Mathematics for Computer Science

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(slides from Luis Soares Barbosa)

What is a logic?

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Logical system

- a set of symbols,
- a set of formulas constructed from the symbols,
- a set of distinguished formulas called axioms, and
- a set of inference rules.

Logical system

Notes

- The set of formulas is called the language of the logic.
- The language is defined syntactically; there is no notion of meaning or semantics in a logic per se.
- Inference rules specify how formulas are derived from other formulas.
- A formula is a theorem of the logic if it is an axiom or if it can be generated from the axioms and already proved theorems using the inference rules.
- A proof that a formula is a theorem is an argument that shows how the inference rules are used to generate the formula.

Logical system

Consistency

A logic is consistent if at least one of its formulas is a theorem and at least one is not; otherwise, the logic is inconsistent.
Adding $\textsc{True} = \textsc{False}$ as an axiom makes the logic inconsistent.

This course examples

Propositional logic

Predicate logic

A less common example

Symbols: $A, B, -$

Formulas: sequences of the form $aAbBc$, where a, b, c denote finite sequeces of zero or more dashes

Axioms:
1. $-A-B--$
2. $--A-B---$

Inference rule:

$$\frac{aAbBc, dAeBf}{adAbeBcf}$$

Models

- The formulas of a logic are intended to be statements about some domain of discourse, that is, some area of interest. We give the formulas a meaning with respect to this domain by defining which formulas are true statements and which are false statements about the domain.
- An interpretation assigns meaning to the operators, constants, and variables of a logic.
- Because a logic is purely a syntactic object, it may have more than one interpretation.
- In a logic in which formulas have variables, an interpretation associates a value with each variable. Conventionally, we split such an interpretation into two parts: one gives a fixed meaning to the operators and constants; the other supplies values for variables, i.e. denotes a state.

A less common example

Interpretation A: A formula $aAbBc$ is mapped to $\#a + \#b = \#c$, where $\#x$ denotes the number of dashes in sequence x

Interpretation B: A formula $aAbBc$ is mapped to $\textsc{True}$ iff $\#a + \#b \leq \#c$

Play!

Compute the value of $-A - B - --$ in both interpretations.

Satisfiability and validity

- Let S be a set of interpretations for a logic and F be a formula of the logic. F is satisfiable (under S) iff at least one interpretation of S maps F to TRUE.
- F is valid (under S) iff every interpretation in S maps F to TRUE.
- An interpretation is a model for a logic iff every theorem is mapped to TRUE by the interpretation.

Soundness and completeness

- A logic is sound iff every theorem is valid. Soundness means that the theorems are true statements about the domain of discourse.
- A logic is complete iff every valid formula is a theorem. Completeness means that every valid formula can be proved.

A sound and complete logic allows exactly the valid formulas to be proved. Failure to prove that a formula is a theorem in such a logic cannot be attributed to weakness of the logic. Unfortunately, many domains of discourse of concern to us do not have sound and complete axiomatizations. This is a consequence of Godel's incompleteness theorem, which states that no formal logical system that axiomatizes arithmetic can be both sound and complete.