

Lecture 2: Superposition and quantum interference

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Quantum computation: The *motto*

Information is a physical entity

Any computation is a physical process

Thus, any abstract computational model has always to reflect what we know about reality, and quantum theory is our current best tool in this road.

Let's revisit 3 previous slides

— on the quantum representation of information ...

Quantum information is a different story

A quantum state holds the information of **both** possible classical states:



A unit of information lives in a 2-dimensional **complex** vector space:

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle$$

and thus possesses a **continuum of possible values**, so potentially, can store lots of classical data.

Quantum information is a different story

However, all this potential is **hidden**:

when **observed** $|v\rangle$ **collapses** into a classic state: $|0\rangle$, with probability $\|\alpha\|^2$, or $|1\rangle$, with probability $\|\beta\|^2$.

(Recall: $\|\alpha\| = \sqrt{\alpha\bar{\alpha}}$ for a complex α)

The outcome of an observation is **probabilistic**, which calls for a restriction to **unit** vectors, i.e. st

$$\|\alpha\|^2 + \|\beta\|^2 = 1$$

to represent quantum states.

Quantum information is a different story

This quantum state is **not** a probabilistic mixture: it is **not** true that the state is really either $|0\rangle$ or $|1\rangle$ and we just do not happen to know which.

Amplitudes are not real numbers (e.g. probabilities) that can only increase when added, but **complex** so that they can **cancel each other or lower their probability**, thus capturing a fundamental **quantum resource**:

interference

Why aren't probabilities enough?

Computation is always a physical process

That's our *motto!*

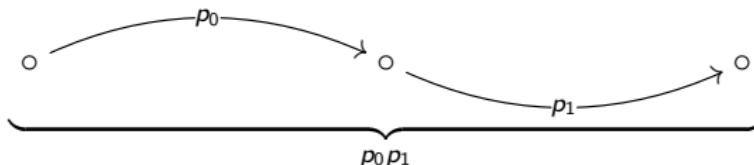
In several cases, the language of **probability theory** can describe the actual **physical evolution of a system**, i.e.

- **Physics** identify the system's structure and assigns numerical probabilities to elementary transition steps.
- **Probability theory**, i.e. the Kolmogorov axioms, ensure mathematical consistency and helps in calculating probabilities along paths of evolution.

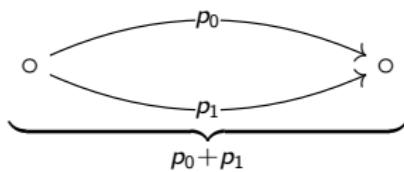
but **not always!**

Probabilistic systems, probabilistic computation

Sequential paths



Alternative, mutually exclusive paths



(Kolmogorov (1903-1987) additivity axiom)

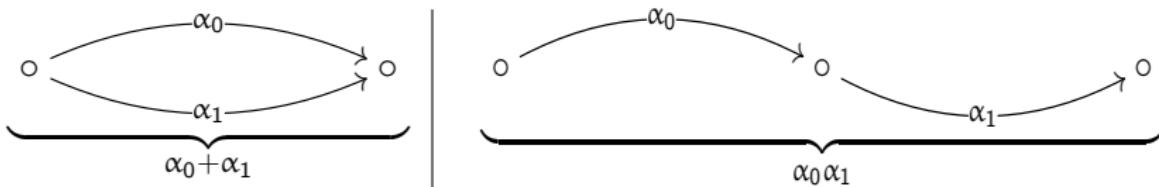
Quantum systems, quantum computation

Many common quantum phenomena, however, cannot be described this way, but are accommodated by a **modified 'probability theory'**:

Transitions are labelled by **complex numbers**, called their **amplitudes**, whose **norms squared** are interpreted as **transition probabilities** through

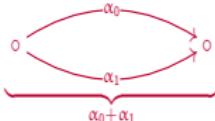
$$\text{Born's rule} \quad p = \|\alpha\|^2$$

(for Max Born, 1882-1970)



Quantum systems, quantum computation

Let's compute the total probability in



$$\begin{aligned} p &= \|\alpha_0 + \alpha_1\|^2 \\ &= \overline{(\alpha_0 + \alpha_1)}(\alpha_0 + \alpha_1) \\ &= (\overline{\alpha_0} + \overline{\alpha_1})(\alpha_0 + \alpha_1) \\ &= \|\alpha_0\|^2 + \|\alpha_1\|^2 + \overline{\alpha_0}\alpha_1 + \alpha_0\overline{\alpha_1} \\ &= p_0 + p_1 + \|\alpha_0\| \|\alpha_1\| \left(e^{i(\varphi_1 - \varphi_0)} + e^{-i(\varphi_1 - \varphi_0)} \right) \\ &= p_0 + p_1 + \underbrace{2\sqrt{p_0 p_1} \cos(\varphi_1 - \varphi_0)}_{\text{interference}} \end{aligned}$$

(expressing α_j in polar form $\|\alpha_j\| e^{i\theta_j}$
and resorting to $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$)

Quantum systems, quantum computation

$$p = p_0 + p_1 + \underbrace{2\sqrt{p_0 p_1} \cos(\varphi_1 - \varphi_0)}_{\text{interference}}$$

- The total probability is the sum of the probabilities of the individual transitions modified by the interference term.
- Depending on term $\varphi_1 - \varphi_0$ the interference can be either negative or positive.
- The important quantity is the relative phase $\varphi_1 - \varphi_0$ rather than individual φ_0, φ_1 .
- If the system's evolution depends only on that difference then the system must have, somehow, experienced both paths.

Quantum systems, quantum computation

Indeed, the **probabilistic** assumption that one or the other transition occurs, or that the system presents one or the other configuration, but **we just do not know which one**, is inconsistent with many experiments:

Typically, a quantum state is a **superposition** of basic states, i.e. states that, mathematically, form a **basis** of the vector space in which such states live — e.g. $\{ |e_i\rangle \mid i = 1 \cdots n \}$

$$|\varphi\rangle = \sum_i \alpha_i |e_i\rangle$$

Recall



Superposition in action: A random number generator

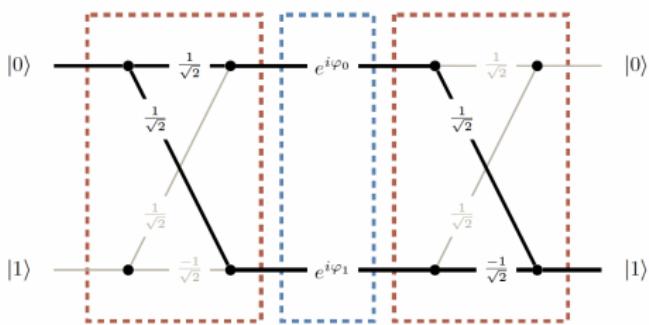
- Prepare quantum state

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

- Measure $|+\rangle$ in the computational basis to obtain either 0 or 1 with equal probability ($\left\|\frac{1}{\sqrt{2}}\right\|^2 = 0.5$)

This algorithm produces a perfect random number even though no randomness has been used inside it.

Ramsey interference experiment

(from [Ekert *et al*, 2024])

Atoms are sent through two separate resonant interaction zones (which attempt to commute the atom between ground and excited states), separated by an intermediate dispersive interaction zone (which performs a phase shift).

The atoms are subsequently measured and found to be in one of the two basic energy states labeled as $|0\rangle$ and $|1\rangle$.

What is the probability of an atom going from $|0\rangle$ to $|1\rangle$?

First compute the **amplitudes**:

$$\begin{aligned} A_{10} &= \frac{1}{\sqrt{2}} e^{i\varphi_0} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} e^{i\varphi_1} \frac{-1}{\sqrt{2}} \\ &= \frac{1}{2} (e^{i\varphi_0} - e^{i\varphi_1}) \\ &= \frac{1}{2} \left(e^{i\frac{\varphi_0+\varphi_1}{2}} e^{i\frac{\varphi_0-\varphi_1}{2}} - e^{i\frac{\varphi_0+\varphi_1}{2}} e^{-i\frac{\varphi_0-\varphi_1}{2}} \right) \\ &= \frac{1}{2} e^{i\frac{\varphi_0+\varphi_1}{2}} \left(e^{i\frac{\varphi_0-\varphi_1}{2}} - e^{-i\frac{\varphi_0-\varphi_1}{2}} \right) \\ &= \frac{1}{2} e^{i\frac{\varphi_0+\varphi_1}{2}} \left(2i \sin \frac{\varphi_0 - \varphi_1}{2} \right) \\ &= -ie^{i\frac{\varphi_0+\varphi_1}{2}} \sin \frac{\varphi_1 - \varphi_0}{2} \end{aligned}$$

resorting to $x = \frac{x+y}{2} + i\frac{x-y}{2}$
and Euler formula $e^{i\varphi} = \cos \varphi + i \sin \varphi$

What is the probability of an atom going from $|0\rangle$ to $|1\rangle$?

Then the **probability**:

$$\begin{aligned} P_{10} &= \|A_{10}\|^2 \\ &= \left\| -ie^{i\frac{\varphi_0 + \varphi_1}{2}} \sin \frac{\varphi_1 - \varphi_0}{2} \right\|^2 \\ &= \left\| \sin \frac{\varphi_1 - \varphi_0}{2} \right\|^2 \\ &= \frac{1}{2} - \underbrace{\frac{1}{2} \cos(\varphi_1 - \varphi_0)}_{\text{interference}} \end{aligned}$$

resorting to $\cos 2\theta = 1 - 2 \sin^2 \theta$

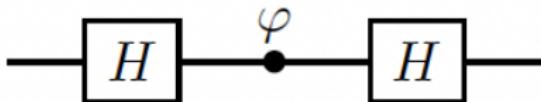
A bulk calculation ...

The results of these calculations for all pairs begin-end states can be expressed as follows:

- the effect of each interaction is described by a **matrix of transition amplitudes**, and
- one resorts to **matrix multiplication** to compose the sequence of independent interactions.

$$\begin{aligned} A &= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} e^{i\varphi_0} & 0 \\ 0 & e^{i\varphi_1} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \\ &= e^{i\frac{\varphi_0 + \varphi_1}{2}} \begin{bmatrix} \cos \frac{\varphi}{2} & -i \sin \frac{\varphi}{2} \\ -i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{bmatrix} \quad \text{making } \varphi = \varphi_1 - \varphi_0 \\ &\equiv \begin{bmatrix} \cos \frac{\varphi}{2} & -i \sin \frac{\varphi}{2} \\ -i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{bmatrix} \quad \text{ignoring global phase, to discuss later ...} \end{aligned}$$

My first quantum circuit



- A **wire** represents a two-dimensional **state** (a **qubit**)
- Three **gates** describing quantum **operations**:

$$H = \underbrace{\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}}_{\text{Hadamard gate}} \quad \text{and} \quad P_\varphi = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{bmatrix}}_{\text{Phase shift gate}}$$

A simple matrix multiplication yields

$$A = HP_\varphi H = \begin{bmatrix} \cos \frac{\varphi}{2} & -i \sin \frac{\varphi}{2} \\ -i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$$

My first quantum circuit

which, expressed in a functional way, gives

$$A|0\rangle = \cos \frac{\varphi}{2}|0\rangle + -i \sin \frac{\varphi}{2}|1\rangle$$

$$A|1\rangle = -i \sin \frac{\varphi}{2}|0\rangle + \cos \frac{\varphi}{2}|1\rangle$$

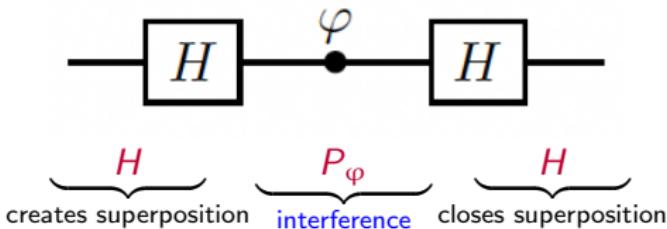
as read from

$$A = \textcolor{red}{H} P_{\varphi} H = \begin{bmatrix} \cos \frac{\varphi}{2} & -i \sin \frac{\varphi}{2} \\ -i \sin \frac{\varphi}{2} & \cos \frac{\varphi}{2} \end{bmatrix}$$

Clearly, for $\varphi = 0$, i.e. in the absence of any phase shift, $A|0\rangle = |0\rangle$ and $A|1\rangle = |1\rangle$, leading to conclude that

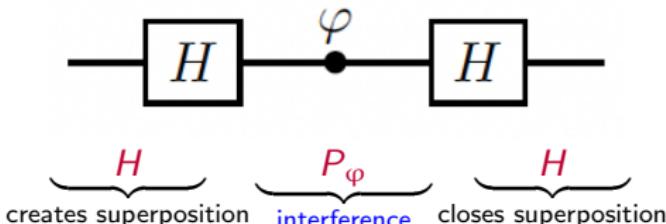
$$\textcolor{blue}{HH} = Id$$

The golden pattern



- H creates/closes a uniform superposition: it is the source of a natural parallelism,
- but the crucial role in controlling interference is located in P_φ .

Lessons learnt for what follows



generalizes to a basic **recipe** in quantum algorithms (to be discussed).

What is quantum computation?

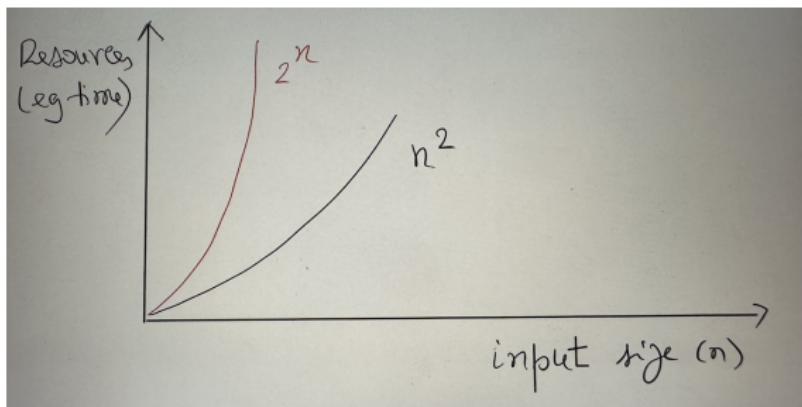
A complex multi-particle quantum **interference involving many computational paths** through a computing device.

Its challenge is **to shape quantum interference** through a sequence of computational steps, enhancing probabilities of the **correct** outputs and suppressing probabilities of the **wrong** ones.

Superposition and interference what for?

Quantum computing has nothing to say on the boundaries of **computability**, but reduces the **complexity** of algorithms to address several problems.

The limits of practicality



Paying a visit to old acquaintances

EXP Problems requiring exponential time to be solved

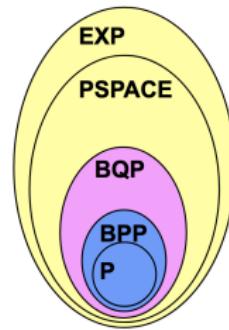
PSPACE Problems requiring polynomial space to be solved

BQP Problems solved in quantum machines in bounded error polynomial time

BPP Problems solved in probabilistic machines in bounded error polynomial time

P Problems solved in deterministic classical machines in polynomial time

$$P \subseteq BPP \subseteq BQP \subseteq PSPACE \subseteq EXP$$

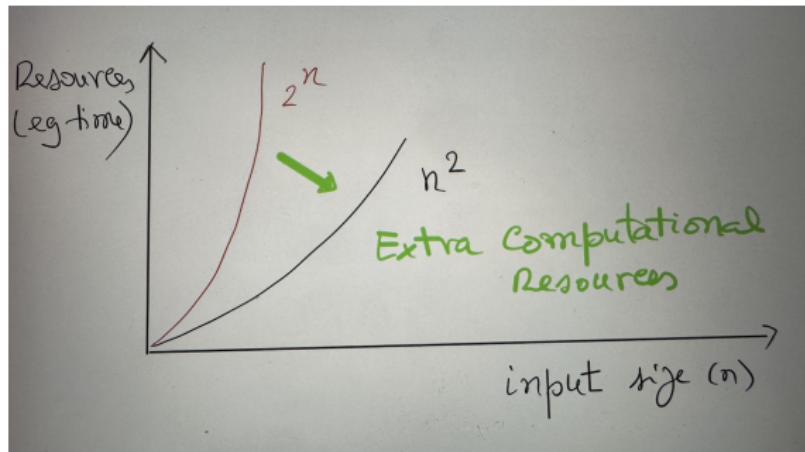


Superposition and interference as computational resources

A problem does not become tractable by increasing computational power:
more of the same keeps similar circumstances.

New ideas & new resources make a difference

The key: Quantum effects as computational resources



Decoherence

The laws of quantum computation just described assume a **perfectly isolated system**.

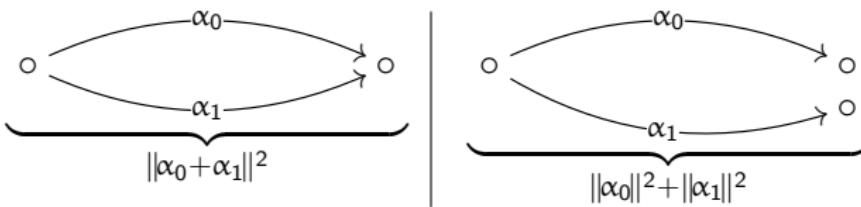
- In practice, any complex quantum system cannot avoid some spurious **interactions with the environment**.
- Thus, when computing the system's evolution, one must take into account not only its internal configurations but also those of its environment.

This is known as **decoherence** and explains why

- we hardly see quantum interference on a daily basis
- and isolation is a main technological challenge in the design/construction of quantum computing devices.

Two scenarios

- Perfect isolation: the environment does not hold any physical record of the path taken to produce an output.
- Total decoherence: the system produces an output but the environment has a record of the path taken: there are no alternative ways to reach that output, but a single path to each of the following situations (joint states):
 1. The output was produced, but the environment knows the path labelled by α_0 was taken,
 2. or similarly for α_1 .



Two scenarios

- Perfect isolation: amplitudes are added and generate a probability for the output to be produced, taking both paths in superposition
- Total decoherence: probabilities are computed for each path and summed as such — interference provided by superposition is lost.

The general scenario

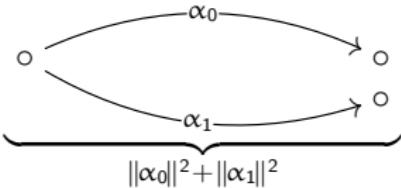
Interference is limited by a visibility parameter — v — ranging from

- 0: total decoherence; no interference
- 1: no decoherence; full interference

The effect of decoherence

$$p_0 + p_1 + 2\textcolor{red}{v}\sqrt{p_0 p_1} \cos(\phi_1 - \phi_0)$$

Parameter $\textcolor{red}{v}$ quantifies the **degree of distinguishability between the disrupted outputs**: *the more the environment knows the less interference is observed more classic becomes the overall computation.*



Fact

Decoherence destroys (quantum) interference.