6.837 - Homework 1

Akhil Raju and Matthew Arbesfeld

September 28, 2014

1 Introduction

The inclusion-exclusion principle is one of the most important results of probability theory. If $\mathbf{A}$ is the union of events $A_1, A_2, \ldots, A_n$, then the inclusion-exclusion principle gives us $\mathbb{P}(\mathbf{A})$ from $\mathbb{P}(A_i), \mathbb{P}(A_i \wedge A_j), \mathbb{P}(A_i \wedge A_j \wedge A_k)$ and all other intersections of the subsets of A.

In probability theory, we often have information about the probability of individual events as well as the probabilities of their intersections. For example, suppose that we wish to find the probability of flipping two fair coins and getting *at least one* heads. If H_1 is the event that the first coin is heads and H_2 is the event that the second coin is heads, then the inclusion-principle tells us that

$$\mathbb{P}(H_1 \vee H_2) = \mathbb{P}(H_1) + \mathbb{P}(H_2) - \mathbb{P}(H_1 \wedge H_2) = \frac{1}{2} + \frac{1}{2} - \frac{1}{4} = \frac{3}{4}.$$

Thus, we can determine the union of the set of events from their individual probabilities and intersection probabilities.

To use the inclusion-exclusion principle on a set of events $A_1, A_2, \ldots, A_n$, we must know the probabilities of the intersections for all subsets of the events:

$$\begin{gathered}
\forall i \in \{1, 2, \ldots, n\} : \mathbb{P}(A_i) \\
\forall (i, j) \in \{1, 2, \ldots, n\}, i \neq j : \mathbb{P}(A_i \wedge A_j) \\
\forall (i, j, k) \in \{1, 2, \ldots, n\}, i \neq j \neq k : \mathbb{P}(A_i \wedge A_j \wedge A_k) \\
\vdots \\
\forall (x_1, x_2, \ldots, x_n) \in \{1, 2, \ldots, n\}, x_1 \neq x_2 \neq \ldots \neq x_n : \mathbb{P}(A_{x_1} \wedge A_{x_2} \wedge \ldots \wedge A_{x_n}).
\end{gathered}$$

In practice, though, we may not have information on all of these intersections. For example, we may only know the probability of individual events and the probability of pairwise intersections of events. In this paper, we will develop an extension of the inclusion-exclusion principle for cases when we are given only a subset of the intersection probabilities. Specifically, we will find upper and lower bounds on the probability of the union of n events when given the intersection probabilities for all subsets of up to k events with $k < n$. These bounds are known as the Bonferroni inequalities. A more comprehensive analysis on inclusion-exclusion with limited information can be found in Linial and Nisan's work [1].

In Section **??**, we formally introduce the inclusion-exclusion principle and define notation that we will use for representing the probability for intersections of events. In Section **??**, we develop an extension of the inclusion-exclusion principle which gives us bounds on the union of a set of events when we have knowledge on only a subset of the intersection probabilities. Finally, in Section **??**, we use our extension of the inclusion-exclusion principle to find the probabilistic bounds of a flip sequence in a coin-flipping game.

References

[1] N. Linial and N. Nisan, "Approximate Inclusion-Exclusion," Combinatorica 10, (1990) 349-365.