Simply Typed Lambda-calculus

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Deductive Reasoning

The essence

Knowledge obtained via assumptions and logical rules

Deductive Reasoning

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Knowledge obtained via assumptions and logical rules

Studied since Aristotle ...

... long before the age of artificial computers

What does it have to do with programming ?

A Basic Deductive System

$\mathbb{A}, \mathbb{B} \ldots$ denote propositions and
1 a proposition that always holds

If $\mathbb{A}$ and $\mathbb{B}$ are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – conjunction of $\mathbb{A}$ and $\mathbb{B}$
- $\mathbb{A} \to \mathbb{B}$ is a proposition – implication of $\mathbb{B}$ from $\mathbb{A}$

A Basic Deductive System

Γ denotes a list of propositions (often called context)

$\Gamma \vdash \mathbb{A}$ reads *"if the propositions in* Γ *hold then* $\mathbb{A}$ *also holds"*

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}}\ (\mathrm{ass}) \qquad \frac{}{\Gamma \vdash 1}\ (\mathrm{trv}) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}}\ (\pi_1) \qquad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}}\ (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \qquad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}}\ (\mathrm{prd}) \qquad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \to \mathbb{B}}\ (\mathrm{cry}) \qquad \frac{\Gamma \vdash \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}}\ (\mathrm{app})$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

New Knowledge From Old

The rules below are derivable from the previous system

$$\frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}} \text{ (exchange)} \qquad\qquad \frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \text{ (weakening)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \qquad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

Proofs (again) by an appeal to your old friend ... induction :-)

Exercises

Derive the following judgements

- $\mathbb{A} \to \mathbb{B}, \mathbb{B} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{C}$
- $\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$

Back to programming . . .

The Bare Essentials of Programming

We should think of what are the basic features of programming ...

- variables
- function application and creation
- pairing ...

and base our study on the simplest language with such features ...

$\boxed{\text{Simply-typed } \lambda\text{-calculus}}$

The basis of `Haskell`, `ML`, `Eff`, `F#`, `Agda`, `Elm` and many other programming languages

Simply-typed λ-Calculus

Types are defined by $\mathbb{A} ::= 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \to \mathbb{A}$

Γ now a non-repetitive list of typed variables $(x_1 : \mathbb{A}_1 \ldots x_n : \mathbb{A}_n)$

Programs built according to the following deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}}\ (\mathrm{ass}) \qquad \frac{}{\Gamma \vdash * : 1}\ (\mathrm{triv}) \qquad \frac{\Gamma \vdash t : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1\, t : \mathbb{A}}\ (\pi_1)$$

$$\frac{\Gamma \vdash t : \mathbb{A} \qquad \Gamma \vdash s : \mathbb{B}}{\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}}\ (\mathrm{prd}) \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}}\ (\mathrm{cry})$$

$$\frac{\Gamma \vdash t : \mathbb{A} \to \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t\, s : \mathbb{B}}\ (\mathrm{app})$$

Examples of λ-terms

$$x : \mathbb{A} \vdash x : \mathbb{A} \qquad \text{(identity)}$$

$$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A} \qquad \text{(duplication)}$$

$$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2\ x, \pi_1\ x \rangle : \mathbb{B} \times \mathbb{A} \qquad \text{(swap)}$$

$$f : \mathbb{A} \to \mathbb{B}, g : \mathbb{B} \to \mathbb{C} \vdash \lambda x : \mathbb{A}.\ g(f\ x) : \mathbb{A} \to \mathbb{C} \qquad \text{(composition)}$$

Exercises

Recall the derivations that lead to the judgement

$$\mathbb{A} \to \mathbb{B}, \mathbb{A} \to \mathbb{C} \vdash \mathbb{A} \to \mathbb{B} \times \mathbb{C}$$

Build the corresponding program

Derive as well the judgement

$$\mathbb{A} \to \mathbb{B} \vdash \mathbb{A} \times \mathbb{C} \to \mathbb{B} \times \mathbb{C}$$

and subsequently build the corresponding program

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A Semantics for Simply Typed λ-calculus

We wish to assign a mathematical meaning to λ-terms

$$[\![-]\!] : \lambda\text{-terms} \longrightarrow \ldots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

A Semantics for Simply Typed λ-calculus

We wish to assign a mathematical meaning to λ-terms

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This is the goal of the next slides. But first . . .

Functions: Basic Facts

For every set X there exists a 'trivial' function

$$! : X \longrightarrow \{\star\} = 1 \qquad !\,(x) = \star$$

We can always pair two functions into $f : X \to A$, $g : X \to B$

$$\langle f, g \rangle : X \to A \times B \qquad \langle f, g \rangle(x) = (f\,x, g\,x)$$

There exist projection functions

$$\pi_1 : X \times Y \to X \qquad \pi_1(x, y) = x$$
$$\pi_2 : X \times Y \to Y \qquad \pi_2(x, y) = y$$

Functions: Basic Facts

We can always 'curry' a function $f : X \times Y \to Z$ into

$$\lambda f : X \to Z^Y \qquad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z. There exists an application function

$$\mathrm{app} : Z^Y \times Y \to Z \qquad \mathrm{app}(f, y) = f\ y$$

Types $\mathbb{A}$ interpreted as sets $[\![\mathbb{A}]\!]$

$$[\![1]\!] = \{\star\}$$
$$[\![\mathbb{A} \times \mathbb{B}]\!] = [\![\mathbb{A}]\!] \times [\![\mathbb{B}]\!]$$
$$[\![\mathbb{A} \to \mathbb{B}]\!] = [\![\mathbb{B}]\!]^{[\![\mathbb{A}]\!]}$$

Typing contexts Γ interpreted as Cartesian products

$$[\![\Gamma]\!] = [\![x_1 : \mathbb{A}_1, \dots, x_n : \mathbb{A}_n]\!] = [\![\mathbb{A}_1]\!] \times \cdots \times [\![\mathbb{A}_n]\!]$$

λ-terms $\Gamma \vdash t : \mathbb{A}$ interpreted as functions

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

Denotational Semantics

λ-term $\Gamma \vdash t : \mathbb{A}$ interpreted as a function

$$[\![\Gamma \vdash t : \mathbb{A}]\!] : [\![\Gamma]\!] \longrightarrow [\![\mathbb{A}]\!]$$

$$\frac{x_i : \mathbb{A} \in \Gamma}{[\![\Gamma \vdash x_i : \mathbb{A}]\!] = \pi_i} \qquad \frac{}{[\![\Gamma \vdash * : 1]\!] = !} \qquad \frac{[\![\Gamma \vdash t : \mathbb{A} \times \mathbb{B}]\!] = f}{[\![\Gamma \vdash \pi_1 t : \mathbb{A}]\!] = \pi_1 \cdot f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A}]\!] = f \qquad [\![\Gamma \vdash s : \mathbb{B}]\!] = g}{[\![\Gamma \vdash \langle t, s\rangle : \mathbb{A} \times \mathbb{B}]\!] = \langle f, g\rangle} \qquad \frac{[\![\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}]\!] = f}{[\![\Gamma \vdash \lambda x : \mathbb{A}.\, t : \mathbb{A} \to \mathbb{B}]\!] = \lambda f}$$

$$\frac{[\![\Gamma \vdash t : \mathbb{A} \to \mathbb{B}]\!] = f \quad [\![\Gamma \vdash s : \mathbb{A}]\!] = g}{[\![\Gamma \vdash t\, s : \mathbb{B}]\!] = \mathrm{app} \cdot \langle f, g\rangle}$$

The Unravelling

$$
\begin{aligned}
&[\![x \vdash \langle \pi_2\, x,\ \pi_1\, x \rangle]\!] && = \ldots \\
&[\![- \vdash \lambda x.\, \langle \pi_2\, x,\ \pi_1\, x \rangle]\!] && = \ldots \\[1em]
&[\![f, g, x \vdash g\, f\, x]\!] && = \ldots \\
&[\![f, g \vdash \lambda x.\, g\, f\, x]\!] && = \ldots \\[1em]
&[\![f, x \vdash \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] && = \ldots \\
&[\![f \vdash \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] && = \ldots \\
&[\![- \vdash \lambda f.\, \lambda x.\, \langle f\, \pi_1\, x,\ \pi_2\, x \rangle]\!] && = \ldots
\end{aligned}
$$

(**N.B.** all types omitted for simplicity)

Show that the following equations hold

$$
\begin{aligned}
[\![x, y \vdash \pi_1 \langle x, y \rangle]\!] &= [\![x, y \vdash x]\!] \\
[\![\Gamma \vdash t]\!] &= [\![\Gamma \vdash \langle \pi_1 \ t, \pi_2 \ t \rangle]\!] \\
[\![x \vdash (\lambda y. \langle x, y \rangle) \ x]\!] &= [\![x \vdash \langle x, x \rangle]\!]
\end{aligned}
$$

Denotational Semantics and Equivalence Revisited

Show that the following equations hold

$$\begin{aligned} [\![x, y \vdash \pi_1 \langle x, y\rangle]\!] &= [\![x, y \vdash x]\!] \\ [\![\Gamma \vdash t]\!] &= [\![\Gamma \vdash \langle \pi_1\ t, \pi_2\ t\rangle]\!] \\ [\![x \vdash (\lambda y. \langle x, y\rangle)\ x]\!] &= [\![x \vdash \langle x, x\rangle]\!] \end{aligned}$$

Show that the (complicated) λ-term below is really just the identity

$$z \vdash \lambda x. \langle \pi_2\, x, \pi_1\, x\rangle \left(\lambda y. \langle \pi_2\, y, \pi_1\, y\rangle\ z\right)$$

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Hard ?

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Logic to the Rescue !

Recall that the rules below are derivable from our logical system

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$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \qquad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

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Filling up the dots will lead us to a fundamental concept

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

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Filling up the dots will lead us to a fundamental concept

Substitution

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The essence

Substitution of variables in a λ-term t by another λ-term s

$t[s/x]$ reads *"replace every occurrence of x in t by s"*

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Example

$$\langle x, x\rangle[s/x] = \langle s, s\rangle$$
$$\langle x, y\rangle[s/x] = \langle s, y\rangle$$
$$\langle y, z\rangle[s/x] = \langle y, z\rangle$$

Substitution More Formally

We define it by induction

$$
\begin{aligned}
x[s/y] &= \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases} \\
*[s/y] &= * \\
\langle t_1, t_2 \rangle [s/y] &= \langle t_1[s/y], t_2[s/y] \rangle \\
(t_1\ t_2)[s/y] &= t_1[s/y]\ t_2[s/y] \\
(\pi_1 t)[s/y] &= \pi_1 t[s/y] \\
(\pi_2 t)[s/y] &= \pi_2 t[s/y] \\
(\lambda x.\, t)[s/y] &= \ \ldots
\end{aligned}
$$

Variable Captures

$\lambda x.\, y$ is a "constant function" (given x return y)

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$(\lambda x.\, y)[x/y]$ is now the identity !?

The problem: variable x "captured" by the construct "$\lambda x.$"

Somehow similar to variable shadowing in programming

Substitution More Formally

$$
\begin{aligned}
x[s/x] &= \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases} \\
*[s/y] &= * \\
\langle t_1, t_2 \rangle[s/y] &= \langle t_1[s/y], t_2[s/y] \rangle \\
(t_1\ t_2)[s/y] &= t_1[s/y]\ t_2[s/y] \\
(\pi_1\ t)[s/y] &= \pi_1\ t[s/y] \\
(\pi_2\ t)[s/y] &= \pi_2\ t[s/y] \\
(\lambda x.\, t)[s/y] &= \lambda z.\, t[z/x][s/y]
\end{aligned}
$$

(where z is fresh (*i.e.* new))

Exercise

Compute the following substitutions

$$
\begin{aligned}
* \, [t/y][s/z] &= \ldots \\
\langle y, z\rangle[t/y][s/z] &= \ldots \\
(\lambda x.\, x)[t/x] &= \ldots \\
(\lambda x.\, \langle x, y\rangle)[z/y] &= \ldots \\
(\lambda x.\, \langle x, y\rangle)[x/y] &= \ldots
\end{aligned}
$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \cdots : \mathbb{B}} \text{ (cut elimination)}$$

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$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \qquad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash t[s/x] : \mathbb{B}} \text{ (cut elimination)}$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)} \qquad \frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

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Substitution also fundamental in the study of equivalence

An Equational System pt. I

$$
\begin{aligned}
\pi_1\langle t, s\rangle &=_{\beta\eta} t & t &=_{\beta\eta} * & &(\text{if } t : 1)\\
\pi_2\langle t, s\rangle &=_{\beta\eta} s & \lambda x.\, t\, s &=_{\beta\eta} t[s/x] & &\\
\langle \pi_1\, t, \pi_2\, t\rangle &=_{\beta\eta} t & \lambda x.\, (t\, x) &=_{\beta\eta} t & &
\end{aligned}
$$

An Equational System pt. II

$$t =_{\beta\eta} t \qquad \frac{t =_{\beta\eta} s}{s =_{\beta\eta} t} \qquad \frac{t =_{\beta\eta} s \qquad s =_{\beta\eta} u}{t =_{\beta\eta} u}$$

$$\frac{t =_{\beta\eta} s}{\pi_1\, t =_{\beta\eta} \pi_1\, s} \qquad \frac{t =_{\beta\eta} s}{\pi_2\, t =_{\beta\eta} \pi_2\, s} \qquad \frac{t =_{\beta\eta} s \qquad u =_{\beta\eta} v}{\langle t, u \rangle =_{\beta\eta} \langle s, v \rangle}$$

$$\frac{t =_{\beta\eta} s \qquad u =_{\beta\eta} v}{t\, u =_{\beta\eta} s\, v}$$

$$\frac{\Gamma \vdash t =_{\beta\eta} s}{\pi\Gamma \vdash t =_{\beta\eta} s} \qquad \frac{u =_{\beta\eta} v \qquad t =_{\beta\eta} s}{u[t/x] =_{\beta\eta} v[s/x]}$$

Equivalence Re-Revisited

Show that the following equations hold

$$\begin{aligned} \pi_1\langle x, y\rangle &=_{\beta\eta} x \\ t &=_{\beta\eta} \langle \pi_1\ t, \pi_2\ t\rangle \\ (\lambda y.\, \langle x, y\rangle)\ x &=_{\beta\eta} \langle x, x\rangle \end{aligned}$$

$$\lambda x.\, \langle \pi_2\, x, \pi_1\, x\rangle \left(\lambda y.\, \langle \pi_2\, y, \pi_1\, y\rangle\ z\right) =_{\beta\eta} z$$

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