

# Introduction to modal logic

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# A logic

## A language

i.e. a collection of well-formed expressions to which meaning can be assigned.

## A semantics

describing how language expressions are interpreted as statements about something.

## A deductive system

i.e. a collection of rules to derive in a purely syntactic way facts and relationships among semantic objects described in the language.

## Note

- a purely syntactic approach (up to the 1940's; the **sacred form**)
- a model theoretic approach (A. Tarski legacy)

# Semantic reasoning: models

- sentences
- models & satisfaction:  $\mathfrak{M} \models \phi$
- validity:  $\models \phi$  ( $\phi$  is satisfied in every possible structure)
- logical consequence:  $\Phi \models \phi$  ( $\phi$  is satisfied in every model of  $\Phi$ )
- theory:  $Th \Phi$  (set of logical consequences of a set of sentences  $\Phi$ )

# Syntactic reasoning: deductive systems

## Deductive systems $\vdash$

- sequents
  - Hilbert systems
  - natural deduction
  - tableaux systems
  - resolution
  - ...
- 
- derivation and proof
  - deductive consequence:  $\Phi \vdash \phi$
  - theorem:  $\vdash \phi$

# Soundness & completeness

- A deductive system  $\vdash$  is **sound** wrt a semantics  $\models$  if for all sentences  $\phi$

$$\vdash \phi \implies \models \phi$$

(every theorem is valid)

- ... **complete** ...

$$\models \phi \implies \vdash \phi$$

(every valid sentence is a theorem)

# Consistency & refutability

For logics with **negation** and a **conjunction** operator

- A sentence  $\phi$  is **refutable** if  $\neg\phi$  is a theorem (i.e.  $\vdash \neg\phi$ )
- A set of sentences  $\Phi$  is **refutable** if some finite conjunction of elements in  $\Phi$  is refutable
- $\phi$  or  $\Phi$  is **consistent** if it is not refutable.

# Examples

$$\mathfrak{M} \models \phi$$

- Propositional logic (logic of **uninterpreted assertions**; models are **truth assignments**)
- Equational logic (formalises **equational** reasoning; models are **algebras**)
- First-order logic (logic of **predicates** and **quantification** over structures; models are **relational structures**)
- Modal logics
- ...

## Modal logic (from P. Blackburn, 2007)

*Over the years modal logic has been applied in many different ways. It has been used as a tool for reasoning about **time**, **beliefs**, **computational systems**, **necessity** and **possibility**, and much else besides.*

*These applications, though diverse, have something important in common: the key ideas they employ (flows of time, relations between epistemic alternatives, transitions between computational states, networks of possible worlds) can all be represented as **simple graph-like structures**.*

Modal logics are

- tools to talk about relational, or graph-like structures.
- fragments of classical ones, with restricted forms of quantification ...
- ... which tend to be **decidable** and described in a pointfree notations.



# The language

## Syntax

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle m \rangle \phi \mid [m] \phi$$

where  $p \in \text{PROP}$  and  $m \in \text{MOD}$

Disjunction ( $\vee$ ) and equivalence ( $\leftrightarrow$ ) are defined by abbreviation. The **signature** of the basic modal language is determined by sets **PROP** of **propositional** symbols (typically assumed to be denumerably infinite) and **MOD** of **modality** symbols.

# The language

## Notes

- if there is only one modality in the signature (i.e., MOD is a singleton), write simply  $\Diamond\phi$  and  $\Box\phi$
- the language has some redundancy: in particular modal connectives are **dual** (as quantifiers are in first-order logic):  $[m]\phi$  is equivalent to  $\neg\langle m\rangle\neg\phi$
- define **modal depth** in a formula  $\phi$ , denoted by  $\text{md } \phi$  as the maximum level of nesting of modalities in  $\phi$

## Example

Models as LTSs over Act.

$MOD = \mathbb{P}Act$  – sets of actions.

$\langle\{a, b\}\rangle\phi$  can be read as “after observing *a* **or** *b*,  $\phi$  must hold.”

$[\{a, b\}]\phi$  can be read as “after observing *a* **and** *b*,  $\phi$  must hold.”

# Semantics

$\mathfrak{M}, w \models \phi$  – what does it mean?

## Model definition

A **model** for the language is a pair  $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ , where

- $\mathfrak{F} = \langle W, \{R_m\}_{m \in \text{MOD}} \rangle$   
is a **Kripke frame**, ie, a non empty set  $W$  and a family  $R_m$  of **binary relations** (called *accessibility relations*) over  $W$ , one for each modality symbol  $m \in \text{MOD}$ . Elements of  $W$  are called **points**, **states**, **worlds** or simply **vertices** in directed graphs.
- $V : \text{PROP} \longrightarrow \mathcal{P}(W)$  is a **valuation**.

## When $\text{MOD} = 1$

- $\Diamond\phi$  and  $\Box\phi$  instead of  $\langle \cdot \rangle \phi$  and  $[\cdot] \phi$
- $\mathfrak{F} = \langle W, R \rangle$  instead of  $\mathfrak{F} = \langle W, \{R_m\}_{m \in \text{MOD}} \rangle$

# Semantics

Satisfaction: for a model  $\mathfrak{M}$  and a point  $w$

$\mathfrak{M}, w \models \text{true}$

$\mathfrak{M}, w \not\models \text{false}$

$\mathfrak{M}, w \models p$                       iff     $w \in V(p)$

$\mathfrak{M}, w \models \neg\phi$                     iff     $\mathfrak{M}, w \not\models \phi$

$\mathfrak{M}, w \models \phi_1 \wedge \phi_2$             iff     $\mathfrak{M}, w \models \phi_1$  and  $\mathfrak{M}, w \models \phi_2$

$\mathfrak{M}, w \models \phi_1 \rightarrow \phi_2$             iff     $\mathfrak{M}, w \not\models \phi_1$  or  $\mathfrak{M}, w \models \phi_2$

$\mathfrak{M}, w \models \langle m \rangle \phi$                 iff    there exists  $v \in W$  st  $wR_mv$  and  $\mathfrak{M}, v \models \phi$

$\mathfrak{M}, w \models [m] \phi$                 iff    for all  $v \in W$  st  $wR_mv$  and  $\mathfrak{M}, v \models \phi$

# Semantics

## Satisfaction

A formula  $\phi$  is

- **satisfiable in a model**  $\mathfrak{M}$  if it is satisfied at some point of  $\mathfrak{M}$
- **globally satisfied** in  $\mathfrak{M}$  ( $\mathfrak{M} \models \phi$ ) if it is satisfied at all points in  $\mathfrak{M}$
- **valid** ( $\models \phi$ ) if it is globally satisfied in all models
- **a semantic consequence** of a set of formulas  $\Gamma$  ( $\Gamma \models \phi$ ) if for all models  $\mathfrak{M}$  and all points  $w$ , if  $\mathfrak{M}, w \models \Gamma$  then  $\mathfrak{M}, w \models \phi$

## Example: Hennessy-Milner logic

### Process logic (Hennessy-Milner logic)

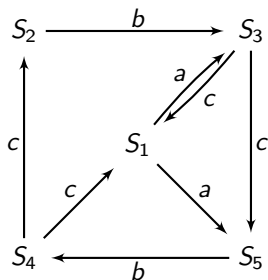
- $\text{PROP} = \emptyset$
- $W = \mathbb{P}$  is a set of states, typically process terms, in a labelled transition system
- each subset  $K \subseteq \text{Act}$  of actions generates a modality corresponding to transitions labelled by an element of  $K$

Assuming the underlying LTS  $\mathfrak{F} = \langle \mathbb{P}, \{p \xrightarrow{K} p' \mid K \subseteq \text{Act}\} \rangle$  as the modal frame, satisfaction is abbreviated as

$$p \models \langle K \rangle \phi \quad \text{iff} \quad \exists_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} \cdot q \models \phi$$

$$p \models [K] \phi \quad \text{iff} \quad \forall_{q \in \{p' \mid p \xrightarrow{a} p' \wedge a \in K\}} \cdot q \models \phi$$

# Example: Hennessy-Milner logic



Prove:

- ①  $S_2 \models [a] (\langle b \rangle tt \wedge \langle c \rangle tt)$
- ②  $S_1 \not\models [a] (\langle b \rangle tt \wedge \langle c \rangle tt)$
- ③  $S_2 \models [b] [c] (\langle a \rangle tt \vee \langle b \rangle tt)$
- ④  $S_1 \models [b] [c] (\langle a \rangle tt \vee \langle b \rangle tt)$

# Proof system **K**

## Minimal modal logic

- all formulas with the form of a **propositional tautology** (including formulas which contain modalities but are truth-functionally tautologous)
- all instances of the axiom schema:

$$\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$$

- two proof rules:

if  $\vdash \phi$  and  $\vdash \phi \rightarrow \psi$  then  $\vdash \psi$  (**modus ponens**)

if  $\vdash \phi$  then  $\vdash \Box\phi$  (**generalization**)



# Variants

*Normal modal logics* are **axiomatic extensions to K**

- different applications of modal logic typically validate different modal axioms;
- a normal modal logic is identified with the set of formulas it generates; it is said to be **consistent** if it does not contain all formulas. This identification immediately induces a lattice structure on the set of all such logics.

# Variants

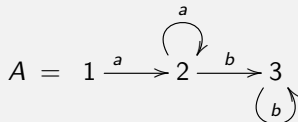
Modal axioms reflect **properties of accessibility relations**:

- **transitive** frames:  $\Box\phi \rightarrow \Box\Box\phi$
- **simple** frames:  $\Diamond\phi \rightarrow \Box\phi$
- frames consisting of **isolated reflexive points**:  $\phi \leftrightarrow \Box\phi$
- frames consisting of **isolated irreflexive points**:  $\Box\text{false}$

But there are classes of frames which are not modally definable, eg, **connected**, **irreflexive**, **containing a isolated irreflexive point**

# Examples I

## An automaton



- two modalities  $\langle a \rangle$  and  $\langle b \rangle$  to explore the corresponding classes of transitions
- note that

$$1 \models \langle a \rangle \cdots \langle a \rangle \langle b \rangle \cdots \langle b \rangle t$$

where  $t$  is a proposition valid only at the (terminal) state 3.

- all modal formulas of this form correspond to the strings accepted by the automaton, i.e. in language  $\mathcal{L} = \{a^m b^n \mid m, n > 0\}$

## Examples II

$(P, <)$  a strict partial order with infimum 0

- $P, x \models \Box \text{false}$  if  $x$  is a maximal element of  $P$
- $P, 0 \models \Diamond \Box \text{false}$  iff ...
- $P, 0 \models \Box \Diamond \Box \text{false}$  iff ...

# Examples III

## Temporal logic

- $\langle T, < \rangle$  where  $T$  is a set of time points (instants, execution states, ...) and  $<$  is the **earlier than** relation on  $T$ .
- Thus,  $\Box\varphi$  (respectively,  $\Diamond\varphi$ ) means that  $\varphi$  holds in all (respectively, some) time points.

# Examples III

$\langle T, < \rangle$

The structure of time is a **strict partial order**  
(i.e., a transitive and asymmetric relation)

For any such structure, a new modality,  $\bigcirc$ , can be defined based on the **cover** relation  $\triangleleft$  for  $<$  (i.e.,  $x \triangleleft y$  if (1) every  $x < y$  and (2) there is no  $z$  such that  $x < z < y$ ). Thus,

$$t \models \bigcirc \phi \quad \text{iff} \quad \forall t' \in \{p' \mid t \triangleleft t'\} . t' \models \phi$$

$$t \models \Box \phi \quad \text{iff} \quad \forall t' \in \{p' \mid t < t'\} . t' \models \phi$$

$$t \models \Diamond \phi \quad \text{iff} \quad \exists t' \in \{p' \mid t < t'\} . t' \models \phi$$

## Examples III

... but typical structures, however, are

### Linear time structures

- **linear**:  $\langle \forall x, y : x, y \in T : x = y \wedge x < y \wedge y < x \rangle$ .
- **discrete**: for each  $t \in T$ , i) if there is a  $u > t$  there is a first such  $u$ ;  
ii) if there is a  $u < t$  there is a last such  $u$ .
- **dense**: if for all  $t, x \in T$ , if  $x < t$  there is a  $v \in T$  such that  $x < v < t$ .
- **Dedekind complete**: if for all  $S \subseteq T$  non-empty and bounded above, there is a least upper bound in  $T$ .
- **continuous**: if it is both dense and Dedekind complete

# Examples IV

## Epistemic logic (J. Hintikka, 1962)

- $W$  is a set of agents
- $\alpha \models i$  means  $i$  is the current knowledge of agent  $i$
- $\alpha \models \Box j$  means the agent knows that  $j$  (in the sense that at each alternative epistemic situation information  $j$  is known)
- $\alpha \models \Diamond j$  means the agent knows that knowledge  $j$  is consistent with what the agent knows (is an epistemically acceptable alternative)



# The first order connection

## From modal logic

$$\phi ::= p \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle m \rangle \phi \mid [m] \phi$$

## To first order logic

$$\phi ::= Px \mid \text{true} \mid \text{false} \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \rightarrow \phi_2 \mid \langle \exists x :: \phi \rangle \mid \langle \forall y :: \phi \rangle$$

# The first order connection

Boxes and diamonds are essentially a **macro notation** to encode quantification over accessible states in a point free way.

## The standard translation

... to first-order logic **expands** these macros:

$$ST_x(p) = P x$$

$$ST_x(\text{true}) = \text{true}$$

$$ST_x(\text{false}) = \text{false}$$

$$ST_x(\neg\phi) = \neg ST_x(\phi)$$

$$ST_x(\phi_1 \wedge \phi_2) = ST_x(\phi_1) \wedge ST_x(\phi_2)$$

$$ST_x(\phi_1 \rightarrow \phi_2) = ST_x(\phi_1) \rightarrow ST_x(\phi_2)$$

$$ST_x(\langle m \rangle \phi) = \langle \exists y :: (xR_my \wedge ST_y(\phi)) \rangle$$

$$ST_x([m] \phi) = \langle \forall y :: (xR_my \rightarrow ST_y(\phi)) \rangle$$

# The first order connection

## Lemma

For any  $\phi$ ,  $\mathfrak{M}$  and point  $w$  in  $\mathfrak{M}$ ,

$$\mathfrak{M}, w \models \phi \quad \text{iff} \quad \mathfrak{M} \models ST_x(\phi)[x \leftarrow w]$$

## Note

Note how the (unique) free variable  $x$  in  $ST_x$  mirrors in first-order the internal perspective: **assigning a value to  $x$  corresponds to evaluating the modal formula at a certain state.**

# The first order connection

The standard translation provides a **bridge** between modal logic and classical logic which makes possible to **transfer** results from one side to the other. For example,

## Compactness

If  $\Phi$  is a set of basic modal formulas and every finite subset of  $\Phi$  is satisfiable, then  $\Phi$  itself is satisfiable.

## Löwenheim-Skolem

If  $\Phi$  is a set of basic modal formulas satisfiable in at least one infinite model, then it is satisfiable in models of every infinite cardinality.

# Summing up

- Propositional modal languages are syntactically simple languages that offer a **pointfree** notation for talking about **relational structures**
- They do this from the **inside**, using the modal operators to look for information at accessible states
- Regarded as a tool for talking about models, any basic modal language can be seen as **a fragment of first-order language**
- The **standard translation** systematically maps modal formulas to first-order formulas (in one free variable) and makes the quantification over accessible states explicit

# Exercise

## Express the following properties in Process Logic

- inevitability of  $a$ :
- progress:
- deadlock or termination:

“ $-$ ” stands for *Act*, and “ $-x$ ” abbreviates  $Act - \{x\}$

# Exercise

## Express the following properties in Process Logic

- inevitability of  $a$ :  $\langle - \rangle \text{ true} \wedge [-a] \text{ false}$
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# Exercise

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- inevitability of  $a$ :  $\langle - \rangle \text{ true} \wedge [-a] \text{ false}$
- progress:  $\langle - \rangle \text{ true}$
- deadlock or termination:  $[-] \text{ false}$
- what about

$\langle - \rangle \text{ false}$  and  $[-] \text{ true}$  ?

“ $-$ ” stands for *Act*, and “ $-x$ ” abbreviates  $\text{Act} - \{x\}$

# Exercise

## Express the following properties in Process Logic

- $\phi_0 =$  *In a taxi network, a car can collect a passenger or be allocated by the Central to a pending service*
- $\phi_1 =$  *This applies only to cars already on service*
- $\phi_2 =$  *If a car is allocated to a service, it must first collect the passenger and then plan the route*
- $\phi_3 =$  *On detecting an emergence the taxi becomes inactive*
- $\phi_4 =$  *A car on service is not inactive*

# Exercise

## Process logic: The taxi network example

- $\phi_0 = \langle \text{rec}, \text{alo} \rangle \text{ true}$
- $\phi_1 = [\text{onservice}] \langle \text{rec}, \text{alo} \rangle \text{ true}$  or  
 $\phi_1 = [\text{onservice}] \phi_0$
- $\phi_2 = [\text{alo}] \langle \text{rec} \rangle \langle \text{plan} \rangle \text{ true}$
- $\phi_3 = [\text{sos}] [-] \text{ false}$
- $\phi_4 = [\text{onservice}] \langle - \rangle \text{ true}$

# Exercise

## Standard translation to FOL

- Explain how propositional symbols and modalities are translated to first-order logic?
- In what sense can modal logic be regarded as a **pointfree** version of a FOL fragment?
- Compute  $ST_x(p \Rightarrow \langle m \rangle p)$

# Bisimulation (of models)

## Definition

Given two models  $\mathfrak{M} = \langle \langle W, R \rangle, V \rangle$  and  $\mathfrak{M}' = \langle \langle W', R' \rangle, V' \rangle$ , a **bisimulation** is a non-empty binary relation  $S \subseteq W \times W'$  st whenever  $wSw'$  one has that

- points  $w$  and  $w'$  satisfy the same propositional symbols
- if  $vRw$ , then there is a point  $v'$  in  $\mathfrak{M}'$  st  $v'Rw'$  and  $vSv'$  (zig)
- if  $v'Rw'$ , then there is a point  $v$  in  $\mathfrak{M}$  st  $vRw$  and  $vSv'$  (zag)

# Invariance and definability

## Lemma (invariance: bisimulation implies modal equivalence)

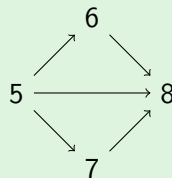
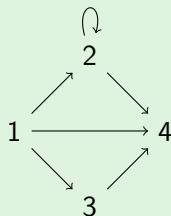
Given two models  $\mathfrak{M} = \langle \langle W, R \rangle, V \rangle$  and  $\mathfrak{M}' = \langle \langle W', R' \rangle, V' \rangle$ , and a **bisimulation**  $S \subseteq W \times W'$ , if two points  $w, w'$  are related by  $S$ , i.e.  $wSw'$ , then  $w, w'$  satisfy the same basic modal formulas.

## Applications

- to prove bisimulation failures
- to show the undefinability of some structural notions, e.g.  
**irreflexivity is modally undefinable**
- to show that typical model constructions are satisfaction preserving
- ...

# Exercise

## Find characterising formulas



e.g., (4) is the only world satisfying  $\Box \perp$

# Frame definability

- A modal formula is valid on a frame if it is true under every valuation at every world (i.e., it cannot be refuted)
- The class of frames defined by a modal formula  $\phi$  are those where  $\phi$  is valid.
- Example:  $\Diamond\Diamond p \rightarrow \Diamond p$  defines transitivity:  
 $\mathfrak{F} = \langle W, R \rangle$  is transitive iff for all  $V$  and  $w$ ,  
 $\langle \mathfrak{F}, V \rangle, w \models \Diamond\Diamond p \rightarrow \Diamond p$



# Exercise

## Exercise: other properties

- Transitivity:  $\Diamond\Diamond p \rightarrow \Diamond p$
- Reflexivity:
- Symmetry:
- Confluence:
- Irreflexibility:

# Exercise

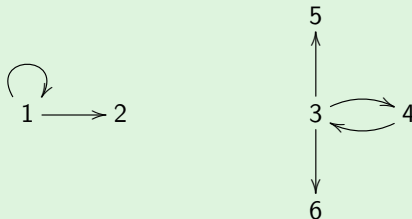
## Exercise: other properties

- Transitivity:  $\Diamond\Diamond p \rightarrow \Diamond p$
- Reflexivity:  $p \rightarrow \Diamond p$
- Symmetry:  $p \rightarrow \Box\Diamond p$
- Confluence:  $\Diamond\Box p \rightarrow \Box\Diamond p$
- Irreflexibility: **Not possible**

# Exercise

## Bisimilarity and modal equivalence

- Consider the following transition systems:



Give a modal formula that can be satisfied at point 1 but not at 3.

- Show that **irreflexivity** is modally undefinable.  
(i.e., no formula that characterises a irreflexive system)

# Invariance and definability

To prove the converse of the invariance lemma requires passing to an **infinitary** modal language with arbitrary (countable) conjunctions and disjunctions. Alternatively, and more usefully, it can be shown for **finite** models:

## Lemma (modal equivalence implies bisimulation)

If two points  $w, w'$  from two finite models  $\mathfrak{M} = \langle \langle W, R \rangle, V \rangle$  and  $\mathfrak{M}' = \langle \langle W', R' \rangle, V' \rangle$  satisfy the same modal formulas, then there is a bisimulation  $S \subseteq W \times W'$  such that  $wSw'$ .

# Invariance and definability

## Note

- The result can be **weakened** to **image-finite** models.
- Combining this result with the invariance lemma one gets the so-called **modal equivalence theorem** stating that, for image-finite models, bisimilarity and modal equivalence coincide. The result is also known as the **Hennessy-Milner theorem** who first proved it for process logics.

## Exercise

- Give an example of modally equivalent states in different Kripke structures which fail to be bisimilar.

# Invariance and definability

## Lemma (modal logic vs first-order)

The following are equivalent for all first-order formulas  $\phi(x)$  in one free variable  $x$ :

- 1  $\phi(x)$  is invariant for bisimulation.
- 2  $\phi(x)$  is equivalent to the standard translation of a basic modal formula.

Therefore:

the basic modal language corresponds to the fragment of their first-order correspondence language that is invariant for bisimulation

# Invariance and definability

- the basic modal language (interpreted over the class of all models) is computationally better behaved than the corresponding first-order language (interpreted over the same models)
- ... but clearly less expressive

	model checking	satisfiability
ML	PTIME	PSPACE-complete
FOL	PSPACE-complete	undecidable

What are the trade-offs? Can this better computational behaviour be lifted to more expressive modal logics?

# mCRL2 - modal logic

## Syntax (simplified)

$$\phi = \text{true} \mid \text{false} \mid \text{forall } x.\phi \mid \text{exists } x.\phi \\ \mid \phi \text{ OP } \phi \mid !\phi \mid [\text{mod}]\phi \mid \langle \text{mod} \rangle \phi \mid \dots$$

$$\text{mod} = \alpha \mid \text{nil} \mid \text{mod} + \text{mod} \mid \text{mod} . \text{mod} \mid \text{mod} * \mid \text{mod} +$$

$$\alpha = a(d) \mid a|b|c \mid \text{true} \mid \text{false} \mid \text{forall } x.\alpha \mid \text{exists } x.\alpha \\ \mid \alpha \text{ OP } \alpha \mid !\alpha \mid \dots$$

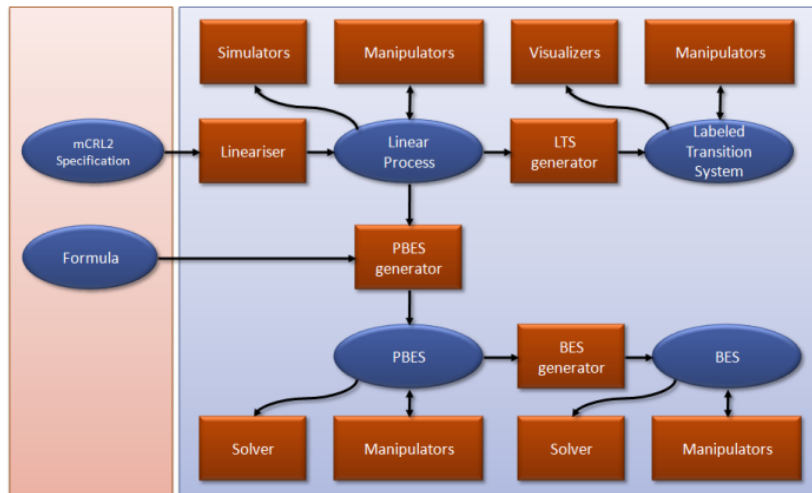
where  $OP = \{\&\&, ||, \Rightarrow\}$

## Example

“ $[\text{true}*.a]\langle b \rangle \text{true}$ ” means “*whenever an  $a$  appears after any number of steps, it must be immediately followed by  $b$* ”.



# mCRL2 toolset overview



– mCRL2 tutorial: Verification part –

# Richer modal logics

can be obtained in different ways, e.g.

- axiomatic extensions
- introducing more complex satisfaction relations
- support novel semantic capabilities
- ...

## Examples

- richer temporal logics
- hybrid logic
- modal  $\mu$ -calculus

# Temporal logics with $\mathcal{U}$ and $\mathcal{S}$

## Until and Since

$\mathfrak{M}, w \models \phi \mathcal{U} \psi$	iff	there exists $v \in W$ st $vRw$ and $\mathfrak{M}, v \models \psi$ , and for all $u$ st $uRw$ and $vRu$ , one has $\mathfrak{M}, u \models \phi$
$\mathfrak{M}, w \models \phi \mathcal{S} \psi$	iff	there exists $v \in W$ st $wRv$ and $\mathfrak{M}, v \models \psi$ , and for all $u$ st $uRv$ and $wRu$ , one has $\mathfrak{M}, u \models \phi$

- note the  $\exists\forall$  qualification pattern: these operators are neither diamonds nor boxes.
- more expressive — e.g. helpful to express **guarantee** properties, e.g. **some event will happen, and a certain condition will hold until then**

# Exercise

## Temporal logics

- Show that  $\mathcal{U}$  is modally undefinable.

*Hint* Consider the following transition structures and formula  $\text{false} \mathcal{U} \text{true}$ :



- Would this be the case if we restrict ourselves to transitive, irreflexive models?

# Linear temporal logic (LTL)

$$\phi := \text{true} \mid p \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \bigcirc\phi \mid \phi_1 \mathcal{U} \phi_2$$

mutual exclusion	$\Box(\neg c_1 \vee \neg c_2)$
liveness	$\Box\Diamond c_1 \wedge \Box\Diamond c_2$
starvation freedom	$(\Box\Diamond w_1 \rightarrow \Box\Diamond c_1) \wedge (\Box\Diamond w_2 \rightarrow \Box\Diamond c_2)$
progress	$\Box(w_1 \rightarrow \Diamond c_1)$
weak fairness	$\Diamond\Box w_1 \rightarrow \Box\Diamond c_1$
eventually forever	$\Diamond\Box w_1$

- First temporal logic to reason about reactive systems [Pnueli, 1977]
- Formulas are interpreted over **execution paths**
- Express **linear-time properties**

# Computational tree logic (CTL, CTL\*)

**state** formulas to express properties of a state:

$$\Phi := \text{true} \mid \Phi \wedge \Phi \mid \neg\Phi \mid \exists\psi \mid \forall\psi$$

**path** formulas to express properties of a path:

$$\psi := \bigcirc\Phi \mid \Phi\mathcal{U}\Psi$$

mutual exclusion	$\forall\Box(\neg c_1 \vee \neg c_2)$
liveness	$\forall\Box\forall\Diamond c_1 \wedge \forall\Box\forall\Diamond c_2$
order	$\forall\Box(c_1 \vee \forall\bigcirc c_2)$

- Branching time structure encode transitive, irreflexive but not necessarily linear flows of time
- flows are **trees**: past linear; branching future

# Hybrid logic

## Motivation

Add the possibility of **naming** points and reason about their **identity**

Compare:

$$\Diamond(r \wedge p) \wedge \Diamond(r \wedge q) \rightarrow \Diamond(p \wedge q)$$

with

$$\Diamond(i \wedge p) \wedge \Diamond(i \wedge q) \rightarrow \Diamond(p \wedge q)$$

for  $i \in \mathbf{NOM}$  (a **nominal**)

## Syntax

$$\phi ::= \dots \mid p \mid \langle m \rangle \phi \mid [m] \phi \mid i \mid @_i \phi$$

where  $p \in \mathbf{PROP}$  and  $m \in \mathbf{MOD}$  and  $i \in \mathbf{NOM}$

# Hybrid logic

## Nominals $i$

- Are special propositional symbols that hold exactly on one state (the state they **name**)
- In a model the **valuation**  $V$  is extended from

$$V : \text{PROP} \longrightarrow \mathcal{P}(W)$$

to

$$V : \text{PROP} \longrightarrow \mathcal{P}(W) \quad \text{and} \quad V : \text{NOM} \longrightarrow W$$

where NOM is the set of nominals in the model

- Satisfaction:

$$\mathfrak{M}, w \models i \qquad \text{iff } w = V(i)$$



# Hybrid logic

## The $@_i$ operator

$\mathfrak{M}, w \models @_i \phi$       iff     $\mathfrak{M}, u \models \phi$  and  $u$  is the state denoted by  $i$

## Standard translation to first-order

$$\begin{aligned} ST_x(i) &= (x = i) \\ ST_x(@_i \phi) &= ST_i(\phi)(x = i) \end{aligned}$$

i.e., hybrid logic corresponds to a first-order language enriched with constants and equality.

# Hybrid logic

## Increased frame definability

- **irreflexivity**:  $i \rightarrow \neg \Diamond i$
- **asymmetry**:  $i \rightarrow \neg \Diamond \Diamond i$
- **antisymmetry**:  $i \rightarrow \Box (\Diamond i \rightarrow i)$
- **trichotomy**:  $@_j \Diamond i \vee @_{i_j} \vee @_i \Diamond j$

# Bisimulation with nominals

## Definition

Given two models  $\mathfrak{M} = \langle \langle W, R \rangle, V \rangle$  and  $\mathfrak{M}' = \langle \langle W', R' \rangle, V' \rangle$ , a **bisimulation** is a non-empty binary relation  $S \subseteq W \times W'$  st whenever  $wSw'$  one has that

- points  $w$  and  $w'$  satisfy the same propositional symbols **and nominals**
- if  $vRw$ , then there is a point  $v'$  in  $\mathfrak{M}'$  st  $v'Rw'$  and  $vSv'$  (**zig**)
- if  $v'Rw'$ , then there is a point  $v$  in  $\mathfrak{M}$  st  $vRw$  and  $vSv'$  (**zag**)
- $V(i) R V'(i)$  for all nominal  $i$  (**name consistency**)

An **invariance** theorem and its **dual** (for image finite models) can also be proved

# Hybrid logic

## Summing up

- basic hybrid logic is a simple notation for capturing the bisimulation-invariant fragment of first-order logic with constants and equality, i.e., a mechanism for equality reasoning in propositional modal logic.
- comes cheap: up to a polynomial, the complexity of the resulting decision problem is no worse than for the basic modal language

# Hybrid logic

## Applications to architectural design

- layout of coordination circuits (e.g. in Reo)
- reconfigurable architectures (parametric on a specification logic)
- hierarchical architectures (e.g. UML statecharts)
- ...

[recent research at HASLab: projects Dali and Nasoni]

# Applications to architectural design

## Structural reasoning over Reo circuits

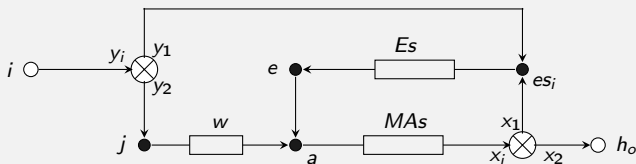
$$\phi ::= p \mid i \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid [K]\phi \mid \llbracket K \rrbracket\phi \mid @_i\phi$$

- modalities are indexed by regular expressions over channel **types**;
- $\langle K \rangle$  and  $[K]$  (resp.,  $\llbracket K \rrbracket$  and  $\llbracket K \rrbracket$ ) express properties of **outgoing** (resp., **incoming**) connections from the node in which they are evaluated.

[Nuno Oliveira PhD thesis (MAP-i, 2015)]

# Applications to architectural design

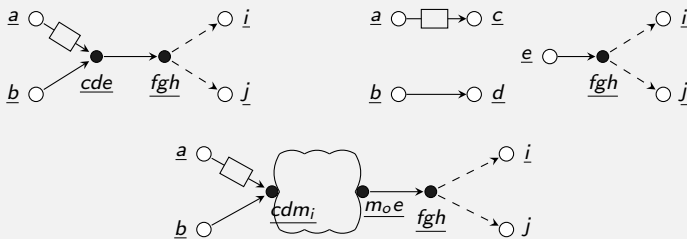
## Structural reasoning over Reo circuits



- 1  $\phi_1 \triangleq @_{t_o} \langle -^* \rangle \text{true} \wedge [-^*] [-MAs] \text{false}$   
(there is a path from triage input port ( $t_o$ ) to a  $MAs$  edge)
- 2  $\phi_2 \triangleq [-] \text{false} \rightarrow [-^*] h_o$   
(all paths from input ports, lead to the billing service ( $h_o$ ) port)

# Applications to architectural design

## Reconfiguration of Reo circuits



Invariant  $\Phi = \langle \text{sync} \rangle (\langle - \rangle \text{true} \wedge [-\text{lossy}] \text{false})$  is displaced along a reconfiguration:

$$@_{\underline{cde}} \Phi \rightsquigarrow @_{\underline{m_o e}} \Phi$$



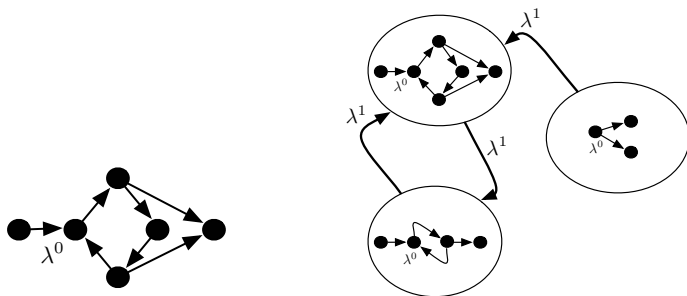
# Applications to architectural design

## Specifying reconfigurable architectures

- Reconfigurable architectures are represented as **structured transition systems** whose
- states are endowed with **local** specifications and
- the **global** transition structure models system's evolution through possible configurations.
- The hybrid language is developed **on top** whichever logic is taken for the local configurations (e.g., equational, first-order, fuzzy, etc.) — by **hybridisation**.

[Alexandre Madeira PhD thesis (MAP-i, 2013)]

# Applications to architectural design



- $\mathcal{H}$ : pure hybrid formulas
- $\mathcal{H}^2$ : hierarchical structures, e.g.

$$@_{j^1} k^0 \wedge^1 [\lambda^1](\rho_1, \dots, \rho_n)$$

# Applications to architectural design

## Hierarchical architectures

- Hierarchical architectures are represented as **hierarchical transition systems** whose states are transition systems themselves
- and (intrusive) transitions between designated states in different local transition systems at different levels of abstraction are allowed.
- Hybrid logic captures this principle which is inherent to well known design formalisms such as statecharts and UML.

