Small-step Semantics

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Semantics for every season

Operational semantics	How a program operates
Denotational semantics	What a program is
Axiomatic semantics	Which logical properties a program satisfies

Small-step operational semantics

How a program operates step-by-step

Example

$\langle \mathrm{x} := 1; \mathrm{x} := \mathrm{x} + 1, v \rangle \longrightarrow \langle \mathrm{x} := \mathrm{x} + 1, 1 \rangle \longrightarrow 2$

A machine with an 'evaluation stack' that is processed at each step

Its uses

Describes how a program operates step-by-step

Describes evaluation techniques (*e.g.* short-circuiting)

A basis for tracing/debugging

Foundations of concurrency, complexity, . . .

Rich notions of equivalence

. . .

Its uses

Describes how a program operates step-by-step

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Foundations of concurrency, complexity, . . .

Rich notions of equivalence

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Thus an essential tool for understanding a programming language

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A propositional language

$$\mathtt{b} ::= \mathtt{x} \mid \mathtt{b} \wedge \mathtt{b} \mid \neg \mathtt{b}$$

Every $\mathtt{x}$ is a proposition (*i.e.* it has either value $\mathtt{tt}$ or $\mathtt{ff}$)

A propositional language

$$\texttt{b} ::= \texttt{x} \mid \texttt{b} \wedge \texttt{b} \mid \neg \texttt{b}$$

Every $\texttt{x}$ is a proposition (*i.e.* it has either value $\texttt{tt}$ or $\texttt{ff}$)

$\boxed{\text{Can we provide a small-step semantics to this language ?}}$

Key points

Uses a memory $\sigma : X \to \mathrm{Bool}$ that assigns to every proposition $\mathtt{x}$ its truth-value $\sigma(\mathtt{x})$

A term $\mathtt{b}$ is evaluated step-by-step until a truth-value v is reached

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow v$$

Focus is on the next step (of the evaluation)

The semantics – a universe of laws

$$\frac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \sigma(\mathtt{x})} \text{ (var)} \qquad\qquad \frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \neg v} \text{ (neg}_1\text{)}$$

The semantics – a universe of laws

$$\frac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \sigma(\mathtt{x})} \text{ (var)}$$

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \neg v} \text{ (neg}_1\text{)}$$

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle}{\langle \neg \mathtt{b}, \sigma \rangle \longrightarrow \langle \neg \mathtt{b}', \sigma' \rangle} \text{ (neg}_2\text{)}$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \mathtt{ff}}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \mathtt{ff}} \text{ (and}_1\text{)}$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \mathtt{tt}}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \langle \mathtt{b}_2, \sigma \rangle} \text{ (and}_2\text{)}$$

$$\frac{\langle \mathtt{b}_1, \sigma \rangle \longrightarrow \langle \mathtt{b}_1', \sigma' \rangle}{\langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \langle \mathtt{b}_1' \wedge \mathtt{b}_2, \sigma' \rangle} \text{ (and}_3\text{)}$$

An example

$\neg\neg \mathtt{x} \longrightarrow ?$

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$$\dfrac{\dfrac{\dfrac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \sigma(\mathtt{x})}\ (\mathsf{var})}{\langle \neg\mathtt{x}, \sigma \rangle \longrightarrow \neg\sigma(\mathtt{x})}\ (\mathsf{neg}_1)}{\langle \neg\neg\mathtt{x}, \sigma \rangle \longrightarrow \neg\neg\sigma(\mathtt{x})}\ (\mathsf{neg}_1)$$

Another example

$(\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2 \longrightarrow ?$

If $\sigma(\mathtt{x}) = \mathtt{ff}$:

$$\dfrac{\dfrac{\overline{\langle \mathtt{x}, \sigma \rangle \longrightarrow \ \mathtt{ff}} \ (\mathsf{var})}{\langle \mathtt{x} \wedge \mathtt{b}_1, \sigma \rangle \longrightarrow \ \mathtt{ff}} \ (\mathsf{and}_1)}{\langle (\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \ \mathtt{ff}} \ (\mathsf{and}_1)$$

Yet another example

$(\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2 \longrightarrow$?

If $\sigma(\mathtt{x}) = \mathtt{tt}$:

$$\dfrac{\dfrac{\dfrac{}{\langle \mathtt{x}, \sigma \rangle \longrightarrow \ \mathtt{tt}}\ (\mathsf{var})}{\langle \mathtt{x} \wedge \mathtt{b}_1, \sigma \rangle \longrightarrow \ \langle \mathtt{b}_1, \sigma \rangle}\ (\mathsf{and}_2)}{\langle (\mathtt{x} \wedge \mathtt{b}_1) \wedge \mathtt{b}_2, \sigma \rangle \longrightarrow \ \langle \mathtt{b}_1 \wedge \mathtt{b}_2, \sigma \rangle}\ (\mathsf{and}_3)$$

Now you try !

$$\mathtt{x} \wedge \neg \mathtt{x} \longrightarrow ?$$

$$\neg(\neg \mathtt{x} \wedge \neg \mathtt{y}) \longrightarrow ?$$

Provide semantics to the Boolean implication $\mathtt{b} \Rightarrow \mathtt{b}$

From one step to many . . .

One often is uninterested on the next step . . .

. . . and rather on the output (that the sequence of steps leads to)

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This multi-step transition $\longrightarrow^n$ is defined by the rules

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \mathtt{b}, \sigma \rangle \longrightarrow^1 v}\ (\mathsf{stp}) \qquad \frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle \qquad \langle \mathtt{b}', \sigma' \rangle \longrightarrow^n v}{\langle \mathtt{b}, \sigma \rangle \longrightarrow^{n+1} v}\ (\mathsf{nxt})$$

What's next ?

Fine, we have an operational semantics; so what ?

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We can now prove cool properties about our language !!

Example (Termination)

It is always the case that $\langle \mathrm{b}, \sigma \rangle \longrightarrow^{n} v$ for some v and n

Exercise 1

Define a ‘complexity function’

$$\begin{aligned} \mathrm{compl}(\mathtt{x}) &= 1 \\ \mathrm{compl}(\neg \mathtt{b}) &= \mathrm{compl}(\mathtt{b}) \\ \mathrm{compl}(\mathtt{b}_1 \wedge \mathtt{b}_2) &= \mathrm{compl}(\mathtt{b}_1) + \mathrm{compl}(\mathtt{b}_2) \end{aligned}$$

Show by induction that $\mathrm{compl}(\mathtt{b}) \geq 1$ for every $\mathtt{b}$

Exercise 2

Show by induction the following implication

$$\text{If } \langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle \text{ then } \mathrm{compl}(\mathtt{b}) > \mathrm{compl}(\mathtt{b}')$$

Exercise 3

Show by induction the following implication

$$\text{If } \langle \mathtt{b}, \sigma \rangle \longrightarrow^{n} v \text{ then } \mathrm{compl}(\mathtt{b}) \geq n$$

When the number of steps does not matter . . .

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. . . and rather just on the output

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This multi-step transition $\longrightarrow^\star$ is defined by the rules

$$\frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow v}{\langle \mathtt{b}, \sigma \rangle \longrightarrow^\star v}\ (\mathsf{stp}) \qquad \frac{\langle \mathtt{b}, \sigma \rangle \longrightarrow \langle \mathtt{b}', \sigma' \rangle \qquad \langle \mathtt{b}', \sigma' \rangle \longrightarrow^\star v}{\langle \mathtt{b}, \sigma \rangle \longrightarrow^\star v}\ (\mathsf{nxt})$$

Exercise 4

Show by induction the following equivalence

$$\langle \mathtt{b}, \sigma \rangle \longrightarrow^{n} v \text{ (for some } n\text{) iff } \langle \mathtt{b}, \sigma \rangle \longrightarrow^{\star} v$$

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A simple while-language

Arithmetic expressions

$e ::= \mathtt{n} \mid \mathtt{e} \cdot \mathtt{e} \mid \mathtt{x} \mid \mathtt{e} + \mathtt{e}$

Programs

$\mathtt{p} ::= \mathtt{x} := \mathtt{e} \mid \mathtt{p} \,;\, \mathtt{q} \mid \mathtt{if}\ \mathtt{b}\ \mathtt{then}\ \mathtt{p}\ \mathtt{else}\ \mathtt{q} \mid \mathtt{while}\ \mathtt{b}\ \mathtt{do}\ \{\ \mathtt{p}\ \}$

A while-language and its semantics

$$\frac{\langle \mathrm{e}, \sigma \rangle \longrightarrow^{\star} v}{\langle \mathtt{x} := \mathrm{e}, \sigma \rangle \longrightarrow \sigma[v/\mathtt{x}]} \ (\mathsf{asg}) \qquad \frac{\langle \mathrm{p}, \sigma \rangle \longrightarrow \sigma'}{\langle \mathrm{p} \mathbin{\texttt{;}} \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{q}, \sigma' \rangle} \ (\mathsf{seq}_1)$$

$$\frac{\langle \mathrm{p}, \sigma \rangle \longrightarrow \langle \mathrm{p}', \sigma' \rangle}{\langle \mathrm{p} \mathbin{\texttt{;}} \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{p}' \mathbin{\texttt{;}} \mathrm{q}, \sigma' \rangle} \ (\mathsf{seq}_2) \qquad \frac{\langle \mathrm{b}, \sigma \rangle \longrightarrow^{\star} \mathtt{tt}}{\langle \texttt{if}\ \mathrm{b}\ \texttt{then}\ \mathrm{p}\ \texttt{else}\ \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{p}, \sigma \rangle} \ (\mathsf{if}_1)$$

$$\frac{\langle \mathrm{b}, \sigma \rangle \longrightarrow^{\star} \mathtt{ff}}{\langle \texttt{if}\ \mathrm{b}\ \texttt{then}\ \mathrm{p}\ \texttt{else}\ \mathrm{q}, \sigma \rangle \longrightarrow \langle \mathrm{q}, \sigma \rangle} \ (\mathsf{if}_2) \qquad \frac{\langle \mathrm{b}, \sigma \rangle \longrightarrow^{\star} \mathtt{ff}}{\langle \texttt{while}\ \mathrm{b}\ \texttt{do}\ \{\ \mathrm{p}\ \}, \sigma \rangle \longrightarrow^{\star} \sigma} \ (\mathsf{wh}_2)$$

$$\frac{\langle \mathrm{b}, \sigma \rangle \longrightarrow^{\star} \mathtt{tt}}{\langle \texttt{while}\ \mathrm{b}\ \texttt{do}\ \{\ \mathrm{p}\ \}, \sigma \rangle \longrightarrow \langle \mathrm{p} \mathbin{\texttt{;}} \texttt{while}\ \mathrm{b}\ \texttt{do}\ \{\ \mathrm{p}\ \}, \sigma \rangle} \ (\mathsf{wh}_1)$$

Exercises

1. Define a semantics for arithmetic expressions
2. Write down the sequence of steps that originates from

$$\langle \texttt{while}\, \texttt{tt}\, \texttt{do}\, \{\, \mathtt{x} := \mathtt{x} + 1 \,\}, \sigma \rangle$$

3. Conclude that the while-language does not enjoy the termination property