Improving on risk parity

Hedging forecast uncertainty

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Asset Management

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Executive summary

So far, the twenty-first century has not been much fun for institutional investors. US equities underperformed not only expectations, but fixed income as well, calling into question the 60/40 split between equities and bonds that has become the baseline for strategic asset allocation. In this harsh environment, the risk parity approach to strategic investing has flourished, not only because its leveraged fixed income exposure has produced high returns, but also because it advances a novel investment principle: the key to asset allocation is to allocate equal shares of portfolio risk to each asset class.

The important question for investors is whether risk parity will work as well in the future. This is more a matter of the investment principles it involves than whether bonds will continue to outperform stocks. The goal of the research project described in the following pages is to understand and evaluate these investment principles and ultimately to incorporate their positive contribution into an improved approach to strategic asset allocation.

Our answer is somewhat surprising. We demonstrate that equal risk shares is meaningless as a guide to investment choices. Instead, we focus on a particular consequence of following the equal risk shares principle: that portfolio choices pay no attention to forecasts of asset class returns. Why might this be a good thing? To the extent that you are sure that the average return on stocks is going to be equal to your 8% forecast, it would not seem wise to ignore it. But if you are uncertain, you should probably tone down your reliance on the forecast, and if you are very uncertain, you could be better off ignoring the forecast altogether, which is what risk parity does.

Plainly, we are uncertain about forecasts and would do well to factor this uncertainty into asset allocation decisions. The forecast hedge asset allocation rule that we propose strikes a balance between the conventional approach to asset allocation (ignore the uncertainty) and risk parity (ignore the forecasts), by measuring the amount of uncertainty and weighting the two accordingly. In our tests across a wide range of return environments, the forecast hedge outperforms the two extremes.

Here, as elsewhere, there is no free lunch. To use the forecast hedge, you need to supply a measure of the confidence or uncertainty you attach to your return forecasts. This is extra work and it may be difficult to do this precisely, but these are not reasons to avoid trying. After all, if you are uncertain about your forecast uncertainty, how can you be certain about your forecast? Moreover, our analysis suggests that precision is not crucial and that the benefits of taking a position on uncertainty can be substantial.

We hope these ideas will help make strategic asset allocation more practical, and as always, welcome your thoughts and comments.

INTRODUCTION

Risk parity is the most prominent of a new group of asset allocation remedies for investors badly bruised by a decade of poor equity returns and high volatility. Proponents advocate allocating equal shares (parity) of portfolio risk to each asset class or groups of asset classes, hence the name.

The equity underweight and fixed income overweight this entails relative to traditional pension or endowment allocations has produced stellar performance, both actual and back tested. The quandary is whether risk parity has found the Holy Grail of asset allocation or has just levered bonds during a long lasting fixed income rally that may soon end.

We argue that 'equal risk shares' is meaningless as an objective. Any portfolio can be shown to satisfy risk parity simply by redefining the asset classes, which changes nothing of substance. This simple but apparently unrecognised fact means that we have to look elsewhere for risk parity's success. We cast risk parity as an exercise in *uncertainty management*, a somewhat neglected part of practical asset allocation. The textbook asset allocation recipe derives an optimal portfolio from known *parameters* describing asset returns: means, volatilities and correlations. Were these parameters indeed known, this would be the end of the story. Simply plug them into the textbook recipe to derive the 'tangent portfolio' (the portfolio with the highest Sharpe ratio) and mix with cash according to (risk) preference. *Practical* asset allocation is the headache that it is precisely because these parameters are unknown and must be estimated, forecast or proxied. In judging risk parity or any other portfolio allocation rule, all we should care about is how effectively it deals with our uncertainty about the true parameters, because uncertainty is the only thing that stands between us and the textbook recipe.

The standard approach to asset allocation ignores uncertainty altogether. It blithely plugs in historical estimates or forecasts of returns where the parameters belong and acts as if they are the real thing. This plug-in approach often translates seemingly incidental features of the inputs into extreme and counterintuitive asset weights.

Viewed as uncertainty management, risk parity finesses the problem of forecasting returns. The risk parity allocation (equal risk shares) and the tangent portfolio are identical in the event that all asset classes have the same Sharpe ratio (and all asset correlations are the same). When these conditions hold, it is extreme folly to build an asset allocation from return forecasts because of the errors forecasting introduces. Instead one should just use the risk parity allocation, which does not require forecasts of mean returns and correlation. In reality, of course, Sharpe ratios for different assets will not be exactly equal, but the closer they are the more sense risk parity makes. So we can recast the question posed at the outset in more practical terms: Which is less serious, the plug-in method's tendency to compound forecast errors or risk parity's error from making an approximation?

We know the verdict of recent history: risk parity wins. For other return environments we might encounter, the answer has been elusive, because risk parity is often cast in terms not readily comparable to the traditional asset allocation framework. The most prominent exposition of risk parity involves choosing asset weights in order to create a portfolio that is balanced with respect to economic factors. An innovation of this paper is to translate the elements of this risk parity story into the traditional asset allocation framework. This allows us to go beyond the historical record and assess risk parity's robustness across a wide range of simulated 'possible worlds,' differentiated by how closely asset Sharpe ratios are clustered. risk parity outperforms the plug-in method in 80% of cases. If it were just a choice between these two rules, basing asset allocation on forecasts would not be a good idea.

We introduce a third method, the forecast hedge rule, or forecast hedge for short, which explicitly recognises that forecasts are uncertain. The forecast hedge rule outperforms risk parity in 70% of our simulations and always outperforms the plug-in method. It manages uncertainty better by using a key piece of information that the other rules ignore, namely how (statistically) different asset Sharpe ratios are. Instead of ditching forecasts altogether (risk parity) or using them indiscriminately (plug-in), the forecast hedge rule strikes a balance, by measuring the uncertainty surrounding forecasts and adjusting appropriately.

Our analysis clearly recommends that uncertainty management should be a central concern of strategic asset allocation. This entails following the forecast hedge rule in combining risk parity and traditional portfolios in proportions that depend on the confidence each warrants, given the information we have. When forecasts warrant extreme confidence, the forecast hedge will approximate the plug-in rule that is standard asset allocation procedure. We believe this will seldom be the case and that it will be worthwhile to go through the exercise of attaching confidence measures to forecasts. When there is little reason for confidence in return forecasts, the forecast hedge defaults to the risk parity allocation. A case in point is provided by a companion paper to this one,[1] in which a risk parity rule is used to allocate among various risk premia that are roughly uncorrelated and about whose returns the only conviction is that they are positive. In this case, the forecast hedge recommendation would be the risk parity allocation.

In the last section of this paper we compare the portfolios currently recommended by the various rules we have examined in our simulations, using J.P. Morgan Asset Management's 2012 *Long-term Capital Market Return Assumptions* as our forecasts. The forecast hedge rule advocates significantly less bonds than does risk parity.

We will have failed, however, if all readers take away is that the forecast hedge makes sense because it avoids bonds, which resonates with current fears that the bond rally cannot last much longer. The benefits of the forecast hedge rule are not likely to be evident over a quarter, a year or even several years. Instead, our analysis shows that on average it is likely to outperform the other rules, because it is built explicitly to manage parameter uncertainty over a wide range of plausible return environments.

The rest of the paper is structured as follows:

- We review textbook portfolio construction and illustrate the issues with the standard approach (see 'Asset allocation by the book').

- We introduce risk parity, highlighting both the positives and the negatives (risk parity).

- We present a unifying framework in which we can test different allocation strategies, including our forecast hedge approach, over a wide range of 'possible worlds' (forecast hedging).

- Finally we present and discuss the results, drawing the link to what this means for portfolios today (a real-world example).

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[1] Romahi, Y., and Santiago, K. (2012, August). 'Diversification — still the only free lunch?' J.P. Morgan Asset Management.

ASSET ALLOCATION BY THE BOOK

Before discussing risk parity, we review briefly what we called the textbook asset allocation recipe.

Each risky asset or portfolio is represented by its return in excess of the riskless asset (cash) and its volatility (risk) as in **Exhibit 1,** where we use the historical data for US stocks and bonds detailed in **Exhibit 2.** In this case, the minimum risk portfolio is 5% in equities, and the other extreme of the efficient frontier is occupied by 100% equities.

An investor who can hold cash as well as risky assets does not care about any portfolio that lies below the steepest slope line connecting the cash point to the frontier, as in Exhibit 1.[2] The frontier portfolio on this *capital market line* is known as the *tangent portfolio.* Since the slope of this line is equal to the tangent portfolio's excess return divided by its risk, the tangent portfolio is the risky portfolio with the highest Sharpe ratio.

An investor who wants to take on more risk but is unable to use leverage is constrained to the portfolios that lie on the frontier above the tangent portfolio. Leverage gives the investor access to the superior risk/return combinations along the capital market line, by investing more than 100% of wealth in the tangent portfolio, with the excess financed by borrowing cash. For instance, in the absence of leverage, a 60% stocks, 40% bonds portfolio was the most efficient way to achieve an excess return of 5% (which corresponded to 8% gross, a common return target). However, the same return could have been achieved at 4% less risk by leveraging the tangent portfolio a little less than twice.

EXHIBIT 1: CAPITAL MARKET LINE, TANGENT PORTFOLIO AND INDIFFERENCE CURVES
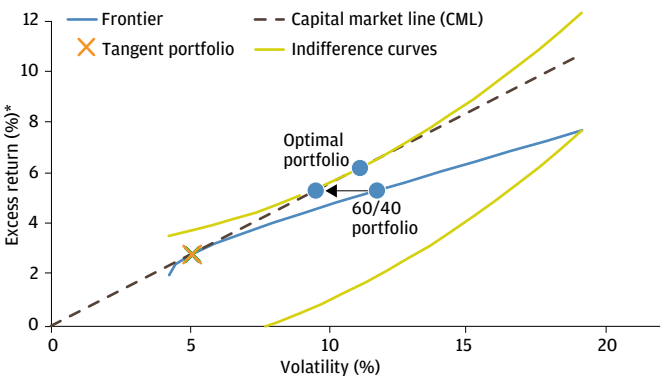
 * Excess over 30 day T-Bill. Past performance is not indicative of future results. Data from 1926 through 2011.

Source: J.P. Morgan Asset Management Strategy Group.

EXHIBIT 2: EQUITY/BOND STATISTICS, 1926–2011*

	Average excess return	Volatility	Sharpe ratio
US equity	7.7%	19.2%	0.40
US gov't bond	1.8%	4.3%	0.43

Source: Ibbotson Associates.

* Correlation during this period was 0.08.

Describing each portfolio by two numbers, risk and return, means it is impossible to rank them unambiguously. It is therefore convenient to combine each portfolio's risk and return into a single measure of its benefit to the investor. Portfolios that offer common amounts of this *utility* can be expected to lie on upward-sloping lines. These *indifference curves* slope upward to reflect the investor's *risk aversion* – extra risk needs to be supplemented by increased return to leave the investor indifferent. To draw these lines, it is necessary to know just how risk averse the investor is. We use a very simple formulation, detailed in the appendix, to back out a representative shape from the two-times leverage often mentioned as investors' choice in discussions of risk parity. Exhibit 1 shows two representative indifference curves drawn under this assumption. Apparently, 100% in equities produces less utility than 100% in bonds, but the best attainable portfolio (with leverage) leverages the 20/80 tangent portfolio twice.[3]

For our purposes, the important feature of the textbook recipe is its use of hindsight. As laid out in **Exhibit 3**, the same realised returns are used to select a portfolio and to evaluate its performance.

Practical asset allocation

Picking a portfolio in advance is a different matter. **Exhibit 4** illustrates how things change as we move from textbook to practical asset allocation. To deal with this more relevant situation, it is useful to imagine that returns are driven by *true parameter* values that investors cannot observe directly.

Nonetheless, investors need to come up with forecasts of average returns and their volatility and correlation over the relevant investment period. For this they typically use some combination of historical returns (that we view as generated by the true parameters), formal and informal economic views and valuation models. Often, volatility and correlation are estimated directly from history, while forecast returns are the product of a more qualitative process. Asset allocation decisions are made using these estimated or forecast parameters, but future returns are still the result of the true parameters. In other words, forecasts are necessarily infected with 'uncertainty' about the true parameters. This forecast uncertainty is distinct from, and additional to, the more familiar version of risk, which refers to the fact that realised returns in any period will not equal average returns.

EXHIBIT 3: TEXTBOOK ASSET ALLOCATION FLOW DIAGRAM
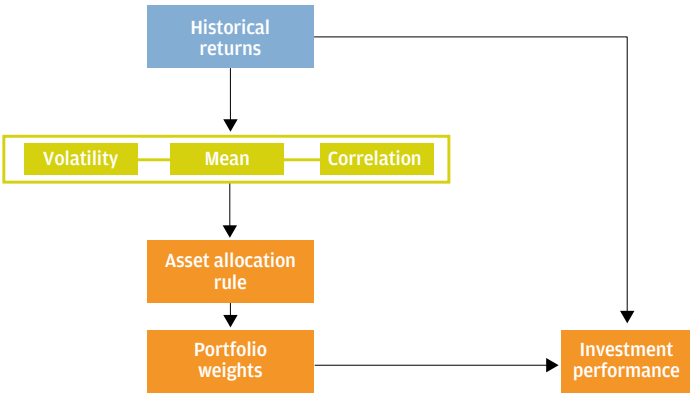
Source: J.P. Morgan Asset Management Strategy

EXHIBIT 4: PRACTICAL ASSET ALLOCATION FLOW DIAGRAM
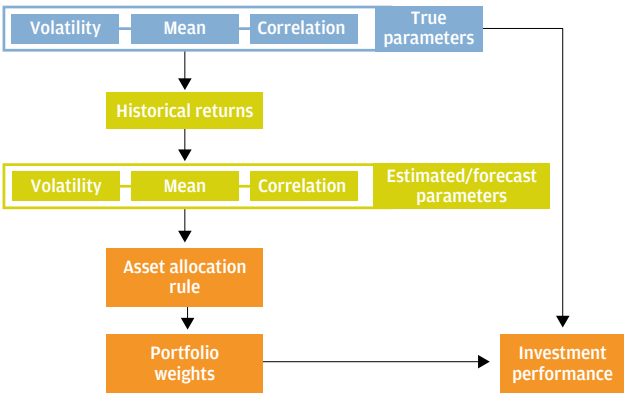
Source: J.P. Morgan Asset Management Strategy Group.

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[3]   We stress that we represent portfolios by their utility for convenience. Our substantive results are unchanged over a wide variety of assumptions concerning investor risk aversion.

In our experience, the usual asset allocation practice makes no allowance for the difference between forecasts and true parameters and simply plugs in forecasts where the textbook model expects true parameters. We should expect that this plug-in approach will affect investment outcomes adversely. Understanding just how adversely requires a little more equipment.

To be able to evaluate practical asset allocation rules, we need to model the qualitative piece of this story. We use a simulation framework, in which a surrogate history of returns (which we call a sample to avoid confusion with actual history) is simulated from the true parameters. The average of this sample serves as the investor's forecast. We will show below how the number of observations in this sample can be linked to the confidence investors have in their forecasts, and so justify using the sample average as a proxy for the forecast.

## The plug-in rule and forecast uncertainty

Exhibit 5A shows how the plug-in method performs when forecast uncertainty is equivalent to having a sample of 50 years of annual data from which to estimate average annual returns. The triangles describe attempts by the plug-in approach to find the tangent portfolio. Each triangle represents the outcome of a particular simulation, in which:

- A 50-year sample is simulated from the true parameters of Exhibit 2;

- The investor uses this sample to estimate the return parameters;

- The investor plugs these estimates into the textbook formula for the tangent portfolio weights; *and*

- The average return and volatility of the portfolio over the subsequent one-year investment period is determined from the same true parameters we started with.

As Exhibit 5A shows, the points are distributed around the true tangent portfolio, but can be quite far away.[4]

Applying leverage to these estimated tangent portfolios to optimise the investor's risk/return tradeoff introduces another layer of error, since the amount of leverage also depends on estimates. The leveraged portfolios (represented in **Exhibit 5B**) miss by quite dramatic amounts. These results suggest that real life asset allocation differs substantially from the textbook model, and that managing the slippage between forecasts and true parameters is essential.[5]

EXHIBIT 5A: PLUG-IN METHOD ATTEMPTS TO FIND THE TANGENT PORTFOLIO
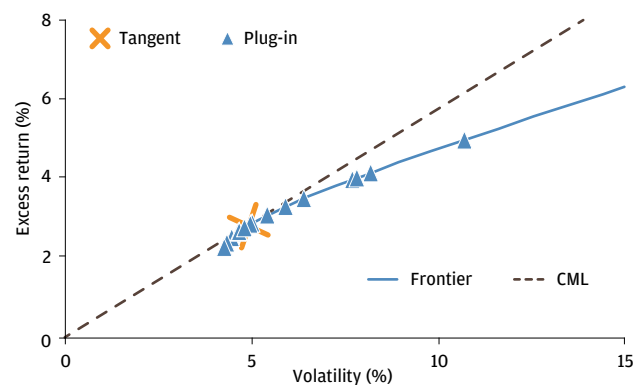
EXHIBIT 5B: PLUG-IN METHOD ATTEMPTS TO FIND THE OPTIMAL PORTFOLIO
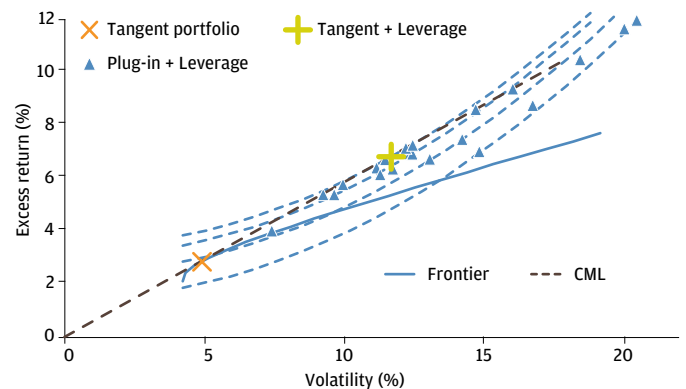
Source: J.P. Morgan Asset Management Strategy Group.

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[4] It is no consolation that the plug-in method always lands on the frontier. This is purely a consequence of there only being two assets in this example, so all possible portfolios lie on the frontier. With more assets, the plug-in rule produces outcomes that lie below the frontier.

[5] This phenomenon has been frequently documented over the last 40 years, seemingly to little avail. Examples are Klein and Bawa, Jorion, Michaud, and Black and Litterman.
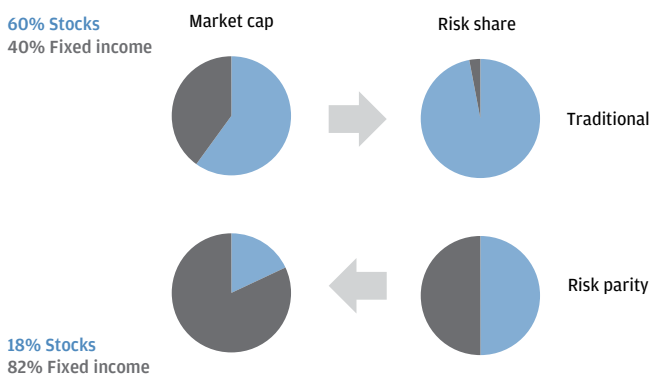
RISK PARITY

Risk parity comes in many different varieties but all derive from the principle that each asset class should be given an equal share of total portfolio risk.

The 60/40 portfolio, for example, is regarded as unbalanced, because more than 90% of its risk derives from equities (**Exhibit 6**). Sharing out the risk equally means that only 20% of the portfolio's value should be in stocks. In general, risk parity portfolios tend to allocate more to low volatility assets than conventional strategic asset allocation.

As long as expected stock returns exceed bonds, the risk parity allocation will have a much lower expected return and risk than the 60/40 portfolio. Typically, the risk parity portfolio is levered to match its risk to a target portfolio. **Exhibit 7** shows that since 1990 a risk parity portfolio levered to match the volatility of the 60/40 *(ex post)* portfolio would have outperformed handily.

**EXHIBIT 6: TRADITIONAL VS RISK PARITY ALLOCATION**

60% Stocks
40% Fixed income

Market cap

Risk share

Traditional

Risk parity

18% Stocks
82% Fixed income

Source: Calculated from Exhibit 2.

EXHIBIT 7: PERFORMANCE OF TRADITIONAL VS RISK PARITY ALLOCATION SINCE 1990

Legend: 60/40 — Risk parity with leverage — Risk parity
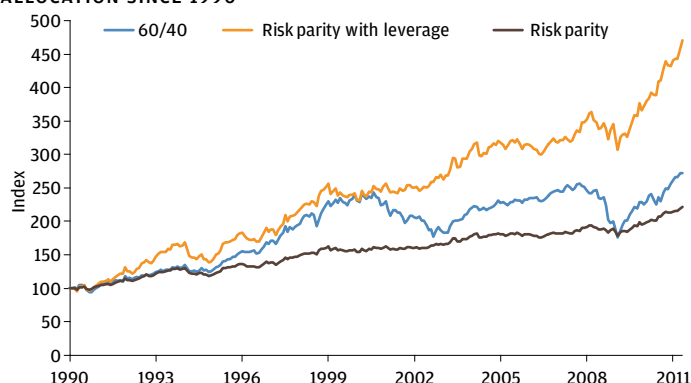
Source: J.P. Morgan Asset Management Strategy Group. 60/40 represented by S&P 500 and Intermediate US Treasuries, respectively. Two times leverage used above. DataSource: Ibbotson Associates.

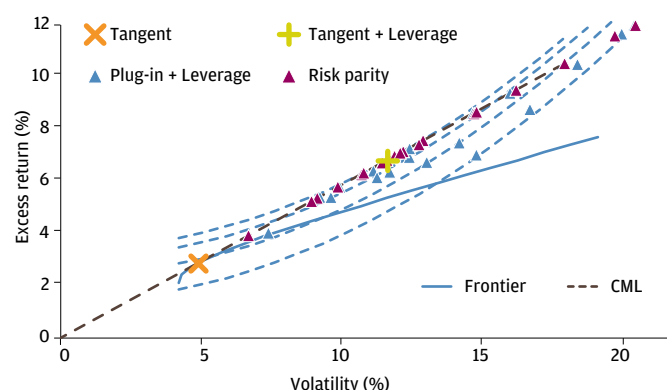**EXHIBIT 8A: RISK PARITY ATTEMPTS TO FIND THE TANGENT PORTFOLIO**

EXHIBIT 8B: RISK PARITY ATTEMPTS TO FIND THE OPTIMAL PORTFOLIO
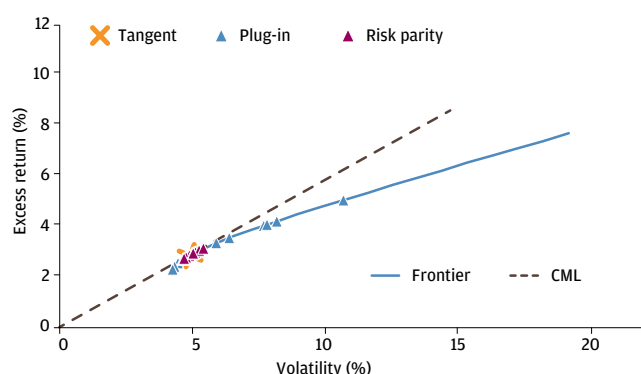
Source: J.P. Morgan Asset Management Strategy Group.

Again, these results are subject to the qualifications that go with hindsight. **Exhibits 8A** and **8B** show how risk parity performs prospectively, using the same simulation framework as we used for the plug-in rule. Apparently, risk parity does a much better job of finding the tangent portfolio – the risk parity triangles are much closer to the tangent portfolio.

Risk parity also produces consistently higher utility than the plug-in rule in aiming for the optimal portfolio (Exhibit 8B). This involves leverage and therefore introduces some additional sources of slippage into the risk parity allocation.[6]

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[6] Retrospective analyses of risk parity typically target the historical volatility of some benchmark portfolio, such as the 60/40. We find that using the equivalent (prospectively) in our simulations – the estimated volatility of the plug-in rule's guess at the optimal portfolio – simply infects the risk parity portfolio with the inaccuracies of the plug-in approach. Instead, the risk parity rule leverages more effectively if the optimal leverage amount is used directly (see appendix). This actually involves a forecast of the return of the risk parity portfolio, so the leveraged version of risk parity makes some use of forecasts and is not based exclusively on volatility estimates. Using the average of asset return forecasts is less prone to error than using each forecast separately, as the plug-in rule does.

Equal risk shares is a red herring

How does this performance come about? Are equal risk shares the answer to asset allocators' prayers or is there something else going on? We believe risk parity offers an important lesson in uncertainty management, but it has nothing to do with equal risk shares. As we shall now explain, the problem with demanding equal risk shares is that: every portfolio satisfies risk parity.

The 60/40 portfolio is often cited as an example of violation of risk parity. Based on the data in Exhibit 2, the discrepancy in risk shares is certainly extreme – 97% is attributable to stocks. For purposes of comparison, we introduce another asset class, called 'bocks.' **Exhibit 9** shows the volatility of bocks and correlations with stocks and bonds. It also shows that a portfolio comprising 31% stocks and 69% bocks has the same total risk as the 60/40 and exhibits risk parity – the contribution of bocks to total portfolio risk is the same as the contribution of stocks. All other things equal, these features would presumably make it more attractive to risk parity devotees than the 60/40.

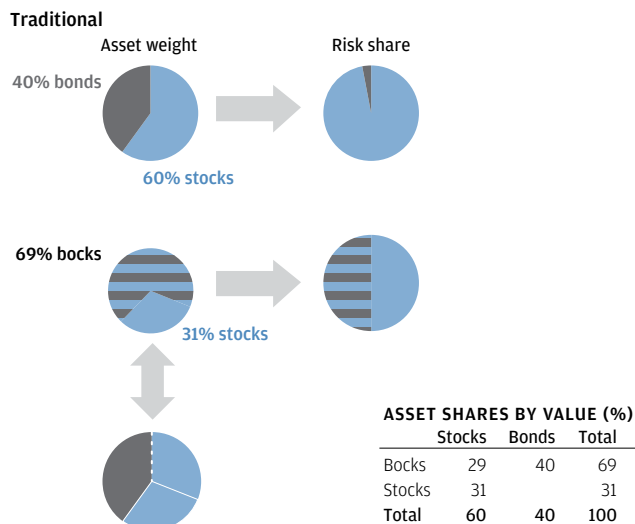**EXHIBIT 9: BUILDING A RISK PARITY PORTFOLIO WITH STOCKS AND BOCKS**

Asset	Volatility	Stocks	Bonds	Bocks	Portfolio share %	Risk contribution*	Portfolio share %	Risk contribution
Stocks	19.2	**1.00**	0.01	0.96	60	133	31	68
Bonds	4.3	0.01	**1.00**	0.30	40	3	0	0
Bocks	8.5	0.96	0.30	**1.00**	0	0	69	68
Total					100	136	100	136

* Risk is measured here as the square of volatility, or variance.
Source: J.P. Morgan Asset Management Strategy Group.

So far so good. But where do bocks come from? They have been cunningly contrived out of stocks and bonds: USD 100 of bocks is equivalent to USD 42 of stocks and USD 58 of bonds (hence the ridiculous name). Put another way, USD 69 of bocks is equivalent to USD 29 of stocks and USD 40 of bonds. Armed with this information, we can translate the 31/69 stocks/bocks portfolio into a stock/bond portfolio. It holds 60% of its value in stocks (31+29) and 40% in bonds. In other words, it's the 60/40 portfolio. So viewed from one perspective, the 60/40 portfolio violates risk parity. From another, it satisfies risk parity. Nothing of substance changes in moving from stocks/bonds to stocks/bocks. We always are exposed to 60% in stocks and 40% in bonds; we just looked at them from the perspective of a different (and purely derivative) asset class, called bocks. We did not buy or sell anything; we just reclassified (**Exhibit 10**).

EXHIBIT 10: EVERY PORTFOLIO IS A RISK PARITY PORTFOLIO

Traditional

Asset weight / Risk share

40% bonds / 60% stocks

69% bocks / 31% stocks

ASSET SHARES BY VALUE (%)			
	Stocks	Bonds	Total
Bocks	29	40	69
Stocks	31		31
Total	60	40	100

Source: J.P. Morgan Asset Management Strategy Group.

In a similar vein, risk parity proponents often adjust the units of existing asset classes to achieve a common level of volatility, emphasising that investors' decisions should not be affected by the way assets come packaged. We agree without reservation. Any investment principle worth its salt should be immune to this kind of cosmetic change.

So which should you go for: risk parity in stocks and bonds, which means an 18/82 mix or risk parity in bocks and stocks, which means 60/40 (stocks/bonds)? The answer is neither. Instead, you should be looking for a portfolio that optimises your risk and return, subject to the uncertainty you face. If the optimal portfolio turns out to exhibit risk parity, then well and good. But that's because it's the outcome, not because you started with equal risk shares as an objective.

Equal Sharpe ratios

If it's not equal risk shares, then what does explain risk parity's ability to target the tangent portfolio? The key lies in understanding the circumstances under which the risk parity and tangent portfolios coincide. A sufficient condition, when there are two asset classes, is that their Sharpe ratios are the same.[7] Exhibit 2 shows this to be approximately the case for the historical data on which we have built our frontier analysis. Stock and bond Sharpe ratios were 0.40 and 0.43, respectively, so the optimal portfolio was very close to the 18/82 mix recommended by risk parity. From 1990 to the present, the optimal portfolio was 14/86 – slightly farther from risk parity, but much closer to it than to the 60/40. So from one perspective, risk parity worked well in the past because it coincided with the tangent portfolio.

It is worth dwelling on what this means for uncertainty management. The risk parity rule got us to the tangent portfolio without requiring us to forecast each asset class's return. In this basic, two-asset-class case, all we needed was an estimate of each asset's return volatility. As the prospective performance of the plug-in rule amply demonstrated in Exhibits 5A and 5B, there is good reason to avoid forecasts, because they introduce significant errors. So the question we should be asking is: How far can we push avoiding forecasts? Specifically, what happens when Sharpe ratios are different? And what happens when there are more than two asset classes?

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7   With more asset classes, some conditions on correlations between asset class returns are required, the simplest of which is that the correlation between each pair of asset classes is the same (see the appendix).

Risk parity and correlation

Addressing these questions requires us to understand how risk parity investment rules deal with correlation. We distinguish two broad versions of risk parity among those we have encountered. Since we view the contribution of risk parity allocation rules to be the uncertainty management they offer, we are principally interested in their use of estimates and forecasts. Neither rule uses average returns or forecasts to approximate the tangent portfolio. One rule, which we call *quantitative risk parity* (RPN), uses historical estimates of correlation and volatility to solve for the list of allocations to each asset class that make risk shares equal.[8] The other, *Qualitative risk parity* (RPL),[9] only uses historical volatility estimates and builds an allocation from hypothesised responses of specific asset classes to 'surprises' in economic drivers of returns. Both need to resort to return forecasts to decide how much leverage to apply to go from the tangent portfolio to the optimal portfolio.

RPN uses a simple formula that is discussed in the appendix. RPL is somewhat more involved. It starts with the view that asset return volatility is dominated by unexpected changes in growth and inflation expectations. Portfolio construction requires understanding how each asset class return moves in response to these surprises. **Exhibit 11** lists the four possible combinations and how major asset classes would be mapped to them. For example, it's expected that equities rise in the event of a positive growth surprise and fall in the event of unexpected inflation.

EXHIBIT 11: ASSET RESPONSES TO GROWTH AND INFLATION

Representative asset class	Equities	Nominal bonds	Commodities	Inflation-linked bonds
Response type	1	2	3	4
Growth surprise up	+	−	+	−
Inflation surprise up	−	−	+	+

Source: J.P. Morgan Asset Management Strategy Group.

Exhibit 12 translates this information into portfolios designed to perform well in each of the four possible environments that result. Each asset class enters the allocation process rescaled to have a common volatility. So, for example, based on the figures in Exhibit 2, bonds and equities would be present in roughly a 5:1 ratio. Within each environment portfolio, these risk-scaled assets are equally weighted. A simple rationale would be that in the event of, say, an inflation shock, bonds and stocks are perfectly positively correlated − both would respond negatively − so any

EXHIBIT 12: TRANSLATION TO ENVIRONMENT-SPECIFIC PORTFOLIOS

		Economic factor	
		Growth	Inflation
Surprise	+	Equities, Commodities	Commodities, IL bonds
	−	Nominal bonds, IL bonds	Equities, Nominal bonds

Source: J.P. Morgan Asset Management Strategy Group.

allocation between them should only depend on relative volatility. Last, since positive and negative surprises are considered equally likely, and growth and inflation shocks are considered equally likely, each of the four environment portfolios gets an equal weight.

Exhibit 12 is the usual way that the RPL asset allocation is presented. Exhibit 11 expresses the same information, in terms of the more conventional correlation relationships among asset classes. It underpins the explicit factor model we use in the simulations below and is the way we make RPL comparable to other portfolio rules in a simulation environment.[10]

In this description of RPL, the only estimated parameters used are each asset's return volatility. Correlations and expected returns do not enter. The factor structure does imply something about correlation. For example, type 1 and type 4 assets will be negatively correlated, although it does not say to what degree. It makes no statements about the correlation between types 1 and 2, which will be negative in the event of a growth surprise and positive in the event of an inflation surprise, so the overall correlation will depend on whether the inflation or growth surprise dominates.

While it is a very neat trick to build a multi-asset portfolio without resorting to correlation or return forecasts, ignoring this information may or may not be sensible uncertainty management. We look at this next.

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[8] See, for example, the paper by Wai Lee listed in references.

[9] Developed by Bridgewater Associates.

[10] Basing correlations on the RPL factor model introduces a new dimension across which outcomes can vary: how much of the variation in returns the factors account for. In our simulations we set these factors shares of return volatility to 0 and 0.33 because higher factor shares result in degrees of leverage that did not seem realistic.

Risk parity versus the plug-in rule

We now expand the simulation framework we used earlier to the case of more than two asset classes. The goal is still to find out how much differences among asset class Sharpe ratios affect the performance of asset allocation rules. Specifically, how much divergence among Sharpe ratios is consistent with risk parity outperforming? We address this question by allowing Sharpe ratios to vary randomly across simulations.

Exhibit 13 is similar to Exhibits 8A and 8B except that we now have six asset classes instead of two, whose Sharpe ratios have been randomly chosen to lie between 0.2 and 0.3, so they are bracketed by a range of 0.1. There is one 'textbook recipe' portfolio, which would be picked by an investor who knew these true parameters. This portfolio levers the tangent portfolio by an amount that leads to a maximum of expected utility. Each of the 100 blue diamonds on the graph represents the investment performance of the plug-in rule in a particular simulation. Similarly, each of the 100 red triangles represents the investment performance of the Qualitative risk parity rule in a simulation. As before, in each simulation we generate at random a sample of asset returns on which estimates of the true parameters are based. Across these simulations the sample of returns changes (because it is simulated at random), causing the

estimated parameters to change, and the portfolio recommended by each rule to vary. What remains the same is the value of the true parameters and therefore what you would do if you were in the situation assumed by the textbook. Then you would know the underlying parameters, which are constant across the histories, and you would always choose the optimal portfolio.[11]

These results relate to a single set of determinants of returns:

- the share of idiosyncratic risk versus underlying economic factors

- the list of asset Sharpe ratios

- the number of observations in the sample (which relates to forecast uncertainty)

Exhibit 14 shows what happens when we change one of these drivers. Specifically, we allow the Sharpe ratios to span a range that at 0.4 is four times greater than the range in Exhibit 13. Now RPL performs worse than the plug-in rule. In general, when Sharpe ratios are tightly bunched it is better to use RPL, while the farther apart they get, the better is the plug-in rule's relative performance.

EXHIBIT 13: RISK PARITY AND PLUG-IN ATTEMPTING TO FIND THE TANGENT PORTFOLIO WITH SHARPE RATIO RANGE 0.2 TO 0.3
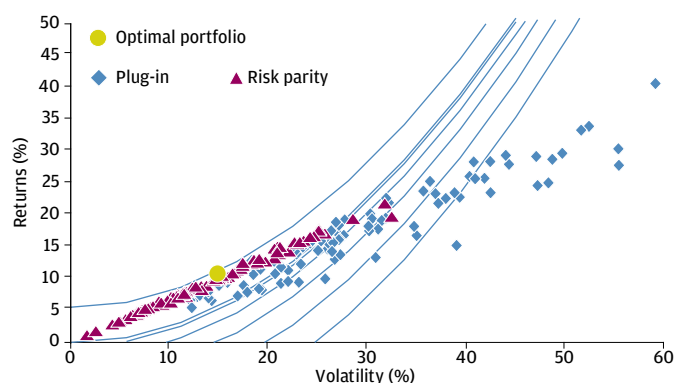
Source: J.P. Morgan Asset Management Strategy Group.

EXHIBIT 14: RISK PARITY AND PLUG-IN ATTEMPTING TO FIND THE TANGENT PORTFOLIO WITH WIDER SHARPE RATIO RANGE (0.4)
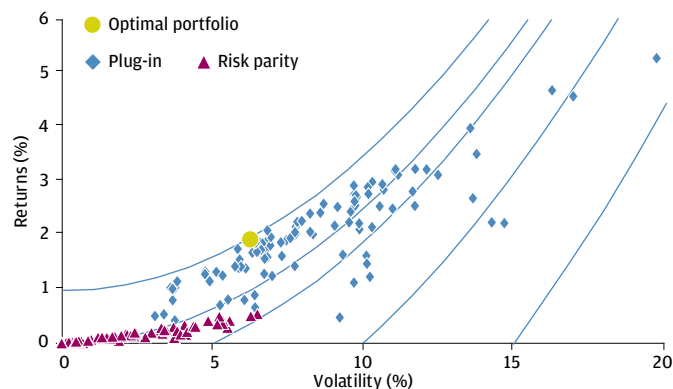
Source: J.P. Morgan Asset Management Strategy Group.

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[11] For the sake of clarity, we only show each rule's levered portfolios and omit the tangent portfolios.

This is illustrated in **Exhibit 15**, which aggregates the results of 2,520 contests like Exhibit 13 (or Exhibit 14), each corresponding to a different set of return drivers. Both RPN and RPL outperformed the plug-in method about 80% of the time. When the range of the Sharpe ratios of the six asset classes was less than 0.1, these figures rise to 100%. These outcomes conform to our uncertainty management perspective. Risk parity is a particularly good idea when Sharpe ratios are similar, in which case using forecasts is counterproductive.

So far, we have categorised simulation results according to their underlying range of asset Sharpe ratios. The other important dimension turns out to be the amount of confidence or uncertainty warranted by the forecasts. Neither plug-in nor risk parity uses uncertainty in its allocation decision, but performance will nevertheless vary depending on it. As a

practical matter, uncertainty is represented by the time span of the samples of data we use.[12] The longer the period the sample spans, the less it matters that the plug-in rule ignores the uncertainty. This trade-off between the range of Sharpe ratios and uncertainty is shown in **Exhibit 16**. As uncertainty increases (that is, as the sample size decreases), risk parity beats the plug-in rule more frequently. More uncertainty is required to achieve each success rate (50%, 70% or 90%) when the range of asset Sharpe ratios is larger. Only when the Sharpe ratios range is wide (>0.5), and the sample size exceeds 50, does the plug-in rule win in more than 50% of cases. (Put another way, with Sharpe ratios diverging by 0.1 to 0.2, risk parity should beat plug-in in 90% of trials at a sample size of 50, but only half the time at a sample size of 80. At the wide end of the Sharpe ratio spectrum, plug-in and risk parity come out even at the 50-observation sample size.)

EXHIBIT 15: WINNING PERCENTAGE COMPARISON OF RISK PARITY AND PLUG-IN METHODS BY SHARPE RATIO RANGE
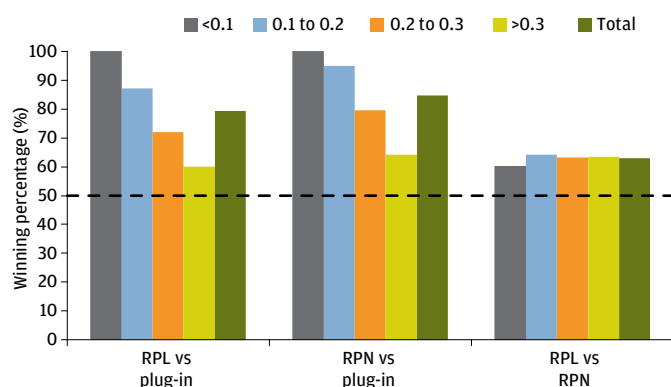
Source: J.P. Morgan Asset Management Strategy Group.

EXHIBIT 16: PERFORMANCE COMPARISON OF RISK PARITY AND PLUG-IN METHODS BY SHARPE RATIO RANGE AND SAMPLE SIZE
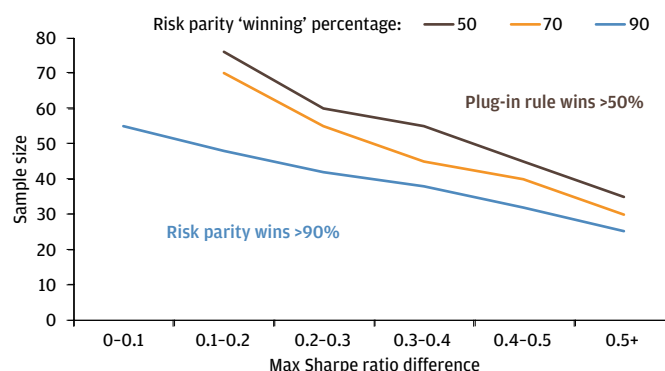
Source: J.P. Morgan Asset Management Strategy Group.

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[12] As we discuss on page 16, the (square root of the) sample time span corresponds to the ratio of the standard deviations of the forecast and asset returns. The forecast standard deviation is inferred from a confidence band around the forecast provided by the investor.
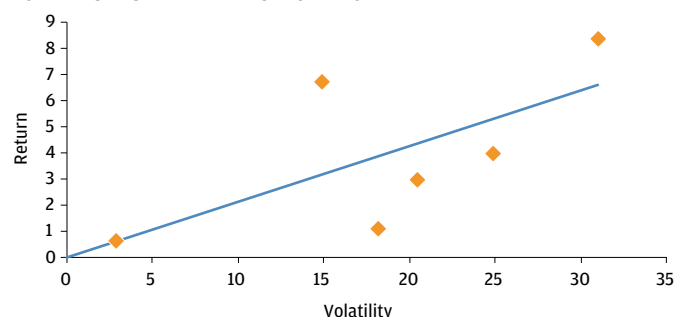
FORECAST HEDGING

These results suggest that it may not be sensible to opt for risk parity or the plug-in rule once and for all, since each works best under different circumstances. The closer asset Sharpe ratios are, the more sense there is in using risk parity.

What if the investor were to estimate the assets' Sharpe ratios first and then decide which rule to use based on how similar these estimates are? As long as estimated and actual Sharpe ratios are dispersed similarly, the fact that the actual Sharpe ratios aren't known may not be too material.

Exhibit 17 provides examples of what is involved. Since we are plotting expected return against volatility, any line through the origin represents combinations that have equal Sharpe ratios. Each dot represents the forecast return and volatility of one of the six asset classes in the simulation in question.

What we care about is how well the six points (one for each asset class) can be fit by such a line. The tighter the fit (for example, simulation 1), the closer are estimated Sharpe ratios, and the more we should lean toward the risk parity allocation. The worse the fit (for example, simulation 2), the more likely we are to benefit from taking account of differences in Sharpe ratios, and so the more we should lean toward the plug-in portfolio.

EXHIBIT 17

TIGHT FIT: SIGNALS CLOSE SHARPE RATIOS, AND TILTS PORTFOLIO TOWARDS RISK PARITY — SIMULATION 1

LOOSE FIT: SIGNALS DISPERSED SHARPE RATIOS, AND TILTS PORTFOLIO TOWARDS PLUG-IN — SIMULATION 2
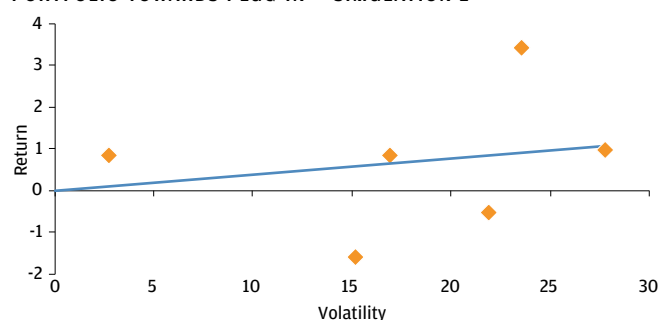
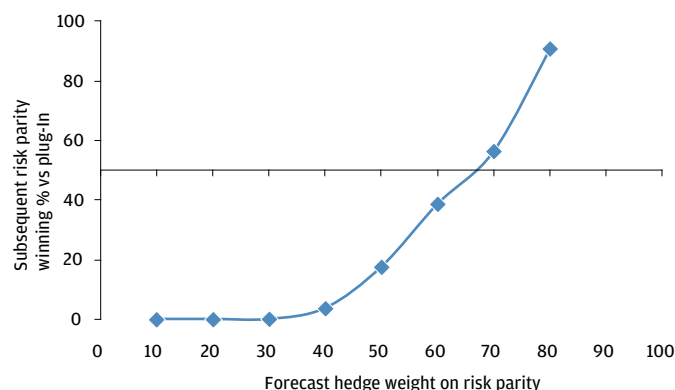Source: J.P. Morgan Asset Management Strategy Group.

The *forecast hedge* rule implements this principle. It combines risk parity and plug-in portfolios in proportions that depend on how tightly the estimated Sharpe ratios hug a straight line through the origin.[13] The key word here is estimated. The forecast hedge rule does not rely on actual Sharpe ratios; it uses a relationship among observed Sharpe ratios, which is a piece of information that both the plug-in rule and risk parity leave on the table. As we shall see, this makes a big difference.

The forecast hedge's choice between plug-in and risk parity rules has some predictive ability, as **Exhibit 18** documents. On the horizontal axis is the weight the Forecast hedge rule places on its risk parity leg. This is chosen as described above, on the basis of the sample data in each simulation. On the vertical axis is the frequency with which the risk parity rule beats the plug-in rule in the *subsequent* investment period. The points trace out a distinct upward slope, implying that the Forecast hedge is able, on average, to predict when risk parity will outperform and tilt towards it accordingly.

Exhibit 19 documents the relative performance of the Forecast hedge rule over the same simulations reported in Exhibit 15. Like the risk parity rules, the Forecast hedge outperforms the plug-in method, except it does so in every case we analyse. In addition, it outperforms both of the risk parity allocation rules: 71% of the time in the case of RPL and 84% of the time in the case of RPN. The risk parity rules turn in their best relative performance – and the Forecast hedge its weakest – when the Sharpe ratios of the six assets lie within a range of 0.1.

Exhibit 20 provides a more comprehensive picture of the relative success of RPL and the Forecast hedge. This time, it takes very close Sharpe ratios and high uncertainty for risk parity to 'win' more than 50% of the time. Only when the sample size falls below 20 years does this happen for Sharpe ratio ranges wider than 0.1.

EXHIBIT 18: FORECAST HEDGE WEIGHT 'PREDICTS' WHICH EXTREME WILL DO BETTER
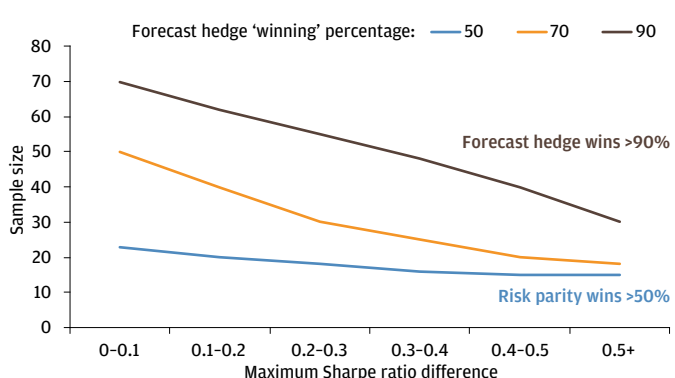

Source: J.P. Morgan Asset Management Strategy Group.

EXHIBIT 19: WINNING PERCENTAGE COMPARISON OF PLUG-IN AND FORECAST HEDGE METHODS BY SHARPE RATIO RANGE RISK PARITY
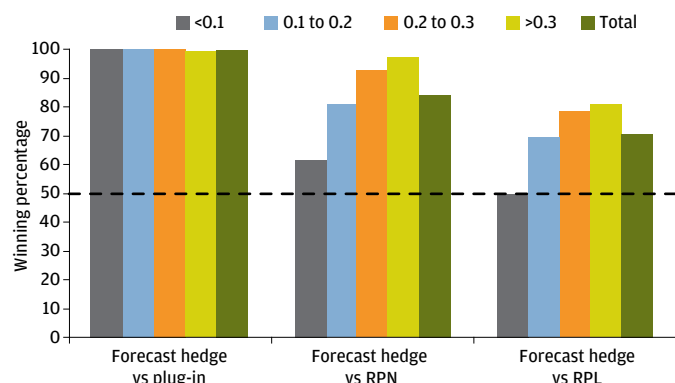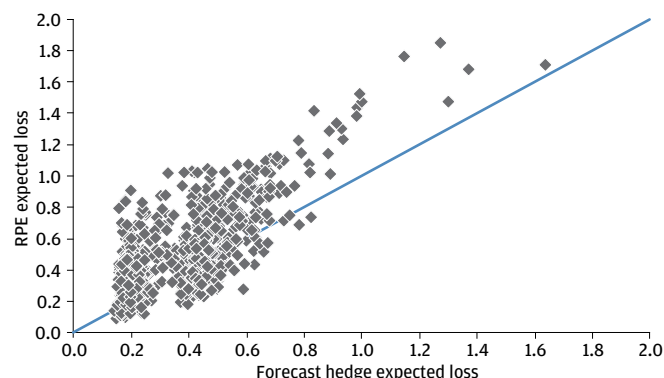

Source: J.P. Morgan Asset Management Strategy Group.

EXHIBIT 20: PERFORMANCE COMPARISON OF RISK PARITY AND FORECAST HEDGE METHODS BY SHARPE RATIO AND SAMPLE SIZE

Source: J.P. Morgan Asset Management Strategy Group.

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[13] In the appendix, we show how the relative weights are related to the statistic one would use to test the hypothesis that asset Sharpe ratios are the same.

EXHIBIT 21: FORECAST HEDGE VS QUALITATIVE RISK PARITY LOSS MAGNITUDE

Source: J.P. Morgan Asset Management Strategy Group.

Aggregating 'wins' across these contests in this way provides a very crude indication of which strategy to choose. However, it could be misleading if wins are insignificant and losses large. **Exhibit 21** shows that the relative efficiency of the Forecast hedge is also better than RPL's, its closest contender. Each dot charts the experience of the two portfolio rules under a particular randomly selected range of asset Sharpe ratios and a particular value of uncertainty. The graph charts each rule's *expected loss* in each scenario. Expected loss is the difference between the maximum attainable utility and the respective portfolio rule's utility (scaled by the maximum). So large values are bad – a value of zero means the rule performs the same as the optimum, and a value of one means it produces zero utility on average, which is to say it would be no better than cash. More points lie above the diagonal than below, consistent with the Forecast hedge's 70% outperformance. In addition, in the scenarios where the forecast hedge loses (those below the diagonal), the difference between the two rules is small relative to the scenarios where RPL loses (above the diagonal).[14]

Our results argue that investors should be using the forecast hedge rule for strategic asset allocation. Using the forecast hedge rule requires one more piece of information than the risk parity and plug-in rules. This is a measure of the investor's forecast uncertainty, which we now explain.

Forecast uncertainty

The focus of our simulation analysis has been the difference between allocating assets using the true mean return and doing so using the average return estimated from a simulated sample. The natural measure of the uncertainty of the estimated average return is the number of periods spanned by the sample, which we abbreviate as T. As T gets very large, forecast uncertainty approaches zero, and we approximate the ideal environment of Exhibit 3 – the sample provides the investor with so much information that amounts to knowing the true parameters. As T approaches zero, the sample average provides less and less information about the location of the true mean. This relationship is embodied in the fact that the standard deviation of an average is $1/\sqrt{T}$ multiplied by the standard deviation of the underlying data.[15] Accordingly, we define forecast uncertainty as the ratio of the standard deviations of the forecast and the underlying data.

To use the forecast hedge rule in real life, where forecasts are not explicitly derived from historical estimates, we need to be able to quantify the investor's forecast uncertainty. There is no single right way of doing this, but one simple approach is for investors to provide a band around each forecast, encompassing the range of true *average* returns they believe likely with, say, 95% confidence.[16] A 9.6% band around a 6% equity return forecast indicates that an investor thinks there is a 95% chance that true *average* annual equity returns could be as low as 1.2% or as high as 10.8%. If we assume a normal distribution for the true return around the forecast, this band is roughly four standard deviations wide, and so corresponds to a standard deviation for the forecast of roughly 2.4% (9.6/4) (**Exhibit 22**). If equity return volatility is 19.2% (as in Exhibit 2), the forecast uncertainty implied by the band is one-eighth (2.4/19.2 = 0.125).

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[14] While expected loss calculated this way is a common measure, scaling by optimum utility can be misleading when it differs widely across scenarios. Presumably, it's utility the investor cares about, not utility relative to the optimum, so losing five units of utility when the optimum is 25 is worse than losing 0.5 when the optimum is 2.5 units. However, our relative measure would treat the two as equal.

[15] When the observations are statistically independent of each other, and irrespective of whether they follow a normal distribution, the standard deviation of the sample average is $\sigma/\sqrt{T}$, where σ is the standard deviation of underlying returns.

[16] Used this way, the forecast hedge is an attempt to capture the same ideas as the Black-Litterman model. Instead of the market portfolio, we are using the risk parity allocation as the default for investors who have no particular view on returns, which we would equate with extreme uncertainty. We also attempt to limit the amount of information additional to forecasts that investors must supply and to couch it in an intuitive notion of uncertainty.

We can view this forecast uncertainty as an equivalent sample size, using the $1/\sqrt{T}$ relationship. A forecast uncertainty of one-eighth corresponds to a sample size of 64 observations. So a sample of 64 years of data would justify the confidence in the forecast described in the example, because it takes 64 years of sample observations to reduce the standard deviation of the sample average to one-eighth of that of the underlying returns.

EXHIBIT 22: DERIVING FORECAST UNCERTAINTY

	Value	Source
Forecast confidence band	1.2% : 10.8%	Supplied by investor
Width of band	9.6%	10.8 - 1.2
Implied forecast standard deviation	2.4%	9.6/4
Asset return standard deviation	19.2%	Exhibit 2
Uncertainty	1/8	2.4/19.2
Equivalent sample length	64 years	8^2

Source: J.P. Morgan Asset Management Strategy Group.

From this perspective, our simulations are the appropriate analysis of alternative asset allocation rules for an investor whose uncertainty corresponds to the sample size we use. To model the behaviour of a forecaster whose uncertainty is described by a 95% confidence band of 9.6%, when the asset return volatility is 19.2%, we would simulate a sample of 64 years of asset returns from the true parameters. We take as the investor's forecast the average asset return of this sample. This estimated average return will be distributed around the true average with a standard deviation equal to one-eighth of the standard deviation of asset returns. It mirrors exactly the forecast uncertainty of the investor we started with.

In all of this, we are taking the investors' reports of the level of their uncertainty on trust. It seems sensible for investors to go through a thought process of the form: 'Given my forecasts and my degree of uncertainty about them, what should my asset allocation look like?' Forecasters might complain that they are uncertain about their uncertainty, but it's hard to understand why this should excuse them from considering it. If they're uncertain about their forecast uncertainty, how can they be certain about their forecasts?

A REAL-WORLD EXAMPLE

To illustrate the forecast hedge rule, we compare the portfolios recommended by the various rules using the four asset classes widely discussed in the context of risk parity: stocks, nominal government bonds, inflation-protected bonds and commodities.

We use J.P. Morgan Asset Management's 2012 *Long-term Capital Market Return Assumptions* as return forecasts and annual historical returns since 1970 to calculate return volatility and correlation. The basic statistics are shown in **Exhibit 23**.

EXHIBIT 23: HISTORICAL DATA FOR FOUR ASSETS

Asset	Excess return estimate (%)	Vol (%)
US TIPS	1.7	8.5
US Gov't	0.0	5.6
S&P 500	7.7	15.7
Commodities	6.0	10.4

Source: J.P. Morgan Asset Management 2012 *Long-term Capital Market Return Assumptions*.

The weights in **Exhibit 24**, represent each rule's version of the tangent portfolio; that is, one that is fully invested in risky assets and whose goal is to maximise the *realised* Sharpe ratio. Each rule also leverages this portfolio in an attempt to maximise the investor's utility.

The plug-in allocation is skewed toward equity and commodities and avoids nominal bonds altogether. This is consistent with nominal bonds' low forecast returns and high correlation with TIPS, which provide higher returns. The plug-in rule advocates leveraging twice and anticipates a mean return of 11.5% and 15.2% volatility, implying a 0.66 Sharpe ratio. The gap between this anticipation and what might be realised has, of course, been the source of our concerns.

The two risk parity rules leverage to a similar degree, but are skewed heavily to fixed income, with a 65% total weight in the case of RPL. RPN has a somewhat higher allocation to commodities than RPL, which may result from its use of estimated correlations – commodities are a stronger diversifier than TIPS.

The forecast hedge rule charts a middle course, leaning more toward risk parity the more tightly asset returns exhibit the same Sharpe ratio. If we had no prohibition on short sales, the forecast hedge rule would put a 70% weight on the risk parity portfolio. The actual weights are in this ballpark. The forecast hedge leverages its version of the tangent portfolio less than the other portfolios because of the hedge it builds in against estimation errors.

EXHIBIT 24: PORTFOLIO ALLOCATIONS RESULTING FROM DIFFERENT METHODS

Legend: US inflation-linked bonds · US equity · Commodities · US nominal bonds

Method	US inflation-linked bonds	US equity	Commodities	US nominal bonds	Leverage (%)
Plug-in	19	28	54		203
RPN	22	14	27	38	200
RPL	26	14	21	39	177
Forecast hedge (high uncertainty)	13	17	37	32	119
Forecast hedge (low uncertainty)	18	22	44	17	172

Percent (x-axis: 0 to 100)
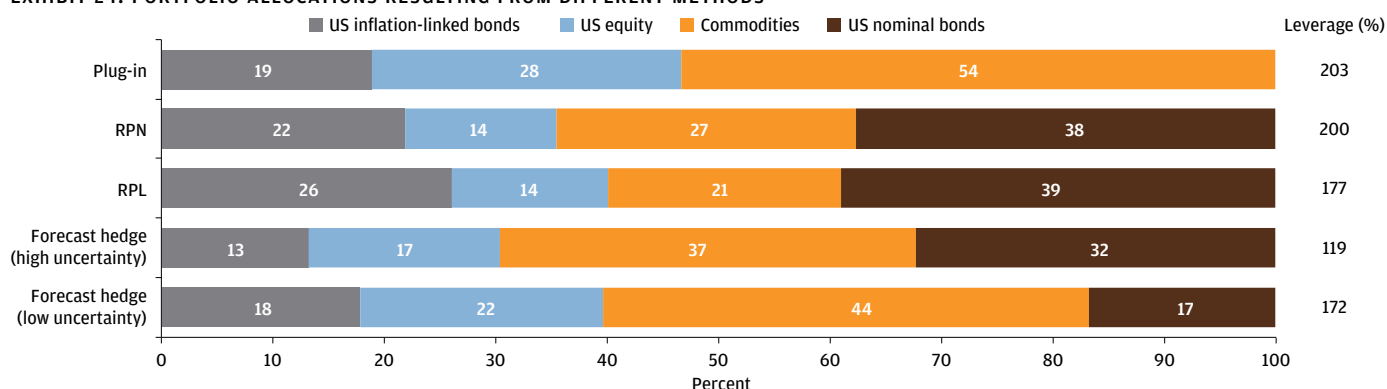
Source: J.P. Morgan Asset Management Strategy Group.

The 'low uncertainty' (high conviction) forecast hedge allocation in Exhibit 24, involves a band around the equity forecast of 9%, implying a forecast uncertainty of 0.14.[17] It occupies the middle ground between plug-in and risk parity, effectively replacing 17% of the plug-in rule's equity and commodity allocations with nominal bonds.

Say we are more uncertain. Specifically, say that we want to entertain the possibility that equities have another long period of near-zero excess returns and bonds have negative returns, and so we increase the confidence band around the equity forecast by two-thirds, from 9% to 15%. Forecast uncertainty therefore rises to 0.21. This makes no difference to the risk parity or plug-in allocations. However, it makes a substantial difference to the forecast hedge portfolio, which moves closer to risk parity. It maintains a significantly larger allocation to commodities, whose high Sharpe ratio and low correlation with equities are still visible even with this much uncertainty. The most significant effect of the increase in uncertainty is the decline in leverage to 119%, or three-fifths of the plug-in and risk parity levels. Evidently, investors who are uncertain enough about the future to entertain near-zero excess equity returns should also be worried about the amount of leverage they use.

The forecast hedge portfolios in this illustration may speak to current fears that accompany the large leveraged bond positions of risk parity portfolios—the forecast-proof risk parity portfolio maintains a 40% exposure to bonds when interest rates are at all-time lows. We stress that this is just an illustration and insist that the substantive case for the forecast hedge derives from our simulations. If the simulation evidence did not point to the strength of the forecast hedge, there would be little reason to consider it, irrespective of how its current prescription lined up with current concerns.

How do we judge among these portfolios? Quite simply each portfolio will look best when viewed under its own lens: plug-in under unvarnished forecasts, risk parity under equal Sharpe ratios and the forecast hedge under a blend. Again, the only way to reliably make any judgment is by using simulations across a wide range of possibilities as we did previously. We have learned from our simulations that the forecast hedge clearly comes out ahead on average, although it is no guarantee that the forecast hedge rule will outperform in every state of the world. The advantages we have documented may not be apparent over the span of months or even a few years. However, these advantages could be material when averaging over the long time horizons important to institutional investors.

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[17] For simplicity, for each asset class we apply the same proportionate amount of uncertainty. So, for example, the implied band for average excess bond returns would be 3.2% (ie (9%/15.7%)*5.6%). It is quite straightforward to accommodate different degrees of uncertainty for each asset class.

CONCLUSION

The big loser in this analysis of portfolio allocation rules is the conventional approach, or plug-in rule. Its lack of awareness to the problems of using forecasts causes it to come in last in just about every simulation we have run.

By the same token, the success of other rules, among them risk parity, comes from the way they manage uncertainty, consciously or not. Thus, risk parity's success occurs not because it assigns equal risk shares to asset classes, but because this translates into the best shortcut to the tangent portfolio in certain cases, namely when asset Sharpe ratios lie very close to each other. Outside this range, other approaches do better.

The forecast hedge asset allocation rule improves on both risk parity and the standard practice. It strikes a balance between the two extremes by measuring uncertainty surrounding forecasts and adjusting appropriately. Our analysis clearly recommends that strategic asset allocation should incorporate this forecast hedge approach.

A P P E N D I X

Utility function

We use a standard quadratic utility function as a way of combining the risk (σ^2) (measured by variance) and mean return (μ) of each portfolio into a single number:

$$U_0(p)=\mu_p-\tfrac{1}{2}\gamma.\sigma^2_p$$

Here, larger values of γ mean greater risk aversion on the investor's part. Utility is calculated both using perceived and 'true' return parameters in the body of the paper.

Returns relative to cash

We think of the investor as having USD 1 to invest in some combination of cash and the N risky assets. The investor chooses the vector of risky asset weights, w, to maximise

$$U_0=(1-\iota'w)r_f + w'\mu_0 - \tfrac{1}{2}\gamma w'\Sigma w,$$

where μ_0 is the vector of absolute asset returns, Σ is the covariance matrix of returns and ι is an N-vector of ones.

As long as the investor can borrow and lend at r_f this is the same as maximising utility by choosing only among the risky assets and substituting their excess returns over cash for their raw returns:

$$U(w)=w'\mu-\tfrac{1}{2}\gamma w'\Sigma w, \text{ where } \mu=\mu_0-r_f$$

All of the analysis in the paper is phrased in terms of excess returns.

Portfolios and leverage

Several of the portfolio rules discussed in the text stipulate the relative shares of each asset class, but not how much should be invested in money terms. For these rules, we determine the scaling of the designated portfolio as the leverage multiple that maximises utility.

So, if a rule designates relative shares of w_r $(\iota'w_r=1)$, then the leverage amount, λ, is the solution to

$$\underset{\lambda}{\text{Max }} \lambda\mu_r-\tfrac{1}{2}\gamma\lambda^2\sigma^2_r,$$

where $\mu_r=w_r'\mu$ and $\sigma^2_r=w_r'\Sigma w_r$

The solution to this problem is

$$\lambda^*=\mu_r/(\gamma\sigma^2_r)$$

So leverage depends on both mean returns and return covariances, even if the asset shares do not. This equation also raises the question of where investors get the information to calculate λ^*. They do not know the true parameters and so have to use estimates of the parameters.

This calculation of leverage also works for rules that use expected returns. Now the relative shares portfolio is the tangent portfolio, subject to the constraint that the weights sum to one. Plugging the mean and variance of this portfolio into the last equation gives the optimal leverage. The dependence on estimated parameters is the same as before.

Calibrating risk aversion

Comparing levels of utility across rules requires that we stipulate a level for the risk aversion parameter, γ. We back this out from anecdotal evidence that risk parity portfolios with a 20/80 equity bond split are leveraged approximately twice. The mean and volatility of the 20/80 unlevered portfolio, the amount of leverage applied to it and the shape of the utility function in combination all entail a level of risk aversion.

- The slope of the line along which leveraged 20/80 portfolios lie is $\mu_{20/80}/\sigma_{20/80}$.

- The slope of an indifference curve (on a graph of mean return against volatility) is $\sigma\gamma$.

- If the optimal portfolio's volatility is σ^*, then $\lambda=\sigma^*/\sigma_{20/80}$.

Plugging in $\lambda=2$, $\mu_{20/80}$ = 3%, and $\sigma_{20/80}$ = 5.4%, γ is then 0.05, which is the value we use. Our results turn out to be not greatly sensitive to changes in γ.

Risk parity allocation rules

Conditions for quantitative risk parity to coincide with the optimal portfolio

The principal result here is that if Sharpe ratios are equal and correlations between asset classes are the same, then RPN coincides with the tangent portfolio.

The RPN weight for asset ι is solved from
$w_i \Sigma w / w' \Sigma w = \iota/N$.[1]

If S is a diagonal matrix with assets' volatilities on the diagonal, then $h = S^{-1}\mu$ is the vector of Sharpe ratios, equal to ιh_0 if they are all equal to h_0.

If R denotes the correlation matrix of asset returns, then $R\iota = \iota m$ for some constant, m, includes all correlations being the same as a special case, and, equivalently, $\iota/m = R^{-1}\iota$

Noting that $\Sigma^{-1} = S^{-1}R^{-1}S^{-1}$, and that the tangent portfolio weights are

$w_t = \Sigma^{-1}\mu / \iota'\Sigma^{-1}\mu$.

The Sharpe ratio and correlation conditions imply that

$w_t = S^{-1}\iota / \iota'S^{-1}\iota$

which is proportionate to the inverse of the volatility of each asset. Plugging our two assumptions into the RPN formula yields the same expression.

Every portfolio satisfies equal risk shares

$\mathcal{A}$ is a list of N assets, from which an arbitrary portfolio, **P**, is constructed. **P**'s weights are described by the N-vector **a**. $\mathcal{B}$ is another list of N assets, each built by combining $\mathcal{A}$ assets in some way. The matrix **D** catalogues the composition of $\mathcal{B}$ assets in terms of $\mathcal{A}$ assets: element d_{ij} is the amount of $\mathcal{A}_j$ that goes into $\mathcal{B}_i$. The goal is to show that we can always find a **D** such that **P** satisfies risk parity in terms of $\mathcal{B}$ assets. Without loss of generality, we set the amount of each $\mathcal{B}$ asset in the new portfolio to be 1. Then, if

(1) $\iota'D = a'$,

the $\mathcal{B}$ portfolio exactly exhausts the $\mathcal{A}$ assets in **P**. In other words, the jth column of **D** sums to the jth element of **a**.

If the covariance matrix of the $\mathcal{A}$ assets is Σ, then the covariance matrix of the $\mathcal{B}$ assets is $D\Sigma D'$. Consequently, equal risk shares from the $\mathcal{B}$-perspective is the condition that

$ID\Sigma D'\iota / \iota'D\Sigma D'\iota = \iota/N$

Here, the ι on the right of numerator and denominator represents the unit weights of the $\mathcal{B}$ assets.

From (1), this is

(2) $Dv = \iota/N$, where $v = \Sigma a / a'\Sigma a$, which is known.

Conditions (1) and (2) represent 2*N linear equations in the N^2 elements of D. One solution is the somewhat silly one in which each $\mathcal{B}$ asset is identical, taking a 1/Nth share of each $\mathcal{A}$ asset, ie $D_s = \iota a'/N$. Any other solution D_o can be built from D_s by noting that the difference between the two, say Δ, must satisfy $\iota'\Delta = 0'$ and $\Delta v = 0$. Pick any (N-1)x(N-1) matrix, C. Define v_1 and ι_1 as the first N-1 elements of v and ι, respectively, and v_N as the N-th element of v. $\Delta v = 0$ implies $Cv_1 + xv_N = 0$ for some N-1 vector, x. So, $x = Cv_1/v_N$.
A similar argument for $\iota'\Delta = 0'$ results in:

$$\Delta = \begin{bmatrix} \square & -\dfrac{\square v_1}{\square_N} \\ -\square_1\square & \dfrac{\square_1\square v_1}{\square_N} \end{bmatrix}$$

To enforce that the $\mathcal{B}$ assets involve no short positions in the $\mathcal{A}$ assets

(3) $d_{ij} > 0,\ i,\ j = 1,...,N$

note that D_s is positive as long as **a** is, so simply scale Δ so that its most negative element is smaller in absolute value that the smallest element of **a**.

In the example in the text, $N=2$ and $D_s = \begin{bmatrix} 0.3 & 0.2 \\ 0.3 & 0.2 \end{bmatrix}$

$a = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, $\Sigma = \begin{bmatrix} 369 & 7 \\ 7 & 18 \end{bmatrix}$, from which we calculate that
$v = \begin{bmatrix} 1.6 \\ 0.08 \end{bmatrix}$.

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[1] See Lee.

So, $\Delta = d * \begin{vmatrix} 1 & -19.7 \\ -1 & 19.7 \end{vmatrix}$, where we are free to choose the value of d, which we set equal to 0.01 to get to the example in the text: {31% stocks and roughly zero bonds} and {29% stocks, roughly 40% bonds}.

Qualitative risk parity asset allocation

We first adjust each asset class so that units of the adjusted asset classes all have the same return volatility. So, for example, using the data in **Exhibit A1**, if we standardise so that a unit of adjusted US government bonds costs USD 1, a unit of TIPS costs USD 0.6588 (USD 5.6/8.5), which is to say that USD 0.6588 of TIPS buys the same volatility as USD 1 of US government bonds.

EXHIBIT A1

Asset	Volatility (σ_i)	Adjusted notational (1/σ_i)	Growth surprise +	Growth surprise −	Inflation surprise +	Inflation surprise −	Scaled adjusted notional 1/σ_i/Σ(1/σ_i)
US TIPS	8.5%	0.66			.0825	.0825	26%
US Gov't	5.6%	1		.125		.125	39%
S&P 500	15.7%	0.36	.045			.045	14%
Commodities	10.4%	0.54	.0575		.0675		21%

Source: J.P. Morgan Asset Management Strategy Group.

Under qualitative risk parity, each of the four subportfolios gets an equal (25%) allocation of adjusted notional. In this example, each contains two asset classes, which in turn receive equal shares of this 25% (ie 1/8). The notional amounts in each subportfolio need not be the same, as the goal is to equalise their contribution to risk. So, for example, USD 0.045 of stocks and USD 0.0675 of commodities is viewed as having the same risk as USD 0.0825 of TIPS and USD 0.125 of government bonds. From the more conventional standpoint, this is consistent with the case where, for example stocks and commodities are perfectly correlated, when there is a growth surprise. The unconditional correlation across assets is determined by the relative frequency of the four surprise combinations, which is not specified by the qualitative risk parity framework.

In this symmetric case, the allocation comes out to be the same as quantitative risk parity with constant correlation, ie asset shares are proportional to $1/\sigma_i$. Since each of the subportfolio notional amounts is one-eighth of each asset's adjusted notional amount, normalising all the asset weights to sum to one gives the same asset shares as the equal-correlation quantitative risk parity recipe.

To expand these ideas to more than four asset classes, as we do in our simulation analysis, we assume that

▪ Every asset class can be placed in one of the four categories in Exhibit 11.

▪ Each of the four buckets receives equal shares of risk-adjusted assets (ie 25%)

The main change from the symmetric four-asset case above is that it is now possible that two or more assets duplicate one of the patterns of response to surprises. In this case, more than two asset classes can turn up in the same subportfolio, whose 25% we accordingly split equally among the constituent asset classes.

Factor model of correlation

The RPL framework outlined above succeeds in producing a portfolio allocation without making explicit quantitative statements about the correlation between asset returns. The other models we examine all require correlation data. To compare performance, we need to subject all the allocation rules to the same return conditions. The simplest way to proceed is to induce a correlation structure out of the RPL framework. This will require one additional parameter to be set in our simulations.

We assume that the N asset returns in a given period, scaled by their respective volatility, (r_t) are driven by two unobservable factors, (f_t) and a random shock (ε_t):

$$r_t = f_t B + \varepsilon_t$$

Here, r and ε are 1xN vectors, and f is a 1x2 vector of 'surprises' to the two factors. B is a 2xN matrix of factor loadings.

The covariance matrix of r is the asset correlation matrix we are after:

$$Er'r = R = B'\Sigma_f B + \Sigma_\varepsilon$$

In order for this decomposition into factors to have any content, we require that the idiosyncratic shocks be uncorrelated, so their covariance matrix (Σ_ε) is diagonal. We also assume that the covariance matrix of the factors (Σ_f) is diagonal. With this assumption, we can go from the response pattern of asset classes to factors specified by RPL to a correlation structure. The assumption that surprises are uncorrelated may or may not be true if the factors are inflation and GDP growth. For quarterly US real GDP and core inflation since 1958, the correlation of

surprises from regressions on four lags of each of the two variables is 0.05, so the zero correlation assumption is not a bad one. It is immaterial whether we think in terms of surprises to GDP and inflation or to those variables scaled by their volatility, so we can assume Σ_f is the identity matrix. Now the factor model is:

$$R=B'B+\Sigma_\varepsilon$$

The single thing we need to specify numerically is the share of idiosyncratic risk in the total variance described by this equation. For example, for the first asset, we have

$$1=b_{21}^2+b_{11}^2+\sigma^2_{\varepsilon 1}$$

If we assume that $\sigma^2_{\varepsilon 1}$ = 0.8, then $b_{11}^2+b_{21}^2$ = 0.2. We simulate b_{11}^2 from a uniform [0, 0.2] distribution, and b_{21}^2 is then b_{21}^2 = 0.2 – b_{11}^2. We can then use the sign pattern of asset 1's factor responses dictated by the RPL model (Exhibit 9 in the text) to set b_{11} and b_{21}. We proceed in the same way for the other N-1 assets, using the same assumed share of idiosyncratic variance. In this way, we can build up the matrix B, and then R, for the simulation in question.

Simulation model overview

Exhibit A2 describes the conventional asset return simulation framework that starts from a set of true parameters known by the investigator and unknown to the investment rules examined by the simulation. Our goal here is to examine the robustness of asset allocation rules across a wide variety of parameters. So we have an 'outer layer' of simulations in which this conventional framework is imbedded. The outer layer simulates across different permutations of asset Sharpe ratios and correlation patterns, to derive a set of the parameters that are the inputs for a standard Exhibit 3-type simulation.

EXHIBIT A2: CONVENTIONAL ASSET RETURN SIMULATION FRAMEWORK – RELATIONSHIP BETWEEN INNER AND OUTER LAYERS
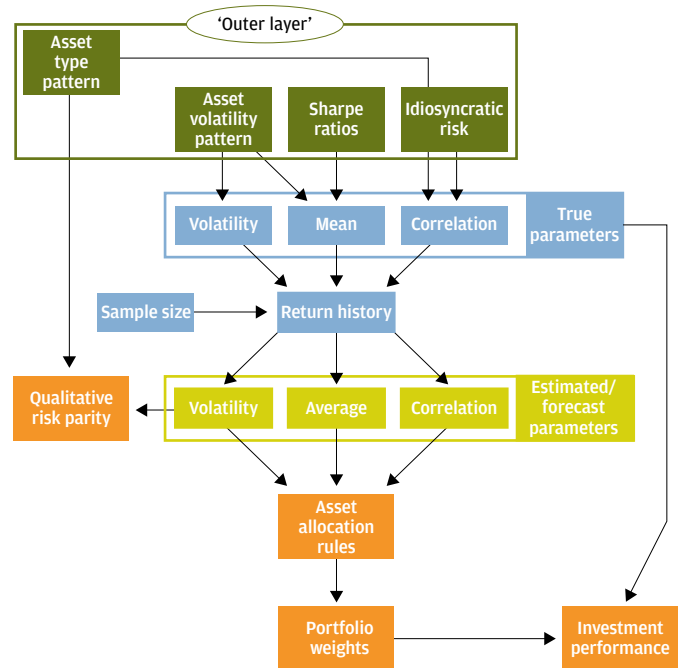
Source: J.P. Morgan Asset Management Strategy Group.

The inputs to an outer layer simulation are –

A. Asset factor response: Each of the N hypothetical asset classes is independently and randomly assigned to one of the four types of factor responses detailed in Exhibit 11 in the text. As long as N>4, at least one type will be duplicated. There is the possibility that there will be no asset classes of some type, since the N types are selected randomly and independently.

B. Share of idiosyncratic risk in asset return correlation: same value for all asset classes.

C. Asset return volatility: We set a lower and upper bound and space the other N-2 volatilities symmetrically between these two limits. Then the volatilities are randomly assigned to the assets.

D. Asset Sharpe ratios: We set bounds eg [0,0.3], or [-0.2, 0.6] and set the N asset Sharpe ratios as N random draws from these intervals. Specifying the intervals allows us to focus on Sharpe ratios in a reasonable range. We throw out any simulation where the average of the N true Sharpe ratios is negative.

Steps A and B give us asset return correlations and C and D give us volatilities and mean returns. We use two values of the share of idiosyncratic risk (0 and 0.33) and nine intervals from which we sample asset Sharpe ratios 20 times each. This results in a total of 360 outer layer simulations of parameters that are input to the inner layer.

The only parameter left to specify in an inner layer simulation is the number of time series observations comprising the sample of returns from which estimates of average returns and covariances are calculated (to input in the various allocation rules). We look at sample sizes of 15, 20, 30, 40, 50, 60 and 70 observations. In each case, we simulate 100 return samples. The numbers we report are averages over these 100 samples. So, we run 252,000 (360 x 7 x 100) samples in all, but the figures reported in, say, Exhibit 13 in the text, span 2,520 (360*7) averages.

Forecast hedge asset allocation rule

The forecast hedge rule is a hybrid of a number of asset allocation proposals that have appeared in the finance literature over the last twenty years. The earliest version is the 'empirical Bayesian' model proposed by Jorion, which starts with a prior distribution over the unknown mean return vector and covariance matrix, and derives the minimum risk (in the Bayesian sense) rule that blends information from historical data and the prior. The paper by Kan and Zhou gets to a very similar result by finding the utility-maximising combination of the plug-in portfolio and minimum variance portfolio. When there are no short-sale (or other inequality) constraints on portfolio weights, the resulting optimal portfolios are in both cases linear combinations of the two underlying portfolios. In both cases, the optimisation problem can be reinterpreted as one based on particular estimates of the mean return and covariance matrix. These estimates are themselves weighted averages of the means and covariances that would make, respectively, the plug-in portfolio and minimum variance portfolio optimal. The weights reflect the amount of confidence in those extreme cases that the data justify (statistically).

The forecast hedge differs from these two by using a risk parity model, instead of the minimum variance portfolio as the alternative to the plug-in rule. It also uses the relatively simple covariance matrix advocated by Kan and Zhou and the formulation of the mean return used by Jorion. These two parameters are the inputs to the optimisation we run for each sample. The resulting forecast hedge allocation is just a weighted average of the plug-in and '$1/\sigma$' allocations in the absence of constraints. Since we impose a no-short-position constraint, our forecast hedge solutions to the optimisation problem do not necessarily satisfy this condition.

The forecast hedge is thus the result of choosing an optimal (utility-maximising) portfolio, subject to a no-short-sales constraint, using the following inputs:

Estimated covariance matrix:

We use $\dfrac{T(T-2)}{(T-N-1)\,(T-N-4)}\,\widehat{\Sigma}$

where $\widehat{\Sigma}$ is the standard estimator of the covariance matrix.[1] The scaling factor exceeds one, so its effect is to scale back leverage, or tilt the portfolio away from risky assets if the no-short-sales constraint is binding.

Expected Returns:

Define m as the investor's forecast of average returns (an Nx1 vector), and s as the Nx1 vector of estimated standard deviations. Define b_{RP} as the (generalised least squares) regression coefficient of m on s. That is

$$b_{RP} = \frac{s'\widehat{\Sigma}^{-1}m}{s'\widehat{\Sigma}^{-1}s}\,,$$

and the fitted part of this regression is $b_{RP}s$. Then the expected return that goes into the forecast hedge optimisation is

$$m_{UH} = (1-\omega)m + \omega b_{RP}s,$$

where ω is a weight between zero and one, related to the regression involving b_{RP}.[2] This regression, illustrated in Exhibit 17 in the text, will fit well if asset Sharpe ratios are the same.

Jorion's formula for ω, which is the one we use, is usually expressed as

$$\omega = \frac{N+2}{N+2+T(m-b_{RP}s)'\widehat{\Sigma}^{-1}(m-b_{RP}s)}$$

which is less than intuitive.

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[1] See Kan and Zhou, p. 636.

[2] See Kan and Zhou, p. 644.

If we ignore the '2' in numerator and denominator and divide both by N, we get

$$\omega \approx \cfrac{1}{1+\cfrac{(m-b_{RP}s)'(\widehat{\Sigma}/T)^{-1}(m-b_{RP}s)}{N}}$$

The term in the denominator is akin to a test statistic for the hypothesis that asset Sharpe ratios are the same. If this hypothesis is true, then $b_{RP}s$ should be a good estimate of the vector of mean returns, and the sum of squared residuals from the corresponding regression, $(m-b_{RP}s)'(m-b_{RP}s)$ should be small, relative to a measure of residual variance that does not impose this hypothesis. One candidate for this is $\widehat{\Sigma}$. The i-th diagonal element of $\widehat{\Sigma}/T$ is the best estimate of the variance of the i-th forecast return under the assumption that there is no relationship among assets' mean returns. The intuition is cleanest if we assume that $\widehat{\Sigma}$ is diagonal. Then

$$\frac{(m-b_{RP}s)'(\widehat{\Sigma}/T)^{-1}(m-b_{RP}s)}{N}$$ is the average of ratios

(one for each asset class) of squared residuals under the equal-Sharpe-ratios hypothesis to squared ratios under an alternative hypothesis. It will be larger, the worse the equal-Sharpe-ratios hypothesis fits, in which case ω, the weight on the risk parity allocation will be smaller.

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FOR FURTHER INFORMATION

For further insight on this topic, look for a related publication, due out in October from our Global Multi-Asset Group: 'Diversification – still the only free lunch? Alternative building blocks for risk parity portfolios' by Yazann Romahi, head of Quantitative Portfolio Strategies and Katherine Santiago, portfolio manager.

Please contact your J.P. Morgan Asset Management client advisor for additional information.

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