Near-Optimal Network Design with Selfish Agents

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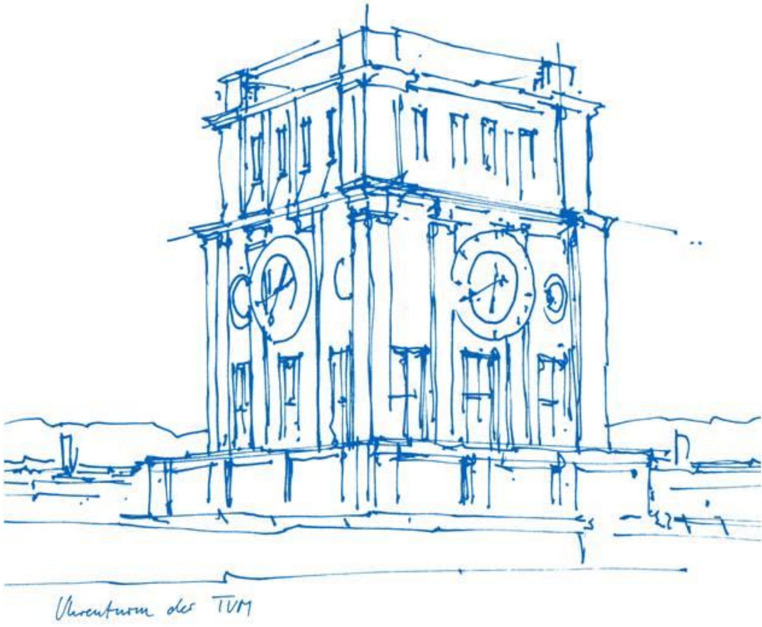
Seminar: Selected Topics in Efficient Algorithms

Informatik Department



Motivations

- Networks are usually administrated by many **agents** that acts in a **non-cooperative** way
  - Internet
  - Transportation Networks
  - Social Networks
  - …
- Often these agents express a limited number of requirements to fulfill

Thus to study the phenomenon of **network creation** :

$$Selfish\ Approach > Centralized\ Approach$$

More Realistic!

Algorithmic Game Theory

- The problem is modelled with a **game**
  - Each player has a set of **rules** that he has to follow
  - Let p_i to be the **strategy** adopted by the player i during the game: we define the **cost function** $c : \mathcal{P} \rightarrow \mathcal{R}_+$ to be the function that assign a value to each valid strategy
  - Each player seek to $\min c(p_i)$

Set of valid strategies

Nash Equilibrium

We say that the system (game) is in the state $p = (p_1, p_2, \ldots, p_N)$ called Nash Equilibrium when **no one** of the players involved has an **incentive to deviate** from his payments.

I.E.: If $\forall j : j \neq i \quad p_j$ is fixed, then p_i is the best possible strategy for player i

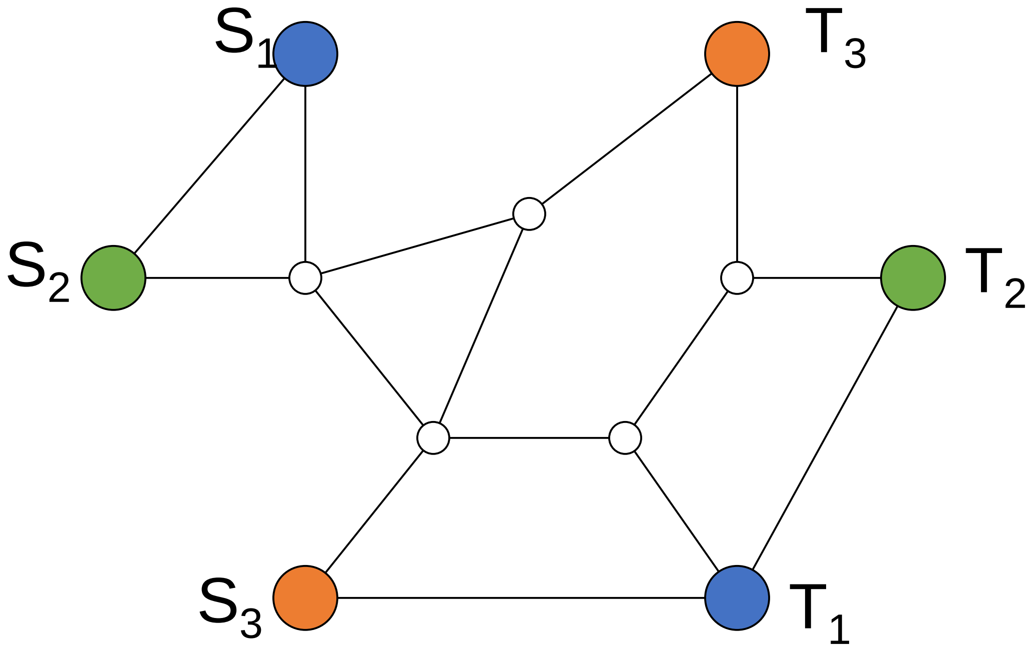
The Connection Game

- We can then adapt the previously introduced concepts to our scenario and define the following **connection game**:
  - Let N be the numbers of players and $G = (V, E)$ an undirected graph w/ non-negative weighted edges $\forall e \in E, c(e) > 0$
  - Each player i has a set of **terminals** nodes S_i that he **needs to connect** $(S_i \cap S_j \neq 0)$
  - The strategy of player i is a payment function $p_i : E \to R_+$, such that $p_i(e)$ is how much he contributes to edge e
  - Edge e is considered **bought** if $\sum_{i=1}^{N} p_i(e) \geq c(e)$ and call G_p the graph composed by all the bought edges (hence the cost of an edge can be **shared** between agents!)
  - A player that does not connect all of his terminals occurs in an infinite penality, otherwise it simply pays the sum of his contributions:

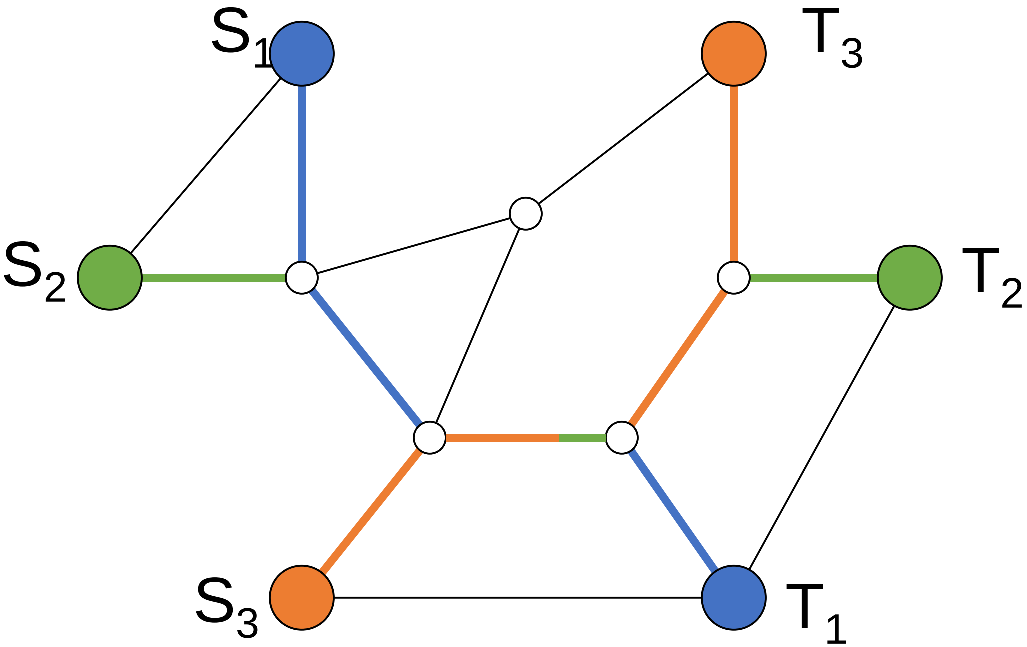
$$\sum_{e \in E} p_i(e)$$

which seeks to **minimize**

An Example



An Example



Some Properties of the Connection Game

- Finding the centralized optimum of the connection game corresponds to the classical network problem of finding the **minimum generalized Steiner Tree**
  - **Note:** It is not always the case that the optimum is reached by the game (later)

- Let $p = (p_1, p_2, \ldots, p_n)$ be a **NE** over an instance of our defined game, then we have:

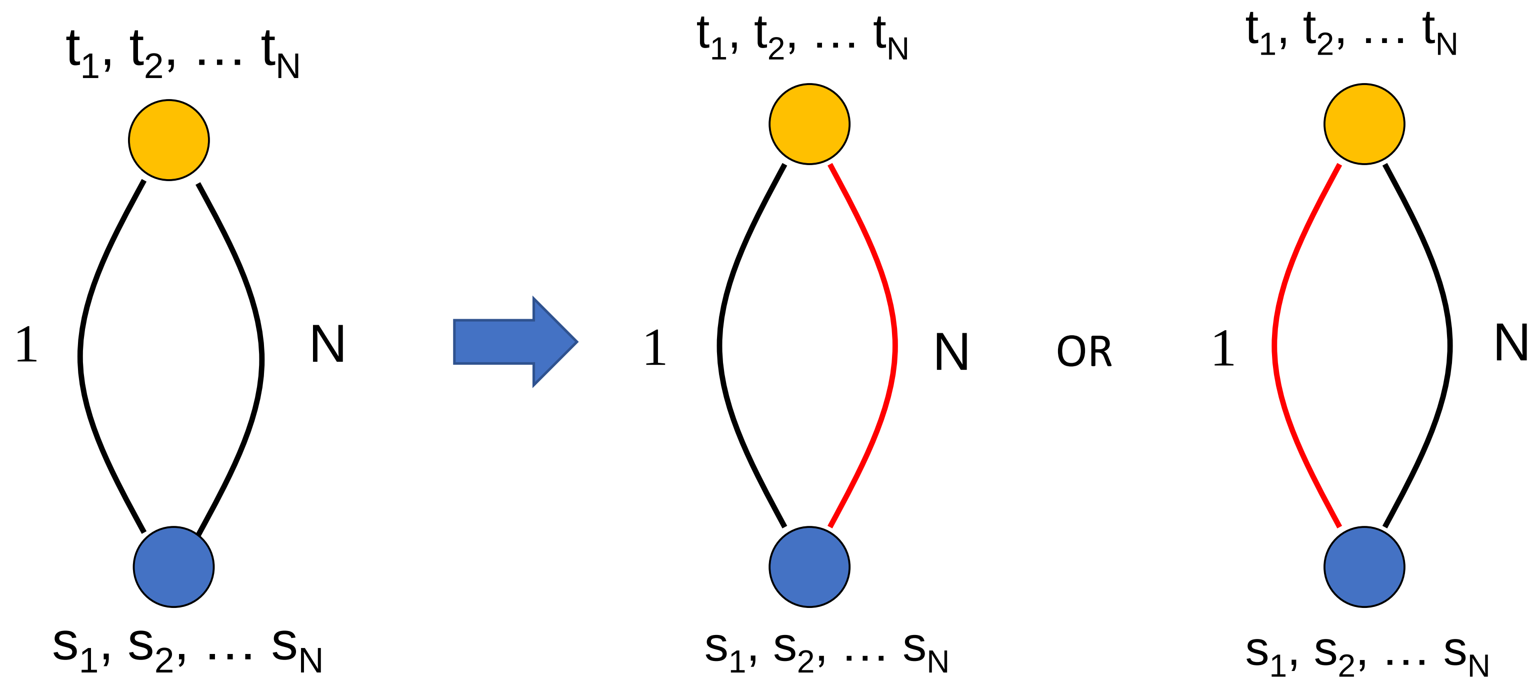
  1. That G_p is a forest (If not then $\exists$ a cycle)

  2. That, let T_i be the minimum spanning Tree over his terminals then i contributes to and only to the edges of T_i (If i stops paying for edges outside T_i, S_i still connected)

  3. Each edge is either fully paid or not at all (Trivial)

- What else can we say about Nash Equilibria? For instance, can there be more than one equilibrium for a given graph?

Another Example



Two Nash Equilibria:

$$\forall i : \sum_{e \in E} p_i(e) = 1$$

Social cost = N

$$\forall i : \sum_{e \in E} p_i(e) = \frac{1}{N}$$

Social cost = 1 = **Cent. Optimum**

Price of Anarchy & Price of Stability

- To underline the gap which can arise between the best centralized solution and the worst NE, researchers introduced the term **Price of Anarchy** which is defined

$$POA := \frac{Worst\ NE}{Central\ Optimum}$$

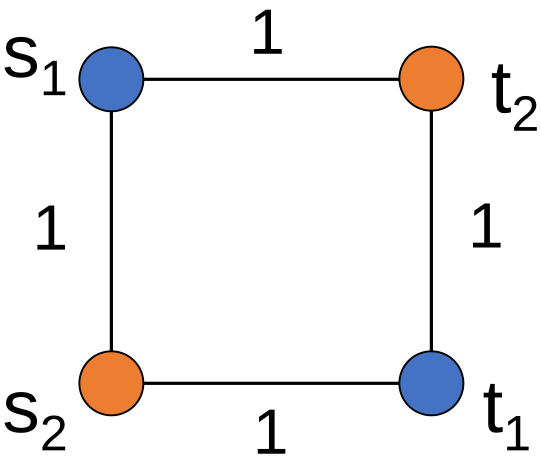
- In their work, instead, the authors focus on the complementary concept, which they call **Price of Stability** defined as

$$POS := \frac{Best\ NE}{Central\ Optimum}$$

Lemma: For the Connection Game: $POA \leq N$

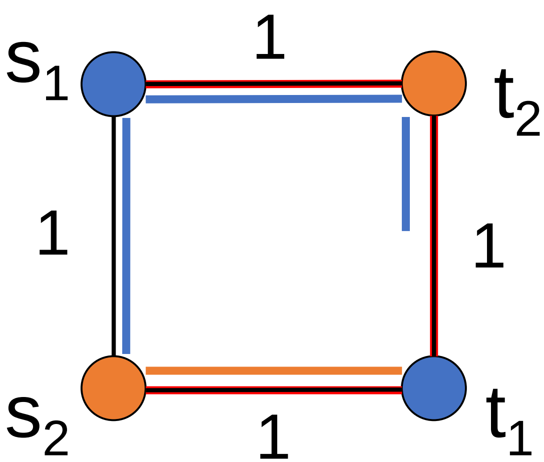
Pf: Assume $POA > N$, then by the pigeon hole principle there is a player i that pays more than the cost of the optimal solution G_{opt}. But then i could deviate by purchasing the whole G_{opt}, which surely satisfies the connection requirement and, as assumed, would reduce his total payments $\Rightarrow\Leftarrow$

NE may not Exist!



- Assume that a NE exists. Then w.l.o.g., by previous forest property let $G_p = \{(s_1,t_2),(t_2,t_1),(t_1,s_2)\}$

  - (s_1,t_2) can be paid only by agent 1, likewise (t_2,t_1) can only be paid by agent 2.
    What about edge (t_2,t_1) ?

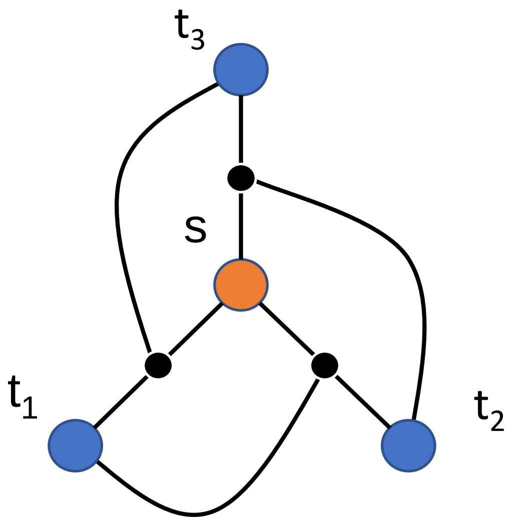


  - **No one would ever paid for it**: by the sake of cont. let agent one start contributing to it, but then..

  - He would prefer to **deviate to** (s_1,s_2) since it allows him to connect his terminals paying 1.
    Hence no equilibria at all in this graph.

SSG

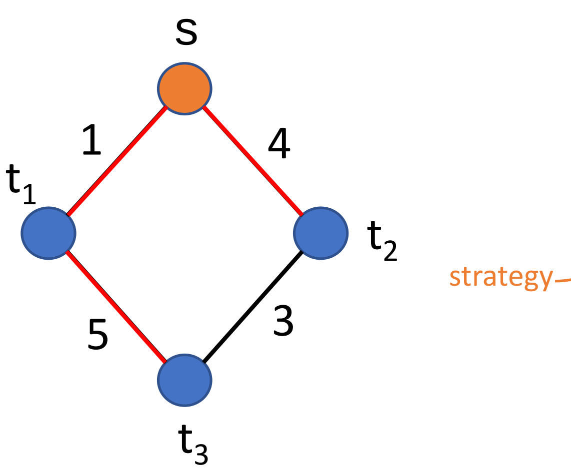
A ***Single Source Game*** is a game (network) in which all players share a **common terminal** S, and in addition, each player has exactly **one other terminal** t_i



- We show that every SSG has a NE of POS=1
  - I.e. we need to find a strategy $p = (p_1, p_2, \ldots, p_N)$ s.t. p is a NE and $G_p = T^*$. Where T^* is the minimum cost Steiner tree for the network

Minimum Spanning Tree

- **Special Case:** every node is a terminal

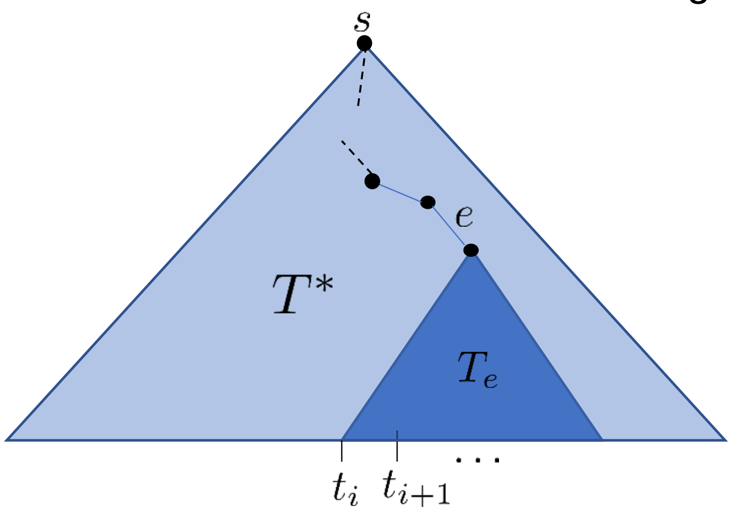


- $T^* = MST$
- Each player pays only for the edge upon him that lies in T^*, then $G_p = T^*$
- G_p is a Nash Equilibrium: since if agent i wants to deviate then it means that there exist an alternative path from t_i to s cheaper than the one in T^*, cont.

- Does this work for any SSG?

A Procedure for p (notation & idea)

- Assume T^* given rooted at s
- Let T_e be the subtree of T^* disconnected from s when edge e is removed



- Let $p_i(T^*) = \sum_{e \in T^*} p_i(e)$
- Let $p(e) = \sum_{i=1}^{n} p_i(e)$

- We present a **procedure to determine payment strategies** p by considering edges in reverse BFS and by paying for the whole T_e before paying for e

- $p_i(e)$ is **upper bounded** by the cost that i would face if deviates in the final solution:
  - $c'(e) = p_i(e) \ \forall e \in T^*$
  - $c'(e) = c(e) \ \forall e \in G \setminus T^*$

A Procedure for p

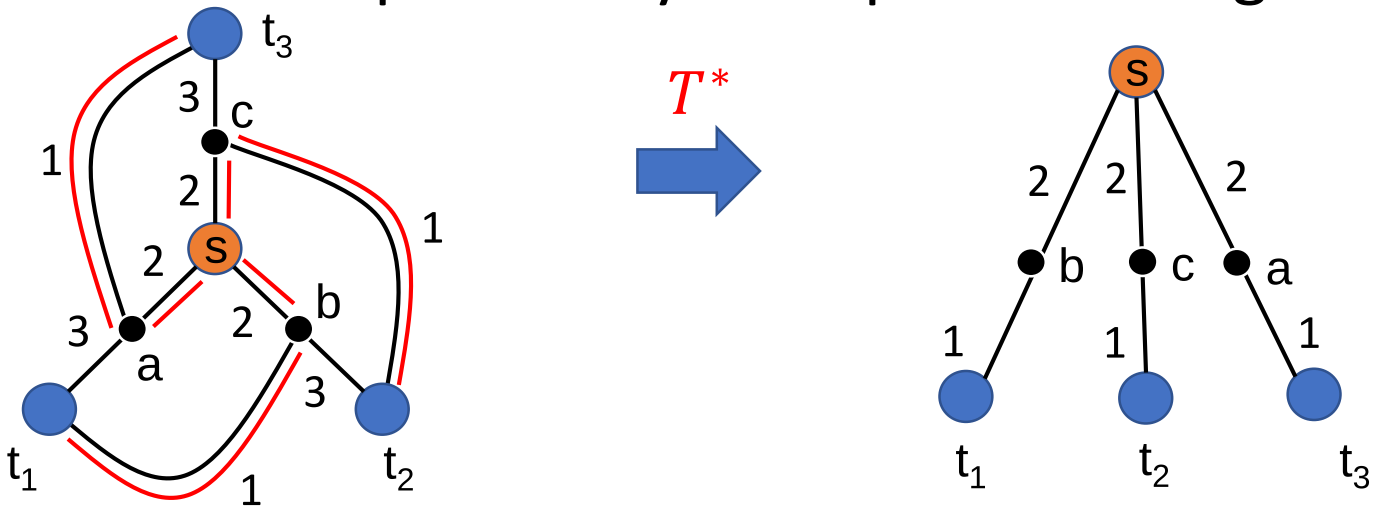
- Initialize $p_i(e) = 0$ for all players and edges
- $\boxed{\text{Loop through all edges } e \text{ in } T^*}$ in reverse BFS order
  - Loop through all players i with $t_i \in T_e$ until e paid for
    - If e is a cut in G set $p_i(e) = c(e)$
    - Else let χ_i be the cost of the cheapest path A from s to t_i in $G \setminus \{e\}$ under the modified costs,
    - Set $\boxed{p_i(e) = \min\{\chi_i - p_i(T^*), c(e) - p(e)\}}$

- **Remark1 :** we never allow player i to contribute so much to e that his total payments exceed the cost of connecting s to i

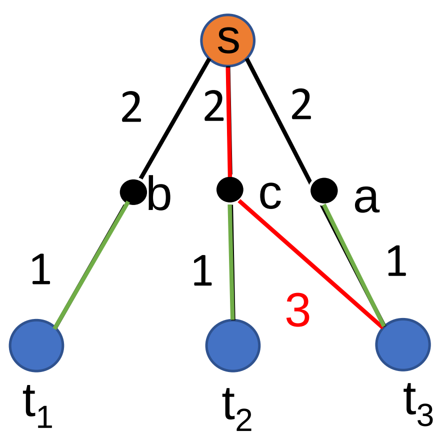
- **Remark2 :** the algorithm constructs payments only to edges in T^*

Example

- Consider the previously example with weighted edges:



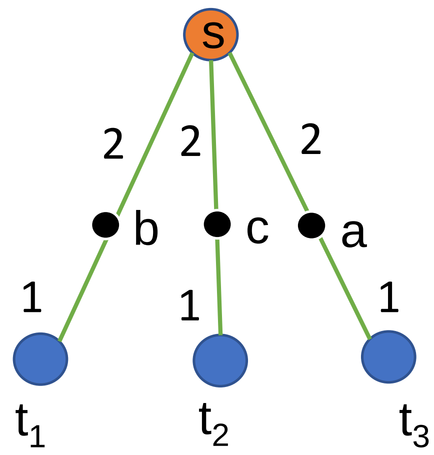
Rev BFS: $e = (t_3, a)$, only t_3 contributes to the edge, not a cut hence compute cheapest alternative path in $G \setminus e$ of cost χ_i under modified cost function c'



- $c'(t_3, c) = 3 \;\; ; \;\; c'(c, s) = 0$
  then $p_3(a, t_3) = \min\{3 - 0, 1\} = 1$
- Hence player 3 fully pays for e
- The same applies to each edge on top of t_i

Example (2)

Now $e = (a, s)$ not a cut set, $T_e = (t_3, a)$ still only player 3 contributes to the edge, the shortest path remains equal in $G \setminus \{(a, s)\}$, thus $p_3(a, S) = \min\{3 - 1, 2\} = 2$. Again, player 3 does buy the edge. The same argument is extended to the other players



- Therefore $G_p = T^*$. Is p a NE? Yes, since a player deviates if and only if there is a shorter path from the terminal to s. Which does not exist by minimality of T^*
- Indeed, **this is true for any SSG**

- **Claim:** The set of payments p produced by the algorithm is a Nash Equilibrium.

- Consider the payment of a player i to edge e. The cost function $c'(e)$ reflects how much would pay if he deviates in the final solution on e: χ_i

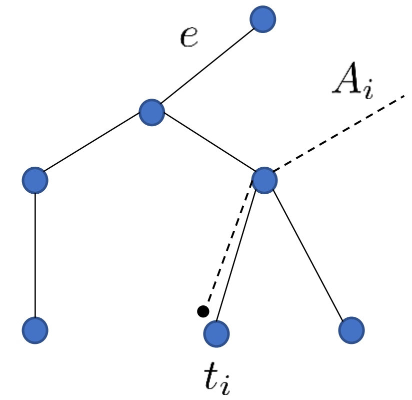
- However, it is the case that $p_i(e) \leq \chi_i - p_i(T_e)$. This is true for any $e \in T^*$, thus $p_i(T^*) \leq \chi_i$: i.e. we never allow e to pay more than χ_i

- Therefore It is never in player i's interest to deviate, this is true for all players and edges

NO! We need to ensure that $G_p = T^*$

To **guarantee connection requirements**

Theorem: the algorithm fully pays for T^*



Idea: Assume by cont. that an edge $e \in T^*$ not completely paid. Than each of the contributors of e has an alt. path which explains why he didn't pay $c(e) - p(e)$

We can use a **carefully selected subset of these paths to construct a tree** which spans all the terminals and the origin and which is **cheaper than** T^*

Proof: Let A_i be the alternative path for player i of cost χ_i. If more of such paths exist, choose the one that has **more ancestors of** t_i **in** T_e .

We need the following result on the structure of A_i

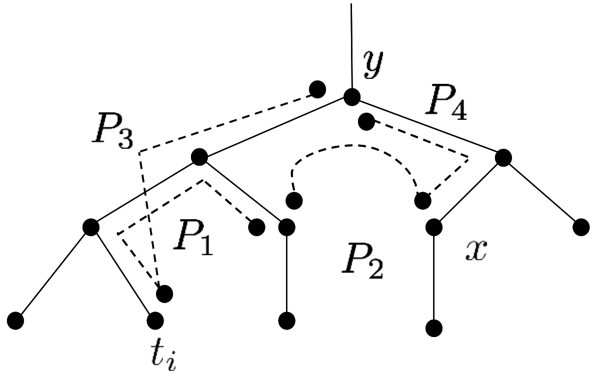
The Structure of A_i

Lemma: Let A_i be one of such paths. Then there are $v, w \in V$ s.t.:

- $t_i A_i v \in T_e$
- $v A_i w \in G \setminus T^*$
- $w A_i s \in T^* \setminus T_e$

Proof: Assume not, then it has to come back into T_e : call the node on the intersection x. Let $y = lca(t_i, x)$ and define: P_1 the first part of A_i in T_e and P_2 the part in G that lands into x. Moreover define $P_3 := t_i T_e y$ and $P_4 := y T_e x$.

W.m.a that P_3 lies over P_1 (othw trivial proof) as showed in the figure:

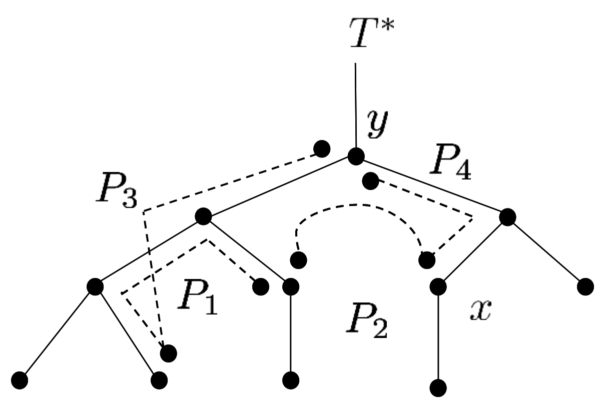


To prove the lemma we need to show that $c'(P_3 \cup P_4) \leq c'(P_1 \cup P_2)$:

- By definition of c' : $c'(P_4) = 0$

- Consider i payments along P_3 : every time $p_i(e) \leq \chi_i - p_i(T_e)$ hence, as shown $c'(P_3) = p_i(P_3) \leq p_i(T_e) \leq \chi_i \leq c'(P_1 \cup P_2 \cup yT^*)$ since it cant cost less than the shortest path

- Since $c'(yT^*) = 0$ we have that $c'(P_3 \cup P_4) = c'(P_3) \leq c'(P_1 \cup P_2)$



- Since $P_1 \cup P_2$ has less ancestors in T_e than $P_3 \cup P_4$ it cant be part of A_i. QED

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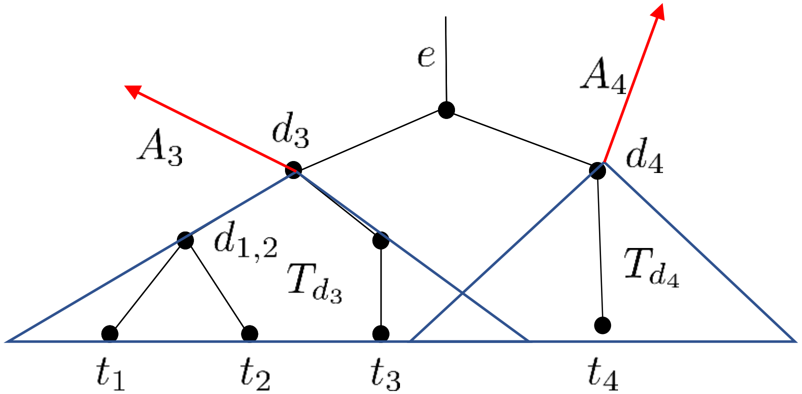
The algorithm fully pays for T^*

- We had supposed that $p(e) < c(e)$
- We show how to rewire T_e to $T^* \setminus T_e$ w/o decreasing agents payments, contradicting minimality of T^*

- For each player i in T_e we call the **deviation point** d_i the highest ancestor of t_i in A_i.
- Let D be the set of deviation points that do not have any other deviation point as their ancestors.

- All players associated w/ a deviation point in D, **deviates** to A_i while the others keep paying the same edges.



- **No players increases his payments**

- Define T_{d_i} the tree rooted at d_i if $d_i \in D$

The algorithm fully pays for T^*

- Consider the Tree composed by every T_{d_i} and T^*, connected by the respective alternative paths. Clearly this tree spans through all terminals

- It remains to **show that this tree is fully bought**:

  - By previous lemma, alternative paths do not re-enter T_e so their cost is fully bought by their respective deviating players

  - Consider any edge $f \in T_{d_i}$. By definition of D, and of how we designed our tree, no terminal of T_{d_i} deviates inside, hence every edge is fully bought.

We have then found a cheaper tree than T^* that connects all terminals $\Rightarrow\Leftarrow$

Approximation

- Finding the minimum cost Steiner Tree T^* is an NP problem
- Our construction, even if mathematically ensure us the existence of NE with POS=1 is **infeasible in practice**

Thm: Given an α-approximated min Steiner Tree T for any $\varepsilon > 0$ there is a polynomial time algorithm which computes a $(1+\varepsilon)$-approximated NE over T' s.t. $c(T') < c(T)$

- **Proof idea:** use the same argument introduced in the last theorem to find an approx. NE over a cheaper tree than T

Summary & Extensions

- We have shown how some of the real world networks are, due to their nature, preferably modeled using a selfish approach instead of a classical one

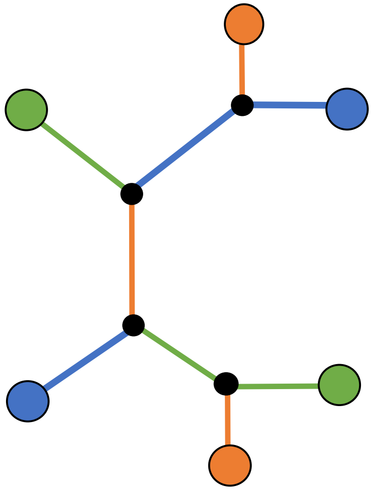
  - Focusing on a specific model: The **Connection Game,** we proved that for some specific instances, namely, the **Single Source Games,** we can always expect the existence of an equilibrium whose cost is **equal to optimal centralized solution**

  - Moreover there exists a computationally tractable procedure that outputs a $(1+\varepsilon)$-approximated NE on a **near optimal centralized solution**

- All these results can be proven to be true for the **directed version** of the SSG and also when each player **would rather stay disconnected** from the source if his payments exceed a given $\max(i)$

Other Extensions? Multi-Source Games

- Authors gives also results for the **general case** of **many sources** and **more than one terminals**



- **Since as shown**, in this scenario NE existence is not guaranteed, we need again to look for approximations, in particular:
  - There exists a 3-approximate NE which costs is **equal to OPT**
  - There exists a poly time algorithm that finds a $(4.65 + \varepsilon)$-approx. NE that costs **at most 2-OTP**

Questions?

Thank for your attention!