Ornstein-Uhlenbeck process

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1 Numerical *simulation* of the OU stochastic process

Let's consider the following stochastic differential equation that defines a Ornstein-Uhlenbeck (OU) process $x(t)$, whose (steady-state) mean and variance are μ and σ^2, respectively:

$$\tau \frac{dx}{dt} = \mu - x + \sigma\sqrt{2\tau}\xi(t)$$

Abusing the notation of ordinary differential equations, $\xi(t)$ is a *Gaussian white-noise* characterised by zero mean, unitary variance, and Dirac's Delta autocorrelation, with Gaussian probability density distribution as follows:

$< \xi(t) >= 0 \quad < \xi(t)\xi(t+T) >= \delta(T) \quad p_\xi(Z) = (\sqrt{2\pi})^{-1}e^{-Z^2/2}$.

$x(t)$ is then a Gauss-distributed non-stationary process, fully characterised by its time-varying mean $m_x(t)$, variance $s(t)$ and autocovariance. The probability density distribution is then $p_x(X) = (\sqrt{2\pi s(t)^2})^{-1}e^{-(X-m(t))^2/(2s(t)^2)}$, with

$$m(t) \ = < x(t) > = \ x_0 e^{-(t-t_0)/\tau} + \mu\left(1 - e^{-(t-t_0)/\tau}\right) \longrightarrow \mu$$

$$s(t)^2 \ = < [x(t) - m(t)]^2 > = \ \sigma^2\left(1 - e^{-2(t-t_0)/\tau}\right) \longrightarrow \sigma^2$$

where x0 indicates the (deterministic) initial condition of x(t) at time t_0, and the autocovariance

$$Cov_x(t,T) \ = < [x(t) - m(t)]\,[x(t+T) - m(t+T)] > = \ \sigma^2 e^{-|T|/\tau}(1 - e^{-2(t-t_0)/\tau})) \longrightarrow \sigma^2 e^{-|T|/\tau}$$

Numerically simulating the continuous-time process $x(t)$ requires iterating the following expression

$$y(t+\Delta t) \approx (1 - \Delta t/\tau)\ y(t) + \mu\Delta t/\tau + \sigma\sqrt{2\Delta t/\tau}\,\hat{\xi}$$

where the (integration) time step Δt must be *small enough*, and where at every iteration $\hat{\xi}$ is generated as a new realisation of a pseudo-random zero mean, unitary variance, uncorrelated Gauss-distributed number:

$$< \hat{\xi} >= 0 \quad < \hat{\xi}_k\hat{\xi}_m >= \delta_{k\ m} \quad p_{\hat{\xi}}(Z) = (\sqrt{2\pi})^{-1}e^{-Z^2/2}$$

with $\delta_{k\ m}$ being the Kronecker's delta.

However, an alternative exact update rule is also available (Gillespie, 1992) that is valid for any finite value of Δt:

$$y(t+\Delta t) = e^{-\Delta t/\tau}\, y(t) + \mu\left(1 - e^{-\Delta t/\tau}\right) + \sigma\sqrt{1 - e^{-2\Delta t/\tau}}\,\hat{\xi}$$

It is easy to prove that, in the limit of small Δt, such an exact update rule reduces to the approximate expression above.

In Julia, one can easily generate a realisation of this OU process, containing N points, by three different style of code:

```
In [89]: # Let's first define the numerical parameters of our numerical example

         μ  = 50.;       # Steady-state mean of x(t)
         σ  = 6.;        # Steady-state standard deviation of x(t)
         τ  = 2.;        # Relaxation time [ms]

         x0 = 0.;        # (Deterministic) initial condition

         Δt = 0.02;      # Discretization time interval [ms]

         N  = 10000000; # No. of samples to generate (beware of transients if x0 is

In [90]: # Let's preallocate the memory for three vectors, which will be used for t

         x1    = zeros(Float64, N);    # Let's preallocate the memory for x1 - note
         x2    = zeros(Float64, N);    # Let's preallocate the memory for x2 - note
         x3    = zeros(Float64, N);    # Let's preallocate the memory for x3 - note
```

The first method does not exploit the true potential of Julia for numerical computation, as no *function* is defined.

```
In [91]: # The first method (slowest) is based on common-sense scripting

         srand(359);     # Set a (fixed) random number seed - for reproducibility pu

         tmp1 = exp(-Δt/τ);                   # Useful constant to slighlty reduce th
         tmp2 = σ * sqrt(1-exp(-2*Δt/τ));     # Useful constant to slighlty reduce th
         tmp3 = μ * (1 - exp(-Δt/τ));         # Useful constant to slighlty reduce th

         x1[1] = x0;                          # We take care of the initial condition

         @time for i=2:N,
             x1[i] = x1[i-1] * tmp1 + tmp3 + tmp2 * randn(); # exact iterative rule
         end

  6.015551 seconds (110.00 M allocations: 1.937 GB, 6.32% gc time)
```

The second method defines a function, calls it with *minimal size* to have it compiled, and then fires it up with full-sized problem.

```
In [92]: # The second method (~30-50 times fastester) is based on Julia's just-in-t

         # We first define a function, which changes one of its input argument (fo
         function simulate_ou!(x, N, x0, Δt, μ, σ, τ)
             tmp1 = exp(-Δt/τ);                    # Useful to slighlty reduce the n.
             tmp2 = σ * sqrt(1-exp(-2*Δt/τ));  # Useful to slighlty reduce the n.
             tmp3 = μ * (1 - exp(-Δt/τ));       # Useful to slighlty reduce the n.

             x[1] = x0;                            # We take care of the initial condi
             for i=2:N,
                 x[i] = x[i-1] * tmp1 + tmp3 + tmp2 * randn();
             end
         end

         # We call the function once, with N = 2, so that it is compiled once for a
         simulate_ou!(x2, 2, x0, Δt, μ, σ, τ);

         # We then run the real call to generate N samples
         srand(359);     # Set a (fixed) random number seed - for reproducibility pu

         @time simulate_ou!(x2, N, x0, Δt, μ, σ, τ);

  0.127130 seconds (4 allocations: 160 bytes)
```

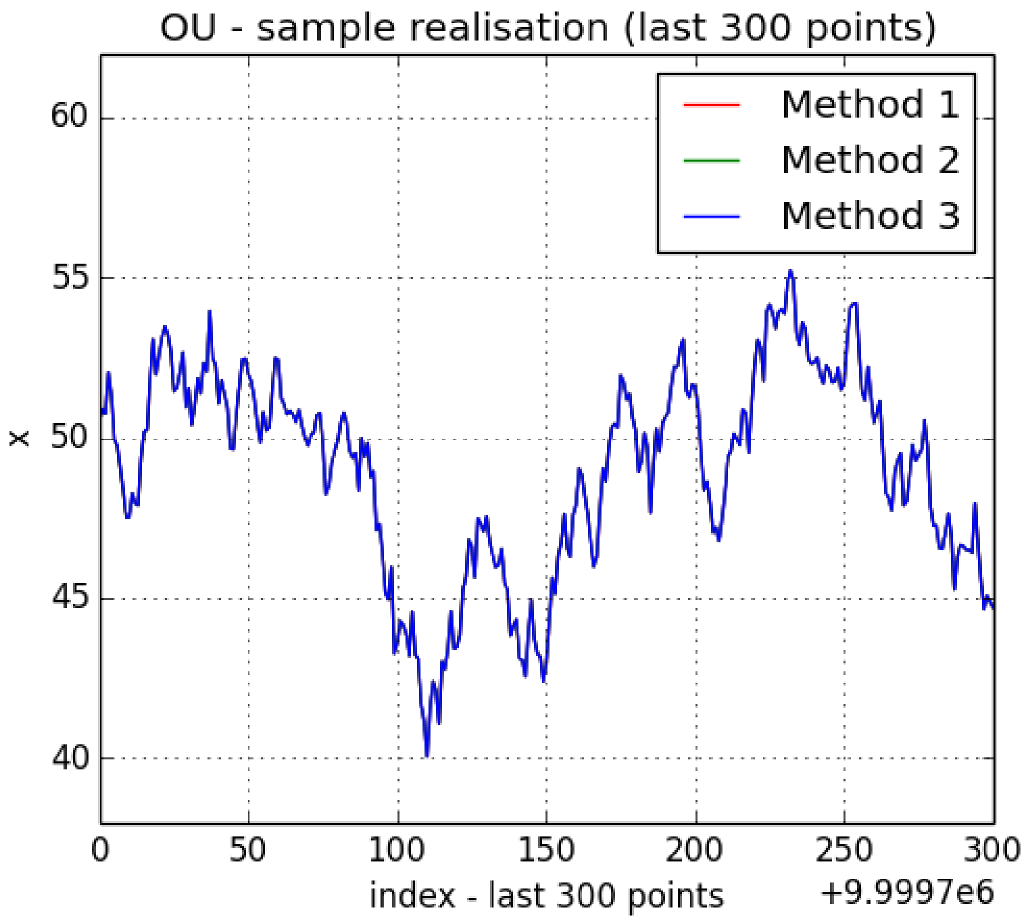
The third method is the vectorised version of the first and employs the function *filt()*. It consumes more memory and it is slower.

```
In [93]: # The third method (slightly slower) is based on Julia's own DSP and filte

         srand(359);     # Set a (fixed) random number seed - for reproducibility pu

         tmp1 = exp(-Δt/τ);                         # Useful to slighlty reduce the n. of
         tmp2 = σ * sqrt(1. - exp(-2*Δt/τ)); # Useful to slighlty reduce the n. of
         tmp3 = μ * (1. - exp(-Δt/τ));          # Useful to slighlty reduce the n. of

         b  = [0, 1];           # Refer to the documentation of "filt!()" - these ar
         a  = [1, -tmp1];       # to efficiently compute a discrete-time iterative e

         @time filt!(x3, b, a, tmp3 + tmp2 * randn(N), [x0]);

  0.310261 seconds (13 allocations: 228.882 MB, 10.60% gc time)
```

By virtue of the fixed random number seed initialisation, produces exactly the same sequence of numerical values, as confirmed below (apart from very minor rounding errors).

```
In [94]: # Let's see whether the numerical sequences are identical (result should b
test1 = 1./N * sqrt(sum(x1.*x1 - x2.*x2));
test2 = 1./N * sqrt(sum(x1.*x1 - x3.*x3));
test3 = 1./N * sqrt(sum(x2.*x2 - x3.*x3));

println("The mean square error comparing x1 and x2 is $test1");
println("The mean square error comparing x1 and x3 is $test2");
println("The mean square error comparing x2 and x3 is $test3");

using PyPlot
fig = figure("Sample realisation",figsize=(6,5)); # Create a new figure, w

index = (N-300):N;
plot(index, x1[index], "r", label="Method 1");
plot(index, x2[index], "g", label="Method 2");
plot(index, x3[index], "b", label="Method 3");

ax = gca()                                          # Get a "handle" on the cur
ax[:set_ylim]((μ-2*σ),(μ+2*σ));

xlabel("index - last 300 points");# Label of the horizontal axis
ylabel("x");                                        # Label of the vertical axi
title("OU - sample realisation (last 300 points)"); # Label of the title,
grid("on");                                         # "Grid" on

legend(borderaxespad=0.5);                          # Add a legend to the curre
```

```
The mean square error comparing x1 and x2 is 0.0
The mean square error comparing x1 and x3 is 1.7638370652718947e-9
The mean square error comparing x2 and x3 is 1.7638370652718947e-9
```

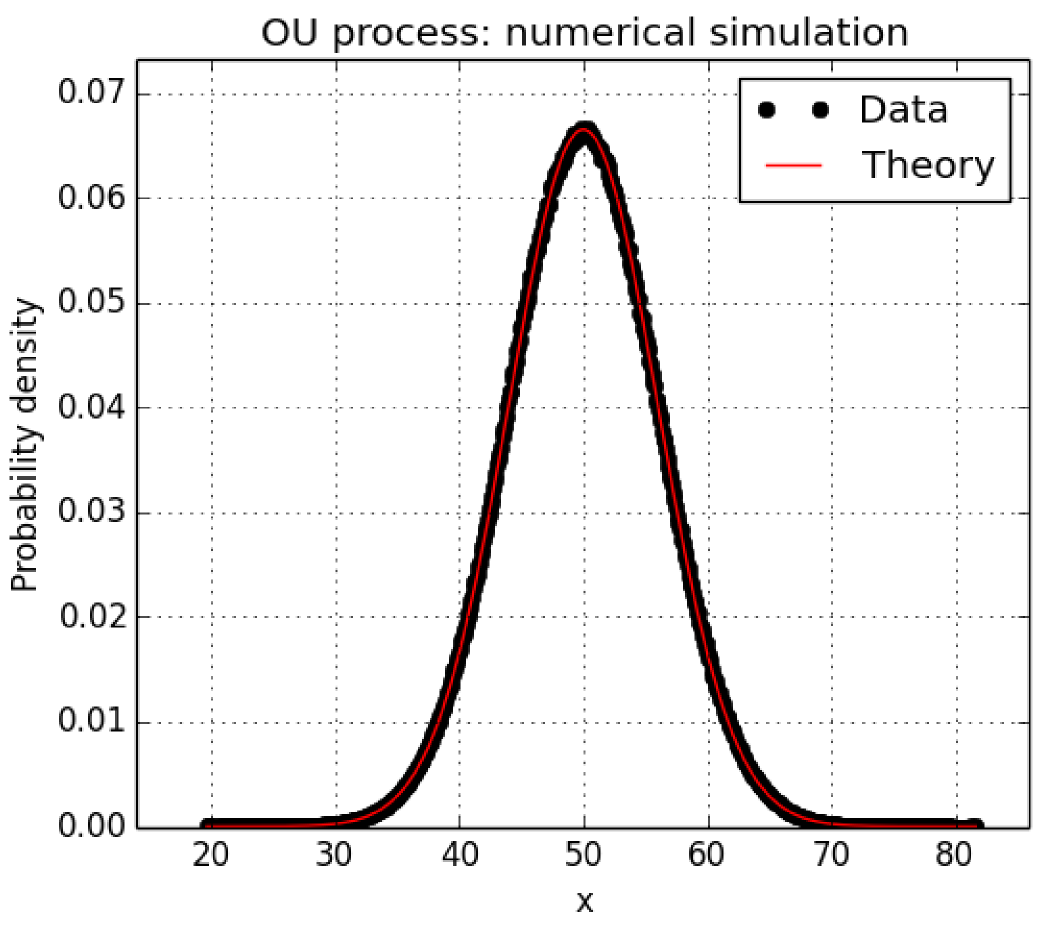


We now analyse (one of) the generated time series, checking for consistency of its statistical properties against the theory. Bear in mind that the larger N, the more accurate will be the agreement between simulation and theory (i.e. large numbers theorem). We first evaluate mean, variance, and then we estimate the probability density function (i.e. by means of the normalised histogram) and compare it to the theoretical Gaussian curve.

```
In [95]: # Let's use x2, since x1, x2, and x3 are identical
         # Note that one should discard a transient because the theoretical PDF is
         # However, if x0 is zero the transient should only affect the estimate of
         transient     = Int64(floor(10. * τ / Δt) + 1);
         x             = x2[transient:end];
         N             = length(x)
         println("Mean is $(mean(x)) [theoretical $μ]");
         println("Stdev is $(std(x)) [theoretical $(σ)]");
```

```
Mean is 49.969602453770236 [theoretical 50.0]
Stdev is 6.014366270070594 [theoretical 6.0]
```

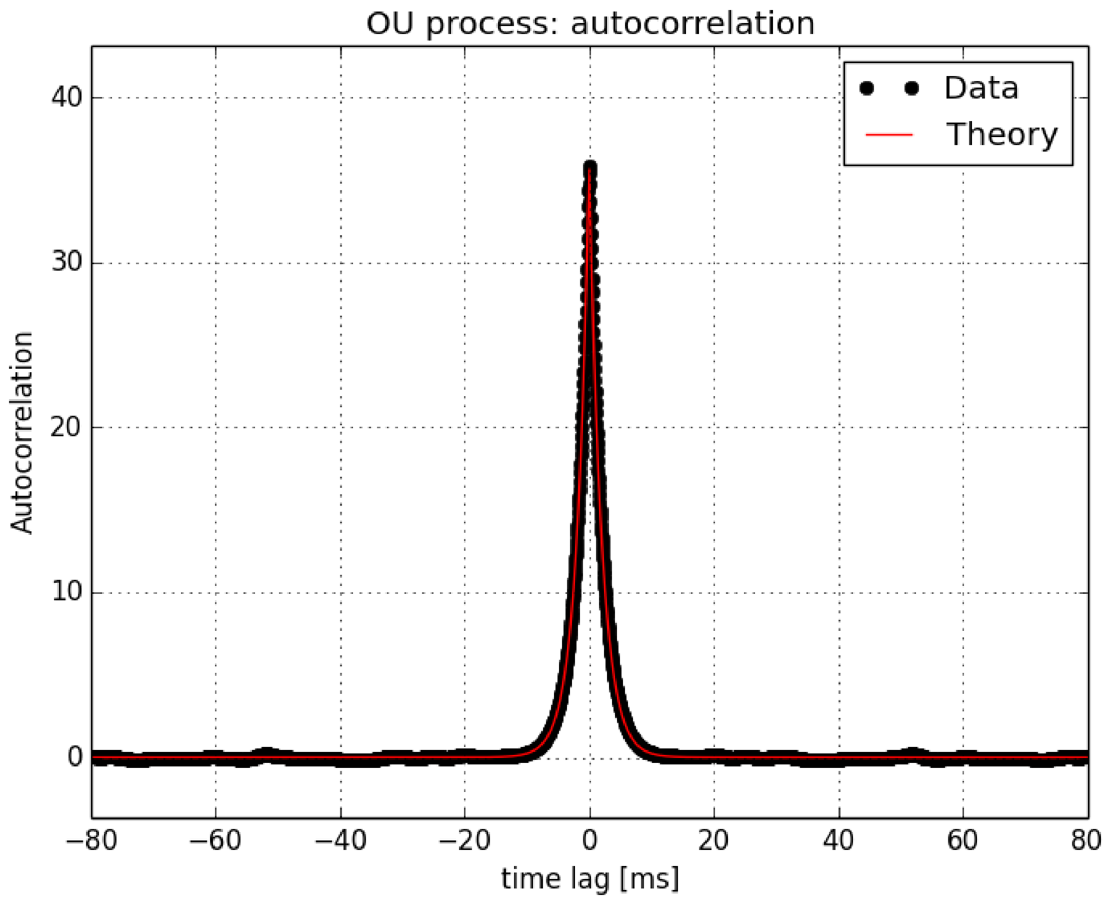
```
In [96]: # Let's now estimate the probability density function (pdf), through the h

         fig = figure("Probability density function",figsize=(6,5)); # Create a new

         Npts          = 1000;
         (edges, counts) = hist(x, Npts);                  # Compute the histogram of x
         xx            = midpoints(edges);                 # Midpoints of the bins with

         ΔE             = edges[3] - edges[2];             # Do not use edges[1] as it
         pdf           = counts / (N * ΔE);                # Normalization, so it becom

         pdf_theory= (1. / sqrt(2 * π * σ^2)) * exp(-(xx-μ).*(xx-μ)/(2 * σ^2)); # A

         plot(xx, pdf, "ko", label="Data");                # Plot in the current figure
         plot(xx, pdf_theory, "r", label="Theory");        #, LineWidth="2");

         ax = gca()                                        # Get a "handle" on the curr
         ax[:set_xlim]((μ-6*σ),(μ+6*σ));                   # For axes "ax" set the hori
         ax[:set_ylim]((0,1.1*maximum(pdf)));           # For axes "ax" set the vertica

         xlabel("x");                                       # Label of the horizontal a
         ylabel("Probability density");                     # Label of the vertical axi
         title("OU process: numerical simulation");         # Label of the title, at th
         grid("on");                                        # "Grid" on

         legend(borderaxespad=0.5);                         # Add a legend to the curre
```



We finally compute the autocorrelation (i.e. the covariance of the process upon removing the *offset*, being the square mean).

```
In [97]: # Let's finally estimate the autocorrelation function (i.e. autocovarianc

         fig = figure("OU - autocorrelation",figsize=(6,5));     # Create a new figu

         # Let's remove the mean from x, prior to compute the correlation
         xc   = xcorr(x - mean(x), x - mean(x)); # xcorr is like the corresponding
         lags = 1.0 * collect((-N+1):(N-1));      # The vector containing the "lags'

         normalised_xc = xc ./ (N-abs(lags));                    # Unbiased normalisat
         xc_theory     = σ^2*exp(-abs(lags*Δt)/(τ));             # Theoretical autoco

PyPlot.Figure(PyObject <matplotlib.figure.Figure object at 0x32f9aee50>)
```

```
In [101]: M              = 3;                                # For reducing the points
          plot(Δt*lags[1:M:end], normalised_xc[1:M:end], "ko", label="Data");  # P
plot(Δt*lags[1:M:end], xc_theory[1:M:end], "r", label="Theory");#, LineW

ax = gca()                                          # Get a "handle" on the cu
ax[:set_xlim](-40*τ,40*τ);                           # For axes "ax" set the h
ax[:set_ylim]((-.1*σ^2, 1.2*σ^2));                   # For axes "ax" set the ve

xlabel("time lag [ms]");                             # Label of the horizontal
ylabel("Autocorrelation");                           # Label of the vertical ax
title("OU process: autocorrelation");                # Label of the title, at t
grid("on");                                          # "Grid" on

legend(borderaxespad=0.5);                           # Add a legend to the curr
```



```
In [ ]:
```