recall that P is in $G \backslash F_i$) but a triangle will be formed with xy and two edges in P. When $r > 3$, we have and a contradiction with Lemma 3.2 because $P + xy$ would form a cycle in which label i is present only once. Thus P is not induced in $G - xy$, $r > 3$, and there are adjacent vertices x_j and x_k with $k > j + 1$. By the minimality of P we have $l(x_j x_k) = i$. We can select j and k such that $k - j$ is minimal among all such vertices x_j and x_k. Then the cycle $C = x_j x_{j+1} \ldots x_{k-1} x_k$ is induced. If C is a triangle we have a contradiction with Condition 1, otherwise we have a contradiction with Lemma 3.2. Hence, we have shown that f is one-to-one.
Let xy be an edge of G with $l(xy) = i$. Then, by the above, x and y are in different components of $G \backslash F_i$. Moreover, they belong to the same component in any of the graphs $G \backslash F_j$, for $j \neq i$. It follows that f maps edges to edges and the claim is proved. Hence, G is isomorphic to the subgraph $G \backslash F_1 \Box G \backslash F_2 \Box \ldots \Box G \backslash F_k$ induced by $f(V(G))$. To complete the proof we show that G is also an induced subgraph of the Hamming graph $k_{|V(G \backslash F_1)|} \Box K_{|V(G \backslash F_2|)} \Box \ldots \Box K_{V(|G \backslash F_k|)}$: Let x and y be non-adjacent vertices of G. Then, by the same reasoning as above, x and y are in different components of at least two graphs $G \backslash F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates which remains valid after adding edges to the factor graphs. $\Box$

Example 3.6. Let us see that C_7 is an induced subgraph of a Hamming graph and that there exists a labeling of C_7 that fulfills Conditions 1 and 2.
C_7 is an induced subgraph of Hamming graph, more specifically of $K_2 \Box K_3 \Box K_2$ in Figure 7 take the cycle $(112, 212, 232, 231, 221, 121, 111)$ and using the color map c, edge $111, 112$ gets label 3, edge $112, 212$ gets label 1, and edge $212, 232$ gets label 2,

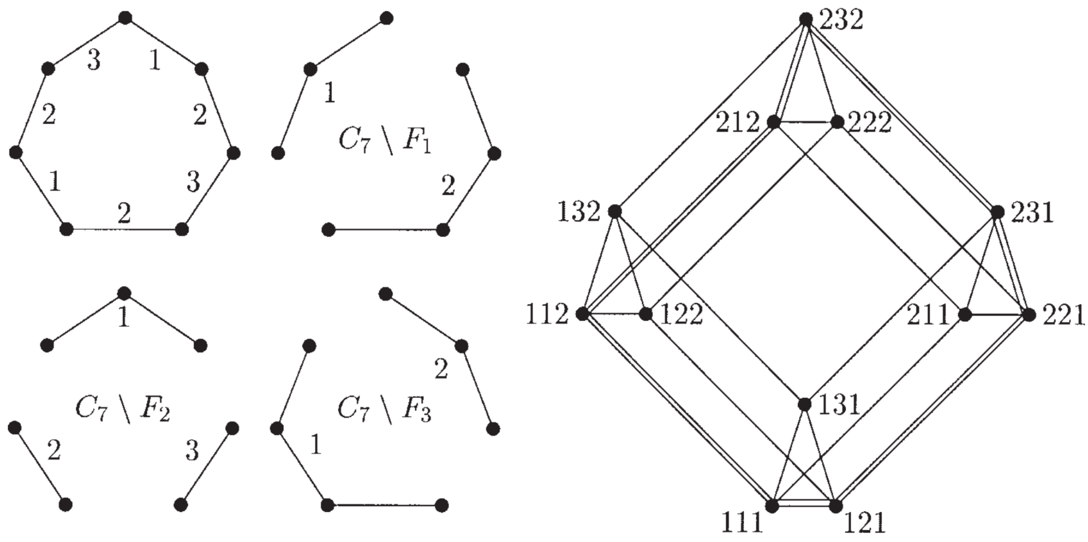

Figure 7: C_7 as an induced subgraph of $K_2 \square K_3 \square K_2$.

edge $232, 231$ gets label 3, edge $231, 221$ gets label 2, edge $221, 121$ gets label 1, edge $121, 111$ gets label 2. We can easily check that edges of a triangle have the same label (here there is no triangle, which is fine) and let u and v be arbitrary vertices of C_7 with $d_{C_7}(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced u, v-path.
C_7 has a labeling in which Conditions 1 and 2 hold as the first graph in Figure 7. Now let us take a look at where does the map f send vertices of C_7. We follow the order clockwise from most top vertex, f maps the first vertex to 112, f maps the second vertex to 212, f maps the third vertex to 232, f maps the fourth vertex to 231, f maps the fifth vertex to 221, f maps the sixth vertex to 121,, f maps the seventh vertex to 111 and as we can see $G = f(G)$ and an induced subgraph of $K_2 \square K_3 \square K_2$.

Note that the quotient graphs obtained in the proof of Theorem 3.5 need not be complete. For instance, consider the path P_4 together with the labeling 1, 2, 1 as we get $G \backslash F_1 \cong P_3$ and $G \backslash F_2 \cong K_2$.

4 Cartesian dimension

Given a positive integer d, a d-realization of a graph $G = (V, E)$ is an injective mapping $\varphi_G : V \to \mathbb{R}^d$ such that two vertices $u, v \in V$ are adjacent if and only if $\varphi_G(u)$ and $\varphi_G(v)$ differ in exactly one coordinate. A graph G is said to be d-realizable if it has a d-realization. Note that G is d-realizable if and only if G has a d-realization $\varphi_G : V \to \mathbb{N}^d$ or any large enough set so that we do not run out of available values for coordinates since $V(G)$ is finite.
Any graph that has a d-realization, it has also a $(d+1)$-realization (simply add one coordinate constantly equal to 0). This naturally leads to a definition: the Cartesian dimension of a graph $G = (V, E)$, denoted $Cdim(G)$, is defined as the minimum non-negative integer d such that G is d-realizable, if such an integer exists, and ∞, otherwise. Note that K_1 is the only graph of Cartesian dimension 0.

For a graph G to have Cartesian dimension 1 it must have a 1-realization, or in other words you can map all the vertices of the graph to $\mathbb{R}$ and becuase the map has to be injective every vertex is mapped to different real number but every two vertices differ in exactly one coordinate which makes them adjacent, therefore the only graphs of Cartesian dimension 1 are exactly the complete graphs of order at least 2.

Example 4.1. C_6 has Cartesian dimension 2. Indeed, since it is not a complete graph it cannot have dimension 1, and we find a $2-realization$ of C_6 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ by the map $\varphi_{C_7} : V \to \mathbb{R}^2$ given by $\varphi(v_1) = (1, 1)$, $\varphi(v_1) = (1, 1)$, $\varphi(v_2) = (3, 1)$, $\varphi(v_3) = (3, 3)$, $\varphi(v_4) = (2, 3)$, $\varphi(v_5) = (2, 2)$ and $\varphi(v_6) = (1, 2)$ (See Figure 8). An easy check shows that indeed for any edge the corresponding values of φ_{C_7} differ in exactly one coordinate, while for any non-edge they differ in both cooridnate.

Lemma 4.2. *If graph G is diamond-free then any two maximal cliques of G intersect in at most one vertex.*

Proof. Suppose two maximal cliques A_1 and A_2 intersect in at least two vertices say v_1, v_2. Because A_1 is a maximal clique there is some vertex v_3 other than v_1, v_2 in A_1 but not in A_2 and since A_1 is clique we have that v_1, v_2, v_3 form a triangle. Similarly, there exists a vertex v_4 in A_2 but not in A_1 such that v_1, v_2, v_4 form triangle. But then vertices v_1, v_2, v_3, v_4 induce a diamond, a contradiction with G being diamond-free. □

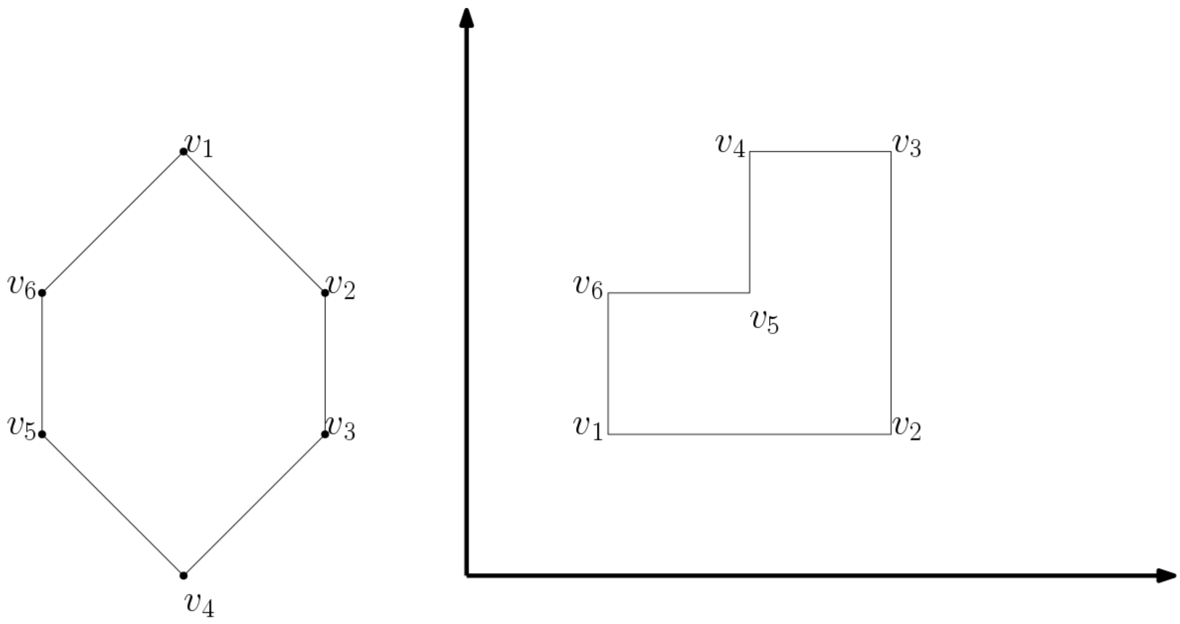

Figure 8: A 2-realization of C_6

Theorem 4.3. *(Staton and Wingard [3]) For every graph G, the following conditions are equivalent:*

1. $Cdim(G) \leq 2$.
2. *G is the line graph of a bipartite graph.*
3. *G is diamond-free and* $K(G)$ *is bipartite.*
4. *G is* $\{claw,\ diamond,\ C_5,\ C_7,\ \ldots\}$*-free.*

Proof. First, we prove that condition 1 implies condition 4. We show that none of the graphs in condition 4 can be an induced subgraph of a graph with $Cdim(G) \leq 2$. Suppose that $G \in \{claw, diamond\} \cup \{C_k | k \geq 5 \text{ and k odd}\}$, $Cdim(G)$ cannot be equal to 1 since only complete graph with Cartesian dimension 1. Now suppose it $Cdim(G) = 2$. First let us consider the diamond. Suppose G has diamond graph, H as induced subgraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(H) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4\}$. Now because v_1, v_2, and v_3 induce a complete graph on 3 vertices, all three together differ in one coordinate, similarly for v_2, v_3 and v_4 so it follows that also v_1 and v_4 differ only in one coordinate but they are not adjacent, a contradiction.
Next suppose G has claw graph, H as induced subgraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(H) = \{v_1v_2, v_1v_3, v_1v_4\}$ and with out loss of generality assume that $\varphi_G(v_1)$ differs only in the first coordinate with $\varphi_G(v_2)$. Because v_2 and v_3 are not adjacent, $\varphi_G(v_1)$ and $\varphi_G(v_3)$ must differ only in the second coordinate(say).

Because v_1 and v_4 are adjacent, they must differ in only one coordinate, but either coordinate is not good as that would imply adjacency between v_4 and v_2, or between v_4 and v_3 but there is no edge among them.
And finally consider ann odd cycle of length more than 3. Let the vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$ with edge set $E(G) = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$ and notice that since there is no triangle, no three vertices lie in the same line so if we color map define in the section 3.1 then by by following the cycle we must alternate by colors 1 and 2, but the cycle is odd so it is not possible.
(4) $\implies$ (3), suppose that G is $\{claw, diamond, C_5, C_7, \ldots\}$-free. We only need to show that $K(G)$ is bipartite which is equivalent to showing that $K(G)$ has no odd cycle. Suppose $K(G)$ has an odd cycle of length greater than 3. First note that no three maximal cliques (vertices in $K(G)$) share one vertice as then a claw would be formed in G (choose any other vertex from each clique and together with common vertex would form the claw) but G is claw-free. Thus we form an odd cycle in G by following the common vertices of cliques of size greater than 3. But G is $\{C_5, C_7, \ldots\}$-free, therefore $K(G)$ has a cycle of length 3. Now if all three cliques have a vertex in common then we can form a claw because those cliques are maximal so there exist some vertex in any of the cliques that other cliques do not have. Let those cliques that form a triangle be A_1, A_2, A_3 and the common vertex of A_1 and A_2 is v_1; the common vertex of A_2 and A_3 is v_2; whereas the common vertex of A_3, A_1 is v_3, because v_1 and v_2 are in A_2 they are adjacent, similiarly for others. Notice that at least one of the cliques must have another vertex v_4 other than v_1, v_2, v_3, which is non-adjacent to one of them, otherwise $K(G)$ would be just a vertex. Now with out loss of generality say $v_4 \in A_1$. Then we have another triangle with $v_1, v_3 \in A_1$, which implies that v_1, v_2, v_3, v_4 induce a diamond because from assumption $v_2v_4 \notin E(G)$ (v_4 is is non-adjacent to one of v_1, v_2, v_3) but G is diamond-free which leads to a contradiction. Therefore $K(G)$ is bipartite.
Now let us show that condition 3 implies condition 1. If G is complete then we know that $Cdim(G) = 1$. Suppose that G is not complete. Since $K(G)$ is bipartite, we can color vertices of $K(G)$, with two colors in natural way. Map vertices of G in $\mathbb{R}^2$ by placing cliques of G having one color on vertical lines and cliques having the other color on horizontal lines. By Lemma 4.2, no two vertices are mapped to same point. This gies a 2-realization of G, showing that $Cdim(G) \leq 2$
Now let us show that condition 1 holds if and only if condition 2 holds. Given a bipartite graph H such that $L(H) = G$, construct the adjacency matrix A of H. Mark points in $\mathbb{R}^2$ at (i, j) if and only if $a_{ij} = 1$ in uppertriangle matrix of A. It is immediate that one can map vertex $i, j \in V(G)$ to (i, j) where $i < j$ and notice that this defines a one to one mapping and if two vertices are adjacent in G then they will defer in exactly on coordinate. This shows that $Cdim(G) \leq 2$ as we found a 2-realization.

Since G can be realized with vertices only at positive integral points (the map φ_G can be defined to $\mathbb{N}^d$), the proof works in reverse. □

Lemma 4.4. *Let G be a d-realizable graph. Let v be a vertex adjacent to vertices x and y. Then x and y are adjacent if and only if both of them differ from v in the same coordinate.*

Proof. ($\Longrightarrow$) If x and y are adjacent then they differ in exactly one coordinate say in kth coordinate. Also say v differs to x exactly in the ith coordinate and to y in the jth coordinate. Now suppose x and y do not differ from v in the same coordinate which means that $i \neq j$, also note that k cannot be equal to both i and j as this would imply $i = j$ so w.l.o.g. $k \neq j$. Now because $i \neq j$, the jth coordinate of x is the same with jth coordinate of v and therefore differs from jth coordinate of y, but also $k \neq j$ which means that x and y differ in more than one coordinate which contradicts the fact that x and y are adjacent.
($\Longleftarrow$) If x and y differ from v in the same coordinate and because both of them are adjacent to v then it means that they differ in exactly that coordinate in which they differ with v, i.e. in one coordinate therefore they are adjacent. □

Theorem 4.5. *Every d-realizable graph is $\{K_{1,d+1}$, diamond, $K_{2,3}$, C_5 $\}$-free.*

Proof. Let G be a d-realizable graph.
For $K_{1,d+1}$: Suppose v is the hub (the vertex with largest degree) of an induced $K_{1,d+1}$ in G. Then v differs from $d + 1$ neighbors in different coordinates by Lemma 4.4, but there are only d coordinates, a contradiction.
For the diamond: If v_1, v_2, x and y are vertices of an induced diamond with v_1 and v_2 the vertices of degree 2, then, invoking Lemma 4.4 we see that v_1 differs from x and y in the same coordinate. Similarly v_2 difers from x and y in the same coordinate. Hence v_1 and v_2 differ from x and y in the same coordinate, and thus either $x = y$ or x is adjacent to y.
For $K_{2,3}$: Let x and y be adjacent to all v_1,v_2,v_3 in an induced $K_{2,3}$. Then by Lemma 4.4, v_1,v_2, v_3 differ from x in 3 distinct coordinates. We may assume that coordinates of x are zeros. Then y is at a distance of 2 from x, so y has exactly 2 non-zero coordinates. Hence y has a zero in coordinate where either v_1,v_2 or v_3 is non-zero, and it follows that y differs from one of these 3 vertices in 3 coordinates, and is therefore not adjacent to that vertex.
For the C_5: Let v_1,v_2,v_3,v_4,v_5 be the vertices of a 5-cycle in cyclic order. We assume that the coordinates of v_1 are all zeros.

$$\varphi(v_1) = (0, 0, 0, \ldots)$$

, where φ is the map from vertex set to $\mathbb{R}^d$.
With out loss of generality we can say

$$\varphi(v_2) = (\alpha, 0, 0, \ldots), \text{ for some } \alpha \in \mathbb{R}$$

By the Lemma 4.4 with out loss of generality

$$\varphi(v_3) = (\alpha, \beta, 0, 0, \ldots), \text{ for some } \beta \in \mathbb{R}$$

Now if v_4 differed from v_3 in any coordinate other than the first two, v_4 would be at a distance at least 3 from v_1 and it would be impossible to find v_5 to complete the 5-cycle. Hence v_4 differs from v_3 in one of the first 2 coordinates. If

$$\varphi(v_4) = (\alpha, \gamma, 0, 0, \ldots), \text{ for some } \gamma \in \mathbb{R}$$

then v_4 is adjacent to v_2. Hence

$$\varphi(v_4) = (\delta, \beta, 0, 0, \ldots), \text{ for some } \delta \in \mathbb{R}$$

and the only choice for v_5 are

$$(\delta, 0, 0, 0, \ldots)$$

and

$$(0, \beta, 0, 0, \ldots)$$

In the first case v_5 is adjacent to v_2, and, in the second one, to v_3; a contradiction.

□

We take a closer look at 3-dimensional case and present two results both are related to the Klavžar-Peterin characterization which is needed for hardness proof for recognizing 3-realizable graphs developed in paper (here reference paper of Prof Martin). In the first result, we will see that the defining properties of a 3-KP-labeling are satisfied for a graph as soon as they are satisfied for the family of all its induced subgraphs isomorphic to a cycle or to a P_3.

Theorem 4.6. *Let G be a graph. A 3-edge-labeling of G is a KP-labeling if and only if it satisfies the following two conditions:*
***Condition 3**: for every induced cycle C of G, the restriction of the labeling to $E(C)$ is a KP-labeling of C.*
***Condition 4**: no induced P_3 is monochromatic.*

Proof. Let us start by showing the necessity of the two conditions. If G is 3-KP-labeling and H is an induced subgraph of G, then the restriction of the labeling to $E(H)$ is a 3-KP-labeling of H because every path in H is still a path in G, hence Condition 3 is

necessary. Condition 4 follows from the fact that every induced P_3 must contain two different labels since it is a path itself.
Now we prove the sufficiency. For Condition 1 it is needed to show that every triangle is monochromatic but from Condition 3 we know that for every induced triangle of G, the restriction to edges, which is the same triangle is KP-labeling but triangle is KP-labeling if and only if it is monochromatic.
Now let's show that Condition 2 also holds by contradiction.
Suppose that there is a 3-edge-labeling $l : E(G) \to \{1, 2, 3\}$ satisfying Conditions 3 and 4, but not Condition 2. Singe Condition 2 does not hold in G, it means that G contains two different induced paths of length at least two, say P and Q, intersecting at their endpoints u and v, such that no pair of different labels appears on both P and Q.
Due to Condition 4, on each of the paths P and Q at least two different labels appear. Since no pair of different labels appears on both P and Q, we may assume that P and Q take alternatingly labels 1, 2 and 1, 3, respectively. Moreover, assume that P and Q were chosen so as to minimize $|V(P)| + |V(Q)|$.
We continue with definition to facilitate our way in the proof. Given a path R and two of its vertices x and y, denote by R_{xy} the subpath of R between x and y, and by $V(R)^{-xy}$ the set $V(R)\backslash\{x, y\}$. Also, say a path is k-labeled if exactly k different labels appear on its edges.
We claim that $V_P^{uv} \cap V_Q^{uv} = \emptyset$. Suppose that there is something in intersection, $w \in V_P^{uv} \cap V_Q^{uv}$. First we see that P_uw and Q_uw cannot be 1-labeled because of Condition 4 none of the paths can have length more than 2, which must follow that u and w are adjacent, but this is in contradiction with minimality of $|V(P)|+|V(Q)|$. So it must be that P_uw and Q_uw are 2-labeled. Since there are only 3 labels it must be the case that $P_{uw} \cup Q_{uw}$ would be 3-labeled, again contrandiction with minimality of $|V(P)|+|V(Q)|$ as we found a path with 3 labels satisfying what we need.
We add also some definitions, which will help us. For $t \in \{u, v\}$ and $xy \in E(G)$ with $(x, y) \in V_P^{uv} \times V_Q^{uv}$, a cycle $C = P_{tx} - xy - Q_{yt}$ such that either $P_{tx} - xy$ or $xy - Q_{yt}$ is an induced path will be called a PQ-cycle. Note that a PQ-cycle cannot be 3-labeled: with out loss of generality say $P' = P_{tx} - xy$ was an induced path, then we can find a shorter path, namely the path P' and Q_{yt} would contradict minimality of $|V(P)| + |V(Q)|$.
Let us investigate the cycle $C_0 = P \cup Q$, from our assumption that not to labels appear in both paths P, Q, from Condition 3 we have in induced cycle, the restriction of the labeling to $E(C_0)$ is a KP-labeling of C_0 it must follow that C_0 is not an induced cycle as the restriction of the labeling to $E(C)$ is obviously not a KP-labeling of C_0. Let xy be a chord in C_0 such that $\{x, y\} \cap \{u, v\} = \emptyset$ such that $x \in V(P)$ is closest to

u(where the distance is measured within P), and y is the neighbor of x in Q closest to v (where the distance is measured within Q). Now we observe two cycles that are formed, namely $C_1 = P_{ux} - xy - Q_{yu}$ and $C_2 = P_{vx} - xy - Q_{yv}$, because of how we selected x and y each of C_1, C_2 is either a PQ-cycle or a triangle, implying that neither of them is 3-labeled. If C_1 is monochromatic then, as $E(C_0) \subset E(C_1) \cup E(C_2)$ while C_1 and C_2 share the label of xy, it would follow that C_2 was 3-labeled, symmetricaly if C_2 was monochromatic. Thus, C_1 and C_2 are 2-labeled.
As C_1 and C_2 are 2-labeled, they share exactly one label. By definition, any PQ-cycle contains a P_3 from either P or Q, hence (recalling that P and Q alternate labels 1, 2 and 1, 3, respectively), C_1 and C_2 share label 1. Such is then the label of xy. However, one of the two edges incident to x in P is also labeled with 1, forming with xy a monochromatic induced P_3 (as part of either C_1 or C_2), which contradicts Condition 4. □

Example 4.7. We will see an example where the Theorem 4.6 does not work for 4-edge-labeling. We find a graph satisfying Condition 3 and 4 but it is not a KP-labeling. Take the graph G with 4 paths of length 6 sharing only the endpoints vertices, say u and v. Label the paths in following way:
path 1: 1, 2, 3, 1, 2, 3
path 2: 2, 3, 4, 2, 3, 4
path 3: 3, 4, 1, 3, 4, 1
path 4: 4, 1, 2, 4, 1, 2
A quick check can show that every induced cycle is a KP-labeling and also no P_3 is monochromatic, which means that Condition 3 and 4 are satisfied. However if we check whether this labeling is a KP-labeling of G, taking non-adjacent vertices u and v, we can see that each pair of the 4 paths share exactly 2 labels, but from pingeonhole principle we cannot find labels i and j which both appear on any induced u, v-path.
This example can be generalized that Theorem 4.6 does not work for any $d > 3$.

From Condition 1 we know that every 3-KP-labeling of 3-cycle is constant(monochromatic). The folliwing result analyzes longer cycles.

Lemma 4.8. *Let C be a cycle of length at least 4. A 3-edge-coloring of C with colors 1, 2, 3 is a KP-labeling if and only if*

- *either it is a 2-edge-coloring of C, or*
- *possibly after permuting the labels 1, 2, 3, cycle C contains a cyclically ordered sequence of 6 distinct (not necessarily consecutive) edges labeled 1, 2, 3, 1, 2, 3, respectively.*

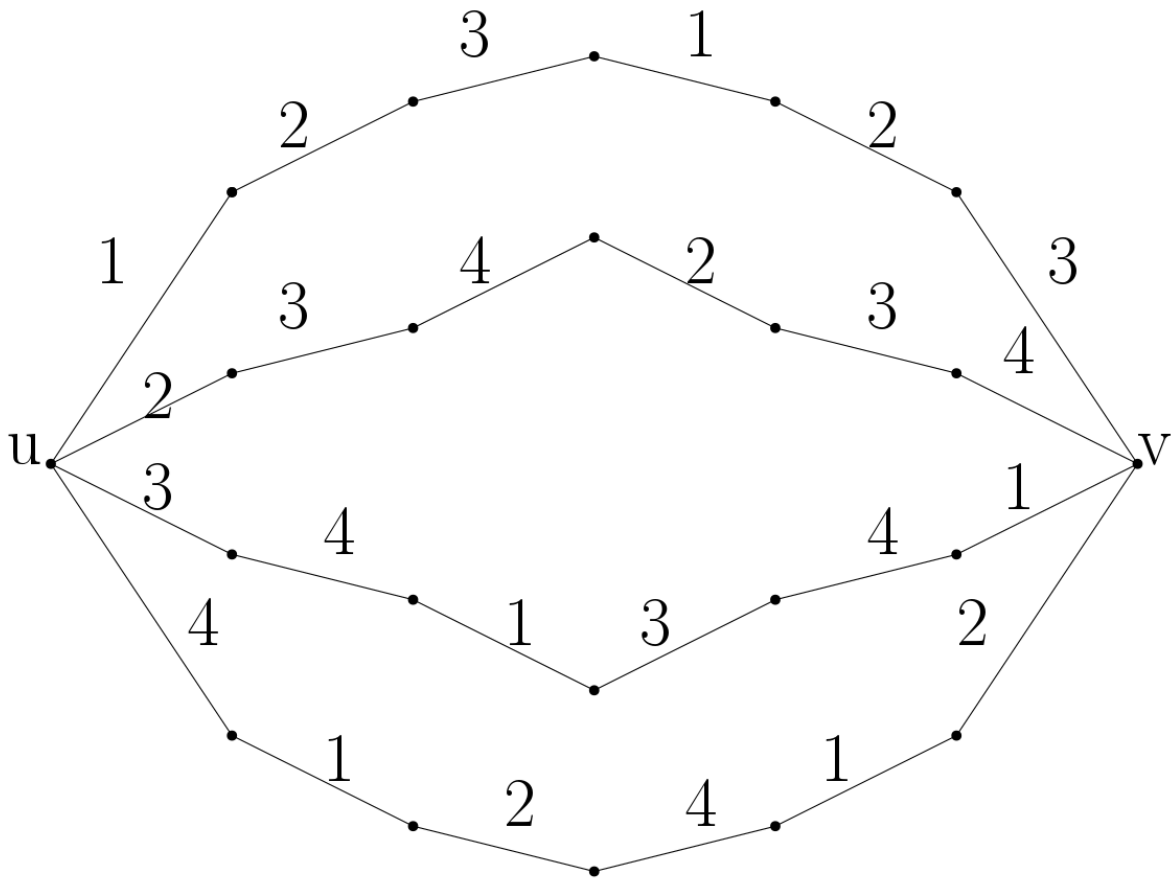

Figure 9: Counterexample of Theorem 4.6 for 4-edge-labeling

Proof. ($\Longrightarrow$) If the label is 2-edge-coloring then it follows, otherwise suppose that C is KP-labeled with an edge-coloring using colors 1,2 and 3 so that all three labels appear on $E(C)$. Fix a cyclic order σ of the edges of C. Without loss of generality, C has a pair of consecutive edges in σ, say e_1 and e_2, that are labeled 1 and 2 respectively. Each label must appear at least twice on C, since otherwise the endpoints of a path formed by two consecutive edges containing a label that appears exactly once would violate Condition 2. Then, the sequence 1,2,3,3 must appear as a sequence of the labels of edges in order σ, we stress it here from the lemma that this order is not necessarily consecutive (we could have something like 1,2,3,1,2,1,2,3). Let P be the maximal subpath of C containing e_1 and e_2 having no edge labeled 3, and let e and e' be the two edges of $E(C)\backslash E(P)$ incident to an edge in P. By the maximality of P, edges e and e' are labeled 3 and, since each each label appears twice they must be distinct. Let P' be the subpath of C formed by the edges not in $E(P) \cup \{e, e'\}$. Since P has only labels 1 and 2, in order not to violate Condition 2 both of them must appear in $E(C)\backslash E(P)$ and since $e, e' \in E(C)\backslash E(P)$ are labeled 3, it follows that labels 1,2 must appear in $E(P')$. If labels 1 and 2 appear in order σ on P' (not necessarily

consecutively), we are done. Suppose then that all occurrences of 2 appear in σ before all occurrences of 1 on $E(P')$. If on P there is an occurrence of label 2 appearing (in σ) before an occurence of 1, we are done again. By way of contradiction ,suppose that this is not the case, that is, suppose that all occurces of 1 appear (in σ) before all occurences of 2 on P.
Then C can be divided in two parts in such a way that the occurrences of 1 appear only in one part and the occurences of 2 in the other part, which will violate Condition 2.
($\Longleftarrow$) Since there are no triangles, Condition 1 is satisfied.
If C is 2-edge-labeled, then clearly Condition 2 holds.
Suppose now, without loss of generality, that C contains a cyclically ordered sequence of 6 distinct (not necessarily consecutive) edges labeled 1, 2, 3, 1, 2, 3, respectively. Let F denote a fixed set of 6 edges with the above property. Consider now an arbitrary pair u, v of non -adjacent vertices of C. Let P and Q be the two $u, v-$subpaths of C. At least on of P and Q, say P, contains at least three consecutive edges from F. Hence, any two distinct labels appearing on Q will appear on every induced $u, v-$path. Since this is true for an arbitrary pair u, v of nonadjacent verices, Condition 2 is satisfied.

□

4.1 NP-completeness of testing realizability in $d \geq 3$ dimension

In this section we will present the result done by Martin Milanič, Peter Muršič, and Marcelo Mydlarz.
In this section we try to understand the question: How difficult is it to determine if a given graph G can be realized in R^d?
First we show that for every $d \geq 3$, determining whether $Cdim(G) \leq d$ is NP-complete.

Theorem 4.9. *(Martin Milanič, Peter Muršič, and Marcelo Mydlarz)*
Given a graph G, determining whether $Cdim(G) \leq 3$ is NP-complete, even for connected bipartite graphs of maximum degree at most 3.

Proof. A polynomially checkable certificate (a problem that can be checked in polynomial time) of the fact that $Cdim(G) \leq 3$ is any 3-realization of G of the form $\varphi_G : V \to N^3$. Therefore, the problem is in NP (on any class of input graphs).
In [16] showed that it is NP complete to determine the chromatic index of an arbitrary graph. The problem remains NP-complete even for cubic graphs, so to show hardness, we make a reduction from the 3-edge-coloring problem in cubic graphs.

Let G be a cubic graph that is the input for the 3-edge-coloring problem. We may assume that G is connected. Construct a graph G' from G by replacing each edge xy of G with the structure shown in Figure 10, such structure we will call XVWY.

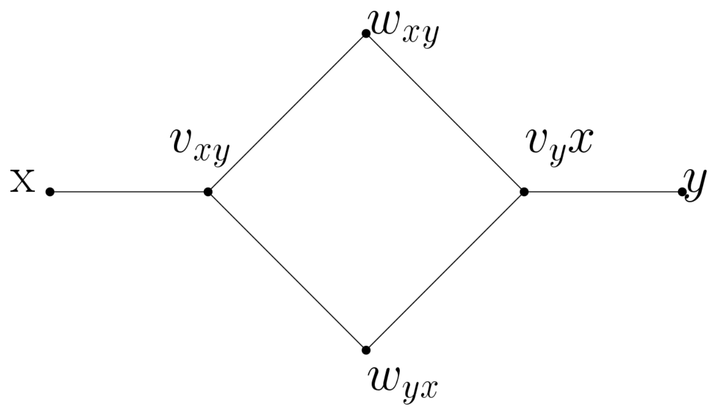

Figure 10: A gadget XVWY replacing each edge xy

Formally,

$$V(G') = V(G) \cup \bigcup_{xy\in E(G)} \{v_{xy}, v_{yx}, w_{xy}, w_{yx}\},$$

$$E(G') = \bigcup_{xy\in E(G)} \{xv_{xy}, v_{xy}w_{xy}, v_{xy}w_{yx}, v_{yx}w_{xy}, v_{yx}w_{yx}, yv_{yx}\}.$$

Now let us show that G' is bipartite. Letting $V_1 = V(G) \cup \bigcup_{xy\in E(G)}\{w_{xy}, w_{yx}\}$ and $V_2 = \bigcup_{xy\in E(G)}\{v_{xy}, v_{yx}\}$, we see that (V_1, V_2) is a bipartition of G' since both V_1 and V_2 are indepedent set and every possible edge in G' is between V_1 and V_2. Thus, G' is a bipartite graph with vertices of degrees 2 and 3 only. We will show that G is 3-edge-colorable if and only if $Cdim(G') \leq 3$.

Let us show the backward direction. Let $Cdim(G') \leq 3$. By Theorem 3.5, G' has a 3-KP-labeling. Then by Lemma 3.2 we can infer that the cycle $C = v_{xy} - w_{xy} - v_{yx} - w_{yx} - v_{xy}$ in G' must be 2-KP-labled. Since we have 3-KP-labeling, this implies that the edges xv_{xy} and yv_{yx} must have the same label l_{xy}, different from labels used in C. Since G' is triangle-free, any KP-labeling of G' is an edge-coloring (otherwise, Condition 4 would be violated). Therefore, by labeling each edge $xy \in E(G)$ with l_xy, we get a 3-edge-coloring of G.

Now suppose that G has a 3-edge-coloring using colors 1, 2, 3. For each edge xy of G labeled $i \in \{1, 2, 3\}$, let $\{j, k\} = \{1, 2, 3\}\backslash\{i\}$ and label the associated edges of G' as follows: edges xv_{xy} and yv_{yx} with i, edges $v_{xy}w_{xy}$ and $v_{yx}w_{yx}$ with j, and edges $v_{xy}w_{yx}$ and $v_{yx}w_{xy}$ with k. We claim that the so obtained labeling of G' is a KP-labeling. Since we have a 3-edge-labeling of G', by Thoerem 4.6, it suffices to check that Conditions 3 and 4 are satisfied. First we check that no induced P_3 is monochromatic. If the edges are stricly from one gadget XVWY then from labeling no P_3 there is monochromatic. If P_3 is formed from two gadgets XVWY having a vertex in common, the labels of this P_3 are taken from the incident edges in G, but the label in G is 3-edge-coloring and it has no monochromatic P_3, therefore the Condition 4 holds. In order to verify that Condition 3 holds, note that G' has two types of induced cycles:

- 4-cycles. They only appear in the gadget XVWY; they are properly 2-edge-colored and hence KP-labeled by Lemma 4.8.

- Cycles of length greater than 4. Each such cycle C has length $4p$ for some $p \geq 3$ because the cycle must close in some vertex from set $V(G)$ and a simple counting we get that it must be of length $4p$, and arises from a (not necessarily induced) p-cycle C' in G. We will show that such cycles satisfy the 123123-condition and apply Lemma 4.8. Let $x_1, x_2, \ldots, x_p$ be a cyclic order of vertices in C'. Without loss of generality, let 1, 2, 3, 1 be the labels (in this order) on some shortest path from x_1 to x_2 in C. We consider what are possible labels on the edges of any shortest path from x_2 to x_3 in C. Since on some shortest path from x_1 to x_2 ends with label 1 then no shortest path from x_2 to x_3 starts with 1, therefore it must starts with 2 or 3. If it starts with 2, it must end with labels 1,3,2 or 3,1,2; else if it starts with 3, it must end with labels 1,2,3 or 2,1,3. In every case we can find labels 2,3 in this order and thus, along cycle C we find 6 distinct edges labeled 1, 2, 3, 1, 2, 3 in order. This shows that C satisfies the 123123-condition.

It follows that Condition 3 is satisfied, hence by Theorem 4.6 G' has a 3-KP-labeling. By Theorem 3.5, we conclude that $Cdim(G') \leq 3$. □

4.2 Cartesian dimension of $P_i \square P_j$

A two-dimensional grid graph is the Cartesian product two path graphs. The results below will demonstrate that when finding the exact Cartesian dimension of these graphs and for paths of length at least 3, the Cartesian dimension is equal for any pair of path graphs. To prove these results, we will use the Theorem 3.5 which can be understood as for a connected graph G and positive integer d, $Cdim(G) \leq d$ if and only if G has a d-KP-labeling.

Theorem 4.10.

$$\mathrm{Cdim}(P_i \square P_j) = \begin{cases} 0, & \textit{if } (i,j) \in \{(1,1)\}; \\ 1, & \textit{if } (i,j) \in \{(1,2),(2,1)\}; \\ 2, & \textit{if } (i,j) \in \{(2,2),(1,k),(k,1)\}, \textit{ where } k \geq 3; \\ 3, & \textit{if } (i,j) \in \{(2,k),(k,2)\}, \textit{ where } k \geq 3; \\ 4, & \textit{if } (i,j) \in \{(k,l)\}, \textit{ where } k,l \geq 3; \end{cases}$$

Proof. Without loss of generality, let us take $i \leq j$.
If $i = 1$, then $P_1 \square P_j = P_j$; if $j = 1$ then $Cdim(P_i \square P_j) = 0$; if $j = 2$, then $Cdim(P_i \square P_j) = 1$; and if $j \geq 2$, then $Cdim(P_i \square P_j) = 2$.
If $i = 2$ and $j = 2$, then $Cdim(P_i \square P_j) = 2$.
If $j \geq 3$, from Theorem 4.5, we have that $Cdim(P_2 \square P_j) \geq 3$, since $P_2 \square P_j$ has $K_{1,3}$ as induced subgraph.
Next, we find a KP-labeling of the graph with 3 labels.

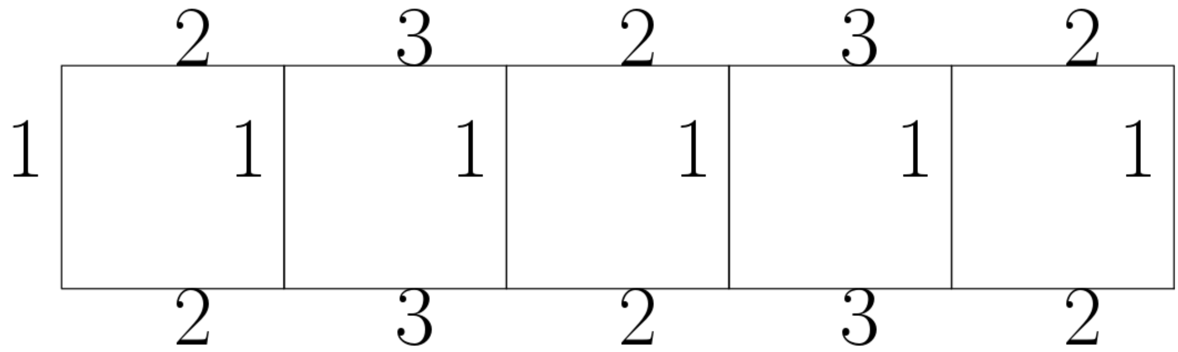

Figure 11: KP-labeling of $P_2 \square P_6$

We label the copies of the first factor (P_2) with label 1 and both copies of P_j the same by alternating between 2 and 3. An example of how would we label $P_2 \square P_6$ is shown in Figure 11.
For any two nonadjacent vertices, u and v, if they belong to a C_4, we take label 1 and the other label from set $\{2,3\}$ to satisfy Condition 2; otherwise take labels 2 and 3 for vertices, u and v, when they do not belong to some C_4 and u and v are in different components when taking the partition of graph with respect to labels 2 or 3, meaning that the edges labeled with i determine a cut-set, for i in $\{2,3\}$.
Therefore, this labeling is a KP-labeling and from Theorem 3.5, we have that $Cdim(P_2 \square P_j) \leq 3$.
Combining both inequalities, we have that $Cdim(P_2 \square P_j) = 3$ if $j \geq 3$

Finally, we will look at when $i \geq 3$.
If $i \geq 3$, from Theorem 4.5, we have that $Cdim(P_i \square P_j) \geq 4$, since $P_i \square P_j$ has $K_{1,4}$ as induced subgraph.

Next, we find a KP-labeling of the graph with 4 labels.

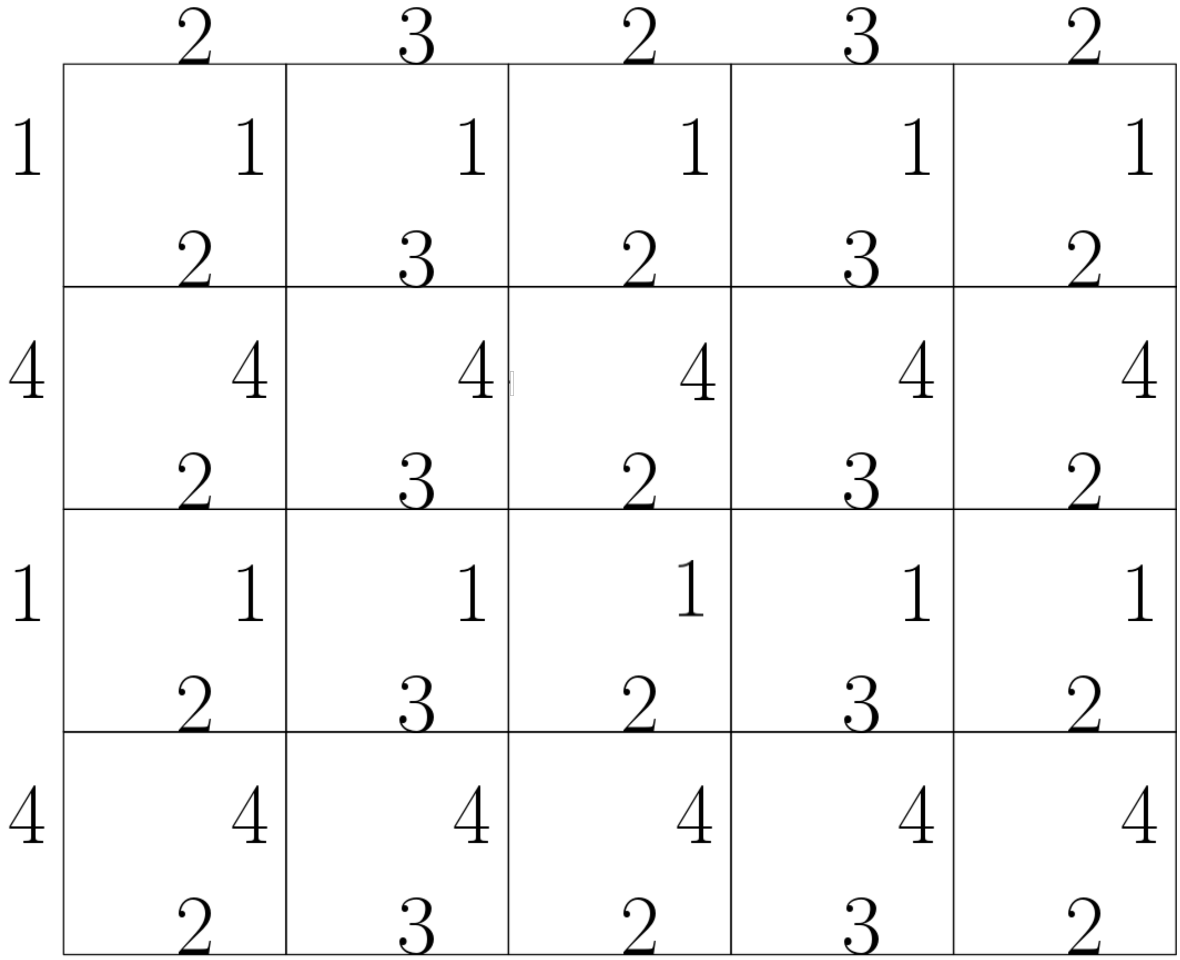

Figure 12: KP-labeling of $P_5 \square P_6$

We label the copies of P_i identically by alternating between 1 and 4; and the copies of P_j identically by alternating between 2 and 3.
For any two nonadjacent vertices, u and v, when they belong to a C_4, we take the labels in this cycle. However, if any two nonadjacent vertices, u with coordinates (a, x) and v with coordinates (b, y), do not belong to a C_4, then either $d_{P_i}(a, b) \geq 2$ or $d_{P_j}(x, y) \geq 2$. If $d_{P_i}(a, b) \geq 2$, we take labels 2 and 3; otherwise, we take labels 1 and 4. Similarly to the previous proposition, vertices u and v would belong to different components when removing either of the choosen labels.
Therefore, this label is a KP-labeling and from Theorem 3.5, we have that $Cdim(P_i \square P_j) \leq 4$.
Combining both inequalities, we have that $Cdim(P_i \square P_j) = 4$, if $i \geq 3$ $\square$

5 Hamming dimesion

Graph products offer a variety of possibilities to introduce the concept of a graph dimension. A classical result of Graham and Winkler [11] shows that any graph can be canonically isometrically embedded into the Cartesian product of graphs. Knowing that, this embedding is unique among all irredundant isometric embeddings with respect to the largest possible number of factors, the latter numberis is defined to be the *isometric dimension* of a graph.
The strong product $G \boxtimes H$ of graphs G and H is a graph such that the vertex set of $G \boxtimes H$ is the Cartesian product $V(G) \times V(H)$; and distinct vertices (u, u') and (v, v) are adjacent in $G \boxtimes H$ if and only if $u = v$ and u' is adjacent to v', or $u' = v'$ and u is adjacent to v, or u is adjacent to v and u' is adjacent to v'. Back in 1938 Schönberg [13] proved that every connected graph admits an isometric embedding into the strong product of paths. It is hence natural to define the *strong isometric dimension*, *idim*(G), of a graph G as the least number k such that G embeds isometrically into the strong product of k paths.
David Eppstein consider the following representation problem: for which unweighted undirected graphs can be assigned integer coordinates in some d-dimensional space Z^d, such that the distance between two vertices in the graph is equal to the L_1-distance(taxicab) between their coordinates? Call the minimum possible dimension d of such an embedding (if one exists) the *lattice dimension* of the graph.
The strong isometric dimension is universal in the sense that as soon as a graph is not the path graph, then its dimension is finite and bigger than 1. A similar conclusion can be stated for the so called direct dimension of a graph. On the other hand, for the most important graph product, the Cartesian one, no such universal dimension is known. While the isometric dimension is useful as soon as the dimension of a graph is more than 1, it was proved in [17] that for almost any graph its isometric dimension is 1. In other words, for almost any graph G, the isometric dimension yields no new insight about G. Also, only partial cubes, a special (although important) subclass of bipartite graphs, have finite lattice dimension.

We say that a graph G is an irredundant subgraph of $\square_i G_i$ if each G_i has at least two vertices and any vertex of G_i appears as a coordinate of some vertex of G.

In order to significantly increase the number of graphs with a non-trivial dimension that comes from the Cartesian product of graphs Sandi Klavžar, Iztok Peterin and Sara Sabrina Zemljič in [20], defined the *Hamming dimension* $Hdim(G)$ of a graph G is introduced as the largest dimension of a Hamming graph into which G embeds as an irredundant induced subgraph. If G is not an induced subgraph of any Hamming graph we set $Hdim(G) = \infty$.
In Figure 13 is shown C_6 induced in two different Hamming graphs, $K_3 \square K_3$ and $K_2 \square K_2 \square K_2$ Clearly, $Hdim(G) = 1$ if and only if G is a complete graph. By Theorem

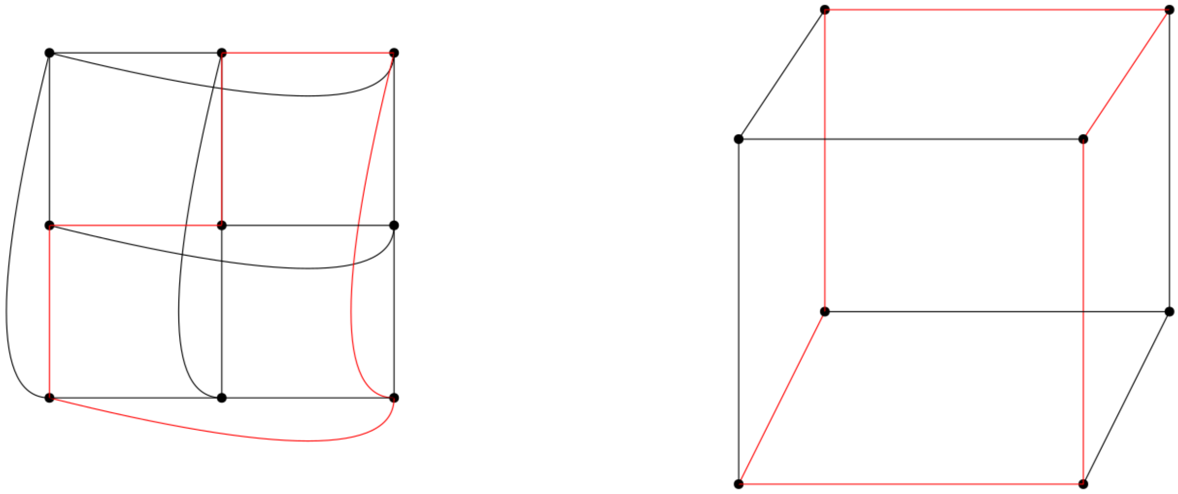

Figure 13: C_6 induced in $K_3 \square K_3$ and $K_2 \square K_2 \square K_2$

3.5, we can understand it as $Hdim(G) < \infty$ if and only if G has a $d - KP$-labeling.

5.1 Hamming dimension of Sierpiski graphs

The general problem of determining the Hamming dimension of a graph seems very demanding, here we will study this concept on Sierpiski graphs. Roughly speaking, in [20], they proved that all Sierpiński graphs (except in the trivial cases) have Hamming dimension bigger than 1 and finite. On the other hand, all but base 3 Sierpiński graphs have isometric dimension 1. Hence the Hamming dimension indeed significantly increases the number of graphs with a non-trivial Cartesian-like dimension.
Graphs whose drawings can be viewed as approximations to the famous Sierpiński triangle have been studied intensely in the past 25 years. The interest for these graphs comes from many different sources such as games like the Chinese rings or the Tower of Hanoi, topology, physics, the study of interconnection networks, and elsewhere.
Sierpiński graphs S_n^k were studied for the first time in [21]. In computer science, a very similar class of graphs (known as WK-recursive networks) was introduced earlier.

The study in [21] was motivated in part by the fact that for $k = 3$ these graphs are isomorphic to the Tower of Hanoi graphs and in part by topological studies.
The graphs S_n^k were investigated from numerous points of view, we recall some of them. These graphs contain unique 1-perfect codes. Metric properties of Sierpiński graphs were investigated in [22]. To determine the chromatic number of these graphs is easy, while in [23] it is proved that they are in edge- and total coloring class 1, except those isomorphic to a complete graph of odd or even order, respectively.
The Sierpiński graph S_n^k , $k, n \geq 1$, is defined on the vertex set $\{1, \ldots, k\}^n$, two different vertices $u = (u1, \ldots, u_n)$ and $v = (v1, \ldots, v_n)$ being adjacent if and only if there exists an $h \in \{1, \ldots, n\}$ such that

(i) $u_t = v_t$, for $t = 1, \ldots, h1$;

(ii) $u_h \neq v_h$; and

(iii) $u_t = v_h$ and $vt = uh$ for $t = h + 1, \ldots, n$.

An example of Sierpiński Graph is shown in Figure 14. Below we will show without proof results from [20]

Theorem 5.1. *For any* $n \geq 2$

$$Hdim(S_n^3) \geq \frac{7}{4} \cdot 3^{n-3} + 3 \cdot 2^{n-4} + \frac{3}{2} \cdot n - \frac{9}{4}$$

Proposition 5.2. *(i)* $Hdim(S_3^2) = 3, Hdim(S_3^3) = 6.$

(ii) For any $k \geq 4$, $Hdim(S_k^2) = 2.$

Theorem 5.3. *(i)* $Hdim(S_3^n) \leq 5 \cdot 3^{n-3} + 1$ $(n \geq 3)$.

(ii) $Hdim(S_k^n) \leq \frac{2}{k-1} k^{n-2} + \frac{2k-4}{k-1}$ $(k \geq 4$ *and* $n \geq 2)$

Corollary 5.4. *For any* $k \geq 4$, $Hdim(S_k^3) = 4$

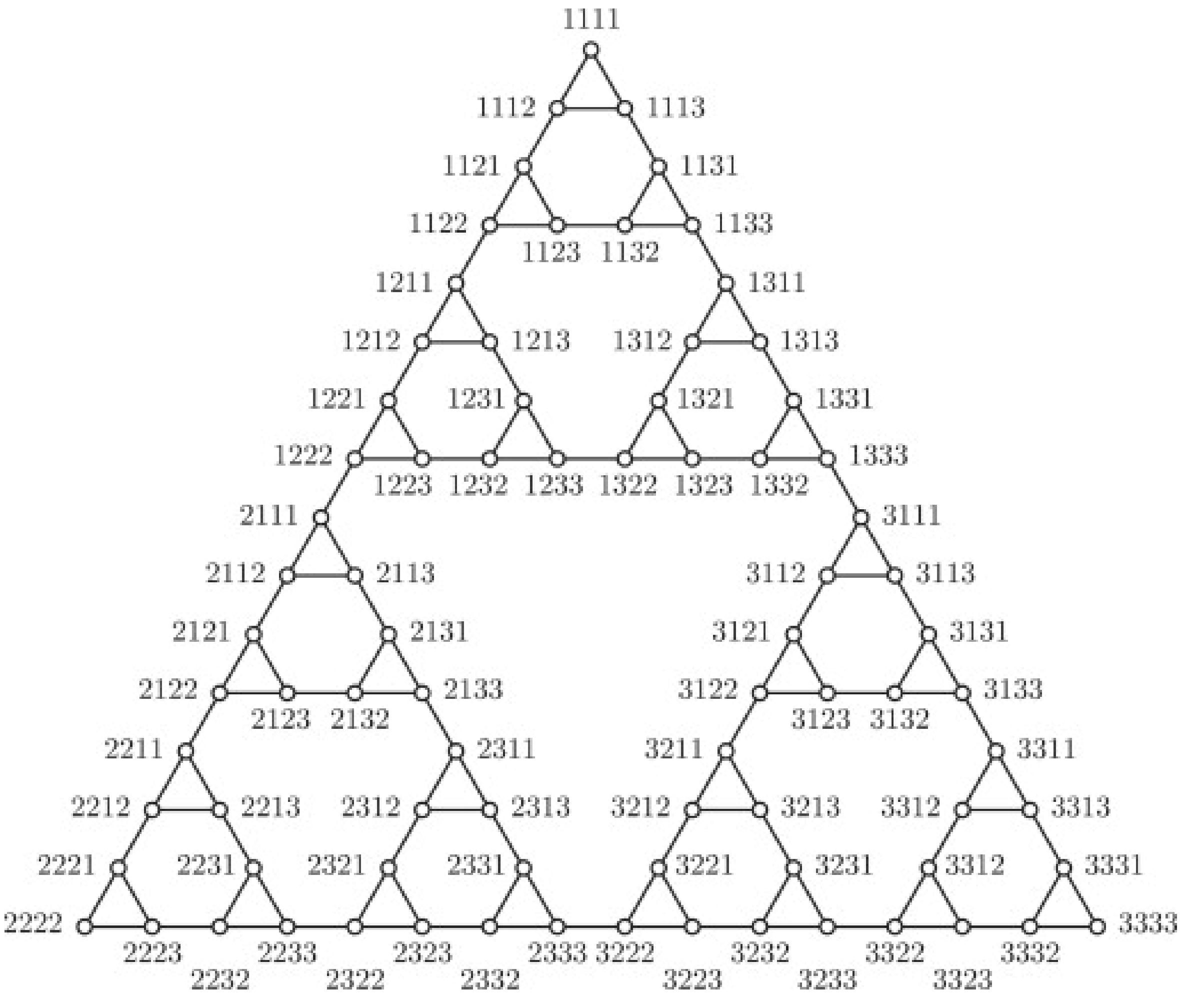

Figure 14: The Sierpiński graph S_4^3

6 Conclusion

This project paper is focused on carefully examining the representations of graphs as induced subgraphs of Hamming graphs. We have seen that for a graph G to be an induced subgraph of a Hamming graph if and only if there exists a labeling of $E(G)$ fulfilling the two conditions:

1. edges of a triangle receive the same label;
2. for any vertices u and v at distance at least two, there exist two labels which both appear on any induced $u;v$-path.

We have noticed that for Cartesian dimension, the two-dimensional case corresponds to the class of line graphs of bipartite graphs and is well-understood and for $d \leq 3$ the problem of determining whether a given graph G is d-realizable is NP-complete.
Shortly, we also defined the Hamming dimension and saw that with the aid of Hamming dimension, we were able to see that number of graphs with a non-trivial dimension that comes from the Cartesian product of graphs increases.
It would be interesting to study the relation of KP-labeling and Cartesian dimension of special graphs such as cycles, Sierpiński graphs, etc.

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Appendices

A Title of First Appendix

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B Title of Second Appendix

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