Mathematics for Computer Science

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Set theory

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Basic concepts

- Definition by enumeration and comprehension

$$\{E(x) \mid x : X \wedge R\}$$

yields the set of values that result from evaluating $E[x := v]$ in the state for each value $v \in X$ such that $R[x := v]$ holds. The notation (similar to the one used for quantifiers) is often simplified to $\{E(x) \mid x : X \wedge R\}$

- Set membership and set equality

(membership)

$$y \in \{E(x) \mid x \in X \wedge R\} \equiv \langle \exists x : X \ : \ R \ : \ y = E(x) \rangle$$

(extensionality) $\quad S = T \ \equiv \ \langle \forall x : X \ :: \ x \in S \equiv x \in T \rangle$

Basic concepts

- Sets and predicates (characteristic predicate)

$$\text{(characteristic)} \qquad y \in \{x \mid R\} \equiv R[x := y]$$

Operations on sets

(cardinal)	$\#S = \langle \Sigma x : X \ : \ X \in S \ : \ 1 \rangle$
(subset)	$S \subseteq T \equiv \langle \forall x : X \ : \ x \in S \ : \ x \in T \rangle$
(complement)	$x \in \ \ S \equiv x \notin S$
(union)	$x \in S \cup T \equiv x \in S \vee x \in T$
(intersection)	$x \in S \cap T \equiv x \in S \wedge x \in T$
(difference)	$x \in S - T \equiv x \in S \wedge x \notin T$
(powerset)	$S \in \mathcal{P} T \equiv S \subseteq T$

Distributed operations on sets

$\bigcup$ $(\mathcal{P}(X), \cup, \emptyset)$ is an Abelian monoid

$\bigcap$ $(\mathcal{P}(X), \cap, \mathit{UNIVERSE})$ is an Abelian monoid

Predicates and sets revisited

$\boxed{\text{Exercise: point out the duality!}}$

(implication vs subset) $\{x \mid P\} \subseteq \{x \mid Q\} \quad \equiv \quad \langle \forall\, x : X \;::\; P \Rightarrow Q \rangle$

Paradoxes in set theory

$$S = \{x \mid x \notin x\}$$

i.e. the set of all sets which do not contain themselves as elements: $S \in S \equiv S \notin S$

$\boxed{\text{Parodox of Zermelo-Russell (1901)}}$

- Variants (e.g. the barber paradox; *this statement is false*), etc.
- self-referencial phenomena vs types