Quantum Systems

(Lecture 3: The principles of quantum computation: information, evolution, composition)

Luís Soares Barbosa

Universidade do Minho

The principles

Quantum computation explores the laws of quantum theory as computational resources.

Thus, the principles of the former are directly derived from the postulates of the latter.

- The state **space** postulate
- The state **evolution** postulate
- The state **composition** postulate
- The state **measurement** postulate

The underlying maths is that of Hilbert spaces.

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product which measures how much two vectors overlap:

$$\langle - | - \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

$$\begin{array}{ll} (1) & \langle v | \sum_i \lambda_i \cdot |w_i\rangle\rangle \;=\; \sum_i \lambda_i \langle v | w_i \rangle \\ (2) & \langle v | w \rangle = \overline{\langle w | v \rangle} \\ (3) & \langle v | v \rangle \geq 0 \;\; (\text{with equality iff } |v\rangle = 0) \end{array}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle \;=\; \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

Notation: $\langle v | w \rangle \;\equiv\; \langle v, w \rangle \equiv (|v\rangle, |w\rangle)$

The underlying maths: Hilbert spaces

Complex, inner-product vector space

A complex vector space with inner product

$$\langle - | - \rangle : V \times V \longrightarrow \mathbb{C}$$

such that

$$\begin{array}{ll} (1) & \langle v | \sum_i \lambda_i \cdot |w_i\rangle\rangle \ = \ \sum_i \lambda_i \langle v | w_i \rangle \\ (2) & \langle v | w \rangle = \overline{\langle w | v \rangle} \\ (3) & \langle v | v \rangle \geq 0 \ \ \text{(with equality iff } |v\rangle = 0) \end{array}$$

Note: $\langle - | - \rangle$ is conjugate linear in the first argument:

$$\langle \sum_i \lambda_i \cdot |w_i\rangle | v \rangle \ = \ \sum_i \overline{\lambda_i} \langle w_i | v \rangle$$

Notation: $\langle v | w \rangle \ \equiv \ \langle v, w \rangle \equiv (|v\rangle, |w\rangle)$

Dirac's notation

Dirac's bra/ket notation is a handy way to represent elements and constructions on an Hilbert space

- $|u\rangle$ A ket stands for a vector in an Hilbert space V. In $\mathcal{C}^n$, it is a column vector of complex entries. Note that the identity for $+$ (the zero vector) is just written 0.
- $\langle u|$ A bra is a vector in the dual space $V^\dagger$, i.e. scalar-valued linear maps in V. In $(\mathcal{C}^n)^\dagger$ it is the adjoint, i.e. the conjugate transpose, of the corresponding ket, therefore a row vector.

There is a bijective correspondence between $|u\rangle$ and $\langle u|$

$$|u\rangle = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \Leftrightarrow \begin{bmatrix} \overline{u}_1 \cdots \overline{u}_n \end{bmatrix} = \langle u|$$

Inner product: examples

In $\mathcal{C}$

$$\langle a+bi|c+di\rangle = (a-bi)(c+di) = ac+adi-bci+bd$$

In $\mathcal{C}^n$: The dot product

Amost useful example of a inner product is the dot product

$$\langle u|v\rangle = \underbrace{\begin{bmatrix}\overline{u_1} & \overline{u_2} & \cdots & \overline{u_n}\end{bmatrix}}_{\langle u|}\begin{bmatrix}v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix} = \sum_{i=1}^{n}\overline{u_i}v_i$$

where $\overline{c} = a - ib$ is the complex conjugate of $c = a + ib$

.

Old friends: The dual space

$\langle u|$ above is the adjoint of $|u\rangle$, i.e a vector in the dual vector space $V^\dagger$

$V^\dagger$

If V is a Hilbert space, $V^\dagger$ is the space of linear maps from V to $\mathcal{C}$.

Elements of $V^\dagger$ are denoted by

$$\langle u| : V \longrightarrow \mathcal{C} \text{ defined by } \langle u|(|v\rangle) = \langle u|v\rangle$$

In a matricial representation $\langle u|$ is obtained as the Hermitian conjugate (i.e. the transpose of the vector composed by the complex conjugate of each element) of $|u\rangle$, therefore the dot product of $|u\rangle$ and $|v\rangle$.

Old friends: Norms and orthogonality

- The inner product measures the *degree of overlapping*: $|v\rangle$ and $|w\rangle$ are orthogonal if $\langle v|w\rangle = 0$
- The "length" of a vector uses the measure of its overlap with itself to yield the (Euclidean) norm:

$$\| \, |v\rangle \, \| = \sqrt{\langle v|v\rangle}$$

  (generalizing the distance between two points)
- $|v\rangle$ is a unit vector if $\| \, |v\rangle \, \| = 1$
- normalization: $\frac{|v\rangle}{\| |v\rangle \|}$
- A set of vectors $\{|i\rangle, |j\rangle, \cdots, \}$ is orthonormal if each $|i\rangle$ is a unit vector and

$$\langle i|j\rangle \; = \; \delta_{i,j} \; = \; \begin{cases} i = j & \Rightarrow 1 \\ \text{otherwise} & \Rightarrow 0 \end{cases}$$

Old friends: Bases

Orthonormal basis

A orthonormal basis for a Hilbert space V of dimension n is a set $B = \{|i\rangle \mid i \in n\}$ of n linearly independent elements of V st

- $\langle i|j\rangle = \delta_{i,j}$ for all $|i\rangle, |j\rangle \in B$
- and B spans V, i.e. every $|v\rangle$ in V can be written as

$$|v\rangle \;=\; \sum_i \alpha_i |i\rangle \quad \text{for some } \alpha_i \in \mathcal{C}$$

Note that the amplitude or coefficient of $|v\rangle$ wrt $|i\rangle$ satisfies

$$\alpha_i \;=\; \langle i|v\rangle$$

Why?

Bases

$\alpha_i \;=\; \langle i|v\rangle$ because

$$\begin{aligned}\langle i|v\rangle &= \langle i|\sum_j \alpha_j j\rangle \\ &= \sum_j \alpha_j \langle i|j\rangle \\ &= \sum_j \alpha_j \delta_{i,j} \\ &= \alpha_i\end{aligned}$$

Note

If $|v\rangle$ is expressed wrt an orthonormal basis $\{|i\rangle \mid i \in n\}$, i.e. $|v\rangle = \sum_i \alpha_i |i\rangle$, then

$$\| \, |v\rangle \, \| = \sum_i \| \, \alpha_i \, \|^2$$

Example: The Hadamard basis

One of the infinitely many orthonormal bases for a space of dimension 2:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$

Check, e. g.

$$\langle +|-\rangle \;=\; \frac{1}{2}(|0\rangle + |1\rangle, |0\rangle - |1\rangle) \;=\; \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right) \;=\; \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} \;=\; 0$$

$$\|\,|+\rangle\,\| \;=\; \sqrt{\langle +|+\rangle} \;=\; \sqrt{\frac{1}{2}(|0\rangle + |1\rangle, |0\rangle + |1\rangle)} \;=\; \sqrt{\frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)} \;=\; 1$$

Bases

A basis for $V^\dagger$

If $\{|i\rangle \mid i \in n\}$ is an orthonormal basis for V, then

$$\{\langle i| \mid i \in n\}$$

is an orthonormal basis for $V^\dagger$

Hilbert spaces

The complete picture

An Hilbert space is an inner-product space V st the metric defined by its norm turns V into a complete metric space, i.e.any Cauchy sequence

$$|v_1\rangle, |v_2\rangle, \cdots$$

$$\forall_{\epsilon>0}\ \exists_N\ \forall_{m,n>N} \quad \| \, |v_m - v_n\rangle \, \| \leq \epsilon$$

converges
(i.e. there exists an element $|s\rangle$ in V st $\forall_{\epsilon>0}\ \exists_N\ \forall_{n>N} \quad \| \, |s - v_n\rangle \, \| \leq \epsilon$)

The completeness condition is trivial in finite dimensional vector spaces

The state space postulate

Postulate 1

The state space of a quantum system is described by a unit vector in a Hilbert space

- In practice, with finite resources, one cannot distinguish between a continuous state space from a discrete one with arbitrarily small minimum spacing between adjacente locations.
- One may, then, restrict to finite-dimensional (complex) Hilbert spaces.

The state space postulate

A quantum (binary) state is represented as a superposition, i.e. a linear combination of vectors $|0\rangle$ and $|1\rangle$ with complex coeficients:

$$|\phi\rangle \;=\; \alpha|0\rangle + \beta|1\rangle \;=\; \begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

When state $|\phi\rangle$ is measured (i.e. observed) one of the two basic states $|0\rangle, |1\rangle$ is returned with probability

$$\|\,\alpha\,\|^2 \quad \text{and} \quad \|\,\beta\,\|^2$$

respectively.
Being probabilities, the norm squared of coefficients must satisfy

$$\|\,\alpha\,\|^2 + \|\,\beta\,\|^2 = \;1$$

which enforces quantum states to be represented by unit vectors.

The state space of a qubit

Global phase

Unit vectors equivalent up to multiplication by a complex number of modulus one, i.e. a phase factor $e^{i\theta}$, represent the same state.

Let

$$|v\rangle = \alpha|u\rangle + \beta|u'\rangle$$

$$\| e^{i\theta}\alpha \|^2 = (\overline{e^{i\theta}\alpha})(e^{i\theta}\alpha) = (e^{-i\theta}\overline{\alpha})(e^{i\theta}\alpha) = \overline{\alpha}\alpha = \| \alpha \|^2$$

and similarly for β.

As the probabilities $\| \alpha \|^2$ and $\| \beta \|^2$ are the only measurable quantities, global phase has no physical meaning.

Representation redundancy

qubit state space $\neq$ complex vector space used for representation

The state space of a qubit

Relative phase

It is a measure of the angle between the two complex numbers.
Thus, it cannot be discarded!

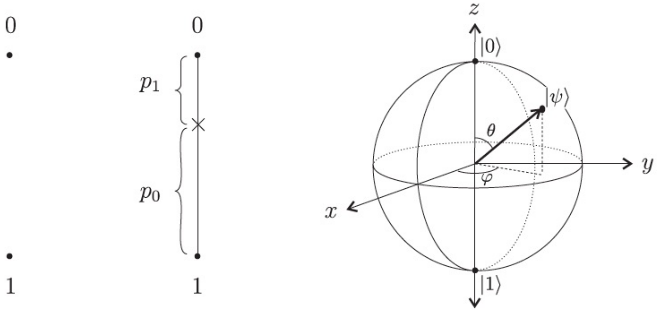
Those are different states

$$\frac{1}{\sqrt{2}}(|u\rangle + |u'\rangle) \quad \frac{1}{\sqrt{2}}(|u\rangle - |u'\rangle) \quad \frac{1}{\sqrt{2}}(e^{i\theta}|u\rangle + |u'\rangle)$$

...

The Bloch sphere

Deterministic, probabilistic and quantum bits



(from [Kaeys *et al*, 2007])

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Express $|\psi\rangle$ in polar form

$$|\psi\rangle = \rho_1 e^{i\varphi_1}|0\rangle + \rho_2 e^{i\varphi_2}|1\rangle$$

- Eliminate one of the four real parameters multiplying by $e^{-i\varphi_1}$

$$|\psi\rangle = \rho_1|0\rangle + \rho_2 e^{i(\varphi_2-\varphi_1)}|1\rangle = \rho_1|0\rangle + \rho_2 e^{i\varphi}|1\rangle$$

making $\varphi = \varphi_2 - \varphi_1$,

which is possible because global phase factors are physically meaningless.

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- Switching back the coefficient of $|1\rangle$ to Cartesian coordinates

$$|\psi\rangle = \rho_1|0\rangle + (a + bi)|1\rangle$$

the normalization constraint

$$\| \rho_1 \|^2 + \| a+ib \|^2 = \| \rho_1 \|^2 + (a-ib)(a+ib) = \boxed{\| \rho_1 \|^2 + a^2 + b^2 = 1}$$

yields the equation of a unit sphere in the real tridimensional space with Cartesian coordinates: (a, b, ρ_1).

The Bloch sphere: Representing $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

- The polar coordinates (ρ, θ, φ) of a point in the surface of a sphere relate to Cartesian ones through the correspondence

$$\begin{aligned} x =& \rho \sin\theta \cos\varphi \\ y =& \rho \sin\theta \sin\varphi \\ z =& \rho \cos\theta \end{aligned}$$

- Recalling $\rho = 1$ (cf unit vector),

$$\begin{aligned} |\psi\rangle &= \rho_1|0\rangle + (a + ib)|1\rangle \\ &= \cos\theta|0\rangle + \sin\theta(\cos\varphi + i\sin\varphi)|1\rangle \\ &= \cos\theta|0\rangle + e^{i\varphi}\sin\theta|1\rangle \end{aligned}$$

which, with two parameters, defines a point in the sphere's surface.

The Bloch sphere

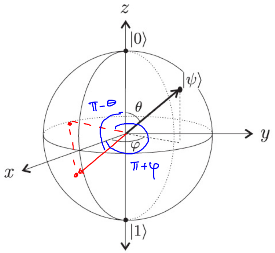
Actually, one may just focus on the upper hemisphere ($0 \leq \theta' \leq \frac{\pi}{2}$) as opposite points in the lower one differ only by a phase factor of -1, as suggested by

$$\theta' = 0 \;\Rightarrow\; |\psi\rangle \;=\; \cos 0|0\rangle + e^{i\varphi}\sin 0|1\rangle \;=\; |0\rangle$$
$$\theta' = \frac{\pi}{2} \;\Rightarrow\; |\psi\rangle \;=\; \cos\frac{\pi}{2}|0\rangle + e^{i\varphi}\sin\frac{\pi}{2}|1\rangle \;=\; e^{i\varphi}|1\rangle \;=\; |1\rangle$$

Note that longitude (φ) is irrelevant in a pole!

The Bloch sphere

Indeed, let $|\psi'\rangle$ be the opposite point on the sphere with polar coordinates $(1, \pi - \theta, \varphi + \pi)$:



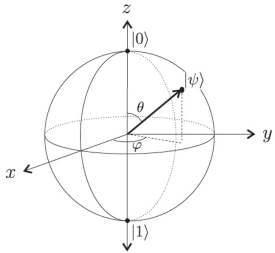
$$
\begin{aligned}
|\psi'\rangle &= \cos(\pi - \theta)|0\rangle + e^{i(\varphi+\pi)} \sin(\pi - \theta)|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} e^{i\pi} \sin\theta|1\rangle \\
&= -\cos\theta|0\rangle + e^{i\varphi} \sin\theta|1\rangle \\
&= -|\psi\rangle
\end{aligned}
$$

The Bloch sphere

which leads to

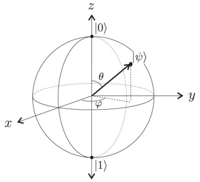
$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

where $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$



The map $\frac{\theta}{2} \mapsto \theta$ is one-to-one at any point but at $\frac{\theta}{2}$:
all points on the equator are mapped into a single point: the south pole.

The Bloch sphere



- The poles represent the classical bits. In general, orthogonal states correspond to antipodal points and every diameter to a basis for the single-qubit state space.
- Once measured a qubit collapses to one of the two poles. Which pole depends exactly on the arrow direction: The angle θ measures that probability: If the arrow points at the equator, there is 50-50 chance to collapse to any of the two poles.
- Rotating a vector wrt the z-axis results into a phase change (φ), and does not affect which state the arrow will collapse to, when measured.

The state evolution postulate

If a quantum state is a ray (i.e. a unit vector in a Hilbert space H up to a global phase), its evolution is specified a certain kind of linear operators $U : H \longrightarrow H$.

Linearity

$$U\left(\sum_j \alpha_j \,|v_j\rangle\right) \;=\; \sum_j \alpha_j \, U(|v_j\rangle)$$

just by itself has an important consequence: quantum states cannot be cloned

The no-cloning theorem

Linearity implies that quantum states cannot be cloned

Let $U(|a\rangle|0\rangle) = |a\rangle|a\rangle$ be a 2-qubit operator and $|c\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle)$ for $|a\rangle$, $|b\rangle$ orthogonal. Then,

$$\begin{aligned} U(|c\rangle|0\rangle) &= \frac{1}{\sqrt{2}}(U(|a\rangle|0\rangle) + U(|b\rangle|0\rangle)) \\ &= \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |b\rangle|b\rangle) \\ &\neq \frac{1}{\sqrt{2}}(|a\rangle|a\rangle + |a\rangle|b\rangle + |b\rangle|a\rangle + |b\rangle|b\rangle) \\ &= |c\rangle|c\rangle \\ &= U(|c\rangle|0\rangle) \end{aligned}$$

As already seen, $|x\rangle|y\rangle = |xy\rangle = |x\rangle \otimes |y\rangle$

The adjoint operator

Given an operator $U : H \longrightarrow H$, its adjoint $U^\dagger : H^\dagger \longrightarrow H^\dagger$ is defined by

$$U^\dagger \langle w| \;\; (|v\rangle) \;\; = \;\; \langle w| \;\; (U|v\rangle) \tag{1}$$

Note that $(UV)^\dagger = V^\dagger U^\dagger$ because

$$\begin{aligned} (UV)^\dagger \langle w| \, (|v\rangle) &= \langle w| \, (UV|v\rangle) \\ &= U^\dagger \langle w| \, (V|v\rangle) \\ &= V^\dagger U^\dagger \langle w| \, (|v\rangle) \end{aligned}$$

The adjoint operator

Using the definition of the application of a transformation in $H^\dagger$ to an element of H,

$$\langle t| \, (|u\rangle) \; = \; (|t\rangle, (|u\rangle) \; = \; \langle t|u\rangle$$

equation (1), boils down to an equality between inner products:

$$\begin{aligned} U^\dagger \langle w| \, (|v\rangle) &= ((U^\dagger \langle w|)^\dagger, |v\rangle) \\ &= (|w\rangle U, |v\rangle) \\ &= (|w\rangle, U|v\rangle) \\ &= \langle w| \, (U|v\rangle) \end{aligned}$$

The inner product $(|w\rangle U, |v\rangle) = (|w\rangle, U|v\rangle)$ can be written without any ambiguity as

$$\langle u|U|v\rangle$$

The matrix representation of $U^\dagger$ is the conjugate transpose of that of U

Exercise: Prove that $\overline{\langle w|U|v\rangle} \; = \; \langle v|U^\dagger|w\rangle$

The state evolution postulate

Postulate 2

The evolution over time of the state of a closed quantum system is described by a unitary operator.

The evolution is linear

$$U\left(\sum_j \alpha_j |v_j\rangle\right) = \sum_j \alpha_j U(|v_j\rangle)$$

and preserves the normalization constraint

$$\text{If } \sum_j \alpha_j U(|v_j\rangle) = \sum_j \alpha'_j |v_j\rangle \text{ then } \sum_j \|\alpha'_j\|^2 = 1$$

The state evolution postulate

Preservation of the normalization constraint means that unit length vectors (and thus orthogonal subspaces) are mapped by U to unit length vectors (and thus to orthogonal subspaces).

It also means that applying a transformation followed by a measurement in the transformed basis is equivalent to a measurement followed by a transformation.

This entails a condition on valid quantum operators: they must preserve the inner product, i.e.

$$(U|v\rangle, U|w\rangle) \;=\; \langle v|U^\dagger U|w\rangle \;=\; \langle v|w\rangle$$

which is the case iff U is unitary, i.e. $U^\dagger = U^{-1}$:

$$U^\dagger U = \; UU^\dagger \; = \; I$$

Unitarity

- Preserving the inner product means that a unitary operator maps orthonormal bases to orthonormal bases.
- Conversely, any operator with this property is unitary.
- If given in matrix form, being unitary means that the set of columns of its matrix representation are orthonormal (because the jth column is the image of $U|j\rangle$). Equivalently, rows are orthonormal (why?)

Unitarity

Unitarity is the only constraint on quantum operators: Any unitary matrix specifies a valid quantum operator.

This means that there are many non-trivial operators on a single qubit (in contrast with the classical case where the only non-trivial operation on a bit is complement).

Finally, because the inverse of a unitary matrix is also a unitary matrix, a quantum operator can always be inverted by another quantum operator

$\boxed{\text{Unitary transformations are reversible}}$

Building larger states from smaller

Operator U in the no-cloning theorem acts on a 2-dimensional state, i.e. over the composition of two qubits.

$\boxed{\text{What does composition mean?}}$

Postulate 3

The state space of a combined quantum system is the tensor product $V \otimes W$ of the state spaces V and W of its components.

Composing quantum states

State spaces in a quantum system combine through tensor: $\otimes$

$$\boxed{n\ m\text{-dimensional vectors} \quad \rightsquigarrow \quad \text{a vector in } m^n\text{-dimensional space}}$$

i.e. the state space of a quantum system grows exponentially with the number of particles: cf, Feyman's original motivation

Example

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \otimes \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ b \begin{bmatrix} d \\ e \\ f \end{bmatrix} \\ c \begin{bmatrix} d \\ e \\ f \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ad \\ ae \\ af \\ bd \\ be \\ bf \\ cd \\ ce \\ cf \end{bmatrix}$$

Composing quantum states

Tensor $V \otimes W$

- $B_{V\otimes W}$ is a set of elements of the form $|v_i\rangle \otimes |w_j\rangle$, for each $|v_i\rangle \in B_V$, $|w_i\rangle \in B_W$ and $\dim(\mathrm{V} \otimes \mathrm{W}) = \dim(\mathrm{V}) \times \dim(\mathrm{W})$
- $(|u_1\rangle + |u_2\rangle) \otimes |z\rangle \;=\; |u_1\rangle \otimes |z\rangle + |u_2\rangle \otimes |z\rangle$
- $|z\rangle \otimes (|u_1\rangle + |u_2\rangle) \;=\; |z\rangle \otimes |u_1\rangle + |z\rangle \otimes |u_2\rangle$
- $(\alpha|u\rangle) \otimes |z\rangle \;=\; |u\rangle \otimes (\alpha|z\rangle) \;=\; \alpha(|u\rangle \otimes |z\rangle)$
- $\langle(|u_2\rangle \otimes |z_2\rangle)|(|u_1\rangle \otimes |z_1\rangle)\rangle \;=\; \langle u_2|u_1\rangle\langle z_2|z_1\rangle$

Composing quantum states

Clearly, every element of $V \otimes W$ can be written as

$$\alpha_1(|v_1\rangle \otimes |w_1\rangle) + \alpha_2(|v_2\rangle \otimes |w_1\rangle) + \cdots + \alpha_{nm}(|v_n\rangle \otimes |w_m\rangle)$$

Example

The basis of $V \otimes W$, for V, W qubits with the computational basis is

$$\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$$

Thus, the tensor of $\alpha_1|0\rangle + \alpha_2|1\rangle$ and $\beta_1|0\rangle + \beta_2|1\rangle$ is

$$\alpha_1\beta_1|0\rangle \otimes |0\rangle \; + \; \alpha_1\beta_2|0\rangle \otimes |1\rangle \; + \; \alpha_2\beta_1|1\rangle \otimes |0\rangle \; + \; \alpha_2\beta_2|1\rangle \otimes |1\rangle$$

i.e., in a simplified notation,

$$\alpha_1\beta_1|00\rangle \; + \; \alpha_1\beta_2|01\rangle \; + \; \alpha_2\beta_1|10\rangle \; + \; \alpha_2\beta_2|11\rangle$$

Bases

The computational basis for a vector space

$$\underbrace{V \otimes V \otimes \cdots \otimes V}_{n}$$

corresponding to the composition of n qubits (each living in V) is the set

$$\{\underbrace{|0\rangle \cdots |0\rangle|0\rangle}_{n}, \underbrace{|0\rangle \cdots |0\rangle|1\rangle}_{n}, \underbrace{|0\rangle \cdots |1\rangle|0\rangle}_{n}, \cdots \underbrace{|1\rangle \cdots |1\rangle|1\rangle}_{n}\}$$

$$\stackrel{abv}{=}$$

$$\{\underbrace{|0 \cdots 00\rangle}_{n}, \underbrace{|0 \cdots 01\rangle}_{n}, \underbrace{|0 \cdots 10\rangle}_{n}, \cdots \underbrace{|1 \cdots 11\rangle}_{n}\}$$

which may be written in a compressed (decimal) way as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, \cdots |2^n - 1\rangle\}$$

Bases

The computational basis for a two qubit system would be

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$$

with

$$|0\rangle = |00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad |1\rangle = |01\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \quad |2\rangle = |10\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \quad |3\rangle = |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

Bases

There are of course other bases ... besides the standard one, e.g.

The Bell basis

$$
\begin{aligned}
|\Phi^+\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\
|\Phi^-\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \\
|\Psi^+\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \\
|\Psi^-\rangle &= \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)
\end{aligned}
$$

Compare with the Hadamard basis for the single qubit systems

Representing multi-qubit states

Any unit vector in a 2^n Hilbert space represents a possible n-qubit state, but for

… a certain level of redundancy

- As before, vectors that differ only in a global phase represent the same quantum state
- but also the same phase factor in different qubits of a tensor product represent the same state:

$$|u\rangle \otimes (e^{i\Phi}|z\rangle) \;=\; e^{i\Phi}(|u\rangle \otimes |z\rangle) \;=\; (e^{i\Phi}|u\rangle) \otimes |z\rangle$$

Actually, phase factors in qubits of a single term of a superposition can always be factored out into a coefficient for that term, i.e. phase factors distribute over tensors

Representing multi-qubit states

Representation

- Relative phases still matter (of course!)

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \quad \text{differs from} \quad \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + |11\rangle)$$

  even if

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(e^{i\Phi}|00\rangle + e^{i\Phi}|11\rangle) = \frac{e^{i\Phi}}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- The complex projective space of dimension 1 (depicted in the Block sphere) generalises to higher dimensions, although in practice linearity makes Hilbert spaces easier to use.

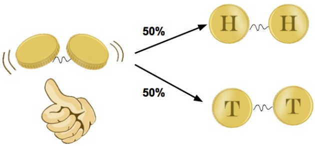
Entanglement

Most states in $V \otimes W$ cannot be written as $|u\rangle \otimes |z\rangle$

For example, the Bell state

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \;=\; \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

is entangled



Entanglement

Actually, to make $|\Phi^+\rangle$ equal to

$$(\alpha_1|0\rangle+\beta_1|1\rangle)\otimes(\alpha_2|0\rangle+\beta_2|1\rangle) \;=\; \alpha_1\alpha_2|00\rangle+\alpha_1\beta_2|01\rangle+\beta_1\alpha_2|10\rangle+\beta_1\beta_2|11\rangle$$

would require that $\alpha_1\beta_2 = \beta_1\alpha_2 = 0$ which implies that either

$$\alpha_1\alpha_2 = 0 \;\text{ or }\; \beta_1\beta_2 = 0$$

Note
Entanglement can also be observed in simpler structures, e.g. relations:

$$\{(a, a), (b, b)\} \subseteq A \times A$$

cannot be separated, i.e. written as a Cartesian product of subsets of A.