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Fast Algorithm For the Markowitz Critical Line Method

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Abstract—The critical line method developed by the Nobel Prize winner H. Markowitz is a classical technique for the construction of a minimum-variance frontier within the paradigm of "the expected return—risk" (mean—variance) and finding minimum portfolios. Considerable interest has recently been attracted to the development of a fast algorithm for the construction of the minimum-variance frontier. In some works, such algorithms have been used to find statistically stable optimal portfolios. An algorithm based on the critical line method has recently been proposed by Andras Niedermayer and Daniel Niedermayer. Its testing showed that it is faster than all similar algorithms known before by several orders of magnitude. In this paper, we present an algorithm for constructing the minimum-variance frontier for the Markowitz problem with the condition of nonnegativity of the optimal portfolio, which requires about half as many operations as the Niedermayers' algorithm. To this end, we had to perform a more thorough analytical and geometric study of the Markowitz problem.

Keywords: portfolio analysis, critical line method, minimum-variance frontier, corner portfolios, corner points, quadratic optimization algorithms.

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1. INTRODUCTION

The critical line method developed by the Nobel Prize winner H. Markowitz (see [1]) is a classical technique for the construction of the minimum-variance frontier within the paradigm of "the expected return—risk" (mean—variance) and finding minimum portfolios [2, 3]. Recently, interest has been aroused in the construction of fast algorithms for the construction of a minimum-variance frontier. In some works, such algorithms have been used to find statistically stable optimal portfolios (see [4—8]).

An algorithm based on the critical line method was proposed by Andras Niedermayer and Daniel Niedermayer (see [9]). Its testing showed that it is faster than all similar algorithms known before by several orders of magnitude (see [10—14]). In this paper, we present an algorithm for constructing the minimum-variance frontier for the Markowitz problem with the condition of nonnegativity of the optimal portfolio (see [2, 3]), which requires about half as many operations as the Niedermayers' algorithm. To this end, we had to perform a more thorough analytical and geometric study of the Markowitz problem.

2. SOLUTION OF THE UNCONSTRAINED MARKOWITZ PROBLEM

Each investor, i.e., a person who invests money in financial assets, such as bonds and stocks, faces a dilemma between choosing an asset with a high return and high risk and a low but guaranteed return. In addition, when there are several assets, what is the best way to allocate them, or in other words, how to form an investment portfolio?

In 1990, the American scientist H. Markowitz won the Nobel Prize in economics for his work on portfolio analysis (see [1]). In a probabilistic model, he proposed for each risky financial asset (for example, a share) to consider its two characteristics, i.e., the expected return, that is, the mathematical expectation of the return, and the square of the risk, that is, the variance of the return. From this, we can derive formulas for the expected return for a portfolio consisting of several risky assets and its variance, which are listed below (see, for example, [2, 3]). In addition, he formulated and solved the optimization problem of minimizing the risk of the portfolio at its fixed expected return, or, equivalently, of maximizing the expected return for a given level of risk. Let there be n risky assets, for which the vector of expected returns $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$[1] and the covariance matrix $\mathbf{V}$ are known. For convenience, we also introduce the col-

[1] T above denotes the transposition of the matrix.

umn vector consisting of units $\mathbf{I} = (1, 1, \ldots, 1)^T$. Let $\mathbf{X} = (x_1, x_2, \ldots, x_n)^T$ be a column vector of shares of the considered assets in the investment portfolio, which for brevity we will call a portfolio. The Markowitz problem with no constraints can be formulated as follows:

Find portfolio **X**, which would minimize the risk σ and would provide the specified value of the expected return μ. Or,

$$\frac{1}{2}\sigma^2 = \frac{1}{2}\mathbf{X}^T\mathbf{V}\mathbf{X} \to \min \tag{1}$$

on the conditions that

$$\boldsymbol{\mu}^T\mathbf{X} = \mu_X, \tag{2}$$

$$\mathbf{I}^T\mathbf{X} = 1. \tag{3}$$

Condition (3) ensures the normalization of the vector of shares; i.e., the sum of shares is equal to unity. The coefficient 1/2 in the objective function is of a technical nature.

We write the solution of problem (1)–(3) under the following assumptions:

(1) The covariance matrix **V** is positively defined, i.e., nondegenerate. Hence, it follows that for any subset of selected assets this property is preserved.

(2) Asset returns are in ascending order, and there are no equal values among them:

$$\mu_1 < \mu_2 < \ldots < \mu_n.$$

In particular, the vector of expected returns μ is not collinear to the vector **I**.

We need the following constants:

$$\alpha = \mathbf{I}^T\mathbf{V}^{-1}\mathbf{I}, \tag{4}$$

$$\beta = \mathbf{I}^T\mathbf{V}^{-1}\boldsymbol{\mu} = \boldsymbol{\mu}^T\mathbf{V}^{-1}\mathbf{I}, \tag{5}$$

$$\gamma = \boldsymbol{\mu}^T\mathbf{V}^{-1}\boldsymbol{\mu}, \tag{6}$$

$$\delta = \alpha\gamma - \beta^2. \tag{7}$$

Since V^{-1} is a positively defined matrix, the constants α, γ are positive numbers. It is easy to show that $\delta > 0$. Under the above assumptions problem (1)–(3) has a unique solution

$$\lambda = (\gamma - \beta\mu_X)/\delta, \quad \tau = (\alpha\mu_X - \beta)/\delta, \quad \mathbf{X} = \mathbf{V}^{-1}(\lambda\mathbf{I} + \tau\boldsymbol{\mu}). \tag{8}$$

At the same time, the components of vector **X** can have an arbitrary sign.

Negative components of the portfolio usually imply borrowing. For example, if the portfolio value is 100 thousand rubles, and the first component is equal to –0.1, this means that the investor borrowed from the stock exchange for a specified term 10 thousand rubles for the purchase of *n* stocks of the first kind. At the end of this term the investor is obliged to redeem the above-mentioned *n* stocks at the current price. Typically, stock exchanges limit the ability of investors to buy stocks on trust by a certain percentage (e.g., 15%) of the investor's investment capital.

Note that the value τ depends linearly on the expected portfolio return $\mu_\mathbf{X}$; moreover, the coefficient of linear dependence α/δ is positive. Consequently, with the increase in $\mu_\mathbf{X}$, the value τ also rises, and vice versa.

The equation of a minimum-variance frontier in this case has the form

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta}. \tag{9}$$

The following equality is satisfied:

$$\left(\frac{1}{2}\sigma^2(\mu)\right)' = \tau(\mu). \tag{10}$$

Since the discriminant of the quadratic polynomial on the right side of (9) is $\frac{4(\beta^2 - \alpha\gamma)}{\delta^2} = -\frac{4}{\delta} < 0,$ the minimum-variance frontier is a convex curve lying above the axis $O\mu$.

It is known that for two assets the equation of a minimum-variance frontier takes the following form:

$$\sigma^2 = \frac{v_{11}(\mu - \mu_2)^2 + v_{22}(\mu - \mu_1)^2 - 2v_{12}(\mu - \mu_1)(\mu - \mu_2)}{(\mu_2 - \mu_1)^2}. \tag{11}$$

Using the previous equality, we can readily show that

$$\tau(\mu_1) = \frac{v_{12} - v_{11}}{\mu_2 - \mu_1}, \tag{12}$$

$$\tau(\mu_2) = \frac{v_{22} - v_{21}}{\mu_2 - \mu_1}, \tag{13}$$

where the numerator comprises the elements of the covariance matrix, and the denominator comprises the expected asset.

3. ADDING A SINGLE ASSET

We further consider a situation when a new asset is added to a portfolio of securities. The following statement describes how this will change the minimum-variance frontier.

Proposition. Let an asset be added to portfolio $\mathbf{X} = (x_1, x_2, \ldots, x_n)$ resulting in portfolio $\tilde{\mathbf{X}} = (\mathbf{X}, x_{n+1})$. Then, for the equations of the minimum-variance frontiers $\sigma_{\mathbf{X}}(\mu)$ and $\sigma_{\tilde{\mathbf{X}}}(\mu)$ for every μ the following inequality is fulfilled:

$$\sigma_{\tilde{X}}(\mu) \leq \sigma_X(\mu). \tag{14}$$

Proof. Indeed, problem (1)–(3) for portfolio $\mathbf{X}$ is a particular case of a similar problem for portfolio $\tilde{\mathbf{X}}$; i.e., it must be assumed that $x_{n+1} = 0$. Hence, there follows inequality (14).

Thus, adding a new asset to the portfolio leads, generally speaking, to a better situation for the investor as the risk for the same return does not increase.

For definiteness, assume that the first asset is added, i.e., the first row and first column are added to this matrix.

Lemma 1. *Let there be a square matrix* $\mathbf{V} = \begin{pmatrix} v & \mathbf{a}^T \\ \mathbf{a} & \mathbf{A} \end{pmatrix}$ *of order n, where* v *is a number,* $\mathbf{a} \in \mathbb{R}^{-1}$ *is a column vector, and* A *is a square matrix of order* $n-1$*, and, at the same time, the following conditions are fulfilled:* $v \neq 0$, $|\mathbf{V}| \neq 0$, $|\mathbf{A}| \neq 0$. *Then, the equality is satisfied:*

$$\mathbf{V}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{A}^{-1} \end{pmatrix} + w\mathbf{c}\mathbf{c}^T, \tag{15}$$

where

$$\mathbf{c} = \begin{pmatrix} 1 \\ -\mathbf{A}^{-1}\mathbf{a} \end{pmatrix}, \quad w = \frac{1}{\begin{pmatrix} v & \mathbf{a}^T \end{pmatrix}\mathbf{c}}. \tag{16}$$

Proof. Direct verification.

It is easy to calculate the number of multiplications required to compute $\mathbf{V}^{-1}$ by formula (15). The calculation of vector $\mathbf{c}$ requires $(n-1)^2$ multiplications and finding the symmetrical matrix $\mathbf{c}\mathbf{c}^T$ requires $\frac{(n-1)n}{2}$ multiplications. Discarding the linear terms, we see that finding $\mathbf{V}^{-1}$ requires $\frac{3}{2}n^2$ multiplications.

Inductive formulas for calculating vectors $\mathbf{C} = \mathbf{V}^{-1}\mathbf{I}$ and $\mathbf{D} = \mathbf{V}^{-1}\boldsymbol{\mu}$, which do not require n^2 multiplications, are given in the following statement using the denotations of Lemma 1.

Lemma 2. We denote $s = \mathbf{I}^T\mathbf{c}$, $t = \boldsymbol{\mu}^T\mathbf{c}$. *Then,*

$$\mathbf{C} = \begin{pmatrix} 0 \\ \mathbf{C}_{n-1} \end{pmatrix} + ws\mathbf{c}, \tag{17}$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ \mathbf{D}_{n-1} \end{pmatrix} + wt\mathbf{c}, \tag{18}$$

where $\mathbf{C}_{n-1} = \mathbf{A}^{-1}\mathbf{I}_{n-1}$, $\mathbf{D}_{n-1} = \mathbf{A}^{-1}\mu_{n-1}$.

Proof. Direct application of formula (15).

It follows from formulas (17) and (18) that the calculation of vectors $\mathbf{C}$ and $\mathbf{D}$ only requires the number of multiplications to be linear in n.

Let the equation of a minimum-variance frontier for the selected n assets have the form

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta},$$

and for $n - 1$ assets (the first asset is removed)

$$\sigma^2 = \frac{\alpha_1\mu^2 - 2\beta_1\mu + \gamma_1}{\delta_1}.$$

It will be assumed that these two curves have a common point at $\mu = \mu_0$. We find formulas relating to the corresponding coefficients.

Lemma 3. *In the notations of Lemma 2* the following relations are satisfied*:*

$$\alpha = \alpha_1 + ws^2, \tag{19}$$

$$\beta = \beta_1 + wst, \tag{20}$$

$$\gamma = \gamma_1 + wt^2. \tag{21}$$

Proof. We present the derivation of (20), and the remaining formulas are obtained in a similar way. We have

$$\beta = \mathbf{I}^T\mathbf{V}^{-1}\mu = \mathbf{I}^T\mathbf{D}.$$

We apply formula (18):

$$\beta = \begin{pmatrix}1 & \mathbf{I}_{n-1}^T\end{pmatrix}\begin{pmatrix} wt \\ \mathbf{D}_{n-1} - wt\mathbf{c}\end{pmatrix} = wt + \mathbf{I}_{n-1}^T\mathbf{D}_{n-1} - wt\mathbf{I}_{n-1}^T\mathbf{c}.$$

Since $\mathbf{I}_{n-1}^T\mathbf{D}_{n-1} = \beta_1$, $\mathbf{I}_{n-1}^T\mathbf{c} = \mathbf{c}^T\mathbf{I}_{n-1} = 1 - s$, then, $\beta = \beta_1 + wt - wt(1-s) = \beta_1 + wst$,as was to be proved.

***Corollary* 1.** *The following relation is fulfilled:*

$$(\alpha - \alpha_1)(\gamma - \gamma_1) = (\beta - \beta_1)^2. \tag{22}$$

This follows immediately from formulas (19)–(21).

***Corollary* 2.** *When an asset is added, the coefficients* α, γ, δ *increase.*

For coefficients α and γ this follows from formulas (19) and (21) because the matrix $\mathbf{V}^{-1}$ along with matrix $\mathbf{V}$ is positively defined, therefore, the element $w = \left(\mathbf{V}^{-1}\right)_{11}$ is positive.

Further,

$$\delta = \alpha\gamma - \beta^2 = \left(\alpha_1 + \frac{k^2}{w}\right)\left(\gamma_1 + \frac{m^2}{w}\right) - \left(\beta_1 + \frac{km}{w}\right)^2 = \delta_1 + \alpha_1\frac{m^2}{w} + \gamma_1\frac{k^2}{w} - 2\beta_1\frac{km}{w}.$$

Thus,

$$\delta - \delta_1 = \alpha_1\frac{m^2}{w} + \gamma_1\frac{k^2}{w} - 2\beta_1\frac{km}{w}.$$

The expression in the right-hand side of the equality is a quadratic function with respect to $\frac{m}{\sqrt{w}}$, $\frac{k}{\sqrt{w}}$ with a negative discriminant equal to $-4\delta_1$, and therefore it only takes on positive values.

However, as we will see later, the coefficients of a minimum-variance frontier decrease when a single asset is added.

Corollary 3. *The minimum-variance frontiers with equations*

$$\sigma^2 = \frac{\alpha\mu^2 - 2\beta\mu + \gamma}{\delta} = A\mu^2 - 2B\mu + C$$

and

$$\sigma^2 = \frac{\alpha_1\mu^2 - 2\beta_1\mu + \gamma_1}{\delta_1} = A_1\mu^2 - 2B_1\mu + C_1,\ A \neq A_1,$$

tangential at point $\mu = \mu_0$.

Proof. First, it should be noted that the two parabolas defined by the equations $y = Ax^2 - 2Bx + C$, $y = A_1x^2 - 2B_1x + C_1$, and $A \neq A_1$, having a common point at $x = x_0$, are tangential at this point if and only if the following condition is fulfilled:

$$(A - A_1)(C - C_1) = (B - B_1)^2. \tag{23}$$

Relation (23) can be reduced to the form

$$AC - B^2 + A_1C_1 - B_1^2 - AC_1 - A_1C + 2B_1B_2 = 0.$$

Substituting the expressions using Greek letters, we obtain

$$\frac{1}{\delta} + \frac{1}{\delta_1} - \frac{\alpha_1\gamma}{\delta\delta_1} - \frac{\alpha\gamma_1}{\delta\delta_1} + 2\frac{\beta\beta_1}{\delta\delta_1} = 0$$

or

$$\delta_1 + \delta - \alpha_1\gamma - \alpha\gamma_1 + 2\beta\beta_1 = 0.$$

Finally,

$$\alpha_1\gamma_1 - \beta_1^2 + \alpha\gamma - \beta^2 - \alpha_1\gamma - \alpha\gamma_1 + 2\beta\beta_1 = 0,$$
$$(\alpha_1 - \alpha)(\gamma_1 - \gamma) = (\beta_1 - \beta)^2,$$

which is what we set out to prove.

Note 1. The inequality condition for the leading coefficients of the equations of the minimum-variance frontiers is essential, as shown by the following example.

Example. Let $\boldsymbol{\mu} = (1, 3/2, 3)^T$, $\mathbf{V} = \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The solution to this problem can readily be found:

$$\mathbf{X} = (5/4 - 1/2\,\mu,\ 1/3,\ -7/12 + 1/2\,\mu)^T.$$

Since $X_2 = 1/3$, the minimum-variance frontier for the three assets does not intersect with the minimum-variance frontiers constructed for the first and second assets. Here are their equations:

$$\sigma^2 = 11/12 - \mu + 1/3\,\mu^2 \quad \text{and} \quad \sigma^2 = 1 - \mu + 1/3\,\mu^2.$$

Note 2. It follows from formula (23) that if $A = A_1$, then $B = B_1$. Therefore, when a single asset is added, the minimum-variance frontiers are either only tangent or do not intersect. In the latter case, they are obtained from each other by a shift along the vertical axis. Our study clarifies the false reasoning from work [15].

***Lemma* 4.** *Let there be two parabolas* $y = Ax^2 - 2Bx + C$ *and* $y = A_1x^2 - 2B_1x + C_1$, *which are tangent at* $x = x_0 > 0$. *Then, if the first parabola lies above the second*, $A > A_1$, $B > B_1$, $C > C_1$.

Proof. Since the first parabola lies above the second, $A > A_1$. The conditions of the parabolas' tangency can be written in the form of equations

$$\begin{cases} Ax_0 - B = A_1x_0 - B_1, \\ Bx_0 - C = B_1x_0 - C_1. \end{cases}$$

Hence,

$$\begin{cases} (A - A_1)x_0 = B - B_1, \\ (B - B_1)x_0 = C - C_1. \end{cases}$$

The statement of the lemma becomes obvious.

Corollary 4. When an asset is added, the coefficients of the equation of the minimum-variance frontier decrease. To be more precise, let $\sigma^2 = A\mu^2 - 2B\mu + C$ be an equation of the minimum-variance frontier under $\mu < \mu_0$, and $\sigma^2 = A_1\mu^2 - 2B_1\mu + C_1$ be the equation of the minimum-variance frontier under $\mu > \mu_0$. Then, the following inequalities will be satisfied:

$$A < A_1,\ B < B_1,\ C < C_1.$$

Indeed, when an asset is added, the minimum-variance frontier lies below the previous frontier because the corresponding problem contains the previous one as a subproblem. In addition, it is typical when $\mu_0 > 0$, as, solving the problem of finding the optimal portfolio with negative expected returns, apparently, does not make sense. Then, the corollary statement follows directly from the Lemma.

4. ASSET REMOVAL

The symmetric situation is the case when one of the assets, we will assume for definiteness that it is the first asset, is removed from the portfolio.

First, we consider a technical result from matrix algebra.

Lemma 5. *Let there be a square matrix* $\mathbf{V} = \begin{pmatrix} v & \mathbf{a}^T \\ \mathbf{a} & A \end{pmatrix}$ *of order n, where* v *is a number,* $\mathbf{a} \in \mathbb{R}^{n-1}$ *is a column vector, and* A *is a square matrix of order* $n-1$*, and, moreover, the following conditions are fulfilled:* $v \neq 0$, $|\mathbf{V}| \neq 0$, $|\mathbf{A}| \neq 0$. *We write the inverse matrix in the form* $\mathbf{V}^{-1} = \begin{pmatrix} w & \mathbf{b}^T \\ \mathbf{b} & \mathbf{B} \end{pmatrix}$. *Then, the equality is satisfied*

$$\mathbf{A}^{-1} = \mathbf{B} - \frac{1}{w}\mathbf{b}\mathbf{b}^T. \tag{24}$$

Proof. We denote by $\mathbf{0}_k$ the column vector of dimensionality k, consisting of zeros, and we denote the unit matrix of size k by $\mathbf{E}_k$. Since $\mathbf{V}\mathbf{V}^{-1} = \begin{pmatrix} 1 & \mathbf{0}_{n-1}^T \\ \mathbf{0}_{n-1} & \mathbf{E}_{n-1} \end{pmatrix}$, we have a system of equations

$$\begin{cases} \mathbf{a}\mathbf{b}^T + \mathbf{A}\mathbf{B} = \mathbf{E}_{n-1}, \\ w\mathbf{a} + \mathbf{A}\mathbf{b} = \mathbf{0}_{n-1}. \end{cases}$$

By eliminating vector $\mathbf{a}$ from the system, we obtain

$$w\mathbf{A}\mathbf{B} - \mathbf{A}\mathbf{b}\mathbf{b}^T = w\mathbf{E}_{n-1},$$

hence $\mathbf{A}\left(\mathbf{B} - \frac{1}{w}\mathbf{b}\mathbf{b}^T\right) = \mathbf{E}_{n-1}$, which completes the proof of the Lemma.

The analogs of formulas (17) and (18) are given in the following lemma.

Lemma 6. Let $k = \left(\mathbf{V}^{-1}\mathbf{I}\right)_1$, $m = \left(\mathbf{V}^{-1}\mu\right)_1$ be the first components of vectors $\mathbf{V}^{-1}\mathbf{I}$, $\mathbf{V}^{-1}\boldsymbol{\mu}$. We assume $w = \left(\mathbf{V}^{-1}\right)_{11}$. Then,

$$\mathbf{C} = \begin{pmatrix} k \\ \mathbf{C}_1 + \frac{k}{w}\mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{C}_1 \end{pmatrix} + \frac{k}{w}\begin{pmatrix} w \\ \mathbf{b} \end{pmatrix}, \tag{25}$$

$$\mathbf{D} = \begin{pmatrix} m \\ \mathbf{D}_1 + \frac{m}{w}\mathbf{b} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{D}_1 \end{pmatrix} + \frac{m}{w}\begin{pmatrix} w \\ \mathbf{b} \end{pmatrix}. \tag{26}$$

Proof is completely analogous to the proof of Lemma 2.

From formulas (25) and (26) the unknown vectors $\mathbf{C}_1$ and $\mathbf{D}_1$ can be found from the known vectors $\mathbf{C}$ and $\mathbf{D}$.

We present formulas relating the values α, β, and γ for the case of an asset removal.

Lemma 7. In the notation of Lemma 6, the following relations hold true:

$$\alpha_1 = \alpha - \frac{k^2}{w}, \tag{27}$$

$$\beta_1 = \beta - \frac{km}{w}, \tag{28}$$

$$\gamma_1 = \gamma - \frac{m^2}{w}. \tag{29}$$

Proof is not given.

5. MARKOWITZ PROBLEM UNDER NONNEGATIVITY CONFITION

It will be additionally required for problem (1)–(3) that the following condition should be fulfilled:

$$\mathbf{X} \geq 0. \tag{30}$$

It is known that in this case the minimum-variance frontier in the coordinates $\left(\mu, \frac{1}{2}\sigma^2\right)$ consists of a finite number of pieces of parabolas divided by the corner points and is a convex curve. We will call portfolio $\mathbf{X}$ a *corner* one if in its sufficiently small vicinity the portfolios located on the minimum-variance frontier have a number of nonzero components that is different from $\mathbf{X}$. Corollary 3 shows that at the corner points, where the corner portfolio is not a single one, the contiguous segments of the minimum-variance frontier are tangential.

Note that if $\mathbf{X}_1$ and $\mathbf{X}_2$ are adjacent corner portfolios, while $\mu_1 = \mu(\mathbf{X}_1)$, $\mu_2 = \mu(\mathbf{X}_2)$, and $\mu_1 < \mu_2$, then on the interval (μ_1, μ_2) all the portfolios located on the minimum-variance frontier have the same number of nonzero components. Let S be a set of indices, for which components $\mathbf{X}$ differ from zero, such variables we will call *internal*, and the rest of the variables will be called the *external* ones. Then, on this interval, the solution of the Markowitz problem with constraints (30) is obtained from the solution of the unconstrained problem (1)–(3) under the conditions

$$x_i = 0, \qquad i \notin S. \tag{31}$$

In addition, with an increase in μ_X, the value $\tau(\mathbf{X})$ increases too.

We present an algorithm for solving the Markowitz problem, based on the analysis performed above under the additional assumption. We will move along the minimum-variance frontier towards increasing μ (from left to right). We assume that in passing through the corner point to the set of the internal variables one asset is added or a single asset is removed from this set.

This assumption does not hold in the general case, as we will see later. Nevertheless, it is satisfied with probability 1, in other words, for a fixed vector of expected returns and fixed asset risks, the set of nondegenerate correlation matrices for which the considered condition is satisfied is dense everywhere.

6. ALGORITHM DESCRIPTION

We sequentially find all the corner points and the corner portfolios corresponding to those while moving along the minimum-variance frontier from left to right towards increasing μ. With the above assumption, it is clear that when the corner point is reached the main task is to find the asset to be added or removed.

The first corner corresponds to the value $\mu_{\min} = \mu_1$, and the corner portfolio has the form $\mathbf{X}_1 = (1, 0, \ldots, 0)$. We assume $S_{cur} = \{1\}$.

We assume that we know the abscissa of the current corner point μ_{tp}, the corresponding corner portfolio $\mathbf{X}_{tp}$, and the set of the internal variables S_{cur}. The algorithm operates differently depending on whether condition $|S_{cur}| = 1$ is satisfied.

1. $|S_{cur}| = 1$. If condition $\mu_{tp} = \mu_{\max}$ is fulfilled, the algorithm finishes its operation. We have reached the right end of the minimum-variance frontier.

Consider the case where this condition is not satisfied, for example, at the very beginning of the algorithm's execution when $\mathbf{X}_1 = (1, 0, \ldots, 0)$ and $\mu(\mathbf{X}_1) = \mu_1$. In order to simplify the notation we consider this case. In this case, the asset can only be added by using the following operations.

We calculate $\tau_{1k} = \dfrac{v_{1k} - v_{11}}{\mu_k - \mu_1}, \quad \mu_k > \mu_1$ and let $\tau_{1i} = \min\{\tau_{1k}\}$. In this case, the asset i is added so the set of the internal variables $S_{cur} = \{1, i\}$ consists of two assets. The equation of the minimum-variance frontier is calculated for the obtained set S_{cur} and then the maximum permissible value is found for : .$\mu_{\max} = \mu_i$

2. $|S_{cur}| > 1$.

We know the set of internal variables S_{cur} and the set of external variables S_{ext}. Assets can be added from set S_{ext} and removed from set S_{cur}. Now, we consider each of these possibilities.

6.1. Asset Addition

For every asset i from the set S_{ext}, let $S = S_{cur} \cup \{i\}$.

Run the procedure of finding an acceptable portfolio consisting of the following steps:

(1) Compute the inverse matrix by the formulas (15) and (16).

(2) Find portfolio $\mathbf{X}(\mu)$ by formulas (8).

(3) Equate to zero the ith component of vector $\mathbf{X}$ and find value $\mu^{(i)}$, i.e., the possible abscissa of the corner point. The numbers $\mu^{(i)}$ must satisfy the inequalities $\mu_{tp} < \mu^{(i)}$.

Out of all such $\mu^{(i)}$ the minimum value must be selected.

Then the system of inequalities $\mathbf{X}(\mu) \geq 0$ must be solved. There are two possible cases:

The system has a solution $\mu_d \leq \mu \leq \mu_u$. We assume $\mu_{tp} = \mu_d$, $\mathbf{X}_{tp} = \mathbf{X}(\mu_{tp})$, and $S_{cur} = S$.

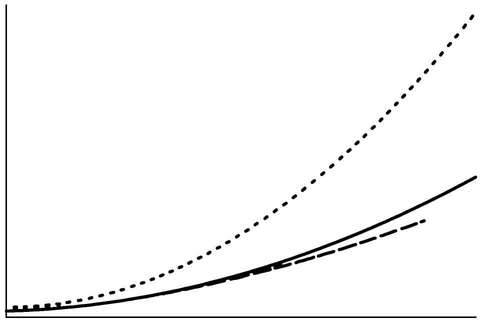

Figure.

Otherwise, the asset is removed.

6.2. Asset Removal

(1) At the previous step, the asset with index i was added.

In this case, it will be assumed $\mu_{tp} = \mu_u$. $\mathbf{X}_{tp} = \mathbf{X}(\mu_u)$. In other words, some asset j must be removed from condition $X_j = 0$. We assume $S_{cur} = S \setminus \{j\}$.

(2) At the previous step, an asset was removed.

In this case, we assume $S = S_{cur}$ and run the procedure of finding an acceptable portfolio; moreover, the inverse matrix is calculated by formula (24).

End of the algorithm description.

The algorithm, is readily justified (see the figure). The figure shows the current minimum-variance frontier as a solid line. If an asset can be added, it should be done, because when an asset is added, the corresponding minimum-variance frontier (dashed line) lies below the current frontier; i.e., there is no need to consider additionally the case of the asset removal. If there is no candidate for an asset addition, this implies that the current minimum-variance frontier is lower than any minimum-variance frontier resulting from the asset removal (dotted line) and, therefore, the rightmost admissible point must be considered.

This variant does not include the additional search for removed assets, which results in the number of multiplications being lower than in algorithm [9] suggested by the Niedermayers.

In order to compare the amount of computation for the two algorithms we consider a model case. Let the number of nonzero components of the corner portfolios increase monotonically from 1 to n–1 from the ends of the minimum-variance frontier to its middle.

Then the Niedermayers' algorithm requires

$$(n-1)2\sum_{k=2}^{n-1}\frac{3}{2}k^2 \sim n^4 \text{ multiplications,}$$

whereas our algorithm requires

$$3\sum_{k=2}^{n-1}(n-k)k^2 \sim \frac{1}{4}n^4 \text{ multiplications,}$$

i.e., a quarter of the number of multiplications required by the Niedermayers' algorithm.

Note 3. If we move away from the asset with the lowest expected return, an asset must be added for which the minimum is achieved from the $n-1$ numbers

$$\min_{2\le k\le n}\left\{\frac{v_{1k}-v_{11}}{\mu_k-\mu_1}\right\}.$$

As a test, Markowitz's example [1] has been selected where there are all these variants.

Note 4. The case is also possible where the values ν_{cur} or μ_{cur} are achieved for several assts, as the following example can show.

Example. For four assets, let the vector of expected returns be

$$\boldsymbol{\mu} = (2, 8, 10, 14)^T,$$

and let the covariance matrix be

$$\mathbf{V} = \begin{pmatrix} 1 & -1 & -2 & -2 \\ -1 & 4 & 4 & 7 \\ -2 & 4 & 9 & 10 \\ -2 & 7 & 10 & 16 \end{pmatrix}.$$

Starting from the rightmost point of the minimum-variance frontier (14, 16), we calculate the values $\nu_{43} = \frac{16-10}{14-10} = \frac{3}{2}$, $\nu_{42} = \frac{16-7}{14-8} = \frac{3}{2}$, and $\nu_{41} = \frac{16-(-2)}{14-2} = \frac{3}{2}$. We can see that the minimum-variance frontiers formed by the fourth and the third, the fourth and the second, and the fourth and the first assets are tangential at the above-specified point. Hence, it follows that the minimum-variance frontier formed by

Table

	Frontier			Niedermayer		
n	*t* (c)	*ntp*	*nit*	*t* (c)	*ntp*	*nit*
60	0.6	37	1314	1.4	37	1999
70	1.0	44	1876	2.2	44	2751
80	1.1	46	2275	2.4	46	3319
90	1.6	50	2818	3.9	50	4128
100	2.0	55	3431	5.2	55	5061

Note: Here, t is the average time of the problem solution, *ntp* is the average number of the corner points, and *nit* is the average number of iterations.

all the four assets will also be tangential to the respective minimum-variance frontiers. We will now present the equations of the entire minimum-variance frontier

$$\sigma^2 = 1.953 - 0.99329\mu + 0.14262\mu^2,\ \mu \in \left[\frac{89}{17}, 14\right],\ S = \{1, 2, 3, 4\};$$

$$\sigma^2 = 3.1087 - 1.4348\mu + 0.18478\mu^2,\ \mu \in \left[\frac{14}{5}, \frac{89}{17}\right],\ S = \{1, 2, 3\};$$

$$\sigma^2 = \frac{27}{8} - \frac{13}{8}\mu + \frac{7}{32}\mu^2,\ \mu \in \left[2, \frac{14}{5}\right],\ S = \{1, 3\}.$$

Therefore, at the rightmost point, three assets must be added to obtain the minimum-variance frontier.

7. COMPARISON OF COMPUTATION AMOUNTS

In order to compare the performance, both algorithms (ours, hereinafter, referred to as the frontier algorithm, and the Niedermayer algorithm) were programmed by means of the customized software MatCalc (Matrix Calculator), developed and implemented by A.V. Brailov in the C++ language (see also http://www.matcalc.ru/). For each dimensionality of the covariance matrix from 10 to 100 with a step of 10, formed by the same rule as in [9], the average number of iterations and the average number of the corner points were calculated. The relevant results are shown in the following table (the results are only shown for dimensions starting from 60).

It is clear from the table that in our algorithm the number of iterations is approximately 70% of the number required by the Niedermayer algorithm and the time of the problem solution is more than halved.

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