Mathematics for Computer Science

José Proença & Alexandre Madeira
(slides from Luis Soares Barbosa)

Functions & Relations
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Function

- concept of function
- composition
- identity
- diagrams and laws

Injective and surjective functions

- Intuition and examples
- Invertibility of a function
- Injectiveness $\equiv$ Left cancellation
- Sobrejectiveness $\equiv$ Right cancellation

Bijections and isomorphism

- Bijective functions
- Alternative characterisations of injective and sobrejective functions

Direct and inverse image of a function

- Definitions
- Examples
- Laws

Isomorphism and cardinality

- Definition: $\#A = \#B \;\equiv\; A \cong B$
- Denumerable, infinite denumerable and uncountable sets; basic examples and results
- Isomorphic sets
- Fundamental constructions on sets: Cartesian product $(A \times B)$, disjoint union $(A + B)$ and function spaces (B^A)

From functions to relations

- (Total) functions: $f : A \longrightarrow B$
- Partial functions: $f : A \rightharpoonup B \;\cong\; f_\perp : A \longrightarrow B + \mathbf{1}$
- Relations: $R \subseteq A \times B \;\cong\; f_R : A \longrightarrow \mathcal{P}(B)$

Relations

- concept of relation
- composition (with diagrammatic notation):

$$R \cdot S = \{(x, y) : \exists z \text{ such that } (x, z) \in R \text{ and } (x, y) \in S\}$$

- composition is monotonic:

$$R \subseteq S \;\wedge\; T \subseteq U \;\;\Rightarrow\;\; (R \cdot T) \subseteq (S \cdot U)$$

- identity and co-reflexives
- diagrams and laws

Relations

Converse

$$
\begin{array}{rcll}
X^\circ \subseteq Y & \equiv & X \subseteq Y^\circ & \boxed{\ ^\circ\text{-universal}} \\
R \subseteq S & \equiv & R^\circ \subseteq S^\circ & (\ ^\circ\text{-monotonicity}) \\
R^{\circ\circ} & \equiv & R & (\ ^\circ\text{-involution}) \\
(R \cap S)^\circ & \equiv & R^\circ \cap S^\circ & (\cap,\ ^\circ\text{-distributivity})
\end{array}
$$

Relations

Kernel and Image

For a relation $A \xrightarrow{R} B$:

$A \xrightarrow{\ker(R)} A$ share same image under R $\qquad \ker(R) \stackrel{\mathrm{def}}{=} R \cdot R^{\circ}$

$B \xrightarrow{\mathrm{img}(R)} B$ are images of same point under R $\qquad \mathrm{img}(R) \stackrel{\mathrm{def}}{=} R^{\circ} \cdot R$

Properties:

$$R \subseteq S \;\Rightarrow\; \ker(R) \subseteq \ker(S)$$
$$R \subseteq S \;\Rightarrow\; \mathrm{img}(R) \subseteq \mathrm{img}(S)$$

Relations

Classification

reflexive:	iff $\mathrm{id}_A \subseteq R$
coreflexive:	iff $R \subseteq \mathrm{id}_A$
transitive:	iff $R \cdot R \subseteq R$
anti-symmetric:	iff $R \cap R^\circ \subseteq \mathrm{id}_A$
symmetric:	iff $R = R^\circ$
connected:	iff $R \cup R^\circ = A \times A$

Relations

Classification

- **Preorders** are reflexive and transitive relations.
- **Partial** orders are anti-symmetric preorders
- **Linear** or **total** orders are connected partial orders
- **Equivalences** are symmetric preorders
- **Entire or Total** relations: $\mathrm{id} \subseteq \ker(R)$
- **Simple or Functional** relations: $\mathrm{img}(R) \subseteq \mathrm{id}$

Back to functions

Classification

Functions are both simple and entire relations:

$$\underbrace{id \subseteq f \cdot f^\circ}_{\text{entire}} \quad \wedge \quad \underbrace{f^\circ \cdot f \subseteq id}_{\text{simple}}$$

Surjectiveness : R° entire

Injectiveness : R° simple

R entire and injective : $\ker(R) = \mathrm{id}$

R simple and surjective : $\mathrm{img}(R) = \mathrm{id}$

f bijective : $\ker(R) = \mathrm{id}$ and $\mathrm{img}(R) = \mathrm{id}$