

Simply Typed Lambda-calculus

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The Calculus

Denotational Semantics

Equational System

Disjunctive Types

The essence

Knowledge obtained via assumptions and logical rules

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Knowledge obtained via assumptions and logical rules

Studied since Aristotle ...

... long before the age of artificial computers

What does it have to do with programming ?

A Basic Deductive System

$\mathbb{A}, \mathbb{B} \dots$ denote propositions and
1 a proposition that always holds



If \mathbb{A} and \mathbb{B} are propositions then

- $\mathbb{A} \times \mathbb{B}$ is a proposition – conjunction of \mathbb{A} and \mathbb{B}
- $\mathbb{A} \rightarrow \mathbb{B}$ is a proposition – implication of \mathbb{B} from \mathbb{A}

A Basic Deductive System

Γ denotes a list of propositions (often called context)

$\Gamma \vdash \mathbb{A}$ reads “if the propositions in Γ hold then \mathbb{A} also holds”

$$\frac{\mathbb{A} \in \Gamma}{\Gamma \vdash \mathbb{A}} \text{ (ass)} \quad \frac{}{\Gamma \vdash 1} \text{ (trv)} \quad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{A}} (\pi_1) \quad \frac{\Gamma \vdash \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \mathbb{B}} (\pi_2)$$

$$\frac{\Gamma \vdash \mathbb{A} \quad \Gamma \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \times \mathbb{B}} \text{ (prd)} \quad \frac{\Gamma, \mathbb{A} \vdash \mathbb{B}}{\Gamma \vdash \mathbb{A} \rightarrow \mathbb{B}} \text{ (cry)} \quad \frac{\Gamma \vdash \mathbb{A} \rightarrow \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (app)}$$

Exercise

Show that $\mathbb{A} \times \mathbb{B} \vdash \mathbb{B} \times \mathbb{A}$

New Knowledge From Old

The rules below are derivable from the previous system

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C} \text{ (exchange)}$$

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A} \text{ (weakening)}$$

$$\frac{\Gamma, A \vdash B \quad \Gamma \vdash A}{\Gamma \vdash B} \text{ (cut elimination)}$$

Proofs (again) by an appeal to your old friend ... induction :-)

Derive the following judgements

- $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$
- $A \rightarrow B, A \rightarrow C \vdash A \rightarrow B \times C$

Back to programming ...

The Bare Essentials of Programming

We should think of what are the basic features of programming ...

- variables
- function application and creation
- pairing ...

and base our study on the simplest language with such features ...

Simply-typed λ -calculus

The basis of Haskell, ML, Eff, F#, Agda, Elm and many other programming languages

Simply-typed λ -Calculus

Types are defined by $\mathbb{A} ::= 1 \mid \mathbb{A} \times \mathbb{A} \mid \mathbb{A} \rightarrow \mathbb{A}$

Γ now a non-repetitive list of typed variables ($x_1 : \mathbb{A}_1 \dots x_n : \mathbb{A}_n$)

Programs built according to the following deduction rules

$$\frac{x : \mathbb{A} \in \Gamma}{\Gamma \vdash x : \mathbb{A}} \text{ (ass)} \qquad \frac{}{\Gamma \vdash * : 1} \text{ (triv)} \qquad \frac{\Gamma \vdash t : \mathbb{A} \times \mathbb{B}}{\Gamma \vdash \pi_1 t : \mathbb{A}} \text{ (\pi}_1\text{)}$$

$$\frac{\Gamma \vdash t : \mathbb{A} \quad \Gamma \vdash s : \mathbb{B}}{\Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B}} \text{ (prd)} \qquad \frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B}}{\Gamma \vdash \lambda x : \mathbb{A}. t : \mathbb{A} \rightarrow \mathbb{B}} \text{ (cry)}$$

$$\frac{\Gamma \vdash t : \mathbb{A} \rightarrow \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash ts : \mathbb{B}} \text{ (app)}$$

Examples of λ -terms

$x : \mathbb{A} \vdash x : \mathbb{A}$ (identity)

$x : \mathbb{A} \vdash \langle x, x \rangle : \mathbb{A} \times \mathbb{A}$ (duplication)

$x : \mathbb{A} \times \mathbb{B} \vdash \langle \pi_2 x, \pi_1 x \rangle : \mathbb{B} \times \mathbb{A}$ (swap)

$f : \mathbb{A} \rightarrow \mathbb{B}, g : \mathbb{B} \rightarrow \mathbb{C} \vdash \lambda x : \mathbb{A}. g(f x) : \mathbb{A} \rightarrow \mathbb{C}$ (composition)

Recall the derivations that lead to the judgement

$$A \rightarrow B, A \rightarrow C \vdash A \rightarrow B \times C$$

Build the corresponding program

Derive as well the judgement

$$A \rightarrow B \vdash A \times C \rightarrow B \times C$$

and subsequently build the corresponding program

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A Semantics for Simply Typed λ -calculus

We wish to assign a mathematical meaning to λ -terms

$$\llbracket - \rrbracket : \lambda\text{-terms} \longrightarrow \dots$$

so that we can reason about them rigorously, and take advantage of known mathematical theories

A Semantics for Simply Typed λ -calculus

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This is the goal of the next slides. But first ...

Functions: Basic Facts

For every set X there exists a 'trivial' function

$$! : X \longrightarrow \{\star\} = 1 \qquad !(x) = \star$$

We can always pair two functions into $f : X \rightarrow A, g : X \rightarrow B$

$$\langle f, g \rangle : X \rightarrow A \times B \qquad \langle f, g \rangle(x) = (f\ x, g\ x)$$

There exist projection functions

$$\pi_1 : X \times Y \rightarrow X \qquad \pi_1(x, y) = x$$

$$\pi_2 : X \times Y \rightarrow Y \qquad \pi_2(x, y) = y$$

Functions: Basic Facts

We can always ‘curry’ a function $f : X \times Y \rightarrow Z$ into

$$\lambda f : X \rightarrow Z^Y \quad \lambda f(x) = (y \mapsto f(x, y))$$

Consider sets X, Y, Z . There exists an application function

$$\text{app} : Z^Y \times Y \rightarrow Z \quad \text{app}(f, y) = f y$$

Types \mathbb{A} interpreted as sets $\llbracket \mathbb{A} \rrbracket$

$$\llbracket 1 \rrbracket = \{\star\}$$

$$\llbracket \mathbb{A} \times \mathbb{B} \rrbracket = \llbracket \mathbb{A} \rrbracket \times \llbracket \mathbb{B} \rrbracket$$

$$\llbracket \mathbb{A} \rightarrow \mathbb{B} \rrbracket = \llbracket \mathbb{B} \rrbracket^{\llbracket \mathbb{A} \rrbracket}$$

Typing contexts Γ interpreted as Cartesian products

$$\llbracket \Gamma \rrbracket = \llbracket x_1 : \mathbb{A}_1, \dots, x_n : \mathbb{A}_n \rrbracket = \llbracket \mathbb{A}_1 \rrbracket \times \dots \times \llbracket \mathbb{A}_n \rrbracket$$

λ -terms $\Gamma \vdash t : \mathbb{A}$ interpreted as functions

$$\llbracket \Gamma \vdash t : \mathbb{A} \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \mathbb{A} \rrbracket$$

λ -term $\Gamma \vdash t : \mathbb{A}$ interpreted as a function

$$\llbracket \Gamma \vdash t : \mathbb{A} \rrbracket : \llbracket \Gamma \rrbracket \longrightarrow \llbracket \mathbb{A} \rrbracket$$

$$\frac{x_i : \mathbb{A} \in \Gamma}{\llbracket \Gamma \vdash x_i : \mathbb{A} \rrbracket = \pi_i}$$

$$\frac{}{\llbracket \Gamma \vdash * : 1 \rrbracket = !}$$

$$\frac{\llbracket \Gamma \vdash t : \mathbb{A} \times \mathbb{B} \rrbracket = f}{\llbracket \Gamma \vdash \pi_1 t : \mathbb{A} \rrbracket = \pi_1 \cdot f}$$

$$\frac{\llbracket \Gamma \vdash t : \mathbb{A} \rrbracket = f \quad \llbracket \Gamma \vdash s : \mathbb{B} \rrbracket = g}{\llbracket \Gamma \vdash \langle t, s \rangle : \mathbb{A} \times \mathbb{B} \rrbracket = \langle f, g \rangle}$$

$$\frac{\llbracket \Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \rrbracket = f}{\llbracket \Gamma \vdash \lambda x : \mathbb{A}. t : \mathbb{A} \rightarrow \mathbb{B} \rrbracket = \lambda f}$$

$$\frac{\llbracket \Gamma \vdash t : \mathbb{A} \rightarrow \mathbb{B} \rrbracket = f \quad \llbracket \Gamma \vdash s : \mathbb{A} \rrbracket = g}{\llbracket \Gamma \vdash t s : \mathbb{B} \rrbracket = \text{app} \cdot \langle f, g \rangle}$$

The Unravelling

$$\llbracket x \vdash \langle \pi_2 x, \pi_1 x \rangle \rrbracket = \dots$$

$$\llbracket - \vdash \lambda x. \langle \pi_2 x, \pi_1 x \rangle \rrbracket = \dots$$

$$\llbracket f, g, x \vdash g f x \rrbracket = \dots$$

$$\llbracket f, g \vdash \lambda x. g f x \rrbracket = \dots$$

$$\llbracket f, x \vdash \langle f \pi_1 x, \pi_2 x \rangle \rrbracket = \dots$$

$$\llbracket f \vdash \lambda x. \langle f \pi_1 x, \pi_2 x \rangle \rrbracket = \dots$$

$$\llbracket - \vdash \lambda f. \lambda x. \langle f \pi_1 x, \pi_2 x \rangle \rrbracket = \dots$$

(**N.B.** all types omitted for simplicity)

Denotational Semantics and Equivalence Revisited

Show that the following equations hold

$$\llbracket x, y \vdash \pi_1 \langle x, y \rangle \rrbracket = \llbracket x, y \vdash x \rrbracket$$

$$\llbracket \Gamma \vdash t \rrbracket = \llbracket \Gamma \vdash \langle \pi_1 t, \pi_2 t \rangle \rrbracket$$

$$\llbracket x \vdash (\lambda y. \langle x, y \rangle) x \rrbracket = \llbracket x \vdash \langle x, x \rangle \rrbracket$$

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$$\llbracket x \vdash (\lambda y. \langle x, y \rangle) x \rrbracket = \llbracket x \vdash \langle x, x \rangle \rrbracket$$

Show that the (complicated) λ -term below is really just the identity

$$z \vdash \lambda x. \langle \pi_2 x, \pi_1 x \rangle \left(\lambda y. \langle \pi_2 y, \pi_1 y \rangle z \right)$$

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Hard ?

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Logic to the Rescue !

Recall that the rules below are derivable from our logical system

$$\frac{\Gamma, \mathbb{A}, \mathbb{B}, \Delta \vdash \mathbb{C}}{\Gamma, \mathbb{B}, \mathbb{A}, \Delta \vdash \mathbb{C}} \text{ (exchange)}$$

$$\frac{\Gamma \vdash \mathbb{A}}{\Gamma, \mathbb{B} \vdash \mathbb{A}} \text{ (weakening)}$$

$$\frac{\Gamma, \mathbb{A} \vdash \mathbb{B} \quad \Gamma \vdash \mathbb{A}}{\Gamma \vdash \mathbb{B}} \text{ (cut elimination)}$$

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)}$$

$$\frac{\Gamma \vdash t : \mathbb{A}}{\Gamma, x : \mathbb{B} \vdash t : \mathbb{A}} \text{ (weak)}$$

$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \dots : \mathbb{B}} \text{ (cut elimination)}$$

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)}$$

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$$\frac{\Gamma, x : \mathbb{A} \vdash t : \mathbb{B} \quad \Gamma \vdash s : \mathbb{A}}{\Gamma \vdash \dots : \mathbb{B}} \text{ (cut elimination)}$$

Filling up the dots will lead us to a fundamental concept

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)}$$

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Filling up the dots will lead us to a fundamental concept

Substitution

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The essence

Substitution of variables in a λ -term t by another λ -term s

$t[s/x]$ reads *"replace every occurrence of x in t by s "*

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Substitution of variables in a λ -term t by another λ -term s

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Example

$$\langle x, x \rangle[s/x] = \langle s, s \rangle$$

$$\langle x, y \rangle[s/x] = \langle s, y \rangle$$

$$\langle y, z \rangle[s/x] = \langle y, z \rangle$$

Substitution More Formally

We define it by induction

$$x[s/y] = \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases}$$

$$*[s/y] = *$$

$$\langle t_1, t_2 \rangle [s/y] = \langle t_1[s/y], t_2[s/y] \rangle$$

$$(t_1 \ t_2)[s/y] = t_1[s/y] \ t_2[s/y]$$

$$(\pi_1 t)[s/y] = \pi_1 t[s/y]$$

$$(\pi_2 t)[s/y] = \pi_2 t[s/y]$$

$$(\lambda x. t)[s/y] = \dots$$

$\lambda x. y$ is a “constant function” (given x return y)

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$(\lambda x. y)[z/y]$ is still a “constant function” (given x return z)

Variable Captures

$\lambda x. y$ is a “constant function” (given x return y)

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$(\lambda x. y)[x/y]$ is now the identity !?

Variable Captures

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$(\lambda x. y)[x/y]$ is now the identity !?

The problem: variable x “captured” by the construct “ $\lambda x.$ ”

Somehow similar to variable shadowing in programming

Substitution More Formally

$$x[s/x] = \begin{cases} s & \text{if } x = y \\ x & \text{otherwise} \end{cases}$$

$$*[s/y] = *$$

$$\langle t_1, t_2 \rangle [s/y] = \langle t_1[s/y], t_2[s/y] \rangle$$

$$(t_1 \ t_2)[s/y] = t_1[s/y] \ t_2[s/y]$$

$$(\pi_1 \ t)[s/y] = \pi_1 \ t[s/y]$$

$$(\pi_2 \ t)[s/y] = \pi_2 \ t[s/y]$$

$$(\lambda x. t)[s/y] = \lambda z. t[z/x][s/y]$$

(where z is fresh (i.e. new))

Compute the following substitutions

$$* [t/y][s/z] = \dots$$

$$\langle y, z \rangle [t/y][s/z] = \dots$$

$$(\lambda x. x) [t/x] = \dots$$

$$(\lambda x. \langle x, y \rangle) [z/y] = \dots$$

$$(\lambda x. \langle x, y \rangle) [x/y] = \dots$$

Via the Programming Lens

$$\frac{\Gamma, x : \mathbb{A}, y : \mathbb{B}, \Delta \vdash t : \mathbb{C}}{\Gamma, y : \mathbb{B}, x : \mathbb{A}, \Delta \vdash t : \mathbb{C}} \text{ (exch)}$$

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