Lecture 6:
Finding the period of a function
(Simon's algorithm and its generalisation)

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Recall: Query algorithms

Input accessed through an oracle

Input provided as a function $f : 2^n \longrightarrow 2^m$ that can be queried by the algorithm, which has, in this way, random way access to segments of the input.

Example: the parity problem

Function f can be thought as a sequence of 2^n bits which can be accessed randomly through its evaluation. For example,

$000 \mapsto$	1
$001 \mapsto$	1
$010 \mapsto$	0
...	...

Recall: Phase kick-back

Typically, the oracle keeps input (in the top qubit) unchaged, e.g.

$$|x\rangle|y\rangle \mapsto |x\rangle|y \oplus f(x)\rangle = |x\rangle\, X^{f(x)}\, |y\rangle$$

Phase kick-back is forced by supplying to the orcle second qubit an eigenvector of X, thus

$$U_f|x\rangle|-\rangle = |x\rangle\, X^{f(x)}\, |-\rangle = (1)^{f(x)}|x\rangle|-\rangle$$

What's for today?

Until now we have discussed examples with moderate gains in performance, typically counting the number of queries as a simple measure of efficiency.

A step ahead

- Another query algorithm,
- not making use of phase kick-back,
- which exhibits an effective quantum advantage, drawing a exponential separation wrt classical computation.

Simon's problem

The problem

Let $f : 2^n \longrightarrow 2^n$ be such that for some $s \in 2^n$,

$$f(x) = f(y) \ \text{ iff } \ x = y \ \text{ or } \ x = y \oplus s$$

Find s.

Exercise

What characterises f if $s = 0$? And if $s \neq 0$?

Simon's problem

Exercise

- f is bijective if $s = 0$, because $y \oplus 0 = y$.
- f is two-to-one otherwise ,because, for a given s there is only a pair of values x, y such that $x \oplus y = s$.

Let us assume $s \neq 0$, and thus f to be two-to-one, and rewrite the problem as follows:

Equivalent formulation as a period-finding problem

Determine the period s of a function f periodic under $\oplus$:

$$f(x \oplus s) \; = \; f(x)$$

Simon's problem

Example

Let $f : 2^3 \longrightarrow 2^3$ be defined as

x	$f(x)$
000	101
001	010
010	000
011	110
100	000
101	110
110	101
111	010

Cleary $s = 110$. Indeed, every output of f occurs twice, and the bitwise XOR of the corresponding inputs gives s.

Simon's problem, classically

The best one can do is to evaluate the function on random inputs and hope to find two distinct values with the same image, i.e.,
Compute f for sequence of values until finding a value x_j such that $f(x_j) = f(x_i)$ for a previous x_i, i.e. a colision. Then

$$x_j \oplus x_i \;=\; x_i \oplus (x_i \oplus s) \;=\; s$$

- Since f is two-to-one, after collecting 2^{n-1} evaluations with no collisions, the next evaluation must cause a collision.
- So in the worst case $2^{n-1} + 1$ evaluations are needed.

Simon's problem, classically

Suppose we made q queries to the oracle, resulting in a sequence of q-tuples $(x, f(x))$. The sequence contains

$$\frac{q(q-1)}{2}$$

possible pairs and the probability that a randomly chosen pair has the same output is

$$\frac{1}{2^{n-1}}$$

and the probability of at least one such pair in the list is

$$\frac{q(q-1)}{2^n} \equiv \frac{q^2}{2^n}$$

which means that ideally the oracle should be queried around $q = \sqrt{2^n}$ times.

Simon's problem, classically

Or, more generically, how many evaluations do we need to have a collision with probability p?

To have a collision with probability $p = \frac{1}{k} \leq \frac{1}{2}$ we need

$$\approx \sqrt{(2 \cdot 2^n) \cdot p} = \sqrt{\frac{2}{k} \cdot 2^n} = \sqrt{\frac{2}{k}} \cdot \sqrt{2^n} \quad \text{evaluations}$$

See the Birthday's problem

The problem query complexity is exponential on the input ...
Simon's algorithm, however, solves the problem in polynomial time with probability $\approx \frac{1}{4}$.

... thus, we are approaching an interesting point ...

Note: The birthday problem

Seeks to determine the probability that, in a set of n randomly chosen people, at least two will share a birthday.

$n = 23$ leads to $p(n) \approx 0.5$

Let the universe be $U = 365$ (days) and $n = 23$.
U^n is the space of birthdays and $V = \frac{U!}{(U-n)!}$ (n permutations of U) the number of birthdays with no repetitions.
Then,

$$p(n) \ = \ 1 - \frac{V}{U^n} \ \approx \ 1 - 0.493 \ \approx \ 0.507$$

Heuristic for cases leading with $p(n) \leq 0.5$

$$p(n) \ \approx \ \frac{n^2}{U} \quad \Rightarrow \quad n \ \approx \ \sqrt{2U * p(n)}$$

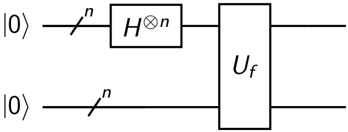
which yields for $p(n) = 0.5$, $n \approx 19$.

Simon's algorithm: The key steps

1. Prepare a superposition $\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)$ for some string x
2. Use interference to find s (indeed, to extract a string y s.t. $y \cdot s = 0$)
3. Repeat previous steps a sufficient number of times to obtain system of equations in the form $y \cdot s = 0$
4. Solve the system for s using Gaussian elimination

Complexity n^3

Simon's algorithm: Preparing the superposition



$U_f\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus f(x)\rangle$

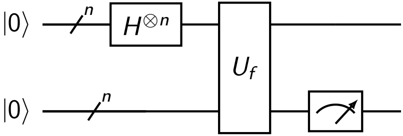
$$U_f(H^{\otimes n} \otimes I)\,|0\rangle\,|0\rangle \;=\; U_f\left(\frac{1}{\sqrt{2^n}}\sum_{x\in 2^n}|x\rangle\,|0\rangle\right) \;=\; \frac{1}{\sqrt{2^n}}\sum_{x\in 2^n}|x\rangle\,|f(x)\rangle$$

The state after the oracle can be rewritten as

$$\frac{1}{\sqrt{2^{n-1}}}\sum_{x\in P}\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)|f(x)\rangle \qquad (1)$$

Set P is composed of one representative of each of the 2^{n-1} sets of strings $\{x, x \oplus s\}$, into which 2^n can be partitioned.

Simon's Algorithm: Preparing the superposition



$U_f |x\rangle |y\rangle = |x\rangle |y \oplus f(x)\rangle$

If the result of measuring the bottom qubits is $f(x)$, then the top ones will contain superposition

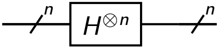
$$\frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)$$

as they are the unique values yielding $f(x)$.

i.e. a measurement of the bottom qubits chooses randomly one of the 2^{n-1} possible outcomes of f ...

as f gives the same output for x and $x \oplus s$, to 2^n possible inputs correspond 2^{n-1} possible outcomes.

Simon's Algorithm: Interference to find s



Recall

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{z \in 2} (-1)^{xz} |z\rangle$$

which extends to a n-qubit as follows

$$\begin{aligned} H^{\otimes n}|x\rangle &= H|x_1\rangle H|x_2\rangle \cdots H|x_n\rangle \\ &= \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} (-1)^{x \cdot z} |z\rangle \end{aligned}$$

where $x \cdot z$ denotes the bitwise product of x and z, modulo 2.

Simon's Algorithm: Interference to find s

Exercise 2 - Q 3.5

$$\begin{aligned}
H^{\otimes n}|x\rangle &= H|x_1\rangle H|x_2\rangle \cdots H|x_n\rangle \\
&= \frac{1}{\sqrt{2}}\sum_{z_1\in 2}(-1)^{x_1 z_1}|z_1\rangle + \frac{1}{\sqrt{2}}\sum_{z_2\in 2}(-1)^{x_2 z_2}|z_2\rangle \cdots \frac{1}{\sqrt{2}}\sum_{z_n\in 2}(-1)^{x_n z_n}|z_n\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{z_1,z_2,\cdots,z_n\in 2}(-1)^{x_1 z_1+x_2 z_2+\cdots+x_n z_n}|z_1 z_2\cdots z_n\rangle \\
&= \frac{1}{\sqrt{2^n}}\sum_{z\in 2^n}(-1)^{x\cdot z}|z\rangle
\end{aligned}$$

Justify the last step.

Simon's Algorithm: Interference to find s

$$H^{\otimes n} \otimes I \left(\frac{1}{\sqrt{2}} (|x\rangle + |x \oplus s\rangle) |f(x)\rangle \right)$$

$$= \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} \frac{1}{\sqrt{2}} ((-1)^{x \cdot z} + (-1)^{(x \oplus s) \cdot z}) |z\rangle |f(x)\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} \frac{1}{\sqrt{2}} ((-1)^{x \cdot z} + (-1)^{(x \cdot z) \oplus (s \cdot z)}) |z\rangle |f(x)\rangle$$

$$= \frac{1}{\sqrt{2^n}} \sum_{z \in 2^n} \frac{1}{\sqrt{2}} ((-1)^{x \cdot z} + (-1)^{(x \cdot z)} (-1)^{(x \cdot s)}) |z\rangle |f(x)\rangle$$

$$= \frac{1}{\sqrt{2^{n+1}}} \sum_{z \in 2^n} \underbrace{(-1)^{x \cdot z} (1 + (-1)^{s \cdot z})}_{(\star)} |z\rangle |f(x)\rangle$$

Simon's Algorithm: Interference to find s

$$\frac{1}{\sqrt{2^{n+1}}} \sum_{z \in 2^n} \underbrace{(-1)^{x \cdot z}(1 + (-1)^{s \cdot z})}_{(\star)} |z\rangle |f(x)\rangle$$

- $s \cdot z = 1 \Rightarrow (\star) = 0$ and the corresponding basis state $|z\rangle$ vanishes
- $s \cdot z = 0 \Rightarrow (\star) \neq 0$: and the corresponding basis state $|z\rangle$ is kept. In this case the probability of geting z upon measurement is $\frac{1}{2^{n-1}}$ (why?)

Simon's Algorithm: Interference to find s

Indeed, this state can be rewritten as follows:

$$\begin{aligned} & \frac{1}{\sqrt{2^{n+1}}} \sum_{z \in 2^n} (-1)^{x \cdot z} (1 + (-1)^{s \cdot z}) \, |z\rangle |f(x)\rangle \\ = & \frac{1}{\sqrt{2^{n+1}}} \sum_{z \in S^\perp} 2(-1)^{x \cdot z} \, |z\rangle |f(x)\rangle \\ = & \frac{1}{\sqrt{2^{n-1}}} \sum_{z \in S^\perp} (-1)^{x \cdot z} \, |z\rangle |f(x)\rangle \end{aligned}$$

where $S^\perp$, for $S = \{0, s\}$ is the orthogonal complement of subspace S, with $\dim(S^\perp) = n - 1$
(because $\dim(S) = 1$, as S is the subspace generated by s)

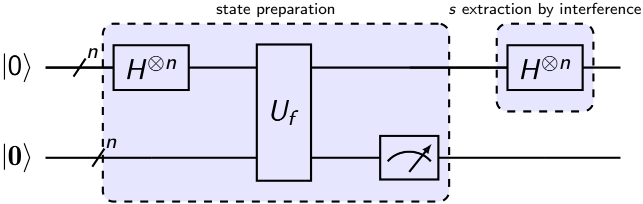
S and $S^{\perp}$

Both are subspaces of the vector space 2^n (often also referred as Z_n^2) whose vectors are strings of length n over $2 = \{0, 1\}$.

- The dimension of 2^n is n; a basis is provided by strings with exactly one 1 in the kth position (for $k = 1, 2, \cdots, n$).
- Two vectors v, u in 2^n are orthogonal iff $v \cdot u = 0$. Thus, a set of strings is linearly independent if no string in it can be expressed as the bitwise sum of other elements in the set.
- Thus, for any subspace F of 2^n, $F^{\perp} = \{u \in 2^n \mid \forall_{v \in F}.\ \ u \cdot v = 0\}$

Warning: to not confuse with the Hilbert space in which the algorithm is executed and whose basis vectors are labeled by elements of 2^n.

Simon's algorithm: The circuit



Simon's Algorithm: Computing s

Running this circuit and measuring the control register results in some z in 2^n satisfying

$$s \cdot z = 0 ,$$

the distribution being uniform over all the strings that satisfy this constraint.

Question

Are we done?

Of course not:
This procedure needs to be repeated until $n-1$ linearly independent such strings $\{z_1, z_2. \cdots , z_{n-1}\}$ are found

Simon's Algorithm: Computing s

Then, it is enough to solve the following set of $n-1$ equations in n unknowns:

$$
\begin{aligned}
z_1 \cdot s &= 0 \\
z_2 \cdot s &= 0 \\
&\vdots \\
z_{n-1} \cdot s &= 0
\end{aligned}
$$

to determine s. Actually,

$\mathrm{span}\{z_1, z_2, \cdots, z_{n-1}\} = S^{\perp}$ and $\{z_1, z_2, \cdots, z_{n-1}\}$ forms a base for $S^{\perp}$

Thus, s is the unique non-zero solution of

$$Z\,s = 0$$

where Z is the matrix whose line i corresponds to vector z_i.

Simon's Algorithm: Computing s

Question

What is the probability of obtaining such a system of equations by running the circuit $n-1$ times (i.e., not having to discard and run again)?

Simon's Algorithm: Probability of success

- Let $Y = \{y_1, \cdots, y_k\}$ be a set of binary strings z linearly independents.
- Y spans a sub-space with 2^k elements with the general form

$$\bigoplus_{i=1..k} b_i y_i \quad \text{for each } b_i \in 2$$

- A new y obtained will be independent of the ones in Y iff it lives out of the subspace generated by Y which occurs with probability

$$1 - \frac{2^k}{2^n}$$

i.e. the probability of failure is $\frac{2^k}{2^n}$

Simon's slgorithm: Probability of success

#	Probability of failure
1	$\frac{2^0}{2^{n-1}}$
2	$\frac{2^1}{2^{n-1}}$
3	$\frac{2^2}{2^{n-1}}$
...	...
$n-1$	$\frac{2^{n-2}}{2^{n-1}}$

This table yields the sequence of probabilities of failure,

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^{n-1}} \qquad \text{(from bottom to top)}$$

Probability of failing in the first $n-2$ steps is thus

$$\frac{1}{4} + \frac{1}{8} + \cdots = \frac{1}{4}\left(1 + \frac{1}{2} + \ldots\right) \leq \frac{1}{4} \cdot \left(\sum_{i \in \mathbb{N}} \frac{1}{2^i}\right) = \frac{1}{2}$$

Geometric series whose sum is equal to two

Simon's algorithm: Probability of success

- Probability of succeeding in the first $n-2$ steps at least $\frac{1}{2}$
- Probability of succeeding in the $(n-1)$-th step is $\frac{1}{2}$
- Thus probability of succeeding in all $n-1$ steps at least $\frac{1}{4}$
- More advanced maths tells us that the probability is slightly higher (around $0.28878\ldots$)

Exponential separation

The period s of f can be computed with some constant probability of error after repeating Simon's algorithm $\mathcal{O}(n)$ times, which witnesses an exponential separation between classical and quantum computation.

The algorithm

1. Prepare the initial state $\frac{1}{\sqrt{2^n}} \sum_{x \in 2^n} |x\rangle|0\rangle$ and make $i := 1$
2. Apply the oracle U_f to obtain the state

   $$\frac{1}{\sqrt{2^n}} \sum_{x \in 2^n} |x\rangle|f(x)\rangle$$

   which can be re-written as

   $$\frac{1}{\sqrt{2^{n-1}}} \sum_{x \in P} \frac{1}{\sqrt{2}}(|x\rangle + |x \oplus s\rangle)|f(x)\rangle$$

   and measure the bottom qubits not strictly necessary but makes the analysis simpler.
3. Apply $H^{\otimes n}$ to the top qubits yielding a uniform superposition of elements of $S^{\perp}$.

The algorithm

4. Measure the first register and record the value observed z_i, which is a randomly selected element of $S^{\perp}$.
5. If the dimension of the span of $\{z_1, z_2, \cdots, z_i\}$ is less than $n-1$, increment i and to go step 2; else proceed.
6. Compute s as the unique non-zero solution of

$$Z\,s = 0$$

The crucial observation is that the set of observed values must form a basis to $S^{\perp}$.

The problem

The problem

Let $f : 2^n \longrightarrow X$, for some X finite, be such that,

$$f(x) = f(y) \text{ iff } x - y \in S$$

for some subspace S of Z_2^n with dimension m.

Find a basis $\{s_1, s_2, \cdots s_m\}$ for S.

In Simon's problem

- $x = y \oplus s$, i.e. $x - y = s$.
- s is a basis for the space S generated by $\{s\}$.

Note

The tuple $(2^n, \oplus, 0)$ forms a group with bitwise negation

Groups

A group (G, θ, u) is a set G with a binary operation θ which is associative, and equipped with an identity element u and an inverse:

$$a^{-1}\theta a = u = a\theta a^{-1}$$

Each set $\{x, x \oplus s\}$ in (1) is a coset of subgroup $S = (\{0, s\}, \oplus, 0)$

Coset

The coset of a subgroup S of a group (G, θ, u) wrt $g \in G$ is

$$gS = \{g\theta s \mid s \in S\}$$

In this case

$$xS = \{x \oplus 0, x \oplus s\} = \{x, x \oplus s\}$$

Generalised Simon's algorithm

If $S = \{0, y_1, \cdots, y_{2^m-1}\}$ is a subspace of dimension m of 2^n, it can be decomposed into 2^{n-m} cosets of the form

$$\{x, x \oplus y_1, x \oplus y_2, \cdots, x \oplus y_{2^m-1}\}$$

Then Step 2 yields

$$\begin{aligned}
&\sum_{x \in 2^n} |x\rangle|f(x)\rangle \\
= &\frac{1}{\sqrt{2^{n-m}}} \sum_{x \in P} \frac{1}{\sqrt{2^m}} (|x\rangle + |x \oplus y_1\rangle + |x \oplus y_2\rangle + \cdots + x \oplus y_{2^m-1})|f(x)\rangle \\
= &\frac{1}{\sqrt{2^{n-m}}} \sum_{x \in P} |x + S\rangle|f(x)\rangle
\end{aligned}$$

where P be a subset of 2^n consisting of one representative of each 2^{n-m} disjoint cosets, and

$$|x + S\rangle = \sum_{s \in S} \frac{1}{\sqrt{2^m}} |s\rangle$$

Generalised Simon's algorithm

- In step 4 the first register is left in a state of the form $|x + S\rangle$ for a random x.
- After applying the Hadamard transformation, the first register contains a uniform superposition of elements of $S^{\perp}$ and its measurement yields a value sampled uniformly at random from $S^{\perp}$.

This leads to the revised algorithm:

5. If the dimension of the span of $\{z_1, z_2, \cdots, z_i\}$ is less than $n - m$, increment i and to go step 2; else proceed.
6. Compute the system of linear equations

$$Z\,s \;=\; 0$$

and let $s_1, s_2, \cdots, s_m$ be the generators of the solution space. They form the envisaged basis.

The hidden subgroup problem

The group S is often called the hidden subgroup.
The (generalised) Simon's algorithm is an instance of a much general scheme, leading to exponential advantage, known as

The hidden subgroup problem

Let (G, θ, u) be a group and $f : G \longrightarrow X$ for some finite set X with the following property:

f is constant on cosets of S and distinct on different cosets

i.e.

there is a subgroup S of G such that for any $x, y \in G$,

$$f(x) = f(y) \text{ iff } x\theta S = y\theta S$$

Characterise S.