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Seminar - uvod v raziskovalno delo
(Seminar - Introduction to Research Work)

Naslov seminarske naloge

(Representations of graphs as induced subgraphs of Hamming graphs)

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i.e. that is
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1 Introduction

How difficult is it to determine if a given graph G can be realized in $\mathbb{R}^d$ so that vertices are mapped to distinct points and two vertices are adjacent if and only if the corresponding points are on a common line that is parallel to some axis? Let us refer to any such mapping as a d-realization of G and say that a graph is d-realizable if it has a d-realization.The class of d-realizable graphs was studied in the literature for decades, under diverse names such as arrow graphs (Cook, 1974), $(d-1)$-plane graphs and $(d-1)$-line graphs of d-partite d-uniform hypergraphs (Bermond etal., 1977), d-dimensional cellular graphs (Gurvich and Temkin, 1992), d-dimensional chessboard graphs (Staton and Wingard, 1998), and d-dimensional gridline graphs (Peterson, 2003).
In order to determine how hard will it be to answer that question we should first take a look at those definitions:
Let A be an algorithm that solves a problem $\mathcal{P}$. The running time of algorithm A is the number of basic calculation steps performed by A (additions, subtractions, multiplications, comparison of two numbers etc.). The time complexity of algorithm A is the function $T_A(n)$ that measures the running time of A in the worst case:

$$T_A : n \mapsto \text{largest running time of } A \text{ on input instances of size} \leq n.$$

Besides time complexity, space complexity of an algorithm might also be important (how much computer memory the algorithm needs). If we have a probability distribution on the input instances, we can also estimate the expected time complexity of the algorithm.
Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ is of order (at most) $O(g)$, if there exists a constant $C > 0$ such that $f(n) \leq C \cdot g(n)$ for all $n \in N$ and we write $f = O(g)$.
An algorithm A is polynomial if its time complexity $T_A(n)$ is of order $O(n^k)$ for some $k \in N$. Polynomial algorithms are also said to be efficient.

A decision problem is a problem in which the set of instances divides into two sets depending on whether the answer is YES or NO. In computational theory, the following three classes of decision problem are of interest:

- P is the set of decision problems that can be solved by a polynomial algorithm. Intuitively: P is the set of problems that can be solved efficiently.

- *NP* is the set of decision problems with the following property: If the answer is YES then there exists a certificate that enables us to verify this fact in polynomial time. Intuitively: NP is the set of problems for which we can quickly verify a positive answer if we are given a solution.
- *co-NP* is the set of decision problems with the following property: If the answer is NO then there exists a certificate that enables us to verify this fact in polynomial time.

Recently, Sangha and Zito studied d-realizable graphs in the more general context of the so-called Line-of-Sight (LoS) networks [49]. Line of Sight (LoS) networks provide a model of wireless communication which incorporates visibility constraints. Vertices of such networks can be embedded in finite d-dimensional grids of size n, and two vertices are adjacent if they share a line of sight and are at distance less than ω where two points $x = (x_1, ..., x_d)$ and $x' = (x'_1, ..., x'_d)$ share a line of sight if there exist $j \in \{1, ..., d\}$ such that $x_i = x'_i$ for all $i \in \{1, ..., d\} \backslash \{j\}$. They studied large independent sets in LoS networks. They proved that the computational problem of finding a largest independent set can be solved optimally in polynomial time for one dimensional LoS networks. However, for $d \geq 2$, the (decision version of) the problem becomes NP-hard for any fixed $\omega \geq 3$ and even if is chosen to be a function of n that is $O(n^{1-\epsilon})$ for any fixed $\epsilon > 0$. In addition they showed that the problem is also NP-hard when $\omega = n$ for $d \geq 3$. This result extends earlier work which showed that the problem is solvable in polynomial time for gridline graphs when $d = 2$.
For the small-dimensional cases, $d \in \{2, 3\}$, Peterson suggested an application of d-realizable graphs to robotics [46]: if the movement of a robot is restricted to be along axis-parallel directions only and turns are allowable only at certain points, then a shortest path in a d-realized graph gives the number of turns required.

Despite many studies on d-realizable graphs in the literature, determining the computational complexity of recognizing d-realizable graphs has been elusive except for $d \in \{1, 2\}$, when d-realizable graphs coincide with complete graphs and with line graphs of bipartite graphs, respectively (and can be recognized in polynomial time).The problem that we study in this final project paper is to analyze the question about recognition complexity of d-realizable graphs for $d \geq 3$, asked explicitly by Peterson in 2003 [46]. We show that for all $d \geq 3$, determining if a given graph is d-realizable is *NP*-complete, even for bipartite graphs.We also identify some tractable cases. We present the work of Martin Milanič, Peter Muršič, and Marcelo Mydlarz d-realizable graphs (for any positive integer d) in the class of HHD-free graphs, a large class of perfect graphs containing chordal graphs and distance-hereditary graphs. The characterization leads to a linear time recognition algorithm.

My approach is based on the fact that a graph G is d -realizable if and only if G is an induced subgraph of a Cartesian product of d complete graphs. Given two graphs G and H, their Cartesian product is the graph $G\Box H$ with vertex set $V(G) \times V(H)$ in which two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1v_1 \in E(G)$ and $u_2 = v_2$, or $u_1 = v_1$ and $u_2v_2 \in E(H)$. The Cartesian product is associative and commutative (in the sense that $G\Box H \equiv H\Box G$ where $\equiv$ denotes the graph isomorphism relation). Another name for Cartesian products of complete graphs is Hamming graphs; a Hamming graph is d-dimensional if it is the Cartesian product of d nontrivial complete graphs. The 3-dimensional Hamming graphs having all factors of the same size were studied in the literature under the name cubic lattice graphs [37, 12, 11, 1, 16], hence, 3-realizable graphs are exactly the induced subgraphs of cubic lattice graphs. Results are based on a characterization of induced subgraphs of d-dimensional Hamming graphs due to Klavžar and Peterin [33], expressed in terms of the existence of a particular edge labeling. For the 3-dimensional case, we develop a more specific characterization based on induced cycles of the graph and use it to prove hardness of recognizing 3-realizable graphs via a reduction from the 3-edge-coloring problem in cubic graphs. The hardness of the 3-dimensional case forms the basis for establishing hardness for all higher dimensions.

Since a d-realizable graph is also $(d + 1)$-realizable, the notion of d-realizability suggests a natural graph parameter. The Cartesian dimension of a graph $G = (V, E)$, denoted $Cdim(G)$, is defined as the minimum non-negative integer d such that G is d-realizable, if such an integer exists, and ∞, otherwise. The infinite case can indeed occur, even some small graphs the diamond, the 5-cycle, and the complete bipartite graph $K_{2,3}$, for example cannot be realized in any dimension. Note that $Cdim(G)$, when finite and strictly positive, is the minimum positive integer d such that G is an induced subgraph of the Cartesian product of d complete graphs. This point of view adds the Cartesian dimension of a graph to the list of graph dimensions studied in the literature related to various embeddings of graphs into Cartesian product graphs [24, 20, 27, 34]. Other dimensions were studied related to the strong product [22, 15, 32, 47] and the direct product of graphs [41, 48].

2 Characterizing Subgraphs of Hamming Graphs

In this chapter we will talk about necessary and sufficent conditions for a graph to be induced subgraph of Hamming Graphs and some results regarding those conditions

2.1 KP-labeling

We take a closer look at the characterization of induced subgraphs of Hamming Graph due to Klavžar and Peterin.
Before going to the result we need to go through couple of definitions and some results.

Corollary 2.1. *An edge $e = \{u, v\}$ of a graph G is a cut-edge iff it doesnt belong to any cycle.*

Proof. Take any edge $e = \{u, v\}$. Remove this edge from our graph: if the graph is still connected, then there is some path from u to v not involving e; consequently, if we add e to the end of this path, we get a cycle. Thus, if e is not a cut-edge, its involved in a cycle.
Conversely: suppose that $e = \{u, v\}$ lies in a cycle. Let P be the path from u to v that doesnt use e (i.e. go the other way around the cycle.) Pick any x, y in G; because G is connected, there is a path from x to y in G. Take this path, and edit it as follows: whenever the edge e shows up, replace this with the path P (or P traced backwards, as needed.) This then creates a walk from x to y; by deleting cycles, this walk will always become a path, and thus G is connected. So if e is involved in a cycle, its not a cut-edge. $\square$

The Cartesian product $G \square H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which the vertex (a, x) is adjacent to the vertex (b, y) whenever $ab \in E(G)$ and $x = y$, or $a = b$ and $xy \in E(H)$. For a fixed vertex a of G, the vertices $\{(a, x) | x \in V(H)\}$ induce a subgraph of $G \square H$ isomorphic to H, called an H-layer of $G \square H$. Analogously we define G-layers. A subgraph of $G \square H$ is called non-trivial if it intersects at least two G-layers and at least two H-layers.

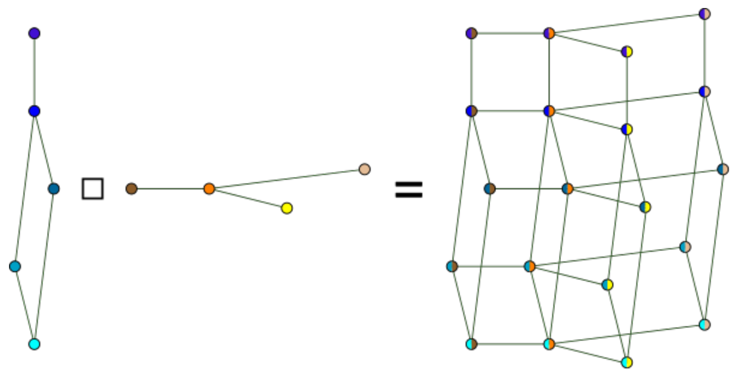

Figure 1: The Cartesian product of graphs.

The map $p_G : G\Box H \mapsto G$ defined by $p_G(a, x) = a$, is called a projection. The image of an edge $(a, x)(b, y)$ under a projection is an edge when $x = y$ or a vertex in when $a = b$.

Cartesian products of complete graphs are known as Hamming graphs. But they can be described also as follows. For $i = 1, 2, \dots n$ let $r_i \geq 2$ be given integers. Let G be the graph whose vertices are the n-tuples $b_1 b_2 \cdots b_n$ with $b_i \in \{0, 1, \dots r_i - 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one placeone coordinate. The vertex set of G is the same as $K_{r_1}\Box K_{r_2}\Box \dots \Box K_{r_n}$ and edges in $K_{r_1}\Box K_{r_2}\Box \dots \Box K_{r_n}$ between two vertices have same vertex in every graph K_{r_i} but one and since in complete graphs we have every possible edge, then it is easy to see that they are isomorphic. For an edge uv of $H = K_{r_1}\Box K_{r_2}\Box \dots \Box K_{r_n}$ we define the color map $c : E(H) \mapsto \{1, 2, \dots n\}$ with $c(uv) = i$, where u and v differ in coordinate i.
Let G be a connected graph and let $\mathcal{F} = \{F_1, F_2, \dots F_k\}$ be a partition of $E(G)$. The *quotient graph* $G\backslash F_i$ has connected components of G
F_i as vertices, two components C and C' being adjacent whenever there exists an edge of F_i connecting a vertex of C with a vertex of C' . For each i, define a map $f_i : V(G) \mapsto V(G\backslash F_i)$ by $f_i(v) = C$, where C is the component of G
F_i containing v.Then let

$$f : G \mapsto G\backslash F_1 \Box G\backslash F_2 \Box \dots \Box G\backslash F_k$$

be the natural coordinate-wise mapping, that is

$$f(v) = (f_1(v), f_2(v), \dots, f_k(v))$$

We call f the *quotient map* of G with respect to F. Note that f need not be one-to-one in general and that it is possible that some quotient graphs are the one vertex graph. However, all the partitions F introduced later will lead to one-to-one mappings with non-trivial quotient graphs.
A partition $\{F_1, F_2, \dots, F_k\}$ of $E(G)$ naturally leads to an edge-labeling $l : E(G) \mapsto$

$\{1, 2, \ldots, k\}$ by setting $l(e) = i$, where $e \in F_i$. Unless stated otherwise, a labeling (or more precisely a k-labeling) of G will mean an edge-labeling (with k labels).
Quotient graph is the central concept in this chapter, so to get a better understanding we also visualize it. We define the partition of $E(C_7)$ by defining F_i having edges with

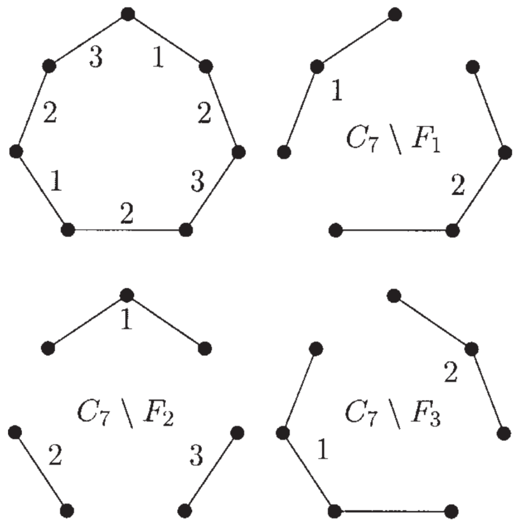

Figure 2: Quotient graph of C_7

label i, as we can see $C_7 \backslash F_1$ and C_7/F_3 have only two vertices and since there is an edge in the correspoding F_i connecting those two components therefore both of $C_7 \backslash F_1$ and C_7/F_3 are simply K_2, while $C_7 \backslash F_2$ has three vertices but each two of compontents are connected with an edge from F_2 so $C_7 \backslash F_2$ is simply K_3.
They defined two conditions to get a better understanding. They actuall named them by letters B and C but since we will not need the condition A will name just by numbers 1 and 2.
We say that a d-edge-labeling of G is a (d-)KP-labeling if it satisfies the following conditions:
Condition 1 Let G be a graph. Then edges of a triangle have the same label.
Condition 2 Let G be a labeled graph and let u and v be arbitrary vertices of G with $d_G(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced u, v-path.

Example 2.2. Let's see two examples of edge labeling of the same graph G as in Figure 2.1.
The label in left satisfy condition 1 as the edges of triangle have the same label, also it satisfies condition 2 as it can be check that for u and v arbitrary vertices of G with

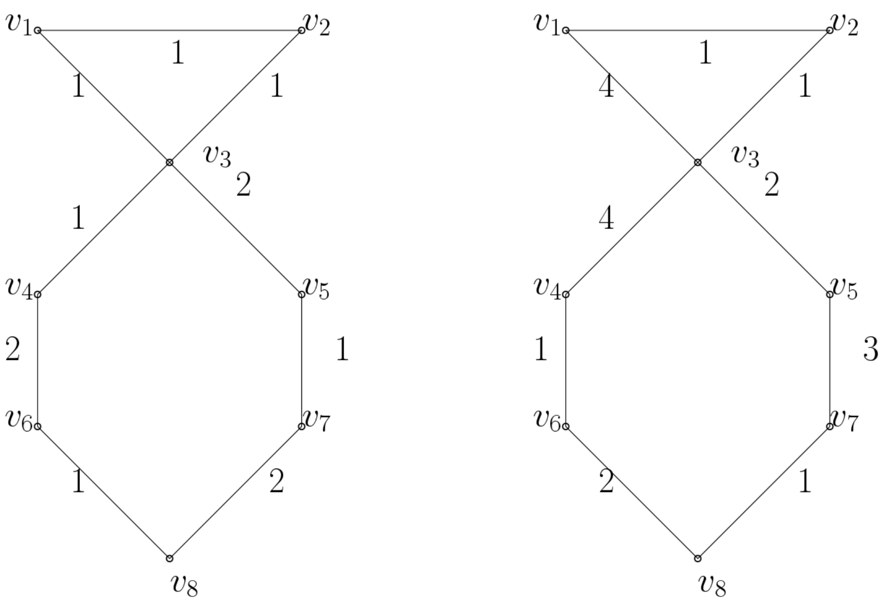

Figure 3: The label in left is a KP-labeling, while the one in right is not.

$d_G(u, v) \geq 2$ then there exist different labels i and j which both appear on any induced u, v-path. Therefore the label in left is a KP-labeling.
The label in right doesn't satisfies neither condition 1 nor 2, as the edges of triangle don't have the same label, and also if we take vertices v_5 and v_8 then paths $P_1 = v_5v_3v_4v_6v_8$ and $P_2 = v_5v_7v_8$ have only one label in their intersecion.

2.2 Induced subgraphs of Hamming Graphs

Next we try to understand those conditions on cycle and we infer the following result

Lemma 2.3. *Let G be a labeled graph fulfilling Condition 2 and let C_k , $k \geq 4$, be an induced cycle of G. Then every label of C_k is presented more than once on C_k.*

Proof. The labels of a u, v-path of length 2 on C_k are different, hence by Condition 2 those two labels should appear also along the other part. So for any label take two vertices that have distance two containg the edge with that label and therefore it is presented more than once on C_k $\square$

Before going to the main result we will need also the following lemma that tells us that we only need to check the induced paths.

Lemma 2.4. *Let G be a labeled graph fulfilling Conditions 1 and 2 and let u, v be vertices of G with $d_G(u, v) \geq 2$. Then, if labels i and j appear on every induced u, v-path, they appear on every u, v-path.*

Proof. Suppose that labels i and j appear on every induced u, v-path. Let $P = x_1x_2...x_r, x_1 = u, x_r = v$, be a u, v-path of minimal length that does not contain both labels i and j. Then P is not induced since every induced path contains both labels, hence we have an edge between some non-consecutive vertices say $e = x_kx_l$ with $l - k > 1$. We may assume that e is selected such that $l - k$ is as small as possible. By the minimality of P, the path $x_1x_2...x_kx_lx_{l+1}...x_r$ which is shorter than P contains both labels i and j. Hence, the label of the edge x_kx_l is either i or j, otherwise P would contain both labels i and j. Assume without loss of generality it is i. Then, using minimality again, label j appears on the path $x_1x_2...x_k$ or on $x_lx_{l+1}...x_r$, note that it cannot happen that $k = 1$ and $l = r$ as $d_G(u, v) \geq 2$. It follows that i does not appear on the path $x_kx_{k+1}...x_l$. But then the label i appears only once on the cycle $C = x_kx_{k+1}...x_lx_k$. If C is a triangle, we have a contradiction with Condition 1, otherwise with previous lemma. □

Proposition 2.5. *Let G be a graph having a KP-labeling and let e be an edge in G. Then G has a KP-labeling in which e receives a unique label if and only if e is a cut-edge.*

Proof. First we will show that if e is a cut-edge then G has a KP-labeling in which e receives a unique label. Fix an arbitrary KP-labeling of G.
Relabel edge e with a new label say i not used before. We will show that the new labeling is also a KP-labeling. We need to verify the following two conditions:
Condition 1: Every triangle is monochromatic.
Condition 1 holds, since e is not part of any cycle and therefore every triangle is still monochromatic.
Condition 2: for every pair of distinct non-adjacent vertices u, v, there exist different labels j and k which both appear on every iduced u, v-path.
To verify condition 2, note that any two u, v-paths either both contain edge e or none (otherwise e would be part of a cycle, but e is a cut-edge)
If none of the induced u, v-paths contains edge e then the new KP-labeling still satisfies Condition 2 for the pair u, v because of the original KP-labeling satisfied for this pair. Otherwise all induced u, v-paths contain e, let the new label of e be i, and original label of e be i_0. We differ two cases:

Case 1 If $i_0 \in \{j, k\}$ then simply take the set $(\{j, k\} \setminus \{i_0\}) \cup \{i\}$.

Case 2 If $i_0 \notin \{j, k\}$ then simply take the set $\{j, k\}$.

Suppose G has a KP-labeling in induced path e receives a unique label (by Corollary). Suppose for contradiction that e is not a cut edge such that e is part of a cycle C. If C is a triangle then it is not monochromatic as then it must satisfy Condition 1 and every edge must label same as e but then e would not be labeled uniquely . Therefore

C cannot be triangle. Let $V(C) = \{v_1, v_2, ..., v_p\}$ in cyclic order and without loss of generality $e = v_1v_2$. Let $u = v_1$ and $v = v_3$. Then since e is labeled uniquely, the two induced paths forming a cycle from u to v. These two paths cannot share any two labels so Condition 2 cannot hold, a contradiction. □

Theorem 2.6. *Let G be a connected graph. Then G is an induced subgraph of a Hamming graph if and only if there exists a labeling of G that fulfills Conditions* 1 *and* 2.

Proof. Let G be an induced subgraph of $H = K_{n_1}\Box K_{n_2}\Box\cdots\Box K_{n_k}$. To make things easier denote $p_i = p_{K_{n_i}}$ which is the projection of complete graph K_{n_i} of H and also consider the labeling $E(G)$ indeuced by the color map c of H. We will show that this labeling satisfies Conditions 1 and 2.
Condition 1 is clear. Indeed, if u, v, and w induce a triangle, then they all lie in the same layer of H and by alternative description of Hamming graphs we have that each pair is adjacent so they differ in exactly on coordinate and therefore color map c will map those edges to the coordinate in which they differ and so the edges uv, uw, and vw receive the same label.
We next show Condition 2 is satisfied also. Let u and v be two vertices of G with $d_G(u, v) \geq 2$. Suppose that there is no label that appears on all induced u, v-paths. Now if we see u and$v$ as $k-tuples$ and we understand paths as changing coordinates and therefore $p_i(u) = p_j(v)$ for all i because if we look at one induced path that doesn't have label i that means than we never change the i^{th} coordinate as we traverse this path and therefore the i^{th} coordinate is the same, and this is contrary to $d_G(u, v) \geq 2$. Suppose now that all induced u, v-paths have exactly one label in common, say i. We have $p_j(u) = p_j(v)$ for all $j \neq i$ and $p_i(u) \neq p_i(v)$ (the case when $p_i(u) = p_i(v)$). Vertices $p_i(u)$ and $p_j(v)$ are adjacent in K_{n_i} , since K_{n_i} is a complete graph. Hence, u and v are adjacent in H and therefore also in G which is impossible.
Conversely, let l be a labeling of G that fulfills Conditions 1 and 2. Let $\mathcal{F} = \{F_1, F_2, \ldots F_k\}$ be the partition of $E(G)$ induced by l and let f be the quotient map of G with respect to $\mathcal{F}$. We claim that f embeds G as an induced subgraph into $G = G\backslash F_1\Box G\backslash F_2\Box\ldots\Box G\backslash F_k$. We show first that f is one-to-one, in this way we show that indeed f embeds G as an induced subgraph into $G = G\backslash F_1\Box G\backslash F_2\Box\ldots\Box G\backslash F_k$. Suppose that vertices x and y are not adjacent in G. Then by Condition C and Lemma 2.4, there exist labels i and j such that on every x, y-path we find labels i and j. So x and y are in different components in both $G\backslash F_i$ and $G\backslash F_j$. Already the first fact assures that $f(x) \neq f(y)$. Let next x and y be adjacent vertices of G and let $l(xy) = i$, and we can see that $f_j(x) = f_j(y)$ for every $j \neq i$, now let us take a look for f_i. Suppose that there exists an x, y-path $P = x_1x_2\ldots x_r$ in $G\backslash F_i$, where $x_1 = x$ and $x_r = y$. We can assume that P is shortest

among all x, y-paths in $G\backslash F_i$. If P is induced in $G - xy$ we have a contradiction with Condition 1 when $r = 3$ since P does not take the label i (P in $G\backslash F_i$) but a triangle will be formed with xy and two edges in P and a contradiction with Lemma 2.3 when $r > 3$ because $P + xy$ would form a cycle and label i is present only once. Thus P is not induced in $G - xy$, $r > 3$, and there are adjacent vertices x_j and x_k with $k > j+1$. By the minimality of P we have $l(x_j x_k) = i$. We can select j and k such that $k - j$ is minimal among all such vertices x_j and x_k. Then the cycle $C = x_j x_{j+1} \ldots x_{k-1} x_k$ is induced. If C is a triangle we have a contradiction with Condition 1, otherwise we have a contradiction with Lemma 2.3. Hence, we have shown that f is one-to-one.
Let xy be an edge with $l(xy) = i$. Then, by the above, x and y are in different components of $G\backslash F_i$. Moreover, they belong to the same component in any of the graphs $G\backslash F_j$, for $j \neq i$. It follows that f maps edges to edges and the claim is proved.
Hence, $G = f(G)$ is an induced subgraph of $G\backslash F_1 \square G\backslash F_2 \square \ldots \square G\backslash F_k$. To complete the proof we show that G is also an induced subgraph of the Hamming graph $k_{|G\backslash F_1|} \square K_{|G\backslash F_2|} \square \ldots \square K_{|G\backslash F_k|}$: Let x and y be non-adjacent vertices of G. Then, by the same reasoning as above, x and y are in different components of at least two graphs $G\backslash F_i$. It follows that $f(x)$ and $f(y)$ differ in at least two coordinates which remains valid after adding edges to the factor graphs. $\square$

Example 2.7. Let's see that C_7 is an induced subgraph of a Hamming graph and there exists a labeling of C_7 that fulfills Conditions 1 and 2.

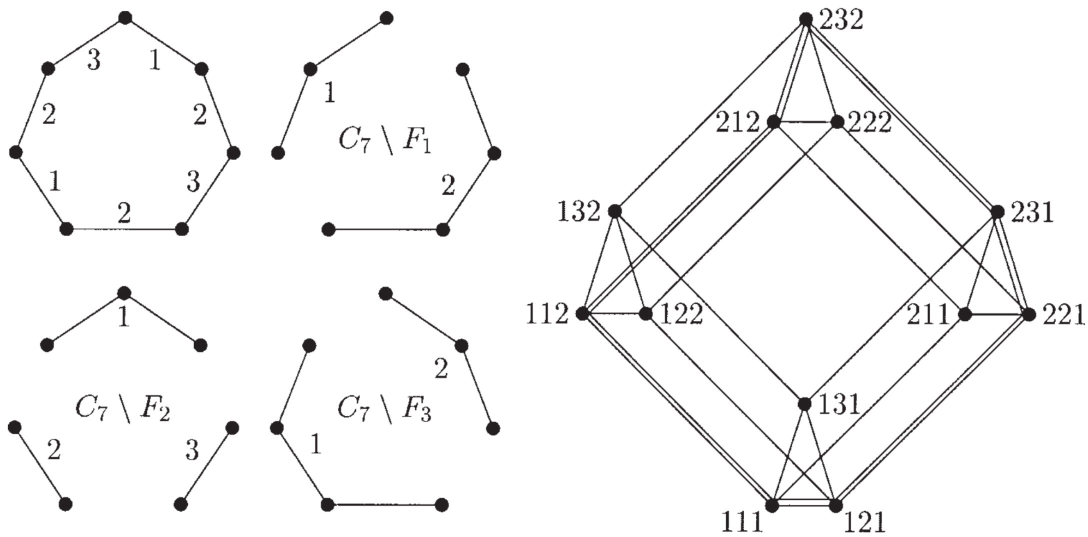

Figure 4: C_7 as an induced subgraph of $K_2 \square K_3 \square K_2$

C_7 is an induced subgraph of Hamming graph more specifically of $K_2 \square K_3 \square K_2$ in the Figure 4 take cycle $(112, 212, 232, 231, 221, 121, 111)$ and using the color map c, edge

$111, 112$ gets label 3, edge $112, 212$ gets label 1, edge $212, 232$ gets label 2, edge $232, 231$ gets label 3, edge $231, 221$ gets label 2, edge $221, 121$ gets label 1, edge $121, 111$ gets label 2. We can easily check that edges of a triangle have the same label (here there is no triangle which is fine) and let u and v be arbitrary vertices of C_7 with $d_{C_7}(u, v) \geq 2$. Then there exist different labels i and j which both appear on any induced u, v-path. C_7 has a labeling in which Conditions 1 and 2 hold as the first graph in Figure 4, now let's take a look at where does the map f sends vertices of C_7. We follow the order clockwise from most top vertice, f maps the first vertice to 112, f maps the second vertice to 212, f maps the third vertice to 232, f maps the fourth vertice to 231, f maps the fifth vertice to 221,, f maps the sixth vertice to 121,, f maps the seventh vertice to 111 and as we can see $G = f(G)$ and induced subgraph of $K_2 \Box K_3 \Box K_2$

Note that the quotient graphs obtained in the proof of Theorem 2.6 need not be complete. For instance, consider the path P_4 together with the labeling 1, 2, 1 as we get $G \backslash F_1 = P_3$ and $G \backslash F_2 = K_2$.

3 Cartesian dimension

Given a positive integer d, a d-realization of a graph $G = (V, E)$ is an injective mapping $\varphi_G : V \mapsto \mathbb{R}^d$ such that two vertices $u, v \in V$ are adjacent if and only if $\varphi G(u)$ and $\varphi G(v)$ differ in exactly one coordinate. A graph G is said to be d-realizable if it has a d-realization. Note that G is d-realizable if and only if G has a d-realization $\varphi_G : V \mapsto \mathbb{N}^d$ or any large enough set of dimension d since $V(G)$ is countable.
Any graph that has a d-realization, it has also a $d + 1$-realization (simply add one coordinate equal to 0). Which naturally leads to a definition: the Cartesian dimension of a graph $G = (V, E)$, denoted $Cdim(G)$, is defined as the minimum non-negative integer d such that G is d-realizable, if such an integer exists, and ∞, otherwise. (Note that K_1 is the only graph of Cartesian dimension 0.)

For a graph G to have Cartesian dimension 1 it must have a 1-realization, or in other words you can map all the vertices of the graph to $\mathbb{R}$ and becuase the map has to be injective every vertice is mapped to different real number but every two vertice differ in exactly one coordinate which makes them adjacent, therefore the only graphs of Cartesian dimension 1 are complete graphs of order at least 2.

Example 3.1. C_6 has Cartesian dimension 2, indeed since it is not a complete graph it cannot have dimension 1, and we find a $2-realization$ of C_6 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}$ by map defining $\varphi_{C_7} : V \mapsto \mathbb{R}^2$ with $\varphi(v_1) = (1, 1)$, $\varphi(v_1) = (1, 1)$, $\varphi(v_2) = (3, 1)$, $\varphi(v_3) = (3, 3)$, $\varphi(v_4) = (2, 3)$, $\varphi(v_5) = (2, 2)$ and $\varphi(v_1) = (1, 2)$. An easy check shows that indeed for any edge the corresponding values of φ differ in exactly one coordinate.

Lemma 3.2. *If graph G is diamond-free then any two cliques of G intersect in at most one vertex.*

Proof. Suppose cliques A_1 and A_2 intersect in at least two vertices say v_1, v_2. Becuase A_1 is maximal clique there is some vertice v_3 other than v_1, v_2 in A_1 but not in A_2 and since A_1 is clique we have that v_1, v_2, v_3 form a triangle, similarly there exists v_4 in A_2 but not in A_1 such that v_1, v_2, v_4 form triangle. But all together v_1, v_2, v_3, v_4 form a diamond, a contradiction with G being diamond-free. □

Theorem 3.3. *For every graph G, the following conditions are equivalent:*

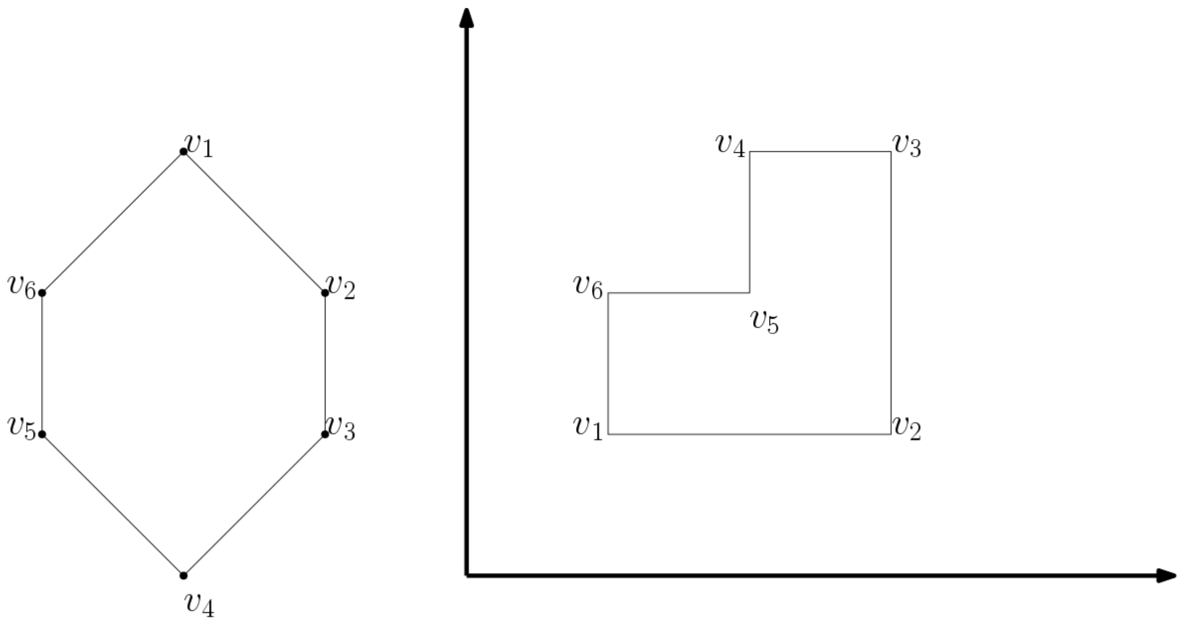

Figure 5: A 2-realization of C_6

1. *$Cdim(G) \leq 2$.*

2. *G is the line graph of a bipartite graph.*

3. *G is diamond-free and $K(G)$ is bipartite.*

4. *G is {claw, diamond, C_5 , C_7 , ...}-free.*

Proof. (1) $\implies$ (4) We show that none of those graphs in (4) can be an induced subgraph of a graph with $Cdim(G) \leq 2$. $Cdim(G)$ cannot be equal to 1 since only complete graph with Cartesian dimension 1. Now suppose it $Cdim(G) = 2$.
First let's show for diamond. Suppose G has diamond as induced subgraph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4\}$, now because v_1, v_2 and v_3 is complete graph on 3 vertices all three together differ in one coordinate, similarly for v_2, v_3 and v_4 so it follows that also v_1 and v_4 differ only in one coordinate but they are not adjacent.
Next suppose G has claw as induced subgraph with vertex set $V(G) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(G) = \{v_1v_2, v_1v_3, v_1v_4\}$ and w.l.o.g. assume that assume that $\varphi_G(v_1)$ differs only in first coordinate with $\varphi_G(v_2)$ and because v_2 and v_3 are not adjacent then $\varphi_G(v_1)$ and $\varphi_G(v_3)$ must differ only in second coordinate. Which bring us to v_4, because v_1 and v_4 are adjacent they must differ only one coordinate, but either coordinate is not good as would imply adjacency between v_4 and v_2 or v_4 and v_3 but there is no edge among them.
And finally for odd-cycle larger than 3. Let the vertex set be $V(G) = \{v_1, v_2, ..., v_n\}$

with edge set $E(G) = \{v_1v_2, v_2v_3, \ldots v_nv_1\}$ and notice that since there is no triangle no three vertices lie in same line so if we color map define in the section 2.1 then by by following the cycle we alternate by colors 1 and 2, but to close the cycle it must be followed by two different colors but cycle is odd so it is not possible.
(4) $\implies$ (3) We only need to show that $K(G)$ is not bipartite which is equivalent that $K(G)$ has an odd cycle. Suppose it has length greater than 3, first note that no three cliques(vertices in $K(G)$) share one vertice as then a claw would be form in G but G is claw-free, so we form an odd cycle by following the common vertice of cliques of size greater than 3 but G is $\{C_5, C_7, \ldots\}$-free, therefore $K(G)$ has a cycle of length 3. Now if all three cliques have a vertice in common then we can form a claw because those cliques are maximal so there exist some vertex that other cliques don't have. Let those cliques that form a triangle be A_1, A_2, A_3 and the common vertex of A_1, A_2 be v_1, A_2, A_3 be v_2 and A_3, A_1 be v_3, because v_1 and v_2 are in A_2 they are adjacent, similiar for others. Notice that one of the cliques must have another vertex v_4 other than v_1, v_2, v_3 otherwise $K(G)$ would be just a vertex. Now w.l.o.g. say $v_4 \in A_1$ then we have another triangle with $v_1, v_3 \in A_1$, which all together v_1, v_2, v_3, v_4 forms a diamond but G is diamond-free which leads to a contradiction, therefore $K(G)$ is bipartite.
(3) $\implies$ (1) If G is complete then we know that $Cdim(G) = 1$, suppose G is not complete. Since $K(G)$ is bipartite we color vertices with two colors in natural way. Map vertices of G in $\mathbb{R}^2$ by placing cliques of G having one color on vertical lines and cliques having the other color on horizontal lines. By Lemma 3.2, no two vertices are mapped to same point.
(1) $\iff$ (2) Given a bipartite graph H such that $L(HG$, construct the adjacency matrix A. Mark points in $\mathbb{R}^2$ at (i, j) iff $a_{ij} = 1$ in uppertriangle matrix of A. It is immediate that one can map vertex $i, j \in G$ to (i, j) where $i < j$ and notice that this defines a one to one mapping and if two vertices are adjacent in G then they will defer in exactly on coordinate. This shows that $Cdim(G) \leq 2$ as we found a 2-realization. Since G can be realized with vertices only at positive integral points (the map φ_G can be defined to $\mathbb{N}^d$), the proof works in reverse. □

Lemma 3.4. *Let G be a d-realizable graph. Let v be a vertex adjacent to vertices x and y. Then x and y are adjacent if and only if they differ from v in the same coordinate.*

Proof. ($\implies$) If x and y are adjacent then they differ in exactly one coordinate say in kth coordinate. Also say v differs to x exactly in ith coordinate and to y in jth coordinate. Now suppose x and y don't differ from v in the same coordinate which means that $i \neq j$, also note that k cannot be equal to both i and j as this would imply $i = j$ so w.l.o.g. $k \neq j$. Now because $i \neq j$, the jth coordinate of x is the same with jth coordinate of v and therefore differs from jth coordinate of y, but also $k \neq j$ which means that x and y differ in more than one coordinate which contradicts the fact that x and y are adjacent. □

Theorem 3.5 (Staton and Wingard 1998). *Every d-realizable graph is* $\{K_{1,d+1}$ *, diamond,* $K_{2,3}$, C_5 $\}$*-free.*

Next we see the relation between KP-labeling and Cartesian dimension:

Theorem 3.6 (Klažvar and Peterin 2005). *For every connected graph G and a positive integer d, we have* $Cdim(G) \leq d$ *if and only if G has a d-KP-labeling.*

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Appendices

A Title of First Appendix

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B Title of Second Appendix

Here we add the second appendix.