2. Transition Systems

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What are Formal Methods?

What are formal methods?

Formal methods are techniques to

model complex systems using

rigorous mathematical models

Specification

Define part of the system using a modelling language

Verification

Prove properties.

Show correctness.

Find bugs.

Implementation

Generate correct code.

All formal models are wrong

All formal models are wrong

... but some of them are usefull!

Syllabus

- CSS: a simple language for concurrency
    - Syntax
    - Semantics
    - Equivalence
- Timed Automata
    - Syntax
    - Semantics (composition, Zeno)
    - Equivalence
    - UPPAAL tool
        - Specification
        - CTL and Verification
- A simple C-like language
    - Syntax
    - Semantics (operational)
- Hybrid-language: adding differential equations
    - Syntax
    - Semantics
    - Lince tool
        - Specification
        - Analysis
- Monads: semantics with computational effects

Why transition systems?

A Sprinkle of Linguistics

During the module we will encounter two linguistic concepts that every programmer should know:

- syntax - the rules used for determining whether a sentence is valid (in a language) or not
- semantics - the meaning of valid sentences

Ex. 2.1: Syntax

The sentence/program $\mathtt{x} := \mathtt{p}\,;\mathtt{q}$ is forbidden by the syntactic rules of most programming languages

Ex. 2.2: Semantics

The sentence/program $\mathtt{x} := 1$ has the meaning "writes 1 in the memory address corresponding to $\mathtt{x}$"

The need for Semantics in Formal Analysis

How can one prove that a program does what is supposed to do if its semantics (i.e. its meaning) is not established *a priori* ?

Ex. 2.3:

What is the end result of running $x := 2\ ; (x := x + 1 \parallel x := 0)$?

parallelism operator

Widely used programming languages still lack a formal semantics

Defining Transition System with Functors

Preliminaries pt. I

Recalling previous modules ...

Definition (Functor)

A functor F sends a set X into a new set FX and a function $f : X \to Y$ into a new function $Ff : FX \to FY$ such that

$$F(\mathrm{id}) = \mathrm{id} \qquad F(g \cdot f) = Fg \cdot Ff$$

Fix a set A. The following two functors then naturally arise

- product - $X \mapsto A \times X, \;\; f \mapsto id \times f$
- exponential - $X \mapsto X^A, \;\; f \mapsto (g \mapsto f \cdot g)$

Preliminaries pt. II - the List and Powerset functors

The list functor - $X \mapsto X^*$, $\ f \mapsto \texttt{map}\ f$

applies f to every element of a given list

The powerset functor - almost like the list functor; the difference is that we do not look at the order in which elements appear and how many times they repeat. Formally,

$$X \mapsto \{A \mid A \subseteq X\}, \qquad f \mapsto (A \mapsto \{f(a) \mid a \in A\})$$

Ex. 2.4: Powerset on Booleans

$\texttt{Bool} \mapsto \{\emptyset, \{\top\}, \{\bot\}, \{\top, \bot\}\}$

A (Generalised) Notion of a Transition System

Definition (Transition system)

Let F be a functor. An F-transition system is a map $X \to FX$

Some famous examples of F-transition systems

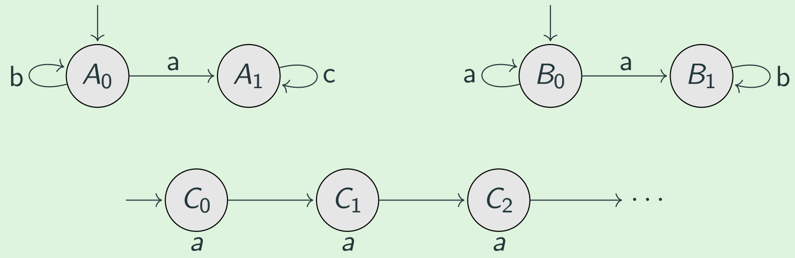
- Moore machine - $X \to N \times X$
- Deterministic automata - $X \to \mathtt{Bool} \times X^N$
- Non-deterministic automata - $X \to \mathtt{Bool} \times \mathrm{P}(X)^N$
- Markov chain - $X \to \mathrm{D}(X)$

Powerset functor

Distribution functor

Exercise

Ex. 2.5: Formalise as an F-transition system



Our First encounter with Coalgebra

Indeed the idea of working at the level of

Functors as Transition Types

is a very fruitful one; and which we only barely grasped (yet) —

in essence, it provides a universal theory of transition systems that can be instantiated to most kinds of transition system we will encounter in our life

Process algebra

Sequential CCS - Syntax

$$\mathcal{P} \ni P, Q ::= K \mid \alpha.P \mid P + Q \mid \mathbf{0} \mid P[f] \mid P\backslash L \mid P|Q$$

where

- $\alpha \in N \cup \{\tau\}$ is an action
- K s a collection of process names or process constants
- $L \subseteq N$ is a set of labels
- f is a function that renames actions s.t. $f(\tau) = \tau$
- notation:

  $[f] = [a_1 \mapsto b_1, \ldots, a_n \mapsto b_n]$

Syntax

$$\mathcal{P} \ni P, Q ::= K \mid \alpha.P \mid P + Q \mid \mathbf{0} \mid P[f] \mid P \backslash L \mid P|Q$$

Ex. 2.6: Which are NOT syntactically correct? Why?

$$a.b.A + B \quad (1)$$

$$(a.\mathbf{0} + b.A) \backslash \{a, b, c\} \quad (2)$$

$$(a.\mathbf{0} + b.A) \backslash \{a, \tau\} \quad (3)$$

$$a.B + [b \mapsto a] \quad (4)$$

$$\tau.\tau.B + \mathbf{0} \quad (5)$$

$$a.(a + b).A \quad (6)$$

$$(a.B + b.B)[a \mapsto a, \tau \mapsto b] \quad (7)$$

$$(a.B + \tau.B)[b \mapsto a, a \mapsto a] \quad (8)$$

$$(a.b.A + b.\mathbf{0}).B \quad (9)$$

$$(a.b.A + b.\mathbf{0}) + B \quad (10)$$

CCS semantics - building a transition system

Every P yields a transition system $X \to$??? with transitions prescribed by the rules below.

$$\text{(act)}\ \frac{}{\alpha.P \xrightarrow{\alpha} P} \qquad \text{(sum-1)}\ \frac{P_1 \xrightarrow{\alpha} P_1'}{P_1 + P_2 \xrightarrow{\alpha} P_1'} \qquad \text{(sum-2)}\ \frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 + P_2 \xrightarrow{\alpha} P_2'}$$

$$\text{(res)}\ \frac{P \xrightarrow{\alpha} P'}{P\backslash L \xrightarrow{\alpha} P'\backslash L}\ \alpha \notin L \qquad \text{(rel)}\ \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]}$$

- Initial states: the process being translated
- Final states: all states are final
- Language: possible sequence of actions of a process

CCS semantics - building a transition system

Every P yields a transition system $X \to$??? with transitions prescribed by the rules below.

$$\text{(act)}\ \frac{}{\alpha.P \xrightarrow{\alpha} P} \qquad \text{(sum-1)}\ \frac{P_1 \xrightarrow{\alpha} P_1'}{P_1 + P_2 \xrightarrow{\alpha} P_1'} \qquad \text{(sum-2)}\ \frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 + P_2 \xrightarrow{\alpha} P_2'}$$

$$\text{(res)}\ \frac{P \xrightarrow{\alpha} P'}{P \backslash L \xrightarrow{\alpha} P' \backslash L}\ \alpha \notin L \qquad \text{(rel)}\ \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]}$$

Ex. 2.7: Build a derivation tree to prove the transitions below

1. $(a.A + b.B) \xrightarrow{b} B$
2. $(a.b.A + (b.a.B + c.a.C)) \xrightarrow{b} a.B$
3. $((a.B + b.A)[a \mapsto c]) \backslash \{a, b\} \xrightarrow{c} (B[a \mapsto c]) \backslash \{a, b\}$

Exercise

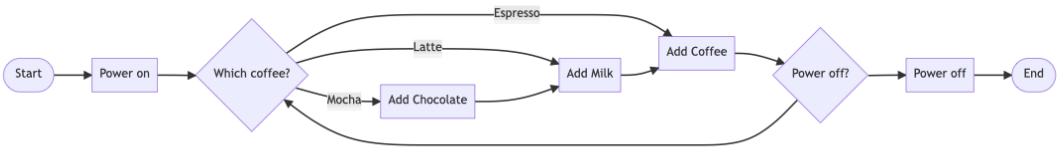
Ex. 2.8: Draw the automata

$$CM = \text{coin}.\text{coffee}.CM$$
$$CS = \text{pub}.(\text{coin}.\text{coffee}.\text{CS} + \text{coin}.\text{tea}.\text{CS})$$

Ex. 2.9: What is the language of the process A?

$$A = \text{goLeft}.\text{A} + \text{goRight}.\text{B}$$
$$B = \text{rest}.\mathbf{0}$$

Exercise



Ex. 2.10: Write the process of the flowchart above

$$
\begin{aligned}
P &= \mathsf{powerOn}.Q \\
Q &= \mathsf{selMocha}.\mathsf{addChocolate}.Mk + \mathsf{selLatte}.Mk + \ldots \\
Mk &= \mathsf{addMilk}\ldots
\end{aligned}
$$

Concurrent Process algebra

Overview

Recall

1. Non-deterministic Finite Automata ($X \to \texttt{Bool} \times \mathtt{P}(X)^N$):

$$\begin{tikzcd} \arrow[r] & q_1 \arrow[r, "a"] & q_2 \arrow[loop right, "b"] \end{tikzcd}$$

2. (Sequential) Process algebra: $P = a.Q \qquad Q = b.Q$
3. Meaning of (2) using (1)

Still missing

- **Interaction** between processes
- Enrich (2) and (3)

CCS - Updated Syntax

$$\mathcal{P} \ni P, Q ::= K \mid \alpha.P \mid P + Q \mid \mathbf{0} \mid P[f] \mid P\backslash L \mid P|Q$$

where

- $\alpha \in N \cup \overline{N} \cup \{\tau\}$ is an action
- K s a collection of process names or process constants
- $L \subseteq N$ is a set of labels
- f is a function that renames actions s.t. $f(\tau) = \tau$ and $f(\bar{a}) = \overline{f(a)}$
- notation:

  $[f] = [a_1 \mapsto b_1, \ldots, a_n \mapsto b_n]$ where $a_i, b_i \in N \cup \{\tau\}$

Process algebras

Syntax

$$\mathcal{P} \ni P, Q ::= K \mid \alpha.P \mid P + Q \mid \mathbf{0} \mid P[f] \mid P\backslash L \mid P|Q$$

Ex. 2.11: Which are syntactically correct?

$$a.\overline{b}.A + B \qquad (11)$$
$$(a.\mathbf{0} + \overline{a}.A)\backslash\{\overline{a}, b\} \qquad (12)$$
$$(a.\mathbf{0} + \overline{a}.A)\backslash\{a, \tau\} \qquad (13)$$
$$(a.\mathbf{0} + \overline{\tau}.A)\backslash\{a\} \qquad (14)$$
$$\tau.\tau.B + \overline{a}.\mathbf{0} \qquad (15)$$
$$(\mathbf{0}|\mathbf{0}) + \mathbf{0} \qquad (16)$$
$$(a.B + b.B)[a \mapsto a, \tau \mapsto b] \qquad (17)$$
$$(a.B + \tau.B)[b \mapsto a, b \mapsto a] \qquad (18)$$
$$(a.B + b.B)[a \mapsto b, b \mapsto \overline{a}] \qquad (19)$$
$$(a.b.A + \overline{a}.\mathbf{0})|B \qquad (20)$$
$$(a.b.A + \overline{a}.\mathbf{0}).B \qquad (21)$$
$$(a.b.A + \overline{a}.\mathbf{0}) + B \qquad (22)$$

CCS semantics - building an NFA

(act)
$$\frac{}{\alpha.P \xrightarrow{\alpha} P}$$

(sum-1)
$$\frac{P_1 \xrightarrow{\alpha} P_1'}{P_1 + P_2 \xrightarrow{\alpha} P_1'}$$

(sum-2)
$$\frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 + P_2 \xrightarrow{\alpha} P_2'}$$

(res)
$$\frac{P \xrightarrow{\alpha} P'}{P\backslash L \xrightarrow{\alpha} P'\backslash L} \quad \alpha, \overline{\alpha} \notin L$$

(rel)
$$\frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]}$$

(com1)
$$\frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q}$$

(com2)
$$\frac{Q \xrightarrow{\alpha} Q'}{P|Q \xrightarrow{\alpha} P|Q'}$$

(com3)
$$\frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'}$$

CCS semantics - building an NFA

$$\text{(act)}\quad \frac{}{\alpha.P \xrightarrow{\alpha} P} \qquad \text{(sum-1)}\quad \frac{P_1 \xrightarrow{\alpha} P_1'}{P_1 + P_2 \xrightarrow{\alpha} P_1'} \qquad \text{(sum-2)}\quad \frac{P_2 \xrightarrow{\alpha} P_2'}{P_1 + P_2 \xrightarrow{\alpha} P_2'}$$

$$\text{(res)}\quad \frac{P \xrightarrow{\alpha} P'}{P\backslash L \xrightarrow{\alpha} P'\backslash L}\ \alpha, \overline{\alpha} \notin L \qquad \text{(rel)}\quad \frac{P \xrightarrow{\alpha} P'}{P[f] \xrightarrow{f(\alpha)} P'[f]}$$

$$\text{(com1)}\quad \frac{P \xrightarrow{\alpha} P'}{P|Q \xrightarrow{\alpha} P'|Q} \qquad \text{(com2)}\quad \frac{Q \xrightarrow{\alpha} Q'}{P|Q \xrightarrow{\alpha} P|Q'} \qquad \text{(com3)}\quad \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P|Q \xrightarrow{\tau} P'|Q'}$$

Ex. 2.12: Draw the transition systems

$$CM = \text{coin}.\overline{\text{coffee}}.CM$$

$$CS = \text{pub}.\overline{\text{coin}}.\text{coffee}.CS$$

$$SmUni = (CM|CS)\backslash\{\text{coin}, \text{coffee}\}$$

Exercises

Ex. 2.13: Let $A = b.a.B$. Show that:

1. $(A \mid \overline{b}.\mathbf{0})\backslash\{b\} \xrightarrow{\tau} (a.B \mid \mathbf{0})\backslash\{b\}$
2. $(A \mid b.a.B) + ((b.A)[b \mapsto a]) \xrightarrow{a} A[b \mapsto a]$

Ex. 2.14: Draw the NFAs A and D

$$A = x.B + x.x.C \qquad D = x.x.x.D + x.E$$

$$B = x.x.A + y.C \qquad E = x.F + y.F$$

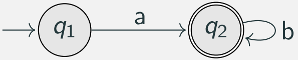
$$C = x.A \qquad F = x.A$$

Observational Equivalence

Overview

Recall

1. F-transition systems, e.g., Non-deterministic Finite Automata:



2. Process algebra: $P = a.Q \quad Q = b.Q \quad P|Q$
3. Interaction between processes
4. Meaning of CCS using transition systems

Still missing

- When is a process P equivalent to a process Q?
- When can a process P be safely replaced by a process Q?

Observational Equivalence Informally

Two programs are observationally equivalent if it is impossible to observe any difference in their behaviour

Here behaviour is described in terms of transition systems

. . . and therefore behaviour/equivalence needs to be pinned down to them

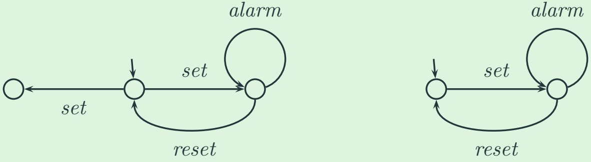
EQ1 – Language equivalence

Language equivalence

Definition

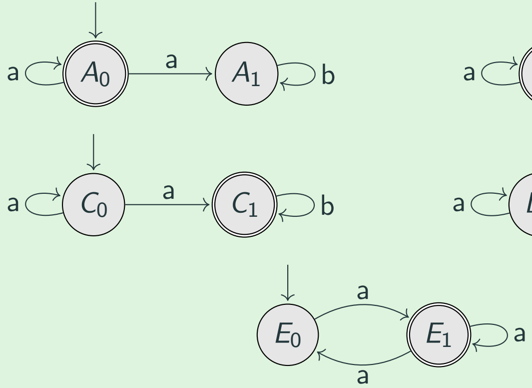
Two automata A, B are language equivalent iff $L_A = L_B$
(i.e. if they can perform the same finite sequences of transitions)

Example



Language equivalence applies when one can neither interact with a system, nor distinguish a slow system from one that has come to a stand still.

Ex. 2.15: Find pairs of automata with the same language



Exercise

Ex. 2.16: Check if the processes are language equivalent

$$P = coin.(\overline{coffee}.P + \overline{tea}.P) \qquad Q = coin.\overline{coffee}.Q + coin.\overline{tea}.Q$$

EQ2 – Similarity

the quest for a behavioural equality:
able to identify states that cannot be distinguished by any realistic form of observation

Simulation

A state q simulates another state p if
every transition from q is corresponded by a transition from p and
this capacity is kept along the whole life of the system to which state space q belongs to.

Simulation of NFA ($X \to \mathcal{P}(X)^N$)

Definition

Given NFA A_1 and A_2 over N with states S_1 and S_2 respectively, a relation $R \subseteq S_1 \times S_2$ is a simulation iff, for all $\langle p, q \rangle \in R$ and $a \in N$,

$$(1) \quad p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$\begin{array}{ccc} p \;\; R \;\; q & & \quad\quad q \\ \downarrow a & \Longrightarrow & \quad\quad \downarrow a \\ p' & & p' \;\; R \;\; q' \end{array}$$

Ex. 2.17: Find simulations

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

Example

Ex. 2.17: Find simulations

$$
\begin{tikzcd}
 & q_1 \arrow[r, "d"] & q_2 \\
q_0 \arrow[ur, "a"] \arrow[dr, "a"'] & & \\
 & q_4 \arrow[r, "e"'] & q_3
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & & p_2 \\
p_0 \arrow[r, "a"] & p_1 \arrow[ur, "d"] \arrow[dr, "e"'] & \\
 & & p_3
\end{tikzcd}
$$

$$q_0 \lesssim p_0 \qquad \text{cf.} \qquad \{\langle q_0, p_0\rangle, \langle q_1, p_1\rangle, \langle q_4, p_1\rangle, \ldots\}$$

Similarity

Definition

$$p \lesssim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a simulation and } \langle p, q\rangle \in R\rangle$$

We say p is simulated by q.

Lemma

The similarity relation is a preorder

(ie, reflexive and transitive)

EQ3 – Bisimilarity

Bisimulation

Definition

Given NFA A_1 and A_2 over N with states S_1 and S_2 respectively, relation $R \subseteq S_1 \times S_2$ is a bisimulation iff both R and its converse R° are simulations.

I.e., whenever $\langle p, q \rangle \in R$ and $a \in N$,

$$(1)\quad p \xrightarrow{a}_1 p' \;\Rightarrow\; \langle \exists\, q' \;:\; q' \in S_2 : \; q \xrightarrow{a}_2 q' \wedge \langle p', q' \rangle \in R \rangle$$

$$(2)\quad q \xrightarrow{a}_2 q' \;\Rightarrow\; \langle \exists\, p' \;:\; p' \in S_1 : \; p \xrightarrow{a}_1 p' \wedge \langle p', q' \rangle \in R \rangle$$

$$\begin{tikzcd}
p \arrow[d, "a"] & R & q & \Rightarrow & & & q \arrow[d, "a"] \\
p' & & & & p' & R & q'
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
p \arrow[d, "a"] & & & \Leftarrow & p & R & q \arrow[d, "a"] \\
p' & R & q' & & & & q'
\end{tikzcd}$$

Examples

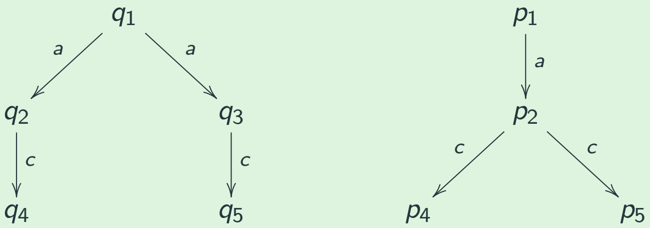
Ex. 2.18: Find bisimulations that include $\langle q_1, m \rangle$

$$\begin{tikzcd} & q_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\ q_2 \arrow[rr, "c"] & & q_3 \arrow[loop right, "c"] \end{tikzcd} \qquad \begin{tikzcd} m \arrow[d, "a"'] \\ n \arrow[loop right, "c"] \end{tikzcd}$$

Ex. 2.19: Find bisimulations that include $\langle q_1, h \rangle$

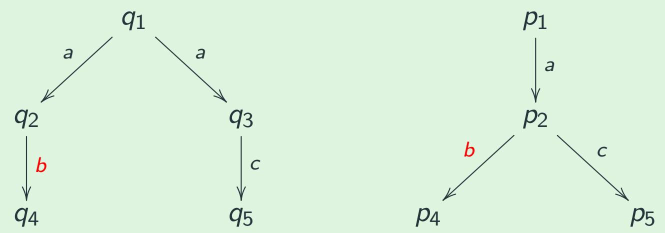
$$q_1 \xrightarrow{a} q_2 \xrightarrow{a} q_3 \xrightarrow{a} \cdots \qquad \begin{tikzcd} h \arrow[loop right, "a"] \end{tikzcd}$$

Ex. 2.20: Check if there is a bisimulation that include $\langle q_1, p_1 \rangle$



Exercises

Ex. 2.21: Check if there is a bisimulation that include $\langle q_1, p_1 \rangle$



Ex. 2.22: Check if there is a bisimulation that include $\langle P, Q \rangle$

$$P = coin.(\overline{coffee}.P + \overline{tea}.P) \qquad Q = coin.\overline{coffee}.Q + coin.\overline{tea}.Q$$

Bisimilarity

Definition

$$p \sim q \;\equiv\; \langle \exists\, R \;::\; R \text{ is a bisimulation and } \langle p, q \rangle \in R \rangle$$

We say p is bisimilar to q.

Lemma

Two processes P and Q are bisimilar if there is a bisimulation that includes $\langle P, Q \rangle$.

Lemma

The bisimilarity relation is an equivalence relation
(ie, symmetric, reflexive and transitive)

Generalising Observational Equivalences

Definition

Fix a functor F and consider two transition systems $f : X \to FX$ and $g : Y \to FY$. Two states $x \in X$, $y \in Y$ are observationally equivalent if there exists a relation $R \subseteq X \times Y$ with $(x, y) \in R$ and there exists a transition system $b : R \to FR$ such that the diagram below commutes

$$
\begin{tikzcd}
X \arrow[d, "f"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "b"'] & Y \arrow[d, "g"] \\
FX & FR \arrow[l, "F\pi_1"] \arrow[r, "F\pi_2"'] & FY
\end{tikzcd}
$$

If such is the case we write $x \sim y$

Observational Equivalence for Moore Automata

Given $\langle o_1, n_1 \rangle : X \longrightarrow A \times X$ and $\langle o_2, n_2 \rangle : Y \longrightarrow A \times Y$ we obtain from the previous slide that $x \sim y$ iff

- $o_1(x) = o_2(y)$
- $n_1(x) \sim n_2(y)$

Observational Equivalence for Labelled Transition Systems

Recall that we used systems of type $X \to \mathrm{P}(X)^L$ for establishing the semantics of CCS processes. This means that ...

notions of observational behaviour/equivalence for such transition systems directly impact our concurrent language

Given $\overline{t_1} : X \to \mathrm{P}(X)^L$ and $\overline{t_2} : Y \to \mathrm{P}(Y)^L$, $x \sim y$ iff for all $l \in L$

- $\forall x' \in t_1(x, l).\ \exists y' \in t_2(y, l).\ x' \sim y'$
- $\forall y' \in t_2(y, l).\ \exists x' \in t_1(x, l).\ x' \sim y'$