Support Vector Machine (SVM)

What is SVM?

A Beginner's Introduction

Imagine you have:

- Red balls and blue balls scattered on a table
- You want to draw a line separating them

SVM does exactly this!

- It finds the **best line** (or boundary) to separate different groups
- The "best" line is the one with **maximum distance** from both groups
- This distance is called the **margin**

Why Use SVM?

Key Advantages

Works great with clear separation

- Finds the optimal boundary between classes

Handles high-dimensional data

- Effective even when you have many features

Memory efficient

- Only uses a subset of training points (support vectors)

Linear SVM

Concept

- SVM finds an **optimal separating hyperplane**
  - In 2D: a line | In 3D: a plane | In higher dimensions: a hyperplane
- The hyperplane is defined by **support vectors**
  - These are the closest points from each class to the boundary
- Only a **few boundary points** decide the classifier
  - Most data points don't affect the decision boundary

Understanding Support Vectors

What Are They?

Support Vectors are:

- The data points **closest** to the decision boundary
- The **critical points** that define where the boundary goes
- Like anchors holding a rope (the boundary) in place

Why are they important?

- Moving a support vector changes the boundary
- Moving other points doesn't affect the boundary

Linear SVM

Example

Positive labelled points

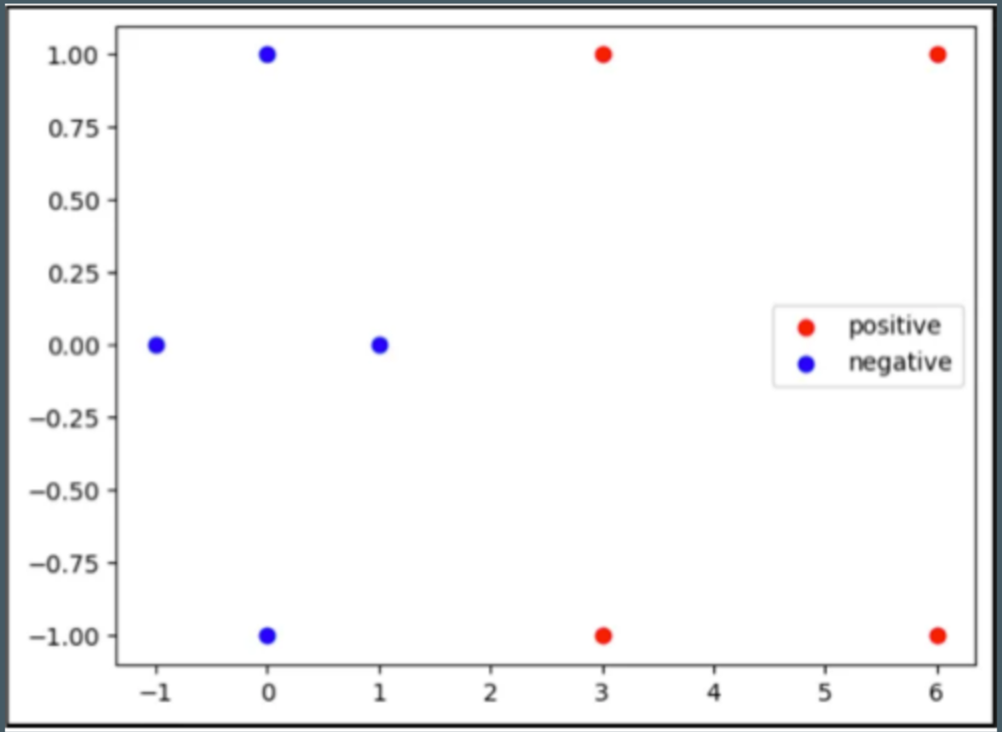
$$(3, 1),\ (3, -1),\ (6, 1),\ (6, -1)$$

Negative labelled points

$$(1, 0),\ (0, 1),\ (0, -1),\ (-1, 0)$$

Visual: Linear SVM Dataset

Plotting the Points



Observation: Positive points are on the right, negative points on the left

Linear SVM

Support Vector Selection

From the graph, we identify:

- **Negative support vector** (closest negative point)

$$s_1 = (1, 0, 1)$$

- **Positive support vectors** (closest positive points)

$$s_2 = (3, 1, 1), \quad s_3 = (3, -1, 1)$$

Note: Last component `1` is added for bias term (intercept)

Visual: Support Vectors & Boundary

Linear SVM Decision Line

```
    y
    |
 1  |    ○     [+]     +      [+] = Support Vector
    |          /               ○ = Support Vector
 0  | ○  [○] /
    |   ○  /
-1  |      /   [+]     +
    |    /_________________ x
   -1   0   1   2   3   4   5   6
        Decision Boundary (y = x - 2)
```

Key: Only boxed points [•] are support vectors!

Linear SVM

Generalization Equations

For each support vector, we create an equation:

$$\alpha_1 s'_1 \cdot s'_1 + \alpha_2 s'_1 \cdot s'_2 + \alpha_3 s'_1 \cdot s'_3 = -1$$

$$\alpha_1 s'_2 \cdot s'_1 + \alpha_2 s'_2 \cdot s'_2 + \alpha_3 s'_2 \cdot s'_3 = 1$$

$$\alpha_1 s'_3 \cdot s'_1 + \alpha_2 s'_3 \cdot s'_2 + \alpha_3 s'_3 \cdot s'_3 = 1$$

The dot product (·) measures similarity between vectors

Step 1: Calculate Dot Products

Recall support vectors:

$s_1 = (1, 0, 1)$, $s_2 = (3, 1, 1)$, $s_3 = (3, -1, 1)$

Compute all dot products:

$$s'_1 \cdot s'_1 = 1^2 + 0^2 + 1^2 = 2$$

$$s'_1 \cdot s'_2 = 1(3) + 0(1) + 1(1) = 4$$

$$s'_1 \cdot s'_3 = 1(3) + 0(-1) + 1(1) = 4$$

Step 2: More Dot Products

$$s_2' \cdot s_1' = 3(1) + 1(0) + 1(1) = 4$$

$$s_2' \cdot s_2' = 3^2 + 1^2 + 1^2 = 11$$

$$s_2' \cdot s_3' = 3(3) + 1(-1) + 1(1) = 9$$

$$s_3' \cdot s_1' = 3(1) + (-1)(0) + 1(1) = 4$$

$$s_3' \cdot s_2' = 3(3) + (-1)(1) + 1(1) = 9$$

$$s_3' \cdot s_3' = 3^2 + (-1)^2 + 1^2 = 11$$

Step 3: Substitute into Equations

Equation 1:

$$2\alpha_1 + 4\alpha_2 + 4\alpha_3 = -1$$

Equation 2:

$$4\alpha_1 + 11\alpha_2 + 9\alpha_3 = 1$$

Equation 3:

$$4\alpha_1 + 9\alpha_2 + 11\alpha_3 = 1$$

Step 4: Solve the System

Simplify Equation 1: Divide by 2

$$\alpha_1 + 2\alpha_2 + 2\alpha_3 = -0.5$$

Subtract Equation 2 from Equation 3:

$$(4\alpha_1 + 9\alpha_2 + 11\alpha_3) - (4\alpha_1 + 11\alpha_2 + 9\alpha_3) = 0$$

$$-2\alpha_2 + 2\alpha_3 = 0 \quad \Rightarrow \quad \alpha_2 = \alpha_3$$

Step 5: Final Substitution

Since $\alpha_2 = \alpha_3$, substitute into simplified Eq 1:

$$\alpha_1 + 2\alpha_2 + 2\alpha_2 = -0.5$$

$$\alpha_1 + 4\alpha_2 = -0.5 \quad ...(\mathrm{A})$$

Substitute into Equation 2:

$$4\alpha_1 + 11\alpha_2 + 9\alpha_2 = 1$$

$$4\alpha_1 + 20\alpha_2 = 1 \quad ...(\mathrm{B})$$

Step 6: Solve for α Values

From (A): $\alpha_1 = -0.5 - 4\alpha_2$

Substitute into (B):

$$4(-0.5 - 4\alpha_2) + 20\alpha_2 = 1$$

$$-2 - 16\alpha_2 + 20\alpha_2 = 1$$

$$4\alpha_2 = 3 \quad \Rightarrow \quad \alpha_2 = 0.75$$

Therefore: $\alpha_3 = 0.75$ and $\alpha_1 = -0.5 - 4(0.75) = -3.5$

Linear SVM

Solving α Values

After solving the equations:

$$\alpha_1 = -3.5, \quad \alpha_2 = 0.75, \quad \alpha_3 = 0.75$$

Linear SVM

Hyperplane Calculation

We combine support vectors with their weights:

$$W' = \sum \alpha_i s_i$$

Substituting our values:

$$W' = -3.5(1,0,1) + 0.75(3,1,1) + 0.75(3,-1,1)$$

Result:

$$W' = (1,0,-2)$$

Linear SVM

Final Equation

The decision boundary equation:

$$y = Wx + b$$

Where:

- $W = (1, 0)$ → weight vector (slope)
- $b = 2$ → bias term (intercept)

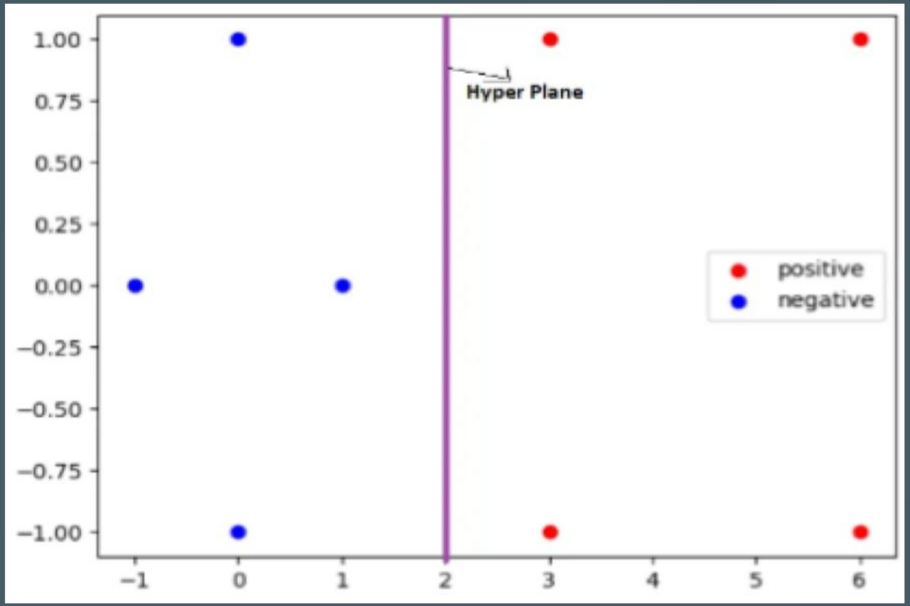
To classify the hyperplane:

$$b - a = 0 | b - 2 = 0$$

$$b = a | b = 2$$

- If $W = (1, 0)$ -> Vertical Line
- If $W = (0, 1)$ -> Horizontal Line

Visual: SVM Margin



Practice Task: Solve Your Own SVM

New Support Vectors

Given three support vectors:

$$S_1 = (2, 1, 1), \quad S_2 = (2, -1, 1), \quad S_3 = (4, 0, 1)$$

Your Task:

1. Calculate all dot products $S_i' \cdot S_j'$
2. Set up the system of equations using:
   - $S_1' \cdot S_1'\alpha_1 + S_1' \cdot S_2'\alpha_2 + S_1' \cdot S_3'\alpha_3 = -1$
   - $S_2' \cdot S_1'\alpha_1 + S_2' \cdot S_2'\alpha_2 + S_2' \cdot S_3'\alpha_3 = 1$
   - $S_3' \cdot S_1'\alpha_1 + S_3' \cdot S_2'\alpha_2 + S_3' \cdot S_3'\alpha_3 = 1$
3. Solve for $\alpha_1, \alpha_2, \alpha_3$

Solution: Practice Task Results

Dot Products Calculation

All dot products:

$$S_1' \cdot S_1' = 2^2 + 1^2 + 1^2 = 6$$

$$S_1' \cdot S_2' = 2(2) + 1(-1) + 1(1) = 4$$

$$S_1' \cdot S_3' = 2(4) + 1(0) + 1(1) = 9$$

$$S_2' \cdot S_1' = 2(2) + (-1)(1) + 1(1) = 4$$

$$S_2' \cdot S_2' = 2^2 + (-1)^2 + 1^2 = 6$$

$$S_2' \cdot S_3' = 2(4) + (-1)(0) + 1(1) = 9$$

$$S_3' \cdot S_1' = 4(2) + 0(1) + 1(1) = 9$$

$$S_3' \cdot S_2' = 4(2) + 0(-1) + 1(1) = 9$$

$$S_3' \cdot S_3' = 4^2 + 0^2 + 1^2 = 17$$

Solution: System of Equations

Substituted Equations

Equation 1:

$$6\alpha_1 + 4\alpha_2 + 9\alpha_3 = -1$$

Equation 2:

$$4\alpha_1 + 6\alpha_2 + 9\alpha_3 = 1$$

Equation 3:

$$9\alpha_1 + 9\alpha_2 + 17\alpha_3 = 1$$

Solution: Final Alpha Values

Results

After solving the system:

$$\alpha_1 = -2, \quad \alpha_2 = 2, \quad \alpha_3 = -0.5$$